Inverse Obstacle Problem: Local Uniqueness for Rougher Obstacles and the Identification of A Ball

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Abstract

In this paper, we study the determination of the shape of an obstacle with rough boundary (not necessary Lipschitz) from its scattering amplitude. Following the technique used in paper [6], we obtain an extension of the local uniqueness result for Lipschitz obstacles to rougher ones. We also give a criterion to identify a ball in $\mathbf{R}^3$ from its scattering amplitude.

1 Introduction and Statements of the Results

Let K be a compact set in $\mathbf{R}^3$ with $\mathbf{R}^3\backslash K$ connected. Consider the scattering problem:

$$(\Delta + k^2)u = 0 \quad \text{in} \quad \mathbf{R}^3\backslash K \tag{1.1}$$

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$$u|_{\partial K} = f \in L^2(\partial K) \tag{1.2}$$

$$r(\frac{\partial u}{\partial r} - iku) \to 0 \quad \text{as} \quad r \to \infty. \tag{1.3}$$

It was shown in book [7] that if f is assumed to be the restriction to ∂K of a function in $C_0^2(\mathbf{R}^3)$, then there is a unique solution u in a weak sense. So for any $k > 0$ and $\omega \in \mathbf{S}^2$, there is a unique solution to the following problem:

$$(\Delta + k^2)u = 0 \quad \text{on} \quad \mathbf{R}^3 \backslash K \tag{1.4}$$

$$u|_{\partial K} = 0 \tag{1.5}$$

$$u = e^{ik\omega \cdot x} + v \tag{1.6}$$

$$|rv(x)| \leq C, \quad r(\frac{\partial v}{\partial r} - ikv) \to 0 \quad \text{as} \quad r \to \infty. \tag{1.7}$$

For large $|x|$,

$$u(x, \omega, k) = e^{ik\omega \cdot x} + \frac{e^{ik|x|}}{|x|} A_K(\hat{x}k, \omega, k) + o(\frac{1}{|x|}) \tag{1.8}$$

where $\hat{x} = \frac{x}{|x|}$ is the unit vector in the direction x, and A_K is the scattering amplitude:

$$A_K(\xi, \omega, k) = \int_{\partial \Omega} e^{-i\xi \cdot y} [\frac{\partial u}{\partial \nu} + i\xi \cdot \nu u](y, \omega, k) d\sigma(y) \tag{1.9}$$

where Ω is a smooth domain containing K. (We denote by ν the unit outward normal to $\partial \Omega$ and $d\sigma$ the surface measure on $\partial \Omega$.) If K is a bounded domain with Lipschitz boundary, it was shown in paper [6] that the scattering amplitude $A_K(\xi, \omega, k)$ given for all $\hat{\xi}$ in an arbitrary open set of $\mathbf{S}^2$ (Since $A_K(\xi, \omega, k)$ is analytic in ξ, it is completely determined at all outgoing directions $\hat{\xi} \in \mathbf{S}^2$ by its values on any open subset of the unit sphere $\mathbf{S}^2$.) at one fixed wave number k and one incident direction ω uniquely determines the k-core of K (see (2.9) in Section 2). A local uniqueness and a uniqueness result for polyhedra were obtained in the same paper as well. Some stability estimates for star-shaped smooth obstacles were also investigated by Isakov in [3]. For an arbitrary compact set K, the boundary might be very wild. It may have some cusps, some fuzzy fringes or even some discrete points. However, we can still define the k-core of K in the same manner as that in [6]. The scattering theory analogue of a theorem of Polya obtained in [6] still holds. Following the same idea in proving the local uniqueness for Lipschitz domains, we obtain a local uniqueness for an arbitrary compact set in the sense given in the following theorem.

Theorem 1.1 *Let K_1 and K_2 be two compact subsets in $\boldsymbol{R}^3$, with $\boldsymbol{R}^3 \setminus K_j$ connected, such that*

$$|\mathcal{O}| < (\frac{12\pi^2}{5})^{3/2} \frac{3}{B_3 k^3}, \tag{1.10}$$

where $\mathcal{O}$ is the interior of the complement of the unbounded connected component of $\boldsymbol{R}^3 \setminus (K_1 \cup K_2)$, $|\mathcal{O}|$ denotes the volume measure of $\mathcal{O}$ and B_3 is the volume of the unit ball. Assume that $A_{K_1}(\xi, \omega, k) = A_{K_2}(\xi, \omega, k)$ for all $\hat{\xi}$ in an open subset set of $\boldsymbol{S}^2$. Then $K_1^\circ = K_2^\circ$, where K_j° is the interior of K_j, $j = 1, 2$.

The proof of the theorem is given in section 2. On account of the wilderness of the boundary of K, the argument is more delicate than that for Lipschitz domains. Some related results, which are extensions of certain results for Lipschitz domains obtained in paper [6], are given in section 2 as well.

In section 3, we study the identification of a ball from scattering amplitude. Back to 1960s, Karp [1] proved that if $A_K(\xi, \omega, k) = A_K(Q\xi, Q\omega, k)$ for all rotations Q and all $\xi, \omega \in \mathbf{S}^2$, then K is a ball. In paper [5], we showed that if the scattering amplitude $A_K(\xi, \omega, k)$ of a Lipschitz domain in $\mathbf{R}^3$ is as same as the scattering amplitude of a ball at two linearly independent directions and one wave number, then this domain must be identical with the ball. The key point to this result is the rotational invariance of the scattering amplitude of a ball. Now we claim the converse is also true.

Theorem 1.2 *Suppose K is a compact set (not necessarily Lipschitz) such that $\boldsymbol{R}^3 \setminus K$ is connected, and for any point p on ∂K, each neighborhood of p in $\boldsymbol{R}^3$ contains interior points of K. If the scattering amplitude of K is rotationally invariant w.r.t. two linearly independent directions ω_1, ω_2 and one wave number; i.e.,*

$$A_K(Q\xi, \omega_j, k) = A_K(\xi, \omega_j, k) \quad \textit{for all} \quad \xi \in \boldsymbol{S}^2$$

where Q is in the set of all rotations Q such that $Q\omega_j = \omega_j$, $j = 1, 2$ respectively, then K must be a ball.

It is clear that this theorem is a much better improvement than Karp's theorem. One of key ingredients in proving this theorem is that in the set K_{iso} of all compact sets with properties in Theorem 1.2 and with a same scattering amplitude at one direction and one wave number, there are only finitely many distinct convex hulls of all sets in K_{iso} (see Proposition 3.1).

Although we restrict attention to the three dimensional case, a similar analysis can be given for obstacles in $\mathbf{R}^n$ when $n \geq 2$ except that the argument for the identification of a ball fails when $n = 2$. Nevertheless, the situation for n even is a little more complicated because of the logarithm branch point at the origin of the outgoing fundamental solution of the Helmholtz equation.

Acknowledgments

It is a pleasure to thank Michael Taylor for many stimulating discussions on related topics.

2 Local Uniqueness

For reader's convenience, we first recall the definition of the k-core of an obstacle, the indicator function of a scattering amplitude and a scattering analogue of a Polya's theorem from paper [6]. Then we give some related propositions and complete the proof of Theorem 1.1.

Given $k > 0$, the solution to (1.1) - (1.3) uniquely exists in the sense: $u \in H^1(B_R \backslash K)$ for any $0 < A \leq R < \infty$ where $B_R = \{x \in \mathbf{R}^3 : |x| < R\}$ and A is such that $B_A \supset (K \cup supp\, f)$, $\chi(u - f) \in H_0^1(\mathbf{R}^3 \backslash K)$ for some $\chi \in C_0^\infty(\mathbf{R}^3)$, chosen so $\chi(x) = 1$ for $|x| \leq A$. The solution u also satisfies the identity

$$u(x) = \int_\Sigma [u(y) \frac{\partial G(x-y,k)}{\partial \nu} - G(x-y,k) \frac{\partial u(y)}{\partial \nu}] d\sigma(y), \ |x| > A_1 \tag{2.1}$$

where $G(x,k) = \frac{e^{ik|x|}}{4\pi|x|}$, $A_1 \geq A$ and $\Sigma = \{|x| = A_1\}$.

Let $A_K(\xi, \omega, k)$ be the scattering amplitude defined in (1.9). Then A_K can be extended to an entire function of $\zeta = \zeta_R + i\zeta_I \in \mathbf{C}^3$ by the following expression.

$$A_K(\zeta, \omega, k) = \int_{\partial\Omega} e^{-i\zeta\cdot y}[\frac{\partial u}{\partial \nu} + i\zeta \cdot \nu u](y, \omega, k) d\sigma(y). \tag{2.2}$$

Since the scattering amplitude A_K can be represented by a series of spherical harmonics:

$$A_K = \sum_{j=0}^{\infty} c_j K_j(\xi), \tag{2.3}$$

another natural way to extend A_K to all $\zeta \in \mathbf{C}^3$ is an entire harmonic extension:

$$\tilde{A}_K(\zeta, \omega, k) = \sum_{j=0}^{\infty} c_j K_j(\zeta). \tag{2.4}$$

These two analytic extensions need not be same. However it was proved (see [2]) that these two extensions are identical on the complex variety $\zeta^2 = |k|^2$ for they agree on the real sphere $\xi^2 = |k|^2$. Since we are primarily interested in the values of A_K on $\zeta^2 = |k|^2$, it is alternative to use either of these two extensions and through all this paper we adopt the first one.

Analogous to the indicator function of an entire function, we define the indicator function h_A of a scattering amplitude A_K by

$$h_A(\hat{\zeta}_I) = \sup_{\omega^\perp} \overline{\lim}_{r\to\infty} \frac{\log |A_K(\zeta, \omega, k)|}{r}, \quad \hat{\zeta}_I \in \mathbf{S}^2 \tag{2.5}$$

where $\zeta \in \mathbf{C}^3$ and $\omega^\perp \in \mathbf{S}^2$ satisfy

$$\omega^\perp \cdot \hat{\zeta}_I = 0 \quad \text{and} \quad \zeta = (\sqrt{r^2 + k^2})\omega^\perp + ir\hat{\zeta}_I$$

(note $\zeta^2 = k^2$).

It is easy to see from (2.2) that

$$|A_K(\zeta, \omega, k)| \leq C e^{h_\Omega(\zeta_I)} \tag{2.6}$$

where $h_\Omega(\zeta_I) = \sup_{x\in\Omega} x \cdot \zeta_I = \sup_{x\in\partial\Omega} x \cdot \zeta_I$. Note that the indicator function h_A of A_K, defined by (2.5), involves only the values of A_K on $\zeta^2 = k^2$. Extend h_A to all $\eta \in \mathbf{R}^3$ by

$$h_A(\eta) = |\eta| h_A(\hat{\eta}). \tag{2.7}$$

Given an exterior solution u of the Helmholtz equation $(\Delta + k^2)u = 0$ for x in $\mathbf{R}^3 \backslash K$, we define a convex set D which we'll call the k-core of K as follows:
For each $\eta \in \mathbf{S}^2$, define

$$j(\eta) = \inf\{a : \quad u \ \text{ solves } \ (\Delta + k^2)u = 0 \ \text{ in } \ x \cdot \eta > a\} \tag{2.8}$$

and

$$D = \cap_{\eta \in \mathbf{S}^2} \{x \cdot \eta \leq j(\eta)\}. \tag{2.9}$$

Obviously, D is a closed convex set and $h_D(\eta) = j(\eta)$ where h_D denotes the supporting function of D. D is also the smallest convex set such that u extends to $\mathbf{R}^3 \backslash D$. In fact, if u can extend to the outside of a convex set D_1 smaller than D, then $D_1 \subsetneq D$. Since both D_1 and D are convex, we can find a plane $x \cdot \eta = a$ tangent to D_1 for some η in $\mathbf{S}^2$ and some $a \in \mathbf{R}$ such that some portion of D is contained in the half space $\{x \cdot \eta > a\}$. This contradicts the definition of D in (2.8) and (2.9). Then the following scattering theory analogue of a classical theorem of Polya [6] still holds.

Theorem 2.1 *Let u be the exterior solution to (1.4)–(1.7) and A_K the corresponding scattering amplitude. Then the indicator function h_A of A_K is equal to the supporting function of the k-core D of K. Therefore, the k-core of K is uniquely determined from A_K (see Figure 1).*

In general, if K is convex and ∂K is smooth, D is strictly contained in K. For example, the k-core of a ball is the center of the ball and the k-core of an ellipsoid obtained from the rotation of an ellipse is the line segment connecting two focuses. If K is non-convex, non-smooth or ∂K has some sharp corners, D might be the convex hull of these sharp corners. Examples are polyhedra. The k-core of any polyhedron is its convex hull. The following proposition is also true.

Proposition 2.2 *From the knowledge of the scattering amplitude A_K for one wave number and one incident direction, one can determine a bounded box Q such that K is contained in Q (see Figure 2).*

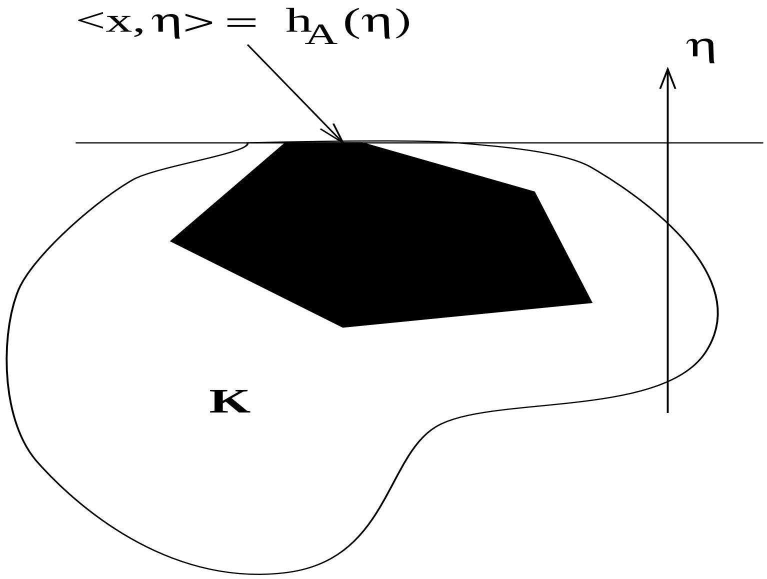

Figure 1: The k-core D and its supporting plane at direction η

Now suppose K_1 and K_2 are two compact sets in $\mathbf{R}^3$ with same scattering amplitude $A(\xi, \omega, k)$ at one incident direction and one wave number $k > 0$. Then two scattered waves v_j agree on the unbounded connected component $\mathcal{U}$ of $\mathbf{R}^3 \backslash (K_1 \cup K_2)$. (Note: $\mathbf{R}^3 \backslash (K_1 \cup K_2)$ might not be connected, see Figure 3.)

In other words, $u_j = e^{ik\omega \cdot x} + v_j(x, \omega, k)$ has the properties

$$(\Delta + k^2) u_j = 0 \text{ on } \ \mathbf{R}^3 \backslash K_j, \ \phi u_j \in H_0^1(\mathbf{R}^3 \backslash K_j) \tag{2.10}$$

$v_j = u_j - e^{ik\omega \cdot x}$ satisfies the radiation condition and $u_1 = u_2$ on $\mathcal{U}$. Here we fix $\phi \in C_0^\infty(\mathbf{R}^3)$ such that $\phi = 1$ on a ball containing $K_1 \cup K_2$ in its interior. Let $\mathcal{R}$ be any connected component of the interior of $(\mathbf{R}^3 \backslash \mathcal{U}) \backslash K_1$ (or $(\mathbf{R}^3 \backslash \mathcal{U}) \backslash K_2$ if $K_2 \subset K_1$). Let $u = u_1 | \mathcal{R}$. Then $u \in H_0^1(\mathcal{R})$. This is a consequence of the following general result.

Lemma 2.3 *Let $\mathcal{R} \subset \Omega$ be open. Then*

$$f \in H_0^1(\Omega) \cap C(\Omega), \ f = 0, \quad on \ \ \partial\mathcal{R} \backslash \partial\Omega \Longrightarrow f|_{\mathcal{R}} \in H_0^1(\mathcal{R}). \tag{2.11}$$

The proof of this lemma can be found in book [7].

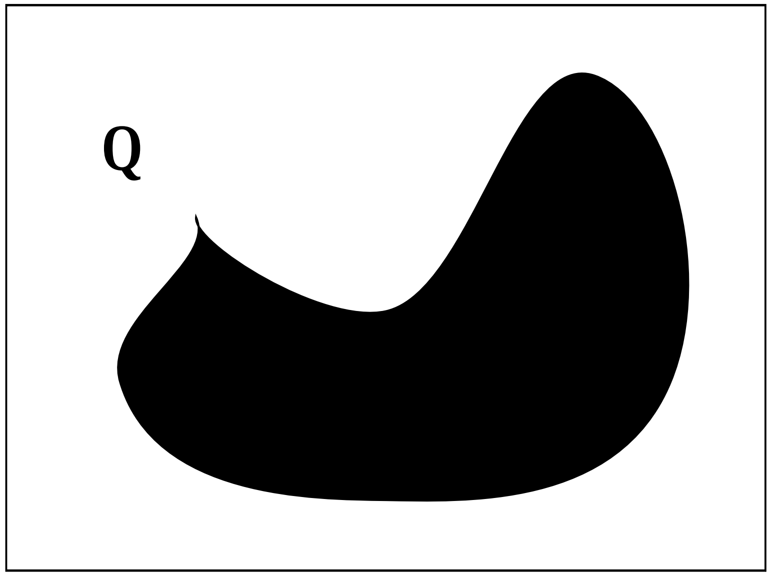

Figure 2: K and its bounding box Q

Now we want to prove the local uniqueness asserted in Theorem 1.1. The following lemma is needed in the proof.

Lemma 2.4 *Let G be a bounded domain in $\boldsymbol{R}^n$. Suppose λ_j denotes the j^{th} eigenvalue on G for the Dirichlet boundary value problem of the operator $-\Delta$. If $|G|$ is the volume of G, then*

$$\sum_{i=1}^{j} \lambda_i \geq \frac{nc_n}{n+2} j^{\frac{2+n}{n}} |G|^{-\frac{2}{n}} \tag{2.12}$$

where $c_n = (2\pi)^2 B_n^{-\frac{2}{n}}$, with $B_n =$ volume of the unit n-ball.

Here, to say v is an eigenfunction means $v \in H_0^1(G)$ satisfying $(\Delta + k^2)v = 0$ on G. The proof of this lemma can be found in paper [4].

Formula (2.12) implies that

$$\#\{\lambda_j \leq \lambda\} \leq B_n |G| (\frac{n+2}{4n\pi^2})^{n/2} \lambda^{n/2}, \tag{2.13}$$

or equivalently,

$$\lambda_N \geq \frac{n}{n+2} 4\pi^2 (\frac{N}{(B_n|G|)^{2/n}}). \tag{2.14}$$

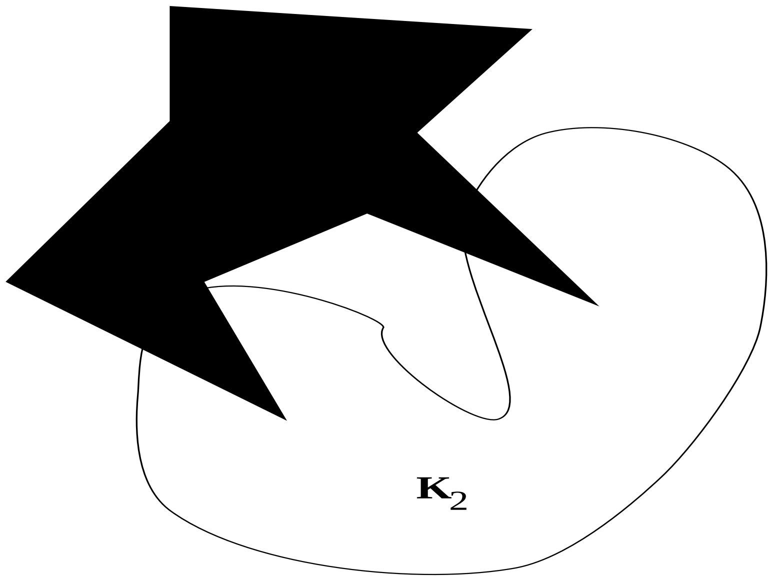

Figure 3: The set $K_1 \cup K_2$

In particular,

$$\lambda_1 \geq \frac{n}{n+2} 4\pi^2 (\frac{1}{(B_n|G|)^{2/n}}). \tag{2.15}$$

Proof of Theorem 1.1: Let u_1 and u_2 be two exterior solutions to (1.4) - (1.7) corresponding to K_1 and K_2. Since

$$A_{K_1}(\xi, \omega, k) = A_{K_2}(\xi, \omega, k) \text{ on } \hat{\xi} \in \mathbf{S}^2$$

two solutions u_1 and u_2 agree in the unbounded connected component $\mathcal{U}$ of $\mathbf{R}^3 \backslash (K_1 \cup K_2)$. It is clear that $K_1 \cap K_2 \neq \emptyset$. Otherwise, both u_1 and u_2 will be extended to the whole space $\mathbf{R}^3$ as entire functions. This is impossible. If $K_1^0 \neq K_2^0$, then $\mathcal{O} \backslash K_1$ or $\mathcal{O} \backslash K_2$ is nonempty, say $\mathcal{O} \backslash K_1$. Let $\mathcal{R}$ be any connected component of $\mathcal{O} \backslash K_1$. Then u_1 satisfies

$$\begin{cases} (\Delta + k^2) u_1 = 0 \text{ in } \mathcal{R} \\ u_1 \in H_0^1(\mathcal{R}) \end{cases}. \tag{2.16}$$

So $-k^2$ is a Dirichlet eigenvalue of Δ on $\mathcal{R}$. By complex conjugation we get that $\bar{u}_1$ is also an eigenfunction with the same eigenvalue $-k^2$. It is easy to see from the asymptotic behavior of u_1 that u_1 and $\bar{u}_1$ are linearly independent. Since the dimension of the eigenspace of Δ with the first negative eigenvalue λ_1 on $\mathcal{R}$ is one, $k^2 \geq \lambda_3$. Then the inequality (2.14) implies $|\mathcal{R}| \geq (\frac{12}{5}\pi^2)^{3/2}\frac{3}{B_3 k^3}$. However, the condition (1.10) implies $|\mathcal{R}| \leq |\mathcal{O}| < (\frac{12}{5}\pi^2)^{3/2}\frac{3}{B_3 k^3}$. This is a contradiction. ♯

Suppose that a scattering amplitude is given for N linearly independent incident directions or N distinct wave numbers. According to Proposition 2.2, one can determine a bounded box Q_j corresponding to each direction ω_j or wave number k_j. Combining this with inequality (2.14) we have the following corollary.

Corollary 2.5 *Suppose that $A_{K_1}(\cdot,\omega,k) = A_{K_2}(\cdot,\omega,k)$ for either N linearly independent directions ω_j with one wave number $k = M$; or N distinct wave numbers k_j with $k_j \leq M$ and one incident direction. If*

$$1 + 2N > B_3 |\cap_{j=1}^{N} Q_j| (\frac{5}{12\pi^2})^{3/2} M^3$$

then $K_1^\circ = K_2^\circ$.

Here, note that $K_1 \cup K_2$ must be contained in $\cap_{j=1}^{N} Q_j$ (see Figure 4).

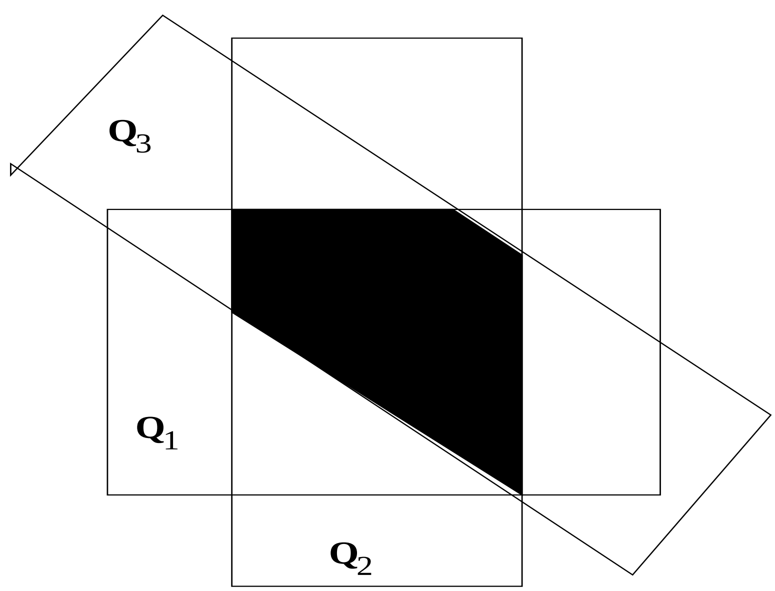

Figure 4: Sets K_1 and K_2 and Q_j

3 The Identification of A Ball

In this section we give the detail of the proof for Theorem 1.2. To do this, we first need to show the following proposition.

Proposition 3.1 *Let K be a compact set in $\boldsymbol{R}^3$ with properties in Theorem 1.2 and let $A_K(\cdot, \omega, k)$ be its scattering amplitude. Let K_{iso} be the set of all compact subsets in $\boldsymbol{R}^3$ with same properties as that of K and with the same scattering amplitude $A_K(\cdot, \omega, k)$. Then the number of distinct convex hulls of all sets from K_{iso} is finite.*

Proof: Let D be the k-core of K. Clearly, from Theorem 2.1, the k-core of each $\tilde{K}$ in K_{iso} is also D and D is contained in the convex hull of $\tilde{K}$. Therefore, D is contained in the intersection of convex hulls of all sets in K_{iso}. Let Q be a box constructed in Proposition 2.2. For each set $\tilde{K}$ in K_{iso}, Proposition 2.2 implies $\tilde{K} \subset Q$. So all sets in K_{iso} are contained in Q. Without loss of generality, we may assume that the convex hulls of all sets in K_{iso} are contained in Q as well. Suppose that there are infinitely many distinct convex hulls. This certainly implies that there are infinitely many distinct sets in K_{iso}. Now we pick a countable sequence of distinct sets $\{\tilde{K}_j\}_{j=1}^{\infty}$ from K_{iso} so that the corresponding convex hulls are also distinct. Then from $\{\tilde{K}_j\}$ we can choose a sequence of disjoint connected domains $\{D_l\}_{l=1}^{\infty}$ in the way that for each l, $D_l \subset \mathbf{R}^3 \backslash D$, $D_l = (\bar{D}_l)^o$ (the interior of $\bar{D}_l$) and $\bar{D}_l = \cap_{m=1}^{\infty} \Omega_m$ where each Ω_m is either $\tilde{K}_j$ or $\overline{(\mathbf{R}^3 \backslash \tilde{K}_j)}$. The existence of this sequence is guaranteed by the existence of infinitely many distinct convex hulls. (However, if $\tilde{K}_j$ doesn't satisfy that for any point p on $\partial \tilde{K}_j$, each neighborhood of p in $\mathbf{R}^3$ contains interior points of $\tilde{K}_j$, the existence of the sequence $\{D_l\}$ may fail, see Figure 5.)

Since $D_l \subset Q$, $|\bar{D}_l| \to 0$. So we can find an integer l^* such that

$$|\bar{D}_{l^*}| < \frac{1}{4}(\frac{12}{5}\pi^2)^{3/2}(\frac{3}{B_3|k|^3}) \stackrel{def}{=} \epsilon. \tag{3.1}$$

Let $\bar{D}_{l^*} = \cap_{m=1}^{\infty} \Omega_m$. Then $|\bar{D}_{l^*}| < \epsilon$ and $\bar{D}_{l^*} \subset \mathbf{R}^3 \backslash D$ imply that $| \cap_{m=1}^{M} \Omega_m| < 2\epsilon$ and $\cap_{m=1}^{M} \Omega_m \subset \mathbf{R}^3 \backslash D$ for some $M \geq 1$. Let $\Omega = \cap_{m=1}^{M} \Omega_m$. Then

$$|\Omega| < 2\epsilon < (\frac{12}{5}\pi^2)^{3/2}(\frac{3}{B_3|k|^3}). \tag{3.2}$$

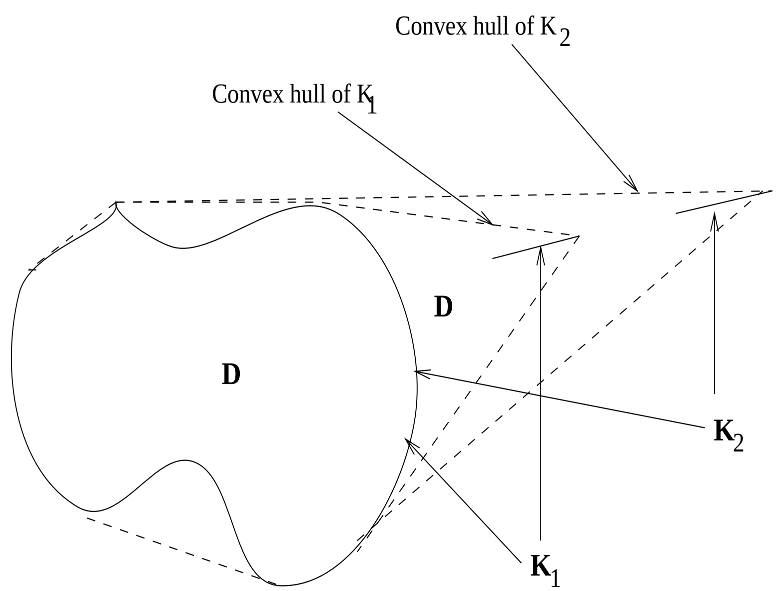

Figure 5: Set $K = K_0 \cup L$ where both are compact and L has empty interior

Obviously all corresponding exterior solutions $\{u_m\}_{m=1}^M$ can be extended to $\mathbf{R}^3 \backslash D$ and these solutions agree on $\mathbf{R}^3 \backslash D$. So u_1 is a Dirichlet eigenfunction on Ω of $-\Delta$ with eigenvalue k^2. On the other hand, by the same reason as that in the proof of Theorem 1.1, we have

$$|\Omega| \geq (\frac{12}{5}\pi^2)^{3/2}(\frac{3}{B_3 k^3}).$$

This is contradictory to (3.2). $\sharp$

Now we are ready to prove Theorem 1.2.

<u>Proof of Theorem 1.2:</u> First we note that the identity

$$A_K(\xi, \omega_j, k) = A_{QK}(Q\xi, Q\omega_j, k), \quad j = 1, 2 \tag{3.3}$$

holds for any rotation Q in $\mathbf{R}^3$ where QK means the transform of K under the rotation Q. For fixed ω_1, let *Rot* be the set of those rotations Q in $\mathbf{R}^3$ such that $Q\omega_1 = \omega_1$ (*Rot* is empty in $\mathbf{R}^2$, this is why the proof breaks down for $n = 2$.). Note that if Q is in *Rot* so is Q^{-1} and $Q^{-1} = Q^*$ where Q^{-1} is the inverse of Q and Q^* is the transpose of Q. Let Q be any rotation in *Rot*. Then from (3.3) and assumptions in Theorem 1.2,

$$A_K(\xi, \omega_1, k) \quad = \quad A_{QK}(Q\xi, Q\omega_1, k)$$

$$= \quad A_{QK}(Q\xi, \omega_1, k) = A_{QK}(\xi, \omega_1, k).$$

Thus $A_K(\xi, \omega_1, k) = A_{QK}(\xi, \omega_1, k)$ for all rotations Q in *Rot*. This implies that K must be invariant under all rotations Q in *Rot*. In fact, if K is not rotationally invariant w.r.t. the direction ω_1, neither is the convex hull of K. Then we can find an infinite sequence $\{Q_j\}_{j=1}^{\infty}$ such that convex hulls of $\{Q_j K\}_{j=1}^{\infty}$ are distinct. This is contradictory to Proposition 3.1.

Replacing ω_1 by ω_2 we obtain that K is rotationally invariant w.r.t. the direction ω_2 as well. It is known that a geometric obstacle being rotationally invariant w.r.t. two linearly independent vectors ω_1 and ω_2 must be a ball. Therefore the proof is completed. $\sharp$

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