Convergence of The Multigrid Method With A Wavelet Coarse Grid Operator

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Abstract

The convergence of the two-level multigrid method with a new coarse grid operator is studied. This new coarse grid operator is constructed using the wavelet transformation. For partial differential equations with highly oscillatory coefficients for which the homogenization theory is applicable, this operator is considered as being close to the corresponding homogenized operator. Under some regularity assumptions between the approximation spaces, a convergence of the method is established. The convergence rate is independent of the grid step size. Furthermore, we show that the convergence rate in general is larger than that of the two-level multigrid method with Garlerkin coarse grid operator.

1 Introduction

For regular elliptic equations it is well known that the multigrid method is practically very efficient. There is a number of papers dealing with the theoretical issue of convergence with Garlerkin coarse grid operator. However, most of these papers focus on the finite element methods. Restricting to the variational form, Garlerkin form is a natural built-in construction under these methods. With the regularity assumption of the partial differential equations, some convergence proofs are established, [2, 7, 8, 9]. The finite element method induces a nested set of smooth function subspaces. The "Aubin-Nitsch" trick has been commonly used in these proofs. The multigrid method generated from finite difference method no longer induces a nested set of smooth subspaces where similar regularity estimate

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can still be established. Discussions on this issue can be found in [8, 10]. For the convergence proof of the multigrid method, the regularity of the partial differential equation is always needed.

For the elliptic differential equation with highly oscillatory coefficients, standard multigrid method is not efficient. Using finite element method with smooth base functions, the convergence proof for the multigrid method in [7] does not require any regularity of the partial differential operator. However, since the finite element method is used there, a nested set of smooth function subspaces are built in. Within such nice subspaces, one obtains easily an approximate solution, but such an approximation could be far away from the true solution of the oscillatory coefficient problem, [1]. For the class of partial differential equations with highly oscillatory coefficients, for which the homogenization theory is applicable, in order to maintain a fast convergence rate for the multigrid method, a new coarse grid operator based on the corresponding homogenized operator is introduced in [3, 4, 5, 6] using finite difference method. This operator has come to be called homogenized coarse grid operator. Since the difference between the oscillatory solution and the homogenized solution is only of first order of the grid step size, a large number of smoothing iteration (which depends on the grid step size) is needed to guarantee the multigrid method to converge in l_2-norm. The difference of the first derivative between both is of a constant order in general, and the use of energy norm is impossible to build a similar "approximate property".

In this paper we study the convergence of the two-level multigrid method by constructing a new coarse grid operator. This operator arises from the wavelet transformation and we call it here the wavelet operator. As already discussed in [3, 4, 5, 6] for the class of partial differential equations, this operator is assumed to approximate to the corresponding homogenized operator. It's therefore natural for us to try this operator instead of the homogenized operator directly in the multigrid method. An interesting property about this operator is that it can be explicitly written as a combination of the original operator and the operators between the grid transformation. Therefore, the use of projection theory is possible to analyze the convergence of the multigrid method with the wavelet coarse gird operator. By assuming the multigrid method with the Garlerkin variational coarse grid operator to satisfy both "approximation property" and "smoothing property" as in [8], we prove that the two-level multigrid method with the wavelet coarse grid operator is convergent independently of the grid step size. We also prove that in general the convergence rate is larger than that with Garlerkin coarse grid operator.

The rest of the paper is organized as follows. In section 2, the wavelet operator is constructed. In section 3, the two-level multigrid method is briefly introduced. The convergence of the two-level multigrid method with the wavelet operator is analyzed in section

4, and in section 5 the convergence rates of the two-level multigrid method with Garlerkin and wavelet coarse grid operators are compared.

2 Wavelet Transformation

Consider the following algebraic equation arising from the discretization of partial differential equation,

$$AU = F, \tag{2.1}$$

where A is symmetric positive definite. To solve equation (2.1) we introduce a transformation

$$M = \begin{bmatrix} L \\ H \end{bmatrix}, \tag{2.2}$$

where

$$L = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & & & \\ 0 & 0 & 1 & 1 & 0 & 0 & \cdots \\ \vdots & & & \ddots & & & \end{pmatrix},$$

and

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 & 0 & 0 & \cdots & & & \\ 0 & 0 & 1 & -1 & 0 & 0 & \cdots \\ \vdots & & & \ddots & & & \end{pmatrix}.$$

Then M satisfies the following properties,

(i)

$$M^T M = M M^T = L^T L + H^T H = I.$$

(ii)

$$L L^T = H H^T = I,$$

$$L H^T = H L^T = 0.$$

Multiplying M to equation (2.1) becomes

$$M A M^T M U = M F.$$

We have

$$(2.3) \qquad \begin{pmatrix} LAL^T & LAH^T \\ HAL^T & HAH^T \end{pmatrix} \begin{pmatrix} LU \\ HU \end{pmatrix} = \begin{pmatrix} LF \\ HF \end{pmatrix}.$$

Denote $T = LAL^T, B = LAH^T, C = HAL^T, D = HAH^T, U_L = LU, U_H = HU, f_L = LF$, and $f_H = HF$. From (2.3),

$$(2.4) \qquad U_L = (T - BD^{-1}C)^{-1}(f_L - BD^{-1}f_H).$$

Equation (2.1) can be written as

$$(2.5) \qquad U = A^{-1}F.$$

Applying the wavelet transformation (2.2) to (2.5), we have

$$MU = MA^{-1}M^TMf.$$

That is,

$$(2.6) \qquad \begin{pmatrix} U_L \\ U_H \end{pmatrix} = \begin{pmatrix} LA^{-1}L^T & LA^{-1}H^T \\ HA^{-1}L^T & HA^{-1}H^T \end{pmatrix} \begin{pmatrix} f_L \\ f_H \end{pmatrix}.$$

Thus

$$(2.7) \qquad U_L = (LA^{-1}L^T)f_L + LA^{-1}H^Tf_H.$$

We are now ready to establish

Lemma 2.1

$$(T - BD^{-1}C)^{-1} = LA^{-1}L^T.$$

Proof.

$$\begin{aligned}
& (T - BD^{-1}C)(LA^{-1}L^T) \\
= \; & LAL^TLA^{-1}L^T - BD^{-1}CLA^{-1}L^T \\
= \; & LA(I - H^TH)A^{-1}L^T - BD^{-1}CLA^{-1}L^T \\
= \; & I - LAH^THA^{-1}L^T - BD^{-1}CLA^{-1}L^T \\
= \; & I - LAH^THA^{-1}L^T - LAH^T(HAH^T)^{-1}HAL^TLA^{-1}L^T \\
= \; & I - LAH^THA^{-1}L^T - LAH^T(HAH^T)^{-1}HA(I - H^TH)A^{-1}L^T \\
= \; & I - LAH^THA^{-1}L^T + LAH^T(HAH^T)^{-1}HAH^THA^{-1}L^T \\
= \; & I - LAH^THA^{-1}L^T + LAH^THA^{-1}L^T = I.
\end{aligned}$$

□

3 Two-Level Multigrid Method

In this section we consider only two-level multigrid method for a one dimensional problem. We discretize the partial differential equation at a fine grid level h and obtain an algebraic equation as (2.1) in a n-dimensional vector space $\Re^n$. For the multigrid method we assume the simple Richardson iteration is used as the smoother G. The coarse grid operator of the Garlerkin variation form on the coarse grid level H is thus denoted by

$$A_G = I_h^H A I_H^h, \tag{3.8}$$

where I_h^H and I_H^h denote the interpolation and prolongation respectively. Therefore, the two-level multigrid operator with pre- and post- smoothing iterations based on these coarse grid operators can be written as

$$M_W = G(A^{-1} - I_H^h A_W^{-1} I_h^H) A G. \tag{3.9}$$

By the analysis of the previous section, we induce the following new coarse grid operator from the wavelet transformation

$$A_W = (I_h^H A^{-1} I_H^h)^{-1}, \tag{3.10}$$

where we set

$$I_h^H = \frac{1}{\sqrt{2}} L, \tag{3.11}$$

L is a $\frac{n}{2} \times n$ matrix defined as in (2.2), and

$$I_H^h = 2(I_h^H)^T = \sqrt{2} L. \tag{3.12}$$

For the rest of the paper we always assume that the interpolation and prolongation are chosen to be of these forms. Furthermore, for simplicity, we only consider the one dimensional case here. However, the analysis can be extended to higher dimensional case. When the partial differential equation belongs to the class mentioned above, the operator (3.10) is considered as an approximation to the corresponding homogenized operator. Hence, by the wavelet coarse grid operator (3.10), the operator of the two-level multigrid method with pre- and post- smoothing iterations can be written as follows,

$$M_G = G(A^{-1} - I_H^h A_G^{-1} I_h^H) A G. \tag{3.13}$$

The following lemma establishes a relationship between A_G and A_W.

Lemma 3.1

$$A_W^{-1} \geq A_G^{-1} > O.$$

Proof. By Lemma 2.1,

$$A_W = A_G - BD^{-1}C.$$

Since A_W, A_G and $BD^{-1}C$ are symmetric and positive,

$$A_G \geq A_W > O.$$

Therefore,

$$A_W^{-1} \geq A_G^{-1} > O.$$

$\square$

4 Convergence Analysis

Set $R = L^T L$ and consider the following two subspaces of the n-dimensional vector space $\Re^n$

(4.14)
$$\Phi = Range(A^{-1/2} R A^{1/2})$$

and

(4.15)
$$\Psi = Range(A^{-1/2}(I - R)).$$

Lemma 4.1 *The two subspaces (4.14) and (4.15) are A-orthogonal and form a direct sum of* $\Re^n$,

$$\Re^n = \Phi \oplus \Psi.$$

Let S and T be the projections of $\Re^n$ onto Φ and Ψ, respectively. Define an energy norm for $\Re^n$ by

$$\|x\|_A = \|A^{1/2}x\|_2, \quad \forall x \in \Re^n.$$

From Lemma 4.1, for any $x \in \Re^n$, $Gx = SGx \oplus TGx$. Further,

(4.16)
$$\|Gx\|_A^2 = \|SGx\|_A^2 + \|TGx\|_A^2.$$

Assumption 1: $\|G\|_2 < 1$ and G and A are commute.

By Assumption 1,

$$\|G\|_A^2 = \max_{\|x\|_A \neq 0} \frac{\|Gx\|_A^2}{\|x\|_A^2} = \max_{\|x\|_A \neq 0} \frac{x^T GAGx}{x^T Ax} = \|G\|_2^2, \tag{4.17}$$

we have

$$\|Gx\|_A \leq \|G\|_A \|x\|_A < \|x\|_A,$$

which implies

$$\|G\|_A < 1.$$

Let operators M_1 and D be defined by

$$M_1 = G(I_H^h A_W^{-1} I_h^H A - I_H^h A_G^{-1} I_h^H A)G, \tag{4.18}$$

and

$$D = A^{1/2} I_H^h A_W^{-1} I_h^H A^{1/2} - A^{1/2} I_H^h A_G^{-1} I_h^H A^{1/2}. \tag{4.19}$$

Then

$$GDG = A^{1/2} M_1 A^{-1/2},$$

and by Lemma 3.1, the operator D defined in (4.19) is symmetric nonnegative definite. We now establish the following lemma.

Lemma 4.2 *Under Assumption 1,*

$$\|M_1\|_A = \max_{\|y\|_2 \neq 0} \frac{y^T GDGy}{y^T y}. \tag{4.20}$$

Proof. Note first that

$$\begin{aligned} \|M_1\|_A^2 &= \max_{\|x\|_A \neq 0} \frac{\|M_1 x\|_A^2}{\|x\|_A^2} = \max_{\|x\|_A \neq 0} \frac{x^T M_1^T A M_1 x}{x^T A x} \\ &= \max_{\|x\|_A \neq 0} \frac{\|A^{1/2} M_1 x\|_2^2}{\|A^{1/2} x\|_2^2} = \|A^{1/2} M_1 A^{-1/2}\|_2^2 = \|GDG\|_2^2 \end{aligned}$$

Since D is symmetric nonnegative definite (spd) and by Lemma 3.1,

$$\|M_1\|_A = \max_{\|y\|_2 \neq 0} \frac{y^T G(A^{1/2} I_H^h A_W^{-1} I_h^H A^{1/2} - A^{1/2} I_H^h A_G^{-1} I_h^H A^{1/2})Gy}{y^T y}.$$

□

Assumption 2: Set $C_G = A^{-1} - I_H^h A_G^{-1} I_h^H$. C_G satisfies the following approximate property,

$$\|C_G\|_2 = \|A^{-1} - I_H^h A_G^{-1} I_h^H\|_2 \le C_1 h^\alpha, \tag{4.21}$$

where α is positive, and C_1 is some constant independent of h.

Remark. For regular differential operator, this assumption is satisfied, [8].

Assumption 3: AG satisfies the smoothing property,

$$\|AG\|_2 = \|AG\|_A \le C_2(\gamma) h^{-\alpha},$$

where α is as in Assumption 2 and $C_2(\gamma)$ goes to 0 as γ increases.

Lemma 4.3

$$\|A^{1/2} S A^{-1/2}\|_2 \le 1.$$

Proof.

$$\begin{aligned} \|A^{1/2} S A^{-1/2}\|_2^2 &= \max_{\|x\|_2 \neq 0} \frac{x^T A^{-1/2} S^T A S A^{-1/2} x}{x^T x} \\ = \quad \max_{\|y\|_A \neq 0} \frac{y^T S^T A S y}{y^T A y} &= \max_{\|y\|_A \neq 0} \frac{\|Sy\|_A^2}{\|y\|_A^2} \le 1. \end{aligned}$$

□

Lemma 4.4

$$\|A^{1/2} S G\|_2^2 \le C_2(\gamma) h^{-\alpha}.$$

Proof. By Lemma 4.3 and Assumptions 1 to 3,

$$\begin{aligned} \|A^{1/2} S G\|_2^2 &\le \|A^{1/2} S A^{-1/2}\|_2^2 \|A^{1/2} G\|_2^2 \\ &\le \|A^{1/2} G\|_2^2 = \max_{\|x\|_2 \neq 0} \frac{x^T A G^2 x}{x^T x} = \|AG^2\|_2 \le C_2(\gamma) h^{-\alpha}. \end{aligned}$$

□

Lemma 4.5

$$\|M_1\|_A^2 \le C_1 C_2(\gamma). \tag{4.22}$$

Proof. We have

$$\begin{aligned}\|C_G\|_2 &= \max_{\|x\|_2\neq 0} \frac{|x^TA^{-1}x - x^TI_H^hA_G^{-1}I_h^Hx|}{x^Tx} \\ &= \max_{\|x\|_2\neq 0} \frac{|x^Tx - x^TA^{1/2}I_H^hA_G^{-1}I_h^HA^{1/2}x|}{x^TAx}.\end{aligned}$$

Hence, by Assumption 2, for all $x \in \Re^n$,

$$|x^Tx - x^TA^{1/2}I_H^hA_G^{-1}I_h^HA^{1/2}x| \leq C_1h^\alpha x^TAx. \tag{4.23}$$

Meanwhile, for all $s \in \Phi$, we can write $s = A^{-1/2}Rx$ for $x \in \Re^n$.

$$\begin{aligned} s^TDs &= x^TRA^{-1/2}(A^{1/2}I_H^hA_W^{-1}I_h^HA^{1/2} - A^{1/2}I_H^hA_G^{-1}I_h^HA^{1/2})A^{-1/2}Rx \\ &= x^TR(I_H^hA_W^{-1}I_h^H - I_H^hA_G^{-1}I_h^H)Rx \\ &= x^T(RA^{-1}R - I_H^hA_G^{-1}I_h^H)x \\ &= s^Ts - x^TA^{1/2}RA^{-1/2}A^{1/2}I_H^hA_G^{-1}I_h^HA^{1/2}A^{-1/2}Rx \\ &= s^Ts - s^TA^{1/2}I_H^hA_G^{-1}I_h^HA^{1/2}s.\end{aligned}$$

Combining with (4.23), we get

$$s^TDs \leq C_1h^\alpha s^TAs, \quad \forall s \in \Phi.$$

Note that for $y_1 = SGx \in \Phi$ and $y_2 = TGx \in \Psi$,

$$(Gx)^TD(Gx) = (y_1^T + y_2^T)D(y_1 + y_2) = y_1^TDy_1.$$

Thus, by Lemma 4.2,

$$\begin{aligned}\|M_1\|_A &= \max_{\|x\|_2\neq 0} \frac{x^TGDGx}{x^Tx} = \max_{\|x\|_2\neq 0} \frac{y_1^TDy_1}{x^Tx} \\ &\leq C_1h^\alpha \max_{\|x\|_2\neq 0} \frac{y_1^TAy_1}{x^Tx} \\ &= C_1h^\alpha \max_{\|x\|_2\neq 0} \frac{(SGx)^TA(SGx)}{x^Tx}. \end{aligned} \tag{4.24}$$

Together with Lemma 4.4 and Assumptions 1 to 3, (4.24) implies

$$\|M_1\|_A \leq C_1 \max_{\|x\|_2\neq 0} \frac{\|A^{1/2}SGx\|_2^2}{\|x\|_2^2}h^\alpha = C_1\|A^{1/2}SG\|_2^2h^\alpha \leq C_1C_2(\gamma). \tag{4.25}$$

□

Lemma 4.6 *Under the Assumptions 2 and 3,*

$$\|M_G\|_A \le C_1 C_2(\gamma). \tag{4.26}$$

Proof. Set $\tilde{M} = G(I - A^{1/2} I_H^h A_G^{-1} I_h^H A^{1/2})G = GQG$. Then

$$\|M_G\|_A = \|A^{1/2} M_G A^{-1/2}\|_2 = \|\tilde{M}\|_2,$$

and $\tilde{M}$ is symmetric. Furthermore

$$\begin{aligned}
\|\tilde{M}\|_2 &= \max_{\|x\|_2 \neq 0} \frac{|x^T \tilde{M} x|}{x^T x} \\
&= \max_{\|x\|_2 \neq 0} \frac{|x^T GQGx|}{x^T x} = \max_{\|x\|_2 \neq 0} \frac{|x^T G A^{1/2}(A^{-1} - I_H^h A_G^{-1} I_h^H) A^{1/2} G x|}{x^T x} \\
&= \max_{\|x\|_2 \neq 0} \frac{|x^T G A^{1/2}(A^{-1} - I_H^h A_G^{-1} I_h^H) A^{1/2} G x|}{x^T GAG x} \frac{x^T GAGx}{x^T x} \\
&\le \|A^{-1} - I_H^h A_G^{-1} I_h^H\|_2 \|A^{1/2} G\|_2^2.
\end{aligned}$$

Thus

$$\|\tilde{M}\|_2 \le C_1 h^{\alpha} C_2(\gamma) h^{-\alpha} = C_1 C_2(\gamma).$$

That is,

$$\|M_G\|_A \le C_1 C_2(\gamma).$$

$\square$

Now, let's state the convergence theorem of the two-level method with wavelet coarse grid operator.

Theorem 4.1 *Under the Assumptions 1 to 3,*

$$\|M_W\|_A \le 2 C_1 C_2(\gamma). \tag{4.27}$$

Proof. By (3.9) and (3.13)

$$M_W = G(A^{-1} - I_H^h A_G^{-1} I_h^H) AG + G(I_H^h A_G^{-1} I_h^H - I_H^h A_W^{-1} I_h^H) AG,$$

and,

$$\|M_W\|_A \le \|M_G\|_A + \|M_1\|_A.$$

The rest of the proof is trivial. $\square$

5 Comparison of Convergence Rates

In this section, we assume that the operators M_W and M_G of the two-level multigrid method under two different choices for the coarse grid operators without post-smoothing iteration G are given by

$$M_W = (I - I_H^h A_W^{-1} I_h^H A)G,$$
$$M_G = (I - I_H^h A_G^{-1} I_h^H A)G.$$

Set

$$\begin{aligned} C_W &= I - A^{1/2} I_H^h A_W^{-1} I_h^H A^{1/2}, \\ C_G &= I - A^{1/2} I_H^h A_G^{-1} I_h^H A^{1/2}. \end{aligned}$$

We are now ready to show that by choosing the coarse grid operator to be the Garlerkin variational form, the two-level multigrid method is always convergent as long as the smoothing iteration operator $\|G\|_2 < 1$. Note that the two-level multigrid method of M_G is convergent if and only if $\|M_G\|_A < 1$. Under the condition that $\|G\|_2 < 1$, to show $\|M_G\|_A < 1$ it suffices to show that $\|C_G\|_A \le 1$.

Lemma 5.1 *C_G is symmetric nonnegative and*

$$\|C_G\|_2 \le 1. \tag{5.28}$$

Proof. Decompose the space $\Re^n$ into two orthogonal subspaces as

$$\Re^n = S_1 \oplus S_2, \tag{5.29}$$

where

$$S_1 = A^{1/2} Range(I_H^h), \quad S_2 = A^{-1/2} null(I_h^H).$$

Thus for any $x \in \Re^n$, it can be decomposed into

$$x = x_1 + x_2,$$

with $x_1 = A^{1/2} I_H^h y_1 \in S_1$ for $y_1 \in \Re^n$ and $x_2 \in S_2$. Simple calculation shows

$$\begin{aligned} C_G x_1 &= 0, \\ C_G x_2 &= x_2. \end{aligned}$$

Hence,

$$\begin{aligned}(x_1)^T(C_G x_1) &= 0,\\ (x_2)^T(C_G x_1) &= 0,\\ (x_2)^T(C_G x_2) &= x_2^T x_2.\end{aligned}$$

Thus

$$x^T C_G x^T = x_2^T x_2 \geq 0.$$

This shows C_G is symmetric nonnegative. Furthermore

$$\|C_G\|_2 = \max_{\|x\|_2\neq 0} \frac{x^T C_G x}{x^T x} = \max_{\|x\|_2\neq 0} \frac{x_2^T x_2}{x_1^T x_1 + x_2^T x_2} \leq 1. \tag{5.30}$$

□

For the Wavelet operator, however, the above result is not true. To see this, decompose $\Re^n$ into two $A-$orthogonal subspaces as

$$\Re^n = S_1 \oplus S_2,$$

where

$$S_1 = A^{-1/2} Range(I_H^h), \quad S_2 = A^{-1/2} null(I_h^H).$$

Then for any $x \in \Re^n$, it can be decomposed into

$$x = x_1 + x_2,$$

where $x_1 = A^{-1/2} I_H^h y_1 \in S_1$ for $y_1 \in \Re^n$ and $x_2 \in S_2$. Simple calculation implies

$$\begin{aligned}C_W x_1 &= x_1 - A^{1/2} R A^{-1/2} x_1 = (I-P)x_1,\\ C_W x_2 &= x_2.\end{aligned}$$

Hence,

$$\begin{aligned}(C_W x_1)^T C_W x_1 &= \|(I-P)x_1\|_2^2,\\ (C_W x_2)^T C_W x_1 &= x_2^T x_1,\\ (C_W x_2)^T C_W x_2 &= x_2^T x_2.\end{aligned}$$

Moreover

$$\|C_W x\|_2^2 = \|(I-P)x_1\|_2^2 + 2x_2^T x_1 + x_2^T x_2, \tag{5.31}$$

where $P = A^{1/2}RA^{-1/2}$ is a projector which is not symmetric. The following property of P is in general not true:

$$\|(I-P)x_1\|_2^2 \leq \|x_1\|_2^2.$$

This is because decomposing $\Re^n$ as above may imply that $x^T C_W x$ is negative for some $x \in \Re^n$. Hence the wavelet operator may not satisfy an analogous property to that in Lemma 5.1.

We now show that the convergence rate of the two-level multigrid method with coarse grid operator of the Garlerkin variational form is always faster than that with the Wavelet operator.

Theorem 5.1

$$\|M_G\|_A \leq \|M_W\|_A. \tag{5.32}$$

Proof. Note first

$$A_W^{-1/2} A_G A_W^{-1/2} + A_W^{1/2} A_G^{-1} A_W^{1/2} \geq 2.$$

Hence,

$$A_W^{-1} A_G A_W^{-1} + A_G^{-1} \geq 2A_W^{-1},$$

which implies

$$A_W^{-1} A_G A_W^{-1} - A_G^{-1} \geq 2(A_W^{-1} - A_G^{-1}) \geq O.$$

This in turn implies

$$A_G A_W^{-1} A_G A_W^{-1} A_G - A_G \geq 2(A_G A_W^{-1} A_G - A_G) \geq O. \tag{5.33}$$

Decompose $\Re^n$ as (5.29) in the proof of Lemma 5.1. As before, for any $x \in \Re^n$,

$$x = x_1 + x_2,$$

with $x_1 = A^{1/2} I_H^h y_1 \in S_1$ for $y_1 \in \Re^n$ and $x_2 \in S_2$. We thus have

$$\begin{aligned} C_W x_1 &= x_1 - A^{1/2} I_H^h A_W^{-1} A_G y_1, \\ C_W x_2 &= x_2. \end{aligned}$$

Hence,

$$\begin{aligned} (C_W x_1)^T (C_W x_1) &= x_1^T x_1 - 2y_1^T A_G A_W^{-1} A_G y_1 + y_1^T A_G A_W^{-1} A_G A_W^{-1} A_G y_1, \\ (C_W x_2)^T (C_W x_1) &= 0, \\ (C_W x_2)^T (C_W x_2) &= x_2^T x_2. \end{aligned}$$

This implies

$$(5.34) \qquad (C_W x)^T (C_W x) = x^T x - 2y_1^T A_G A_W^{-1} A_G y_1 + y_1^T A_G A_W^{-1} A_G A_W^{-1} A_G y_1 \geq 0.$$

Similarly,

$$(5.35) \qquad (C_G x)^T (C_G x) = x^T x + y_1^T A_G y_1 - 2y_1^T A_G y_1 \geq 0.$$

(5.33) together with (5.34) and (5.35) implies

$$(C_W x)^T (C_W x) \geq (C_G x)^T (C_G x).$$

Thus

$$\begin{aligned} \|M_W\|_A &= \|A^{1/2} M_W A^{-1/2}\|_2 = \|C_W G\|_2 \\ \geq \|M_G\|_A &= \|A^{1/2} M_G A^{-1/2}\|_2 = \|C_G G\|_2. \end{aligned}$$

□

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