Spatial patterns described by the
Extended Fisher-Kolmogorov (EFK) equation: Periodic solutions

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1. Introduction

In recent studies in phase transitions, for instance near a Lifshitz point [HLS,Z], and in studies of spatial and temporal pattern formation in bi-stable systems [CER,DS], the *Extended Fisher- Kolmogorov* (EFK)-equation,

$$\frac{\partial u}{\partial t} = -\gamma\frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial x^2} + u - u^3, \quad \gamma > 0, \tag{1.1}$$

has been proposed as a higher order model equation and a generalization of the classical Fisher-Kolmogorov (FK) equation. In two earlier papers [PT,PTV] we investigated the existence and qualitative properties of *Kinks* or *Domain Walls*. They are stationary solutions which represent transition layers between the two stable uniform solutions $u = -1$ and $u = +1$. Kinks were found to exist for all $\gamma > 0$ and to inherit many of the properties of the kinks of the Fisher-Kolmogorov equation ($\gamma = 0$), such as monotonicity, when $\gamma \le \frac{1}{8}$.

In the present paper we shall be interested in stationary *periodic solutions* of the EFK-equation. Thus we look for periodic solutions of the equation

$$\gamma u^{iv} = u'' + u - u^3 \quad \text{on} \quad \mathbf{R}. \tag{1.2}$$

It is readily verified that, as for the FK-equation, equation (1.2) has a constant of integration: introducing the functional

$$\mathcal{E}(u) \stackrel{\text{def}}{=} 2\gamma u'u''' - \gamma(u'')^2 - (u')^2 + \frac{1}{2}(1-u^2)^2, \tag{1.3}$$

we find that if u is a solution of (1.2), then

$$\mathcal{E}(u(x)) = \text{ constant } \stackrel{\text{def}}{=} \frac{\mu}{2}. \tag{1.4}$$

For kinks, which are solutions of (1.2) which converge to $u = 1$ as $x \to \infty$ and to $u = -1$ as $x \to -\infty$, we have $\mu = 0$, whilst for the trivial solution $u = 0$ we have $\mu = 1$. In this paper we shall investigate the existence and multiplicity of periodic solutions for different values of γ and μ.

It is well known that the FK-equation ($\gamma = 0$) has periodic solutions if and only if $0 < \mu < 1$. These solutions are all symmetric with respect to their extrema and antisymmetric with respect to their zeros. In particular, if u is a periodic solution, which we have normalized so that

$$u(0) = 0, \quad u'(0) > 0 \quad \text{and} \quad u''(0) = 0, \tag{1.5}$$

then it has the following symmetry properties:

$$u(-x) = -u(x) \tag{1.6a}$$

and if we set

$$\xi = \sup\{x > 0 : u' > 0 \text{ on } [0, x)\},$$

then

$$u(\xi - x) = u(\xi + x). \tag{1.6b}$$

Note that these symmetry properties imply that the period of the solution equals 4ξ.

Throughout this paper we shall only dicuss these *single hump* periodic solutions which have the symmetry properties (1.5) and (1.6). As we shall see in a forthcoming paper, there also exist many other periodic solutions, with multiple humps.

We shall find that, as with the FK equation, the single hump periodic solutions of the EFK equation which have the properties (1.5) and (1.6) are all strictly concave on the intervals on which they are positive:

$$u''(x) < 0 \quad \text{when} \quad u(x) > 0. \tag{1.7}$$

For the EFK-equation ($\gamma > 0$) we find that there appears a critical value of γ:

$$\gamma = \frac{1}{8}.$$

Roughly speaking, for values of γ *below* $\frac{1}{8}$, the solution set of the EFK equation is similar to that of the FK equation, whilst for values of γ *above* $\frac{1}{8}$ there are significant differences, one of the important ones being that for $\mu = 0$ there appear two new branches of periodic solutions. On one of these branches the relative maxima of the solutions are *larger* than 1, and on the other branch the relative maxima of the solutions are *smaller* than 1.

Specifically, we shall prove the following existence and nonexistence theorems for periodic solutions of the type described above.

For each $\mu \in (0, 1)$, the FK equation is known to have a unique periodic solution with maxima smaller than 1. These periodic solutions continue to exist for all $\gamma > 0$.

Theorem A. *Let $0 < \mu < 1$ and $\gamma > 0$. Then there exists a periodic solution $u(x)$ of (1.2) such that*

$$\max\{|u(x)| : x \in \mathbf{R}\} < 1.$$

For either $\mu \geq 1$ or $\mu \leq 0$, the FK equation has no periodic solutions. In the following two theorems we show how this generalizes when $\gamma > 0$.

Theorem B. *Let $\mu \geq 1$ and $\gamma > 0$. Then there exists no periodic solution.*

Theorem C. *Let $\mu = 0$.*
(a) *If $0 < \gamma \le \frac{1}{8}$, then there exist no periodic solutions.*
(b) *If $\gamma > \frac{1}{8}$, then there exists a periodic solution $u_1(x)$ such that*

$$\max\{|u_1(x)| : x \in \mathbf{R}\} < 1$$

and a periodic solution $u_2(x)$ such that

$$\max\{|u_2(x)| : x \in \mathbf{R}\} > 1.$$

Thus, here we encounter a family of periodic solutions which exceed unity. This is new, because it is well-known that the FK equation has no such periodic solutions. For the EFK equation, this is only true when $\gamma \le \frac{1}{8}$.

Theorem D. *Let $0 \le \mu \le 1$ and $0 < \gamma \le \frac{1}{8}$. Then there exists no periodic solution $u(x)$ such that*

$$\max\{|u(x)| : x \in \mathbf{R}\} > 1.$$

Thus, if $\mu = 0$, two new branches of solutions appear when γ becomes larger than $\frac{1}{8}$, one with maxima larger than 1 and one with maxima smaller than 1, and we see that the solution set becomes richer. This is consistent with [CEP], where it is suggested that complicated patterns appear when γ passes through $\frac{1}{8}$. Our next result shows that these new solutions bifurcate from the unique monotone symmetric kink at $\gamma = \frac{1}{8}$, which was found in [PT].

Theorem E. *Let $\mu = 0$ and let $\{\gamma_i\}$ be a sequence such that*

$$\gamma_i \searrow \frac{1}{8} \quad \text{as} \quad i \to \infty.$$

For each $i \ge 1$, let u_i be a periodic solution corresponding to γ_i. Then

$$u_i(x) \to U(x) \quad \text{as} \quad i \to \infty,$$

where U is the unique monotone symmetric kink corresponding to $\gamma = \frac{1}{8}$. The convergence is uniform on compact intervals.

An analogous result holds for periodic solutions when $0 < \mu < 1$ and $\gamma \to 0$. They converge to the periodic solution of the FK equation for the given value of μ.

For the global behaviour of the branches of periodic solutions we have several bounds. In particular we recall from [PTT] the upper bound:

Theorem F. *Let $\mu = 0$ and $\gamma > 0$. Then any periodic solution satisfies*

$$|u(x, \gamma)| < \sqrt{1 + \frac{2\sqrt{2\gamma}}{3\sqrt{3}}}.$$

In this paper we prove the lower bound:

Theorem G. *Let $0 \le \mu < 1$ and $\gamma > (\frac{2}{5})^4$. Then any periodic solution satisfies*

$$\max\{|u(x,\gamma)| : x \in \mathbf{R}\} > \frac{1}{50}\sqrt{\frac{1-\mu}{\log 2}}.$$

For periodic solutions u which do not exceed unity, i.e. which satisfy

$$\max\{|u(x)| : x \in \mathbf{R}\} < 1,$$

we prove an upper and a lower bound for the slope $u'(0)$ at the origin:

$$\frac{\sqrt{1-\mu}}{5} < \gamma^{1/4} u'(0,\gamma) < \{4(1-\mu)\log 2\}^{1/4} \tag{1.8}$$

for $\gamma > \frac{1}{8}$ if $\mu = 0$, and for $\gamma > (\frac{2}{5})^4$ if $0 < \mu < 1$. These bounds enable us to obtain information about the behaviour of periodic solutions for large values of γ. The description of their limiting behaviour involves the reduced problem,

$$\begin{cases} v^{iv} = v - v^3 & \text{(1.9a)} \\ v(0) = 0, \quad v''(0) = 0 & \text{(1.9b)} \\ v'(0) = \omega, \quad v'''(0) = -\dfrac{1-\mu}{4\omega}. & \text{(1.9c)} \end{cases}$$

in which ω is a positive number.

Theorem H. *Let $0 \le \mu < 1$. Suppose that $\{\gamma_i\}$ is a sequence which tends to infinity, and $\{u_i\}$ is a sequence of periodic solutions which corresponds to $\{\gamma_i\}$, such that*

$$\max\{u_i(x) : x \in \mathbf{R}\} < 1.$$

Then there exists a subsequence and a periodic solution V of the Problem (1.9) such that

$$\max\{V(s) : s \in \mathbf{R}\} < 1$$

and

$$u_i(\gamma_i^{1/4} s, \gamma_i) \to V(s) \quad \text{as} \quad i \to \infty,$$

uniformly on compact sets.

The organization of the paper is the following. The existence theorems A and C(b) will be proved by analyzing equation (1.2) with a topological shooting technique. In Section 2 we set up the problem and give some important preliminary results. In Sections 3 and 5 we then prove the existence theorems. The proofs of the nonexistence theorems B, C(a) and D are given in Section 4. Finally the qualitative properties of periodic solutions, formulated in theorems E - H are proved in Section 6.

2. Preliminaries

Our basic method in proving existence of periodic solutions will be a shooting technique and so we consider the initial value problem

$$\begin{cases} \gamma u^{iv} = u'' + u - u^3, \qquad x > 0 & (2.1\text{a}) \\ u(0) = 0,\ u'(0) = \alpha,\ u''(0) = 0,\ u'''(0) = \beta & (2.1\text{b}) \end{cases}$$

in which $\alpha > 0$. If we fix $\mu \geq 0$, then β and α are related through the energy identity (1.4) by

$$\beta = \beta(\alpha) = \frac{1}{2\gamma\alpha}\left\{\alpha^2 - \frac{1-\mu}{2}\right\}. \tag{2.2}$$

We seek a value of α such that the resulting solution $u(x, \alpha)$ has the properties

$$u'(x, \alpha) > 0 \qquad \text{for} \quad 0 \leq x < \xi \tag{2.3a}$$

$$u'(\xi, \alpha) = 0 \quad \text{and} \quad u'''(\xi, \alpha) = 0 \tag{2.3b}$$

for some finite $\xi = \xi(\alpha) > 0$. It is easily verified that a solution defined on $[0, \xi]$, which satsifies (2.1-3) can be extended to yield a periodic solution of period 4ξ. Thus, we define

$$\xi(\alpha) = \sup\{x > 0 : u'(\cdot, \alpha) > 0 \ \text{ on } \ [0, x)\}.$$

In this section we shall show that for all values of $\alpha > 0$, except those for which the corresponding solution u is a monotone kink, $\xi(\alpha)$ is finite and that u has horizontal slope at ξ. Therefore, in order to satisfy (2.3) it then remains to determine $\alpha > 0$ so that

$$u'''(\xi(\alpha), \alpha) = 0.$$

At times we shall find it convenient to adopt a different formulation for the initial value problem (2.1). Since we construct periodic solutions from strictly monotone segments, defined on $[0, \xi]$, we may introduce u as an independent variable, as was done in [PT] for the study of kinks. Denoting the inverse function of $u(x)$ by $x(u)$, we set

$$t = u \quad \text{and} \quad z(t) = (u')^2(x(t)). \tag{2.4}$$

This yields

$$z'(t) = 2u''(x) \quad \text{and} \quad z''(t) = 2\frac{u'''(x)}{u'(x)}, \tag{2.5}$$

and hence upon substitution into (1.4),

$$\begin{cases} zz'' = \dfrac{(z')^2}{4} + \dfrac{1}{\gamma}\{z - f_\mu(t)\}, \quad z > 0 & (2.6\text{a}) \\ z(0) = \alpha^2 \quad \text{and} \quad z'(0) = 0, & (2.6\text{b}) \end{cases}$$

where

$$f_\mu(t) = \frac{1}{2}\{(t^2-1)^2 - \mu\}. \tag{2.7}$$

We denote the solution by $z(t,\alpha)$ and write

$$\tau(\alpha) = \sup\{t > 0 : z(\cdot,\alpha) > 0 \ \text{ on } \ [0,t)\}.$$

Plainly

$$\tau(\alpha) = u(\xi(\alpha),\alpha).$$

Thus, for z to correspond to a periodic solution we must have

$$\sqrt{z(\tau)}\, z''(\tau) = 2u'''(\xi) = 0. \tag{2.8}$$

Lemma 2.1. *Suppose that $\mu \geq 0$ and $\gamma > 0$.*
(a) *For any $\alpha \in \mathbf{R}^+$ we have*

$$u(\xi(\alpha),\alpha) < \infty \qquad \textit{and} \qquad u'(\xi(\alpha),\alpha) = 0.$$

(b) *If $\mu > 0$, then*

$$\xi(\alpha) < \infty \quad \textit{for any} \quad \alpha \in \mathbf{R}^+.$$

(c) *If $\mu = 0$, then*

$$\xi(\alpha) < \infty \quad \begin{cases} \textit{for any} \quad \alpha \in \mathbf{R}^+ & \textit{if} \quad \gamma > \dfrac{1}{8} \\ \textit{for any} \quad \alpha \in \mathbf{R}^+ \setminus \{\alpha_0\} & \textit{if} \quad 0 < \gamma \leq \dfrac{1}{8}. \end{cases}$$

Here $\alpha_0 = U'(0)$, and U is the unique odd monotone kink found in [PT].

Proof. (a) We write (2.6a) as

$$(z^{3/4})'' = \frac{3}{4\gamma}\frac{z-f}{z^{5/4}}, \tag{2.9}$$

where we have suppressed the subscript μ from f, and we define

$$\tau_0 = \sup\{t > 0 : z' < 0 \ \text{ on } \ (0,t)\},$$

if $z' < 0$ in a right-neighbourhood of the origin and $\tau_0 = 0$ otherwise.
We distinguish two cases:

$$(i) \ \ \tau_0 = \tau \quad \text{and} \quad (ii) \ \ \tau_0 < \tau.$$

(i) In this case, $z(t) < \alpha^2$ for $0 < t < \tau$. Suppose that $\tau = \infty$. Then, since $f(t) \sim \frac{1}{2}t^4$ as $t \to \infty$, there exists a $T > 0$ such that

$$(z^{3/4})'' < -1 \quad \text{for} \quad t > T, \tag{2.10}$$

which implies that $\tau < \infty$, a contradiction.

(ii) As we saw in [PT], when $\tau_0 < \tau$, then

$$z''(\tau_0) > 0 \quad \text{and} \quad z(\tau_0) > f(\tau_0).$$

Let

$$\tau_1 = \sup\{t > \tau_0 : z' > 0 \ \text{ on } \ (\tau_0, t)\}.$$

We shall show that

$$\tau_1 < \infty \quad \text{and} \quad z'(\tau_1) = 0, \quad z''(\tau_1) < 0. \tag{2.11}$$

Suppose to the contrary that $\tau_1 = \infty$. Then, since $f(t) > 0$ for $t > \tau_+(\mu) = \sqrt{1+\sqrt{\mu}}$, it follows from equation (2.9) that

$$(z^{3/4})'' < \frac{3}{4\gamma} z^{-1/4} \quad \text{for} \quad t > \tau_+,$$

or

$$y'' < \frac{3}{4\gamma} y^{-1/3} \quad \text{for} \quad t > \tau_+, \tag{2.12}$$

where we have set $y = z^{3/4}$. If we now multiply (2.12) by $2y'$ and integrate over (τ_+, t), we find eventually that

$$y' < \frac{3}{2\sqrt{\gamma}} \sqrt{y^{2/3} + C} \quad \text{for} \quad t > \tau_+,$$

where C is a positive constant. If we set $w = y^{2/3} = z^{1/2}$ this inequality translates into

$$w' < \frac{1}{\sqrt{\gamma}} \sqrt{\frac{w+C}{w}} \quad \text{for} \quad t > \tau_+.$$

Thus, since w is increasing, w' is uniformly bounded on (τ_+, ∞), so that

$$z(t) < A(1+t)^2 \quad \text{for} \quad t > 0, \tag{2.13}$$

for some positive constant A.

Remembering that $f(t) \sim \frac{1}{2}t^4$ as $t \to \infty$, it follows from (2.13) that $z(t) - f(t) \sim -\frac{1}{2}t^4$ as $t \to \infty$ and hence, since z is increasing, there exists a constant $K > 0$ such that

$$(z^{3/4})'' < -K(1+t)^{3/2} \quad \text{for} \quad t > t_1,$$

where t_1 is some sufficiently large number. This means that z cannot keep increasing indefinitely and hence, that $\tau_1 < \infty$, a contradiction.

Plainly, $z'(\tau_1) = 0$ and it is clear from equation (2.6a) that $z''(\tau_1) < 0$.

It follows from (2.6a) and (2.11) that $z' < 0$ on (τ_1, τ). Thus, $z - f < 0$ on (τ_1, τ). From this and (2.9) we conclude that$(z^{3/4})' < 0$ and $(z^{3/4})'' < 0$ on (τ_1, τ) and it easily follows that $\tau < \infty$ and $z(\tau) = 0$. This proves Part (a).

(b) & (c). Since u is increasing and bounded on $[0,\xi)$ it follows that

$$\lim_{x\to\xi} u(x) \text{ exists } \stackrel{\text{def}}{=} u(\xi).$$

It follows from the differential equation (2.1a) for u that $\xi < \infty$ if $u(\xi) \neq 1$. If $u(\xi) = 1$ and $\xi = \infty$, then u is a monotone kink. However, such a kink can only exist if $\mu = 0$ and in addition $\gamma \leq \frac{1}{8}$ [PT]. Therefore $\xi < \infty$. This proves Parts (b) and (c) of the lemma.

3. Existence and uniqueness of periodic solutions: $0 < \mu < 1,\ \gamma > 0$

In this section we focus our attention on the parameter range $0 < \mu < 1,\ \gamma > 0$. We prove Theorem A and a uniqueness theorem for a more restricted range of values of μ, i.e. $\mu \in (0, \frac{4}{9}]$.

Theorem 3.1. *Let $\mu \in (0,1)$ and $\gamma > 0$. Then there exists a periodic solution $u(x)$ such that*

$$\max\{|u(x)| : x \in \mathbf{R}\} < 1.$$

The proof proceeds via a sequence of lemmas.

We define the shooting set

$$\mathcal{S} = \{\alpha_0 > 0 : u(\xi(\alpha),\alpha) < 1,\ u''(\xi(\alpha),\alpha) < 0 \text{ and } u'''(\xi(\alpha),\alpha) < 0 \text{ for } 0 < \alpha \leq \alpha_0\}.$$

Lemma 3.2. *We have*
(a) $\xi \in C^1(\mathcal{S})$.
(b) $\mathcal{S}$ *is an open interval.*

(c) $$u(\xi(\alpha),\alpha) < \sqrt{1-\sqrt{\mu}} \quad \text{if} \quad \alpha \in \mathcal{S}.$$

Proof. (a) Let $\alpha \in \mathcal{S}$. Then we have at $\xi = \xi(\alpha)$,

$$u'(\xi(\alpha),\alpha) = 0 \quad \text{and} \quad u''(\xi(\alpha),\alpha) < 0.$$

Hence, by the Implicit Function Theorem, $\xi \in C^1(\mathcal{S})$.

(b) Since the inequalities in the definition of $\mathcal{S}$ are strict, the assertion follows immediately from Part (a) and the continuous dependence of solutions on initial data.

Part (c) follows at once from the energy identity (1.4).

In the following lemma we show that $\mathcal{S}$ is nonempty. Define

$$\overline{\alpha} = \min\Big\{\frac{\sqrt{1-\mu}}{2}, \sqrt{\frac{3(1-\mu)}{24\gamma+7}}\Big\}.$$

Lemma 3.3. $(0,\overline{\alpha}) \subset \mathcal{S}$.

Proof. Let $\alpha \in (0,\overline{\alpha})$. Observe that $u'''(0) < 0$. As we increase x, then as long as $u''' < 0$, it follows that $u'' < 0$, $u' < \alpha$ and $u(x) < \alpha x$. Thus, as long as $u > 0$ and $u''' < 0$, it follows from equation (2.1a) that

$$u^{iv}(x) < \frac{\alpha}{\gamma}x. \tag{3.1a}$$

We integrate this inequality three times to obtain

$$u'''(x) < \beta + \frac{\alpha}{2\gamma}x^2, \tag{3.1b}$$

$$u''(x) < \beta x + \frac{\alpha}{6\gamma}x^3, \tag{3.1c}$$

$$u'(x) < \alpha + \frac{\beta}{2}x^2 + \frac{\alpha}{24\gamma}x^4. \tag{3.1d}$$

Since we assume that $\alpha < \frac{1}{2}\sqrt{1-\mu}$, we have

$$\beta < -\frac{1-\mu}{8\gamma\alpha}.$$

Hence, the right hand sides of (3.1b) and (3.1c) are negative for $0 < x \leq 1$ and, since $\alpha < 1$, it follows that $u < 1$ on $[0,1]$ as long as $u' \geq 0$.

On the other hand, because we have chosen $\alpha < \sqrt{\frac{3(1-\mu)}{24\gamma+7}}$, the right hand side of (3.1d) is negative at $x = 1$. Thus, there must exist a first zero ξ of u' on $(0,1)$, where $u < 1$, $u'' < 0$ and $u''' < 0$, so that $\alpha \in \mathcal{S}$.

Define

$$\alpha^* = \sup \mathcal{S}.$$

Lemma 3.4. *We have*

$$\alpha^* \leq \sqrt{\frac{1-\mu}{2}}.$$

Proof. It follows from (1.4) that

$$\begin{aligned} 2\gamma u'u''' &\geq (u')^2 - \frac{1}{2}\{(1-u^2)^2 - \mu\} \\ &> (u')^2 - \frac{1-\mu}{2} \quad \text{for} \quad 0 < u < 1. \end{aligned}$$

Thus, if $\alpha^2 > (1-\mu)/2$, then $u''' > 0$, $u'' > 0$ and $u' > \alpha > 0$ as long as $0 < u \leq 1$, so that u' cannot have a first zero ξ such that $u(\xi) < 1$.

Lemma 3.5. *We have*

$$u(\xi(\alpha^*),\alpha^*) < 1, \qquad u''(\xi(\alpha^*),\alpha^*) < 0 \quad \textit{and} \quad u'''(\xi(\alpha^*),\alpha^*) = 0.$$

Proof. Suppose that $u(\xi(\alpha^*),\alpha^*) \geq 1$. Then by the energy identity (1.4), $u(\xi(\alpha^*),\alpha^*) \geq \sqrt{1+\sqrt{\mu}}$ and hence $u'(x,\alpha^*) > 0$ when $0 \leq x \leq x_1$, where $x_1(\alpha^*)$ is the unique root of the equation $u(x,\alpha^*) = 1$ on $[0,\xi(\alpha^*)]$. Hence, by the continuous dependence of $u(\cdot,\alpha)$ on α on compact intervals, it follows that $u(\xi(\alpha),\alpha) > 1$ for all α in a small enough neighbourhood of α^*. Since $(0,\alpha^*) \subset \mathcal{S}$ this cannot be the case, and we conclude that

$$u(\xi(\alpha^*),\alpha^*) < 1. \tag{3.2}$$

Thus, as α increases towards α^*, the first inequality in the definition of $\mathcal{S}$ continues to hold, and we wish to prove that the second one continues to hold as well. Thus, suppose that this inequality fails first. It follows from the definition of ξ that $u''(\xi(\alpha^*),\alpha^*) \leq 0$. Hence, we suppose that

$$u''(\xi(\alpha^*),\alpha^*) = 0. \tag{3.3}$$

In what follows we shall write $\xi^* = \xi(\alpha^*)$ and $u^* = u(\xi^*,\alpha^*)$.

To show that (3.3) leads to a contradiction we proceed via a series of steps.

<u>STEP 1.</u> We show that (3.3) implies that

$$u'''(\xi^*,\alpha^*) > 0. \tag{3.4}$$

Suppose that $u'''(\xi^*,\alpha^*) < 0$. Then $u'' > 0$ and $u' < 0$ in a left-neighbourhood of ξ^*, contradicting the definition of ξ^*.

Next, suppose that $u'''(\xi^*,\alpha^*) = 0$. Then, since $u^* \in (0,1)$ by (3.2), it follows from the differential equation that $u^{iv}(\xi^*,\alpha^*) > 0$ and so $u''' < 0$, $u'' > 0$ and $u' < 0$ in a left neighbourhood of ξ^*. This means that u' has a zero on $(0,\xi^*)$ contradicting again the definition of ξ^*.

Thus, if (3.3) holds, then so does (3.4).

<u>STEP 2.</u> From (3.3) and (3.4) we deduce that ξ is continuous at α^*.

It follows from (3.3), (3.4) and the differential equation that $u''' > 0$, $u'' > 0$ and $u' > 0$ for $x > \xi^*$ until $u = 1$ which must happen at some finite value $x_0 > \xi^*$. Since $u'(\cdot,\alpha)$ has a zero for every $\alpha \in (0,\alpha^*)$, continuity implies that

$$\xi(\alpha) \to \xi(\alpha^*) \quad \text{as} \quad \alpha \to \alpha^*, \quad \alpha \in \mathcal{S} \tag{3.5}$$

<u>STEP 3.</u> The contradiction.

It follows from (3.5) that

$$u'''(\xi(\alpha),\alpha) \to u'''(\xi(\alpha^*),\alpha^*) \quad \text{as} \quad \alpha \to \alpha^*, \quad \alpha \in \mathcal{S}.$$

Because $u'''(\xi(\alpha),\alpha) < 0$ for all $\alpha \in \mathcal{S}$, this implies that

$$u'''(\xi(\alpha^*),\alpha^*) \leq 0,$$

which contradicts (3.4) and we conclude that

$$u''(\xi(\alpha^*), \alpha^*) < 0. \tag{3.6}$$

To complete the proof we suppose that

$$u'''(\xi(\alpha^*), \alpha^*) < 0.$$

Then $\alpha^* \in \mathcal{S}$ and since $\mathcal{S}$ is open, α^* cannot be the supremum of $\mathcal{S}$. Therefore

$$u'''(\xi(\alpha^*), \alpha^*) = 0$$

and the lemma is proved.

Corollary 3.6. *We have*

$$u(\xi(\alpha^*), \alpha^*) < \sqrt{1 - \sqrt{\mu}} \quad \textit{and} \quad \alpha^* < \sqrt{\frac{1-\mu}{2}}.$$

Proof. The first inequality follows as in Lemma 3.2(c) from the energy identity (1.4). However, because we know from Lemma 3.5 that $u''(\xi^*, \alpha^*) < 0$, we now obtain strict inequality.

The second inequality is proved if we can rule out equality from Lemma 3.4. Thus, suppose that $(\alpha^*)^2 = (1-\mu)/2$. Then $u^{(i)}(0) = 0$ for $i = 2, 3, 4$ and $u^{(5)}(0) > 0$. Hence, since $u^{(i)} > 0$, $i = 2, 3, 4$ in a right neighbourhood of the origin, we concude that $u' > 0$ as long as $u \leq 1$, so that $u(\cdot, \alpha^*)$ cannot yield a periodic solution such that $u(\xi^*, \alpha^*) < 1$.

Proof of Theorem 2.1. It follows from Lemma 3.5 that the solution $u(x, \alpha^*)$ of Problem (2.1) satisfies the conditions (2.3) at $\xi = \xi(\alpha^*)$, and so can be continued to yield a periodic solution with period $4\xi(\alpha^*)$.

Concerning *uniqueness* we can give the following partial result.

Lemma 3.7. *Let $\gamma > 0$ and $\frac{4}{9} \leq \mu < 1$. Then there exists a unique periodic solution u with the symmetry properties (1.5) and (1.6), such that* $\max|u| < 1$.

Proof. Suppose that there exist values $\gamma > 0$ and $\mu \in [\frac{4}{9}, 1)$ such that there exist two distinct periodic solutions u_1 and u_2 with $\max|u_i| < 1$ $(i = 1, 2)$. Let α_1 and α_2 be their respective slopes at $x = 0$. Since

$$\frac{d\beta}{d\alpha} = \frac{1}{2\gamma} + \frac{1-\mu}{4\gamma\alpha^2} > 0,$$

it follows that

$$\alpha_1 < \alpha_2 \quad \Rightarrow \quad u_1'''(0) < u_1'''(0) < 0.$$

Let $w = u_1 - u_2$. Then, by the mean value theorem,

$$\begin{cases} \gamma w^{iv} = w'' + (1 - 3\tilde{u}^2)w & \text{(3.7a)} \\ w(0) = 0, \quad w'(0) < 0, \quad w''(0) = 0 \quad \text{and} \quad w'''(0) < 0, & \text{(3.7b)} \end{cases}$$

where $\tilde{u}$ is a function whose values lie between those of u_1 and u_2. Since $\frac{4}{9} \le \mu < 1$ and $\max|u_i| < \sqrt{1-\sqrt{\mu}}$, it follows that

$$|\tilde{u}(x)| \le \frac{1}{\sqrt{3}} \quad \text{for} \quad x \in \mathbf{R}$$

and hence

$$1 - 3\tilde{u}^2(x) \ge 0 \quad \text{for} \quad x \in \mathbf{R}.$$

Thus, we conclude from (3.7a,b) that

$$w^{iv} < 0,\ w''' < 0,\ w'' < 0 \ \text{ and } \ w' < 0 \quad \text{for} \quad x \in \mathbf{R}.$$

This implies that $w(x) \to -\infty$ as $x \to \infty$. Because $|w(x)| \le |u_1(x)| + |u_2(x)| \le \frac{2}{\sqrt{3}}$ for all $x \in \mathbf{R}$, this is not possible and we have a contradiction.

We conjecture that for every $\gamma > 0$ and $0 < \mu < 1$ there exists a unique periodic solution with maximum less than one.

4. Nonexistence of periodic solutions

In this section we show that, like the FK equation, when $\mu = 1$, the EFK equation admits no stationary periodic solutions for any $\gamma > 0$. When $\mu = 0$, the situation is more delicate and we shall show that periodic solutions do not exist for $0 < \gamma \le \frac{1}{8}$. This range is optimal: in the next section we shall show that they do exist for $\gamma > \frac{1}{8}$.

We also recall that the FK equation possesses no periodic solutions which exceed unity. In this section we show that this remains true for the EFK equation for any $0 \le \mu < 1$ provided $0 < \gamma \le \frac{1}{8}$. This range is also optimal, as we shall see in the next section.

We emphasize again that by a periodic solution we shall mean one which has the symmetry properties (1.5) and (1.6).

Theorem 4.1. *Let $\mu = 1$ and $\gamma > 0$. Then there exists no periodic solution.*

Proof. It is sufficient to prove that there exists no value of $\alpha > 0$ such that the solution $z(t, \alpha)$ of (2.6) satisfies

$$(\sqrt{z}z'')\Big|_{t=\tau} = 0. \tag{2.8}$$

Let $\alpha > 0$ be arbitrary. It follows from (2.6a) that because $\mu = 1$,

$$z''(0) = \frac{1}{\gamma} > 0.$$

Hence

$$z' > 0 \ \text{ for } \ t > 0 \quad \text{as long as} \quad z > f,$$

where we have written $f = f_1$. Since $\tau < \infty$ and $f < 0$ for $0 < t < 2$, there must exist a point $t_0 > 2$ where

$$z(t_0) < f(t_0), \quad z'(t_0) = 0 \quad \text{and} \quad z''(t_0) < 0. \tag{4.1}$$

When we differentiate (2.6a) and divide by $\sqrt{z}$, we obtain

$$(\sqrt{z}z'')' = \frac{z' - f'}{\gamma\sqrt{z}}, \tag{4.2}$$

which yields upon integration over (t_0, τ):

$$(\sqrt{z}z'')\Big|_{t=\tau} = (\sqrt{z}z'')\Big|_{t=t_0} + \int_{t_0}^{\tau} \frac{z' - f'}{\gamma\sqrt{z}}\, ds.$$

It follows from (4.1) that $z' < 0$ in a right-neighbourhood of t_0. Since $f'(t) > 0$ for $t > 1$, we deduce from (4.2) that $z'' < 0$ and hence $z' < 0$ for $t \in (t_0, \tau]$. Thus, the integrand remains negative for all $t \in (t_0, \tau]$ and we conclude that

$$(\sqrt{z}z'')\Big|_{t=\tau} < 0,$$

so that (2.8) is not satisfied. Because α was arbitrary, the lemma is proved.

Lemma 4.2. *Let $0 \le \mu < 1$ and $\gamma > 0$. If z corresponds to a periodic solution, then*

(a)
$$z(0) < \frac{1-\mu}{2},$$

(b)
$$z'(t) < 0 \quad \textit{for} \quad 0 < t < \tau.$$

Proof. (a) If $z(0) > \frac{1}{2}(1-\mu)$, then $z''(0) > 0$ and we proceed as in the proof of Theorem 4.1 to show that z cannot satisfy (2.8). If $z(0) = \frac{1}{2}(1-\mu)$, then $z''(0) = 0$, $z'''(0) = 0$ and $z^{iv}(0) > 0$, so that $z > f_\mu$ near the origin and we can complete the proof again as before.

(b) By Part (a), we have $z''(0) < 0$ and hence $z' < 0$ in a right-neighbourhood of the origin. Suppose that z' has a first zero at a point $\tau_0 \in (0, \tau)$. Then

$$z(\tau_0) > 0, \quad z'(\tau_0) = 0, \quad z''(\tau_0) \ge 0$$

and hence, according to the equation (2.6a) for z,

$$z(\tau_0) \ge f_\mu(\tau_0).$$

Since $z'' > 0$ when $z > f_\mu$, this implies that there exists a point $t_0 \ge \tau_0$ where

$$z(t_0) \le f_\mu(t_0), \quad z'(t_0) = 0, \quad z''(t_0) \le 0.$$

Proceeding as in the proof of Theorem 4.1, integrating (4.2) over (t_0, τ), we find that (2.8) cannot be satisfied and z cannot correspond to a periodic solution.

Before establishing the main nonexistence theorems for $0 < \gamma \le \frac{1}{8}$, we prove two auxiliary lemmas.

Lemma 4.3. *Suppose that z corresponds to a periodic solution and that $f_\mu(\tau) \neq 0$. Then*

$$z''(\tau) = \lim_{t\to\tau} z''(t) = \frac{2}{\gamma}\Big\{1 + \frac{\sqrt{\gamma}}{2}\frac{f'_\mu(\tau)}{\sqrt{f_\mu(\tau)}}\Big\}. \tag{4.3}$$

Proof. Because u' and u''' both vanish as $x \to \xi$, or $t \to \tau$, it follows from l'Hôpital's rule that

$$z''(\tau) = 2\lim_{x\to\xi}\frac{u'''(x)}{u'(x)} = 2\lim_{x\to\xi}\frac{u^{iv}(x)}{u''(x)} = \frac{2}{\gamma}\Big(1 + \frac{u-u^3}{u''}\Big), \tag{4.4}$$

where the last term is evaluated at $x = \xi$. By the energy identity (1.4) we have

$$(u'')^2 = \frac{1}{\gamma} f_\mu(u) \quad \text{at} \quad x = \xi,$$

so that

$$u'' = -\frac{1}{\sqrt{\gamma}}\sqrt{f_\mu(u)} \quad \text{at} \quad x = \xi.$$

If we substitute this expression for u'' into (4.4), and remember that $u(\xi) = \tau$, the assertion follows.

Define

$$H = z\Big(z'' - \frac{1}{\gamma}\Big) - \frac{\mu}{2\gamma}. \tag{4.5}$$

and

$$\tau_0 = \sup\{t \in (0,\tau) : z' < 0 \ \text{ on } \ (0,t)\}.$$

Lemma 4.4. *Let $0 \le \mu < 1$ and $0 < \gamma \le \frac{1}{8}$ and let z be the solution of (2.6). Then*

$$H(t) < 0 \quad \textit{for} \quad 0 \le t < \tau^* = \min\{\tau_0, 1\}.$$

Proof. Observe that we can write H as

$$H = \frac{(z')^2}{4} - \frac{(t^2-1)^2}{2\gamma}. \tag{4.6}$$

Hence

$$H(0) = -\frac{1}{2\gamma} < 0$$

and it follows that $H < 0$ in neighbourhood of the origin. Suppose that H first vanishes at a point $t_0 \in (0, \tau^*)$. Then

$$H(t_0) = 0 \quad \text{and} \quad H'(t_0) \ge 0. \tag{4.7}$$

We deduce from the differential equation for z that

$$z'(t_0) = -\sqrt{\frac{2}{\gamma}}(1 - t_0^2). \tag{4.8}$$

For H' we obtain from (4.5) that

$$\begin{aligned} H' &= zz''' + z'\Big(z'' - \frac{1}{\gamma}\Big) \\ &= \frac{1}{2}z'z'' - \frac{1}{\gamma}f' \\ &= -(1 - t_0^2)\Big\{\frac{z''}{\sqrt{2\gamma}} - \frac{2t_0}{\gamma}\Big\} \end{aligned} \tag{4.9}$$

in view of (4.8). Since $\mu \geq 0$ and H vanishes at t_0, it follows from (4.5) that

$$z\Big(z'' - \frac{1}{\gamma}\Big) = \frac{\mu}{2\gamma} \geq 0 \quad \text{at} \quad t = t_0.$$

Hence $z''(t_0) \geq \frac{1}{\gamma}$, and we conclude from (4.9) that

$$H'(t_0) \leq -\frac{1}{\gamma}(1 - t_0^2)\Big\{\frac{1}{\sqrt{2\gamma}} - 2t_0\Big\} < 0$$

because $0 < t_0 < 1$ and $\gamma \leq \frac{1}{8}$. This contradicts (4.7) and the lemma is proved.

We are now ready to prove the main nonexistence theorems for $0 < \gamma \leq \frac{1}{8}$. From the previous section we know that there exist periodic solutions for these values of γ when $0 < \mu < 1$ and that they do not exceed $u = 1$. In the first theorem we show that such periodic solutions no longer exist when $\mu = 0$. In the second theorem we show that if $\mu \in [0, 1)$, then there exist no periodic solutions which exceed $u = 1$.

Theorem 4.5. *Let $\mu = 0$ and $0 < \gamma \leq \frac{1}{8}$. Then there exists no periodic solution $u(x)$ such that*

$$\max\{|u(x)| : x \in \mathbf{R}\} < 1.$$

Proof. Suppose, to the contrary, that there exists a periodic solution u whose maximum is less than 1. Let z correspond to u. Then $\tau < 1$, Lemma 4.2 implies that $z' < 0$ on $(0, \tau]$, and we deduce from Lemma 4.4 that $H < 0$ on $(0, \tau)$. Thus

$$z''(t) < \frac{1}{\gamma} \quad \text{for} \quad 0 < t < \tau \tag{4.10}$$

and in particular

$$z''(\tau) \leq \frac{1}{\gamma}.$$

Remembering the expression for $z''(\tau)$ from Lemma 4.3, we conclude that

$$f'(\tau) \leq -\frac{1}{\sqrt{\gamma}}\sqrt{f(\tau)},$$

and therefore

$$\tau \geq \frac{1}{\sqrt{8\gamma}}.$$

Since $\gamma \leq \frac{1}{8}$, this means that we must have $\tau \geq 1$, a contradiction.

Theorem 4.6. *Let $0 \leq \mu < 1$ and $0 < \gamma \leq \frac{1}{8}$. Then there exists no periodic solution $u(x)$ such that*

$$\max\{|u(x)| : x \in \mathbf{R}\} > 1.$$

Proof. Suppose now that there exists a periodic solution u whose maximum is greater than 1. Let z correspond to u. Then $\tau > 1$, Lemma 4.2 implies that $z' < 0$ on $(0, \tau)$, and so in particular $z'(1) < 0$.

To force a contradiction we shall show that $z'(1) > 0$. Plainly, this is the case when $\alpha > \alpha_\mu = \sqrt{(1-\mu)/2}$, and by continuity this will remain so until $z'(1) = 0$ for some $\tilde{\alpha} < \alpha_\mu$. When $z'(1) = 0$, it follows from (4.6) that

$$H(1) = 0 \quad \text{and} \quad H'(0) = 0.$$

In addition

$$H''(1) \geq \frac{1-8\gamma}{2\gamma^2},$$

where inequality holds if $\mu > 0$ and equality if $\mu = 0$. Thus

$$H''(1) > 0 \quad \text{if} \quad (\mu, \gamma) \neq (0, \frac{1}{8}),$$

in which case $H' < 0$ and $H > 0$ in a left-neighbourhood of $t = 1$. Since $H < 0$ on $(0, 1)$ by Lemma 4.4, we have a contradiction.

On the other hand, if $\mu = 0$ and $\gamma = \frac{1}{8}$, then $H''(1) = 0$ and we have to consider higher derivatives. We find that

$$H'''(1) = -\frac{12}{\gamma} < 0.$$

Therefore, in this case, $H'' > 0$, $H' < 0$ and $H > 0$ in a left-neighbourhood of $t = 1$ and by Lemma 4.4, we have once again arrived at a contradiction.

5. Existence of periodic solutions: $\mu = 0$, $\gamma > \frac{1}{8}$

In the previous section we saw that if $\mu = 0$, then there are no periodic solutions for $0 < \gamma \leq \frac{1}{8}$. In this section we shall show that there do exist periodic solutions when $\gamma > \frac{1}{8}$, both with a maximum less than one and with a maximum greater than one. The method of proof is the similar to the one used in Section 3.

Theorem 5.1. *Let $\mu = 0$ and $\gamma > \frac{1}{8}$. Then there exists a periodic solution u such that*

$$\max\{|u(x)| : x \in \mathbf{R}\} < 1.$$

As in Section 3, we define

$$\xi(\alpha) = \sup\{x > 0 : u'(\cdot, \alpha) > 0 \quad \text{on} \quad [0, x)\}.$$

and we recall from Lemma 2.1 that if $\mu = 0$, then

$$\xi(\alpha) < \infty \quad \text{and} \quad u'(\xi(\alpha), \alpha) = 0 \quad \text{for every} \quad \alpha > 0.$$

Continuing as in Section 3, we set

$$\mathcal{S} = \{\alpha_0 > 0 : u(\xi(\alpha), \alpha) < 1,\ u''(\xi(\alpha), \alpha) < 0 \text{ and } u'''(\xi(\alpha), \alpha) < 0 \text{ for } 0 < \alpha \leq \alpha_0\}.$$

Reproducing the proofs of Lemmas 3.2, 3.3 and 3.4, we establish the following properties of ξ and $\mathcal{S}$:

Lemma 5.2. *We have*
(a) $\xi \in C^1(\mathcal{S})$.
(b) *The set $\mathcal{S}$ is a nonempty, open interval of the form $(0, \alpha^*)$.*

(c)
$$\alpha^* \leq \frac{1}{\sqrt{2}}.$$

It remains to determine the properties of $u(\cdot, \alpha^*)$. This will be done in the next lemma.

Lemma 5.3. *We have*

$$u(\xi(\alpha^*), \alpha^*) < 1, \quad u''(\xi(\alpha^*), \alpha^*) < 0 \quad \textit{and} \quad u'''(\xi(\alpha^*), \alpha^*) = 0.$$

Proof. We first show that $u''(\xi^*, \alpha^*) < 0$, where we have written $\xi^* = \xi(\alpha^*)$.
From the definition of ξ we conclude that $u''(\xi^*, \alpha^*) \leq 0$. We claim that $u''(\xi^*, \alpha^*) < 0$. Thus, suppose to the contrary, that

$$u''(\xi^*, \alpha^*) = 0. \tag{5.1}$$

Then, by the energy identity (1.4),

$$u(\xi^*, \alpha^*) = 1.$$

We assert that (5.1) implies that

$$u'''(\xi^*, \alpha^*) > 0. \tag{5.2}$$

Suppose that $u'''(\xi^*, \alpha^*) < 0$. Then $u'' > 0$ and $u' < 0$ in a left-neighbourhood of ξ^*, contradicting the definition of ξ^*. On the other hand, if $u'''(\xi^*, \alpha^*) = 0$, then by uniqueness, $u(x) = 1$ for all $x \in \mathbf{R}$, a contradiction. Thus, indeed, (5.2) holds.

As in the proof of Lemma 3.5 we can now show that

$$\xi(\alpha) \to \xi(\alpha^*) \quad \text{as} \quad \alpha \to \alpha^*, \quad \alpha \in \mathcal{S},$$

which means that

$$u'''(\xi(\alpha), \alpha) \to u'''(\xi(\alpha^*), \alpha^*) \quad \text{as} \quad \alpha \to \alpha^*, \quad \alpha \in \mathcal{S}.$$

Because $u'''(\xi(\alpha), \alpha) < 0$ for all $\alpha \in \mathcal{S}$, it follows that

$$u'''(\xi(\alpha^*), \alpha^*) \leq 0, \tag{5.3}$$

which contradicts (5.2). Thus (5.1) must be false and hence

$$u''(\xi(\alpha^*), \alpha^*) < 0. \tag{5.4}$$

It follows from (5.4) and the energy identity (1.4) that

$$u^* = u(\xi(\alpha^*), \alpha^*) \neq 1$$

and that $\xi(\alpha)$ is continuous at $\alpha = \alpha^*$. Therefore, by continuous dependence on initial data, if $u^* > 1$, then $u(\xi(\alpha), \alpha) > 1$ for α in a left neighbourhood of α^*. Since $(0, \alpha^*) \subset \mathcal{S}$, the definion of $\mathcal{S}$ shows that this is impossible. Thus

$$u(\xi(\alpha^*), \alpha^*) < 1. \tag{5.5}$$

Finally, as regards u''', we must have equality in (5.3). For if

$$u'''(\xi(\alpha^*), \alpha^*) < 0$$

then continuity implies that $\alpha^* < \sup \mathcal{S}$, a contradiction.

Thus, we have shown that the solution $u(x, \alpha^*)$ of Problem (2.1) satisfies the properties (2.3) at $x = \xi(\alpha^*)$ and this yields a periodic solution of which, by (5.5), the maximum is less than 1. This completes the proof of Theorem 5.1.

In the next theorem we find periodic solutions of which the maxima exceed unity.

Theorem 5.4. *Let $\mu = 0$ and $\gamma > \frac{1}{8}$. Then there exists a periodic solution u such that*

$$\max\{|u(x)| : x \in \mathbf{R}\} > 1.$$

We now take the shooting set from those values of α for which the maximum of u *exceeds* 1. Specifically we define

$$\mathcal{T} = \{\alpha_0 > 0 : u(\xi(\alpha), \alpha) > 1, \; u''(\xi(\alpha), \alpha) < 0 \text{ and } u'''(\xi(\alpha), \alpha) < 0 \text{ for } \alpha > \alpha_0\}.$$

Lemma 5.5. *We have*

$$\left(\frac{1}{\sqrt{2}},\infty\right) \subset \mathcal{T}.$$

Proof. It follows from (1.4) that

$$2\gamma u'u''' \geq (u')^2 - \frac{1}{2}(1-u^2)^2.$$

Thus, if $\alpha^2 > \frac{1}{2}$, then $u''' > 0$, $u'' > 0$ and $u' > 0$ as long as $0 < u \leq 1$. Hence u first reaches 1 at a finite value x_1, where

$$u'(x_1) > 0, \quad u''(x_1) > 0, \quad u'''(x_1) > 0.$$

Therefore, at $x = \xi$ we have

$$u(\xi) > 1, \quad u'(\xi) = 0, \quad u''(\xi) < 0, \tag{5.6}$$

where the last inequality is strict because of the energy identity (1.4). Hence, u'' has a first zero at a point $x_2 \in (x_1, \xi)$. At his point we have

$$u(x_2) > 1, \quad u''(x_2) = 0, \quad u'''(x_2) \leq 0.$$

Since, by equation (2.1a), $u^{iv} < 0$ when both $u > 1$ and $u'' \leq 0$, it follows that

$$u'''(\xi) < 0. \tag{5.7}$$

From (5.6) and (5.7) we deduce that for any $\alpha > \frac{1}{\sqrt{2}}$,

$$u(\xi(\alpha),\alpha) > 1, \quad u''(\xi(\alpha),\alpha) < 1 \ \text{ and } \ u'''(\xi(\alpha),\alpha) < 0,$$

and so $(\frac{1}{\sqrt{2}},\infty) \subset \mathcal{T}$.

As with Lemma 5.2, we can prove

Lemma 5.6. *We have*
(a) $\xi \in C^1(\mathcal{T})$.
(b) *The set $\mathcal{T}$ is an open interval of the form (α_*, ∞).*

(c)
$$\alpha_* \geq \alpha^*,$$

where $\alpha^* = \sup \mathcal{S}$ as defined in Lemma 5.2, part (b).

In the next lemma we list again the important properties of $u(\xi(\alpha),\alpha)$ at $\alpha = \alpha_*$.

Lemma 5.7. *We have,*

$$u(\xi_*,\alpha_*) > 1, \quad u''(\xi_*,\alpha_*) < 0 \quad \textit{and} \quad u'''(\xi_*,\alpha_*) = 0,$$

where we have written $\xi_* = \xi(\alpha_*)$.

Proof. From the definition of ξ we conclude that $u''(\xi_*, \alpha_*) \le 0$. Let us first suppose that

$$u''(\xi_*, \alpha_*) = 0. \tag{5.8}$$

Then, by the energy identity (1.4)

$$u(\xi_*, \alpha_*) = 1.$$

We assert that (5.8) implies that

$$u'''(\xi_*, \alpha_*) > 0. \tag{5.9}$$

For if $u'''(\xi_*, \alpha_*) < 0$, then $u'' > 0$ and $u' < 0$ in a left-neighbourhood of ξ_*, which contradicts the definition of ξ_*. If $u'''(\xi_*, \alpha_*) = 0$, it follows from uniqueness that $u(x) = 1$ for all $x \in \mathbf{R}$, which contradicts the condition at $x = 0$. Therefore (5.9) holds (see also [PT, Lemma 3.10]).

In order to complete the proof of Lemma 5.7, we need the following lemma in which we establish continuity of ξ at α_* under the above conditions.

Lemma 5.8. *Suppose that for some $\alpha_0 > 0$ we have*

$$u(\xi(\alpha_0), \alpha_0) = 1, \quad u''(\xi(\alpha_0), \alpha_0) = 0 \;\; \textit{and} \;\; u'''(\xi(\alpha_0), \alpha_0) > 0.$$

Then

$$\xi(\alpha) \to \xi(\alpha_0) \quad \text{as} \quad \alpha \to \alpha_0.$$

Accepting Lemma 5.8 for the moment, we conclude that

$$u'''(\xi(\alpha), \alpha) \to u'''(\xi(\alpha_*), \alpha_*) \quad \text{as} \quad \alpha \to \alpha_*, \quad \alpha \in \mathcal{T}.$$

Because $u'''(\xi(\alpha), \alpha) < 0$ for all $\alpha \in \mathcal{T}$, it follows that

$$u'''(\xi(\alpha_*), \alpha_*) \le 0, \tag{5.10}$$

which contradicts (5.9) and we have shown that

$$u''(\xi(\alpha_*), \alpha_*) < 0. \tag{5.11}$$

It follows from (5.11) and the energy identity (1.4) that

$$\textit{either} \;\; u(\xi_*, \alpha_*) > 1 \quad \textit{or} \quad u(\xi_*, \alpha_*) < 1$$

and that $\xi(\alpha)$ is continuous at $\alpha = \alpha_*$. Hence, by continuous dependence on initial data, if $u(\xi_*, \alpha_*) < 1$, then $u(\xi(\alpha), \alpha) < 1$ for α in a right neighbourhood of α_*. Since $(\alpha_*, \infty) \subset \mathcal{T}$ the definition of $\mathcal{T}$ implies that this is impossible. Thus

$$u(\xi_*, \alpha_*) > 1. \tag{5.12}$$

Finally, as regards u''', we must have equality in (5.10). For if

$$u'''(\xi(\alpha_*), \alpha_*) < 0,$$

then continuity implies that $\alpha_* > \inf \mathcal{T}$, a contradiction. Therefore

$$u'''(\xi(\alpha_*), \alpha_*) = 0. \tag{5.13}$$

We conclude from (5.13) that the solution $u(x, \alpha_*)$ of Problem (2.1) satisfies the properties (2.3) at $x = \xi(\alpha_*)$ and thus yields a periodic solution of which, by (5.12), the maximum is greater than 1. This completes the proof of Lemma 5.7.

The proof of Theorem 5.4 is complete once we have proved Lemma 5.8.

Proof of Lemma 5.8. Fix $\varepsilon > 0$ and small. Then by the assumptions on $u(\cdot, \alpha_0)$ there exists a $\delta > 0$ such that

$$u(\xi_0 - \varepsilon, \alpha_0) < 1 - 2\delta, \quad u(\xi_0 + \varepsilon, \alpha_0) > 1 + 2\delta, \tag{5.14a}$$

and

$$u'(x, \alpha_0) > \delta \text{ for all } x \in [0, \xi_0 - \varepsilon]. \tag{5.14b}$$

We wish to prove that there exists a $\nu > 0$ such that if $|\alpha - \alpha_0| < \nu$, then $u'(\cdot, \alpha)$ has a zero on $(\xi_0 - \varepsilon, \xi_0 + \varepsilon)$.

By the continuous dependence on initial data it follows from (5.14) that there exists a $\nu_1 > 0$ such that

$$u(\xi_0 - \varepsilon, \alpha) < 1 - \delta, \quad u(\xi_0 + \varepsilon, \alpha) > 1 + \delta, \text{ and } u'(x, \alpha) > 0 \text{ for all } x \in [0, \xi_0 - \varepsilon] \tag{5.15}$$

if $|\alpha - \alpha_0| < \nu_1$ and so

$$\xi(\alpha) > \xi_0 - \varepsilon \text{ if } |\alpha - \alpha_0| < \nu_1. \tag{5.16}$$

To show that $\xi(\alpha) < \xi_0 + \varepsilon$ for α sufficiently close to α_0 it suffices to prove that

$$\tau(\alpha) \to \tau(\alpha_0) = 1 \quad \text{as} \quad \alpha \to \alpha_0, \tag{5.17}$$

where we recall that $\tau(\alpha) = u(\xi(\alpha), \alpha)$. For (5.17) implies that there exists a $\nu_2 > 0$ such that

$$\tau(\alpha) < 1 + \delta \quad \text{if} \quad |\alpha - \alpha_0| < \nu_2,$$

and hence, because $u' > 0$ on $(0, \xi)$, we conclude from (5.15) that

$$\xi(\alpha) < \xi_0 + \varepsilon \quad \text{if} \quad |\alpha - \alpha_0| < \nu = \min\{\nu_1, \nu_2\}. \tag{5.18}$$

Thus, (5.15) and (5.18) yield the continuity of $\xi(\alpha)$ at α_0.

Let us now turn to proving (5.17). Let $z_0(t) = z(t, \alpha_0)$ be the solution of Problem (2.6) which corresponds to $u(x, \alpha_0)$. Then

$$z_0(t) \to 0 \text{ and } \sqrt{z_0(t)} z_0''(t) \to A \qquad \text{as } t \to 1^-,$$

where by assumption $A = 2u'''(\xi(\alpha_0), \alpha_0)$ is a positive constant. It is readily shown that this implies that the function $y_0(t) = z_0^{3/4}(t)$ has the properties

$$\frac{y_0(t)}{1-t} \to B \quad \text{and} \quad y_0'(t) \to -B \quad \text{as} \quad t \to 1^-, \tag{5.19}$$

where $B = \frac{3}{2}\sqrt{A}$.

We can write the equation (2.6a) for z as

$$(z^{-1/4} z')' = \frac{z - f}{\gamma z^{5/4}},$$

so that since $f(t) \geq 0$ for all $t \geq 0$, the function $y(t) = z^{3/4}(t)$ satisfies

$$y'' \leq \frac{3}{4\gamma} y^{-1/3}. \tag{5.20}$$

Fix $\rho \in (0,1)$. Then $y_0(1-\rho) > 0$ and it follows from the continuous dependence on initial data on $[0, \rho]$ that there exists a $\vartheta_1 > 0$ such that $\tau(\alpha) > 1 - \rho$ when $|\alpha - \alpha_0| < \vartheta_1$. Since ρ may be chosen as small as we wish, we conclude that

$$\liminf_{\alpha \to a_0} \tau(\alpha) \geq 1. \tag{5.21a}$$

It remains to prove that

$$\limsup_{\alpha \to a_0} \tau(\alpha) \leq 1. \tag{5.21b}$$

Fix $\varepsilon > 0$ and $t_0 \in (0,1)$. By (5.19), it is possible to choose t_0 so close to 1 that

$$y_0'(t_0) \leq -\frac{\sqrt{3}}{2} B \quad \text{and} \quad y_0(t_0) \leq \frac{B}{8}\,\varepsilon. \tag{5.22}$$

By continuity we can find a constant $\vartheta_2 > 0$ so small that if $|\alpha - \alpha_0| < \vartheta_2$, then

$$y'(t_0) \leq -\frac{B}{2} \quad \text{and} \quad 0 < y(t_0) \leq \frac{B}{4}\varepsilon.$$

Thus, in a neighbourhood of t_0 we have $y' < 0$, so that when we multiply (5.20) by y' we obtain

$$(y'^2)' \geq \frac{9}{4\gamma}(y^{2/3})'$$

for $t > t_0$ as long as $y > 0$ and $y' < 0$. This yields upon integration over (t_0, t),

$$\begin{aligned} y'^2(t) &\geq y'^2(t_0) + \frac{9}{4\gamma}\{y^{2/3}(t) - y^{2/3}(t_0)\} \\ &\geq y'^2(t_0) - \frac{9}{4\gamma} y^{2/3}(t_0) \\ &\geq \frac{B^2}{4} - \frac{9}{4\gamma}\Big(\frac{B\varepsilon}{4}\Big)^{2/3} > \frac{B^2}{16}, \end{aligned}$$

if we choose $\varepsilon < \varepsilon_0 = \frac{1}{2}(\frac{\gamma}{3})^{3/2}B^2$. Therefore

$$y'(t) < -\frac{B}{4} \quad \text{for} \quad t_0 \le t \le \tau.$$

Thus, when $0 < \varepsilon < \varepsilon_0$, integration over (t_0, τ) yields

$$\tau \le t_0 + \frac{4}{B}y(t_0) < t_0 + \varepsilon < 1 + \varepsilon,$$

where we have used (5.22). Since ε can be chosen arbitrary small, this proves (5.21b) and the proof of Lemma 5.8 is complete.

6. Qualitative properties

In this section we prove several qualitative properties of periodic solutions. We begin with a convexity lemma and then we establish universal global bounds for periodic solutions. This is followed by an analysis of the behaviour of periodic solutions as $\gamma \to 0$ (when $0 < \mu < 1$), as $\gamma \to \frac{1}{8}$ (when $\mu = 0$), and as $\gamma \to \infty$.

We begin with a convexity property.

Lemma 6.1. *Let $u(x)$ be a periodic solution which has a single critical point between zeros and has the symmetry properties (1.5) and (1.6). Then*

$$u''(x) < 0 \quad \textit{when} \quad u(x) > 0.$$

Proof. By Lemma 4.2, if $z(t)$ is the solution of Problem (2.6) which corresponds to $u(x)$, then $z'(t) < 0$ for $0 < t < \tau$ and hence

$$u''(x) = \frac{1}{2}z'(t(x)) < 0 \quad \text{for} \quad 0 < x < \xi.$$

Since $u''(\xi) < 0$ by the energy identity, the assertion follows.

In the following lemma we recall a global bound for periodic solutions proved in [PTT], which is valid when $\mu = 0$. Let

$$M = \sqrt{1 + \frac{2\sqrt{2\gamma}}{3\sqrt{3}}}. \tag{6.1}$$

Lemma 6.2. *Let $\mu = 0$ and $\gamma > 0$, and let $u(x)$ be a periodic solution. Then for all $x \in (-\infty, \infty)$,*

$$|u(x)| < M, \qquad |u'(x)| < \frac{M^2}{\sqrt{2}},$$
$$|u''(x)| < M^3, \qquad |u'''(x)| < \frac{3}{\sqrt{2}}M^4.$$

We now turn to a discussion of the behaviour of periodic solutions for values of γ close to $\gamma = 0$, or $\gamma = \frac{1}{8}$ when $\mu = 0$, and for large values of γ.

Lemma 6.3. *Let $\{\gamma_i\}$ be a sequence such that*

$$\gamma_i \searrow \theta = \begin{cases} \frac{1}{8} & \text{when } \mu = 0 \\ 0 & \text{when } 0 < \mu < 1 \end{cases} \qquad \text{as } i \to \infty.$$

For each $i \geq 1$, let u_i be a periodic solution corresponding to γ_i. Then

$$u_i(x) \to U(x) \quad \text{as} \quad i \to \infty, \qquad \text{uniformly on compact intervals},$$

where,
(i) *if $\mu = 0$, then U is the unique monotone symmetric kink corresponding to $\gamma = \frac{1}{8}$;*
(ii) *if $0 < \mu < 1$, then U is the unique periodic solution of the FK-equation with energy μ.*

Proof. Let $\alpha_i = u_i'(0)$. Then, by Lemmas 3.3 and 3.4,

$$\sqrt{\frac{1-\mu}{4}} \leq \alpha_i \leq \sqrt{\frac{1-\mu}{2}}, \tag{6.2}$$

for i suffiently large and $\gamma_i - \theta > 0$ sufficiently small. Hence there exists a convergent subsequence, which we also denote by $\{\gamma_i\}$, such that

$$\alpha_i \to \hat{\alpha} \quad \text{as} \quad i \to \infty,$$

where $\hat{\alpha}$ satisfies (6.2).

We consider the cases $\mu = 0$ and $0 < \mu < 1$ in succession.

Case I. Let $\mu = 0$ and let $\alpha_0 = U'(0)$, where U is the kink for $\gamma = \theta = \frac{1}{8}$.

Suppose that $\hat{\alpha} > \alpha_0$. Then, by [PT, Lemma 3.6 and Theorem 3.7],

$$u(\xi(\hat{\alpha}, \frac{1}{8}), \hat{\alpha}, \frac{1}{8}) > 1 \quad \text{and} \quad u''(\xi(\hat{\alpha}, \frac{1}{8}), \hat{\alpha}, \frac{1}{8}) < 0. \tag{6.3}$$

Since by Theorem 4.6, $u(\cdot, \hat{\alpha}, \frac{1}{8})$ cannot be a periodic solution, it follows that in addition

$$u'''(\xi(\hat{\alpha}, \frac{1}{8}), \hat{\alpha}, \frac{1}{8}) \neq 0. \tag{6.4}$$

The inequality in (6.3) implies that the function $\xi(\alpha, \gamma)$ is continuous at $(\hat{\alpha}, \frac{1}{8})$ and so, by the continuous dependence of $u(\cdot, \alpha, \gamma)$ on α and γ, it follows that for i large enough,

$$u'''(\xi(\alpha_i, \gamma_i), \alpha_i, \gamma_i) \neq 0 \tag{6.5}$$

as well. However, since $u(\cdot, \alpha_i, \gamma_i)$ is a periodic solution for every i, we must have

$$u'''(\xi(\alpha_i, \gamma_i), \alpha_i, \gamma_i) = 0$$

for every i, which contradicts (6.5).

If $\hat{\alpha} < \alpha_0$, then by [PT, Lemma 3.6 and Theorem 3.7],

$$u(\xi(\hat{\alpha}, \frac{1}{8}), \hat{\alpha}, \frac{1}{8}) < 1 \quad \text{and} \quad u''(\xi(\hat{\alpha}, \frac{1}{8}), \hat{\alpha}, \frac{1}{8}) < 0.$$

It follows from Theorem 4.5 that (6.4) holds again and, as before, we arrive at a contradiction.

Thus, for the subsequence we have $\alpha_i \to \alpha_0$ as $i \to \infty$. Because the limit is uniquely determined, it follows that the entire sequence $\{\alpha_i\}$ converges to α_0. Therefore $u_i \to U$ uniformly on compact sets of the form $[0, L]$, $L > 0$.

Case II. Let $0 < \mu < 1$ and let $\alpha_\mu = U'(0)$ where U is the periodic solution of the FK-equation with energy equal to μ. From the energy identity (1.4) in which we set $\gamma = 0$, we conclude that

$$\alpha_\mu = \sqrt{\frac{1-\mu}{2}}.$$

It follows from Corollary 3.6 that

$$\hat{\alpha} \le \alpha_\mu.$$

In the remainder of the proof we shall show that

$$\hat{\alpha} \ge \alpha_\mu$$

as well. This then proves that $\alpha_i \to \alpha_\mu$ as $i \to \infty$, so that $u_i \to U$ uniformly on compact sets, as asserted.

We shall show that for each $\varepsilon > 0$ there exists a $\gamma_\varepsilon > 0$ such that if $0 < \gamma < \gamma_\varepsilon$ and $u(\cdot, \alpha, \gamma)$ is a periodic solution, then $\alpha > \alpha_\mu - \varepsilon$ and so

$$\liminf_{i \to \infty} \alpha_i \ge \alpha_\mu.$$

Remember that the initial conditions for u are

$$u(0) = 0, \quad u'(0) = \alpha, \quad u''(0) = 0, \quad u'''(0) = \beta(\alpha) = \frac{\alpha^2 - \alpha_\mu^2}{2\alpha\gamma}.$$

Because $\beta'(\alpha) > 0$ it follows that

$$\beta(\alpha) \le \beta(\alpha_\mu - \varepsilon) = -\frac{\delta(\varepsilon)}{\gamma} \quad \text{for} \quad 0 < \alpha \le \alpha_\mu - \varepsilon,$$

where $\delta(\varepsilon) \sim \varepsilon$ as $\varepsilon \to 0$.

We now proceed as in the proof of Lemma 3.3. Because $u(0) = 0$ and $u'''(0) < 0$ it follows that $u < 1$ and $u''' < 0$ in a neighbourhood of the origin. As long as these inequalities do not change it follows from the equation for u that

$$u^{iv}(x) < \frac{\alpha}{\gamma}x \tag{6.6a}$$

$$u'''(x) < -\frac{\delta}{\gamma} + \frac{\alpha}{2\gamma}x^2 \tag{6.6b}$$

$$u''(x) < -\frac{\delta}{\gamma}x + \frac{\alpha}{6\gamma}x^3 \tag{6.6c}$$

$$u'(x) < \alpha - \frac{\delta}{2\gamma}x^2 + \frac{\alpha}{24\gamma}x^4. \tag{6.6d}$$

Set

$$x_\gamma = \gamma^{1/4}.$$

Then, when $\gamma < \gamma_1 = (2\delta/\alpha_\mu)^2$, it follows from (6.6b) that $u''' < 0$ on $(0, x_\gamma]$. Moreover, the right hand side of (6.6d) will be negative at x_γ if $\gamma < \gamma_2 = \{12\delta/(25\alpha_\mu)\}^2$.

Thus, if we set $\gamma_\varepsilon = \min\{\gamma_1, \gamma_2\}$ and denote as usual the first zero of u' by ξ, then

$$\xi \in (0, x_{\gamma_\varepsilon}) \quad \text{and} \quad u'''(\xi) < 0 \quad \text{if} \quad 0 < \gamma < \gamma_\varepsilon.$$

This means that if $\alpha \le \alpha_\mu - \varepsilon$ and $\gamma \in (0, \gamma_\varepsilon)$, then $u(\cdot, \alpha, \gamma)$ cannot be a periodic solution. Therefore, if it is given that $u(\cdot, \alpha, \gamma)$ is a periodic solution, then we must conclude that $\alpha > \alpha_\mu - \varepsilon$, and the proof is complete.

To determine the behaviour of periodic solutions as $\gamma \to \infty$ we first need an upper and a lower bound for the slope at the origin. We shall confine ourselves here to the family of solutions which do not exceed 1.

Lemma 6.4. *Let $0 \le \mu < 1$ and $\gamma > 0$, and let $u(x)$ be a periodic solution such that*

$$\max\{|u(x)| : x \in \mathbf{R}\} < 1,$$

then

$$u'(0) \le \{4(1-\mu)\log 2\}^{1/4}\gamma^{-1/4} \quad \textit{for} \quad \gamma > \theta,$$

where $\theta = 0$ if $0 < \mu < 1$ and $\theta = \frac{1}{8}$ if $\mu = 0$.

Proof. It is convenient to prove this lemma using the function $z(t)$ introduced in Section 3. We then need to show that

$$z(0) \le \{4(1-\mu)\log 2\}^{1/2}\gamma^{-1/2}. \tag{6.7}$$

It follows from (2.6a) that

$$zz'' = \frac{(z')^2}{4} + \frac{1}{\gamma}\{z - f_\mu(t)\} > -\frac{1-\mu}{2\gamma} \quad \text{for} \quad 0 \le t < \sqrt{2}.$$

Because $\tau < 1$, it follows that

$$z'' > -\frac{1-\mu}{2\gamma z} \quad \text{for} \quad 0 \le t < \tau.$$

We multiply by $z' < 0$, integrate over $(0,t)$ and obtain

$$z'(t) > -\Big\{\frac{1-\mu}{\gamma}\log\frac{z(0)}{z(t)}\Big\}^{1/2} \quad \text{for} \quad 0 \le t < \tau. \tag{6.8}$$

Let

$$t_0 = \sup\{t > 0 : z > \frac{1}{2}z(0) \ \text{ on } \ [0,t)\}.$$

Then

$$0 < t_0 < 1 \quad \text{and} \quad \frac{z(0)}{z(t)} \le 2 \quad \text{for} \quad 0 \le t \le t_0.$$

Hence, by (6.8),

$$z'(t) > -\Big(\frac{(1-\mu)\log 2}{\gamma}\Big)^{1/2} \quad \text{for} \quad 0 < t < t_0$$

and we obtain after an integration over $(0,t_0)$,

$$z(t_0) - z(0) > -\Big(\frac{(1-\mu)\log 2}{\gamma}\Big)^{1/2} t_0 \tag{6.9}$$

Since $t_0 < 1$, this implies that

$$\frac{1}{2}z(0) < \Big(\frac{(1-\mu)\log 2}{\gamma}\Big)^{1/2},$$

from which (6.7) follows.

By a simple modification of the proof of Lemma 6.4, we can prove the following generalization.

Corollary 6.4a. *Let $0 \le \mu < 1$ and $\gamma > 0$, and let $u(x)$ be a periodic solution such that*

$$\max\{|u(x)| : x \in \mathbf{R}\} < \sqrt{2},$$

then

$$u'(0) \le \{8(1-\mu)\log 2\}^{1/4}\gamma^{-1/4} \quad \textit{for} \quad \gamma > \theta,$$

where $\theta = 0$ if $0 < \mu < 1$ and $\theta = \frac{1}{8}$ if $\mu = 0$.

Next we establish a lower bound for $u'(0)$.

Lemma 6.5. *We have*

$$u'(0) > \frac{\sqrt{1-\mu}}{5}\gamma^{-1/4} \quad \textit{for} \quad \gamma > \Big(\frac{2}{5}\Big)^4.$$

Proof. In light of the upper bound obtained in Lemma 6.3, we scale the variables and set

$$s = \frac{x}{\gamma^{1/4}} \quad \text{and} \quad v(s) = u(x). \tag{6.10}$$

We then obtain the problem

$$\begin{cases} v^{iv} = \dfrac{v''}{\sqrt{\gamma}} + v - v^3 & \text{(6.11a)} \\ v(0) = 0, \quad v''(0) = 0 & \text{(6.11b)} \\ v'(0) = \omega, \quad v'''(0) = \dfrac{1}{2\omega}\Big(\dfrac{\omega^2}{\sqrt{\gamma}} - \dfrac{1-\mu}{2}\Big) & \text{(6.11c)} \end{cases}$$

in which

$$\omega = \alpha\gamma^{1/4},$$

and we need to prove that

$$\omega > \frac{\sqrt{1-\mu}}{5}.$$

Suppose, to the contrary, that

$$\omega \leq \frac{\sqrt{1-\mu}}{5}.$$

Then for $\gamma > (2/5)^4$, we have

$$v'''(0) < -\frac{1-\mu}{8\omega}.$$

As long as $v > 0$ and $v''' < 0$, we have

$$v(s) < \omega s$$

and so

$$v^{iv}(s) < \omega s,$$

which yields upon integration over $(0, s)$,

$$v'''(s) < -\frac{1-\mu}{8\omega} + \frac{1}{2}\omega s^2 \tag{6.12}$$

One verifies that the right hand side of (6.12) is negative for all $s \in [0,1]$. Two more integrations yield

$$v'(s) < \omega - \frac{1-\mu}{16\omega}s^2 + \frac{1}{24}\omega s^4$$

and it follows that the first zero $\sigma = \sigma(\omega, \gamma)$ of v' has the properties

$$0 < \sigma < 1 \quad \text{and} \quad v'''(\sigma) < 0,$$

so that v and hence u cannot be a periodic solution, a contradiction.

With the lower bound we now have in hand we can return to the argument used in the proof of Lemma 6.1 to obtain a lower bound for the maximum of $|u(x)|$ on $\mathbf{R}$.

Lemma 6.6. *Let $u(x,\gamma)$ be a periodic solution. Then*

$$\max\{|u(x,\gamma)| : x \in \mathbf{R}\} > \frac{1}{50}\sqrt{\frac{1-\mu}{\log 2}} \qquad \textit{if} \quad \gamma > \Big(\frac{2}{5}\Big)^4.$$

Proof. If u is a periodic solution, then by Lemma 6.5,

$$z(0) > \frac{1-\mu}{25\sqrt{\gamma}} \quad \text{if} \quad \gamma > \Big(\frac{2}{5}\Big)^4.$$

and hence, by (6.9),

$$\Big(\frac{(1-\mu)\log 2}{\gamma}\Big)^{1/2} t_0 > \frac{1-\mu}{50\sqrt{\gamma}}.$$

This means that

$$t_0 > \frac{1}{50}\sqrt{\frac{1-\mu}{\log 2}}.$$

and hence, because

$$\max\{|u(x,\gamma)| : x \in \mathbf{R}\} = \tau(\gamma) > t_0,$$

the assertion follows.

From Lemmas 6.4 and 6.5 we conclude that if u is a periodic solution such that $\max\{|u(x)| : x \in \mathbf{R}\} < 1$, then for γ large enough,

$$\frac{\sqrt{1-\mu}}{5} < \omega < \{4(1-\mu)\log 2\}^{1/4}.$$

Let $\{\gamma_i\}$ be a sequence tending to infinity, and let u_i be a corresponding sequence of periodic solutions, with initial slopes α_i. Let v_i and ω_i be the solutions of Problem (6.11) corresponding to these periodic solutions. Then by compactness there exists a subsequence, which we also denote by $\{\omega_i\}$, which converges to a number $\omega^* < \infty$ as $i \to \infty$. Plainly, $v_i \to V$ as $i \to \infty$, where V satisfies

$$\begin{cases} V^{iv} = V - V^3 & (6.13a) \\ V(0) = 0, \quad V''(0) = 0 & (6.13b) \\ V'(0) = \omega^*, \quad V'''(0) = -\dfrac{1-\mu}{4\omega^*}. & (6.13c) \end{cases}$$

We assert that

$$\limsup_{i\to\infty} \sigma(\omega_i, \gamma_i) < \infty.$$

For if not, then there exists a subsequence along which $\sigma_i = \sigma(\omega_i, \gamma_i)$ tends to infinity and hence

$$V'(s) > 0 \quad \text{and} \quad 0 < V(s) < 1 \quad \text{for} \quad 0 \le s < \infty,$$

which is impossible. Therefore $\{\sigma_i\}$ is bounded and there exists a subsequence which converges to some $\sigma^* < \infty$ as $i \to \infty$. Since $v_i'''(\sigma_i) = 0$ for every i, it easily follows that $V'''(\sigma^*) = 0$, and so V is a periodic solution of Problem (6.13).

Thus we have shown:

Lemma 6.7. *Let $0 \le \mu < 1$. Suppose that $\{\gamma_i\}$ is a sequence which tends to infinity, and $\{u_i\}$ is a sequence of periodic solutions which corresponds to $\{\gamma_i\}$, such that*

$$\max\{u_i(x) : x \in \mathbf{R}\} < 1.$$

Then there exists a subsequence and a periodic solution V of Problem (6.13) such that

$$\max\{V(x) : s \in \mathbf{R}\} < 1.$$

and

$$u_i(\gamma_i^{1/4} s, \gamma_i) \to V(s) \quad \text{as} \quad i \to \infty, \tag{6.14}$$

uniformly on compact sets.

The above argument yields as a byproduct the existence of a periodic solution V of Problem (6.13) which does not exceed unity, for every $\mu \in [0, 1)$. This result can also be proved by means of the method used in Sections 3 and 5, and we can use the same ideas to prove the existence of a periodic solution which does exceed unity, when $\mu = 0$. Since the proofs are very close to those already presented, we omit them here. Summarizing, we have:

Theorem 6.8. *If $0 \le \mu < 1$, then Problem (6.13) has a periodic solution V such that*

$$\max\{|V(x)| : x \in \mathbf{R}\} < 1.$$

If $\mu = 0$, then Problem (6.13) has a periodic solution V such that

$$\max\{|V(x)| : x \in \mathbf{R}\} > 1.$$

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