Convergence analysis of the semi-implicit Euler method for abstract evolution equations

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Abstract

The *semi-implicit* Euler discretization method is studied for abstract evolution equations in a Hilbert space $\mathcal{H}$, like $\dot{u} = f(t,u,u)$, $t \in (0,T]$, $u(0) = u_0$, where $f(t,\cdot,v)$ is one-sided Lipschitz and $\mathcal{R}(I - hf(t,\cdot,v)) = \mathcal{H}$ for $h > 0$ sufficiently small, and $f(t,u,\cdot)$ is Lipschitz-continuous. Extension to Banach spaces is then pointed out. *Ordinary* and *partial*, differential and integro-differential *equations* or *systems* are included. For instance, $\dot{u} = A(t,u) + B(t,u)$, where $A(t,\cdot)$ is [strongly] dissipative and maximal, and $B(t,\cdot)$ is Lipschitz-continuous, fall into the previous class. The scheme is $u_{n+1} = u_n + \Delta t\, f((n+1)\Delta t, u_{n+1}, u_n)$, $n = 0, 1, ..., N-1$, where $\Delta t := T/N$. Two main computational advantages with respect to *fully implicit* methods are: (a) *linearization* of semilinear problems, and (b) *decoupling* of systems into lower-dimensional (stationary) subsystems, at each time step. In the latter case, *parallelization* becomes possible. A full error analysis is performed: *consistency* and *stability* are estabilished, and precise *convergence estimates* are obtained. Several applications, including reaction-diffusion and hyperbolic systems, are finally given.

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1 Introduction.

In this paper we are concerned with the numerical treatment of abstract *nonlinear* evolution problems of a rather general class that *includes* as a special case

$$\dot{u} = A(t,u) + B(t,u)\ , \quad 0 < t \leq T\ ; \quad u(0) = u_0\ . \tag{1}$$

Here $A(t,\cdot) : (D_t \subseteq \mathcal{H}) \to \mathcal{H}$, $t > 0$, denotes a [strongly] dissipative and maximal operator [33], in general nonlinear, $\mathcal{H}$ being a real or complex Hilbert space, and $B(t,\cdot) : (D \subseteq \mathcal{H}) \to \mathcal{H}$ is a uniformly Lipschitz continuous operator, with Lipschitz constant $K > 0$, $\bigcup_{t>0} D_t \subseteq D \subseteq \mathcal{H}$. The method consists in reducing the numerical solution of (1) to the solution of a sequence of stationary problems, arising from a semi-implicit time discretization of (1), as

$$u_{n+1} = u_n + hA((n+1)h, u_{n+1}) + hB((n+1)h, u_n)\ , \quad n = 0, 1, ..., N-1\ , \quad h := T/N\ , \tag{2}$$

$h > 0$ denoting the time discretization step. Such problems are linear when $A(t,\cdot) \equiv A(t)\cdot$ is a linear operator: This is the case, for instance, when

$$A(t,\cdot) = \sum_{i,j} a_{ij}(x,t)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_i b_i(x,t)\frac{\partial}{\partial x_i} + c(x,t)$$

is a linear elliptic operator, i.e. when (1) is a semilinear partial differential equation of parabolic type. Existence, uniqueness and regularity of solutions to equations like (1) have been studied by several authors (see [20, 23], e.g.). Treating by the *semi-implicit* Euler method (2) an equation like (1) may be advantageous, in that it leads to *linearization*, in case of semilinear equations; to *decoupling* into lower-dimensional stationary subsystems in case that (1) represents a system of ordinary [28, 29] or partial differential equations; or to equations characterized by *local* rather than nonlocal (e.g. integral) operators. The latter is typical of linear partial integro-differential equations (where integration is over the space domain), in which case the method leads, at each time step after space-discretization, to sparse or band-structured, instead of full, systems. When (2) decouples into subsystems, the ensuing computational advantage is clear, as *parallel* implementation becomes feasible (see [28, 29], e.g., for systems of ordinary differential equations). These observations show the importance of adopting the semi-implicit Euler method, while only the fully implicit method seems to have been widely used so far (see [19, 22, 24, 26] and references therein), at least in the infinite-dimensional case. Other semi-implicit methods for special evolutionary PDE problems do appear in the literature. Concerning nonlinear parabolic equations, one should mention [30], where, however, the equation was *first* discretized *in space* by a Galerkin finite element scheme, thus obtaining a *linear* system of ODEs. In another

direction, semi-implicit finite difference discretizations (both in space and in time) have been used to solve shallow water equations, see [6].

In this paper, we are actually able to study stability and consistency (and thus convergence, cf. [7]) of certain semi-implicit schemes for (abstract) evolution problems in Hilbert space, like

$$\dot{u} = f(t, u, u) \ , \quad 0 < t \leq T \ ; \quad u(0) = u_0 \ , \tag{3}$$

where f has the (uniform) Lipschitz property with respect to the scalar product (one-sided Lipschitz condition) in the second argument, and is (uniformly) Lipschitz-continuous in the third. Equation (1) above is a special case of (3). As the consistency analysis is based on some regularity conditions, throughout the paper we assume, besides existence and uniqueness of a strong solution u to (3), that $u \in C^1[0,T]$, or $\dot{u} \in BV[0,T]$ (and $u \in AC[0,T]$).

The procedure we follow to analyze the implicit Euler method for (3) is based on the general theory of convergence for numerical methods developed by B. Chartres and R. Stepleman [7, 8]. Such a theory has the feature of allowing to take into account *any kind* of "perturbations" (e.g. truncation as well as round-off errors), at the same time. In Section 2, we review briefly such a theory, and in Section 3 the semi-implicit discretization scheme is presented. Stability and consistency are here estabilished, along with precise convergence estimates. Extension to evolutionary problems in Banach spaces is then pointed out. In Section 4, several applications are given, which include equations and systems, differential and integro-differential, of the reaction-diffusion type, as well as semilinear hyperbolic equations and systems. In Section 5, finally, we collect some general remarks.

2 A brief account of the Chartres-Stepleman theory.

In [7], B. Chartres and R. Stepleman introduced an abstract framework to analyze general numerical methods, see also [8]. Their approach, which originated by earlier basic ideas of I. Babuska et al. [2], is quite flexible and general. A "numerical method" is defined as a sextuple

$$\{X, Y, F, H, \{F_h\}, \{E_h\}\} \ , \tag{4}$$

X denoting the "data space", Y the "solution space", H the "discretization parameters set". $F : X \to Y$ is the mapping that associates the data to the solution of the underlying problem. Usually, Y is a linear normed space, equipped with the norm $\|\cdot\|_Y$, but it could be merely a metric space or even a topological vector space. H is a set of "discretization steps" where a convergence to 0 is defined, though $0 \notin H$.

The approximating procedure is described by means of the family of operators $\{F_h\}_{h\in H}$, $F_h : X \times E_h \to Y$, $(x,e) \mapsto F_h(x,e)$, where $\{E_h\}_{h\in H}$ denotes a family of "perturbation spaces". The space E_h contains, for each $h \in H$, an element denoted by 0, and a topology is given in E_h by means of a function $\phi_h : E_h \to R_0^+$, $\phi_h(0) = 0$. A basis of neighborhoods of 0 is defined as $N_0(\delta) := \{e \in E_h : \phi_h(e) < \delta\}$, $\delta > 0$. The function $\phi_h(e)$ measures, for every fixed h, the perturbation e. In most cases, E_h is a linear normed space, and ϕ_h is a norm on E_h.

The method (4) is termed *convergent* at $x \in X$ if

$$\|F_h(x,e) - F(x)\|_Y \to 0 \ , \quad \text{as} \quad h \to 0 \ , \quad \phi_h(e) \to 0 \ , \tag{5}$$

and is termed *stable* at $x \in X$ if

$$\|F_h(x,e) - F_h(x,0\|_Y \to 0 \ , \quad \text{as} \quad h \to 0 \ , \quad \phi_h(e) \to 0 \ . \tag{6}$$

Moreover, it is said to be *consistent* if, for every $h \in H$, there exists a special perturbation, say $\hat{e}_h(x)$, depending in general on x and h, such that

$$\|F_h(x, \hat{e}_h(x)) - F(x)\|_Y \to 0 \ , \quad \text{as} \quad h \to 0 \ , \tag{7}$$

with

$$\phi_h(\hat{e}_h(x)) \to 0 \ , \quad \text{as} \quad h \to 0 \ . \tag{8}$$

It is then easy to prove (cf. [7]) that a general form of the celebrated Lax Equivalence Theorem [17] holds: *A numerical method (defined as in* (4)*) is convergent* (*as in* (5)) *if and only if it is stable* (*as in* (6)) *and consistent* (*as in* (7), (8)).

It is worth noting that the present definition of convergence is more general than the usual one. In fact, in the literature it is often required only that $F_h(x,0) \to F(x)$ as $h \to 0$, i.e. that the "exact" (unperturbed) scheme converges to the solution of the original problem. The notion of convergence in (5) takes into account, however, the presence of unavoidable perturbations. Therefore, the usual convergence concept can be seen as a special case of consistency (taking $\hat{e}_h(x) \equiv 0$), and is implied by stability and consistency as above, in view of Lax theorem.

3 Stability and consistency of the semi-implicit Euler method.

In this Section we prove stability (§3.1) and consistency (§3.2) according to the definitions given in Section 2, for a semi-implicit Euler scheme applied to

$$\dot{u} = f(t,u,u) \ , \quad 0 < t \leq T \ ; \quad u(0) = u_0 \ . \tag{9}$$

Here $f(t,\cdot,v) : (D_t \subseteq \mathcal{H}) \to \mathcal{H}$, $\mathcal{H}$ being a real Hilbert space, $t > 0$, $v \in D$, with $\bigcup_{t>0} D_t \subseteq D \subseteq \mathcal{H}$, satisfies a *Lipschitz* condition with respect to the scalar product $\langle \cdot\,, \cdot \rangle$ in $\mathcal{H}$ (one-sided Lipschitz condition), i.e., for every fixed $t \in (0,T]$ and $v \in D$,

$$\langle f(t,u_1,v) - f(t,u_2,v), u_1 - u_2 \rangle \leq K_1 \|u_1 - u_2\|^2 \ , \ \ \forall\, u_1, u_2 \in D_t \ , \tag{10}$$

with a constant $K_1 \in R$, and $f(t,u,\cdot) : (D \subseteq \mathcal{H}) \to \mathcal{H}$, $t > 0$, $u \in D_t$, is uniformly *Lipschitz* continuous *in the classical sense*, with a constant $K_2 > 0$

$$\|f(t,u,v_1) - f(t,u,v_2)\| \leq K_2 \|v_1 - v_2\| \ . \tag{11}$$

Moreover, we assume that for each $h > 0$ sufficiently small $\mathcal{R}(I - hf(t,\cdot,v)) = \mathcal{H}$, namely that the equation

$$u = hf(t,u,v) + b \tag{12}$$

has a unique solution u in D_t, for each fixed $t \in (0,T]$, $v \in D$, $b \in \mathcal{H}$, and $h > 0$ sufficiently small. When f is *dissipative* in u ($K_1 = 0$), the solvability of (12) is equivalent to that of the same equation *with* $h = 1$, since the operator $f(t,\cdot,v)$ is *maximal* in this case; cf. [4, 20, 33].

In particular, $f(t,u,v) = A(t,u) + B(t,v)$, where $A(t,\cdot)$ is *dissipative*, or *strongly dissipative* ($K_1 < 0$) and *maximal* [33], and $B(t,\cdot)$ is Lipschitz continuous (cf. §1), as it occurs in typical cases arising within the context of parabolic differential or integrodifferential equations. In these cases, equation (12) is an elliptic partial differential or integrodifferential equation, and its solvability must be explicitly required.

By a solution to (9) we mean a function $u \in C^0([0,T];\mathcal{H})$, such that $u(t) \in D_t$ for every $t \in (0,T]$, T being a fixed positive number, strongly differentiable in $(0,T]$, satisfying (9). Under assumptions (10), (11), uniqueness can be estabilished in a standard way by using differential inequalities (cf. [16], e.g.), $f(t,u,u)$ being one-sided Lipschitz as a function of u, with constant $K_1 + K_2$. We assume that further hypotheses are made, when necessary, ensuring existence of solutions $u \in C^1[0,T]$, *or* $u \in AC[0,T]$, $\dot{u} \in BV[0,T]$ (cf. [13, Ch.3] concerning the definitions of AC and BV functions in the abstract context). Indeed, our algorithm is able to approximate solutions in these classes in the C^0-norm.

According to the formalism introduced in Section 2, here we take for X the set of all initial data $u_0 = x \in D$ for which a solution with the properties above exists (*necessarily* $u_0 \in cl(\bigcup_{t>0} D_t)$); $Y = C^0([0,T];\mathcal{H})$, since we shall approximate the solution u by *linear interpolation*; F maps u_0 into the unique solution u to (9); $H = \{h = T/N \ , \ N \in N^+\}$. The approximating procedure is based on the following (ideal) iterative algorithm: For $n = 0, 1, 2, ..., N$, set $t_n = nh$, and

$$u_{n+1} = u_n + hf(t_{n+1}, u_{n+1}, u_n) \ . \tag{13}$$

As we have to take into account all unavoidable errors introduced on each step, we consider the *perturbed* scheme

$$v_{n+1} = \tilde{u}_n + hf(t_{n+1}, v_{n+1}, \tilde{u}_n) + \delta_{n+1} \ ,$$

$$\tilde{u}_{n+1} = v_{n+1} + \sigma_{n+1} \ , \quad \tilde{u}_{n+1} \in D \ . \tag{14}$$

Here $\delta_0 = \tilde{u}_0 - u_0$ represents the *error* on the initial data, $\tilde{u}_0 \in D$, and $\sigma_n \in \mathcal{H}$, $n = 1, 2, ..., N-1$, is the *error* made in solving approximately the first equation in (14). The perturbation term σ_{n+1} *cannot*, in general, be absorbed in δ_{n+1}, as it is possible in the case of ordinary differential equations, where f is Lipschitz continuous (in the classical sense) also with respect to its second argument. The scheme (14) is well-defined since, for every n, the first equation in (14) *has a unique* solution v_{n+1} in view of (10), (11) (when $K_1 > 0$, uniqueness is guaranteed at least for h sufficiently small, which fact, in turn, modifies the set H above). From the scheme in (14) it is clear that one should require explicitely that $\tilde{u}_n \in D$ at each time step. In other words, this means that the method to be used for solving the first equation in (14) must give an approximate solution, $\tilde{u}_{n+1}$, *in* D.

Finally, the "perturbation space", E_h (cf. §2), consists of all admissible "perturbations vectors" $(\delta,\sigma) = (\delta_0, \delta_1, ..., \delta_N; \sigma_1, \sigma_2, ..., \sigma_N)$ described above, and thus $E_h \subseteq \mathcal{H}^{2N+1}$. The approximating mapping, F_h (cf. §2), is taken to be the *linear interpolant* among the values $\tilde{u}_n$, $n = 0, 1, 2, ..., N$. The function ϕ_h (cf. §2) that introduces the topology on E_h, will be defined in §3.1.

3.1 Stability of the semi-implicit Euler method.

From (13), (14), we obtain

$$\tilde{u}_{n+1} - u_{n+1} = v_{n+1} - u_{n+1} + \sigma_{n+1} \ , \tag{15}$$

and

$$\begin{aligned} v_{n+1} - u_{n+1} &= \tilde{u}_n - u_n + h[f(t_{n+1}, v_{n+1}, \tilde{u}_n) - f(t_{n+1}, u_{n+1}, u_n)] + \delta_{n+1} \\ &= \tilde{u}_n - u_n + h[f(t_{n+1}, v_{n+1}, \tilde{u}_n) - f(t_{n+1}, v_{n+1}, u_n)] \\ &\quad + h[f(t_{n+1}, v_{n+1}, u_n) - f(t_{n+1}, u_{n+1}, u_n)] + \delta_{n+1} \ . \end{aligned} \tag{16}$$

Setting

$$\varepsilon_n := v_n - u_n \ , \tag{17}$$

and taking the scalar product of both sides of (16) by ε_{n+1}, we obtain, using repeatedly the Schwarz inequality and (10), (11),

$$\|\varepsilon_{n+1}\| \leq (1 + hK_2)\|\tilde{u}_n - u_n\| + hK_1\|\varepsilon_{n+1}\| + \|\delta_{n+1}\| \ , \tag{18}$$

and then, for $hK_1 < 1$ (which is only required when $K_1 > 0$),

$$\|\varepsilon_{n+1}\| \leq \frac{1+hK_2}{1-hK_1}\|\tilde{u}_n - u_n\| + \frac{\|\delta_{n+1}\|}{1-hK_1} \ . \tag{19}$$

Therefore, by (15),

$$\|\tilde{u}_{n+1} - u_{n+1}\| \leq \frac{1+hK_2}{1-hK_1}\|\tilde{u}_n - u_n\| + \frac{\|\delta_{n+1}\|}{1-hK_1} + \|\sigma_{n+1}\| \ . \tag{20}$$

Setting, for short,

$$a_n := \|\tilde{u}_n - u_n\| \ , \quad \rho := \frac{1+hK_2}{1-hK_1} \ , \quad \tau := \frac{1}{1-hK_1} \ , \tag{21}$$

(20) becomes

$$a_{n+1} \leq \rho a_n + \tau\|\delta_{n+1}\| + \|\sigma_{n+1}\| \ , \quad n = 0, 1, ..., N-1 \ , \tag{22}$$

where $a_0 := \|\delta_0\|$. Then, inductively obtains

$$a_n \leq \rho^n\|\delta_0\| + \tau\sum_{k=1}^{n}\rho^{n-k}\|\delta_k\| + \sum_{k=1}^{n}\rho^{n-k}\|\sigma_k\| \ . \tag{23}$$

Recalling that $K_1 \in R$, $K_2 > 0$, it is convenient to list separately two cases:

a) $K_1 + K_2 \leq 0$, then $0 < \rho \leq 1$, and from (23)

$$\max_{0\leq n\leq N}\|\tilde{u}_n - u_n\| \leq \|\delta_0\| + \tau\sum_{k=1}^{N}\|\delta_k\| + \sum_{k=1}^{N}\|\sigma_k\| =: \phi_h(e) \ , \tag{24}$$

$\phi_h(e)$ being that introduced in §2;

b) $K_1 + K_2 > 0$, then $\rho > 1$, and

$$\max_{0\leq n\leq N}\|\tilde{u}_n - u_n\| \leq \rho^N(\|\delta_0\| + \tau\sum_{k=1}^{N}\|\delta_k\| + \sum_{k=1}^{N}\|\sigma_k\|) =: \phi_h(e) \ . \tag{25}$$

Here

$$\rho^N = \left(\frac{1+\frac{K_2T}{N}}{1-\frac{K_1T}{N}}\right)^N \leq c\exp\{(K_1+K_2)T\} \ , \tag{26}$$

($\rho^N \sim \exp\{(K_1+K_2)T\}$, $N \to \infty$) where the constant c can be taken equal to 1 when $K_1 > 0$.

Finally, the map F_h (cf. §2) being a linear interpolant, we have

$$\|\|F_h(x,e) - F_h(x,0)\|\| \le \max_{0\le n\le N} \|\tilde{u}_n - u_n\| \le \phi_h(e) \ , \quad hK_1 < 1 \ , \tag{27}$$

where $x = u_0$, and $\|\|u\|\| = \|u\|_Y := \sup_{0\le t\le T} \|u(t)\|$, which shows *stability*, according to §2 (such an estimate is referred to as "order stability" in [7]).

Remark 3.1 In the special case that $f(t,\cdot,v)$ is Lipschitz continuous in the classical sense with constant $K_3 > 0$, the error σ_{n+1} in (14) due to the numerical solution of the first equation in (14), can be adsorbed in δ_{n+1}. In fact, one may think that $\tilde{u}_{n+1}$ is the (unique) solution to

$$z = \tilde{u}_n + hf(t_{n+1}, z, \tilde{u}_n) + h[f(t_{n+1}, v_{n+1}, \tilde{u}_n) - f(t_{n+1}, \tilde{u}_{n+1}, \tilde{u}_n)] + \delta_{n+1} \ . \tag{28}$$

Note that here one should require that $\tilde{u}_n \in D_{t_n}$ for every $n \ge 1$, that is that the method followed to solve the first equation in (14) yields an approximate solution, $\tilde{u}_{n+1}$, *in* $D_{t_{n+1}}$. Comparing with the first equation in (14), we see that δ_{n+1} there is replaced by

$$\delta^*_{n+1} = \delta_{n+1} + h[f(t_{n+1}, v_{n+1}, \tilde{u}_n) - f(t_{n+1}, \tilde{u}_{n+1}, \tilde{u}_n)] \ , \tag{29}$$

and hence a scheme like (14) is obtained with $\sigma_{n+1} = 0$. Then,

$$\|\delta^*_{n+1}\| \le \|\delta_{n+1}\| + hK_3\|v_{n+1} - \tilde{u}_{n+1}\| = \|\delta_{n+1}\| + hK_3\|\sigma_{n+1}\| \ , \tag{30}$$

which shows that in the estimates (24)-(25) absorbing σ_{n+1} in δ_{n+1} is advantageous provided that $\tau hK_3 < 1$, i.e. $(K_1 + K_3)h < 1$. When $K_1 + K_3 > 0$, this is an actual condition on the time step. Note that, in any case, there is a contraction by a factor $\tau hK_3 \sim hK_3$ as $h \to 0$.

The occurrence described above is typical of many systems of *ordinary* differential equations, while it is ruled out in case of *partial* differential equations (for which $f(t,\cdot,v)$ is a differential operator).

3.2 Consistency of the semi-implicit Euler method.

To prove consistency (§2), we write

$$u(t_{n+1}) = u(t_n) + hf(t_{n+1}, u(t_{n+1}), u(t_n)) + \omega_{n+1}(h) \ , \tag{31}$$

that is (14) with the perturbation vector $\hat{e}_h(x) = (\delta,\sigma) = (0, \omega_1(h), ..., \omega_N(h); 0, ..., 0)$, where $x = u_0$, cf. (7). Here $u(t)$ is the unique solution to (9), with $u \in C^1[0,T]$ *or* $u \in AC[0,T]$, $\dot{u} \in BV[0,T]$. Clearly, (7) is automatically satisfied by the uniform

continuity of $u(t)$ on $[0,T]$, $F_h(u_0, \hat{e}_h(u_0))$ being the *linear interpolant* among the values $u(t_n)$, $n = 0, 1, ..., N$. What remains to be proved is (8).

We have to show that $\phi_h(\hat{e}_h(u_0)) \to 0$ as $h \to 0$, that is equivalent to show that $\sum_{k=1}^{N} \|\omega_k(h)\| \to 0$, as $h \to 0$, for u_0 fixed (cf. (13) with $\delta_0 = 0$, $\sigma_k = 0$). Now, from (31)

$$\|\omega_{k+1}(h)\| = \|u(t_{k+1}) - u(t_k) - hf(t_{k+1}, u(t_{k+1}), u(t_k))\|$$

$$= \| \int_{t_k}^{t_{k+1}} \dot{u}(t)dt - h\dot{u}(t_{k+1}) - h[f(t_{k+1}, u(t_{k+1}), u(t_k)) - f(t_{k+1}, u(t_{k+1}), u(t_{k+1}))\|$$

$$\leq \|h \sum_{i=1}^{m_k} \beta_{k,i}[\dot{u}(\tau_{k,i}) - \dot{u}(t_{k+1})] + w_\varepsilon\| + hK_2\|u(t_{k+1}) - u(t_k)\| \ . \tag{32}$$

Here the fact that $u \in AC[0,T]$ has been used. Besides, as $\dot{u}(t)$ is *Riemann integrable* on $[0,T]$ (cf. [13, Ch.3]), for every $\varepsilon > 0$, and for every k, $k = 1, 2, ..., N$, the nodes $\tau_{k,i}$, and the weights $\beta_{k,i} > 0$, $i = 1, 2, ..., m_k$, with $\sum_{i=1}^{m_k} \beta_{k,i} = 1$, exist so as to approximate the integral of $\dot{u}$ by the finite sum above, up to an error w_ε, $\|w_\varepsilon\| < \varepsilon$. Then

$$\|\omega_{k+1}(h)\| \leq h \sum_{i=1}^{m_k} \beta_{k,i}\|\dot{u}(\tau_{k,i}) - \dot{u}(t_{k+1})\| + \varepsilon + hK_2\|u(t_{k+1}) - u(t_k)\| \ . \tag{33}$$

Now, when $u' \in C^0[0,T]$, we get

$$\sum_{k=1}^{N} \|\omega_k(h)\| \leq hN[\mathrm{osc}(\dot{u}; h) + K_2 \mathrm{osc}(u; h)] \ , \tag{34}$$

where $hN = T$, and

$$\mathrm{osc}(u; h) := \sup \{\|u(t_1) - u(t_2)\| : \ t_1, t_2 \in [0,T], |t_1 - t_2| \leq h\} \ , \tag{35}$$

is a certain definition of "oscillation" of stepsize h (modulus of continuity). Hence $\phi_h(\hat{e}_h(u_0)) = o(1)$, being $\mathrm{osc}(u; h) = O(h)$ and $\mathrm{osc}(\dot{u}; h) = o(1)$.

When $\dot{u} \in BV[0,T]$, we can estimate $\sum \|\omega_k(h)\|$ in a different way. In fact, choosing (as is permissible) m_k and $\beta_{k,i}$ independent of k in (33), and interchanging the sums (over k and i), obtains, instead of (34),

$$\sum_{k=1}^{N} \|\omega_k(h)\| \leq hV_0^T(u') + TK_2 \mathrm{osc}(u; h) \ , \tag{36}$$

V_0^T denoting *the variation* on $[0,T]$. Therefore $\phi_h(\hat{e}_h(u_0)) = O(h)$ (as $\mathrm{osc}(u; h) = O(h)$, $\dot{u}$ being bounded), and thus *consistency* is proved.

Remark 3.2 (*order of consistency*) If $\dot{u}$ is Lipschitz continuous in $[0,T]$, then the r.h.s of (34) is $O(h)$. If $\dot{u} \in BV[0,T] \cap C^0[0,T]$, (36) is in general a better estimate than (34), being always of $O(h)$, in view of the boundedness of $\dot{u}$.

3.3 Convergence estimates.

From the results of §3.1, §3.2, convergence of the semi-implicit Euler method (cf. (5)) follows by a Lax-type equivalence theorem [7]. Following the proof of the theorem in [7], we obtain by (27)

$$|||F_h(u_0,e)-u||| \le |||F_h(u_0,e)-F_h(u_0,0)||| + |||F_h(u_0,0)-F_h(u_0,\hat{e}_h(u_0))|||$$

$$+ |||F_h(u_0,\hat{e}_h(u_0))-u||| \le \phi_h(e)+\phi_h(\hat{e}_h(u_0))+\mathrm{osc}(u;h)\ , \tag{37}$$

where $|||u||| = \|u\|_Y := \sup_{0\le t\le T}\|u(t)\|$, since the oscillation term, $\mathrm{osc}(u;h)$, estimates the error made in the linear interpolation of the exact solution, u. Now, the fact that $\dot{u}$ is bounded (implied by either $u \in C^1[0,T]$, or by $\dot{u} \in BV[0,T]$) entails that $\mathrm{osc}(u;h) = O(h)$. For convenience, we display the convergence estimates obtained from (37) and (24)-(25). Setting

$$r_N := \begin{cases} 1\ , & \text{if } K_1+K_2 \le 0 \\ \rho^N \equiv \left(\frac{1+K_2T/N}{1-K_1T/N}\right)^N\ , & \text{if } K_1+K_2>0\ , \end{cases} \tag{38}$$

we have

$$|||F_h(u_0,e)-u||| \le r_N\{\|\delta_0\| + \tau\sum_{k=1}^{N}\|\delta_k\| + \sum_{k=1}^{N}\|\sigma_k\|$$

$$+ \tau T(\mathrm{osc}(u';h)+K_2\mathrm{osc}(u;h))\} + \mathrm{osc}(u;h)\ , \tag{39}$$

when $\dot{u} \in C^0[0,T]$, and

$$|||F_h(u_0,e)-u||| \le r_N\{\|\delta_0\| + \tau\sum_{k=1}^{N}\|\delta_k\| + \sum_{k=1}^{N}\|\sigma_k\|$$

$$+ \tau(hV_0^T(u')+TK_2\mathrm{osc}(u;h))\} + \mathrm{osc}(u;h)\ , \tag{40}$$

when $\dot{u} \in BV[0,T]$.

From the computational viewpoint, in (39), (40) it is reasonable to take $e = (\delta_0,0,...,0;\sigma_1,...,\sigma_N)$ where δ_0 is the error on the initial value and σ_k represents the overall error made in the numerical solution of the k-th equation in (14). Such an error takes into account, typically, *round-off* errors and the error inherent to the method itself used to solve the k-th equation in (14). In case of partial differential equations, for instance, the latter takes into account, in turn, the error in the *boundary data* (if any) and the *truncation error* with respect to the space variables. In the theory above, perturbation terms $\delta_k \ne 0$ $(k = 1,2,...,N)$ have been included in studying stability. In fact they appear, in a natural way, as local truncation errors, $\omega_k(h)$ (cf. (31), §3.2).

Therefore, (40), as well as (39) when $\dot{u}$ is Lipschitz continuous in $[0,T]$, yields the following qualitative convergence estimate:

$$|||F_h(u_0,e)-u||| = O(\|\delta_0\|) + O(\sum_{k=1}^{N}\|\sigma_k\|) + O(h)\ , \tag{41}$$

since $r_N \to \exp\{(K_1+K_2)T\}$ and $\tau \to 1$ as $h \to 0$ (cf. (21). The practical use of the method rests on having at hand accurate algorithms to solve the first equation in (14) (with $\delta_{n+1}=0$), since, roughly speaking, one should guarantee that $h^{-1}\max_{1\le k\le N}\|\sigma_k\|$ be small when h is small.

Remark 3.3 Everything done in this section holds in case of a *complex* Hilbert space, just taking the real part on the l.h.s. of (10) and similarly after then.

Remark 3.4 All results obtained in this section hold when $\mathcal{H}$ is replaced by a general (real or complex) Banach space $\mathcal{B}$, provided that $f(t,u,\cdot)$ is Lipschitz continuous as before, $\mathcal{R}(I-hf(t,\cdot,v)) = \mathcal{B}$ for $h>0$ sufficiently small, and $f(t,\cdot,v)$ satisfies, for every fixed $t\in(0,T]$ and $v\in D$, the condition

$$\|u_1-u_2-h[f(t,u_1,v)-f(t,u_2,v)]\| \ge (1-hK_1)\|u_1-u_2\|\ ,\ \ \forall\, u_1,u_2\in D_t\ . \tag{42}$$

This property is equivalent to that in (10) above when the space is a Hilbert space [20]. and has beeen extensively used in the numerical treatment of ordinary differential systems, (cf. [9, 29], e.g.). When $K_1=0$, the operator $f(t,\cdot,v)$ is termed "dissipative"; in general, in (42) K_1 can be positive, negative, or zero. The straightforward modifications in §3.1 are left to the reader.

Remark 3.5 Note that the analysis developed in Section 3 embodies the case of the *fully implicit* Euler method. In fact, if $K_1\in R$ denotes the one-sided Lipschitz constant of $p(t,u)$ with respect to u, where $p(t,u):=f(t,u,u)$, with $\mathcal{R}(I-hp(t,\cdot)) = \mathcal{H}$ for $h>0$ sufficiently small, the theory of Section 3 can be applied with $K_2=0$. For an application of the theory of [7] to the fully implicit Euler method with $K_1=0$, see [31]. We stress that the case $K_1>0$ is also included, which occurrence seems to be not recorded in the literature for the infinite-dimensional case.

4 Examples.

In this Section, we present some applications of the semi-implicit Euler method, in both the finite and the infinite-dimensional case. The computational advantages in using such a method will be pointed out. Below, letters denoting m-dimensional vectors will be boldfaced, for clarity.

A) *Systems of ordinary differential equations.*

Consider equation (9) with $\mathcal{H} = R^m$, that is an m-dimensional system of ordinary differential equations (ODEs). Assume, moreover, that all hypotheses accompanying equation (9) in Section 3 are satisfied. Note that such hypotheses are classical in the context of the numerical treatment of systems of ODEs (cf. [9], e.g.). Our approach is closely related to the so-called "backward Euler multirate method", used to decouple system (9) into stationary sub-systems, in view of *parallel implementation*, cf. [28, 29]. The latter represents an approach alternative to the "waveform relaxation (WR) method" [11] used to solve very large, possibly stiff systems. An advantage of such an approach with respect to the WR method is that no iteration is required in the continuous time domain. Indeed, the convergence of such an iteration can be very slow, and memory occupacy for storing the waveforms may be large.

An important example treated in [29] is the "decoupled implicit Euler method" for $\dot{u} = F(t,u) \equiv f(t,u,u)$, $u(0) = u_0$, where

$$f(t,u,v) = (F_1(t,u_1,v_2,...,v_r),...,F_r(t,v_1,...v_{r-1},u_r)) \ . \tag{43}$$

Here F_k, u_k are m_k-dimensional vectors, and v_j is m_j-dimensional; $\sum_{k=1}^{r} m_k = m$. In this way one is facing, at each time step, r independent nonlinear algebraic systems, which can be solved concurrently (cf. (14) above, with $\delta_{n+1} = 0$).

As shown in [29], when F is sufficiently smooth, extrapolation can be used to obtain an higher order of integration accuracy. In the opposite direction, our analysis also allows $f(t,\cdot,v)$ to be discontinuous. This feature is essential in solving evolutionary *partial* differential equations and systems.

B) *A single reaction-diffusion equation in* $L^2(\Omega)$.

Consider the problem

$$w_t = a\Delta w + g(t,x,w) \ , \quad \text{for } x \in \Omega \subset R^d \ , \ \Omega \text{ open smooth bounded} \ , \ 0 < t \leq T \ ,$$

$$w = w_1(x,t) \text{ on } \partial\Omega \ ; \quad w(x,0) = w_0(x) \ , \ x \in \Omega \ , \tag{44}$$

where Δ denotes the Laplace operator in R^d, g is globally Lipschitz-continuous with respect to w, uniformly in $x \in \Omega$ and $t \in (0,T]$. Moreover, assume that for every $t \in (0,T]$, $g(t,\cdot,0) \in L^2(\Omega)$ and w_0 and w_1 are suitable functions. The problem (44) can be recast into the form of equation (1) with $\mathcal{H} = L^2(\Omega)$, $A(t,\cdot) \equiv A(\cdot) = a\Delta$, $D_t = \{u \in H^2(\Omega) : u = w_1 \text{ on } \partial\Omega\}$, and $B(t,\cdot) : u(t) = w(\cdot,t) \mapsto g(t,\cdot,w(\cdot,t))$, $D = L^2(\Omega)$.

The hypotheses on g above imply that $g(t,\cdot,w(\cdot,t)) \in L^2(\Omega)$, and that $B(t,\cdot)$ is Lipschitz-continuous as a map from $L^2(\Omega)$ into $L^2(\Omega)$ (see [4], e.g.). On the other hand, A is a maximal dissipative operator, as known (cf. [5]).

We assume, further, that additional conditions are fulfilled ensuring existence of $C^1[0,T]$ solutions; this is the case, for instance, when $w_1 \equiv 0$ (and hence $D_t \equiv D(A) = H^2(\Omega) \cap H_0^1(\Omega)$), B is a C^1-function of its arguments, and $u_0 = w_0(\cdot) \in D(A)$, cf. [23, Thm.1.5, p.187]. Unfortunately, the substitution operator $B(t,\cdot)$ is *differentiable if and only if* the scalar function $g(t,x,w)$ in (44) is *linear* in w (cf. [15, Thm. 20.1, p. 419]). This restriction, however, can be overcome when g is Lipschitz continuous in both t and w, uniformly with respect to $x \in \Omega$. In fact, the substitution operator turns out to be Lipschitz continuous in both t and u, which ensures the existence of a "strong" solution in the sense of [23, Thm.1.6, p.189]. Such a strong solution is actually *continuously differentiable* in $[0,T]$, which fact can be seen as follows. One considers first the linear inhomogeneous problem $\dot{u} = Au + b(t)$, $t > 0$, $u(0) = u_0 \in D(A)$, with b Lipschitz continuous in $[0,T]$. It is known [12, Remark 9.8, p.129] that there is a unique solution which is continuously differentiable in $(0,T]$. By standard arguments of linear semigroups theory one then shows that $\dot{u}$ can be continued up to $t = 0$ from the right. Finally, combining this argument with the proof of Theorem 1.6 of [23, p.189], extension can be made to the *semilinear* problem we are considering.

We observe that the restriction that g be globally Lipschitz-continuous with respect to w can be removed whenever a priori estimates on the solution, say $|w(x,t)| \leq M$ for all $x \in \overline{\Omega}$ and $t \in [0,T]$, are available, and g is Lipschitz-continuous in $[-M,M]$ uniformly in $x \in \overline{\Omega}$ and $t \in (0,T]$. In this case, in fact, problem (44) can be replaced by a similar one where g is replaced by $g^*(t,x,w) = g(t,x,w)$ in $(0,T] \times \Omega \times [-M,M]$ and $g^*(t,x,w) = g(t,x,M)$ for $w > M$, $g^*(t,x,w) = g(t,x,-M)$ for $w < -M$, cf. [14].

The computational advantages of the method are clear. At each time step, the semi-implicit Euler method applied to problem (44) requires solving the d-dimensional *linear* inhomogeneous Helmholtz equation

$$v_{n+1}(x) = \tilde{u}_n(x) + ha\Delta v_{n+1}(x) + hg(t_{n+1}, x, \tilde{u}_n(x)) \ , \quad x \in \Omega \ ,$$

$$v_{n+1}(x) = w_1(x, t_{n+1}) \ , \quad x \in \partial\Omega \ , \tag{45}$$

(cf. (14) with $\delta_{n+1} = 0$). A unique solution, v_{n+1}, to (45) does exist as $a\Delta$ is a maximal dissipative operator from $\{u \in H^2(\Omega) : u(x) = w_1(x, t_{n+1}) \text{ on } \partial\Omega\}$ in $L^2(\Omega)$. The approximate solution to (45), say $\tilde{u}_{n+1}(x)$, should be seeked in $D = L^2(\Omega)$, cf. (14). The equation in (45) being linear, its numerical treatment can be based on a variety of methods, besides the widely used finite difference and finite element methods. For instance, when Ω can be decomposed into rectangular subdomains, equations like (45) (as well as similar linear elliptic systems, cf. Example (D) below) can be treated by *spectral* collocation methods. These methods are known to possess an extremely high convergence rate, when the solution is smooth, cf. [25]. When the spatial dimension d is comparatively large, one might apply *probabilistic* methods. We

stress, incidentally, that in this approach *parallelization* is trivial, since the solution at every point $x \in \Omega$ is evaluated as an average over Brownian trajectories which can be generated independently (and hence on several different processors), cf. [21], e.g.. The direct application of such methods to the nonlinear evolution equation (44) is not straightforward, cf. [27].

C) *A single semilinear parabolic equation in* $C^0(\overline{\Omega})$.

The following example shows an application in a Banach space which is *not* a Hilbert space (cf. Remark 3.4):

$$w_t = Lw + g(t,x,w)\ ,\quad x \in \Omega \subset R^d\ ,\quad 0 < t \le T\ ,$$

$$Lu := a_{ij}(x)u_{x_ix_j} + b_i(x)u_{x_i} - c(x)u\ ,$$

$$w(x,t) \equiv 0\ ,\quad x \in \partial\Omega\ ,\quad 0 < t \le T\ ;\quad w(x,0) = w_0(x)\ ,\quad x \in \Omega\ . \tag{46}$$

Here, $a_{ij}, b_i, c \in C^0(\overline{\Omega})$, $c(x) \ge 0$ in Ω, and L is strictly elliptic, i.e. there is $\theta > 0$ such that $a_{ij}\xi_i\xi_j \ge \theta|\xi|^2$ for every d-vector ξ and x in Ω; g is Lipschitz-continuous in w uniformly in $(x,t) \in \Omega \times (0,T]$, and $g(t,x,0) = 0$ for each pair $(x,t) \in \partial\Omega \times (0,T]$. In this case, the parabolic equation in (46) is of type (1) with $\mathcal{H} = C_0^0(\overline{\Omega})$ endowed with the sup-norm, $A(t,\cdot) \equiv L$, $D_t \equiv D(L) = \{u \in W^{2,p}(\Omega)\, \forall p < \infty : Lu \in C^0(\overline{\Omega}), u \equiv 0 \text{ on } \partial\Omega\}$, and $B(t,\cdot) : u(t) = w(\cdot,t) \mapsto g(t,\cdot,w(\cdot,t))$, $D = C_0^0(\overline{\Omega})$. Besides, assume that $g \in C^1([0,T] \times \Omega \times R)$. Then $B \in C^1([0,T] \times C_0^0(\overline{\Omega}); C_0^0(\overline{\Omega}))$ (cf. [32, p.167]) and thus, if $w_0 \in D(L)$ by Theorem 1.5, p.187 of [23] the abstract problem (1) corresponding to (46) has a unique $C^1([0,T]; C_0^0(\overline{\Omega}))$ solution. Therefore the semi-implicit Euler method can be applied, and the computational advantage of facing a *linear* elliptic problem at each time step again follows.

D) *Systems of reaction-diffusion equations.*

Consider the system

$$w_t = M\Delta w + g(t,x,w)\ ,\quad x \in \Omega \subset R^d\ ,\quad 0 < t \le T\ ,$$

$$w(x,0) = w_0(x)\ ,\quad x \in \Omega\ ;\quad w = w_1(x,t) \text{ on } \partial\Omega\ , \tag{47}$$

for the m-dimensional vector w, where M is a diagonal constant positive definite matrix. Here, $g(t,x,\cdot)$ is supposed to be a Lipschitz-continuous function from R^m into R^m, uniformly in (t,x), and $g_i(t,\cdot,0) \in L^2(\Omega)$, $i = 1,2,...,m$. To cast the problem into the abstract form (1), we choose $\mathcal{H} = (L^2(\Omega))^m$, $A(t,\cdot) = M\Delta$, $B(t,\cdot) : u(t) = w(\cdot,t) \mapsto g(t,\cdot,w(\cdot,t))$, $D_t = \{u \in (H^2(\Omega))^m : u_i = w_{1,i} \text{ on } \partial\Omega,\ i = 1,2,...,m\}$, $D = \mathcal{H}$. As in Example (B), it follows that $g(t,\cdot,w(\cdot,t)) \in \mathcal{H}$, and $B(t,\cdot)$ is Lipschitz-continuous from $\mathcal{H}$ into $\mathcal{H}$. Moreover, we assume further that additional conditions are satisfied to ensure existence of $C^1[0,T]$ solutions. Similarly to Example (B), for

instance, this is the case when g is Lipschitz continuous in both t and w uniformly in $x \in \Omega$.

When the semi-implicit Euler method is applied to (47), one has to solve numerically m *linear* (scalar) elliptic (Helmholtz) *uncoupled* equations at each time step, i.e.

$$v_{i,n+1}(x) = \tilde{u}_{i,n}(x) + hM_i \Delta v_{i,n+1}(x) + hg_i(t_{n+1}, x, \tilde{u}_n(x)) \ ,$$
$$v_{i,n+1}(x) = w_{1,i}(x, t_{n+1}) \ \text{ on } \partial\Omega \ , \ \ i = 1, 2, ..., m \ . \tag{48}$$

Note that, in this case, besides linearization, decoupling takes place of the m-dimensional system into m independent Helmholtz problems, which fact allows for a trivial *parallel implementation*. In particular, one may consider adopting different methods for each of the m components, e.g. when the M_i's have very different size. This suggests also the following

Remark 4.1 The semi-implicit Euler method can be applied to systems of semilinear parabolic equations like those in (47) with $M = M_1 \bigoplus M_2 \bigoplus ... \bigoplus M_k$, where M_j is an $m_j \times m_j$ constant positive definite matrix, $\sum_{j=1}^{k} m_j = m$. In this case, the method reduces to the solution of k *linear* elliptic *uncoupled* subsystems, at each time step.

E) *An integro-differential parabolic equation.*

Consider

$$w_t = a\Delta w + \int_\Omega K(x, \xi) w(\xi, t) d\xi \ , \ \ x \in \Omega \subset R^d \ , \ \ 0 < t \leq T \ ,$$
$$w = w_1 \text{ on } \partial\Omega \ ; \ \ w(x, 0) = w_0(x) \ , \ x \in \Omega \ , \tag{49}$$

where $a > 0$ and $K(\cdot, \cdot) \in L^2(\Omega \times \Omega)$. Again, equation (49) can be cast into the abstract form (1) with $\mathcal{H} = L^2(\Omega)$, $A(t, \cdot) \equiv A(\cdot) = a\Delta$, $D_t \equiv D(A) = \{u \in H^2(\Omega) : u = w_1 \text{ on } \partial\Omega\}$, and $B(t, \cdot) \equiv B(\cdot) : u(t) = w(\cdot, t) \mapsto \int_\Omega K(\cdot, \xi) w(\cdot, \xi) d\xi$, $D = L^2(\Omega)$. Note that the integral operator $B(t, \cdot)$ is linear *bounded* from $L^2(\Omega)$ into $L^2(\Omega)$ (since $K \in L^2(\Omega \times \Omega)$) and hence has Lipschitz constant $K_2 = \|K\|_{L^2(\Omega \times \Omega)}$, cf. (11). Existence of solutions $u \in C^1([0, T]; L^2(\Omega))$ is guaranteed, for instance, when $w_1 \equiv 0$ by Theorem 1.5, p.187 of [23], as $B(t, \cdot) \in C^1(L^2(\Omega); L^2(\Omega))$. Therefore the semi-implicit Euler method can be applied, and requires solving the linear inhomogeneous elliptic (Helmholtz) equation

$$v_{n+1}(x) = \tilde{u}_n(x) + ha\Delta v_{n+1}(x) + h \int_\Omega K(x, \xi) \tilde{u}_n(\xi) d\xi \ , \ \ x \in \Omega \ ,$$
$$v_{n+1}(x) = w_1(x, t_{n+1}) \ , \ \ x \in \partial\Omega \ , \tag{50}$$

at each time step.

If, in addition, the integral operator in (49) is *dissipative* on $L^2(\Omega)$ (e.g., when the kernel $K(x,\xi)$ is symmetric negative semi-definite Hilbert-Schmidt [4, Example 7.3.1, p.234]), the fully implicit Euler method is also applicable (cf. 3.5). In fact, the whole operator $A(\cdot)+B(\cdot)$ is dissipative, in this case (cf. [19, 26]). This method exhibits better stability properties, since the "propagation factor" in (26), ρ^N, is replaced by 1, but one must face, after space discretization, the solution of *full* high-dimensional linear systems. Following the semi-implicit Euler method, instead, one has to solve merely *band structured* (using finite differences), or *sparse* systems (finite elements), since only the differential operator is discretized at each time-step. Therefore, the latter approach requires handling only *local* operators versus *integral* operators.

Remark 4.2 We chose the linear scalar equation (49) for simplicity. All could be done for more general nonlinear reaction-diffusion systems like

$$w_t = M\Delta w + g(t,x,w) + \int_\Omega K(t,x,\xi,w(\xi,t))d\xi \ , \tag{51}$$

for the m-vector w, with suitable initial-boundary data. Such problem generalizes all the other examples described above. Clearly, in discretizing numerically equation (51) by the semi-implicit Euler method *all* the advantages pointed out above (linearization, decoupling, local versus nonlocal structure) are obtained at the same time. It is worth noting that in the block-diagonal matrix M in (51) some blocks are allowed to vanish. This means that in the integro-differential system (51), some subsystems may reduce to ordinary (integro-differential) systems. This happens in several models, e.g. concerning competing species dynamics (cf. [18]).

F) *A semilinear hyperbolic equation.*

The problem

$$\begin{gathered} w_{tt} = a\Delta w + g(t,x,w) \ , \quad \text{in } \Omega\times(0,T] \ , \\ w(x,0)=w_0(x) \ , \ w_t(x,0)=w_1(x) \ , \quad x\in\Omega \ ; \quad w(x,t)=0 \ , \quad x\in\partial\Omega \ , \quad t\in(0,T] \ , \end{gathered} \tag{52}$$

can be written as as an abstract evolutionary problem for the 2-vector $u(t)=(u_1(t),u_2(t))$ $:=(w(\cdot,t),w_t(\cdot,t))$

$$\dot u = A(u)+B(t,u) \ , \quad t\in(0,T] \ ; \quad u(0)=(w_0(\cdot),w_1(\cdot)) \ , \tag{53}$$

where

$$A := \begin{pmatrix} 0 & I \\ a\Delta & 0 \end{pmatrix} \ , \quad B(t,\cdot):((u_1(t),u_2(t))\mapsto(0,g(t,\cdot,w(\cdot,t))) \ . \tag{54}$$

Here, $\mathcal{H}=H_0^1(\Omega)\times L^2(\Omega)$, $D_t\equiv D(A)=(H^2(\Omega)\cap H_0^1(\Omega))\times H_0^1(\Omega)$, $D=\mathcal{H}$, and g satisfies the same hypotheses of Example (B). As is known [5], in this case A is

a maximal dissipative operator, and $B(t,\cdot)$ is Lipschitz continuous. Moreover, we assume that suitable assumptions are made to ensure existence of $C^1[0,T]$ solutions. Hence, the semi-implicit Euler method can be applied (see [24] for the fully implicit), yielding the system

$$\begin{cases} v_{1,n+1}(x) = \tilde{u}_{1,n}(x) + hv_{2,n+1}(x) \\ v_{2,n+1}(x) = \tilde{u}_{2,n}(x) + ha\Delta v_{1,n+1}(x) + hg(t_{n+1}, x, \tilde{u}_{1,n+1}(x)) \ , \quad x \in \Omega \ , \end{cases}$$

$$v_{1,n+1}(x) = 0 \ , \ v_{2,n+1}(x) = 0 \ , \quad x \in \partial\Omega \ , \tag{55}$$

for $(v_{1,n+1}(x), v_{2,n+1}(x))$, at each time step (cf. 14) with $\delta_{n+1} = 0$). Such a system is equivalent to

$$\begin{cases} v_{1,n+1}(x) = h^2 a\Delta v_{1,n+1}(x) + \tilde{u}_{1,n}(x) + h\tilde{u}_{2,n}(x) + h^2 g(t_{n+1}, x, \tilde{u}_{1,n}(x)) \\ hv_{2,n+1}(x) = v_{1,n+1}(x) - \tilde{u}_{1,n}(x) \ , \end{cases} \tag{56}$$

with the same data on $\partial\Omega$. Therefore, one should solve a *linear* inhomogeneous Helmholtz equation at each time step.

An interesting example of a semilinear hyperbolic equation with *globally* Lipschitz nonlinear term is given by the Sine-Gordon equation, $w_{tt} = \Delta w - \lambda \sin(w)$, $\lambda > 0$.

For hyperbolic semilinear systems for the m-vector w, such as

$$w_{tt} = M\Delta w + g(t,x,w) \ , \quad M \text{ diagonal } m \times m \text{ matrix} \tag{57}$$

as well as for the corresponding integro-differential systems (cf. equation (51) for the parabolic case), with initial-boundary data as in (52), the semi-implicit Euler method reduces essentially to the numerical solution of m *independent* Helmholtz equations, at each time step.

5 Final observations.

In closing, some remarks are in order.

Remark 5.1 The assumption that $f(t,u,\cdot)$ be *globally* Lipschitz-continuous *may be* not restrictive in practice, *when* equation (9) is equivalent to an ordinary or to a partial differential equation or system. For several reaction-diffusion systems like those in (47) with $g(t,x,\cdot)$ *locally* Lipschitz-continuous in R^m, for instance, *a priori estimates* on w are known (cf. [18], e.g.). In this case, the original problem can be transformed into an equivalent problem with the required property (cf. [14]).

Remark 5.2 Examples (B)-(F) above are *semilinear*, but this was done merely for the purpose of illustration. The semi-implicit method analyzed in Section 3 can actually be applied to *fully nonlinear* equations. For instance, the method is still convenient when $f(t,u,u)$ has the additive structure, $A(t,u)+B(t,u)$, but A is *nonlinear*, and the stationary equations, $v_{n+1} = \tilde{u}_n + hA(t_{n+1}, v_{n+1}) + hB(t_{n+1}, \tilde{u}_n)$, do present computational advantages.

In the remark below, the following results, valid in an arbitrary *Banach* space, will be used:

(i) If A is maximal dissipative, and B is everywhere defined, dissipative and Lipschitz-continuous (and thus maximal dissipative), then $A+B$ is maximal dissipative;

(ii) If A is maximal dissipative, B is everywhere defined and Lipschitz-continuous, *and* $A+B$ is dissipative, then $A+B$ is *maximal* dissipative.

Proof. The part (i) is proved, e.g., in [1], even for B everywhere defined, dissipative and merely continuous. As for (ii), it suffices to show that

$$u = h(A+B)(u) + b \tag{58}$$

has a solution for every b in $\mathcal{H}$ and for $h>0$ sufficiently small. In fact, the same will be true for *every* $h>0$ since $A+B$ is dissipative.

Consider the sequence of equations

$$u^{n+1} = hA(u^{n+1}) + hB(u^n) + b\ , \quad n = 0,1,2,..., \tag{59}$$

$u^0 \in \mathcal{H}$ arbitrary. They have a (unique) solution, $u^{n+1} \in D(A)$, since A is maximal dissipative. It is easily seen by the same technique used in §3.1 to prove stability, that $\{u^n\}$ is a Cauchy sequence, and thus $u^n \to u^* \in \mathcal{H}$. Consider then

$$z = hA(z) + hB(u^*) + b\ . \tag{60}$$

It has a (unique) solution, $z \in D(A)$, for the same reason above. Again, by a procedure similar to that in §3.1, one can show subtracting (60) from (59) that $u^n \to z$. Therefore $z = u^*$. **Q.E.D.**

Remark 5.3 When $K_1 + K_2 \le 0$ (see (10), (11)), and thus the operator $p(t,u) = A(t,u) + B(t,u)$ in (1) is *globally dissipative* in u, and hence *maximal* by (ii), the semi-implicit method has the same stability properties as the fully implicit one, since the propagation factor is $r_N = 1$ in both cases (cf. (11) and Remark 3.5). When $K_1 + K_2 > 0$, but $K_1 \le 0$ and $B(t,\cdot)$ is also *dissipative* (and then $p(t,\cdot)$ is *maximal* dissipative by (i)), the fully implicit method exhibits *better* stability properties. One

should notice, however, that the semi-implicit approach still retains the computational advantages emphasized in the Examples above. Indeed, in Examples (B)-(F) above we have $K_1 + K_2 = K_2 > 0$, with $A(t,\cdot)$ dissipative, but $B(t,\cdot)$ *may* also *be* dissipative (for instance, this happens in Example (B) when g is a decreasing function of w, see [4, Ch.3], besides being Lipschitz-continuous in w). A case when $K_1 + K_2 \leq 0$, instead, occurs in Example (C) with $c(x) \geq c_0 > 0$, and the Lipschitz constant of g in (46) $K_2 \leq c_0$. In fact, the operator L is now strongly dissipative with $K_1 = -c_0$.

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