Approximations for singularly perturbed parabolic equations of arbitrary order

MICHAEL A KOURITZIN*
514 VINCENT HALL
INSTITUTE FOR MATHEMATICS AND ITS APPLICATIONS
UNIVERSITY OF MINNESOTA
MINNEAPOLIS, MINNESOTA 55455-0436
KOURITZI@IMA.UMN.EDU

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ABSTRACT. Approximations for singularly perturbed parabolic equations

$$\partial_\tau v^\varepsilon(\tau,x) = \varepsilon \sum_{|k|\le 2p} A_k(\tau,x)\,\partial_x^k v^\varepsilon(\tau,x), \quad v^\varepsilon(0,x)=\varphi(x) \tag{1}$$

are studied over regions $(0,T/\varepsilon]\times\mathbb{R}^d$. Time scaling reduces this equation to

$$\partial_t u^\varepsilon(t,x) = \sum_{|k|\le 2p} A_k(t/\varepsilon,x)\,\partial_x^k u^\varepsilon(t,x), \quad u^\varepsilon(0,x)=\varphi(x) \tag{2}$$

on $(0,T]\times\mathbb{R}^d$. Under mild conditions, (2) has a $\mathbb{C}^N$-valued solution whose arbitrary order x-derivatives are compared (as $\varepsilon \searrow 0$ on $(0,T]\times\mathbb{R}^d$) to like derivatives of the unique solution of an "averaged" equation

$$\partial_t u(t,x) = \sum_{|k|\le 2p} A_k^0(x)\,\partial_x^k u(t,x), \quad u(0,x)=\varphi(x). \tag{3}$$

We show that $\partial_x^m(u^\varepsilon - u)$ tends to zero (as $\varepsilon\to 0$) uniformly over $(0,T]\times\mathbb{R}^d$ in a weighted Hölder-continuity norm or, in the unscaled case, that $\partial_x^m[v^\varepsilon(\tau,x) - u(\varepsilon\tau,x)]$ converges to 0 uniformly on $(0,T/\varepsilon]\times\mathbb{R}^d$. Coincidently, we develop some new bounds for the fundamental solutions of arbitrary order parabolic equations. Moreover, this work applies to stochastic partial differential equations and fluctuation results for parabolic equations with random coefficients.

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1. Introduction

According to Sanders and Verhulst [13] p. 181 ff., the method of averaging developed slowly out of formal, eighteenth century perturbation techniques for finding approximate solutions to nonlinear ordinary differential equations. In fact, even in the periodic (for $s \to F(x,s)$) case, the first proof of asymptotic (as $\varepsilon \to 0$) validity for the *classical averaging principle*, stating that the unique continuous solution to

$$\dot{X}_t^\varepsilon = F(X_t^\varepsilon, t/\varepsilon), \qquad X_0^\varepsilon = x_0 \tag{4}$$

can be approximated over $[0,T]$, $T>0$ by an "averaged" differential equation

$$\dot{X}_t = \overline{F}(X_t), \qquad X_0 = x_0, \qquad \overline{F}(x) \doteq \lim_{T\to\infty} \frac{1}{T}\int_0^T F(x,t)\,dt, \tag{5}$$

had to await the 1928 work of Fatou [4]. The general case was not rigorously proven until the 1945 work of Bogoliubov [2]. However, more recently, averaging theories have also been developed for many branches of nonlinear analysis, stochastic differential equations, and partial differential equations. In this note, we are interested in averaging for arbitrary-order parabolic partial differential equations. Suppose $m = (m_1, ..., m_d) \in \mathbb{N}^d$ is arbitrary; u^ε is a $\mathbb{C}^N$-valued continuous, bounded solution of

$$\partial_t u^\varepsilon(t,x) = \sum_{|k|\le 2p} A_k(t/\varepsilon, x)\partial_x^k u^\varepsilon(t,x) \quad \text{subject to } u^\varepsilon(0,x) = \varphi(x); \tag{6}$$

and u is the unique continuous bounded solution of the "averaged" parabolic equation

$$\partial_t u(t,x) = \sum_{|k|\le 2p} A_k^0(x)\partial_x^k u(t,x) \quad \text{subject to } u(0,x) = \varphi(x). \tag{7}$$

Then, we develop a rigorous theory for comparing $\partial_x^m u^\varepsilon(t,x)$ uniformly over $(0,T]\times\mathbb{R}^d$ to $\partial_x^m u(t,x)$ in a weighted Hölder continuity norm as $\varepsilon \to 0$.

Apparently, the first rigorous averaging theorem for partial differential equations was developed by Khas'minskii [8]. Indeed, in Theorem 2 of [8] he considered averaging for systems of scalar, second order parabolic equations of the form

$$\begin{aligned}\partial_t u^\varepsilon(t,x) &= \sum_{i,j=1}^d a_{ij}(t/\varepsilon,x)\partial_{x_i x_j} u^\varepsilon(t,x) + \sum_{j=1}^d b_j(t/\varepsilon,x)\partial_{x_j} u^\varepsilon(t,x) \\ &\quad + c(t/\varepsilon,x)u^\varepsilon(t,x) + d(t/\varepsilon,x)\end{aligned} \tag{8}$$

over $(0,T)\times\mathbb{R}^d$ and subject to some final condition. The proof was far from direct but rather the averaging result was inferred from a theorem on continuous dependence which in turn was proved using probabilistic representations and methods.

Later, Bensoussan et. al. [1] pp. 516-533, Kurtz (as an application of an abstract theorem) [12], and Henry [7] pp. 218-222 established averaging principles for (8) under alternative conditions and using different methods. More recently, Zhikov et. al. [14] consider averaging for arbitrary order parabolic equations but in a setting much different than ours. Most notably, they consider weak solutions to $\mathbb{R}$-valued equations instead of strong solutions to $\mathbb{C}^N$-valued equations, weak and $\mathcal{L}^1$ types of convergence in place of uniform convergence in a Hölder continuity norm, and specific parabolic operators instead of arbitrary order derivatives. Finally, Kouritzin [9] established a general averaging principle for the fundamental solutions to the system of Equations (6) as well as for their x-derivatives up to order $2p-1$. This could be used to prove $|\partial_x^m[u^\varepsilon(t,x)-u(t,x)]| \le \gamma(\varepsilon)\, t^{-|m|/2p}\ (1+|x|^2)^\nu$ for any $\nu > 0$ and some function $\gamma(\varepsilon) \to 0$ as $\varepsilon \to 0$ provided $0 \le |m| \doteq m_1 + m_2 + ... + m_d < 2p$. However; this bound explodes as $t \to 0$ and for many applications in e.g. stochastic partial differential equations, parabolic equations with random coefficients, and stochastic diffusion equations; it is necessary to establish $\partial_x^m(u^\varepsilon - u) \overset{\varepsilon\to 0}{\rightarrow} 0$ uniformly over $(0,T]\times\mathbb{R}^d$ in a Hölder continuity norm *when* $|m| \ge 2p$ (see Theorem 13 of Dawson and Kouritzin [3]). We establish this convergence result within for arbitrary $|m| \ge 0$ under general conditions.

It is well established (see Theorems 9.4.2 and 9.4.3 vis-à-vis Theorem 9.6.7 of Friedman [6]) that the relatively wieldy methods and mild conditions required in substantiating existence and bounds for the lower order derivatives of solutions to parabolic equations are deficient for the higher order derivatives. Therefore, apropos of Kouritzin [9], one can almost foresee that neither its methods will adapt nor its conditions on the coefficients of (6) and (7) be adequate for our current problem. Still, our current development does retain a certain resemblance to the classical methods and, hereby, we can be relatively certain that our conditions are general. Indeed, the coefficients of (6) and (7) need not satisfy conditions more encumbering than those required to show existence and uniqueness to the Cauchy problem for (6) and (7) and to establish the correct derivative bounds on their fundamental solutions. Moreover, we only impose one additional very mild approximation hypothesis stating that

$$\frac{1}{M}\int_0^M \left[A_k(s,y) - A_k^0(y)\right]\, ds \to 0 \ \text{ as } M \to \infty, \tag{9}$$

for each $y \in \Re^d$ (i.e. pointwise).

In the next section we explain our notation, list our conditions, and state our main result. Section 3 contains an outline of our proof which becomes rigorous and full with the addition of the results in Section 4. Naturally, these details are referenced when required throughout the main proof. Actually, these subsidiary results contain the most involved steps and are of independent interest. In fact, the bounds for

u^ε and for the fundamental solutions Γ^ε, Γ derived in Lemma 2 and Proposition 3 respectively have already been used in Dawson and Kouritzin [3] and Kouritzin [10].

2. Notation, Conditions, and Result

Throughout this note; p, N, and d are fixed positive integers; $(l_1, ..., l_d)$ and $(b_1, ..., b_d)$ are fixed vectors of non-negative integers such that $0 \le l_1 + l_2 + \cdots + l_d \le 2p$; and $|\cdot|$ denotes absolute value as well as modulus. For technical reasons it will be most convenient to define our norms on $\mathcal{C}^N$ and $\mathcal{C}^d$ via

$$|\zeta| \doteq \left[\sum_{j=1}^{N} |\zeta_j|^\chi\right]^{\frac{1}{\chi}} \qquad \text{and} \qquad |x| \doteq \left[\sum_{j=1}^{d} |x_j|^\chi\right]^{\frac{1}{\chi}}, \qquad \chi = \frac{2p}{2p-1} \tag{10}$$

for $\zeta \in \mathcal{C}^N$ and $x \in \mathcal{C}^d$. Then, $||\cdot||$ will be used for the $|\cdot|$−induced norm for $\mathcal{C}^{N\times N}$ matrices. $a \vee b$ and $a \wedge b$ denote the maximum respectively minimum of two real numbers a, b, and $a_{m,n} \overset{n,m}{\ll} b_{m,n}$ implies that there is a constant $c > 0$ such that $|a_{m,n}| \le c\,|b_{m,n}|$ for all n, m. The latest notation extends the Vinogradov symbol.

For vectors $k = (k_1, k_2, ..., k_d)$ of non-negative integers, we define

$$|k| \doteq k_1 + k_2 + \cdots + k_d \tag{11}$$

and let "$\sum_{|k|\le 2p}$" denote the summation over all possible d-tuples k of non-negative integers such that $|k| \le 2p$. (It will always be clear from the context whether $|\cdot|$ is being used as absolute value, modulus, norm in $\mathcal{C}^N$, norm in $\mathcal{C}^d$, or the sum of non-negative integers). Moreover, we also introduce a partial ordering on this set of non-negative integer vectors by letting $k \le m$ denote the relation $k_i \le m_i$ for all $i = 1, 2, ..., d$ and $k < m$ mean $k \le m$ and $k \ne m$. Then, we can define for all $k^1 + k^2 + k^3 = k$ with $k^i = (k_1^i, ..., k_d^i)$

$$\binom{k}{k^1\, k^2\, k^3} = \frac{k_1!}{k_1^1!\, k_1^2!\, k_1^3!}\, \frac{k_2!}{k_2^1!\, k_2^2!\, k_2^3!} \cdots \frac{k_d!}{k_d^1!\, k_d^2!\, k_d^3!} \tag{12}$$

Next, using the Schwartz multi-index notation with k as above, we define

$$\partial_x^k \doteq \partial_{x_1}^{k_1}\, \partial_{x_2}^{k_2} \cdots \partial_{x_d}^{k_d}\,. \tag{13}$$

Letting f denote any continuous function on $(0, T] \times \mathbb{R}^d$, we define for each $\gamma, \nu > 0$

$$|f|_{l,\gamma} \doteq \sup_{\substack{0<t\le T\\ x\in\mathbb{R}^d}} \frac{|t^{\frac{(|l|-\gamma/2)\vee 0}{2p}} f(t,x)|}{(1+|x|^2)^\nu} + \sup_{\substack{t,t'\\ 0<|x-x'|\le 1}} \frac{|t^{\frac{(|l|-\gamma/2)\vee 0}{2p}} f(t,x) - t'^{\frac{(|l|-\gamma/2)\vee 0}{2p}} f(t',x')|}{(1+|x|^2)^\nu \left[|t-t'|^{\frac{\gamma}{2p}} + |x-x'|^\gamma\right]}. \tag{14}$$

The following Conditions will be assumed throughout this note:

(C1) The system (6) is uniformly parabolic in the sense that

$$-\sup_{t\geq 0}\sup_{x\in\mathbb{R}^d}\max_j\sup_{|\xi|=1}\ \lambda_j(\xi;x,t)>0, \tag{15}$$

where $\{\lambda_j(\xi;x,t)\}_{j=1}^{2N}$ are the (real) roots of the polynomial

$$\det\left(\sum_{|k|=2p}\left[\begin{array}{cc}\mathrm{Re}[A_k(x,t)+A_k^T(x,t)] & -\mathrm{Im}[A_k(x,t)-A_k^T(x,t)]\\ \mathrm{Im}[A_k(x,t)-A_k^T(x,t)] & \mathrm{Re}[A_k(x,t)+A_k^T(x,t)]\end{array}\right](i\xi)^k-\lambda\, I_{2N}\right) \tag{16}$$

for all $\xi, x\in\mathbb{R}^d$, and $t\geq 0$, I_{2N} being the identity matrix in $\mathbb{R}^{2N\times 2N}$.

(C2) (7) is uniformly parabolic in the sense that

$$-\sup_{x\in\mathbb{R}^d}\max_j\sup_{|\xi|=1}\mathrm{Re}\left\{\lambda_j^0(\xi;x)\right\}>0, \tag{17}$$

where $\left\{\lambda_j^0(\xi;x)\right\}_{j=1}^{N}$ are the roots of the polynomial

$$\det\left(\sum_{|k|=2p}A_k^0(x)\,(i\xi)^k-\lambda\, I_N\right) \tag{18}$$

for all $\xi, x\in\mathbb{R}^d$, I_N being the identity matrix in $\mathcal{C}^{N\times N}$.

(C3) For each $|k|\leq 2p$ and $|m|\leq|b|$, $\partial_x^m A_k$ and $\partial_x^m A_k^0$ exist and are continuous and uniformly bounded on $\mathbb{R}^d\times[0,\infty)$ respectively $\Re^d$.

(C4) When $|k|\leq 2p$ and $|m|=|b|$, $\partial_x^m A_k$ and $\partial_x^m A_k^0$ are Hölder continuous in x with exponent $0<\varsigma\leq 1$ uniformly on $\mathbb{R}^d\times[0,\infty)$ respectively $\mathbb{R}^d$.

(C5) For each $m\leq b$, $\partial_x^m\varphi$ exists and is a continuous, bounded function on $\mathbb{R}^d$. Furthermore, $\partial_x^b\varphi$ is Hölder continuous with exponent $0<\varsigma\leq 1$ uniformly on $\mathbb{R}^d$.

Under Conditions (C1-C4) there exist fundamental solutions Γ^ε, Γ to (6) and (7) (see Theorem 9.4.2 of Friedman [6] and Theorem A of Kouritzin [9]) satisfying the bounds in Proposition 3 of Section 4 and by Condition (C5) we can define

$$u^\varepsilon(t,x)\doteq\int_{\mathbb{R}^d}\Gamma^\varepsilon(x,t;\xi,0)\,\varphi(\xi)\,d\xi\quad\forall t\in[0,T],\ x\in\mathbb{R}^d. \tag{19}$$

However, our proof also requires a uniqueness condition. In the case where $|b|\geq q$ this uniqueness follows from Conditions (C2-C4) and Theorem 9.5.6 of Friedman [6]. Still, in the general case we must impose it as an assumption.

(C6) There is at most one continuous bounded solution of

$$\partial_t v(t,x) = \sum_{|k|\le 2p} A_k^0(x)\partial_x^k v(t,x) + f(t,x) \quad \text{subject to } v(0,x)=\varphi(x) \tag{20}$$

for any continuous f on $(0,T]\times\mathbb{R}^d$.

When f is bounded and Hölder continuous in x uniformly on bounded subsets of $[a,T]\times\mathbb{R}^d$ for each $a>0$; and

$$\lim_{a\to 0}\sup_{a\le t\le T}\int_0^a\int_{\mathbb{R}^d}(t-\tau)^{-d/2p}\exp\left\{-c\left[|x-\xi|^{2p}/|t-\tau|\right]^{1/(2p-1)}\right\}|f(\tau,\xi)|\,d\xi d\tau = 0 \tag{21}$$

for each $x\in\mathbb{R}^d$, $c>0$; it follows easily from Friedman p. 257 and standard bounds for fundamental solutions that this solution is given by

$$v(t,x) \doteq \int_{\mathbb{R}^d}\Gamma(x,t,\xi)\varphi(\xi)d\xi - \int_0^t\int_{\mathbb{R}^d}\Gamma(x,t-\tau,\xi)f(\tau,\xi)d\xi d\tau. \tag{22}$$

Now, in preparation of our main result (to follow immediately), we let u be the unique solution to (20) when $f\equiv 0$, set $\gamma\doteq\varsigma/12$, fix $\nu\in(0,1/4)$, and define

$$\alpha_{k,m}^{\varepsilon}(t,x) \doteq \int_0^t \partial_x^m[A_k(\tau/\varepsilon,x)-A_k^0(x)]\,d\tau \quad \forall t\in[0,\infty), x\in\mathbb{R}^d,\ |k|\le 2p,\ m\le b. \tag{23}$$

Theorem 1. *Suppose Regularity Conditions (C1-C6) hold and*

$$\alpha_{k,m}^{\varepsilon}(1,x) \overset{\varepsilon\to 0}{\to} 0 \tag{24}$$

for each (fixed) $x\in\mathbb{R}^d$, $|k|\le 2p$, and $m\le b$. Then, it follows that

$$\left|\partial_x^{l+b}(u^\varepsilon - u)\right|_{l,\gamma} \overset{\varepsilon\to 0}{\to} 0. \tag{25}$$

3. Proof of Theorem 1.

Insomuch as it suffices to prove (25) for an arbitrary subsequence of ε and the value of T does not significantly alter our proof, we fix a subsequence $\{\varepsilon_r\}_{r=1}^{\infty}\subset(0,1]$ such that $\varepsilon_r\to 0$ as $r\to\infty$ and take $T=1$ in the sequel. Moreover, to ease the notation, we define $B\doteq\left\{y\in\mathbb{R}^d : |y_1|^\chi+|y_2|^\chi+\cdots+|y_d|^\chi<1\right\}$, $I\doteq(0,1]$, $\beta\doteq\gamma/2$, $q\doteq 2p$,

$$\overline{A}_{k,m}^{r}(s,y) \doteq \partial_y^m\overline{A}_k^r(s,y),\ \overline{A}_k^r(s,y)\doteq A_k(\frac{s}{\varepsilon_r},y)-A_k^0(y) \tag{26}$$

for all $s\in I$, $y\in\mathbb{R}^d$, $r=1,2,...$, $|k|\le q$, $m\le b$, and

$$u_m^r(s,y)\doteq\partial_y^m u^r(s,y),\ u^r(s,y)\doteq u^{\varepsilon_r}(s,y)\ \forall s\in I,\ y\in\mathbb{R}^d, m\in\mathbb{N}^d,\ r=1,2,... \tag{27}$$

Finally, the case $l=0$ is proved in a similar yet simpler manner to $l>0$ so we will dispense with the first case and assume $l>0$. First, by (6) and (26-27) we find that

$$\partial_t u^r(t,x)=\sum_{|k|\le q} A_k^0(x)u_k^r(t,x)+\sum_{|k|\le q}\overline{A}_k^r(t,x)u_k^r(t,x),\ u^r(0,x)=\varphi(x) \tag{28}$$

so it follows from Condition (C6), (22), (7) and Lemma 2 (i) that

$$u^r(t,x)=\sum_{|k|\le q} B_k^r u^r(t,x)+u(t,x)\quad \forall t\in I,\ x\in\mathbb{R}^d, \tag{29}$$

where for any $g\in C^{0,q}(I\times\mathbb{R}^d)$, the functions having continuous (possibly unbounded) derivatives on $I\times\mathbb{R}^d$ in ξ up to order q,

$$B_k^r g(t,x)\doteq -\int_0^t\int_{\mathbb{R}^d}\Gamma(x,t-\tau,\xi)\overline{A}_k^r(\tau,\xi)\,\partial_\xi^k g(\tau,\xi)\,d\xi d\tau\quad \forall (t,x)\in I\times\mathbb{R}^d. \tag{30}$$

Hence, defining

$$y^r(t,x)\doteq u^r(t,x)-u(t,x)\quad \forall t\in I,\ x\in\mathbb{R}^d,\ r=1,2,..., \tag{31}$$

we find by (29) that

$$y_{b+l}^r(t,x)\doteq \partial_x^{b+l}y^r(t,x)=\sum_{|k|\le q}\partial_x^{l+b}B_k^r\,u^r(t,x)\quad \forall t\in I,\ x\in\mathbb{R}^d. \tag{32}$$

Moreover, recalling $\beta\doteq\gamma/2>0$ and letting

$$q_k^r(t,x)\doteq t^{\frac{|l|-\beta}{q}}\int_0^t\int_{\mathbb{R}^d}\partial_x^{b^1}\Gamma_l(x,t-\tau,x+\xi)\overline{A}_{k,b^2}^r(\tau,x+\xi)\,u_{k+b^3}^r(\tau,x+\xi)\,d\xi d\tau, \tag{33}$$

where $b=b^1+b^2+b^3$ and

$$\Gamma_l(y,t-\tau,z)\doteq\partial_y^l\Gamma(y,t-\tau,z), \tag{34}$$

we note by (33), (32) and (30) that

$$-\sum_{\substack{0\le b^1,b^2,b^3\le b\\ b^1+b^2+b^3=b}}\binom{b}{b^1\,b^2\,b^3}\sum_{|k|\le q}q_{k;b^1b^2b^3}^r(t,x)=t^{\frac{|l|-\beta}{q}}y_{b+l}^r(t,x). \tag{35}$$

(Pausing momentarily to justify the interchanges of integration and differentiation in (35), we first use Lemma 2, Conditions (C3,C4), the top of p. 256 of Friedman [6], and the argument in Section 1.3 of [6] to discover that

$$\partial_x^l B_k^r u^r(t,x)=\int_0^t\int_{\mathbb{R}^d}\Gamma_l(x,t-\tau,\xi)\overline{A}_k^r(\tau,\xi)\,u_k^r(\tau,\xi)\,d\xi d\tau. \tag{36}$$

Next, considering $0<\tau<t$, $x\in\mathbb{R}^d$, $j\le b$; factoring $\partial_x^l=\partial_{x_i}\partial_x^n$, $n\ge 0$; defining

$$J\left(x,t,\tau\right)\doteq\int_{\mathbb{R}^d}\Gamma_l(x,t-\tau,x+\xi)\overline{A}_k^r(\tau,x+\xi)\,u_k^r(\tau,x+\xi)\,d\xi; \tag{37}$$

and using Lemma 2, Proposition 3 (i,iv) as well as the divergence theorem; one finds

$$\begin{aligned}
&\left|\partial_x^j J\left(x,t,\tau\right)\right|=\left|\int_{\mathbb{R}^d}\partial_x^j\left\{\Gamma_l(x,t-\tau,x+\xi)\overline{A}_k^r(\tau,x+\xi)\,u_k^r(\tau,x+\xi)\right\}\,d\xi\right| \qquad (38)\\
&\le\left|\int_{B^c}\partial_x^j\left\{\Gamma_l(x,t-\tau,x+\xi)\overline{A}_k^r(\tau,x+\xi)\,u_k^r(\tau,x+\xi)\right\}\,d\xi\right|\\
&+\sum_{j^1,j^2,j^3}\binom{j}{j^1\,j^2\,j^3}\left\{\int_B|\partial_x^{j^1}[\tilde{\Gamma}_l(-\xi,t-\tau,x+\xi)-\tilde{\Gamma}_l(-\xi,t-\tau,x)]|d\xi\left|\overline{A}_{k,j^2}^r(\tau,x)\,u_{k+j^3}^r(\tau,x)\right|\right.\\
&+\int_B\left|\partial_x^{j^1}\Gamma_l(x,t-\tau,x+\xi)[\overline{A}_{k,j^2}^r(\tau,x+\xi)\,u_{k+j^3}^r(\tau,x+\xi)-\overline{A}_{k,j^2}^r(\tau,x)\,u_{k+j^3}^r(\tau,x)]\right|d\xi\\
&\left.+\left|\int_{\partial B}\partial_\xi^n\partial_x^{j^1}\tilde{\Gamma}(-\xi,t-\tau,x)d\xi_1\wedge..\wedge d\xi_{i-1}\wedge d\xi_{i+1}\wedge..\wedge d\xi_d\right|\left|\overline{A}_{k,j^2}^r(\tau,x)\,u_{k+j^3}^r(\tau,x)\right|\right\}\\
&\overset{t,\tau,x}{\ll}(t-\tau)^{\frac{2\gamma}{q}-1}\tau^{\frac{\gamma}{q}-1},
\end{aligned}$$

where we used (41-42) to follow. Next, one finds by the arguments in (55-63) to follow that $x\to\int_0^t\int_{\mathbb{R}^d}\partial_x^j\left\{\Gamma_l(x,t-\tau,x+\xi)\overline{A}_k^r(\tau,x+\xi)\,u_k^r(\tau,x+\xi)\right\}\,d\xi\,d\tau$ is continuous for $j\le b$. (35) ensues from (38), Fubini, and the fundamental theorem of calculus.)

At this point, it may be beneficial to the reader to highlight our approach. We first show that $\{q_k^r\}_{r=1}^\infty$ is uniformly bounded and equicontinuous on $I\times\mathbb{R}^d$ from which we can deduce (Lemma 4) that $\{q_k^r(t,x)/(1+|x|^2)^\nu\}_{r=1}^\infty$ is relatively compact in $C_B(I\times\mathbb{R}^d)$, the space of continuous, bounded functions with supremum norm. Convergence in $C_B(I\times\mathbb{R}^d)$ then follows by showing pointwise convergence for each t,x,k. Finally, convergence of y_{b+l}^r in norm $|\cdot|_{l,\gamma}$ follows by a stronger form of equicontinuity (called $(0,\gamma)$-equicontinuity in Kufner et. al. [11] terminology), (14), and (35).

It follows by (33), the substitution $\overline{A}_{k,b^2}^r(\tau,x+\xi)\,u_{k+b^3}^r(\tau,x+\xi)=\overline{A}_{k,b^2}^r(\tau,x)\,u_{k+b^3}^r(\tau,x)$ $+\left\{\overline{A}_{k,b^2}^r(\tau,x+\xi)\,u_{k+b^3}^r(\tau,x+\xi)-\overline{A}_{k,b^2}^r(\tau,x)\,u_{k+b^3}^r(\tau,x)\right\}$ for $\xi\in B$, Conditions (C3,C4) and Lemma 2 (i,ii) that

$$\begin{aligned}
|q_k^r(t,x)|&\overset{r,t,x}{\ll}t^{\frac{|l|-\beta}{q}}\int_0^t\tau^{\frac{\gamma-q}{q}}\left\{\int_{B^c}\|\partial_x^{b^1}\Gamma_l(x,t-\tau,x+\xi)\|\,d\xi\right. \qquad (39)\\
&\left.+\|\int_B\partial_x^{b^1}\Gamma_l(x,t-\tau,x+\xi)\,d\xi\|\,+\int_B\|\partial_x^{b^1}\Gamma_l(x,t-\tau,x+\xi)\|\,|\xi|^\gamma d\xi\right\}d\tau.
\end{aligned}$$

Now, using Proposition 3 (i) with $y=x$, $w=\xi$ and the bound $|t-\tau|^{-\frac{\gamma}{q}}|\xi|^\gamma\overset{\tau,t,\xi}{\ll}$ $\exp\left\{a(|\xi|^q/|t-\tau|)^{1/(q-1)}\right\}$ for any $a>0$, and recalling $\beta=\gamma/2$, one easily finds that the first and third terms in (39) are majorized by

$$t^{\frac{|l|-\beta}{q}}\int_0^t \tau^{\frac{\gamma-q}{q}}+\tau^{\frac{\gamma-q}{q}}(t-\tau)^{\frac{\gamma-|l|}{q}}d\tau \overset{t}{\ll} t^{\frac{3\beta}{q}} \quad \forall t\in I. \tag{40}$$

To handle the second term in (39), we define

$$\tilde{\Gamma}(y,s,z)\doteq\Gamma(y+z,s,z);\quad \tilde{\Gamma}_l(y,s,z)\doteq\partial_y^l\tilde{\Gamma}(y,s,z)\ \forall y,z\in\mathbb{R}^d,\ s\in I \tag{41}$$

and note by (34) that for $x,\xi\in\mathbb{R}^d$, $s\in I$

$$\partial_x^{b^1}\Gamma_l(x,s,x+\xi)=\partial_x^{b^1}\left[\partial_y^l\tilde{\Gamma}(y-x-\xi,s,x+\xi)|_{y=x}\right]=\partial_x^{b^1}\tilde{\Gamma}_l(-\xi,s,x+\xi). \tag{42}$$

Moreover, we have by (42) and Proposition 3 (iv) with $z=x-\xi$, $y=x$, $w=\xi$ that

$$\|\int_B\partial_x^{b^1}[\tilde{\Gamma}_l(-\xi,s,x+\xi)-\tilde{\Gamma}_l(-\xi,s,x)]d\xi\| \overset{x,s}{\ll}\int_B\frac{|\xi|^\gamma}{s^{\frac{d+|l|}{q}}}\exp\left\{-c\left|\frac{|\xi|^q}{s}\right|^{\frac{1}{q-1}}\right\}d\xi \overset{x,s}{\ll} s^{\frac{\gamma-|l|}{q}} \tag{43}$$

and by (41) and the Divergence theorem (Fleming [5] p. 359) with $\partial_\xi^l=\partial_{\xi_i}\partial_\xi^n$ that

$$\|\int_B\partial_x^{b^1}\tilde{\Gamma}_l(-\xi,s,x)d\xi\|=\|\int_{\partial B^0}\partial_\xi^n\partial_x^{b^1}\tilde{\Gamma}(-\xi,s,x)\,d\xi_1\wedge...\wedge d\xi_{i-1}\wedge d\xi_{i+1}\wedge...\wedge d\xi_d\|, \tag{44}$$

where ∂B^0 denotes the boundary of B with either orientation. Hence, noting by (42) and Proposition 3 (i) with $y=x-\xi$, $w=\xi$, $k=n$ that

$$\left\|\partial_\xi^n\partial_x^{b^1}\tilde{\Gamma}(-\xi,s,x)\right\| \overset{\xi,s,x}{\ll} s^{-(d+|n|)/q}\exp\left\{-c's^{-1/(q-1)}\right\}\overset{\xi,s,x}{\ll} 1 \tag{45}$$

for some $c'>0$ and all $\xi\in\partial B$, $s\in I$, $x\in\mathbb{R}^d$, we find from (4) on the top of p. 357 of Fleming [5], (44), and (45) that

$$\|\int_B\partial_x^{b^1}\tilde{\Gamma}_l(-\xi,s,x)d\xi\|\overset{x,s}{\ll}1\quad\forall x\in\mathbb{R}^d,\ s\in I. \tag{46}$$

Therefore, by (39), (40), (43), and (46) we find that

$$|q_k^r(t,x)|\overset{r,t,x}{\ll} t^{\frac{3\beta}{q}}\quad\forall x\in\mathbb{R}^d,\ t\in I,\ r=1,2,3,... \tag{47}$$

Now, we consider equicontinuity, fix arbitrary $0<t'<t$, and find by (33), Proposition 3 (iii), and Fubini that

$$\begin{aligned}
&|q_k^r(t,x)-q_k^r(t',x)| \qquad\qquad (48)\\
\le\ &\left|t^{\frac{|l|-\beta}{q}}-t'^{\frac{|l|-\beta}{q}}\right|\left|\int_0^t\int_{\mathbb{R}^d}\partial_x^{b^1}\Gamma_l(x,t-\tau,x+\xi)\overline{A}_{k,b^2}^r(\tau,x+\xi)\,u_{k+b^3}^r(\tau,x+\xi)\,d\xi d\tau\right|\\
&+t'^{\frac{|l|-\beta}{q}}\left\{\left|\int_{t'}^t\int_{\mathbb{R}^d}\partial_x^{b^1}\Gamma_l(x,t-\tau,x+\xi)\overline{A}_{k,b^2}^r(\tau,x+\xi)\,u_{k+b^3}^r(\tau,x+\xi)\,d\xi d\tau\right|\right.\\
&\left.+\left|\int_0^{t'}\int_{t'}^t\int_{\mathbb{R}^d}\partial_\sigma\partial_x^{b^1}\Gamma_l(x,\sigma-\tau,x+\xi)\,\overline{A}_{k,b^2}^r(\tau,x+\xi)u_{k+b^3}^r(\tau,x+\xi)d\xi d\sigma d\tau\right|\right\}
\end{aligned}$$

for $x \in \mathbb{R}^d$, $r = 1, 2, ...$ One finds by (47) that the first term in (48) is majorized by

$$t^{\frac{2\gamma-|l|}{q}} \left| t^{\frac{|l|-\beta}{q}} - t'^{\frac{|l|-\beta}{q}} \right| \leq t^{\frac{2\gamma-|l|}{q}} (t-t')^{\frac{|l|-\beta}{q}} \leq (t-t')^{\frac{3\beta}{q}} \tag{49}$$

and, by repeating the arguments (39-47) with $|\xi|^{2\gamma}$ replacing $|\xi|^{\gamma}$ in (39), (43), that

$$\begin{aligned} & t'^{\frac{|l|-\beta}{q}} \left| \int_{t'}^{t} \int_{\mathbb{R}^d} \partial_x^{b^1} \Gamma_l(x, t-\tau, x+\xi) \overline{A}^r_{k,b^2}(\tau, x+\xi) u^r_{k+b^3}(\tau, x+\xi)\, d\xi d\tau \right| \\ & \overset{t,t',x,r}{\ll} t'^{\frac{|l|-\beta}{q}} \int_{t'}^{t} \tau^{\frac{\gamma-q}{q}} (t-\tau)^{\frac{2\gamma-|l|}{q}} d\tau \overset{t,t',x,r}{\ll} \int_{t' \vee \frac{t}{2}}^{t} (t-\tau)^{\frac{2\gamma-q}{q}} d\tau + \int_{t'}^{t' \vee \frac{t}{2}} \tau^{\frac{5\beta-q}{q}} d\tau \\ & \overset{t,t',x,r}{\ll} (t-t')^{\frac{3\beta}{q}} \quad \forall x \in \mathbb{R}^d,\ r = 1, 2, 3, ... \end{aligned} \tag{50}$$

Next, we bound the third term in (48) by dividing its inner integral as in (39)

$$\begin{aligned} & t'^{\frac{|l|-\beta}{q}} \int_{t'}^{t} \int_{0}^{t'} \tau^{\frac{\gamma-q}{q}} \left\{ \int_{B^c} \| \partial_\sigma \partial_x^{b^1} \Gamma_l(x, \sigma-\tau, x+\xi) \|\, d\xi \right. \\ & \left. + \| \int_B \partial_\sigma \partial_x^{b^1} \Gamma_l(x, \sigma-\tau, x+\xi)\, d\xi \| \ + \int_B \| \partial_\sigma \partial_x^{b^1} \Gamma_l(x, \sigma-\tau, x+\xi) \|\, |\xi|^{2\gamma} d\xi \right\} d\tau d\sigma \end{aligned} \tag{51}$$

and find that the first and third terms of (51) can be handled as in (40) with Proposition 3 (iii) replacing Proposition 3 (i)

$$\begin{aligned} & t'^{\frac{|l|-\beta}{q}} \int_{t'}^{t} \left\{ \int_{0}^{\frac{\sigma}{2} \wedge t'} + \int_{\frac{\sigma}{2} \wedge t'}^{t'} \right\} \tau^{\frac{\gamma-q}{q}} |\sigma-\tau|^{\frac{2\gamma-|l|-q}{q}} d\tau d\sigma \\ & \overset{t,t'}{\ll} \int_{t'}^{t} \sigma^{\frac{5\beta-q}{q}} + \sigma^{\frac{|l|+\beta-q}{q}} |\sigma-t'|^{\frac{2\gamma-|l|}{q}} d\sigma \overset{t,t'}{\ll} |t-t'|^{\frac{3\beta}{q}} \quad \forall x \in \mathbb{R}^d. \end{aligned} \tag{52}$$

Now, by (42), Proposition 3 (iii,vi) with $z = x$, $w = \xi$, $y = x - \xi$, the argument in (43-46), and (52) we find that the second term in (51) is bounded by

$$\begin{aligned} & t'^{\frac{|l|-\beta}{q}} \int_{t'}^{t} \int_{0}^{t'} \tau^{\frac{\gamma-q}{q}} \int_B \| \partial_\sigma \partial_x^{b^1} [\widetilde{\Gamma}_l(-\xi, \sigma-\tau, x+\xi) - \widetilde{\Gamma}_l(-\xi, \sigma-\tau, x)] \|\, d\xi d\tau d\sigma \\ & + t'^{\frac{|l|-\beta}{q}} \int_{t'}^{t} \int_{0}^{t'} \tau^{\frac{\gamma-q}{q}} \left\| \int_B \partial_\xi^l \partial_\sigma \partial_x^{b^1} \widetilde{\Gamma}(-\xi, \sigma-\tau, x) d\xi \right\| d\tau d\sigma \\ & \overset{x,t,t'}{\ll} t'^{\frac{|l|-\beta}{q}} \int_{t'}^{t} \int_{0}^{t'} \tau^{\frac{\gamma-q}{q}} \left\{ |\sigma-\tau|^{\frac{2\gamma-|l|-q}{q}} + 1 \right\} d\tau d\sigma \overset{x,t,t'}{\ll} |t-t'|^{\frac{3\beta}{q}} \quad \forall x \in \mathbb{R}^d. \end{aligned} \tag{53}$$

Hence, by (48-53), one finds that

$$|q_k^r(t,x) - q_k^r(t',x)| \overset{t,t',x,r}{\ll} |t-t'|^{\frac{3\beta}{q}} \quad \forall r = 1, 2, 3, ..., \ x \in \mathbb{R}^d,\ t, t' \in I. \tag{54}$$

Next, we fix $x, x' \in \mathbb{R}^d$ such that $|x - x'| \le 1$ and use (33) to discover

$$
\begin{aligned}
&|q_k^r(t,x) - q_k^r(t,x')| \le \qquad (55)\\
&t^{\frac{|l|-\beta}{q}} \Big| \int_0^t \int_{\mathbb{R}^d} \partial_x^{b^1} [\Gamma_l(x, t-\tau, x+\xi) - \Gamma_l(x', t-\tau, x'+\xi)] \overline{A}_{k,b^2}^r(\tau, x+\xi)\, u_{k+b^3}^r(\tau, x+\xi)\\
&+ \partial_x^{b^1} \Gamma_l(x', t-\tau, x'+\xi)[(\overline{A}_{k,b^2}^r\, u_{k+b^3}^r)(\tau, x+\xi) - (\overline{A}_{k,b^2}^r\, u_{k+b^3}^r)(\tau, x'+\xi)] d\xi d\tau \Big|
\end{aligned}
$$

for all $t \in I$, $r = 1, 2, 3, \ldots$ However, expanding the first term in (55) as in (39) and availing ourselves of (42) we find this term is majorized by

$$
\begin{aligned}
&t^{\frac{|l|-\beta}{q}} \int_0^t \tau^{\frac{\gamma-q}{q}} \Big\{ \int_{B^c} \Big\| \partial_x^{b^1} [\tilde{\Gamma}_l(-\xi, t-\tau, x+\xi) - \tilde{\Gamma}_l(-\xi, t-\tau, x'+\xi)] \Big\| d\xi \qquad (56)\\
&+ \Big\| \int_B \partial_x^{b^1} [\tilde{\Gamma}_l(-\xi, t-\tau, x+\xi) - \tilde{\Gamma}_l(-\xi, t-\tau, x'+\xi)] d\xi \Big\|\\
&+ \int_B \Big\| \partial_x^{b^1} [\tilde{\Gamma}_l(-\xi, t-\tau, x+\xi) - \tilde{\Gamma}_l(-\xi, t-\tau, x'+\xi)] \Big\| |\xi|^\gamma d\xi \Big\} d\tau.
\end{aligned}
$$

Now, we can use Proposition 3 (iv) with $z = x$, $w = \xi$, $y = x'$ to discover that

$$
\Big\| \partial_x^{b^1} [\tilde{\Gamma}_l(-\xi, s, x+\xi) - \tilde{\Gamma}_l(-\xi, s, x'+\xi)] \Big\| \overset{\xi,s,x,x'}{\ll} \frac{|x-x'|^{3\beta}}{s^{\frac{d+|l|}{q}}} \exp\left\{ -c \left| \frac{|\xi|^q}{s} \right|^{\frac{1}{q-1}} \right\}. \qquad (57)
$$

Hence, the method of (40) can be used to bound the first and third terms in (56) above by a constant times

$$
|x-x'|^{3\beta} \cdot t^{\frac{|l|-\beta}{q}} \int_0^t \tau^{\frac{\gamma-q}{q}} (t-\tau)^{\frac{\gamma-|l|}{q}} d\tau \overset{x,x',t}{\ll} |x-x'|^{3\beta} \cdot t^{\frac{3\beta}{q}} \quad \forall t \in I. \qquad (58)
$$

Moreover, for the second term in (56) we first note by Proposition 3 (iv) with $z = x - \xi$, $w = \xi$, $y = x' - \xi$ (in place of (45)) and the argument in (44-46) that

$$
\Big\| \int_B \partial_x^{b^1} [\tilde{\Gamma}_l(-\xi, s, x) - \tilde{\Gamma}_l(-\xi, s, x')] d\xi \Big\| \overset{x,x',s}{\ll} |x-x'|^{3\beta} \quad \forall\, s \in I. \qquad (59)
$$

On the other hand, using Proposition 3 (iv) four times (with z, w, y as in the previous two applications; $z = x$, $w = \xi$, $y = x - \xi$; and $z = x'$, $w = \xi$, $y = x' - \xi$) and using the fact that $|x-x'|^{2\gamma} \wedge |\xi|^{2\gamma} \le |x-x'|^{3\beta} \cdot |\xi|^\beta$, one finds that

$$
\begin{aligned}
&\int_B \Big\| \partial_x^{b^1} [\tilde{\Gamma}_l(-\xi, s, x+\xi) - \tilde{\Gamma}_l(-\xi, s, x'+\xi) - \tilde{\Gamma}_l(-\xi, s, x) + \tilde{\Gamma}_l(-\xi, s, x')] \Big\| d\xi \qquad (60)\\
&\overset{x,x',s}{\ll} \int_B \frac{|x-x'|^{2\gamma} \wedge |\xi|^{2\gamma}}{s^{\frac{d+|l|}{q}}} \exp\left\{ -c \left| \frac{|\xi|^q}{s} \right|^{\frac{1}{q-1}} \right\} d\xi \overset{x,x',s}{\ll} |x-x'|^{3\beta}\, s^{\frac{\beta-|l|}{q}} \quad \forall\, s \in I.
\end{aligned}
$$

Therefore, combining (59) and (60) with the method of (40), we discover that

$$t^{\frac{|l|-\beta}{q}}\int_0^t \tau^{\frac{\gamma-q}{q}}\left\|\int_B \partial_x^{b^1}[\widetilde{\Gamma}_l(-\xi,t-\tau,x+\xi)-\widetilde{\Gamma}_l(-\xi,t-\tau,x'+\xi)]d\xi\right\|d\tau \quad (61)$$
$$\overset{x,x',t}{\ll} |x-x'|^{3\beta}\cdot t^{\frac{|l|-\beta}{q}}\int_0^t \tau^{\frac{\gamma-q}{q}}(t-\tau)^{\frac{\beta-|l|}{q}}d\tau \overset{x,x',t}{\ll} |x-x'|^{3\beta} \quad \forall t\in I.$$

Next, we consider the second term in (55), and repeat (39-47) with Lemma 2 (ii) instead of (i) to discover that this term is majorized by

$$|x-x'|^{3\beta}t^{\frac{|l|-\beta}{q}}\int_0^t \tau^{\frac{\gamma-q}{q}}\left\{\int_{B^c}\left\|\partial_x^{b^1}\Gamma_l(x',t-\tau,x'+\xi)\right\|d\xi\right. \quad (62)$$
$$\left.+\left\|\int_B \partial_x^{b^1}\widetilde{\Gamma}_l(-\xi,t-\tau,x'+\xi)d\xi\right\|+\int_B\|\partial_x^{b^1}\Gamma_l(x',t-\tau,x'+\xi)\|\,|\xi|^\beta d\xi\right\}d\tau$$
$$\overset{x,x',t}{\ll} |x-x'|^{3\beta} \quad \forall t\in I, \quad x,x'\in\mathbb{R}^d, |x-x'|\le 1.$$

Thus, it follows by (47), (54), (55), (56), (58), (61), and (62) that

$$|q_k^r(t,x)-q_k^r(t',x')| \overset{t,t',x,x',r}{\ll} |t-t'|^{\frac{3\beta}{q}}+|x-x'|^{3\beta} \quad (63)$$

for $r=1,2,...;\ 0\le|k|\le q;\ t,t'\in I$; and $x,x'\in\mathbb{R}^d$. Hence, one finds by (47), (63) and Lemma 4 that $(t,x)\to q_k^r(t,x)/(1+|x|^2)^\nu$ is relatively compact in $C_B(I\times\mathbb{R}^d)$.

Now, we show that the only possible limit is 0 by showing $q_k^r(t,x)/(1+|x|^2)^\nu \overset{r\to\infty}{\to} 0$ on a dense set of $(t,x)\in I\times\mathbb{R}^d$. Fixing $(t,x)\in I\times\mathbb{R}^d$ and repeating the arguments in (39-47), we find that

$$\left|\left\{\int_{t-\delta}^t+\int_0^\delta\right\}\int_{\mathbb{R}^d}\partial_x^{b^1}\Gamma_l(x,t-\tau,x+\xi)\,\overline{A}_{k,b^2}^r(\tau,x+\xi)\,u_{k+b^3}^r(\tau,x+\xi)\,d\xi d\tau\right| \quad (64)$$
$$\overset{\delta,r}{\ll}\left\{\int_{t-\delta}^t+\int_0^\delta\right\}\tau^{\frac{\gamma-q}{q}}|t-\tau|^{\frac{\gamma-|l|}{q}}d\tau \overset{\delta,r}{\ll} t^{\frac{\gamma-q}{q}}\delta^{\frac{\gamma}{q}} \quad \forall\delta\in(0,t/2).$$

We have removed the singularities at $\tau=t$ and $\tau=0$ and so by Condition (C3), Lemmas 2 (i), and Proposition 3 (i) with $y=x$, $w=\xi$

$$\left|\int_\delta^{t-\delta}\partial_x^{b^1}\Gamma_l(x,t-\tau,x+\xi)\,\overline{A}_{k,b^2}^r(\tau,x+\xi)\,u_{k+b^3}^r(\tau,x+\xi)\,d\tau\right|\le C'\exp\left\{-c'\,|\xi|^{\frac{q}{q-1}}\right\} \quad (65)$$

for some $c'>0$ and all $r=1,2,...,$ $\xi\in\mathbb{R}^d$, where $C' \overset{t,\delta,\gamma,x,\xi}{\ll} \int_\delta^{t-\delta}\tau^{(\gamma-q)/q}(t-\tau)^{-(d+q)/q}d\tau$. Furthermore, we find by integration by parts, (23), and (26) that

$$\int_\delta^{t-\delta}\partial_x^{b^1}\Gamma_l(x,t-\tau,x+\xi)\overline{A}_{k,b^2}^r(\tau,x+\xi)\,u_{k+b^3}^r(\tau,x+\xi)\,d\tau \quad (66)$$

$$
\begin{aligned}
= \;& \partial_x^{b^1}\Gamma_l(x,\delta,x+\xi)\,\alpha^r_{k,b^2}(t-\delta,x+\xi)\,u^r_{k+b^3}(t-\delta,x+\xi) \\
& -\partial_x^{b^1}\Gamma_l(x,t-\delta,x+\xi)\,\alpha^r_{k,b^2}(\delta,x+\xi)\,u^r_{k+b^3}(\delta,x+\xi) \\
& -\int_\delta^{t-\delta}\partial_\tau\partial_x^{b^1}\Gamma_l(x,t-\tau,x+\xi)\,\alpha^r_{k,b^2}(\tau,x+\xi)\,u^r_{k+b^3}(\tau,x+\xi)\,d\tau \\
& -\int_\delta^{t-\delta}\partial_x^{b^1}\Gamma_l(x,t-\tau,x+\xi)\,\alpha^r_{k,b^2}(\tau,x+\xi)\,\partial_\tau u^r_{k+b^3}(\tau,x+\xi)\,d\tau
\end{aligned}
$$

for all $\delta\in(0,t/2)$ and by Condition (C3)

$$
\left\|\alpha^r_{k,b^2}(\tau,x+\xi)\right\| = \left\|\int_0^\tau \overline{A}^r_{k,b^2}(s,x+\xi)\,ds\right\| \overset{r,\tau,\xi}{\ll} 1. \tag{67}
$$

Thus, one finds by the fact $\alpha^r_{k,b^2}(\tau,y)\overset{r\to\infty}{\rightrightarrows}0$ (c.f. Hypothesis (24)), Proposition 3 (i,iii), Lemma 2 (i), (64-67), and two nested dominated convergence applications that

$$
\left|\int_0^t\int_{\mathbb{R}^d}\partial_x^{b^1}\Gamma_l(x,t-\tau,x+\xi)\,\overline{A}^r_{k,b^2}(\tau,x+\xi)\,u^r_{k+b^3}(\tau,x+\xi)\,d\xi d\tau\right| \overset{r\to\infty}{\rightrightarrows} 0. \tag{68}
$$

It follows by (33) that $q^r_k(t,x)/(1+|x|^2)^\nu \overset{r\to\infty}{\rightrightarrows} 0$ for each $x\in\mathbb{R}^d$, $t\in I$. This combined with our relative compactness of the previous paragraph surely establishes that

$$
\sup_{t,x}|q^r_k(t,x)|\,(1+|x|^2)^{-\nu}\overset{r\to\infty}{\rightrightarrows}0. \tag{69}
$$

Now, we established more than equicontinuity for $\{q^r,\ r=1,2,...\}$. Indeed, recalling $\beta\doteq\gamma/2$, we showed in (63) that given $\kappa>0$ there is a $0<\delta<1$ such that

$$
|q^r_k(t,x)-q^r_k(t',x')|/[|t-t'|^{\gamma/q}+|x-x'|^\gamma]\le\kappa \tag{70}
$$

for all $r=1,2,...$, whenever $|t-t'|\vee|x-x'|<\delta$. Thus, it follows easily by (69), (70), and the definition $\frac{0}{0}\doteq 0$ that there exists an $R_k>0$ such that

$$
\begin{aligned}
& \sup_{t,t'\in I;|x-x'|\le 1}\frac{|q^r_k(t,x)-q^r_k(t',x')|}{(1+|x|^2)^\nu\left[|t-t'|^{\frac{\gamma}{q}}+|x-x'|^\gamma\right]} && (71)\\
\le\; & \sup_{\substack{|t-t'|<\delta\\|x-x'|<\delta}}\frac{|q^r_k(t,x)-q^r_k(t',x')|}{(1+|x|^2)^\nu\left[|t-t'|^{\frac{\gamma}{q}}+|x-x'|^\gamma\right]}+\delta^{-\gamma}\sup_{\substack{\delta<|t-t'|\\ \delta<|x-x'|\le 1}}\frac{|q^r_k(t,x)|+|q^r_k(t',x')|}{(1+|x|^2)^\nu} \\
\le\; & \kappa+5\delta^{-\gamma}\sup_{t\in I;x\in\mathbb{R}^d}|q^r_k(t,x)|/(1+|x|^2)^\nu\le 2\kappa\quad\forall r>R_k.
\end{aligned}
$$

The result follows from (69), (71), (35), and (14). ■

4. Subsidiary Results

Lemma 2. *Under Conditions (C1-C6), u^r is continuously differentiable and*

$$\text{(i)} \quad \left|\partial_x^{m+k} u^r(t,x)\right| \overset{r,t,x}{\ll} t^{\frac{(\gamma-|k|)\wedge 0}{q}} \tag{72}$$

$$\text{(ii)} \quad \left|\partial_x^{m+k}[u^r(t,x') - u^r(t,x)]\right| \overset{r,t,x,x'}{\ll} |x'-x|^{2\gamma} \cdot t^{\frac{(\gamma-|k|)\wedge 0}{q}} \tag{73}$$

for all $0 \le m \le b$, $0 \le |k| \le q$; $x, x' \in \mathbb{R}^d$; $t \in I$; and $r = 1, 2, 3, \ldots$

Proof. Inasmuch as (i) follows by an argument similar to yet simpler than the proof of (ii) and the case $|k| > 0$ is more difficult than $|k| = 0$ we only supply the proof of (ii) when $|k| > 0$. Moreover, to ease the notation in the sequel, we define

$$\varphi_m(y) \doteq \partial_y^m \varphi(y);\ \widetilde{\Gamma}_k^r(y,s,x) \doteq \partial_y^k \widetilde{\Gamma}^r(y,s,x);\ \widetilde{\Gamma}^r(y,s,x) \doteq \Gamma^r(y+x,s;x,0) \tag{74}$$

for all $y, x \in \mathbb{R}^d$, $s \in I$. Then, it follows from (19) and the argument in (42) that

$$\partial_x^k u^r(t,x) = \int_{\mathbb{R}^d} \widetilde{\Gamma}_k^r(-\xi,t,x+\xi)\,\varphi(x+\xi)\,d\xi \tag{75}$$

for all $r = 1, 2, \ldots$, $t \in I$, and $x \in \mathbb{R}^d$. Therefore, for such r, t, and x, x' we have that

$$\begin{aligned}
&|u^r_{m+k}(t,x') - u^r_{m+k}(t,x)| \\
\le\ & \left|\int_{B^c} \partial_x^m [\widetilde{\Gamma}_k^r(-\xi,t,x'+\xi)\varphi(x'+\xi) - \widetilde{\Gamma}_k^r(-\xi,t,x+\xi)\varphi(x+\xi)] d\xi\right| \\
+ & \left|\int_B \partial_\xi^k \left\{\partial_x^m [\widetilde{\Gamma}^r(-\xi,t,x')\varphi(x') - \widetilde{\Gamma}^r(-\xi,t,x)\varphi(x)]\right\} d\xi\right| \\
+ & \left|\int_B \partial_x^m [\{\widetilde{\Gamma}_k^r(-\xi,t,x'+\xi) - \widetilde{\Gamma}_k^r(-\xi,t,x')\}\varphi(x') - \{\widetilde{\Gamma}_k^r(-\xi,t,x+\xi) - \widetilde{\Gamma}_k^r(-\xi,t,x)\}\varphi(x)] d\xi\right| \\
+ & \left|\int_B \partial_x^m [\widetilde{\Gamma}_k^r(-\xi,t,x'+\xi)\{\varphi(x'+\xi) - \varphi(x')\} - \widetilde{\Gamma}_k^r(-\xi,t,x+\xi)\{\varphi(x+\xi) - \varphi(x)\}]\, d\xi\right|.
\end{aligned} \tag{76}$$

We use Condition (C5) and Proposition 3 (ii,v) with $y = x'$, $w = \xi$, $z = x$ to find that the first term in (76) is bounded by all m^1, m^2-combinations ($m = m^1 + m^2$) of

$$\begin{aligned}
&\int_{B^c} \left|\partial_x^{m^1} \widetilde{\Gamma}_k^r(-\xi,t,x'+\xi)\,[\varphi_{m^2}(x'+\xi) - \varphi_{m^2}(x+\xi)]\,\right| d\xi \\
&+ \int_{B^c} \left|\partial_x^{m^1} [\widetilde{\Gamma}_k^r(-\xi,t,x'+\xi) - \widetilde{\Gamma}_k^r(-\xi,t,x+\xi)]\varphi_{m^2}(x+\xi)\,\right| d\xi \\
&\overset{x,x',t}{\ll} \int_{B^c} \frac{|x'-x|^{2\gamma}}{t^{(d+|k|)/q}} \exp\left\{-c\left|\frac{|\xi|^q}{t}\right|^{\frac{1}{q-1}}\right\} d\xi \overset{x,x',t}{\ll} |x'-x|^{2\gamma}\ \ \forall |x-x'| \le 1, t \in I.
\end{aligned} \tag{77}$$

Moreover, for the second term in (76) we divide $\partial_\xi^k = \partial_{\xi_i}\partial_\xi^n$ and employ the divergence theorem as in (44-46) to find this term is equal to

$$\left|\int_{\partial B^0}\partial_\xi^n\Big\{\partial_x^m[\tilde\Gamma^r(-\xi,t,x')\varphi(x') - \tilde\Gamma^r(-\xi,t,x)\varphi(x)]\Big\}\,d\xi_1\wedge ...d\xi_{i-1}\wedge d\xi_{i+1}\wedge ...d\xi_d\right| \tag{78}$$
$$\stackrel{x,x',t}{\ll} |x'-x|^{2\gamma}\ \ \forall |x-x'|\le 1, t\in I,$$

since by Condition (C5) and Proposition 3 (v) with $y = x'-\xi$, $w=\xi$, $z=x-\xi$

$$\left|\partial_\xi^n\left\{\partial_x^m[\tilde\Gamma^r(-\xi,t,x')\varphi(x') - \tilde\Gamma^r(-\xi,t,x)\varphi(x)]\right\}\right| \stackrel{\xi,x,x',t}{\ll} |x'-x|^{2\gamma}\ \forall\xi\in\partial B. \tag{79}$$

Next, letting $m = m^1 + m^2$, and using Condition (C5) as well as four applications of Proposition 3 (v), one bounds the third term of (76) by expressions like

$$\left|\int_B \partial_x^{m^1}[\tilde\Gamma_k^r(-\xi,t,x'+\xi) - \tilde\Gamma_k^r(-\xi,t,x') - \tilde\Gamma_k^r(-\xi,t,x+\xi) + \tilde\Gamma_k^r(-\xi,t,x)]\varphi_{m^2}(x)d\xi\right| \tag{80}$$
$$+\left|\int_B \partial_x^{m^1}\{\tilde\Gamma_k^r(-\xi,t,x'+\xi) - \tilde\Gamma_k^r(-\xi,t,x')\}[\varphi_{m^2}(x') - \varphi_{m^2}(x)]\,d\xi\right|$$
$$\stackrel{x,x',t}{\ll} \int_B \frac{|x'-x|^{3\gamma}\wedge|\xi|^{3\gamma}}{t^{(d+|k|)/q}}\exp\left\{-c\left|\frac{|\xi|^q}{t}\right|^{\frac{1}{q-1}}\right\}d\xi \stackrel{x,x',t}{\ll} t^{\frac{\gamma-|k|}{q}}|x'-x|^{2\gamma}\ \ \forall\,|x'-x|\le 1.$$

Finally, we turn to the fourth term in (76) and find by Condition (C5) and Proposition 3 (ii,v) that it is bounded above by m^1, m^2 combinations of

$$\left|\int_B \partial_x^{m^1}[\tilde\Gamma_k^r(-\xi,t,x'+\xi) - \tilde\Gamma_k^r(-\xi,t,x+\xi)]\{\varphi_{m^2}(x'+\xi) - \varphi_{m^2}(x')\}\,d\xi\right| \tag{81}$$
$$+\left|\int_B \partial_x^{m^1}\tilde\Gamma_k^r(-\xi,t,x+\xi)\{\varphi_{m^2}(x'+\xi) - \varphi_{m^2}(x') - \varphi_{m^2}(x+\xi) + \varphi_{m^2}(x)\}d\xi\right|$$
$$\stackrel{x,x',t}{\ll} \int_B \frac{|x'-x|^{3\gamma}\wedge|\xi|^{3\gamma}}{t^{(d+|k|)/q}}\exp\left\{-c\left|\frac{|\xi|^q}{t}\right|^{\frac{1}{q-1}}\right\}d\xi \stackrel{x,x',t}{\ll} t^{\frac{\gamma-|k|}{q}}|x'-x|^{2\gamma}\ \ \forall\,|x'-x|\le 1.$$

The lemma follows by (76-81). ■

The following proposition establishes the fundamental solution bounds which were required throughout the proofs of Theorem 1 and Lemma 2. The notation in the statement of the Proposition is taken from (34) and

$$\Gamma_l^r(y,s,\xi) \doteq \partial_y^l\Gamma^r(y,s,\xi,0)\quad \forall y,\xi\in\mathbb{R}^d,\ s\in I. \tag{82}$$

The proof is necessarily longer and more involved than the previous developments.

Proposition 3. *Under Conditions (C1-C4) of Section 2, all indicated derivatives exist, all objects enclosed in* $\|\cdot\|$*'s are continuous in all their variables, and there exists constants* $c, C > 0$ *independent of* y, z, w, s*, and* r *such that*

$$(i)\ \left\|\partial_y^m \Gamma_k(y,s,y+w)\right\| \le C\, s^{-(d+|k|)/q} h_c(w,s) \tag{83}$$

$$(ii)\ \left\|\partial_y^m \Gamma_k^r(y,s,y+w)\right\| \le C\, s^{-(d+|k|)/q} h_c(w,s) \tag{84}$$

$$(iii)\ \|\partial_s \partial_y^m \Gamma_k(y,s,y+w)\| \le C\, s^{-(d+|k|+q)/q} h_c(w,s) \tag{85}$$

$$(iv)\ \|\partial_z^m \Gamma_k(z,s,z+w) - \partial_y^m \Gamma_k(y,s,y+w)\| \le C\, |z-y|^{\overline{\gamma}} s^{-(d+|k|)/q} h_c(w,s) \tag{86}$$

$$(v)\ \|\partial_z^m \Gamma_k^r(z,s,z+w) - \partial_y^m \Gamma_k^r(y,s,y+w)\| \le C\, |z-y|^{\overline{\gamma}} s^{-(d+|k|)/q} h_c(w,s) \tag{87}$$

$$(vi)\ \|\partial_s[\partial_z^m \Gamma_k(z,s,z+w) - \partial_y^m \Gamma_k(y,s,y+w)]\| \le C\, |z-y|^{\overline{\gamma}} s^{-(d+|k|+q)/q} h_c(w,s) \tag{88}$$

for all $s \in I$, $y, z, w \in \mathbb{R}^d$, $0 \le |k| \le q$, *and* $0 \le |m| \le |b|$, *where* $\overline{\gamma} \doteq 3\gamma$ *and*

$$h_c(\xi,s) \doteq \exp\left\{-c\, s^{-1/(q-1)}\, |\xi|^{q/(q-1)}\right\}. \tag{89}$$

Proof. The proofs of (i) through (vi) in the two cases $|k| < q$ and $|k| = q$ all have a similar basic structure with added difficulties due to: (a) Differentiation with respect to s, (b) the difference structure in $(iv - vi)$, (c) the r-dependence in (ii) and (v), and (d) the extra complexity when $|k| = q$. Hence, we will only prove (iii) and half of (iv) when $|k| = q$. We still establish the bounds on $\widehat{Z}, \widehat{Z}^r, \Phi$, and Φ^r (to be defined below) which are required for the other parts and note that the difficulties due to the r-dependence in (ii) and (v) arise only through these bounds.

Recalling Friedman p. 252 for motivation and letting $A_k^0(y,w) \doteq A_k^0(y) - A_k^0(w)$, $A_k(\frac{t}{\varepsilon_r}, y, w) \doteq A_k(\frac{t}{\varepsilon_r}, y) - A_k(\frac{t}{\varepsilon_r}, w)$, we define: $\widehat{Z}(y, t-\tau, w)$ and $\widehat{Z}^r(y,t,\tau,w)$ to be the fundamental solutions to the auxiliary parameter parabolic equations

$$\partial_t v(t,y) = \sum_{|k|=q} A_k^0(w) \partial_y^k v(t,y) \quad \text{subject to } v(0,y) = \delta(y-w), \tag{90}$$

$$\partial_t v^r(t,y) = \sum_{|k|=q} A_k(t/\varepsilon_r, w) \partial_y^k v^r(t,y) \quad \text{subject to } v^r(0,y) = \delta(y-w); \tag{91}$$

$$K(y,t-\tau,w) \doteq \sum_{|k|=q} A_k^0(y,w)\, \partial_y^k \widehat{Z}(y,t-\tau,w) + \sum_{|k|<q} A_k^0(y)\, \partial_y^k \widehat{Z}(y,t-\tau,w), \tag{92}$$

$$K^r(y,t,w,\tau) \doteq \sum_{|k|=q} A_k(\frac{t}{\varepsilon_r}, y, w)\, \partial_y^k \widehat{Z}^r(y,t,\tau,w) + \sum_{|k|<q} A_k(\frac{t}{\varepsilon_r}, y)\, \partial_y^k \widehat{Z}^r(y,t,\tau,w); \tag{93}$$

and Φ and Φ^r via the integral equations

$$\Phi(y,t-\tau,w) = K(y,t-\tau,w) + \int_\tau^t \int_{\mathbb{R}^d} K(y,t-\sigma,\eta)\Phi(\eta,\sigma-\tau,w)\,d\eta\,d\sigma, \tag{94}$$

$$\Phi^r(y,t,w,\tau) = K^r(y,t,w,\tau) + \int_\tau^t \int_{\mathbb{R}^d} K^r(y,t,\eta,\sigma)\Phi^r(\eta,\sigma,w,\tau)\,d\eta\,d\sigma. \tag{95}$$

(It is established in Friedman Ch. 9 that these definitions make sense.) Moreover, to lighten the notation we let ς be as in Conditions (C4,C5) and define

$$\overline{\varsigma} \doteq \varsigma/2 = 2\overline{\gamma} = 6\gamma. \tag{96}$$

Now, suppose $c, C > 0$ are constants (different than those in the proposition statement) and we established that the quantities on the left of the following thirteen inequalities exist, are continuous in all variables, and satisfy the indicated bounds:

$$\|\partial_x^n \widehat{Z}_k(x,s,x+\xi)\| \vee \|\partial_x^n \widehat{Z}_k^r(x,s+\tau,\tau,x+\xi)\| \le C\, s^{-(d+|k|)/q} h_c(\xi,s) \tag{97}$$

$$\|\partial_s\partial_x^n \widehat{Z}_k(x,s,x+\xi)\| \le C\, s^{-(d+|k|+q)/q} h_c(\xi,s) \tag{98}$$

$$\|\partial_x^n \widehat{Z}_k(x,s,x+\xi) - \partial_v^n \widehat{Z}_k(v,s,v+\xi)\| \le C\,|x-v|^{\varsigma} s^{-(d+|k|)/q} h_c(\xi,s) \tag{99}$$

$$\|\partial_x^n \widehat{Z}_k^r(x,s+\tau,\tau,x+\xi) - \partial_v^n \widehat{Z}_k^r(v,s+\tau,\tau,v+\xi)\| \le C\,|x-v|^{\varsigma} s^{-(d+|k|)/q} h_c(\xi,s) \tag{100}$$

$$\|\partial_s[\partial_x^n \widehat{Z}_k(x,s,x+\xi) - \partial_v^n \widehat{Z}_k(v,s,v+\xi)]\| \le C\,|x-v|^{\varsigma} s^{-(d+|k|+q)/q} h_c(\xi,s) \tag{101}$$

$$\|\partial_x^n \Phi(x+\xi,s,x)\| \vee \|\partial_x^n \Phi^r(x+\xi,s+\tau,x,\tau)\| \le C\, s^{-(d+q-\varsigma)/q} h_c(\xi,s) \tag{102}$$

$$\|\partial_s\partial_x^n \Phi(x+\xi,s,x)\| \le C\, s^{-(d+2q-\varsigma)/q} h_c(\xi,s) \tag{103}$$

$$\|\partial_x^n \Phi(x+\xi,s,x) - \partial_v^n \Phi(v+\xi,s,v)\| \le C\,|x-v|^{\overline{\varsigma}} s^{(\overline{\varsigma}-d-q)/q} h_c(\xi,s) \tag{104}$$

$$\|\partial_x^n \Phi^r(x+\xi,s+\tau,x,\tau) - \partial_v^n \Phi^r(v+\xi,s+\tau,v,\tau)\| \le C\,|x-v|^{\overline{\varsigma}} s^{(\overline{\varsigma}-d-q)/q} h_c(\xi,s) \tag{105}$$

$$\|\partial_s[\partial_x^n \Phi(x+\xi,s,x) - \partial_v^n \Phi(v+\xi,s,v)]\| \le C\,|x-v|^{\overline{\varsigma}} s^{(\overline{\varsigma}-d-2q)/q} h_c(\xi,s) \tag{106}$$

$$\|\partial_x^n[\Phi(x+\xi,s,x) - \Phi(x+v,s,x)]\| \le C\,|\xi-v|^{\overline{\varsigma}} s^{(\overline{\varsigma}-d-q)/q} h_c(\xi,v,s) \tag{107}$$

$$\|\partial_x^n \Phi^r(x+\xi,s+\tau,x,\tau) - \partial_v^n \Phi^r(x+v,s+\tau,x,\tau)\| \le C\,|\xi-v|^{\overline{\varsigma}} s^{(\overline{\varsigma}-d-q)/q} h_c(\xi,v,s) \tag{108}$$

$$\|\partial_s\partial_x^n[\Phi(x+\xi,s,x) - \Phi(x+v,s,x)]\| \le C\,|\xi-v|^{\overline{\varsigma}} s^{(\overline{\varsigma}-d-2q)/q} h_c(\xi,v,s) \tag{109}$$

for all $0 \le |k| \le 2q$ (say), $0 \le |n| \le |b|$, $s \in I$, $\tau \in [0, 1-s]$, and $x, v, \xi \in \mathbb{R}^d$, where $\widehat{Z}_k(y,\sigma,z) = \partial_y^k \widehat{Z}(y,\sigma,z)$, $\widehat{Z}_k^r(y,t,\sigma,z) = \partial_y^k \widehat{Z}^r(y,t,\sigma,z)$, and, in (107-109),

$$h_c(\xi,v,s) \doteq h_c(\xi,s) + h_c(v,s). \tag{110}$$

Then, all (i-vi) of the lemma would follow from (97-109) and the following method.

(iii) Fixing s, y, w, we find from Friedman p. 252, our definition

$$\widehat{Z}(x,s,\xi) \doteq Z(x-\xi,s,\xi) \doteq Z(x-\xi,s,\xi,0), \tag{111}$$

formal differentiation through the integrals, and changes of variables that

$$\Gamma_k(y,s,y+w) - \widehat{Z}_k(y,s,y+w) = \int_0^s \int_{\mathbb{R}^d} \widehat{Z}_k(y,s-\sigma,y+\eta)\Phi(y+\eta,\sigma,y+w)d\eta d\sigma \tag{112}$$

$$= \int_0^{\frac{s}{2}} \int_{\mathbb{R}^d} \widehat{Z}_k(y,\sigma,y+\eta)\Phi(y+\eta,s-\sigma,y+w) + \widehat{Z}_k(y,s-\sigma,y+\eta)\Phi(y+\eta,\sigma,y+w)d\eta d\sigma.$$

or (formally) differentiating with respect to s and y

$$\begin{aligned}\partial_s\partial_y^m\Gamma_k(y,s,y+w) = {} & \partial_s\partial_y^m\widehat{Z}_k(y,s,y+w) + \int_{\mathbb{R}^d}\partial_y^m\{\widehat{Z}_k(y,\tfrac{s}{2},y+\eta)\Phi(y+\eta,\tfrac{s}{2},y+w)\}d\eta \\ & + \int_0^{\frac{s}{2}}\int_{\mathbb{R}^d}\partial_y^m\{\widehat{Z}_k(y,\sigma,y+\eta)\partial_s\Phi(y+\eta,s-\sigma,y+w) \\ & + \partial_s\widehat{Z}_k(y,s-\sigma,y+\eta)\Phi(y+\eta,\sigma,y+w)\}d\eta d\sigma.\end{aligned} \tag{113}$$

(The interchange of integration and differentiation in (112,113) must be justified: Suppose we fix $a > 0$, let s range in $[a,1]$, and consider the last term of (112). Letting

$$g(\eta,\sigma) \doteq \sigma^{(3\varsigma/2-q)/q}\,|\eta|^{-d-\varsigma/2}\,1_{\{|\eta|>1\}} + \sigma^{(\varsigma/2-q)/q}\,|\eta|^{-d+\varsigma/2}\,1_{\{|\eta|\le 1\}}, \tag{114}$$

we find from (97) and (102) that

$$\left\|\widehat{Z}_l(y,s-\sigma,\eta+z)\Phi(\eta+z,\sigma,z)1_{\{\sigma\le s/2\}}\right\| \overset{s,\sigma,\eta,y,z}{\ll} (2/a)^{1+d/q}\,g(\eta,\sigma) \tag{115}$$

which is $\int_0^{1/2} d\sigma \int_{\mathbb{R}^d} d\eta$-integrable for all $|l| \le q$. Therefore, since $s_j \to s$ implies $1_{\{\sigma\le s_j/2\}} \to 1_{\{\sigma\le s/2\}}$ for almost all σ, one finds by repeated use of dominated convergence, Fubini, and the fundamental theorem of calculus

$$\partial_y^k\int_0^{s/2}\int_{\mathbb{R}^d}\widehat{Z}(y,s-\sigma,\eta)\,\Phi(\eta,\sigma,z)\,d\eta\,d\sigma = \int_0^{s/2}\int_{\mathbb{R}^d}\widehat{Z}_k(y,s-\sigma,\eta)\,\Phi(\eta,\sigma,z)\,d\eta\,d\sigma \tag{116}$$

is continuous on (s,y,z) in $[a,1]\times\mathbb{R}^{2d}$. Moreover, we find by (97) and (102) that

$$\left\|\partial_y^n\left\{\widehat{Z}_k(y,s-\sigma,y+\eta+w)\Phi(y+\eta+w,\sigma,y+w)\right\}1_{\{\sigma\le s/2\}}\right\| \overset{s,\sigma,\eta,y,w}{\ll} g(\eta,\sigma) \tag{117}$$

for all $n \le m$ and it follows from the procedure used to derive (116) that

$$\begin{aligned}&\partial_y^m\int_0^{s/2}\int_{\mathbb{R}^d}\widehat{Z}_k(y,s-\sigma,\eta)\,\Phi(\eta,\sigma,y+w)\,d\eta\,d\sigma \\ = {} & \int_0^{s/2}\int_{\mathbb{R}^d}\partial_y^m\left\{\widehat{Z}_k(y,s-\sigma,\eta+y+w)\,\Phi(\eta+y+w,\sigma,y+w)\right\}\,d\eta\,d\sigma\end{aligned} \tag{118}$$

is also continuous. Next, we follow the procedure outlined in (162) to find

$$\begin{aligned} & \partial_s \int_0^{s/2} \partial_y^m \left\{ \widehat{Z}_k(y, s-\sigma, \eta+y+w)\, \Phi(\eta+y+w, \sigma, y+w) \right\} \, d\sigma \qquad (119) \\ = \ & \frac{1}{2} \partial_y^m \left\{ \widehat{Z}_k(y, s/2, \eta+y+w)\, \Phi(\eta+y+w, s/2, y+w) \right\} \\ & + \int_0^{s/2} \partial_y^m \left\{ \partial_s \widehat{Z}_k(y, s-\sigma, \eta+y+w)\, \Phi(\eta+y+w, \sigma, y+w) \right\} \, d\sigma \end{aligned}$$

and it follows by (98) and (102) that

$$\left\| \partial_y^m \left\{ \partial_s \widehat{Z}_k(y, s-\sigma, y+\eta+w) \Phi(y+\eta+w, \sigma, y+w) \right\} 1_{\{\sigma \le s/2\}} \right\| \overset{s,\sigma,\eta,y,w}{\ll} g(\eta, \sigma) \qquad (120)$$

Hence, one finds by (117-120), and the procedure to derive (116) that

$$\begin{aligned} & \partial_s \partial_y^m \int_0^{s/2} \int_{\mathbb{R}^d} \widehat{Z}_k(y, s-\sigma, \eta) \Phi(\eta, \sigma, y+w)\, d\eta\, d\sigma \qquad (121) \\ = \ & \frac{1}{2} \int_{\mathbb{R}^d} \partial_y^m \left\{ \widehat{Z}_k(y, s/2, \eta+y)\, \Phi(\eta+y, s/2, y+w) \right\} \, d\eta \\ & + \int_0^{s/2} \int_{\mathbb{R}^d} \partial_y^m \left\{ \partial_s \widehat{Z}_k(y, s-\sigma, \eta+y)\, \Phi(\eta+y, \sigma, y+w) \right\} d\eta \; d\sigma \end{aligned}$$

is continuous on $[a,1] \times \mathbb{R}^{2d}$, $\forall a > 0$ whence on $I \times \mathbb{R}^{2d}$. Returning to (112), we find by (107) that

$$\| \partial_x^n \left[\Phi(x+\xi, s-\sigma, x) - \Phi(x+v, s-\sigma, x) \right] \| \overset{s,\sigma,\xi,v,x}{\ll} |\xi - v|^{\varsigma} \qquad (122)$$

for all $0 \le n \le m$, $s \in [a,1]$ and $\sigma \in [0, s/2]$, for any $a > 0$. This is enough to conclude from the arguments in (36-38) that

$$\begin{aligned} & \partial_y^m \left[\partial_y^k \int_0^{s/2} \int_{\mathbb{R}^d} \widehat{Z}(y, \sigma, \eta+z) \Phi(\eta+z, s-\sigma, z)\, d\eta\, d\sigma \right]_{z=y+w} \qquad (123) \\ = \ & \int_0^{s/2} \int_{\mathbb{R}^d} \partial_y^m \left\{ \widehat{Z}_k(y, \sigma, \eta+y) \Phi(\eta+y, s-\sigma, y+w) \right\} d\eta\, d\sigma \end{aligned}$$

for $s \in [a,1]$. Now, we use (109) to establish that

$$\| \partial_u \partial_x^n \left[\Phi(x+\xi, u-\sigma, x) - \Phi(x+v, u-\sigma, x) \right] \| \overset{u,\sigma,\xi,v,x}{\ll} |\xi - x|^{\overline{\varsigma}} \qquad (124)$$

for all $0 \le n \le m$, $x, \xi, v \in \mathbb{R}^d$, $u \in [a,1]$, and $\sigma \in [0, u/2]$. From this, the divergence theorem, (97), (99), and (103), one can determine that

$$\begin{aligned}
& \partial_u \int_{\mathbb{R}^d} \partial_y^m \left\{ \widehat{Z}_k(y,\sigma,\eta+y)\Phi(\eta+y,u-\sigma,y+w) \right\} d\eta \qquad (125) \\
= & \int_{B^c} \partial_u \partial_y^m \left\{ \widehat{Z}_k(y,\sigma,\eta+y)\Phi(\eta+y,u-\sigma,y+w) \right\} d\eta \\
& + \int_{\partial B} \partial_\eta^l \partial_u \partial_y^m \left\{ Z(-\eta,\sigma,y)\Phi(y,u-\sigma,y+w) \right\} d\eta_1 \wedge \ldots \wedge d\eta_{i-1} \wedge d\eta_{i+1} \wedge \ldots \wedge d\eta_d \\
& + \int_B \partial_u \partial_y^m \left\{ \left[Z_k(-\eta,\sigma,\eta+y) - Z_k(-\eta,\sigma,y) \right] \Phi(y,u-\sigma,y+w) \right\} d\eta \\
& + \int_B \partial_u \partial_y^m \left\{ \widehat{Z}_k(y,\sigma,\eta+y) \left[\Phi(\eta+y,u-\sigma,y+w) - \Phi(y,u-\sigma,y+w) \right] \right\} d\eta
\end{aligned}$$

is continuous in $y, w \in \mathbb{R}^d$, $u \in [a,1]$, and $\sigma \in [0, u/2]$ and

$$\left\| \partial_u \int_{\mathbb{R}^d} \partial_y^m \left\{ \widehat{Z}_k(y,\sigma,\eta+y)\Phi(\eta+y,u-\sigma,y+w) \right\} d\eta \, 1_{\left\{\sigma \le \frac{s'}{2}\right\}} \right\| \overset{s',u,\sigma,y,w}{\ll} \sigma^{\frac{\overline{\varsigma}-q}{q}} \qquad (126)$$

for all $y, w \in \mathbb{R}^d$, $s' \in [a,1]$, $u \in [s',1]$, and $\sigma \in [0,1/2]$. Hence, it follows by dominated convergence, Fubini, and the fundamental theorem of calculus that

$$\begin{aligned}
& \partial_u \int_0^{s'/2} \int_{\mathbb{R}^d} \partial_y^m \left\{ \widehat{Z}_k(y,\sigma,\eta+y)\Phi(\eta+y,u-\sigma,y+w) \right\} d\eta d\sigma \qquad (127) \\
= & \int_0^{s'/2} \int_{\mathbb{R}^d} \partial_y^m \left\{ \widehat{Z}_k(y,\sigma,\eta+y)\partial_u \Phi(\eta+y,u-\sigma,y+w) \right\} d\eta d\sigma
\end{aligned}$$

is continuous on $y, w \in \mathbb{R}^d$, $s' \in [a,1]$, $u \in [s',1]$. Thus,

$$\begin{aligned}
& \int_0^{s/2} \int_{\mathbb{R}^d} \partial_y^m \left\{ \widehat{Z}_k(y,\sigma,\eta+y)\Phi(\eta+y,s-\sigma,y+w) \right\} d\eta d\sigma \qquad (128) \\
& - \int_0^{s'/2} \int_{\mathbb{R}^d} \partial_y^m \left\{ \widehat{Z}_k(y,\sigma,\eta+y)\Phi(\eta+y,s'-\sigma,y+w) \right\} d\eta d\sigma \\
= & \int_{s'}^{s} \left[\frac{1}{2} \int_{\mathbb{R}^d} \partial_y^m \left\{ \widehat{Z}_k(y,u/2,\eta+y)\Phi(\eta+y,s-u/2,y+w) \right\} d\eta \right. \\
& \left. + \int_0^{s'/2} \int_{\mathbb{R}^d} \partial_y^m \left\{ \widehat{Z}_k(y,\sigma,\eta+y)\partial_u \Phi(\eta+y,u-\sigma,y+w) \right\} d\eta d\sigma \right] du
\end{aligned}$$

for $a \le s' < s \le 1$ and any $a > 0$ whence for $0 < s' < s \le 1$. (113) follows from the proof of the fundamental theorem of calculus and (121).) The first two terms on the right hand side of (113) are trivially bounded (see the more complicated development

(131) to follow) by (98) with $x = y, \xi = w$; (97) with $x = y, \xi = \eta$; and (102) with $x = y + w$, $\xi = \eta - w$; and the fact

$$\left|\frac{|a|^q}{\alpha}\right|^{\frac{1}{q-1}} + \left|\frac{|b|^q}{\beta}\right|^{\frac{1}{q-1}} \geq \left|\frac{|a+b|^q}{\alpha+\beta}\right|^{\frac{1}{q-1}} \quad \forall a, b \in \mathbb{R}^d,\ \alpha, \beta > 0. \tag{129}$$

This fact follows inter alia from the proof of Theorem 9.4.2 (see the equation following (9.4.15) on p. 254) in Friedman [6] and is the reason we used a norm other than Euclidean distance. Next, recalling (111) and the argument in (42), one finds that the third term of (113) is bounded above by all m^1, m^2-combinations of

$$\begin{aligned}
&\int_0^{\frac{s}{2}} \int_{B^c} \left\|\partial_y^{m^1} \widehat{Z}_k(y,\sigma,y+\eta)\right\| \left\|\partial_s \partial_y^{m^2} \Phi(y+\eta, s-\sigma, y+w)\right\| d\eta d\sigma \qquad (130)\\
&+ \int_0^{\frac{s}{2}} \left\|\int_B \partial_\eta^k \partial_y^{m^1} Z(-\eta,\sigma,y) d\eta\right\| \left\|\partial_s \partial_y^{m^2} \Phi(y, s-\sigma, y+w)\right\| d\sigma\\
&+ \int_0^{\frac{s}{2}} \int_B \left\|\partial_y^{m^1} [Z_k(-\eta,\sigma,y+\eta) - Z_k(-\eta,\sigma,y)]\right\| d\eta \left\|\partial_s \partial_y^{m^2} \Phi(y, s-\sigma, y+w)\right\| d\sigma\\
&+ \int_0^{\frac{s}{2}} \int_B \left\|\partial_y^{m^1} \widehat{Z}_k(y,\sigma,y+\eta)\right\| \left\|\partial_s \partial_y^{m^2} [\Phi(y, s-\sigma, y+w) - \Phi(y+\eta, s-\sigma, y+w)]\right\| d\eta d\sigma.
\end{aligned}$$

Furthermore, letting $c = c' + c"$ with $c', c" > 0$ and using (97) with $x = y$, $\xi = \eta$, (103) with $x = y + w$, $\xi = \eta - w$, (129), and the fact $s - \sigma \geq s/2$, one finds that the first term in (130) is majorized by

$$\begin{aligned}
&s^{-\frac{d+2q-\varsigma}{q}} \int_0^{\frac{s}{2}} \int_{B^c} \sigma^{-\frac{d+|k|}{q}} \exp\left\{-c\left|\frac{|\eta|^q}{\sigma}\right|^{\frac{1}{q-1}} - c\left|\frac{|w-\eta|^q}{s-\sigma}\right|^{\frac{1}{q-1}}\right\} d\eta\, d\sigma \qquad (131)\\
&\overset{s,w}{\ll} s^{-\frac{d+2q-\varsigma}{q}} h_{c'}(w,s) \int_0^{\frac{s}{2}} \int_{\left\{|\eta| \geq \sigma^{-1/q}\right\}} \sigma^{-\frac{|k|}{q}} \exp\left\{-c"|\eta|^{q/(q-1)}\right\} d\eta\, d\sigma\\
&\overset{s,w}{\ll} s^{(\varsigma-d-q)/q} h_{c'}(w,s) \qquad \forall w \in \mathbb{R}^d,\ s \in I.
\end{aligned}$$

Next, (97) with $x = y - \eta$, $\xi = \eta$, a now standard divergence theorem application followed by (103) with $x = y + w$, $\xi = -w$ reduces the second term of (130) to

$$\int_0^{\frac{s}{2}} \left\|\partial_s \partial_y^{m^2} \Phi(y, s-\sigma, y+w)\right\| d\sigma \overset{s,w}{\ll} s^{(\varsigma-d-q)/q} h_c(w,s). \tag{132}$$

Moreover, the same use of (103) and (99) with $x = y$, $v = y - \eta$, $\xi = \eta$ can be used to bound the third term of (130) by a constant times

$$\int_0^{\frac{s}{2}} \int_B |\eta|^{\overline{\gamma}} \sigma^{-(d+|k|)/q} h_c(\eta,\sigma) d\eta d\sigma\, s^{(\varsigma-d-2q)/q} h_c(w,s) \overset{s,w}{\ll} s^{(\varsigma+\overline{\gamma}-d-q-|k|)/q} h_c(w,s). \tag{133}$$

Finally, we find by (97) with $x=y$, $\xi=\eta$, (109) with $x=y+w$, $\xi=-w$, $v=\eta-w$, (110), and (129) that for any $c'+c"=c$ the last term in (130) is majorized by

$$s^{(\overline{\varsigma}-d-2q)/q}\int_0^{\frac{s}{2}}\int_B \sigma^{-(d+|k|)/q}h_c(\eta,\sigma)\,|\eta|^{\overline{\gamma}}[h_c(w,s-\sigma)+h_c(w-\eta,s-\sigma)]d\eta d\sigma \quad (134)$$
$$\overset{s,w}{\ll} s^{(\overline{\varsigma}-d-2q)/q}h_{c'}(w,s)\int_0^{\frac{s}{2}}\int_B \sigma^{-(d+|k|)/q}h_{c"}(\eta,\sigma)\,|\eta|^{\overline{\gamma}}d\eta d\sigma \overset{s,w}{\ll} s^{(3\overline{\gamma}-d-q-|k|)/q}h_{c'}(w,s).$$

Fortunately, the last term in (113) does not have any non-integrable singularities and we can bound this term by (98), (102), and (129) as

$$\left\|\int_0^{\frac{s}{2}}\int_{\mathbb{R}^d}\partial_y^m\{\partial_s\widehat{Z}_k(y,s-\sigma,y+\eta)\Phi(y+\eta,\sigma,y+w)\}d\eta d\sigma\right\| \quad (135)$$
$$\overset{s,w}{\ll} s^{-(d+q+|k|)/q}\int_0^{\frac{s}{2}}\int_{\mathbb{R}^d}h_c(\eta,s-\sigma)\frac{h_c(w-\eta,\sigma)}{\sigma^{(d+q-\varsigma)/q}}d\eta d\sigma \overset{s,w}{\ll} s^{-(d+q+|k|-\varsigma)/q}h_{c'}(w,s).$$

(iii) follows from (113), (130-134), (135), and bounds for the first two terms of (113).

(iv) Suppose we fix s,z,y,w. Then, it follows from (112) and (99) that

$$\|\partial_z^m\Gamma_k(z,s,z+w)-\partial_y^m\Gamma_k(y,s,y+w)\|\le C\,|z-y|^{\overline{\gamma}}s^{-(d+|k|)/q}h_c(w,s) \quad (136)$$
$$+\int_0^{\frac{s}{2}}\|\int\partial_z^m\{\widehat{Z}_k(z,\sigma,z+\eta)\Phi(z+\eta,s-\sigma,z+w)\}-\partial_y^m\{\widehat{Z}_k(y,\sigma,y+\eta)\Phi(y+\eta,s-\sigma,y+w)\}d\eta\|d\sigma$$
$$+\int_0^{\frac{s}{2}}\int\|\partial_z^m\{\widehat{Z}_k(z,s-\sigma,z+\eta)\Phi(z+\eta,\sigma,z+w)\}-\partial_y^m\{\widehat{Z}_k(y,s-\sigma,y+\eta)\Phi(y+\eta,\sigma,y+w)\}\|d\eta d\sigma.$$

There are no singularities in the last term so we can use (99) with $x=z$, $\xi=\eta$, $v=y$, (102) $x=z+w$, $\xi=\eta-w$, (97) $x=y$, $\xi=\eta$, (104) $x=z+w$, $\xi=\eta-w$, $v=y+w$, and (129) to conclude that this term is less than m^1,m^2-combinations of

$$\int_0^{\frac{s}{2}}\int\|\partial_z^{m^1}\widehat{Z}_k(z,s-\sigma,z+\eta)-\partial_y^{m^1}\widehat{Z}_k(y,s-\sigma,y+\eta)\|\|\partial_z^{m^2}\Phi(z+\eta,\sigma,z+w)\|d\eta d\sigma \quad (137)$$
$$+\int_0^{\frac{s}{2}}\int\|\partial_y^{m^1}\widehat{Z}_k(y,s-\sigma,y+\eta)\|\|\partial_z^{m^2}\Phi(z+\eta,\sigma,z+w)-\partial_y^{m^2}\Phi(y+\eta,\sigma,y+w)\|d\eta d\sigma$$
$$\overset{z,y,s}{\ll} |z-y|^{\overline{\gamma}}s^{-(d+|k|)/q}\int_0^{\frac{s}{2}}\sigma^{(\overline{\varsigma}-q)/q}\int_{\mathbb{R}^d}h_c(\eta,s-\sigma)\sigma^{-d/q}h_c(w-\eta,\sigma)d\eta\, d\sigma$$
$$\overset{z,y,s}{\ll} |z-y|^{\overline{\gamma}}s^{(\overline{\varsigma}-d-|k|)/q}h_{c'}(w,s)\quad\forall\,0<c'<c.$$

For second term of (136), we use the procedure (130-134) with $\partial_y^{m^2}\Phi(y+\eta,s-\sigma,y+w)$ and $[\partial_z^{m^1}\widehat{Z}_k(z,\sigma,z+\eta)-\partial_y^{m^1}\widehat{Z}_k(y,\sigma,y+\eta)]$ in place of $\partial_s\partial_y^{m^2}\Phi(y+\eta,s-\sigma,y+w)$ and

$\partial_y^{m^1} \widehat{Z}_k(y, \sigma, y+\eta)$, (99,102,107) instead of respectively (97,103,109), and the bound

$$\begin{aligned} &\|\partial_z^{m^1}[\widehat{Z}_k(z,\sigma,z+\eta)-\widehat{Z}_k(z-\eta,\sigma,z)]-\partial_y^{m^1}[\widehat{Z}_k(y,\sigma,y+\eta)-\widehat{Z}_k(y-\eta,\sigma,y)]\| \\ \leq\ & 2C\,|\eta|^{\overline{\varsigma}} \wedge |z-y|^{\overline{\varsigma}}\,\sigma^{-(d+|k|)/q}\,h_c(\eta,\sigma) \overset{z,y,\eta,\sigma}{\ll} |\eta|^{\overline{\gamma}} \cdot |z-y|^{\overline{\gamma}}\,\sigma^{-(d+|k|)/q}\,h_c(\eta,\sigma) \end{aligned} \tag{138}$$

(which is obtained from (99)) in lieu of (99) to find that

$$\begin{aligned} &\int_0^{\frac{s}{2}} \Big\| \int [\partial_z^{m^1}\widehat{Z}_k(z,\sigma,z+\eta)-\partial_y^{m^1}\widehat{Z}_k(y,\sigma,y+\eta)]\partial_y^{m^2}\Phi(y+\eta,s-\sigma,y+w)d\eta\Big\| d\sigma \\ &\overset{z,y,s}{\ll} |z-y|^{\overline{\gamma}} s^{-(d+|k|-\overline{\varsigma})/q} h_{c'}(w,s) \quad \forall\, 0<c'<c. \end{aligned} \tag{139}$$

Furthermore, we can employ the procedure (130-134) a third time with $\partial_z^{m^1}\widehat{Z}_k(z,\sigma,z+\eta)$ and $[\partial_z^{m^2}\Phi(z+\eta,s-\sigma,z+w) - \partial_y^{m^2}\Phi(y+\eta,s-\sigma,y+w)]$ in place of $\partial_y^{m^1}\widehat{Z}_k(y,\sigma,y+\eta)$ and $\partial_s\partial_y^{m^2}\Phi(y+\eta,s-\sigma,y+w)$, (105) instead of (103), and the bound

$$\begin{aligned} &\|\partial_z^{m^2}[\Phi(z,\tau,z+w)-\Phi(z+\eta,\tau,z+w)]-\partial_y^{m^2}[\Phi(y,\tau,y+w)-\Phi(y+\eta,\tau,y+w)]\| \\ \leq\ & 2C\,|\eta|^{\overline{\varsigma}} \wedge |z-y|^{\overline{\varsigma}}\,\tau^{\frac{\overline{\varsigma}-d-q}{q}} h_c(w,w-\eta,\tau) \overset{z,y,\eta,\sigma}{\ll} |\eta|^{\overline{\gamma}}\,|z-y|^{\overline{\gamma}}\,\tau^{\frac{\overline{\varsigma}-d-q}{q}} h_c(w,w-\eta,\tau) \end{aligned} \tag{140}$$

(which is obtained from (107) and (104)) in lieu of (109) to find that

$$\begin{aligned} &\int_0^{\frac{s}{2}} \Big\| \int \partial_z^{m^1}\widehat{Z}_k(z,\sigma,z+\eta)[\partial_z^{m^2}\Phi(z+\eta,s-\sigma,z+w)-\partial_y^{m^2}\Phi(y+\eta,s-\sigma,y+w)]d\eta\Big\| d\sigma \\ &\overset{z,y,s}{\ll} |z-y|^{\overline{\gamma}} s^{(\overline{\varsigma}-d-|k|)/q} h_{c'}(w,s) \quad \forall\, 0<c'<c. \end{aligned} \tag{141}$$

(iv) follows immediately from (136), (137), (139), and (141).

It remains to prove (97-109). However, the first bound in (97) and (98) follow from the relation $Z(x-\xi,s,\xi,0) = \widehat{Z}(x,s,\xi)$, (90), Condition (C3) and (9.3.15) of Friedman [6]. Moreover, (99) and (101) follow from Conditions (C3,C4) and (9.3.16) of Friedman. Finally, the first bound in (102) is just (9.6.11) of Friedman. (There appears to be a small inconsistency which propagates through Theorem 9.6.7 of Friedman [6] in the case $a \equiv 0$ and $|b| = r$: In order for (9.6.4) to hold in this case Lipschitz continuity for $\partial_x^h A_k$ with $|h| = r$ and $|k| = 2p$ would have to be assumed. However, this was not assumed in (C) on p. 260 of [6]. Still, under our conditions, especially (C4), the proof of Theorem 9.6.7 is correct and the first factor on the right hand side of (9.6.11) becomes $\text{const}/(t-\tau)^{(n+2p-\varsigma)/2p}$ as we require.)

We now establish (100) and the second bounds in (97) and (102). Unfortunately, the constants $C, c > 0$ in these bounds must be independent of r which precludes immediate application of the classical theory. Still, we do not require a completely new theory. In fact, if we used the notation $\zeta^k \doteq \zeta_1^{k_1}\zeta_2^{k_2}...\zeta_d^{k_d}$, and defined

$$V^r(t,\tau;y,\zeta) = I + \int_\tau^t \sum_{|k|=q} A_k(\frac{s}{\varepsilon_r},y)\,(i\zeta)^k\,V^r(s,\tau;y,\zeta)\,ds \tag{142}$$

for all $0 \le \tau \le t \le 1$, $y \in \mathbb{R}^d$, $\zeta \in \mathbb{C}^d$, and $r = 1,2,...$ Then, it follows from the proof of Lemma 14 of Dawson and Kouritzin [3] that

$$\left\|\partial_y^a V^r(t,\tau;y,\zeta)\right\| \le C\,\exp\left\{[\lambda\,|\beta|^q - \delta\,|\alpha|^q](t-\tau)\right\} \tag{143}$$

for some constants $C,\lambda,\delta > 0$ and all $0 \le |a| \le |b|$, $0 \le \tau \le t \le 1$, $y \in \mathbb{R}^d$, $\zeta = \alpha + i\beta \in \mathbb{C}^d$ and $r = 1,2,...$ Consequently, the second bound in (97) follows from the development on p. 249 (top half) and pp. 245-6 of Friedman. Moreover, we can continue by using (143) plus the considerations on pp. 249-250 with $A_k(y,t) = A_k(\frac{t}{\varepsilon_r},x+\xi)$, $B_k(y,t) = A_k(\frac{t}{\varepsilon_r},v+\xi)$ for $|k| = q$ and otherwise 0 to establish (100). Now, the second bound in (97) also constitutes a uniform (in r) version of Friedman (9.3.15) with $(x,\xi,y) = (0,\xi,x+\xi)$. Hence, the second bound in (102) follows from either a simplified version of the development for (103) (to follow) or from development of (9.6.11) on pp. 261-2 of Friedman.

Next, we establish (107-109) using (97-102) and assuming (103). Considering (109), one finds from (94) and the method of (112-113) that at least formally

$$\begin{aligned}\partial_s\partial_y^n\Phi(y+w,s,y) \;=\; & \partial_s\partial_y^n K(y+w,s,y) + \int_{\mathbb{R}^d}\partial_y^n\{K(y+w,\frac{s}{2},\eta+y)\Phi(\eta+y,\frac{s}{2},y)\}d\eta \\ & + \int_0^{s/2}\int_{\mathbb{R}^d}\partial_y^n\{\partial_s K(y+w,s-\sigma,\eta+y)\Phi(\eta+y,\sigma,y)\}\,d\eta\,d\sigma \\ & + \int_0^{s/2}\int_{\mathbb{R}^d}\partial_y^n\{K(y+w,\sigma,\eta+y)\partial_s\Phi(\eta+y,s-\sigma,y)\}\,d\eta\,d\sigma\end{aligned} \tag{144}$$

for all $s \in I$, $y,w \in \mathbb{R}^d$. We show below that $\partial_s\partial_y^n\Phi(y+w,s,y)$is the unique solution to this integral equation. (It is not establishable a priori that Leibniz's rule applies or that ∂_s commutes with the integrals and ∂_y^n.) Now, we know from (92), (97), the fact $|w-\eta|^\varsigma/s^{\frac{\varsigma}{q}} \overset{w,s,\eta}{\ll} \exp\left\{a\,s^{-\frac{1}{q-1}}|w-\eta|^{\frac{q}{q-1}}\right\}$ for any $a > 0$, and Condition (C3) that

$$\begin{aligned}\|\partial_y^n K(y+w,s,y+\eta)\| = & \sum_{|k|=q}\left\|\partial_y^n\{A_k^0(y+w,y+\eta)\,\widehat{Z}_k(y+w,s;y+\eta)\}\right\| \\ + & \sum_{|k|<q}\left\|\partial_y^n\{A_k^0(y+w)\,\widehat{Z}_k(y+w,s;y+\eta)\}\right\| \overset{y,w,s,\eta}{\ll} s^{(\varsigma-d-q)/q}\,h_{c'}(w-\eta,s)\end{aligned} \tag{145}$$

for all $0 < c' < c$, $|n| \leq |b|$. Similarly, it follows with (98) instead of (97) that

$$\|\partial_y^n \partial_s K(y+w,s,y+\eta)\| \overset{y,w,s,\eta}{\ll} s^{(\varsigma-d-2q)/q}\, h_{c'}(w-\eta,s) \quad \forall 0 < c' < c. \tag{146}$$

Hence, in the case where $|w-z|^q > s$, one finds that

$$\begin{aligned}\|\partial_y^n[K(y+w,s,y+\eta) - K(y+z,s,y+\eta)]\| &\overset{y,z,w,s,\eta}{\ll} s^{(\varsigma-d-q)/q}\, h_{c'}(w-\eta,z-\eta,s) \\ &\overset{y,z,w,s,\eta}{\ll} |w-z|^{\bar\varsigma}\, s^{(\bar\varsigma-d-q)/q}\, h_{c'}(w-\eta,z-\eta,s)\end{aligned} \tag{147}$$

$$\|\partial_s\partial_y^n[K(y+w,s,y+\eta)-K(y+z,s,y+\eta)]\| \overset{y,z,w,s,\eta}{\ll} \frac{|w-z|^{\bar\varsigma}}{s^{(d+2q-\bar\varsigma)/q}}\, h_{c'}(w-\eta,z-\eta,s). \tag{148}$$

Conversely, when $|w-z|^q \leq s$, we have by (92) and Conditions (C3,C4) that

$$\begin{aligned}&\|\partial_s\partial_y^n[K(y+w,s,y+\eta) - K(y+z,s,y+\eta)]\| \\ \leq\; &\sum_{n^1+n^2=n} \binom{n}{n^1\, n^2} \Bigg\{ \sum_{|k|\leq q} |w-z|^{\varsigma} \left\|\partial_y^{n^2}\partial_s \widehat{Z}_k(y+w,s;y+\eta)\right\| \\ &+ \sum_{|k|=q} |\eta-z|^{\varsigma} \left\|\partial_y^{n^2}\partial_s[\widehat{Z}_k(y+w,s;y+\eta) - \widehat{Z}_k(y+z,s;y+\eta)]\}\right\| \\ &+ \sum_{|k|<q} \left\|\partial_y^{n^2}\partial_s[\widehat{Z}_k(y+w,s;y+\eta) - \widehat{Z}_k(y+z,s;y+\eta)]\}\right\| \Bigg\}\end{aligned} \tag{149}$$

and by the mean value theorem and (98) with $x = y+v$, $\xi = \eta - v$ that

$$\left\|\partial_y^{n^2}\partial_s[\widehat{Z}_k(y+w,s;y+\eta)-\widehat{Z}_k(y+z,s;y+\eta)]\right\| \overset{y,z,w,s,\eta}{\ll} \frac{|w-z|}{s^{(d+q+|k|+1)/q}}\, h_c(v-\eta,s) \tag{150}$$

for some v on the line connecting w and z. Moreover, using (89) and the fact $|z-\eta| \leq |z-w| + |v-\eta| \leq s^{1/q} + |v-\eta|$, one finds that

$$\frac{|\eta-z|^{\varsigma}\,|w-z|}{s^{(d+2q+1)/q}}\, h_c(v-\eta,s) \ll \frac{|\eta-z|^{\varsigma}\,|w-z|^{\bar\varsigma}}{s^{(d+2q+\bar\varsigma)/q}}\, h_{c_1}(z-\eta,s) \ll \frac{|w-z|^{\bar\varsigma}}{s^{(d+2q-\bar\varsigma)/q}}\, h_{c'}(z-\eta,s) \tag{151}$$

for $c_1 \doteq c\,2^{-\frac{q}{q-1}}$ and some $0 < c' < c_1$. It follows by (149), (98), (150), and (151) that

$$\|\partial_s\partial_y^n[K(y+w,s,y+\eta) - K(y+z,s,y+\eta)]\| \overset{y,z,w,s,\eta}{\ll} \frac{|w-z|^{\bar\varsigma}}{s^{(d+2q-\bar\varsigma)/q}}\, h_{c'}(z-\eta,s) \tag{152}$$

when $|w-z|^q \leq s$. Similarly, when $|w-z|^q \leq s$

$$\|\partial_y^n K(y+w,s,y+\eta) - \partial_y^n K(y+z,s,y+\eta)\| \overset{y,z,w,s,\eta}{\ll} \frac{|w-z|^{\bar\varsigma}}{s^{(d+q-\bar\varsigma)/q}}\, h_{c'}(z-\eta,s). \tag{153}$$

Hence, it follows from (144), (152), (148), (153), (147), (102), (103) and (129) that

$$\|\partial_s\partial_y^n[\Phi(y+w,s,y)-\Phi(y+z,s,y)]\| \overset{y,z,w,s,\eta}{\ll} \frac{|w-z|^{\overline{\varsigma}}}{s^{(d+2q-\overline{\varsigma})/q}}\,h_{c"}(z,w,s) \tag{154}$$

for any $0<c"<c'$ which establishes (109). (107) and (108) follow similarly.

Now, we must establish that $\partial_s\partial_x^n\Phi(x+\xi,s,x)$ exists, is continuous and satisfies (103) and (144). This will be done inductively. Suppose either $n\equiv 0$ or $\partial_s\partial_x^m\Phi(x+\xi,s,x)$ is continuous and satisfies (103) for all $m<n$. Then, it follows from (94), (102), (145), and dominated convergence that

$$\partial_x^n\Phi(x+\xi,s,x)=\mathcal{A}(s,\xi)+\int_0^s\int_{\mathbb{R}^d}\mathcal{E}(s,\xi,\sigma,\eta)\partial_x^n\Phi(x+\eta,\sigma,x)d\eta d\sigma, \tag{155}$$

where, $\mathcal{A}(s,\xi;x)$ and $\mathcal{E}(s,\xi,\sigma,\eta;x)$ are defined by

$$\mathcal{A}\doteq\partial_x^nK(x+\xi,s,x)+\sum_{\substack{i+j=n\\0\le j<n}}\binom{n}{i\,j}\int_0^s\int_{\mathbb{R}^d}\partial_x^iK(x+\xi,s-\sigma,\eta+x)\partial_x^j\Phi(x+\eta,\sigma,x)d\eta d\sigma \tag{156}$$

$$\overset{s,\xi;x}{\ll}\ s^{(\varsigma-d-q)/q}\,h_c(\xi,s),$$

$$\mathcal{E}\doteq K(x+\xi,s-\sigma,\eta+x)\overset{s,\xi,\sigma,\eta;x}{\ll}(s-\sigma)^{(\varsigma-d-q)/q}h_c(\xi-\eta,s-\sigma). \tag{157}$$

Then, it follows by Lemma 6 uniqueness (with $t-\tau=s$, $\sigma=\sigma-\tau$), (102), and the inductive hypothesis that $\partial_x^n\Phi(x+\xi,s,x)$ is continuous on x, $\xi\in\mathbb{R}^d$, $s\in I$ and

$$\partial_x^n\Phi(x+\xi,s,x)=\sum_{m=0}^{\infty}K_m^n(x+\xi,s,x), \tag{158}$$

where

$$\begin{aligned}K_0^n(x+\xi,s,x)&=\partial_x^nK(x+\xi,s,x) \qquad (159)\\ &+\sum_{\substack{n^1+n^2=n\\0\le n^2<n}}\binom{n}{n^1\,n^2}\left\{\int_0^{\frac{s}{2}}\int_{\mathbb{R}^d}\partial_x^{n^1}K(x+\xi,s-\sigma,x+\eta)\partial_x^{n^2}\Phi(x+\eta,\sigma,x)\,d\eta d\sigma\right.\\ &\left.+\int_0^{\frac{s}{2}}\int_{\mathbb{R}^d}\partial_x^{n^1}K(x+\xi,\sigma,x+\eta)\partial_x^{n^2}\Phi(x+\eta,s-\sigma,x)\,d\eta d\sigma\right\},\end{aligned}$$

$$\begin{aligned}K_m^n(x+\xi,s,x)\ &=\ \int_0^{\frac{s}{2}}\int_{\mathbb{R}^d}K(x+\xi,s-\sigma,x+\eta)K_{m-1}^n(x+\eta,\sigma,x)\,d\eta d\sigma \qquad (160)\\ &+\int_0^{\frac{s}{2}}\int_{\mathbb{R}^d}K(x+\xi,\sigma,x+\eta)K_{m-1}^n(x+\eta,s-\sigma,x)\,d\eta d\sigma.\end{aligned}$$

First, we fix $x,\xi,\eta\in\mathbb{R}^d$, $a>0$, n^1, n^2 and consider

$$H(s)\doteq\int_0^{\frac{s}{2}}\partial_x^{n^1}K(x+\xi,s-\sigma,\eta+x)\,\partial_x^{n^2}\Phi(x+\eta,\sigma,x)\,d\sigma\quad\forall s\in(0,1].\tag{161}$$

Then, for all $\{\lambda_i\}_{i=1}^\infty$, s such that $\frac{s}{2}-\lambda_i,\frac{s}{2}+\lambda_i,\ s,\ s+\lambda_i\in[a,1]$ it follows by the (uniform) continuity of $\partial_x^{n^2}\Phi$, $\partial_x^{n^1}K$, $\partial_s\partial_x^{n^1}K$ on $[a,1]$ and dominated convergence that

$$\begin{aligned}
[H(s+\lambda_i)-H(s)]/\lambda_i&=\frac{1}{\lambda_i}\int_{\frac{s}{2}}^{\frac{s+\lambda_i}{2}}\partial_x^{n^1}K(x+\xi,s-\sigma,\eta+x)\,\partial_x^{n^2}\Phi(x+\eta,\sigma,x)\,d\sigma \\
&+\frac{1}{\lambda_i}\int_{\frac{s}{2}}^{\frac{s+\lambda_i}{2}}\partial_x^{n^1}[K(x+\xi,s+\lambda_i-\sigma,\eta+x)-K(x+\xi,s-\sigma,\eta+x)]\partial_x^{n^2}\Phi(x+\eta,\sigma,x)d\sigma \\
&+\int_0^{\frac{s}{2}}\frac{1}{\lambda_i}\int_0^{\lambda_i}\partial_t\partial_x^{n^1}K(x+\xi,s+t-\sigma,\eta+x)\,dt\,\partial_x^{n^2}\Phi(x+\eta,\sigma,x)\,d\sigma \\
&\overset{i\to\infty}{\to}\frac{1}{2}\partial_x^{n^1}K(x+\xi,s/2,\eta+x)\,\partial_x^{n^2}\Phi(x+\eta,s/2,x) \\
&+\int_0^{\frac{s}{2}}\partial_s\partial_x^{n^1}K(x+\xi,s-\sigma,\eta+x)\partial_x^{n^2}\Phi(x+\eta,\sigma,x)\,d\sigma.
\end{aligned}\tag{162}$$

Next, using (162), one finds $\partial_sH(s)$ exists and is continuous on $(2a,1]$. Moreover, it follows easily from (145), (146), (102), and (129) that for $c"\doteq c-c'>0$

$$\begin{aligned}
\|\partial_sH(s;x,\xi,\eta)\|&\overset{\eta,s}{\ll}a^{\frac{\varsigma-d-2q}{q}}h_{c'}(\xi,s)\,[a^{\frac{\varsigma-d}{q}}h_{c"}(\eta,\frac{s}{2})+\int_0^{\frac{s}{2}}\sigma^{\frac{\varsigma-d-q}{q}}h_{c"}(\eta,\sigma)d\sigma] \\
&\overset{\eta,s}{\ll}h_{c"}(\eta,\frac{1}{2})+\int_0^{\frac{s}{2}}\sigma^{(\frac{3\varsigma}{2}-q)/q}|\eta|^{-d-\varsigma/2}d\sigma\,1_{|\eta|\geq1}+\int_0^{\frac{s}{2}}\sigma^{(\varsigma/2-q)/q}|\eta|^{-d+\varsigma/2}d\sigma\,1_{|\eta|<1} \\
&\overset{\eta,s}{\ll}|\eta|^{-d-\varsigma/2}1_{|\eta|\geq1}+|\eta|^{-d+\varsigma/2}1_{|\eta|<1}\qquad\forall s\in(2a,1],\ \eta\in\mathbb{R}^d
\end{aligned}\tag{163}$$

so by dominated convergence $s\to\int_{\mathbb{R}^d}\partial_sH(s;x,\xi,\eta)d\eta$ is continuous on $(2a,1]$ and

$$\int_{\mathbb{R}^d}[H(t+\lambda;x,\xi,\eta)-H(t;x,\xi,\eta)]\,d\eta=\int_t^{t+\lambda}\int_{\mathbb{R}^d}\partial_sH(s;x,\xi,\eta)\,d\eta\,d\sigma\tag{164}$$

for all $t,t+\lambda\in[2a,1]$ by Fubini. Hence, by (159), (161), (162), (164), the fundamental theorem of calculus, the inductive hypothesis and a development similar to (162-164) for the last term in (159) one finds that

$$\begin{aligned}
&\partial_sK_0^n(x+\xi,s,x)=\partial_s\partial_x^nK(x+\xi,s,x) \\
&+\sum_{\substack{n^1+n^2=n\\0\leq n^2<n}}\binom{n}{n^1n^2}\Big\{\int_{\mathbb{R}^d}\partial_x^{n^1}K(x+\xi,s/2,x+\eta)\partial_x^{n^2}\Phi(x+\eta,s/2,x)\,d\eta
\end{aligned}\tag{165}$$

$$+\int_0^{\frac{s}{2}}\int_{\mathbb{R}^d}\partial_s\partial_x^{n^1}K(x+\xi,s-\sigma,x+\eta)\partial_x^{n^2}\Phi(x+\eta,\sigma,x)\,d\eta d\sigma$$
$$\left.+\int_0^{\frac{s}{2}}\int_{\mathbb{R}^d}\partial_x^{n^1}K(x+\xi,\sigma,x+\eta)\partial_s\partial_x^{n^2}\Phi(x+\eta,s-\sigma,x)\,d\eta d\sigma\right\}$$

is continuous on $(2a,1]$ (any a) whence on $(0,1]$. Moreover, it follows from (146), (145), (102), (103) with $n^2<n$ and (129) that there exists a $c>0$ such that

$$\|\partial_s K_0^n(x+\xi,s,x)\| \overset{x,\xi,s}{\ll} s^{(\varsigma-d-2q)/q}h_c(\xi,s)\,. \tag{166}$$

Now, initiating induction, we assume $s\to\partial_s K_{m-1}^n(x+\xi,s,x)$ is continuous on $(0,1]$ and

$$\left\|K_{m-1}^n(x+\xi,s,x)\right\|\vee s\cdot\left\|\partial_s K_{m-1}^n(x+\xi,s,x)\right\| \overset{x,\xi,s}{\ll} s^{(m\varsigma-d-q)/q}h_{c_{m-1}}(\xi,s) \tag{167}$$

for all $x,\xi\in\mathbb{R}^d$, $s\in I$, and some $\{c_i\}_{i=0}^\infty\subset(0,1]$, $m\in\mathbb{N}$. Then, one finds by (160) and the method of (161-166) that $s\to\partial_s K_m^n(x+\xi,s,x)$ is continuous on $(0,1]$,

$$\begin{aligned}\partial_s K_m^n(x+\xi,s,x) &= \int_{\mathbb{R}^d}K(x+\xi,s/2,x+\eta)K_{m-1}^n(x+\eta,s/2,x)\,d\eta \\ &+\int_0^{\frac{s}{2}}\int_{\mathbb{R}^d}\partial_s K(x+\xi,s-\sigma,x+\eta)K_{m-1}^n(x+\eta,\sigma,x)\,d\eta d\sigma \\ &+\int_0^{\frac{s}{2}}\int_{\mathbb{R}^d}K(x+\xi,\sigma,x+\eta)\partial_s K_{m-1}^n(x+\eta,s-\sigma,x)\,d\eta d\sigma\end{aligned} \tag{168}$$

on $(0,1]$ and (with (160), (168), (167), (145), (146), and (129))

$$\|K_m^n(x+\xi,s,x)\|\vee s\cdot\|\partial_s K_m^n(x+\xi,s,x)\| \overset{x,\xi,s}{\ll} s^{((m+1)\varsigma-d-q)/q}h_{c_m}(\xi,s) \tag{169}$$

for all $x,\xi\in\mathbb{R}^d$, $s\in I$, where $c_m=b>0$ for all $m>d/\varsigma$. Indeed, with care (see proof of Lemma 5 for the method) one finds that the constants in (169) are such that

$$\sum_{m=0}^\infty\|K_m^n(x+\xi,u,x)\| \overset{\xi,u}{\ll} u^{(\varsigma-d-q)/q}h_b(\xi,u)\quad\forall\xi\in\mathbb{R}^d,\ u\in(0,1], \tag{170}$$

$$\sum_{m=0}^\infty\|\partial_u K_m^n(x+\xi,u,x)\| \overset{\xi,u}{\ll} u^{(\varsigma-d-2q)/q}h_b(\xi,u)\quad\forall\xi\in\mathbb{R}^d,\ u\in(0,1]. \tag{171}$$

Thus, by (158), dominated convergence, another application of Fubini, and the fundamental theorem of calculus it follows that

$$\partial_s\partial_x^n\Phi(x+\xi,s,x)=\sum_{m=0}^\infty\partial_s K_m^n(x+\xi,s,x)\quad\forall s\in[a,1],\ a>0. \tag{172}$$

Moreover, from (170), (171), (172), (168), (145), (146), and Fubini it follows that

$$\begin{aligned}\partial_s\partial_x^n\Phi(x+\xi,s,x) &= \partial_s K_0^n(x+\xi,s,x) \\ &+\int_{\mathbb{R}^d} K(x+\xi,s/2,x+\eta)\partial_x^n\Phi(x+\eta,s/2,x)\,d\eta \\ &+\int_0^{\frac{s}{2}}\int_{\mathbb{R}^d}\partial_s K(x+\xi,s-\sigma,x+\eta)\partial_x^n\Phi(x+\eta,\sigma,x)\,d\eta d\sigma \\ &+\int_0^{\frac{s}{2}}\int_{\mathbb{R}^d} K(x+\xi,\sigma,x+\eta)\partial_s\partial_x^n\Phi(x+\eta,s-\sigma,x)\,d\eta d\sigma \quad \forall s\in(0,1]\end{aligned} \tag{173}$$

and by easily establishable bounds on the first three terms (see (166), (145), (146), (102), and (129)) it follows by Lemma 5 that $\partial_s\partial_x^n\Phi(x+\xi,s,x)$ is the unique solution to (173) satisfying (103). Therefore, $\partial_s\partial_x^n\Phi(x+\xi,s,x)$ is continuous on $\mathbb{R}^d\times\mathbb{R}^d\times I$ and combining (173) with (165), one finds that (144) is indeed true.

Finally, it remains to show (104-106). We will inductively show (105) in detail. Suppose either $n\equiv 0$ or (105) holds for all $m<n$. We note by (95), a bound akin to (145) for $\partial_x^n K^r$, (102), and dominated convergence that

$$\begin{aligned}\partial_x^n\Phi^r(x+\xi,t,x,\tau) &= \partial_x^n K^r(x+\xi,t,x,\tau) \\ &+\int_{\frac{t+\tau}{2}}^t\int_{\mathbb{R}^d}\partial_x^n\left\{K^r(x+\xi,t,x+\eta,\sigma)\Phi^r(x+\eta,\sigma,x,\tau)\right\}d\eta d\sigma \\ &+\int_\tau^{\frac{t+\tau}{2}}\int_{\mathbb{R}^d}\partial_x^n\left\{K^r(x+\xi,t,x+\eta,\sigma)\Phi^r(x+\eta,\sigma,x,\tau)\right\}d\eta d\sigma.\end{aligned} \tag{174}$$

Hence, $\Theta^r(t,\xi,\tau;x,x')\doteq\partial_x^n[\Phi^r(x+\xi,t,x,\tau)-\Phi^r(x'+\xi,t,x',\tau)]$ satisfies

$$\Theta^r(t,\xi,\tau)=\mathcal{A}^r(t,\xi,\tau)+\int_\tau^t\int_{\mathbb{R}^d}\mathcal{E}^r(t,\xi,\sigma,\eta)\Theta^r(\sigma,\eta,\tau)d\eta d\sigma, \tag{175}$$

where

$$\begin{aligned}\mathcal{A}^r(t,\xi,\tau;x,x') &\doteq \partial_x^n[K^r(x+\xi,t,x,\tau)-K^r(x'+\xi,t,x',\tau)] \\ &+\int_\tau^t\int_{\mathbb{R}^d}\partial_x^n\{[K^r(x+\xi,t,\eta+x,\sigma)-K^r(x'+\xi,t,\eta+x',\sigma)]\Phi^r(\eta+x',\sigma,x',\tau)\}d\eta\, d\sigma \\ &+\sum_{\substack{n^1+n^2=n\\0\le n^2<n}}\binom{n}{n^1\,n^2}\int_\tau^t\int_{\mathbb{R}^d}\partial_x^{n^1}K^r(x+\xi,t,\eta+x,\sigma)\partial_x^{n^2}[\Phi(\eta+x,\sigma,x,\tau)-\Phi(\eta+x',\sigma,x',\tau)]d\eta d\sigma\end{aligned} \tag{176}$$

$$\mathcal{E}^r(t,\xi,\sigma,\eta;x)\doteq K^r(x+\xi,t,\eta+x,\sigma)\overset{t,\xi,\sigma,\eta;x}{\ll}(t-\sigma)^{(\varsigma-d-q)/q}h_c(\xi-\eta,t-\sigma). \tag{177}$$

Now, it follows from (93), (97), (100), and the bound (c.f. Condition (C3,C4))

$$\left\|\partial_x^m[A_k(\frac{t}{\varepsilon_r}, x+\xi, x+\eta) - A_k(\frac{t}{\varepsilon_r}, x'+\xi, x'+\eta)]\right\| \overset{t,r,x,x',\xi,\eta}{\ll} |\xi-\eta|^{\varsigma}\cdot|x-x'|^{\varsigma} \quad (178)$$

for all $|m| \le |b|$; $x, x', \xi, \eta \in \mathbb{R}^d$; $t \in I$ that for $c' < c$

$$\|\partial_x^m[K^r(x+\xi, t, \eta+x, \sigma) - K^r(x'+\xi, t, \eta+x', \sigma)]\| \overset{t,r,\sigma,x,x',\xi,\eta}{\ll} \frac{|x-x'|^{\overline{\varsigma}}\, h_{c'}(\xi-\eta, t-\sigma)}{(t-\sigma)^{(d+q-\overline{\varsigma})/q}}. \quad (179)$$

Hence, one finds by (176), (179), (102), a bound like (145) for K^r, (105) with $m < n$, and (129) that there is a $c_1 > 0$ such that

$$\|\mathcal{A}^r(t,\xi,\tau;x,x')\| \overset{t,r,\tau,x,x',\xi}{\ll} |x-x'|^{\overline{\varsigma}}(t-\tau)^{(\overline{\varsigma}-d-q)/q}h_{c_1}(\xi, t-\tau) \quad (180)$$

so (105) with $m = n$ follows by (175), (180), (177), Lemma 6 uniqueness and (102). Recalling (144), one finds that the proofs of (104) and (106) are very similar to that of (105). ■

The following lemma is used following (63) of Section 3.

Lemma 4. *Suppose $\{q^r\}_{r=1}^{\infty}$ is a sequence of functions on $I \times \mathbb{R}^d$ such that*

$$(i)\ |q^r(t,x)| \overset{t,r,x}{\ll} 1\ \forall t \in I,\ x \in \mathbb{R}^d,\ r = 1,2,... \quad (181)$$

$$(ii)\ |q^r(t,x) - q^r(t',x')| \overset{t,t',r,x,x'}{\ll} |t-t'|^{\alpha} + |x-x'|^{\alpha}\ \forall t,t',\ x,x',\ r=1,2,... \quad (182)$$

Then, for any $\nu > 0$, $(t,x) \to q^r(t,x)/(1+|x|^2)^{\nu}$, $r = 1,2,...$ is relatively compact in $C_B(I \times \mathbb{R}^d)$.

Proof. Clearly, each q^r can be extended onto $[0,1] \times \mathbb{R}^d$ such that (i) and (ii) hold on $[0,1] \times \mathbb{R}^d$. Now, the lemma follows from the argument in lines (66-77) of Kouritzin [9] with the definition $\phi^r(t,x) \doteq \frac{q^r(t,x)}{(1+|x|^2)^{\nu}}$. ■

The following two lemmas are at the heart of the parametrix method. The first lemma is used in (173) above and in establishing (106).

Lemma 5. *Suppose that $\mathcal{A}, \mathcal{B}$ are $\mathbb{C}^{N\times N}$-valued continuous functions which satisfy*

$$\|\mathcal{A}(s,\xi)\| \doteq \|\mathcal{A}(s,\xi;x,v)\| \le A\, s^{-(d+a)/q}h_{c'}(\xi, s) \quad (183)$$

$$\|\mathcal{B}(\xi,\sigma,\eta)\| \doteq \|\mathcal{B}(\xi,\sigma,\eta;x,v)\| \le B\, \sigma^{-(d+b)/q}h_c(\xi-\eta,\sigma) \quad (184)$$

for some $a, b, c, c', A, B > 0$ *with* $\alpha \doteq q - b > 0$ *and* $c > c'$; *all* $\sigma, s \in I$; *and all* $x, v, \eta, \xi \in \mathbb{R}^d$. *Then, there exists a solution* $\Theta(s,\xi) = \Theta(s,\xi;x,v)$ *to*

$$\Theta(s,\xi) = \mathcal{A}(s,\xi) + \int_0^{\frac{s}{2}} \int_{\mathbb{R}^d} \mathcal{B}(\xi,\sigma,\eta)\Theta(s-\sigma,\eta)d\eta d\sigma \tag{185}$$

which satisfies

$$\|\Theta(s,\xi;x,v)\| \leq C' A\, s^{-(d+a)/q} h_{c'}(\xi,s), \tag{186}$$

for some $C' > 0$ *not depending on* $A; s, \xi; x, v$. *This solution is continuous on* $(s,x,\xi,v) \in I \times \mathbb{R}^{3d}$, *and is given by* $\Theta(s,\xi;x,v) \doteq \sum_{m=0}^{\infty} K_m(s,\xi;x,v)$ *where*

$$K_0(s,\xi) = \mathcal{A}(s,\xi),\ K_m(s,\xi) = \int_0^{\frac{s}{2}} \int_{\mathbb{R}^d} \mathcal{B}(\xi,\sigma,\eta) K_{m-1}(s-\sigma,\eta)\, d\eta d\sigma,\ m \geq 1. \tag{187}$$

Moreover, there is only one solution to (185) which satisfies

$$\|\Theta(s,\xi;x,v)\| \leq K\, s^{-(d+a)/q} h_k(\xi,s) \tag{188}$$

for any $K, k > 0$.

Proof. For existence, we show inductively that $\{K_m\}_{m=1}^{\infty}$ is well defined, and

$$\|K_m(s,\xi)\| \leq A\, C^m\, \Gamma^{-1}\left(1 + \frac{\alpha m}{q}\right) s^{(\alpha m - d - a)/q} h_{c'}(\xi,s) \quad \forall m \geq 0, \tag{189}$$

where $C = C(c') = 2^{(d+a)/q} B\, \Gamma(\alpha/q) \int_{\mathbb{R}^d} h_{c"}(\eta,1)d\eta$ and $c" = c - c'$. It follows from (183) that (189) is satisfied for $m = 0$ so we will prove it is true for $m = n+1$. However, one finds by (187), (184), (189), (129) and gamma function properties that

$$\begin{aligned} \|K_{n+1}(s,\xi)\| &\leq s^{-\frac{d+a}{q}} \int_0^{\frac{s}{2}} \frac{A\, C^n B (s-\sigma)^{\alpha n/q}}{\Gamma\left(\frac{q+\alpha n}{q}\right) \sigma^{b/q}} \int_{\mathbb{R}^d} \frac{h_{c'+c"}(\xi-\eta,\sigma)}{\sigma^{d/q}} h_{c'}(\eta, s-\sigma) d\eta d\sigma \\ &\leq A\, C^{n+1} s^{(\alpha(n+1)-d-a)/q} h_{c'}(\xi,s) \qquad \forall s \in I,\ \xi \in \mathbb{R}^d \end{aligned} \tag{190}$$

so (189) is satisfied for $m = n+1$. To show continuity, we define $G(\sigma,s,\xi,\eta;x,v) \doteq \mathcal{B}(\xi,\sigma,\eta;x,v)\, K_n(s-\sigma,\eta;x,v)$, fix a $(s,\xi,x,v) \in I \times \mathbb{R}^{3d}$ and let $(s_j,\xi_j,x_j,v_j) \to (s,\xi,x,v)$. Then, it follows that there exists an $R > 0$ such that $|\xi| \vee |x| \vee |v| \vee |\xi_j| \vee |x_j| \vee |v_j| \leq R$ for all j and we can define the compact sets

$$\Delta_r \doteq \left\{(\sigma,s,\xi,\eta,x,v)' :\ \frac{1}{r} \leq \sigma' \leq \frac{s'}{2} \leq \frac{1}{2},\ |\xi'| \vee |\eta'| \vee |x'| \vee |v'| \leq r\right\},\ r \geq R. \tag{191}$$

Moreover, for $r \ge 2R \vee \frac{4}{s}$ and j large enough that $|s - s_j| \le \frac{2}{r}$, we have

$$\left\| \int_0^{\frac{s_j}{2}} \int_{\mathbb{R}^d} G(\sigma, s_j, \xi_j, \eta; x_j, v_j)\, d\eta\, d\sigma - \int_0^{\frac{s}{2}} \int_{\mathbb{R}^d} G(\sigma, s, \xi, \eta; x, v)\, d\eta\, d\sigma \right\| \tag{192}$$

$$\begin{aligned} \le\ & \int_{\frac{1}{r}}^{\frac{s}{2}-\frac{1}{r}} \int_{rB} \| G(\sigma, s_j, \xi_j, \eta; x_j, v_j) - G(\sigma, s, \xi, \eta; x, v) \|\, d\eta\, d\sigma \\ & + \left\{ \int_0^{\frac{1}{r}} \int_{rB} + \int_{\frac{s}{2}-\frac{1}{r}}^{\frac{s_j}{2}} \int_{rB} + \int_0^{\frac{s_j}{2}} \int_{(rB)^c} \right\} \| \mathcal{B}(\xi_j, \sigma, \eta; x_j, v_j) K_n(s_j - \sigma, \eta; x_j, v_j) \|\, d\eta\, d\sigma \\ & + \left\{ \int_0^{\frac{1}{r}} \int_{rB} + \int_{\frac{s}{2}-\frac{1}{r}}^{\frac{s}{2}} \int_{rB} + \int_0^{\frac{s}{2}} \int_{(rB)^c} \right\} \| \mathcal{B}(\xi, \sigma, \eta; x, v) K_n(s - \sigma, \eta; x, v) \|\, d\eta\, d\sigma. \end{aligned}$$

However, using (184), (189), (129), and the fact $s_j \ge s - \frac{2}{r} \ge \frac{s}{2}$, one finds that the second term of (192) is majorized by

$$\left(\frac{2}{s_j}\right)^{\frac{d+a}{q}} h_{c'}(\xi_j, s_j) \left\{ \int_0^{\frac{1}{r}} \int_{\mathbb{R}^d} + \int_{\frac{s}{2}-\frac{1}{r}}^{\frac{s_j}{2}} \int_{\mathbb{R}^d} + \int_0^{\frac{s_j}{2}} \int_{(rB)^c} \right\} \sigma^{-(d+b)/q} h_{c''}(\xi_j - \eta, \sigma) d\eta d\sigma \tag{193}$$

$$\overset{r,j}{\ll} \left(\frac{4}{s}\right)^{\frac{d+a}{q}} \left\{ \left(\frac{1}{r}\right)^{\alpha/q} + \int_0^{\frac{s_j}{2}} \sigma^{1-b/q} d\sigma \int_{(rB)^c} |\xi_j - \eta|^{-q-d}\, d\eta \right\}$$

$$\overset{r,j}{\ll} r^{-\alpha/q} + \int_{\frac{r}{2}}^{\infty} t^{d-1} t^{-q-d}\, dt \overset{r,j}{\ll} r^{-\alpha/q} \quad \forall r \ge 2R \vee \frac{4}{s},\ j \ni |s - s_j| \le \frac{2}{r}.$$

The third term of (193) is analogously bounded so continuity of K_{n+1} on $I \times \mathbb{R}^{3d}$ follows from the continuity of G on each Δ_r. Now, we note that by Stirling's formula, (189), and dominated convergence (take $s \in [\gamma, 1]\ \forall \gamma > 0$)

$$\Theta(s, \xi) \doteq \sum_{m=0}^{\infty} K_m(s, \xi) \overset{A;s,\xi;x,v}{\ll} A\, s^{-(d+a)/q} h_{c'}(\xi, s) \tag{194}$$

is well defined and continuous on $I \times \mathbb{R}^{3d}$. Fubini's theorem leads to

$$\begin{aligned} \Theta(s, \xi) &= \mathcal{A}(s, \xi) + \sum_{m=1}^{\infty} \int_0^{\frac{s}{2}} \int_{\mathbb{R}^d} \mathcal{B}(\xi, \sigma, \eta) K_{m-1}(s - \sigma, \eta)\, d\eta d\sigma \\ &= \mathcal{A}(s, \xi) + \int_0^{\frac{s}{2}} \int_{\mathbb{R}^d} \mathcal{B}(\xi, \sigma, \eta) \Theta(s - \sigma, \eta)\, d\eta d\sigma \quad \forall s \in I,\ \xi, x, v \in \mathbb{R}^d. \end{aligned} \tag{195}$$

Turning to uniqueness, we suppose without loss of generality that $k < c$, Θ_1,Θ_2 are two solutions, and $\Psi \doteq \Theta_1 - \Theta_2$. Then, it follows by (185) and (184) that

$$\|\Psi(s, \xi)\| \le \int_{\frac{s}{2}}^{s} \int_{\mathbb{R}^d} B\, (s - \sigma)^{-(d+b)/q} h_c(\xi - \eta, s - \sigma)\, \|\Psi(\sigma, \eta)\|\, d\eta d\sigma. \tag{196}$$

Now, using (196) recursively with (188), one finds that

$$\|\Psi(s,\xi)\| \le 2K\, C^n\, \Gamma^{-1}\left(1+\frac{\alpha n}{q}\right) s^{(\alpha n-d-a)/q} h_k(\xi,s), \tag{197}$$

where $C = C(k)$ is defined in (189). Hence, letting $n \to \infty$ on the right of the above equation, we find $\Psi(s,\xi) = 0$ for all s, ξ. ∎

The next lemma is used in (158), following (180), and in validating (104). Its proof follows easily form the considerations on pp. 252-255 of Friedman [6] and the foregoing proof of Lemma 5.

Lemma 6. *Suppose that $\mathcal{A}, \mathcal{E}$ are $\mathbb{C}^{N\times N}$-valued continuous functions which satisfy*

$$\|\mathcal{A}(t,\xi,\tau)\| \doteq \|\mathcal{A}(t,\xi,\tau;x,v)\| \le A\,(t-\tau)^{-(d+a)/q} h_{c'}(\xi, t-\tau) \tag{198}$$

$$\|\mathcal{E}(t,\xi,\sigma,\eta)\| \doteq \|\mathcal{E}(t,\xi,\sigma,\eta;x,v)\| \le E\,(t-\sigma)^{-(d+e)/q} h_c(\xi-\eta, t-\sigma) \tag{199}$$

for some $a, c', c, e, A, E > 0$ with $\alpha \doteq q - e > 0$, $a < q$, $c > c'$; all $\sigma, t, \tau \in I$ with $\tau < \sigma < t$; and all $x, v, \eta, \xi \in \mathbb{R}^d$. Then, there is a solution $\Theta(t,\xi,\tau) = \Theta(t,\xi,\tau;x,v)$ to

$$\Theta(t,\xi,\tau) = \mathcal{A}(t,\xi,\tau) + \int_\tau^t \int_{\mathbb{R}^d} \mathcal{E}(t,\xi,\sigma,\eta)\Theta(\sigma,\eta,\tau)\, d\eta d\sigma \tag{200}$$

on $0 \le \tau < t \le 1$, $x, \xi, v \in \mathbb{R}^d$ which satisfies

$$\|\Theta(t,\xi,\tau;x,v)\| \le C'\, A\,(t-\tau)^{-(d+a)/q} h_{c'}(\xi, t-\tau), \tag{201}$$

for some $C' > 0$ not depending on $A; s, \xi, \tau; x, v$. This solution is continuous and is given by $\Theta(t,\xi,\tau;x,v) \doteq \sum_{m=0}^\infty K_m(t,\xi,\tau;x,v)$ where

$$K_0 = \mathcal{A},\ K_m(t,\xi,\tau) = \int_\tau^t \int_{\mathbb{R}^d} \mathcal{E}(t,\xi,\sigma,\eta) K_{m-1}(\sigma,\eta,\tau)\, d\eta d\sigma,\ m \ge 1. \tag{202}$$

Moreover, there is only one solution to (200) which satisfies

$$\|\Theta(t,\xi,\tau;x,v)\| \le K\,(t-\tau)^{-(d+a)/q} h_k(\xi, t-\tau) \tag{203}$$

for any $K, k > 0$.

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