A NOTE ON TENSOR PRODUCTS OF q-ALGEBRA REPRESENTATIONS AND ORTHOGONAL POLYNOMIALS

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ABSTRACT. We work out examples of tensor products of distinct generalized $s\ell_q(2)$ algebras with a factor from the positive discrete series of representations of one algebra and a factor from the negative discrete series of the other. We show that the equation for the common eigenfunctions of the Casimir operator and the Cartan subalgebra generator is just the three term recurrence relation corresponding to orthogonality for special cases of the Askey-Wilson polynomials, and this connection yields an almost immediate resolution of the tensor product representation into a direct integral of irreducible representations. An identity for the matrix elements of the "group representation operators" with respect to the tensor product and the reduced bases follows easily. Cases where the measures for the orthogonal polynomials are not unique correspond to cases where the tensor products and their resolutions are also nonunique.

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1. Introduction. Zhedanov and others have introduced a product of generalized $s\ell_q(2)$ algebras that allows one to take tensor products of representations corresponding to two distinct algebras, [1,2]. Here we work out examples of tensor products with a factor from the positive discrete series of representations of one algebra and a factor from the negative discrete series of the other. We show that the equation for the common eigenfunctions of the Casimir operator and the Cartan subalgebra generator is just the three term recurrence relation corresponding to orthogonality for special cases of the Askey-Wilson polynomials, and this connection yields an almost immediate resolution of the tensor product representation into a direct integral of irreducible representations. Furthermore, an identity for the matrix elements of the "group representation operators" with respect to the tensor product and the reduced bases follows immediately. Cases where the measures for the orthogonal polynomials are not unique correspond to cases where the tensor products and their resolutions are also nonunique.

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The notation used for q-series and q-integrals in this paper follows that of Gasper and Rahman [3].

2. A generalization of $s\ell_q(2)$. We consider a generalization of $s\ell_q(2)$, denoted by $[v,u]$. This is an algebra with generators H, E_+, E_- which obey the commutation relations

$$[H,E_+]=E_+,\quad [H,E_-]=-E_-,\quad [E_+,E_-]=-uq^{-H}-vq^{H}. \tag{2.1}$$

Here, u and v are real numbers and $0<q<1$. For $uv\neq 0$ this algebra is isomorphic to one of the true $s\ell_q(2)$ type algebras, for $uv=0,\ u^2+v^2>0$ it is isomorphic to a special realization of the q-oscillator algebra, and for $u=v=0$ it is isomorphic to the Euclidean Lie algebra $m(2)$, [1,4-6]. This algebra has an invariant element

$$C=E_+E_-+\frac{vq^H-uq^{1-H}}{1-q},\quad [C,A]=0,\ \forall A\in[v,u]. \tag{2.2}$$

As pointed out by Zhedanov and others [1,2], the family of algebras admits a multiplication $[v,u]\otimes[-u,w]\cong[v,w]$, defined by

$$\begin{aligned} F_+ =&\Delta(E_+)=E_+\otimes q^{\frac12 H}+q^{-\frac12 H}\otimes E_+\\ F_- =&\Delta(E_-)=E_-\otimes q^{\frac12 H}+q^{-\frac12 H}\otimes E_-\\ L =&\Delta(H)=H\otimes I+I\otimes H. \end{aligned} \tag{2.3}$$

The operators $F_\pm$, L satisfy the commutation relations (2.1). Using (2.3) we can easily define the tensor product $\rho\otimes\mu$ of a representation ρ of $[v,u]$ and the representation μ of $[-u,w]$, thereby obtaining a representation of $[v,w]$. This construction yields a convenient generalization of the tensor product computations in, for example, [5,6].

We consider the family of algebraically irreducible representations $\uparrow_\lambda$, of the algebra $[-u,w]$, where $\lambda<0$, $w<0$ and $w-u<0$, defined as follows. A convenient orthonormal basis for the representation space is $\{e_n : n=0,1,\cdots\}$ where

$$\begin{aligned} E_-e_n &= -\left[\frac{(1-q^n)(wq^{\lambda-n+1}+uq^{-\lambda})}{1-q}\right]^{\frac12} e_{n-1}\\ E_+e_n &= \left[\frac{(1-q^{n+1})(-wq^{\lambda-n}+uq^{-\lambda})}{1-q}\right]^{\frac12} e_{n+1}\\ He_n &= (-\lambda+n)e_n. \end{aligned} \tag{2.4}$$

We have $E_+=(E_-)^*$ and $H^*=H$. The invariant element $C=-(wq^{\lambda+1}+uq^{-\lambda})(1-q)^{-1}I$ for this representation, where I is the identity operator.

In analogy with a standard relationship between special functions and the representations of Lie groups we can compute the "matrix elements" of q-analogs of the group operators $e^{\beta E_+}e^{\alpha E_-}$ with respect to the $\{e_n\}$ basis, in the representation $\uparrow_\lambda$. Of course there are many q-analogs of the exponential mapping, none of which have all the properties needed to ensure that there is a true "group" associated with the q-algebra. Among the q-analogs we shall limit ourselves to the two that are most important, [3, page 9]:

$$e_q(z)=\sum_{k=0}^{\infty}\frac{z^k}{(q;q)_k},\quad E_q(z)=\sum_{k=0}^{\infty}\frac{q^{k(k-1)/2}}{(q;q)_k}z^k.$$

If z is a complex number, the first series converges to $1/(z;q)_\infty$ for $|z|<1$ and the second series converges to $(-z;q)_\infty$ for all z.

Among the 8 possibilities, [7], we consider the matrix elements

$$(E+,e-):\quad E_q(\beta E_+)e_q(\alpha E_-)e_n=\sum_{n'}T^{(\lambda)}_{n'n}(\alpha,\beta)e_{n'}.$$

(2.5)

$$\begin{aligned}&T^{(\lambda)}_{n'n}(\alpha,\beta)\\ &=\frac{(\sqrt{-w}\beta q^{-\lambda/2})^{n'-n}q^{(n'-n)(n'-3n-1)/4}}{(q;q)_{n'-n}}\left[\frac{(\frac{u}{w}q^{-2\lambda};q)_{n'}(q;q)_{n'}}{(\frac{u}{w}q^{-2\lambda};q)_n(q;q)_n(1-q)^{n'-n}}\right]^{1/2}\\ &\times\ {}_2\phi_1\left(\begin{matrix}q^{-n}, & \frac{w}{u}q^{1-n+2\lambda}\\ q^{n'-n+1} & \end{matrix};q;\frac{\alpha\beta u q^{n'-\lambda}}{1-q}\right).\end{aligned}$$

A second family of irreducible representations $\downarrow_\mu$ of $[v,u]$ where $\mu<0$, $v>0$ and $v+u>0$, is defined as follows. A convenient orthonormal basis for the representation space is $\{j_m : m=0,1,\cdots\}$ where

$$\begin{aligned}E_+j_m&=-\left[\frac{(1-q^m)(vq^{\mu-m+1}+uq^{-\mu})}{1-q}\right]^{\frac{1}{2}}j_{m-1}\\ E_-j_m&=\left[\frac{(1-q^{m+1})(vq^{\mu-m}+uq^{-\mu})}{1-q}\right]^{\frac{1}{2}}j_{m+1}\qquad(2.6)\\ Hj_m&=(\mu-m)j_m.\end{aligned}$$

We have $E_+=(E_-)^*$ and $H^*=H$. The invariant element $C=(vq^{\mu+1}-uq^{-\mu})(1-q)^{-1}I$ for this representation, where I is the identity operator.

For this representation we consider the matrix elements [7]

$$(E+,e-):\quad E_q(\beta E_+)e_q(\alpha E_-)j_n=\sum_{n'}S^{(\mu)}_{n'n}(\alpha,\beta)j_{n'}.$$

(2.7)

$$\begin{aligned}&S^{(\mu)}_{n'n}(\alpha,\beta)\\ &=\frac{(\sqrt{v}\alpha q^{\mu/2})^{n'-n}q^{-(n'-n)(n'+n-1)/4}(\frac{\alpha\beta u q^{n-\mu}}{1-q};q)_\infty}{(q;q)_{n'-n}(\frac{-\alpha\beta v q^{\mu-n'}}{1-q};q)_\infty(1-q)^{n'-n}}\left[\frac{(\frac{-u}{v}q^{-2\mu};q)_{n'}(q;q)_{n'}}{(\frac{-u}{v}q^{-2\mu};q)_n(q;q)_n}\right]^{1/2}\\ &\times\ {}_2\phi_1\left(\begin{matrix}q^{-n}, & \frac{-v}{u}q^{1-n+2\mu}\\ q^{n'-n+1} & \end{matrix};q;\frac{\alpha\beta u q^{n-\mu}}{1-q}\right).\end{aligned}$$

A third family of irreducible representations $< z, \xi >$, of the algebra $[v, w]$, where z is a complex number of absolute value one and ξ is real, is defined in terms of the orthonormal basis for the representation space $\{f_k : k = 0, \pm 1, \pm 2, \cdots\}$ and operators

$$(2.8)\qquad \begin{aligned} E_+ f_k &= (1-q)^{-\frac{1}{2}} \left(\sqrt{-w}\, q^{(\xi-k)/2} - \sqrt{v}\, q^{(-\xi+k+1)/2} z\right) f_{k+1} \\ E_- f_k &= (1-q)^{-\frac{1}{2}} \left(\sqrt{w}\, q^{(\xi-k+1)/2} + \frac{\sqrt{v}}{z} q^{(-\xi+k)/2}\right) f_{k-1} \\ H f_k &= (-\xi + k) f_k . \end{aligned}$$

We have $E_+ = (E_-)^*$, $H^* = H$ and the invariant element $C = \frac{\sqrt{-wvq}}{1-q}(z + 1/z)I$ for this representation.

Here we consider the matrix elements [7]

$$(E+, e-):\quad E_q(\beta E_+) e_q(\alpha E_-) f_k = \sum_{k'} R^{(\xi)}_{k' k}(\alpha, \beta) f_{k'} .$$

$$(2.9)\qquad \begin{aligned} &R^{(\xi)}_{k' k}(\alpha, \beta) \\ &= \frac{(-\sqrt{-w}\, q^{\xi/2})^{k-k'} q^{-(k-k')(k+k'-1)/4} \left(-\frac{q^{k'-\xi+1/2}}{z}\sqrt{-\frac{v}{w}}, q^{k-k'+1}; q\right)_\infty}{(1-q)^{(k-k')/2} \left(-\frac{q^{k-\xi+1/2}}{z}\sqrt{-\frac{v}{w}}, q; q\right)_\infty} \\ &\times\ {}_2\phi_1 \left(\begin{matrix} -z q^{-k'+\xi+1/2}\sqrt{-\frac{w}{v}}, & \frac{q^{-k'+\xi+3/2}}{z}\sqrt{-\frac{w}{v}} \\ q^{k-k'+1} & \end{matrix} ; q; \frac{\alpha\beta v q^{k'}}{1-q} \right) . \end{aligned}$$

Now we form the tensor product representation

$$(2.10)\qquad \downarrow_\mu [v, u] \otimes \uparrow_\lambda [-u, w]$$

of $[v.w]$. In this case the invariant operator is

$$C = F_+ F_- + \frac{v q^L - w q^{1-L}}{1-q} .$$

To decompose this representation we compute the common eigenfunctions of L and C. Clearly, eigenfunctions of L with eigenvalue $\alpha \geq 0$ are just those linear combinations of the basis vectors $J^\alpha_m = j_m \otimes e_n$ where $n = \alpha + m$, $m = 0, 1, \cdots$. For $\alpha < 0$, they are linear combinations of the basis vectors $J^\alpha_m = j_m \otimes e_n$ where $n = \alpha + m$, $n = 0, 1, \cdots$. Taking the case $\alpha \geq 0$ and applying C to the ON set $\{J^\alpha_m\}$ we find

$$(2.11)\qquad \begin{aligned} (1-q) C J^\alpha_m =& \sqrt{-wvq} \left[(1-q^{m+1})(1-q^{\alpha+m+1})(1+\frac{u}{v}q^{-2\mu+m})(1-\frac{u}{w}q^{-2\lambda+\alpha+m})\right]^{\frac{1}{2}} J^\alpha_{m+1} \\ &+ \sqrt{-wvq} \left[(1-q^m)(1-q^{\alpha+m})(1+\frac{u}{v}q^{-2\mu+m-1})(1-\frac{u}{w}q^{-2\lambda+\alpha+m-1})\right]^{\frac{1}{2}} J^\alpha_{m-1} \\ &+ [-w q^{\lambda-\mu-\alpha} + v q^{\mu-\lambda+\alpha+1} - v q^{\mu-\lambda+\alpha+1}(1-q^m)(1+\frac{u}{v}q^{-2\mu+m-1}) \\ &+ w q^{\lambda-\mu-\alpha}(1-q^{m+\alpha+1})(1-\frac{u}{w}q^{-2\lambda+m+\alpha})] J^\alpha_m . \end{aligned}$$

The operator C is self-adjoint. If we introduce the spectral transform of this operator so that C corresponds to multiplication by the transform variable x, then (2.11) takes the form of a three term recurrence relation for orthogonal polynomials $J^\alpha_m(x)$ of order m in x. Indeed, comparing (2.11) with the three term recurrence relation for the continuous Askey-Wilson polynomials

(2.12)
$$p_m(x) \equiv p_m(x;a,b,c,d|q) = (ab,ac,ad;q)_m a^{-m} {}_4\phi_3\left(\begin{matrix} q^{-m}, & abcdq^{m-1}, & ae^{i\theta}, & ae^{-i\theta} \\ ab, & ac, & ad & \end{matrix}; q, q\right),$$

[3, page 173], we get a match with

$$a = (-\frac{v}{w})^{\frac{1}{2}} q^{-\lambda+\mu+\alpha+1/2},\ b = (-\frac{v}{w})^{\frac{1}{2}} q^{\lambda-\mu+1/2},\ c = \frac{u}{\sqrt{-vw}} q^{-\lambda-\mu-1/2},\ d = 0,$$

and $C \sim -2\sqrt{-vwq}x/(1-q)$.

Making the identification

$$J^\alpha_m(x,t) = \left[\frac{(q^{m+1}, q^{\alpha+m+1}, \frac{u}{w}q^{\alpha-2\lambda+m}, \frac{u}{v}q^{-2\mu+m}; q)_\infty}{2\pi}\right]^{\frac{1}{2}} \frac{p_m(x)t^\alpha(-1)^m}{(\sqrt{-\frac{v}{w}}q^{-\lambda+\mu+\alpha+\frac{1}{2}}e^{i\theta};q)_\infty},$$

where $t = e^{i\phi}$ and $x = \cos\theta$, we can verify that (2.11) holds, as well as the orthogonality relations

(2.13)
$$< J^\alpha_m, J^{\alpha'}_{m'} >= \delta_{m,m'}\delta_{\alpha,\alpha'}, \quad < f,g >= \frac{1}{2\pi}\int_{-1}^{1} \rho(x)\ dx \int_0^{2\pi} d\phi\ f(x,t)\overline{g(x,t)}$$

$$\rho(x) = (1-x^2)^{-1/2} \times \frac{(e^{i\theta}, e^{-i\theta}, -e^{i\theta}, -e^{-i\theta}, q^{\frac{1}{2}}e^{i\theta}, q^{\frac{1}{2}}e^{-i\theta}, -q^{\frac{1}{2}}e^{i\theta}, -q^{\frac{1}{2}}e^{-i\theta}; q)_\infty}{(\sqrt{-\frac{w}{v}}q^{\frac{1}{2}+\lambda-\mu}e^{i\theta}, \sqrt{-\frac{w}{v}}q^{\frac{1}{2}+\lambda-\mu}e^{-i\theta}, \frac{u}{\sqrt{-vw}}q^{-\frac{1}{2}+\lambda-\mu}e^{i\theta}, \frac{u}{\sqrt{-vw}}q^{-\frac{1}{2}+\lambda-\mu}e^{-i\theta}; q)_\infty}.$$

For $\alpha = -\beta$, $\beta = 0, 1, \cdots$, the expression for $J^\alpha_m(x,t)$ can be obtained by analytic continuation and a limiting procedure:

(2.14)
$$\begin{aligned} J^{-\beta}_m(x,t) =& \frac{(-1)^{n+\beta}(\sqrt{-\frac{w}{v}}zq^{\lambda-\mu+\frac{1}{2}};q)_\beta(\frac{u}{w}q^{-2\lambda};q)_n(q^{\beta+1};q)_\infty}{z^\beta[\sqrt{-\frac{v}{w}}q^{\mu-\lambda+\frac{1}{2}}]^n(\sqrt{-\frac{v}{w}}q^{\mu-\lambda+\frac{1}{2}}z;q)_\infty} \\ &\times \left[\frac{(q^{n+1}, \frac{u}{w}q^{-2\lambda+n}, -\frac{u}{v}q^{-2\mu+n+\beta}; q)_\infty}{2\pi(q^{\beta+n+1};q)_\infty}\right]^{\frac{1}{2}} \\ &\times {}_3\phi_2\left(\begin{matrix} q^{-n}, & \sqrt{-\frac{v}{w}}q^{\mu-\lambda+\frac{1}{2}}z, & \sqrt{-\frac{v}{w}}q^{\mu-\lambda+\frac{1}{2}}z^{-1} \\ q^{\beta+1}, & \frac{u}{w}q^{-2\lambda} & \end{matrix}; q, q\right) t^{-\beta}, \end{aligned}$$

where $z=e^{i\theta}$ and $m=n+\beta$.

Furthermore, it is straightforward, though tedious, to verify that in terms of the new variables z,t the action of the operators $F_\pm, L$ is

$$F_+ = \frac{t}{\sqrt{1-q}}\left(\sqrt{-w}q^{(\lambda-\mu)/2}T_t^{-1/2} - \sqrt{v}zq^{(\mu-\lambda+1)/2}T_t^{1/2}\right),$$
$$F_- = \frac{1}{t\sqrt{1-q}}\left(-\sqrt{-w}q^{(\lambda-\mu+1)/2}T_t^{-1/2} + \frac{\sqrt{v}}{z}q^{(\mu-\lambda)/2}T_t^{1/2}\right),$$
$$L = \mu-\lambda+t\frac{\partial}{\partial t}, \quad C = \frac{\sqrt{-wvq}}{1-q}(z+\frac{1}{z}). \tag{2.15}$$

Comparing (2.8) and (2.15), we have proved the direct integral decomposition

$$\downarrow_\mu [v,u]\otimes \uparrow_\lambda [-u,w] \cong \int_0^{2\pi} \oplus < z, \lambda-\mu > \ d\theta \tag{2.16}$$

where $z=e^{i\theta}$. The functions $J^\alpha_m(x,t)$ are the Clebsch-Gordan coefficients for this decomposition; the orthogonality and completeness relations for the corresponding Askey-Wilson polynomials are the unitarity conditions for the C-G coefficients.

The decomposition (2.16) can be used to obtain an identity relating the matrix elements (2.5), (2.7) and (2.9). We can compute the matrix element

$$T_{m'n',mn}(\alpha,\beta) = < E_q(\beta F_+)e_q(\alpha F_-)j_m\otimes e_n, j_{m'}\otimes e_{n'} >$$

in two diferent ways. On one hand we have the integral representation

$$T_{m'n',mn}(\alpha,\beta) = < E_q(\beta F_+)e_q(\alpha F_-)J^s_m, J^{s'}_{m'} > = < R^{(\lambda-\mu)}_{s's}(\alpha,\beta)K^s_m, K^{s'}_{m'} >, \tag{2.17}$$

where $J^s_m(x,t) = K^s_m(x)t^s$, and $n=s+m, n'=s'+m'$. On the other hand we can use the fact that for linear operators X and Y such that $YX=qXY$, the formal identities

$$e_q(X+Y) = e_q(X)e_q(Y), \quad E_q(X+Y) = E_q(Y)E_q(X)$$

hold,[3, page 28; 5, 7], so that

$$\begin{aligned}
&E_q(\beta F_+)e_q(\alpha F_-)\\
=&E_q(\beta E_+\otimes q^{\frac{1}{2}H})E_q(\beta q^{-\frac{1}{2}H}\otimes E_+)e_q(\alpha E_-\otimes q^{\frac{1}{2}H})e_q(\alpha q^{-\frac{1}{2}H}\otimes E_-)\\
=&E_q(\beta E_+\otimes q^{\frac{1}{2}H})e_q(\alpha E_-\otimes q^{\frac{1}{2}H})E_q(\beta q^{-\frac{1}{2}H}\otimes E_+)e_q(\alpha q^{-\frac{1}{2}H}\otimes E_-).
\end{aligned}$$

Thus,

$$T_{m'n',mn}(\alpha,\beta) = T^{(\lambda)}_{n'n}(\alpha q^{(m-\mu)/2},\beta q^{(m-\mu)/2})S^{(\mu)}_{m'm}(\alpha q^{(n'-\lambda)/2},\beta q^{(n'-\lambda)/2}). \tag{2.18}$$

Note that the matrix elements in a tensor product basis actually factor. Equating (2.17) and (2.18), we have the desired identity.

3. A non-unique tensor product. The q-oscillator algebra provides a particularly interesting illustration of the ideas presented in the last section. It is the associative algebra generated by the four elements H, E_+, E_-, $\mathcal{E}$ that obey the commutation relations

$$\begin{aligned}&[H,E_+]=E_+,\quad [H,E_-]=-E_-,\\ &[E_+,E_-]=-q^{-H}\mathcal{E},\quad [\mathcal{E},E_\pm]=[\mathcal{E},H]=0.\end{aligned} \tag{3.1}$$

It admits a class of algebraically irreducible representations $\uparrow_{\ell,\lambda}$ where ℓ,λ are real numbers and $\ell>0$, [1,4,5]. These are defined on a vector space with basis $\{e_n : n=0,1,2,\cdots\}$, such that

$$\begin{aligned}&E_+e_n=\ell\sqrt{\frac{q^{-n-1}-1}{1-q}}e_{n+1},\ E_-e_n=\ell\sqrt{\frac{q^{-n}-1}{1-q}}e_{n-1}\\ &He_n=(\lambda+n)e_n,\quad \mathcal{E}e_n=\ell^2q^{\lambda-1}e_n.\end{aligned} \tag{3.2}$$

Since $\mathcal{E}$ is a constant for the representations $\uparrow_{\ell,\lambda}$, they can be considered as representations of the algebras $[0,\ell q^{\lambda-1}]$.

Similarly, the q-oscillator algebra admits a class of algebraically irreducible representations $\downarrow_{\ell,\lambda}$ where ℓ,λ are real numbers and $\ell>0$. These are defined on a vector space with basis $\{h_m : -m=0,1,2,\cdots\}$, such that

$$\begin{aligned}&E_+h_m=\ell\sqrt{\frac{q^{m}-1}{1-q}}h_{m+1},\ E_-h_m=\ell\sqrt{\frac{q^{m-1}-1}{1-q}}h_{m-1}\\ &Hh_m=(-\lambda+m)h_m,\quad \mathcal{E}h_m=-\ell^2q^{\lambda-1}h_m.\end{aligned} \tag{3.3}$$

Note that the $\downarrow_{\ell,\lambda}$ can be considered as representations of $[-\ell q^{\lambda-1},0]$.

Now we consider the tensor product representation

$$\uparrow_{\ell,\lambda}[0,\ell q^{\lambda-1}]\otimes\downarrow_{\ell,\lambda}[-\ell q^{\lambda-1},0] \tag{3.4}$$

of the Euclidean Lie algebra [0.0]. In this case the invariant operator is $C=F_+F_-$.

To decompose this representation we compute the common eigenfunctions of L and C. Clearly, eigenfunctions of L with eigenvalue $s\geq 0$ are just those linear combinations of the basis vectors $H^s_n=e_n\otimes h_m$ where $n+m=s,\ -m=0,1,\cdots$. For $s<0$, they are linear combinations of the basis vectors $H^s_n=e_n\otimes h_m$ where now $n+m=s,\ n=1,2,\cdots$. Taking the case $s<0$ and applying C to the ON set $\{H^s_n\}$ we find

$$\begin{aligned}(1-q)CH^s_n=&q^{-\lambda-n+(s-1)/2}\ell^2\left[(1-q^{-n-1})(1-q^{s-n-1})\right]^{\frac12}H^s_{n+1}\\ &+q^{-\lambda-n+(s+1)/2}\ell^2\left[(1-q^{s-n})(1-q^{-n})\right]^{\frac12}H^s_{n-1}\\ &-[q^{-\lambda+s-n}\ell^2(1-q^{-n-1})+q^{-\lambda-n}\ell^2(1-q^{s-n})]H^s_n.\end{aligned} \tag{3.5}$$

The operator C is symmetric. If we introduce a spectral transform for a self-adjoint extension of this operator so that C corresponds to multiplication by the transform variable

x, then (3.5) takes the form of a three term recurrence relation for orthogonal polynomials $H_n^s(x)$ of order n in x. Indeed, comparing (3.5) with the three term recurrence relation for the q-Laguerre polynomials

$$L_n^{(\alpha)}(x;q) = \frac{(q^{\alpha+1};q)_n}{(q;q)_n} {}_1\phi_1 \left(\begin{matrix} q^{-n} \\ q^{\alpha+1} \end{matrix} ; q, -(1-q)xq^{n+\alpha+1} \right), \tag{3.6}$$

[3, page 194], we get a match with $C = \ell^2 q^{-\lambda} x$ and the identification

$$H_n^s(x,t) = (-1)^n x^{-s/2} t^s \left[\frac{(q;q)_n q^n (1-q)^{1-s}}{(q^{1-s};q)_n (q;q)_{-s} q^{(s-s^2)/2} \log q^{-1}} \right]^{\frac{1}{2}} L_n^{(-s)}(x;q), \tag{3.8}$$

where $t = e^{i\phi}$ and $x = \cos\theta$, we can verify that (3.5) holds, as well as the orthogonality relations

(3.9)

$$< H_n^s, H_{n'}^{s'} > = \delta_{n,n'} \delta_{s,s'}, \quad < f,g > = \frac{1}{2\pi} \int_0^\infty \rho(x)\ dx \int_0^{2\pi} d\phi\ f(x,t)\overline{g(x,t)}$$

$$\rho(x) = \frac{1}{(-(1-q)x;q)_\infty}.$$

For $s \geq 0$, the expression for $H_n^s(x,t)$ can be obtained by analytic continuation and a limiting procedure:

$$H_n^s(x,t) = (-1)^M x^{s/2} t^s \left[\frac{(q;q)_M q^M (1-q)^{1+s}}{(q^{1+s};q)_M (q;q)_s q^{-(s+s^2)/2} \log q^{-1}} \right]^{\frac{1}{2}} L_M^{(s)}(x;q), \tag{3.10}$$

where $M = -m = 0, 1, \cdots$, and $n = M + s$. Furthermore, it is straightforward to verify that in terms of the new variables x, t the action of the operators $F_\pm, L$ is

$$F_+ = q^{-\lambda/2} \ell t x^{1/2}, \quad F_- = q^{-\lambda/2} \ell t^{-1} x^{1/2}, \quad L = t\frac{\partial}{\partial t}. \tag{3.11}$$

Operators (3.11) acting on the space of square integrable functions of the periodic variable ϕ define the unitary irreducible representation $(\sqrt{q^{-\lambda}x}\ell)$ of the Euclidean Lie algebra $[0,0]$, [6]. Thus, we have derived a direct integral decomposition

$$\uparrow_{\ell,\lambda} [0, \ell q^{\lambda-1}] \otimes \downarrow_{\ell,\lambda} [-\ell q^{\lambda-1}, 0] \cong \int_0^\infty \oplus (\sqrt{q^{-\lambda}x}\ell)\ \rho(x)\ dx. \tag{3.12}$$

The functions $H_n^s(x,t)$ are the Clebsch-Gordan coefficients for this decomposition. However, as is well known [3, 8], the measure (3.7) for which the q-Laguerre polynomials are orthogonal is not unique. Indeed the symmetric operator C has no unique self-adjoint extension (the deficiency indices are (1,1)). Thus, there is a multiplicity of possible self-adjoint extensions for C and each such extension defines a different tensor product (3.4).

For each of these cases the $H_n^s(x,t)$ can be thought of as the Clebsch-Gordan coefficients, with the proviso that the coefficients satisfy orthogonality but not completeness relations.

A well-known example of alternate orthogonality relations for the q-Laguerre polynomials is

(3.13a)

$$H_n^s(x,t) = (-1)^n x^{-s/2} t^s \left[\frac{(q;q)_n q^n (-\frac{q}{c(1-q)}, -c(1-q);q)_\infty}{(q^{1-s};q)_n c^{-s} (q, -cq^{1-s}(1-q), \frac{-1}{cq^{-s}(1-q)};q)_\infty} \right]^{\frac{1}{2}} L_n^{(-s)}(x;q),$$

where $s<0$, and

(3.13b)

$$H_n^s(x,t) = (-1)^M x^{s/2} t^s \left[\frac{(q;q)_M q^M (-\frac{q}{c(1-q)}, -c(1-q);q)_\infty}{(q^{1+s};q)_M c^{s} (q, -cq^{1+s}(1-q), \frac{-1}{cq^{s}(1-q)};q)_\infty} \right]^{\frac{1}{2}} L_n^{(s)}(x;q),$$

for $s \geq 0$ and $M = -m = 0, 1, \cdots, n = M + s$. Here, $c > 0$ The orthogonality relations hold with the inner product defined by

$$< f, g >' = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \rho(cq^k) q^k \int_0^{2\pi} d\phi \ f(cq^k, t) \overline{g(cq^k, t)}$$

$$\rho(x) = \frac{1}{(-(1-q)x;q)_\infty}.$$

Thus, we have the direct sum decompositions

$$\uparrow_{\ell,\lambda} [0, \ell q^{\lambda-1}] \otimes \downarrow_{\ell,\lambda} [-\ell q^{\lambda-1}, 0] \cong \sum_{k=-\infty}^{\infty} \oplus (\sqrt{q^{-\lambda} c q^k} \ell) \ . \tag{3.14}$$

These decompositions can be used just as in §2 to obtain identities relating the matrix elements of the operator $E_q(\beta F_+) e_q(\alpha F_-)$ with respect to the tensor product basis and the reduced basis. The identities express products of q-Laguerre polynomials as integrals or sums over Hahn-Exton q-Bessel functions, with expansion coefficients that are themselves products of q-Laguerre polynomials.

4. Final remarks. Closely related results have been derived in the following papers. In [1] the Clebsch-Gordan coefficients for the tensor product of representations in the positive discrete series is calculated for the case $[v,u] \otimes [-u,w]$. The results are expressed in terms of q-Hahn polynomials. In this case the eigenspace of L with eigenvalue α is finite dimensional, so one is considering finite discrete Askey-Wilson polynomials.

In [5] we considered the tensor product representation

$$(\omega)[0,0] \otimes \uparrow_{\ell,\lambda} [0, \ell q^{\lambda-1}]$$

of the q-oscillator algebra $[0.\ell q^{\lambda-1}]$ and worked out the related matrix element identity. The direct sum decomposition corresponded to the orthogonality relations for the polynomials

$$\begin{aligned} H_n(x) =& {}_2\phi_1 \left(\begin{matrix} q^{-n}, & \frac{\ell^2}{x} \\ 0 & \end{matrix} ; q, \frac{-xq}{\omega^2(1-q)} \right) \\ =& q^n {}_2\phi_1 \left(\begin{matrix} q^{-n}, & \frac{-\omega^2(1-q)}{x} \\ 0 & \end{matrix} ; q, \frac{xq}{\ell^2} \right). \end{aligned}$$

These polynomials are orthogonal with respect to a measure with support at the points

$$x = q^m \ell^2, \; -(1-q)q^m\omega^2, \quad m = 0, 1, 2, \cdots .$$

In [6] the authors considered the tensor product representation

$$(\omega)[0,0] \otimes (\omega')[0,0]$$

of the Euclidean Lie algebra. In this case the three-term recurrence relation for the common eigenfunctions of L and $C = F_+F_-$ is unbounded above and below. Thus the solutions do not correspond to polynomials. Again in this case the tensor product decompositon is not unique. One of the associated identities for the matrix elements generalizes Koelink's addition formula for Hahn-Exton q-Bessel functions, [9].

In [10] we considered tensor products of various discrete analogs of the Euclidean and oscillator algebras.

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