

THE BLOW-UP PROBLEM FOR EXPONENTIAL NONLINEARITIES

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ABSTRACT. We give a solution of the blow-up problem for equation $\square u = e^u$, with data close to constants, in any number of space dimensions: there exists a blow-up surface, near which the solution has logarithmic behavior; its smoothness is estimated in terms of the smoothness of the data. More precisely, we prove that for any solution of $\square u = e^u$ with Cauchy data on $t = 1$ close to $(\ln 2, -2)$ in $H^s(\mathbb{R}^n) \times H^{s-1}(\mathbb{R}^n)$, s is a large enough integer, must blow-up on a space like hypersurface defined by an equation $t = \psi(x)$ with $\psi \in H^{s-146-9[n/2]}(\mathbb{R}^n)$. Furthermore, the solution has an asymptotic expansion $\ln(2/T^2) + \sum_{j,k} u_{jk}(x)T^{j+k}(\ln T)^k$, where $T = t - \psi(x)$, valid upto order $s - 151 - 10[n/2]$. Logarithmic terms are absent if and only if the blow-up surface has vanishing scalar curvature. The blow-up time can be identified with the infimum of the function ψ . Although attention is focused on one equation, the strategy is quite general; it consists in applying the Nash-Moser IFT to a map from “singularity data” to Cauchy data.

CONTENTS

1. Introduction	3
1.1. What is known?	
1.2. Background	
2. Overview of the argument	8
2.1. Motivation	
2.2. Basic definitions	
3. Step 1: Construction of S	12
3.1. Three systems	
3.2. Solving Fuchsian systems	
4. Step 2: Smoothness of S	18
5. Step 3: Smoothness of Z and definition of K	20
6. Step 4: Linearization of K	21
6.1. Characterization of K'	
6.2. First expansion of U	
7. Step 5: Inversion of K'	25
7.1. Fuchsian equation for U	
7.2. Definition of the inverse of K'	
7.3. Is this a right inverse?	
8. Step 6: Application of the Nash-Moser theorem, end of proof	29
8.1. A version of the Nash-Moser theorem	
8.2. Application and end of proof	
9. Concluding remarks	35
References	36

1. INTRODUCTION

The problem of describing the solutions of nonlinear evolution equations near the time when they become infinite has received intense attention in the last few years. For nonlinear hyperbolic equations, this question is particularly interesting because the simplest examples of such singularities do not occur on surfaces which are characteristic for the linear part. In fact, these singularities have remained beyond the reach of the standard methods for studying propagation of singularities. They are also not accessible to the theory of shock waves, which corresponds (for first-order systems) to the spontaneous formation of finite discontinuities.

A general scenario which would both describe the formation of such strong singularities and bring this phenomenon within the reach of microlocal methods has been proposed and refined in a series of papers which show how to construct large classes of singular solutions with a prescribed blow-up surface. The present paper shows that this scenario is the correct one by showing, conversely, how to produce the blow-up surface corresponding to a given set of Cauchy data. In this sense, this approach gives, whenever applicable, a detailed description of the solutions near blow-up singularities. It turns out that these solutions can be continued in a meaningful way beyond the singularity surface.

The present paper limits itself to one example, the equation

$$\square u = e^u,$$

which in our earlier papers was the simplest equation for this type of investigation, because it leads to the minimal amount of computation. The method is however quite robust, and application to other examples such as polynomial nonlinearities will be given elsewhere. We consider solutions defined in $\mathbb{R}^n \times \mathbb{R}$, where $n \geq 1$, which are close to the reference solution $\ln(2/t^2)$. As

usual, $H^s(\mathbb{R}^n)$ denotes the Sobolev space of order s , with norm

$$|u|_s := \|(1 + |\xi|^2)^{s/2} \hat{u}\|_{L^2(\mathbb{R}^n)},$$

where \hat{u} denotes the Fourier transform.

The main result may be stated as follows:

Theorem 1. *Assume that u is the solution of the Cauchy problem for $\square u = e^u$ with Cauchy data on $\{t = 1\}$ satisfying*

$$(u(1, x) - \ln 2, u_t(1, x) + 2) \in H^s \times H^{s-1}, \text{ where } s > 167 + 10[n/2].$$

If the data are sufficiently close to $(\ln 2, -2)$, there is a function $\psi \in H^r$, with $r = s - 146 - 9[n/2]$ such that $u - \ln[2/(t - \psi(x))^2]$ is bounded near $\Sigma := \{t = \psi(x)\}$. In fact, the difference is in H^{r-4} and is defined on both sides of Σ .

In other words, blow-up occurs along a hypersurface (close to $t = 0$ in this case), the regularity of which is estimated in terms of the regularity of the data. Note also that we are giving the Cauchy data for $t = 1$ and solving backwards in time, for convenience. The solution continues beyond the blow-up time for $t \geq -1/2$ at least, as will be clear from the proof.

The rest of the paper is essentially devoted to the proof of Theorem 1.

1.1. What is known? For a more thorough discussion of blow-up and more complete references, see the monographs by Strauss [21], John [12], Hörmander [9], Alinhac's notes [3] and Zuily [22]. The last two references discuss very recent work (up to 1994); see also [16]. These papers also give references on the complementary issue of global existence.

F. John [10] proved that all solutions of $\square u = u^2$ with compactly supported data in three space dimensions must become singular in finite time. There are related results for other power nonlinearities and other dimensions which can be found in [21]. Yet, the solution $6/(t + 1)^2$ is free of singularities for all

positive time. Several other results for powers nonlinearities can be found in [21].

Caffarelli and Friedman [4] showed that under appropriate restrictions on the data, in 1, 2 or 3 space dimensions, there exists a C^1 space-like blow-up surface on which the solution becomes infinite. Thus, u is finite if and only if $t < \psi(x)$. For the solutions considered in this paper, we prove that the blow-up surface is more regular if the data are.

Several authors (see [9, 11, 12, 21] and their references) focused on the estimation of the *blow-up time*, which is the time of the first singularity. This time is equal to the infimum of the function defining the blow-up surface:

$$T_* = \inf_x \psi(x).$$

Two lines of thought led to precise estimates of the asymptotics of T_* in the limit of small data. First, John, (with refinements by Hörmander) proved that for a class of *quasi-linear* equations, the blow-up time could be computed explicitly in the limit of small data (in three dimensions), and that indeed it becomes infinite when the equation satisfies the null condition. On the other hand (Lindblad [18]), for the *semi-linear* equation, $\square u = u^2$, if data are proportional to ε , then $T_* \approx \varepsilon^{-2}$; furthermore, one can define a rescaling of the solution which converges as $\varepsilon \rightarrow 0$, and solves a limiting equation which can be written down explicitly. In such a limit of course, the singular set is rejected at future infinity.

As explained for instance in [9], these methods generalize the argument leading to the universality of the equation $u_t + uu_x = 0$ as a model for shock wave formation. A different light on some of these results was shed by the work of Alinhac [2], see below. Some results on the classification of possible singularities are due to Caffisch *et al.*, see [5].

Littman and the author have proved on the other hand, in [13] and [14], that one can, for all the semi-linear equations considered in these works, construct solutions with a prescribed analytic blow-up set. This rested on a Cauchy-

Kowalewska theorem for a “generalized Fuchsian system.” This system is not symmetric, and so it is not immediately clear that there are solutions for non-analytic blow-up surfaces. However, in [15], a general existence result for Fuchsian hyperbolic systems in Sobolev spaces was presented, and as an application, the existence of solutions with prescribed H^s blow-up surfaces was obtained for s large. The results in the analytic case had another application: it proved the convergence of the WTC (or generalized Painlevé) expansions of the theory of solitons, introduced in 1983 by Weiss, Tabor and Carnevale, and, since then, extensively studied at the formal level. The structure of these expansions was further analyzed in [17], with several applications not relevant to the present paper. As far as blow-up was concerned, these results suggested that the singularity could be eliminated by introducing a new unknown, which can be thought of as the “regular part” of the solution. This regular part w is now finite on the blow-up surface; furthermore, it solves a characteristic initial-value problem where the blow-up surface is characteristic. The value $w^{(0)}$ of w on the blow-up surface can be prescribed arbitrarily.

For example, for $\square u = e^u$, these singular solutions are such that e^{-u} is continuous across the blow-up surface. Even more, if the blow-up surface has vanishing scalar curvature, our solutions are even such that e^{-u} is analytic.

The main technical tool is the analysis of nonlinear Fuchsian PDE. For references to the extensive literature on Fuchsian equations, we refer to [15].

In this perspective, it is more natural to label the solution by a pair of “singularity data,” namely the function ψ of which the blow-up surface is the graph, and the value of the regular part w on the blow-up surface.

The next step in this approach is to show that one can compute singularity data from Cauchy data.

This is what the present paper accomplishes for exponential nonlinearities.

The present approach, although applied to semi-linear equations only, has

parallels for quasi-linear equations, which have been proposed independently by Alinhac and Caffisch *et al.* [2, 5]. These papers propose to analyze the onset of shock waves by introducing uniformizing changes of variables, and are, when successful, capable of providing a detailed picture of singularity formation in the quasi-linear case.

Thus, in these papers as in the present one, the singularity is removed by an appropriate change of variables leading to a degenerate initial-value problem. The situation is quite comparable to that of a Puiseux expansion, which is obtained from a perfectly regular function by inserting a singular change of variables. We know that Puiseux expansions have been generalized into series solutions for nonlinear Fuchsian ODEs of first order (Briot-Bouquet), and of second order and first degree (Painlevé *et al.*), which gave birth to a large body of results which is still growing. Formal extensions to PDEs have appeared in the theory of solitons, prompting a revival of these techniques. We can now show that these expansion techniques are not limited to ODEs, but actually provide very precise models for general types of singularity formation.

1.2. Background. The present paper is the fifth of a series [13, 14, 15, 17]. We recall here the results of the previous four parts, mostly as they apply to the equation $\square u = e^u$. The reader is referred to these papers for more general equations.

This series places itself as a natural continuation and extension to PDE of well-known work of Briot and Bouquet, Painlevé and his school, ..., as recalled above.

In [13], singular solutions which are analytic except for a blow-up singularity on an analytic spacelike hypersurface with zero scalar curvature were constructed. For such solutions, e^{-u} is analytic through the blow-up surface. This argument suffices to prove the convergence of the WTC expansions.

In [14], the curvature condition was removed, at the expense of allowing logarithmic terms in the expansion of the singular solution near the blow-up

surface.

In [15], the analyticity assumption was replaced by a Sobolev smoothness condition. One of the difficulties one has to deal with in this case is that the Fuchsian system used in [14] is not in “symmetric form,” and has to be replaced by another one obtained by expanding the components of the unknown to different orders before energy estimates can be used. Other technical difficulties must also be overcome. The correct system is recalled in §3.

In [17], the structure of the singular expansion was studied in more detail, and was shown to be conveniently studied using a particular representation of $\mathfrak{sl}(2)$ related to the invariant theory of binary forms.

The proof of Th. 1 being quite technical, we present an overview first, concluded by a statement of the main result, and then present the five steps of the proof in order.

2. OVERVIEW OF THE ARGUMENT

2.1. Motivation. Stated briefly, the argument consists in establishing, via an inverse function theorem, a correspondence between the Cauchy data and a pair of “singularity data” which completely describe the blow-up.

More precisely, let us consider a solution of $\square u = e^u$ which becomes singular for $t = \psi(x)$. We first label the solution in two ways: first by a pair of Cauchy data on $t = 1$; second by a pair of “singularity data” $(w^{(0)}, \psi)$. To define the function $w^{(0)}$ entering in the singularity data, we introduce coordinates (X, T) by $T = t - \psi(x)$, $X^i = x^i$, and define in Eq. (4) below (§2.2) a rescaled function $w(X, T)$; we then let $w^{(0)} = w(X, 0)$.

It is usually convenient to use the same letter to denote a function in the (x, t) or the (X, T) coordinates. Whenever this may lead to confusion, we distinguish them by using tildes in the (x, t) coordinates: $u(X, T) = \tilde{u}(x, t)$. The same convention applies to other functions.

Now u and w can be thought of as the first components of suitable first-order

systems for vector-valued unknowns \vec{u} and \vec{w} respectively. These systems are discussed in §3. The system for \vec{u} is the usual symmetric-hyperbolic system associated with $\square u = e^u$ in the coordinates (X, T) . The system for \vec{w} is a Fuchsian system, to which the existence theorem of [15] applies. This means that given the singularity data $(w^{(0)}, \psi)$, we construct a singular solution, and then read off its Cauchy data (u_0, u_1) . Note that w is, as a function of $(X, T, T \ln T)$, as smooth as the data permit, even on the blow-up surface.

We write $K(w^{(0)}, \psi) = (u_0, u_1)$.

We wish now to *invert this process*, constructing singularity data from Cauchy data. It will follow from Eq. (4) that blow-up takes place precisely on $t = \psi(x)$, and, using the Taylor expansion of w , the existence of the first few terms of an asymptotic expansion of the solution near the blow-up surface will follow.

We achieve this for data close to the reference solution $u(x, t) = \ln(2/t^2)$, which means that Cauchy data on $t = 1$ are close to $(\ln 2, -2)$, and that the singularity data are close to $(0, 0)$. Other nearly-constant data can be handled in a similar fashion.

This set-up suggests the use of an implicit function theorem. We use the Nash-Moser theorem, in a form recalled in §8.

The main point is the proof of the invertibility of the linearization of the map K from singularity data to Cauchy data. The inverse of this linearization is computed by comparing two expansions of a solution to the linearization of (1)

REMARK: A key point which makes this proof possible is that despite what one might surmise at first, u and the solution of the linearized equation both blow-up exactly on the same surface. The only difference is that the linearized solution is *more singular* than u .

2.2. Basic definitions. We define here the function w and the map K . The mapping properties of the latter are discussed in §§3 and 4. We also introduce

notation for the linearizations of the various maps used in the proof. Note that we consistently use capital letters for the arguments of differentials: U and W solve the linearized u and w equations, while $(W^{(0)}, \Psi)$ represents a generic tangent vector to the space of singularity data, and (U_0, U_1) are linearized Cauchy data.

Throughout the paper, $\psi \in H^r(\mathbb{R}^n)$, and $\|\psi\|_\infty < 1/4$. All Sobolev indices will be rounded off to their integer part, for simplicity, and will be assumed large enough. The specific bounds given in Th. 1 will be obtained in §8.

Since we need to deal with two sets of coordinates, (x, t) and (X, T) , we will use tildes to distinguish the expressions of a given function in one or the other set of coordinates; thus, $u(X, T) = \tilde{u}(x, t)$. However, the tilde may be omitted whenever the meaning is clear from the context. Note that $\tilde{u}(x, t) = u(x, t - \psi(x))$.

2.2.1. *Singularity data.* If $\tilde{u}(x, t)$ solves

$$(1) \quad \square \tilde{u} = e^{\tilde{u}},$$

and if $T = t - \psi(x)$, $X^i = x^i$, we have

$$(2) \quad \gamma u_{TT} - \Delta u + 2\psi^i \partial_i u_T + (\Delta \psi) u_T = e^u,$$

where $\gamma = 1 - |D\psi|^2$. Note that $\partial_t = \partial_T$ and $\nabla_x = \nabla_X - (\nabla \psi) \partial_T$.

We now let R denote the scalar curvature of the hypersurface $t = \psi(x)$, so that (see the appendix of [13] for the detailed calculation¹)

$$(3) \quad R = [\psi^{il} \psi_{il} - (\Delta \psi)^2] / \gamma + 2 \left\{ (\psi^\rho \psi_{i\rho}) (\psi_\sigma \psi^{i\sigma}) - \psi^\rho \psi^\sigma \psi_{\rho\sigma} \Delta \psi \right\} / \gamma^2.$$

We define the new unknown w by

$$(4) \quad u = \ln \frac{2}{T^2} + v^{(0)} + v^{(1)} T + R_1 T^2 \ln T + T^2 w(X, T).$$

where

$$v^{(0)} = \ln \gamma, \quad v^{(1)} = -\gamma^{-1} \Delta \psi, \quad R_1 = -2R / (3\gamma).$$

¹Indices are raised and lowered using the Kronecker δ ; subscripts denote derivatives; the summation convention applies.

We also define $g = G(\psi, w) := u - \ln(2/t^2)$. In that case, w solves the Fuchsian equation

$$\begin{aligned}
& \gamma(T\partial_T)(T\partial_T + 3)w \\
& + T \left[(\Delta\psi)(R_1 + (T\partial_T)w) + 2\psi^i \delta_i^{i'} \partial_{i'}(R_1 + (T\partial_T)w) \right. \\
& \qquad \qquad \qquad \left. - T \ln T \Delta R_1 - T \Delta w \right] \\
(5) \quad & + 4\psi^i \delta_i^{i'} \partial_{i'}(R_1 T \ln T + Tw) + 2(\Delta\psi)(R_1 T \ln T + Tw) - T \Delta v^{(1)} \\
= & (1 - |D\psi|^2) \left\{ (v^{(1)} + R_1 T \ln T + Tw)^2 - [v^{(1)}]^2 \right. \\
& + T(v^{(1)} + R_1 T \ln T + Tw)^3 \\
& \qquad \qquad \qquad \left. \int_0^1 (1 - \sigma)^2 \exp(T\sigma(v^{(1)} + R_1 T \ln T + Tw)) d\sigma \right\}.
\end{aligned}$$

The singularity data are $\psi(X)$ and $w^{(0)}(X) := w(X, 0)$. Solving (5) and substituting into (4), we see that they determine u uniquely.

2.2.2. Definition of K . We define the operator K , from Cauchy to singularity data, and the spaces on which it acts. It will be understood that the operators of this paper are only defined in a neighborhood of the origin in their respective spaces; recall that $\|\psi\|_{L^\infty} < 1/4$. The proof that the mappings K , S , G , Z and E introduced below do map into the indicated Sobolev spaces is given in §§3 to 6.

Let us first define an operator S which gives us the solution u on $1/4 \leq T \leq 2$. It is obtained by composition of the operator mapping $(w^{(0)}, \psi)$ to the solution of Eq. (5) with initial condition $w^{(0)}$ for $T = 0$, with the operator G defined by (4), followed by subtraction of the reference solution $\ln(2/T^2)$:

$$\begin{aligned}
S & : H^{r-3} \times H^r \rightarrow H^{r-6-n/2}((1/4, 2) \times \mathbb{R}^n) \\
& (w^{(0)}, \psi) \mapsto G(\psi, w).
\end{aligned}$$

Its detailed construction and regularity properties are studied in §§3 and 4, where restrictions on r can be found.

Let

$$Z : (\psi, g) \mapsto \tilde{g}.$$

Z expresses the conversion from the coordinates (X, T) to the coordinates (x, t) .

We then compose $Z \circ S$ with the evaluation of the Cauchy data of \tilde{u} on $t = 1$:

$$(6) \quad E : H^s((1/2, 3/2) \times \mathbb{R}^n) \rightarrow H^{s-1/2}(\mathbb{R}^n) \times H^{s-3/2}(\mathbb{R}^n)$$

$$\tilde{g} \mapsto (\tilde{g}(\cdot, 1), \tilde{g}_t(\cdot, 1)) = (\tilde{u}(\cdot, 1) - \ln 2, \tilde{u}_t(\cdot, 1) + 2).$$

Note that E is independent of ψ . By construction, $K(w^{(0)}, \psi) + (\ln 2, -2)$ represents Cauchy data on $t = 1$, for a solution which is close to the reference solution $\ln(2/t^2)$.

Finally, we let

$$(7) \quad K = E \circ Z \circ S : H^{r-3} \times H^r \rightarrow H^{r-9-n/2} \times H^{r-10-n/2}.$$

The goal is to invert K .

3. STEP 1: CONSTRUCTION OF S

To define S , we begin by recasting equation (2) in the form of a Fuchsian system, for which we set-up an initial-value problem. The first component of the unknown in this Fuchsian system is the unknown w in (5). In view of the complexity of (5), it is not advisable to start from (5) directly. We find in this way solutions \tilde{u} of $\square \tilde{u} = e^{\tilde{u}}$ which are continuous in T with values in some Sobolev space.

3.1. Three systems. We define three vector-valued functions \vec{u} , \vec{v} and \vec{w} , the components of which are defined in terms of u and ψ . In particular, the first component of \vec{u} is the function u of §2, while the first component of \vec{w} coincides with the function w defined by (4).

Each of these functions solves a first-order system. Furthermore, \vec{v} can be computed from \vec{w} , and \vec{u} from \vec{v} .

The algebra having been detailed in [14] and [15], only the final formulae are given here.

The system for \vec{u} is simply the symmetric-hyperbolic system associated with Eq. (2).

3.1.1. *System for \vec{u} .* Let

$$\vec{u} := (u, u_0, u_i)$$

where i runs from 1 to n . The following system implies (1) if the u_i are, for $T = 0$, the components of the spatial gradient of u :

$$(Q\partial_T - A^i\partial_i)\vec{u} = \varphi(X, T, u) := \begin{pmatrix} u_0 \\ e^u - (\Delta\psi)u_0 \\ 0 \end{pmatrix};$$

here Q is diagonal, and the A^i are symmetric, both $(n+2) \times (n+2)$. They are defined by

$$Q(x) = \begin{pmatrix} 1 & & \\ & \gamma & \\ & & I_n \end{pmatrix},$$

and A^i has only three non-zero entries, namely

$$(A^i)_{2,2} = -2\psi_i; \quad (A^i)_{2,i+2} = (A^i)_{i+2,2} = 1.$$

3.1.2. *Fuchsian system for \vec{v} .* We subtract from \vec{u} a few terms from its expansion, and obtain a Fuchsian equation:

Define \vec{v} by

$$(8) \quad \begin{cases} u = \ln(2/T^2) + v^{(0)} + v^{(1)}T + vT^2 \\ u_0 = -(2/T) + v^{(1)} + v_0T \\ u_i = v_i^{(0)} + v_iT, \end{cases}$$

where $\vec{v} = (v, v_0, v_i)$ is a new unknown, and

$$\exp(v^{(0)}) = \gamma; \quad v^{(1)}\gamma + \Delta\psi = 0; \quad v_i^{(0)} = \partial_i v^{(0)},$$

which is consistent with the notation of the introduction. The system for \vec{u} now becomes

$$(9) \quad Q[T\partial_T + A]\vec{v} = \tilde{\varphi}(X) + TA^i\partial_i\vec{v} + TF(X, \vec{v})$$

where

$$\tilde{\varphi} = \begin{pmatrix} 0 \\ -2R(X) \\ \partial_i v^{(1)} \end{pmatrix}; \quad F = \begin{pmatrix} 0 \\ b_0 \\ 0 \end{pmatrix},$$

R is defined in (3), and

$$A = \begin{pmatrix} 2 & -1 & & \\ -2 & 1 & & \\ & & & \\ & & & I_n \end{pmatrix}.$$

This matrix has eigenvalues 0, 3 and 1, with multiplicities 1, 1, n . Its null-space is generated by $(1, 2, 0, \dots, 0)^T$. The function b_0 is given by

$$b_0(X, T, \vec{v}) = -v_0\Delta\psi + \gamma(2v^{(1)}v + Tv^2) + \gamma h(T, v^{(1)} + Tv),$$

where $h(T, z) = z^3 \int_0^1 (1-\sigma)^2 \exp[\sigma Tz] d\sigma$. Since we are only interested in small \vec{v} , we will truncate the nonlinear part of b_0 , namely $\gamma[h(T, v^{(1)} + Tv) + Tv^2]$, so that it is smooth, identically zero for $|v| > 2$, and given by this expression for $|v| < 1$.

REMARK: The principal part of (9) is equal to the principal part of the equation for \vec{u} multiplied by T .

3.1.3. Fuchsian system for \vec{w} . Since \vec{v} is not free from logarithmic terms unless $R = 0$, we now view it as obtained from a function of the variables X , t_0 and t_1 after the substitution $t_0 = T$; $t_1 = T \ln T$. It will also be necessary to further subtract a few more terms from \vec{v} , leading to the introduction of a third unknown \vec{w} , normalized eventually in such a way that its value for $t_0 = t_1 = 0$ be zero.

Let $t_0 = T$, $t_1 = T \ln T$ and introduce a third unknown $\vec{w} = (w, w_0, w_i)$ by the formulae

$$(10) \quad \begin{cases} u &= \ln(2/t_0^2) + v^{(0)} + v^{(1)}t_0 + R_1 t_0 t_1 + w(t_0, t_1, X)t_0^2 \\ u_0 &= -(2/t_0) + v^{(1)} + (t_0 + 2t_1)R_1 + w_0 t_0 \\ u_i &= v_i^{(0)} + v_i^{(1)}t_0 + w_i t_0, \end{cases}$$

where

$$R_1 = -\frac{2R}{3\gamma}.$$

In other words, we are defining \vec{w} by

$$\begin{cases} v &= w + R_1 t_1 / t_0 \\ v_0 &= w_0 + R_1 (1 + 2t_1 / t_0) \\ v_i &= w_i + v_i^{(1)} \end{cases}$$

This definition of w is consistent with (4).

REMARK: Note that w_0 should not be confused with $w^{(0)}$, which is the initial value of w .

Equation (9) now takes the form

$$(11) \quad Q(N + A)\vec{w} = t_0 A^i \partial_i \vec{w} + t_0 g_0(X, t_0, t_1, \vec{w}) + t_1 g_1(X, t_0, t_1, \vec{w}),$$

where Q , A , A^i are as before,

$$N = t_0 \partial / \partial t_0 + (t_0 + t_1) \partial / \partial t_1,$$

and

$$g_0 = \begin{pmatrix} 0 \\ -(w_0 + R_1)\Delta\psi + \Delta v^{(1)} - \sum_i 2\psi_i \partial_i R_1 \\ \quad + \gamma[h(v^{(1)} + t_0 w + t_1 R_1) + (2v^{(1)} + t_0 w + t_1 R_1)w] \\ \partial_i R_1 \end{pmatrix}$$

and

$$g_1 = \begin{pmatrix} 0 \\ \gamma(2v^{(1)} + t_0 w + t_1 R_1)R_1 - 2R_1 \Delta\psi - 4\sum_i \psi_i \partial_i R_1 \\ -2v_i^{(1)} \end{pmatrix}.$$

Note that since the kernel of A is one-dimensional, the solutions of (11) are determined entirely by one function, namely the value of the first component of \vec{w} for $t_0 = t_1 = 0$. We let

$$\vec{w}^{(0)} = w(t_0 = t_1 = 0),$$

and replace \vec{w} by $\vec{w} - \vec{w}^{(0)}$. This doesn't change the form of the equation, but reduces us to the case of a Fuchsian system with vanishing initial values.

By abuse of notation, we will sometimes write t for (t_0, t_1) , the meaning being clear from the context.

REMARK: Since the equations for \vec{w} include those deduced from $u_0 = \partial_T u$ and $\partial_T u_i = \partial_i u_0$ (which implies in particular that $\partial_T(\partial_i u - u_i) = 0$), we see that if \vec{w} solves (11), the function $\vec{w}(T, T \ln T, X)$ satisfies

$$w_0 = 2w + T\partial_T w,$$

and

$$\partial_T[w_i T - T^2 \partial_i w - (T^2 \ln T) \partial_i R_1] = 0.$$

It follows that we have in fact

$$w_i = T\partial_i w + (T \ln T) \partial_i R_1.$$

This shows that w determines precisely one solution of (11), which has the property that the u_i are the components of the gradient of u , so that this construction does produce solutions of $\square u = e^u$. This is noteworthy since not all solutions of the equation for \vec{w} correspond to solutions of (1).

We summarize the results in the following theorem:

Theorem 2. *There are symmetric matrices Q and A^j , and a constant matrix A as well as a function f such that if $t = (t_0, t_1)$, and \vec{w} solves*

$$(12) \quad Q(N + A)\vec{w} = t_0 A^j \partial_j \vec{w} + t \cdot f(t, X, \vec{w}),$$

with $w \equiv 0$ for $t_0 = t_1 = 0$, then the first component w of \vec{w} generates, via (4), a singular solution u of (1) which blows up for $T = 0$.

Furthermore, if $\psi \in H^r(\mathbb{R}^n)$, we have Q and A^j in H_{loc}^{r-1} , while f maps H^{r-1} to H^{r-4} smoothly if $r > n/2 + 4$, and is smooth in t .

3.2. Solving Fuchsian systems. We now need to solve the equation for w . We recall below a general set-up for this purpose, which will also enable us to solve the linearization of (21).

We will denote by (u, v) both the Euclidean scalar product on \mathbb{R}^{n+2} and the associated L^2 scalar product.

Assumptions on Q , A , A^j and f which enable one to solve the initial-value problem are as follows:

- (H1) A is constant, while multiplication by Q , Q^{-1} and A_j are bounded operators in H^s ; all the eigenvalues of A have nonnegative real parts.
- (H2) $f(t_0, t_1, u)$ is a C^∞ function in u and t and defines a map from $\mathbb{R}^2 \times H^s$ to H^s ; furthermore, $f \equiv 0$ if $\|u\|_{L^\infty}$ or $|t|$ is large enough.
- (H3) There is a positive-definite matrix-valued function V , which commutes with Q and A_j , and such that $(u, VQAu) \geq 0$, and (u, VQu) is equivalent to the L^2 norm. In addition, $VA_j = A_j$, and multiplication by V is a bounded operator in H^s .

Here, we may take $V = \text{diag}(2\gamma, 1, I_n)$ to satisfy (H3). The other assumptions are readily checked if $s > n/2$.

The following existence theorem is a simple consequence of the results in [15], where a much more general result (tailored to more general blow-up problems) can be found. The statement on the domain of existence is proved in the following section.

Theorem 3. *Assume that the coefficients and nonlinearity satisfy the assumptions (H1)–(H3). For $s > n/2 + 1$, (12) has exactly one solution $(X, T) \mapsto \vec{w}(T, T \ln T, X)$, continuous in T , with values in H^s , which vanishes for $T = 0$. The solution is defined for $|T| \leq 2$ if ψ and $w^{(0)}$ are small enough.*

In particular, \vec{w} is therefore defined for $|t - 1| \leq 1/2$ at least, since $\|\psi\|_{L^\infty} < 1/2$.

4. STEP 2: SMOOTHNESS OF S

There are three issues that must be settled about S : On which spaces does it act in such a way that u is defined upto $t = 1$? Does it produce functions of high Sobolev regularity in both space and time variables? Is it a C^2 mapping on these spaces?

For the first question, we note that the nonlinearities g_1 and g_2 satisfy, if $s \leq r - 4$,

$$|g_i|_s \leq C|\psi|_r(1 + |w|_s + |w_0|_s) + C|t||w|_s.$$

It follows that the time of existence in T can be made arbitrarily large if ψ and $w^{(0)}$ are small enough, by the energy estimates of [15]. Let us therefore assume that the solutions are all defined for $|T| < 2$ throughout the rest of the paper.

For the second question, we must consider the smoothness of the time derivatives of the solution. We are only interested in what happens near $t = 0$, so we do not need information on the T -derivatives of w near $T = 0$, although they could be investigated as in [15]. More simply, we note that the Fuchsian system for $\vec{w}(t_0, t_1)$, can be viewed as a system for $\vec{w}(T, T \ln T)$, by replacing N by $t\partial_T$ and t_0 and t_1 in terms of T ; it therefore contains information on $(T\partial_T)w$. By applying $T\partial_T$ repeatedly, we see that for any k such that $s - k > n/2$, the derivative $(T\partial_T)^k w$ belongs to H^{s-k} . This will ensure that for $1/4 < |T| < 2$, and therefore for $1/2 \leq t \leq 3/2$, w belongs to $H^{s-n/2-1}$ in space and time.

For the third question, we need to study the first two derivatives of w with respect to $w^{(0)}$ and ψ . Since the mapping G defined by (4) is manifestly smooth, the real question is the smoothness of \vec{w} . We will bound its differentials upto third order, thereby ensuring that it is twice continuously differentiable.

Two observations are helpful here: First, if an operator P between Banach spaces is such that for any u and h the function $P(u + \varepsilon h)$ is C^1 as a function of ε , and its derivative is uniformly bounded by $C\|h\|$ uniformly in u , it follows that P is continuous and Fréchet differentiable at u . This argument can be transposed immediately to obtain a criterion for P to be C^1 , C^2 , \dots . We are therefore led to the consideration of $S(w^{(0)} + \varepsilon W^{(0)}, \psi + \varepsilon \Psi)$. The second observation is that we may take ε as an additional space variable in our Fuchsian equations, and consider instead the function

$$\varphi(\varepsilon)S(w^{(0)} + \varepsilon W^{(0)}, \psi + \varepsilon \Psi),$$

where φ is a smooth cutoff function equal to 1 near $\varepsilon = 0$. It is immediate that the corresponding Fuchsian system has a solution \vec{w} of class $H^{r-n/2-5}$ in (X, T, ε) . In particular, if $r > n + 7$, say, it will be a function of class C^1 in ε , and we may differentiate the equation with respect to ε . We have therefore proved that the Gâteaux differential of S may be computed by formal differentiation of the equation.

Now, the linearization of the Fuchsian system for w , is another Fuchsian system, with coefficients of the same degree of regularity. Since it is a linear system, we are guaranteed that the solution exists upto $T = 2$. Similarly, the second variation is computed by linearizing once more. Since $\vec{w}(\varepsilon) - \vec{w}(\varepsilon = 0)$ can be expressed using Taylor's formula, we obtain a bound on the first differential of S . We then repeat the argument for the second and third differential. The final result is that if the solution is in H^q , the differential of S is defined with values in H^{q-1} , and so on, because each linearization of the equation leads to a Fuchsian equation where the time derivative of the previously computed differentials occur, leading to a loss of one derivative for each linearization. If we allow for a loss of 3 derivatives, we see that we can achieve S of class C^2 . (In the presence of further local smoothing properties, the regularity of the differential may perhaps be improved.)

To summarize, $w \in H^{s-4}$ for fixed T , but is in $H^{r-6-n/2}((1/4, 2) \times \mathbb{R}^n)$. It

is however of class C^2 with values in $H^{r-9-n/2}((1/4, 2) \times \mathbb{R}^n)$.

5. STEP 3: SMOOTHNESS OF Z AND DEFINITION OF K

K is obtained from the solution u given by S by (i) changing variables from (X, T) to (x, t) ; (ii) restricting \tilde{u} and \tilde{u}_t to $t = 0$. We study the domain and smoothness of the first operation, namely Z . We let $s = r - 9 - n/2$, which we assume to be greater than $n/2$.

Theorem 4. Z maps $H^r(\mathbb{R}^n) \times H^s(\mathbb{R}^n \times (1/4, 2))$ to $H^s(\mathbb{R}^n \times (-1/2, 1/2))$. It is of class C^2 with values in $H^{s-2}(\mathbb{R}^n \times (-1/2, 1/2))$.

Proof: We need to estimate the space-time regularity of $\tilde{u}(x, t) = u(x, t - \psi(x))$. The idea is as usual (see e.g. [8] for a review of this type of argument) to differentiate and estimate the products of derivatives of ψ using the Gagliardo-Nirenberg inequalities. Since there is an asymmetry between the x and t variables here, we provide the details in a form convenient for the rest of the argument.

Let us therefore find bounds on the derivatives of \tilde{u} . We first note that for any function f ,

$$\iint_{1/2 \leq t \leq 3/2} |f(x, t - \psi(x))|^2 dx dt \leq \iint_{1/4 \leq T \leq 2} |f(X, T)|^2 dX dT,$$

since $\|\psi\|_\infty \leq 1/4$.

We therefore find

$$\|\partial_t^k u(x, t - \psi(x))\|_{L^2(1/2 < t < 3/2)} \leq \|\tilde{u}(X, T)\|_{H^s(1/4 < T < 2)}$$

for $k \leq s$. It therefore suffices to estimate pure x derivatives of \tilde{u} , and then apply the above argument to estimate mixed space-time derivatives.

Now, for any integer s , a generic derivative in x of order s has the form

$$D_x^s [u(x, t - \psi(x))] = \sum c_{a,b,k_1,\dots,k_j} (D_X^a D_T^b u)(x, t - \psi(x)) D^{k_1} \psi \dots D^{k_j} \psi,$$

where $k_1 + \dots + k_j = b = s - a$ (this statement can be checked by induction on s). The Gagliardo-Nirenberg inequality gives quite generally, for functions

of $n + 1$ variables,

$$\|D^j \psi\|_{L^{2s/j}} \leq C |\psi|_s^{j/s} \|\psi\|_{L^\infty}^{(s-j)/s},$$

provided that $s > (n + 1)/2$.

There are now two cases. Either $j \geq 1$ and the Gagliardo-Nirenberg inequality gives that

$$\int |D_x^s [u(x, t - \psi(x))]|^2 dx \leq C |\psi|_s (1 + \|\psi\|_\infty^{s-1}),$$

uniformly in t , so that the integral of this quantity over time is *a fortiori* bounded, or otherwise $j = 0$ and we are dealing with the pure x derivative $D^s u$, which is estimated in the same way as the time derivatives, since $D_x = D_X$.

To complete the construction of K , it suffices to compose $Z \circ S$ with E , which is linear. Since we know that $u(x, t - \psi(x))$ is of class H^s , we compute $u(x, 1 - \psi(x)) \in H^{s-1/2}$ and

$$\partial_t [u(x, t - \psi(x))](t = 1) = u_T(x, 1 - \psi(x)) \in H^{s-3/2},$$

by the trace theorem.

The twice continuous differentiability of K follows again from the consideration of Gâteaux derivatives upto order 3. We find therefore that K is defined near the origin and is C^2 with values in $H^{r-9-n/2-1/2} \times H^{r-10-n/2-1/2}$.

This completes the construction of the evaluation at $t = 1$. Note that one could obtain more detailed results in H_{loc}^s using the properties of paracomposition [1] (one would of course need to avoid any preliminary cutoff in space).

6. STEP 4: LINEARIZATION OF K

Since K is the composition of three C^2 maps, it is itself C^2 .

6.1. Characterization of K' . We must now characterize solutions of the linearization of K in order to be able to identify its inverse in the following section. We first characterize the linearization in two different ways:

Theorem 5. *Let $K'(w^{(0)}, \psi)[W^{(0)}, \Psi] = (U_0, U_1)$ and consider $U = (Z \circ S)'(w^{(0)}, \psi)[W^{(0)}, \Psi]$. Then U can be computed in two different ways:*

(1) *compute the solution of the linearization of (5), linearized with respect to ψ and w , and substitute into the linearization of Eq. (4);*

(2) *let u be the solution $S(w^{(0)}, \psi)$; then U solves the equation*

$$\square \tilde{U} = e^{\tilde{u}} \tilde{U}$$

with data (U_0, U_1) .

As usual, $\tilde{U}(x, t) = U(X, T)$ and $\tilde{u}(x, t) = u(X, T)$.

Proof: Since S is the composition of the solution operator associated with the Fuchsian equation (5) with the operator G , its differential is simply the composition of the differentials of these two operators. We have already seen that since all differentials can be computed as Gâteaux derivatives, we are allowed to compute them in the natural way, by linearizing all the equations used to compute u . This proves (1).

For statement (2), we note that the functions $\ln(2/t^2) + (Z \circ S)(w^{(0)}, \psi)$, as $w^{(0)}$ and ψ vary, all solve the same equation, namely $\square u = e^u$. The differential can again be evaluated as a Gâteaux derivative. Since the reference solution $\ln(2/t^2)$ is independent of $(w^{(0)}, \psi)$, the second statement follows.

We now compute an expansion of U in powers of T and $T \ln T$ by each of the two methods. By comparing the results, we will be able to define the inverse of K' and to estimate its regularity.

6.2. First expansion of U .

6.2.1. *Computation of w .* Let us first clarify the notation of the various maps involved.

The linearization S' of S is obtained by linearizing the Fuchsian equation for w . We write this linearization as

$$(13) \quad S'(w^{(0)}, \psi) : (W^{(0)}, \Psi) \mapsto (W, W_T) \mapsto (U, U_T),$$

using capitals for solutions of linearized equations.

Similarly, it follows from the definition of Z that

$$(14) \quad Z'(\psi, u) : (\Psi, U) \mapsto \tilde{U} - \tilde{u}_t \Psi,$$

using the fact that $u_T(x, t - \psi(x)) = \tilde{u}_t(x, t)$.

Finally, E is linear.

Now, the successive derivatives of w with respect to t_0 and t_1 exist upto order three if we assume, say, $r - 11 - n/2 > 0$. In fact, if we let $\vec{w} = t_0 \vec{w}' + t_1 \vec{w}''$, one can, as in the previous paper in this series [15], define a Fuchsian system for (\vec{w}', \vec{w}'') which implies the original system for \vec{w} . These systems contain derivatives of w . By iterating the process, we establish the existence of an expansion of the solution in powers of T and $T \ln T$, at least as long as the nonlinearities in these “derived” Fuchsian systems continue to act on a Sobolev space of order greater than $n/2 + 1$:

$$w = w^{(0)} + w^{(1)}T + w^{(1,1)}T \ln T + \dots + w^{(j,k)}T^j(\ln T)^k + \dots,$$

with $k \leq j$. The same considerations apply to the linearization of (5), or rather the associated Fuchsian system, and its solution W , corresponding to the initial value $W^{(0)}$. giving

$$W = W^{(0)} + W^{(1)}T + W^{(1,1)}T \ln T + \dots + W^{(j,k)}T^j(\ln T)^k + \dots$$

Since each term in these series entails a loss of one derivative, these expansions remain valid upto order j as long as $r - 4 - j > n/2$.

It should be kept in mind that the coefficients $w^{(j,k)}$ are *known*, since they are the coefficients of the expansion of the reference solution. They can be computed by substitution of the expansion into (5). We give the result for $w^{(1)}$ and $w^{(1,1)}$, for later use:

$$\begin{aligned} w^{(1,1)} &= -\gamma^{-1} \sum_i \partial_i (R_1 \partial_i \psi); \\ 4\gamma w^{(1)} &= 4 \sum_i (\psi_i \partial_i - \partial_i \psi_i) w^{(0)} + \sum_i (3\psi_i \partial_i + 4\partial_i \psi_i) R_1 + \Delta v^{(1)} + \frac{1}{3} \gamma [v^{(1)}]^3. \end{aligned}$$

6.2.2. *Computation of U .* We now compute U by linearization of (4). This is accomplished in two steps: first, we linearize w , which produces

$$\begin{aligned} U &= V^{(0)} + V^{(1)}T + R_1' T^2 \ln T \\ &\quad + T^2(W^{(0)} + W^{(1)}T + W^{(1,1)}T \ln T) + \mathcal{O}(T^2(\ln T)^2), \end{aligned}$$

where $V^{(0)}$, $V^{(1)}$ and R_1' are the linearizations of $v^{(0)}$, $v^{(1)}$ and R_1 respectively, with respect to ψ . We are interested in

$$\tilde{U} = (Z \circ S)'(w^{(0)}, \psi)[W^{(0)}, \Psi].$$

By (14), the linearization of $Z \circ S$ is computed by replacing T by $t - \psi$ in the above expression for U , and by adding

$$-\Psi\left(-\frac{2}{T} + v^{(1)} + R_1 T(1 + 2 \ln T) + 2T w + T^2 w_T\right)$$

to the result. Therefore, we obtain an expression for \tilde{U} of the form

$$\begin{aligned} \tilde{U} &= \frac{U^{(-1)}}{T} + U^{(0)} + U^{(1,1)}T \ln T + U^{(1)}T \\ (15) \quad &\quad + U^{(2,1)}T^2 \ln T + U^{(2)}T^2 + \dots \end{aligned}$$

where the higher-order terms have at least a factor of T^3 (possibly multiplied by powers of $\ln T$). The first few coefficients are

$$\begin{aligned} (16) \quad U^{(-1)} &= 2\Psi \\ U^{(0)} &= V^{(0)} - \Psi v^{(1)} \\ U^{(1,1)} &= -2\Psi R_1 \\ U^{(1)} &= V^{(1)} - \Psi(R_1 + 2w^{(0)}) \\ U^{(2,1)} &= R_1' - 3\Psi w^{(1,1)} \\ (17) \quad U^{(2)} &= W^{(0)} - \Psi(3w^{(1)} + w^{(1,1)}). \end{aligned}$$

Observe that Ψ and $W^{(0)}$ can be recovered from $U^{(-1)}$ and $U^{(2)}$ if w and ψ are known. Note also the absence of a pure $\ln T$ term.

7. STEP 5: INVERSION OF K'

To compute the inverse of K' , we must consider a reference solution $u = S(w^{(0)}, \psi)$, such that $(u_0, u_1) = K(w^{(0)}, \psi)$, and a pair (U_0, U_1) . We must then find a pair $(W^{(0)}, \Psi)$ such that $K'(w^{(0)}, \psi)[W^{(0)}, \Psi] = (U_0, U_1)$. We assume that these data are in $H^\sigma \times H^{\sigma-1}$, where $\sigma = r - 10 - n/2$.

From the characterization of Step 4, we know that we must first define U by solving $\square \tilde{U} = e^{\tilde{u}} \tilde{U}$ with data (U_0, U_1) . We must then study the behavior of the function $U = \tilde{U}(X, T + \psi(X))$ as $T \rightarrow 0$.

We first show that the linearization of (1) is itself again a Fuchsian equation. We next show that a solution of a linear Fuchsian equation cannot have a singularity worse than a power of T . We then prove iteratively that the solution of the linearized equation has in fact an expansion in powers of T and $T \ln T$ to all orders. Finally, by comparing this expansion with the result of the previous step, and more specifically Eqs. (16) and (17), we identify the desired values of $W^{(0)}$ and Ψ .

7.1. Fuchsian equation for U .

7.1.1. *Expansion of e^u .* We provide below the expansion of e^u in terms of T ; its existence is a consequence of the representation (4), combined with the properties of w :

$$\begin{aligned} e^u &= \frac{2\gamma}{T^2} \{v^{(1)}T + R_1 T^2 \ln T + T^2 w\} \\ (18) \quad &= \frac{2\gamma}{T^2} \{1 + v^{(1)}T + R_1 T^2 \ln T + T^2(w + \frac{1}{2}[v^{(1)}]^2) + \mathcal{O}(T^3 \ln T)\}. \end{aligned}$$

Note that there is no $T \ln T$ term in the curly brackets.

The equation $\square U = \exp(u)U$ therefore reads, in the (X, T) variables,

$$T^2(\gamma U_{TT} - \Delta U + 2\psi^i \partial_i u_T + (\Delta \psi)U_T) = \gamma(2 + a_1 T + a_{21} T^2 \ln T + \dots)U.$$

7.1.2. *Fuchsian system for the determination of U .* The equation for U is converted into Fuchsian form by letting

$$\vec{U} = (U, U_0, U_i) := (U, T \partial_T U, T \nabla_x U).$$

We find

$$Q(T\partial_T + B)\vec{U} = TA^i\partial_i\vec{U} + T \begin{pmatrix} 0 \\ [\exp(Tv^{(1)} + T^2w) - 1]U/T - U_0\Delta\psi \\ 0 \end{pmatrix},$$

where

$$B = \begin{pmatrix} 0 & -1 & \\ -2 & -1 & \\ & & 0_n \end{pmatrix}.$$

7.1.3. A general property of Fuchsian systems. We now show that a solution of a linear Fuchsian equation cannot have a singularity worse than a power of T . We then prove iteratively that the solution of the linearized equation has in fact an expansion in powers of T and $T \ln T$ to all orders. Finally, we identify from the terms of this expansion the desired values of $W^{(0)}$ and Ψ .

Let us therefore start with a general Fuchsian system, and show that its solutions have only power singularities in T . We then apply the argument to U . This generalizes results of Tahara for the linear, C^∞ case. In our case, we need in addition to track the number of derivatives involved carefully.

Let \vec{w} solve

$$Q(T\partial_T + A)\vec{w} = T(\mathcal{B}\vec{w} + f(\vec{w}))$$

for $T > 0$, where $\mathcal{B} = \sum_j A_j \partial_j$, f is linear (or sublinear), and is only assumed to be continuous in T (it might therefore involve terms in $T \ln T$). The dependence of f on space and time coordinates is suppressed.

We find by multiplication that if $\epsilon(T) = (\vec{w}, VQ\vec{w})(T)$,

$$T\partial_T\epsilon + \alpha\epsilon \geq -CT(1 + \epsilon),$$

where α can always be taken to be positive. In this particular example, $\alpha = 1$, but the following applies quite generally. It follows that

$$(T^\alpha\epsilon)_T \geq -CT^\alpha(1 + \epsilon) \geq -C(1 + T^\alpha\epsilon),$$

so that we get, by integration, say from T to 1,

$$1 + \epsilon(T)T^\alpha \leq \text{const.}$$

Therefore, we see that $\|\vec{w}\|_{L^2}$ cannot grow faster than a power of T .

In fact, we know [15] that $(1 - \Delta)^{\sigma/2}$ solves again an equation of the same form as \vec{w} , and therefore, we also know that $T^\alpha \|\vec{w}\|_{H^\sigma}$ remains bounded.

7.2. Definition of the inverse of K' .

7.2.1. *Existence of an expansion for U .* We now apply these general facts to the Fuchsian equation for U . We find we may take $\alpha = 1$; indeed, α is determined as the smallest value which makes the inequality $(VQB\vec{w}, \vec{w}) \leq \alpha(VQ\vec{w}, \vec{w})$ hold.

REMARK: It is fortunate that $\alpha = 1$ here. But quite generally, it is always possible to reduce oneself to this case: Assume for example that

$$|U(T)|_\sigma + |TU_T(T)|_\sigma \leq CT^{-a}.$$

The equation for U takes the form

$$(T\partial_T + 1)(T\partial_T - 2)U = O(T)[U],$$

where the r.h.s. contains products of T by bounded expressions involving second derivatives of ψ derivatives of U of the form D^2U , DU or $DT\partial_T U$ at most, where D stands for any space derivative. This r.h.s. is therefore estimated by CT^{1-a} in $H^{\sigma-2}$ if $\sigma - 2 \geq r - 2 > n/2$. By solving this equation for U , we find that U is in fact $\mathcal{O}(T^{1-a})$ in $H^{\sigma-2}$ unless $1 - a \geq -1$. TU_T satisfies a similar estimate.

Iterating the process, we see that there is an integer r_1 such that

$$|U|_{\sigma-r_1} \leq CT^{-3/2}.$$

This remark would be useful in handling more general equations. It is implemented in the next paragraph to produce the existence of an expansion of U in powers of T and $T \ln T$, to be identified with the expansion of §6.2.

Since one can solve $(T\partial_T + 1)(T\partial_T - 2)U = g(t)$ as

$$U = \frac{c_{-1}}{T} + c_2 T^2 + \int^T \frac{T^3 - s^3}{3s^3 T} g(s) ds,$$

we see that whenever $g(s) = O(s^a)$, where a is not an integer, there is a particular solution which is $O(T^a)$ as well. Indeed, for $a < 0$, we take the lower limit of integration to be 1, and for $a > 0$, we split the integrand into two parts, one of order s^{a-3} , and the other of order s^a ; we then choose different constants of integration for these two terms.

It follows that there is a function U_0 in $H^{\sigma-2}$ such that the linearization U satisfies

$$U = \frac{U^{(-1)}(X)}{T} + \mathcal{O}(1).$$

We may now insert $U = U^{(-1)}T^{-1} + V$ in the l.h.s. of the equation for U . Integrating again produces the next term, so that

$$U = U^{(-1)}T^{-1} + U^{(0)} + \mathcal{O}(T \ln T).$$

The T^{-1} term obtained at this step must clearly be the same as that obtained before. In addition, we find that $U^{(0)} \in H^{\sigma-1}$. The process can clearly be iterated. Quite generally, since the second derivative terms are multiplied by T^2 , we find that terms at level j have two derivatives less than those at level $j - 2$.

One can immediately conclude to the existence of a logarithmic series for U ; the source of the logarithmic terms is to be found already in the logarithms in the expansion of e^u . we also note that this series has only $T \ln T$ as its first logarithmic term (and not $\ln T$), in accordance with the expansion of U found earlier.

7.2.2. Definition of K^{l-1} . We have therefore found two different ways of computing the expansion of U in powers of T and $T \ln T$.

Comparing with (16) and (17), we find that

$$\Psi = U^{(-1)}/2$$

and

$$W^{(0)} = U^{(2)} + \Psi(3w^{(1)} + w^{(1,1)}).$$

Note that $\Psi \in H^{\sigma-2}$, and $U^{(2)} \in H^{\sigma-8}$.

7.3. Is this a right inverse? The above procedure gave us a left inverse of K' . In fact, this operator is also a right inverse, as we proceed to show. Let us apply our inverse to a given pair (U_0, U_1) . We obtain a pair $(W^{(0)}, \Psi)$. We want to show that $K'(w^{(0)}, \psi)[W^{(0)}, \Psi]$ coincides with (U_0, U_1) . Now the given pair (U_0, U_1) generaltes a solution U of the linearized equation, and it would suffice to show that this U coincides with the solution U' generated by $(W^{(0)}, \Psi)$ as in Theorem 5. However, both generate solutions of the Fuchsian equation for \vec{U} , and they have the same expansion upto order two at least, because these coefficients are completely determined by the value of $(W^{(0)}, \Psi)$. But the coefficients of the expansion of U are determined recursively after order T^2 , and therefore U and U' coincide to all orders. We then note that we may, by the process already used to derive the system for \vec{w} from that satisfied by \vec{u} , write $U = U^{-1}/T + \dots + (U^{(2)} + Y)T^2$, where Y is the first component of a generalized Fuchsian system. Similarly, U' is associated with a function Y' *which solves the same equation*. Since this system will has [15] only one solution which vanishes for $t_0 = t_1 = 0$, we conclude that $U' = U$, as desired.

8. STEP 6: APPLICATION OF THE NASH-MOSER THEOREM, END OF PROOF

We wish to use the Nash-Moser theorem with smoothing to invert the mapping K . To this end, we recall the statement and proof of an appropriate version, based on [19]. We then describe its application. There are several versions of the Nash-Moser theorem in the literature, and many references can be found in Hamilton [6] and Hörmander [7] in particular. The former contains a very general set-up for the inversion of smooth maps on scales of Banach spaces, while the latter deals with the case when the injections in the

target scale of spaces is compact, but deals with C^2 maps. Hörmander's argument would therefore apply here in the case of periodic boundary conditions. Similarly, Th. 1.1.1 of [6] could perhaps be adapted to show that the inverse of K is twice continuously differentiable. In view of these points, a short, self-contained proof of a simple and convenient version has been included.

8.1. A version of the Nash-Moser theorem. We wish to solve the equation $F[u] = 0$, where F acts from a scale $\{X^s\}_{s>0}$ of Banach spaces to another scale $\{Y^s\}_{s>0}$. It is important to note that we will be using in the statements and proofs only *four* spaces in the scale $\{X^s\}$; they will be denoted by X^{s-a} , X^s , X^{s+b} and X^{s+a+b} . These spaces will be products of Sobolev spaces, so the reader should think of s as a measure of differentiability. In this section, there is no relation between s and r .

For clarity, the norms in X^s are denoted by double bars, and those in Y^s by single bars.

8.1.1. *Assumptions.* There are two sets of hypotheses: one pertaining to the smoothing, the other to F . In all, M , a and s are fixed constants; $M \geq 1$, and $a > 0$.

Let us assume that there is a family $S(t)$ of smoothing operators such that

$$(S1) \quad \|S(t)u\|_{s+\sigma} \leq Mt^\sigma \|u\|_s \text{ if } u \in X^s \text{ and } \sigma \geq 0.$$

$$(S2) \quad \|(I - S(t))u\|_{s-\sigma} \leq Mt^{-\sigma} \|u\|_s \text{ if } u \in X^s \text{ and } 0 \leq \sigma \leq s.$$

For some versions of the theorem, it is convenient to require a bound on $dS(t)/dt$, so as to ensure convexity estimates on the norms involved.

We also assume that the map F has the following properties:

$$(F1) \quad F \text{ is a mapping of class } C^2 \text{ from the open ball of radius } R \text{ in } X^s, \text{ with values in } Y^s. \text{ Its first and second derivatives are bounded by } M \geq 1.$$

$$(F2) \quad F'(u) \text{ has a right inverse } L(u) \text{ which sends } Y^s \text{ to } X^{s-a}, \text{ and is bounded uniformly by } M \text{ if } \|u\|_{X^s} < R. \text{ Furthermore, there is a number } b > 8a$$

such that for every u with $\|u\|_{X^s} < R$,

$$\|L(u)F(u)\|_{s+b} \leq M(1 + \|u\|_{s+a+b}).$$

Thus, F' fails to be invertible because $L(u)$ has its range in X^{s-a} instead of X^s . Note also that $L(u)$ is not required to depend continuously on u .

8.1.2. *Result.* The result is as follows:

Theorem 6. *Under assumptions (S1)–(S2) and (F1)–(F2), equation $F(u) = 0$ has a solution in X^s if $F(0)$ is small enough in Y^s .*

REMARKS: 1) The assumptions are stronger than requiring that F' have an unbounded inverse. In fact, this latter assumption alone would lead to an incorrect result, as the case of the mapping $F(u) = u - u_0$ from H^s to H^{s-1} shows, if $u_0 \notin H^s$. What happens here is that (F2) requires $L(u)F(u)$ to be smoother and smoother if u is. This excludes the counter-example we just gave. Also, in practice, it is much easier to find $L(0)$ than it is to construct $L(u)$ for $u \neq 0$. At the formal level, the existence of $L(0)$ suffices to construct a formal series to all orders. This series may however be completely meaningless.

2) We can construct a family of smoothing operators by choosing a function $\phi(\xi)$ which equals 1 for $|\xi| < 1$ and 0 for $|\xi| > 2$, with $0 \leq \phi \leq 1$, and by letting $S(t)u = \mathcal{F}^{-1}\phi(\xi/t)\mathcal{F}u$, where \mathcal{F} denotes the Fourier transform. Indeed, we then have $\phi(\xi/t)(1+|\xi|^2)^\sigma \leq (1+4t^2)^\sigma$ and $(1-\phi(\xi/t))(1+|\xi|^2)^{-\sigma} \leq (1+t^2)^{-\sigma}$.

8.1.3. *Proof.* Let $q_k = \exp[\lambda\rho^k]$ and $S_k = S(q_k)$, where $\rho = 3/2$ for definiteness (the reader is invited to examine the rest of the proof for $\rho \in (1, 2)$); λ will be determined later.

We define $\{u_k\}_{k \geq 0}$ by $u_0 = 0$ and

$$(19) \quad u_{k+1} = u_k - S_k L(u_k) F(u_k).$$

We prove by induction that there exist positive constants μ and ν such that

$$(20) \quad \|u_k - u_{k-1}\|_s \leq q_k^{-\mu a}; \quad 1 + \|u_k\|_{s+a+b} \leq q_k^{\nu a}.$$

These estimates will ensure that the iteration is well-defined, and converges in the s -norm.

We first estimate the difference of two consecutive approximations:

$$\begin{aligned}
& \|u_{k+1} - u_k\|_s \\
& \leq Mq_k^a \|L(u_k)F(u_k)\|_{s-a} \\
& \leq M^2q_k^a |F(u_k)|_s \\
& = M^2q_k^a \left[|F(u_{k-1}) - F'(u_{k-1})S_{k-1}L(u_{k-1})F(u_{k-1})|_s \right. \\
& \quad \left. + \left| \int_0^1 (1-\sigma)F''((1-\sigma)u_{k-1} + \sigma u_k) \cdot (u_k - u_{k-1}, u_k - u_{k-1}) d\sigma \right|_s \right] \\
& \leq M^2q_k^a \left\{ |F'(u_{k-1})(I - S_{k-1})L(u_{k-1})F(u_{k-1})|_s + Mq_k^{-2a\mu} \right\}.
\end{aligned}$$

Now, using (S2) and (F2), we find

$$\|L(u_{k-1})F(u_{k-1})\|_{s+b} \leq M(1 + \|u_{k-1}\|_{s+b+a}) \leq Mq_{k-1}^{a\nu},$$

while (F1) and (S2) imply, for any $v \in X^{s+b}$,

$$|F'(u_{k-1})(I - S_{k-1})v|_s \leq M^2q_{k-1}^{-b}\|v\|_{s+b}.$$

Therefore

$$\|u_{k+1} - u_k\|_s \leq M^5q_k^a q_{k-1}^{a\nu-b} + M^3q_k^{a(1-2\mu)}.$$

Since $M \geq 1$, it suffices to have

$$(21) \quad M^5(q_k^a q_{k-1}^{a\nu-b} + q_k^{a(1-2\mu)}) \leq q_{k+1}^{-a\mu}$$

to ensure the desired estimate on $\|u_{k+1} - u_k\|_s$.

As for the other bound, we estimate

$$\begin{aligned}
1 + \|u_{k+1}\|_{s+b+a} & \leq 1 + \sum_{j=0}^k \|S_j L(u_j)F(u_j)\|_{s+b+a} \\
& \leq 1 + M \sum_j q_j^a \|L(u_j)F(u_j)\|_{s+b} \\
& \leq 1 + M^2 \sum_j q_j^a (1 + \|u_k\|_{s+b+a}) \\
& \leq 1 + M^2 \sum_j q_j^{a(1+\nu)}.
\end{aligned}$$

We therefore need

$$(22) \quad 1 + M^2 \sum_j q_j^{a(1+\nu)} \leq q_{k+1}^{a\nu}.$$

Since $q_{k+1} \geq q_{j+1}$, and $q_0 = e^{\lambda a}$ this follows from

$$1 \geq q_{k+1}^{-a\nu} (1 + M^2 e^{\lambda a(1+\nu)}) + M^2 \sum_{j=1}^k e^{\lambda \rho^j a[(1+\nu) - \rho\nu]}$$

which, since $\rho = 3/2$, holds if $\nu > 2$ and λ is large enough.

As for (21), we may satisfy it by requiring

$$\lambda \rho^{k-1} (a\nu - b + a\rho + \mu\rho^2) \leq -5 \ln M - \ln 2,$$

and

$$\lambda \rho^k (a(1 - 2\mu) + a\mu\rho) \leq -5 \ln M - \ln 2.$$

These hold as well for λ large, provided that

$$\mu > 2 \quad \text{and} \quad b > a\nu + 3a/2 + 9\mu/4.$$

This estimate also ensures, by taking λ still larger, that $\|u_k\|_s$ remains less than $R/2$, so that the iterations are well-defined.

To summarize, all we need is to be able to choose $\mu > 2$, $\nu > 2$, and $b > a\nu + 3a/2 + 9a\mu/4$. This is certainly possible if b is greater than $2a + 3a/2 + 18a/4 = 8a$, which is the case by assumption.

To start the induction, we need to consider $u_1 = -S_0 L(0)F(0)$. Since $\|S_0 L(0)F(0)\|_{s+a+b} \leq M q_0^a \|L(0)F(0)\|_{s+b} \leq M^2 q_0^a$ (from (F2)), we require

$$(23) \quad 1 + M^2 q_0^a \leq q_1^{\nu a}.$$

For the first part of (21), we write

$$\|u_1\|_s \leq M q_0^a \|L(0)F(0)\|_{s+a} \leq M^2 q_0^a |F(0)|_s,$$

which leads to the condition

$$(24) \quad |F(0)|_s \leq q_0^{-a} q_1^{-\mu a} / M^2.$$

We therefore choose λ large enough to satisfy (23), and then check the smallness condition (24) on $F(0)$.

If $F(0)$ is small enough, we see that the iterations remain in a small neighborhood of 0 (and are therefore well-defined), and converge in the X^s norm. It follows from the continuity of F and the existence of a uniform bound on $L(u)$ that $L(u_k)F(u_k)$ converges in X^{s-a} , and since the smoothing operators approximate the identity, we may write

$$\|(I - S_k)L(u_k)F(u_k)\|_{s-a-1} \leq Cq_k^{-1} \rightarrow 0$$

as $k \rightarrow \infty$. We conclude that $u_\infty = \lim_{k \rightarrow \infty} u_k$ solves

$$L(u_\infty)F(u_\infty) = 0,$$

in the space X^{s-a-1} . Applying $F'(u_\infty)$, we conclude that

$$F(u_\infty) = 0,$$

QED.

8.2. Application and end of proof. We now take $X^r = H^{r-3} \times H^r$ and

$$Y^r = H^{r-10-[n/2]} \times H^{r-11-[n/2]}.$$

For simplicity, all Sobolev spaces will be taken to have integer order. $F(w^{(0)}, \psi) := K(w^{(0)}, \psi) - (u_0 - \ln 2, u_1 + 2)$, and $a = 6 + [n/2]$. We also assume $r > 11 + [n/2]$. We want to apply the Nash-Moser theorem of the previous section with $s = r$.

We have seen in §5 that

$$K \in C^2(X^r; Y^r).$$

Note that the solution w generated by a pair $(w^{(0)}, \psi)$ in X^r belongs to H^{r-4} for fixed T , and we have seen that the coefficients $w^{(1)}$ and $w^{(1,1)}$ of the expansion of w are in H^{r-6} at least.

On the other hand, the inverse of K' sends $H^\sigma \times H^{\sigma-1}$ to $(W^{(0)}, \Psi)$ as given in §7; it therefore takes values in $H^{r-18-[n/2]} \times H^{r-10-[n/2]}$. We conclude that $L(u)$ exists and is bounded from Y^r to X^{r-a} , where

$$a = 15 + [n/2].$$

Since we want $b > 8a$, we take $b = 8a + 1 = 121 + 8[n/2]$ to fix ideas. To guarantee (F2), we need F to map X^{r+a+b} to Y^{r+a+b} ; this imposes a regularity condition on u_0 and u_1 , namely

$$(u_0 - \ln 2, u_1 + 2) \in H^{r+146+9[n/2]} \times H^{r+145+9[n/2]}.$$

To ensure that all Sobolev indices appearing in the calculations are greater than $n/2$, we require $r > 11 + [n/2]$.

In terms of the Cauchy data, it means that they are taken in $H^s \times H^{s-1}$ with

$$s > 167 + 10[n/2].$$

The Nash-Moser theorem ensures that if Cauchy data have this regularity and are close to $(\ln 2, -2)$, the corresponding solution must blow-up on a spacelike hypersurface of class H^r with $r = s - 146 - 9[n/2]$.

This proves the announced result.

9. CONCLUDING REMARKS

We have therefore proved that any solution with data close to those of $\ln(2/t^2)$ must blow-up on a spacelike hypersurface near which it has logarithmic behavior. It is in fact described by the first few terms of the formal expansion derived in [14], truncated to allow for the limited regularity of the solution. From the knowledge of ψ , one can read off the blow-up time, which may not be attained at any finite x , as the case of a bell-shaped ψ shows. Also, because the Fuchsian equation (5), or the associated Fuchsian system, can be solved in a full neighborhood of $T = 0$, the singular solutions are at once defined on both sides of the blow-up surface. One therefore reaches the conclusion

that singular solutions have a meaningful continuation after blow-up. This continuation procedure is similar to the regularization of collisions in the three-body problem [20].

The present approach applies whenever we are given a reference solution (other than $\ln(2/t^2)$), and consider data close to those of this solution. The reason is that our argument for the invertibility of the linearization of K did not use in any essential way the properties of this reference solution. This suggests that the set of data leading to blow-up is *open*.

It is also possible that one may allow for more general blow-up surfaces, which do not become flat at infinity, by working in uniformly local Sobolev spaces.

One should stress in conclusion that the above method is not limited in scope to the particular example treated here because (i) the exact form of the reference solution is not used anywhere: what matters is that it should have a logarithmic expansion; (ii) such expansions have been shown to exist for very large classes of nonlinear equations [13, 14, 15, 17].

REFERENCES

1. S. Alinhac, Paracomposition et opérateurs paradifférentiels, *Comm. PDE*, **11** (1986) 87–121.
2. S. Alinhac, Explosion géométrique pour les systèmes quasi-linéaires, preprint #93-84, Orsay, Dec. 1993. See also: Temps de vie précisé et explosion géométrique pour des systèmes quasi-linéaires en dimension un d'espace, preprint #94-18, Orsay, March 1994.
3. S. Alinhac, *Blow-up for nonlinear Hyperbolic Equations*, Lecture notes of a graduate course, Orsay 1994.
4. L. Caffarelli and A. Friedman, The blow-up boundary for nonlinear wave equations, *Trans. AMS* **297** (1986) 223–241, and Differentiability of the blow-up curve for one-dimensional wave equations, *Arch. Rat. Mech. Anal.* **91** (1985) 83–98.

5. R. E. Caflisch, Singularity formation for complex solutions of the 3D incompressible Euler equations, *Physica D*, **67** (1993) 1–18; see also the paper by Caflisch et al. in *Comm. Pure and Appl. Math*, **46** (1993) 453–499, and their references.
6. R. S. Hamilton, The inverse function theorem of Nash and Moser, *Bull. AMS*, **7** (1982) 65–222
7. L. Hörmander, The boundary problem of physical geodesy, *Arch. Rat. Mech. Anal.*, **62** (1976) 1–52; see also: On the Nash-Moser implicit function theorem, *Ann. Acad. Sci. Fenn.*, **10** (1985) 255–259.
8. L. Hörmander, On Sobolev spaces associated with some Lie algebras, Mittag-Leffler Institute Report # 4, 1985.
9. L. Hörmander, The lifespan of classical solutions of non-linear hyperbolic equations, *Springer Lect. Notes in Math.*, **1256** (1986) 214–280.
10. F. John, Blow-up of solutions of nonlinear wave equations in three space dimensions, *Manus. Math.*, **18** (1979) 235–268.
11. F. John, Existence for large times of strict solutions of nonlinear wave equations in three space dimensions for small data, *Comm. Pure Appl. Math.*, **40** (1987) 79–109.
12. F. John, *Nonlinear wave equations, formation of singularities*, University Lecture Series, vol. 2, (1990) AMS.
13. S. Kichenassamy and W. Littman, Blow-up surfaces for nonlinear wave equations, I, *Comm. PDE*, **18** No. 3&4 (1993) 431–452.
14. S. Kichenassamy and W. Littman, Blow-up surfaces for nonlinear wave equations, II: *Comm. PDE*, **18** No. 11 (1993) 1869–1899.
15. S. Kichenassamy, Fuchsian equations in Sobolev spaces and blow-up, to appear in *J. Diff. Eq.*
16. S. Kichenassamy, *Nonlinear Wave Equations*, (1995) to appear.
17. S. Kichenassamy and G. K. Srinivasan, The structure of WTC expansions and applications, *J. Phys. A: Math. Gen.* **28** : 7 (1995) 1977–2004.

18. H. Lindblad, Blow-up of solutions of $\partial_t^2 u - \Delta u = |u|^p$ with small data, *Comm. PDE*, **15** No. 6 (1990) 757–821.
19. J. T. Schwartz, *Nonlinear Functional Analysis*, Gordon and Breach, 1969.
20. C. L. Siegel and J. Moser, *Lectures on Celestial Mechanics*, Springer (1971).
21. W. Strauss, *Nonlinear Wave Equations*, CBMS lecture notes # 73.
22. C. Zuily, Bourbaki Seminar, 1994.

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