Remarks on $W^{2,p}$-Solutions of Bilateral Obstacle Problems

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Abstract

The well known theorem on $W^{2,p}$-regularity of the solution of the bilateral obstacle problem is proved as a consequence of a necessary condition for an appropriate nonsmooth variational problem. Equivalence of the bilateral obstacle problem to a certain semilinear elliptic equation is proved, as well.

1 Introduction

Let Ω be a bounded domain in R^n. Let $g \in H^1(\Omega)$ be a boundary value, and let $\psi_1, \psi_2 \in H^1(\Omega)$ be obstacles, given functions such that $\psi_1 \leq g \leq \psi_2$.

It is well known (see e.g. [5]) that the following variational inequality has the unique solution:

Find

$$z \in K = \{v \in H^1(\Omega); \psi_1 \leq v \leq \psi_2, v - g \in H_0^1(\Omega)\}, \tag{1}$$

such that

$$\int_\Omega \nabla z \cdot \nabla(v - z) \geq \langle -u, v - z\rangle \quad \forall v \in K. \tag{2}$$

In (2), $-u \in H^{-1}(\Omega)$ is given.

Moreover, it is well known that if data has more regularity (to be specified later in another context), then solution belongs to $W^{2,p}(\Omega)$ (see [5, 1, 2]).

In this paper we introduce another technique of proving $W^{2,p}$-regularity of solutions of bilateral obstacle problems. The technique is based on another variational characterization of the obstacle problem which takes full advantage of the regularity of the right-hand side, and of a variant of the maximum principle, and gives $W^{2,p}$-regularity of the solution simply, as a consequence of the necessary condition for the minimizer (Theorem 1).

This method is introduced by the author in [4] in the case of the single obstacle problem (zero-obstacle). In that case it is enough to consider the variational functional

$$F(z) = \int_\Omega \left(|\nabla z|^2 + 2uz^+\right) dx, \tag{3}$$

where $z = z^+ - z^-, z^\pm \geq 0$, and where u is a given function such that

$$u \in L^p(\Omega) \quad (2 \leq p < \infty), \tag{4}$$

on an affine set in $H^1(\Omega)$

$$A = \{z \in H^1(\Omega); z - g \in H_0^1(\Omega)\}, \tag{5}$$

where $g \in H^1(\Omega), g \geq 0$, and to consider the problem:

Find $z \in A$ such that

$$F(z) = \min_{v \in A} F(v). \tag{6}$$

The purpose of the present paper is to adapt these new ideas in the case of the bilateral obstacle problem.

There are various formulations of the obstacle problem: variational inequality, complementary problem, minimum problem (6), etc. But whatever the formulation of the problem is, in the end the solution obtained either satisfies the equation

$$\Delta z = u \tag{7}$$

or it coincides with (one or) the (other) given obstacle ψ_i, in which case

$$\Delta z = \Delta \psi_i. \tag{8}$$

So, the most natural, or the most explicit, formulation of the (bilateral) obstacle problem seems to be elliptic *equation* (39), which happens to be semilinear, with discontinuous nonlinearity. The small problem is that equation (39) by itself, unless further assumptions are made (see (60)), is not complete description of the problem, since uniqueness does not hold. One has to add condition (41), i.e., (42), to get uniqueness, i.e., to identify the solution of (39) with minimal energy. Hence, the complete characterization of the obstacle problem is: an equation (39) summarizing (7) and (8), and an inequality (41), i.e., (42), granting the uniqueness. The key result here is the uniqueness (Lemma 4).

Our approach leads naturally to such characterization of the bilateral obstacle problem. The semilinear elliptic equation obtained, can be thought of as an Euler equation for the minimum problem discussed in this paper, in a way in which the variational inequality (2) is an Euler "equation" of the corresponding minimum problem on K.

In addition, this *equation* structure of the obstacle problem has been exploited recently by the author in deriving the perturbation formula for regular free boundaries [3].

2 $W^{2,p}$-Regularity Via Nonsmooth Analysis

We assume, in addition to (4),

$$\psi_1, \psi_2 \in W^{2,p}(\Omega) \tag{9}$$

$$\left.\begin{array}{c} \psi_1 < \psi_2 \\ \psi_1 \le g \le \psi_2 \end{array}\right\} \quad \text{a.e. in } \Omega. \tag{10}$$

We shall use the notation $a \vee b = \max(a,b)$, $a \wedge b = \min(a,b)$.

Consider the variational functional

$$\begin{aligned} G(z) = \int_\Omega \frac{1}{2}|\nabla z|^2 + u(\psi_1 \vee z \wedge \psi_2) + (\Delta\psi_1)(z \wedge \psi_1) \\ +(\Delta\psi_2)(\psi_2 \vee z) \end{aligned} \tag{11}$$

for any $z \in A$. Notice that because of (10), $\psi_1 \vee z \wedge \psi_2$ is well defined. Also notice that G is locally Lipschitz continuous on A, but not differentiable.

Remark 1 *If $z \in K$ then*

$$\begin{aligned} G(z) &= \int_\Omega \frac{1}{2}|\nabla z|^2 + uz + (\Delta\psi_1)\psi_1 + (\Delta\psi_2)\psi_2 \\ &= \int_\Omega \left(\frac{1}{2}|\nabla z|^2 + uz\right) + const. \end{aligned} \tag{12}$$

Consider the problem:

Find $z \in A$ such that

$$G(z) = \min_{v \in A} G(v). \tag{13}$$

Lemma 1 *Problem (13) has a solution.*

Proof: We note only that, regarding last three terms in (11), by passing to subsequences if necessary, weak convergence in $H^1(\Omega)$ implies strong convergence in $L^2(\Omega)$, and hence convergence a.e. in Ω, so that we can pass the limit along a minimizing sequence.

Lemma 2 *Let z be a minimizer in Problem (13). Then*

$$\psi_1 \le z \le \psi_2 \tag{14}$$

a.e. in Ω, i.e., $z \in K$.

Proof: Since $z \wedge \psi_2 \in A$, we have

$$G(z) \le G(z \wedge \psi_2). \tag{15}$$

Hence, it is easy to see that

$$\int_\Omega |\nabla z|^2 + 2(\Delta\psi_2)(z \vee \psi_2) \leq \int_\Omega |\nabla(z \wedge \psi_2)|^2 + 2(\Delta\psi_2)\psi_2, \tag{16}$$

which implies, since $z \vee \psi_2 - \psi_2 \in H_0^1(\Omega)$,

$$\begin{aligned} &\int_\Omega \nabla(z - z \wedge \psi_2) \cdot \nabla(z + z \wedge \psi_2) \leq \\ &\qquad 2\int_\Omega \nabla\psi_2 \cdot \nabla(z \vee \psi_2 - \psi_2). \end{aligned} \tag{17}$$

This implies

$$\begin{aligned} &\int_\Omega |\nabla(z - z \wedge \psi_2)|^2 + 2\int_\Omega \nabla(z - z \wedge \psi_2) \cdot \nabla(z \wedge \psi_2) \leq \\ &\qquad 2\int_\Omega \nabla\psi_2 \cdot \nabla(z \vee \psi_2 - \psi_2), \end{aligned} \tag{18}$$

and hence, since $\nabla(z - z \wedge \psi_2) \cdot \nabla(z \wedge \psi_2) = \nabla\psi_2 \cdot \nabla(z \vee \psi_2 - \psi_2)$,

$$\int_\Omega |\nabla(z - z \wedge \psi_2)|^2 \leq 0, \tag{19}$$

i.e., $z = z \wedge \psi_2$, i.e., $z \leq \psi_2$. Similarly, we can prove that $z \geq \psi_1$.

Remark 2 *No explicit obstacles are introduced, but a variant of a maximum principle forces a minimizer to stay above ψ_1 and below ψ_2.*

Remark 3 *Lemma 2 will follow also from the necessary condition of minimality (more precisely, from Lemma 3).*

Corollary 1 *A minimizer in Problem (13) is unique.*

Proof: By Lemma 2, $z \in K$. But by Remark 1, G is strictly convex on K, which implies that a minimizer is unique.

Corollary 2 *Problems (13) and (2) are equivalent.*

For $D \subset R^n$ let

$$I_D(x) = \begin{cases} 1 & x \in D \\ 0 & x \notin D \end{cases} \tag{20}$$

be the characteristic function of the set D.

In the next Theorem we shall not use the knowledge obtained in Lemma 2, i.e., we shall do computations on sets $\{z < \psi_1\}$ and $\{z > \psi_2\}$, as well. That should help understanding the relationship between necessary and sufficient conditions of minimality to follow.

Theorem 1 *The necessary and sufficient condition for z to be a minimizer for Problem (13) is*

$$(21)\qquad \begin{cases} \displaystyle\int_\Omega \left(-uI_{\{\psi_1\le z<\psi_2\}} - (\Delta\psi_1)I_{\{z<\psi_1\}} - (\Delta\psi_2)I_{\{z\ge\psi_2\}}\right) h \le \\ \displaystyle\qquad\qquad \le \int_\Omega \nabla z\cdot\nabla h \le \\ \displaystyle \le \int_\Omega \left(-uI_{\{\psi_1<z\le\psi_2\}} - (\Delta\psi_1)I_{\{z\le\psi_1\}} - (\Delta\psi_2)I_{\{z>\psi_2\}}\right) h \\ \qquad\qquad\qquad\qquad \forall h\in H_0^1(\Omega), h\ge 0 \\ \qquad\qquad z-g\in H_0^1(\Omega). \end{cases}$$

Proof: We will first prove the necessity. The sufficiency will follow from the consideration in the next Section. Suppose first that $h\in C_0^\infty(\Omega)$. Since

$$(22)\qquad \frac{(\psi_1+\lambda h)\vee\psi_1-\psi_1}{\lambda} \to \left\{\begin{array}{ll} 0, & \{h<0\} \\ h, & \{h\ge 0\}\end{array}\right\} = h^+$$

as $\lambda\downarrow 0$, a.e. in Ω, we have

$$(23)\qquad \frac{(z+\lambda h)\vee\psi_1 - z\vee\psi_1}{\lambda} \to \begin{cases} h, & \{z>\psi_1\} \\ h^+, & \{z=\psi_1\} \\ 0, & \{z<\psi_1\}\end{cases}$$

as $\lambda\downarrow 0$, a.e. in Ω. Similarly,

$$(24)\qquad \frac{(z+\lambda h)\wedge\psi_2 - z\wedge\psi_2}{\lambda} \to \begin{cases} h, & \{z<\psi_2\} \\ -h^-, & \{z=\psi_2\} \\ 0, & \{z>\psi_2\}\end{cases} .$$

Hence,

$$(25)\qquad \begin{array}{l} \dfrac{((z+\lambda h)\vee\psi_1)\wedge\psi_2 - (z\vee\psi_1)\wedge\psi_2}{\lambda} \longrightarrow \\ \longrightarrow \begin{cases} h, & \{\psi_1<z<\psi_2\} \\ h^+, & \{z=\psi_1\} \\ -h^-, & \{z=\psi_2\} \\ 0, & \{z<\psi_1\}\cup\{z>\psi_2\}\end{cases}\end{array}$$

as $\lambda\downarrow 0$, a.e. in Ω. Also,

$$(26)\qquad \frac{(z+\lambda h)\wedge\psi_1 - z\wedge\psi_1}{\lambda} \to \begin{cases} 0, & \{z>\psi_1\} \\ -h^-, & \{z=\psi_1\} \\ h, & \{z<\psi_1\}\end{cases}$$

and

$$(27)\qquad \frac{(z+\lambda h)\vee\psi_2 - z\vee\psi_2}{\lambda} \to \begin{cases} 0, & \{z<\psi_2\} \\ h^+, & \{z=\psi_2\} \\ h, & \{z>\psi_2\}\end{cases}$$

as $\lambda \downarrow 0$, a.e. in Ω. Notice also, for example, that

$$\left\| \frac{((z+\lambda h) \vee \psi_1) \wedge \psi_2 - (z \vee \psi_1) \wedge \psi_2}{\lambda} \right\|_{L^2(\Omega)} \leq \|h\|_{L^2(\Omega)}, \tag{28}$$

and similar inequalities hold for other relevant quantities. Hence, if z is a minimizer, using uniform integrability, we can compute the one-sided directional derivative, and

$$\begin{aligned} 0 \leq G'(z;h) = \lim_{\lambda \downarrow 0} \frac{G(z+\lambda h) - G(z)}{\lambda} \\ = \int_\Omega \nabla z \cdot \nabla h + u \left(I_{\{\psi_1 < z < \psi_2\}} h + I_{\{z=\psi_1\}} h^+ + I_{\{z=\psi_2\}} h^- \right) \\ +(\Delta\psi_1) \left(I_{\{z=\psi_1\}}(-h^-) + I_{\{z<\psi_1\}} h \right) \\ +(\Delta\psi_2) \left(I_{\{z=\psi_2\}} h^+ + I_{\{z>\psi_2\}} h \right), \end{aligned} \tag{29}$$

for all $h \in C_0^\infty(\Omega)$. This implies, by first taking $h \geq 0$, and then also $-h$,

$$\begin{aligned} \int_\Omega \nabla z \cdot \nabla h \geq \\ \int_\Omega \left(-u I_{\{\psi_1 \leq z < \psi_2\}} - (\Delta\psi_1) I_{\{z<\psi_1\}} - (\Delta\psi_2) I_{\{z \geq \psi_2\}} \right) h, \end{aligned} \tag{30}$$

and

$$\begin{aligned} \int_\Omega \nabla z \cdot \nabla h \leq \\ \int_\Omega \left(-u I_{\{\psi_1 < z \leq \psi_2\}} - (\Delta\psi_1) I_{\{z \leq \psi_1\}} - (\Delta\psi_2) I_{\{z > \psi_2\}} \right) h, \end{aligned} \tag{31}$$

for all $h \in C_0^\infty(\Omega), h \geq 0$, from which (21) follows.

Corollary 3 *Let z be a minimizer in Problem (13). Then*

$$z \in W^{2,p}_{loc}(\Omega). \tag{32}$$

If $g \in W^{2,p}(\Omega)$, then $z \in W^{2,p}(\Omega)$, and the following apriori estimate holds

$$\|z\|_{W^{2,p}(\Omega)} \leq c \left(\|u\|_{L^p(\Omega)} + \|\Delta\psi_1\|_{L^p(\Omega)} + \|\Delta\psi_2\|_{L^p(\Omega)} + \|g\|_{W^{2,p}(\Omega)} \right). \tag{33}$$

Proof: Consider the linear functional

$$H_0^1(\Omega) \ni h \mapsto \int_\Omega \nabla z \cdot \nabla h. \tag{34}$$

By Theorem 1, this functional can be extended to $L^2(\Omega)$, i.e., there exists $f \in L^2(\Omega)$ such that

$$\int_\Omega \nabla z \cdot \nabla h = \int_\Omega f h \quad \forall h \in H_0^1(\Omega), \tag{35}$$

and then

$$\int_\Omega \left(-uI_{\{\psi_1\le z<\psi_2\}} - (\Delta\psi_1)I_{\{z<\psi_1\}} - (\Delta\psi_2)I_{\{z\ge\psi_2\}}\right) h$$
$$\le \int_\Omega fh \le$$
$$\int_\Omega \left(-uI_{\{\psi_1< z\le\psi_2\}} - (\Delta\psi_1)I_{\{z\le\psi_1\}} - (\Delta\psi_2)I_{\{z>\psi_2\}}\right) h$$
$$\forall h \in L^2(\Omega), h \ge 0 \tag{36}$$

which implies

$$-uI_{\{\psi_1\le z<\psi_2\}} - (\Delta\psi_1)I_{\{z<\psi_1\}} - (\Delta\psi_2)I_{\{z\ge\psi_2\}}$$
$$\le f \le$$
$$-uI_{\{\psi_1< z\le\psi_2\}} - (\Delta\psi_1)I_{\{z\le\psi_1\}} - (\Delta\psi_2)I_{\{z>\psi_2\}} \tag{37}$$

a.e. in Ω. From (35) and (37), Elliptic L^p-estimates imply (32) and (33).

Corollary 4 *Let z be a minimizer in Problem (13). Then*

$$-uI_{\{\psi_1\le z<\psi_2\}} - (\Delta\psi_1)I_{\{z<\psi_1\}} - (\Delta\psi_2)I_{\{z\ge\psi_2\}}$$
$$\le -\Delta z \le$$
$$-uI_{\{\psi_1< z\le\psi_2\}} - (\Delta\psi_1)I_{\{z\le\psi_1\}} - (\Delta\psi_2)I_{\{z>\psi_2\}}. \tag{38}$$

Proof: Estimate (38) follows from (35) and (37).

3 Bilateral Obstacle Problem as a Semilinear Equation

Consider following equations and inequalities:

$$\begin{array}{c} z \in W^{2,p}_{loc}(\Omega) \cap H^1(\Omega) \\ -\Delta z + uI_{\{\psi_1<z<\psi_2\}} + (\Delta\psi_1)I_{\{z\le\psi_1\}} + (\Delta\psi_2)I_{\{z\ge\psi_2\}} = 0 \quad \text{a.e. in } \Omega \\ z - g \in H^1_0(\Omega); \end{array} \tag{39}$$

$$\begin{array}{c} z \in W^{2,p}_{loc}(\Omega) \cap H^1(\Omega) \\ -\Delta z + uI_{\{\psi_1<z<\psi_2\}} + (\Delta\psi_1)I_{\{z=\psi_1\}} + (\Delta\psi_2)I_{\{z=\psi_2\}} = 0 \quad \text{a.e. in } \Omega \\ z - g \in H^1_0(\Omega); \end{array} \tag{40}$$

$$\left.\begin{array}{c} (u-\Delta\psi_1)^- I_{\{z=\psi_1\}} = 0 \\ (u-\Delta\psi_2)^+ I_{\{z=\psi_2\}} = 0 \end{array}\right\} \quad \text{a.e. in } \Omega. \tag{41}$$

Remark 4 *More explicitly, (41) can be written as*

$$\begin{array}{l} u \ge \Delta\psi_1 \quad \textit{in } \Omega\cap\{z=\psi_1\} \\ u \le \Delta\psi_2 \quad \textit{in } \Omega\cap\{z=\psi_2\}. \end{array} \tag{42}$$

We have

Theorem 2 ***(a)*** *Necessary conditions of minimality for the variational Problem (13) are (14), (39), (40), and (41).*

(b) *A function z is a solution of (39) if and only if it is a solution of (40) and (14).*

(c) *Sufficient conditions of minimality for the variational Problem (13) are (39) and (41), or alternatively (14), (40), and (41).*

In particular, Problem (39) and (41) admits a unique solution.

Proof: From (38) we can see:

on $\{z < \psi_1\}$ we have $-\Delta\psi_1 \leq -\Delta z \leq -\Delta\psi_1$, i.e.,

$$\Delta z = \Delta\psi_1 \quad \text{a.e. on } \{z < \psi_1\}; \tag{43}$$

on $\{z = \psi_1\}$ we have $-u \leq -\Delta\psi_1 \leq -\Delta\psi_1$, i.e.,

$$u \geq \Delta\psi_1, \quad \Delta z = \Delta\psi_1 \quad \text{a.e. on } \{z = \psi_1\}; \tag{44}$$

on $\{\psi_1 < z < \psi_2\}$ we have $-u \leq -\Delta z \leq -u$, i.e.,

$$\Delta z = u \quad \text{a.e. on } \{\psi_1 < z < \psi_2\}; \tag{45}$$

on $\{z = \psi_2\}$ we have $-\Delta\psi_2 \leq -\Delta\psi_2 \leq -u$, i.e.,

$$u \leq \Delta\psi_2, \quad \Delta z = \Delta\psi_2 \quad \text{a.e. on } \{z = \psi_2\}; \tag{46}$$

and on $\{z > \psi_2\}$ we have $-\Delta\psi_2 \leq -\Delta z \leq -\Delta\psi_2$, i.e.,

$$\Delta z = \Delta\psi_2 \quad \text{a.e. on } \{z > \psi_2\}. \tag{47}$$

So, we see that (39) and (41) hold.

Lemma 3 *If z is a solution of (39) then $\psi_1 \leq z \leq \psi_2$.*

Proof: If z is solution of (39) then

$$\begin{aligned} \int_\Omega \nabla z \cdot \nabla\varphi \\ + \left(uI_{\{\psi_1<z<\psi_2\}} + (\Delta\psi_1)I_{\{z\leq\psi_1\}} + (\Delta\psi_2)I_{\{z\geq\psi_2\}}\right)\varphi = 0 \\ \forall\varphi \in H_0^1(\Omega). \end{aligned} \tag{48}$$

Set $\varphi = (z-\psi_2)^+$ in (48). Since

$$\int_\Omega \nabla z \cdot \nabla(z-\psi_2)^+ = \int_\Omega |\nabla(z-\psi_2)^+|^2 + \nabla\psi_2 \cdot \nabla(z-\psi_2)^+, \tag{49}$$

and

$$\int_\Omega \Delta\psi_2 I_{\{z\geq\psi_2\}}(z-\psi_2)^+ = -\int_\Omega \nabla\psi_2\cdot\nabla(z-\psi_2)^+, \tag{50}$$

we get

$$\int_\Omega |\nabla(z-\psi_2)^+|^2 = 0 \tag{51}$$

and hence, $z\leq\psi_2$. Similarly we can deduce $z\geq\psi_1$, and Lemma follows.

This proves (40) and reproves (14). So, *(a)* holds. Also, we have proved *(b)*. To prove *(c)*, it suffices to prove the uniqueness, for example, for Problem (14), (40), (41).

Lemma 4 *Solution of Problem (14), (40), (41) is unique.*

Proof: Let z_1 and z_2 be two solutions. Then

$$\begin{aligned} -\Delta(z_1-z_2) + u\left(I_{\{\psi_1<z_1<\psi_2\}} - I_{\{\psi_1<z_2<\psi_2\}}\right)& \\ +(\Delta\psi_1)\left(I_{\{z_1=\psi_1\}} - I_{\{z_2=\psi_1\}}\right)& \\ +(\Delta\psi_2)\left(I_{\{z_1=\psi_2\}} - I_{\{z_2=\psi_2\}}\right) &= 0. \end{aligned} \tag{52}$$

We need to use some kind of monotonicity of the nonlinear term. To this end, notice

$$u = (u-\Delta\psi_1)^+ - (u-\Delta\psi_1)^- + \Delta\psi_1. \tag{53}$$

From (52) we have

$$\begin{aligned} -\Delta(z_1-z_2) + (u-\Delta\psi_1)^+\left(I_{\{\psi_1<z_1\leq\psi_2\}} - I_{\{\psi_1<z_2\leq\psi_2\}}\right)& \\ -u\left(I_{\{z_1=\psi_2\}} - I_{\{z_2=\psi_2\}}\right) + (\Delta\psi_2)\left(I_{\{z_1=\psi_2\}} - I_{\{z_2=\psi_2\}}\right)& \\ = (u-\Delta\psi_1)^-\left(I_{\{z_2=\psi_1\}} - I_{\{z_1=\psi_1\}}\right) - (\Delta\psi_1)\left(I_{\{z_2=\psi_1\}} - I_{\{z_1=\psi_1\}}\right)& \\ -(\Delta\psi_1)\left(I_{\{z_1=\psi_1\}} - I_{\{z_2=\psi_1\}}\right)&. \end{aligned} \tag{54}$$

In (54) we have used also

$$I_{\{\psi_1<z_1\leq\psi_2\}} - I_{\{\psi_1<z_2\leq\psi_2\}} = I_{\{z_2=\psi_1\}} - I_{\{z_1=\psi_1\}}. \tag{55}$$

From (54) we get

$$\begin{aligned} -\Delta(z_1-z_2) + (u-\Delta\psi_1)^+\left(I_{\{\psi_1<z_1\leq\psi_2\}} - I_{\{\psi_1<z_2\leq\psi_2\}}\right)& \\ +(\Delta\psi_2-u)^+\left(I_{\{z_1=\psi_2\}} - I_{\{z_2=\psi_2\}}\right)& \\ = (u-\Delta\psi_1)^-\left(I_{\{z_2=\psi_1\}} - I_{\{z_1=\psi_1\}}\right)& \\ +(\Delta\psi_2-u)^-\left(I_{\{z_1=\psi_2\}} - I_{\{z_2=\psi_2\}}\right)& \\ &= 0, \end{aligned} \tag{56}$$

because of (41), and since $(\Delta\psi_2 - u)^- = (u - \Delta\psi_2)^+$. Finally, notice the monotonicity (on K)

$$\int_\Omega (u - \Delta\psi_1)^+ \left(I_{\{\psi_1 < z_1 \le \psi_2\}} - I_{\{\psi_1 < z_2 \le \psi_2\}}\right)(z_1 - z_2) \ge 0, \tag{57}$$

and

$$\int_\Omega (\Delta\psi_2 - u)^+ \left(I_{\{z_1 = \psi_2\}} - I_{\{z_2 = \psi_2\}}\right)(z_1 - z_2) \ge 0. \tag{58}$$

So, multiplying (56) by $(z_1 - z_2)$ and integrating, we get

$$\int_\Omega |\nabla(z_1 - z_2)|^2 \le 0. \tag{59}$$

This completes the proof of the Lemma and of the Theorem.

Corollary 5 *If*

$$\Delta\psi_1 \le u \le \Delta\psi_2 \tag{60}$$

then a solution of (39) is unique.

Example 1 *In general, solution of (39) is not unique. Let* $\Omega = (-2\pi, 2\pi) \subset R^1, g = 0, \psi_1 = -1, \psi_2 = 1, u(x) = -\sin x$. *Then (60), obviously, does not hold. Let*

$$z_1(x) = \sin x \tag{61}$$

$$z_2(x) = \begin{cases} -1 & x \in (-\frac{\pi}{2}, \frac{3\pi}{2}) \\ \sin x & x \in \Omega \setminus (-\frac{\pi}{2}, \frac{3\pi}{2}) \end{cases} \tag{62}$$

$$z_3(x) = \begin{cases} 1 & x \in (-\frac{3\pi}{2}, \frac{\pi}{2}) \\ \sin x & x \in \Omega \setminus (-\frac{3\pi}{2}, \frac{\pi}{2}) \end{cases} . \tag{63}$$

Then all three z_1, z_2, *and* z_3 *are solutions of (39), while only* z_1 *is the solution of (39), (41).*

Example 2 *In general, (40) does not imply (39). Let* $\Omega = (-\frac{5}{4}, \frac{5}{4}) \subset R^1, g = 0, \psi_1 = -2, \psi_2(x) = x^2 - 1, u = 0$. *Then (60), obviously, holds. Let*

$$z_1(x) = \begin{cases} x^2 - 1 & x \in (-\frac{1}{2}, \frac{1}{2}) \\ |x| - \frac{5}{4} & x \in \Omega \setminus (-\frac{1}{2}, \frac{1}{2}) \end{cases} \tag{64}$$

$$z_2(x) = 0. \tag{65}$$

Then both z_1 *and* z_2 *are solutions of (40), while only* z_1 *is the solution of (39).*

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