# Fuchsian Equations in Sobolev Spaces and Blow-up

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**Short title:** Fuchsian equations and blow-up.

**Abstract**

We construct solutions of $\square u = e^u$ which blow-up precisely on a given space-like hypersurface of class $H^s$. For this purpose, we prove a general existence theorem for Fuchsian PDE in Sobolev spaces. The precise relation between the regularity of the data and that of the solution is shown to involve logarithmic symbols, in a model situation. A few further results on power nonlinearities are also included.

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## 1. Introduction.

We prove in this paper that the equation

$$\Box u = e^u \tag{1}$$

has solutions defined in the neighborhood of any space-like hypersurface $\Sigma = \{t = \psi(x)\}$ with $\psi \in H^s(\mathbf{R}^n)$, which blow-up precisely on it. More precisely, if $T = t - \psi(x)$,

$$u(x,t) = \ln(2/T^2) + v(T, T\ln T, x), \tag{2}$$

where $v$ is of class $H^{s-4}$ with respect to $x$; it is $C^\infty$ in its arguments if $\psi$ is. There are infinitely many such solutions, depending on the choice of one function on $\Sigma$. The number of space dimensions is arbitrary. Note that the singularity surface is not characteristic for the linear part of the equation. The procedure consists in constructing, and then solving, a PDE of the general form

$$N\vec{u} + A\vec{u} = f(t_0, \dots, t_l, x, \vec{u}, D_x\vec{u}) \tag{3}$$

where

1. $N$ is a first-order operator of the form, $\sum_{0\le i,j\le l} m_{ij} t_j \partial/\partial t_i$,
2. $x \in \mathbf{R}^n$, $\vec{u} \in \mathbf{R}^P$, and the matrices $A$ and $M := (m_{ij})$ are constant,
3. $f = O(|t|)$ near $t = 0$.

Such a system is called (generalized) Fuchsian. The set $t_0 = \cdots = t_l = 0$ is characteristic. For problem (1), it is sufficient to take $l = 1$, but more complicated choices of $N$ occur naturally for other equations, as detailed in [8].

The function $v$ in (2) is then recovered from the first component of $\vec{u}(T, T\ln T, \dots, T(\ln T)^l, x)$.

### 1.1. Relation to the literature.

This result is motivated by the recent proof of this result in the analytic case [10], which proved that one could prescribe with great freedom the blow-up set for (1). The assumption of analyticity may give the impression that the result of [10] implies a certain "rigidity" of the blow-up surface; we show here that such is not the case, by proving the natural analogue in the $H^s$ framework. Further motivation is given in §1.2.

The only other results on the blow-up surface for non-linear hyperbolic equations are due to Caffarelli and Friedman [4].

Two recent perspectives on singularity formation, which further justify the approach taken here, may be mentioned; in all three approaches, one finds a remarkable convergence of views: by performing a suitable change of variable, one can replace the original problem by an "unfolded" system with smooth solutions. The new system is usually degenerate (Fuchsian, or non-strictly hyperbolic for instance), but can be solved by a Cauchy-Kowalewska theorem.

The first idea (see the survey by Caflisch [5]) consists in tracing the singularities to complex singularities in the initial data, which travel and become real at some later time. This approach has been successful in the Rayleigh-Taylor problem and the Kelvin-Helmholtz problem, among others. It then becomes natural to expect that the most common singularities should be poles or branch points, *and can therefore be removed by the introduction of suitable uniformizing variables*. And indeed, classification of generic singularities can sometimes be effected, in the sense of catastrophe theory, by constructing an unfolding the equations.

The second approach (see Alinhac [2]) also consists in performing an unfolding, but only on a "time" variable. This leads to a "blown-up system" which can be again solved in the analytic case (and, sometimes, in the smooth case as well).

Such unfoldings appear to be the exact analogue, for shock waves, of the change of variable and unknown in (2) with the difference that the equation of the blow-up surface cannot be prescribed arbitrarily, but is rather part of the unknown; thus, in [2], its evolution is related to one of the characteristic speeds.

Several authors have studied the blow-up *time* for nonlinear hyperbolic equations with small data (John, Hörmander, Lindblad and others; see the surveys by John (1990) and by Strauss (1989)). These works imply that the blow-up is governed, in the limit of small data, by a few universal rescaled equations. We are however interested here in solutions with possibly large data (the basic model being a solution independent of $x$, and therefore, not square-summable). One general result in this direction is F. John's proof that all solutions of $\Box u = u^2$ with compactly supported data in three space dimensions must blow up in finite time.

There has also been much recent activity on the question of ruling out blow-up for $\Box u + u^p = 0$, but this has no bearing on the question at hand here.

On the other hand, Fuchsian equations have been studied for their own sake by many authors, in view of the numerous contexts in which they occur naturally; let us recall recall two.

First, Fuchsian equations occur as models of operators with double characteristics (in the "non-involutive" case); propagation of singularities in the linear, $C^\infty$ case has been studied by several authors (see for instance the monograph of Bove, Lewis and Parenti (1983)). The extension of their results will be considered elsewhere.

Second, if $u_t + A(D_x)u = g$ is, say, a hyperbolic equation, and $t \in \mathbf{R}$, $u\,|_{t=0} = u_0$, then $v := (u - u_0)/t$ solves a Fuchsian equation. Quite generally, Fuchsian equations arise whenever one attempts to substract off the first terms of the Taylor expansion of the unknown. This is why results on Fuchsian equations always must contain as special cases results on the Cauchy problem. Thus, the results of [10] imply a version of the Cauchy-Kowalewska theorem. The class of Fuchsian equations is also invariant under changes of variables of the form $t \mapsto t^\alpha$.

(Note also that certain classes of elliptic problems degenerating at the boundary have been studied in the context of Fuchsian equations.)

This is by no means intended to be an exhaustive list of applications of Fuchsian equations.

There are fairly complete existence results for $C^\infty$ solutions of linear Fuchsian equations due to Tahara (1978) (see also Alinhac (1974), and the references of these papers; we have not repeated references to the literature in the analytic case). Among the additional difficulties of the present situation, which are not found in the smooth, linear case let us mention two: (i) the coefficients have limited regularity, which is important, since they are related to the equation of the blow-up surface; (ii) it is not possible to obtain differentiability in the "time" variables by substracting terms from the Taylor expansion of the solution, because this causes a loss of derivatives in the coefficients—two, in the case of a second order equation—each time one term is substracted; we cannot afford this if the coefficients are not $C^\infty$.

The existence of $H^s$ solutions of Fuchsian equations should be, in principle, an adaptation of the energy method. There are however a number of new technical difficulties:

1. It is not clear from [10] whether the Fuchsian equations one has to solve can be cast in symmetric form; this requires a modification of the formal procedure, described in §2.

2. For symmetric-hyperbolic systems, one classical way of using energy estimates to prove existence in $H^s$ spaces is to use Friedrichs mollification to define approximate solutions with good regularity estimates, and then to prove their convergence in a weak norm such as $L^2$. *A priori* estimates in $H^s$, based on commutator estimates, give a solution in $L^\infty(\mathbf{R}, H^s)$. (see e.g. Taylor (1991). One must then use a special argument to prove the differentiability and continuous dependence of the solution on the data. For example, one can show the continuity in time of the $H^s$ norm and use a uniform convexity argument to deduce from it that the solution is actually in $C(\mathbf{R}, H^s)$ as desired. However, this argument uses the fact that $u_t$ is changed into $-u_t$ by time-reversal. This property does not hold for $tu_t$, which is the simplest example of the expression $Nu$ appearing in (3). Nevertheless, this argument is still useful if $l = 1$, on the set $\{t \neq 0\}$, where the equations remain symmetric-hyperbolic: one can simply divide the equation by $t$.

3. It is not possible to use the equation itself to obtain easy bounds on the time derivatives: a bound on $Nu$ does not imply a bound on all the $t$-derivatives of t$u$ if $l > 0$.

4. Another argument consists in using a Yosida regularization on the spatial derivatives, and proving the *uniform* convergence of a sequence of approximate solutions. This argument is usually more abstract, but usually requires minimal regularity assumptions, and makes continuity relatively easy to obtain, since it constructs uniformly convergent sequences of approximate solutions. It relies on density arguments for semi-groups associated with the linear part (Gronwall inequalities only give bounds, not convergence.) Since Fuchsian equations do not define semi-groups in a simple way, we find that this procedure also leads to difficulties. This is the procedure used in the survey by Kato [8];

its success is well-known, and we will adapt some of its arguments to derive energy estimates anyway.

5. It might be said that since the operator $t\,d/dt$ goes into $d/d\tau$ by the exponential change of variable $\tau = \exp t$, we are merely dealing with the large time behavior of solutions of a familiar-looking hyperbolic problem. However, the study of large-time behavior of semi-groups does not provide the existence results we need (there is some information in the parabolic case, see Pazy (1983)). On the contrary, it seems fruitful to relate asymptotic questions to degenerate initial-value problems.

6. As we pointed out earlier, the limited regularity of the coefficients creates additional difficulties; we will see on an example (§4) that the relation between regularity of the solution and that of the data is very different from the familiar case of the Cauchy problem.

Also, as was noted by Kato, one cannot expect to obtain a modulus of continuity of the solution, as a function of the initial data, even for the simplest examples; this difficulty is not special to the Fuchsian case, however.

Before motivating and discussing further the results of this paper, let us review the strategy in the analytic situation, since it will be used repeatedly in the sequel; the procedure consists in reducing the problem to the solution of a system for which $\Sigma$ is characteristic.

**1.2. The role of Fuchsian equations.**

Performing the change of variables $T = t - \psi(x)$, $X^i = x^i$, we obtain the equation

$$(1 - |D\psi|^2)u_{TT} - \Delta u + 2\psi^i \partial_i u_T + (\Delta\psi)u_T = e^u. \tag{4}$$

We now seek a solution of the form

$$\ln(2/T^2) + \sum_{j\geq 0} u^{(j)}(X)T^j$$

near $T = 0$, since $\ln(2/T^2)$ is an exact solution for $\psi \equiv 0$. One finds (see [10] for details) that

1. $\exp(u^{(0)}) = 1 - |D\psi|^2$, which forces $\Sigma$ to be space-like;

2. $u^{(0)}$ and $u^{(1)}$ must satisfy a constraint, which expresses that $\Sigma$ has vanishing scalar curvature for the (Riemannian) metric induced by Minkowski space.

3. If the curvature constraint holds, the coefficients $u^{(j)}$ can be computed recursively for $j \geq 0$, except for $u^{(2)}$ which can be chosen arbitrarily.

However, even if the curvature condition does not hold, one can show that there is always a formal solution of the form

$$\ln(2/T^2) + \sum_{j,k\geq 0} u^{(jk)}(X)T^j(T\ln T)^k, \tag{5}$$

Assuming only that $\Sigma$ is space-like.

To prove that these formal series converge if $\psi$ is analytic, one re-casts the equation into a first-order system, for which the initial value is determined by the choice of $u^{(2)}$. More precisely, the new unknown

$$z = (w, T\partial_T w, T\partial_i w),$$

where $w = [u - \ln T - u^{(0)} - u^{(1)}T]/T^2$, solves a system of the form

$$(T\partial_T + A)z = r(X) + Tf(X, T, z, D_X z). \tag{6}$$

$A$ is a constant matrix. Such a system is said to be Fuchsian, by analogy with ODEs with regular singular points. The condition that $Az(0) = r(X)$ be solvable is the curvature condition.

If the curvature condition does not hold, one writes

$$z(X, T) = q(X)\ln T + h(X, T, Y),$$

where $Y = T\ln T$. One then finds that for a suitable choice of $q$, all logarithms group themselves into powers of $Y$, and $h$ satisfies a "generalized Fuchsian equation" of the form

$$(N + A)h = Tf_1(X, T, Y, h, D_X h) + Yf_2(X, T, Y, h, D_X h), \tag{7}$$

where $N = T\partial_T + (T + Y)\partial_Y$. Since the null-space of $A$ is non-trivial, $h$ can be prescribed with some freedom for $T = Y = 0$. This translates into the arbitrariness of the coefficient of $T^2$ in the expansion of $u$. The convergence of the formal series then follows by solving the initial-value problem for (6). Note that $\Sigma$, which is defined by $T = Y = 0$, is now characteristic for this system, whereas it was not for the original equation (1).

It was also proved in [10] that the same procedure can be generalized to any singular expansion for very general PDE, provided that its first term satisfies simple properties (see §6 of [10], part 2). The solution is then an analytic function of $(x, T, T \ln T, \ldots, T(\ln T)^l)$ for some integer $l$ depending on the equation, in which case we must replace $N$ by

$$\sum_{k=0}^{l}(t_k + kt_{k-1})\partial/\partial t_k,$$

where $t_k = T(\ln T)^k$. This very general set-up gives in particular the proof of the convergence of the WTC expansions.

We now turn to the extension of these considerations to non-analytic solutions.

**1.3. Further motivation and applications.** We already mentioned that the analyticity assumption raises the question whether there is any necessary "rigidity" in the blow-up set: whether prescribing it in a portion of space determines it completely in others. As a consequence of this paper, this is not the case.

The passage from $C^\omega$ to $H^s$ has several other uses.

First, it shows a new estimate of the relation between the regularity of the blow-up surface and that of the solution: if the blow-up surface is the graph of a function of class $H^r$, $r > n/2 + 5$, then the solutions are of class $H^{r-4}$ in space; therefore, we may use the traces of this solution on a nearby space-like hypersurface as Cauchy data, and produce solutions with a prescribed blow-up surface of class $H^r$. Recall that Caffarelli and Friedman require data to be in $C^4$ and prove the $C^1$ regularity of the blow-up surface.

Second, one can prove, as will be reported elsewhere, that there is a natural mapping which, for small $\psi$, associates to the pair $\{\psi, u^{(20)}\}$ the Cauchy data $\{u, u_t\}$ on a plane $t =$const., and that this mapping is *invertible* in suitable Sobolev spaces, in the limit of small $\psi$ and $u^{(20)}$. (Here $u^{(20)}$ is one of the coefficients in the series (5), which must now be interpreted as a finite Taylor series.) This suggests strongly that the solutions that we construct are not only examples, but actually represent the blow-up behavior of all solutions with nearly constant Cauchy data. But to make this idea precise, it is imperative to have an existence theorem for Fuchsian systems in Sobolev spaces.

Next, since the "regular part" $v$ of the solution is now very smooth in $x$, the present results open up the possibility of being able to use s tandard techniques on the propagation of weak singularities to study the finer structure of blow-up. This study will also be detailed elsewhere.

Another noteworthy application of [10] was the possibility to analytically continue solutions after blow-up, by continuation of the "regular part" $v$. A similar case can be made here too: by taking $r$ large enough, we may produce solutions for which $v$ is of class $C^k$ in $(X, T, Y)$, with $k$ arbitrarily large, and which are again defined on both sides of the blow-up surface. The first terms of the series expansion (5) are still useful, since they provide the Taylor expansion of $v$.

It is then natural to ask whether one can obtain better regularity results by choosing the arbitrary function in the expansion of $v$ (namely, $u^{(20)}$) suitably. We analyze completely this question on a model example in §4; the answer is somewhat complicated, and may have some independent interest since it brings about symbols closely related to the Unterberger-Bokobza classes (see [15]).

We now describe in more detail the organization of this paper.

§2 gives the reduction of (1) to a Fuchsian system with symmetric coefficients.

§3 states and proves an existence theorem for the system introduced in §2.

§4 analyzes in detail the question of optimal regularity on a model example.

§5 contains a few further results on the case of $\Box u = u^p$ where $p$ is irrational, complementing those of [10] in the rational case, and illustrating the advantages of allowing for rather general matrices $M$ in the definition of the operator $N$ in (3).

## 2. Fuchsian systems.

We construct a Fuchsian system, the solutions of which provide singular solutions to

$$\Box u = \exp(u) \tag{8}$$

with a prescribed singular set. The result is stated as Theorem 1 at the end of the section.

Let us start with a space-like hypersurface $\Sigma := \{t = \psi(x)\}$, where $\psi \in H^r(\mathbf{R}^n)$, and let us seek a solution which blows up precisely on $\Sigma$. We know from [10] that we should assume $\Sigma$ to be space-like. We perform a reduction in three steps, leading to a generalized Fuchsian equation, solved in §3.

Step 1. First of all, we perform the change of variables $T = t - \psi(x)$, $X^i = x^i$, which transforms (7) into (4).

Letting $\gamma = 1 - |D\psi|^2$ (which is positive since $\Sigma$ is space-like) we introduce a system for a new unknown

$$\vec{u} := (u, u_0, u_i)$$

where $i$ runs from 1 to $n$, which implies (1) for $u$. This system is the usual symmetric-hyperbolic first-order system associated with (4):

$$\left\{\begin{array}{rcl} \partial_T u &=& u_0 \\ \gamma \partial_T u &=& \sum_i (\partial_i u_i - 2\psi_i \partial_i u_0) - (\Delta\psi) u_0 + e^u \\ \partial_T u_i &=& \partial_i u_0. \end{array}\right. \tag{9}$$

(We wrote $\psi_i$ for $\partial_i \psi$.)

This system can be written

$$(Q\partial_T + A^i \partial_i)\vec{u} = \varphi(X, T, u) := \begin{pmatrix} u_0 \\ e^u - (\Delta\psi) u_0 \\ 0 \end{pmatrix},$$

where $Q$ is diagonal, and the $A^i$ are symmetric:

$$Q(x) = \begin{pmatrix} 1 & & \\ & \gamma & \\ & & I_n \end{pmatrix},$$

and $A^i$, which is also an $(n+2)\times(n+2)$ matrix, has only three non-zero entries, namely

$$(A^i)_{2,2} = -2\psi_i; \quad (A^i)_{2,i+2} = (A^i)_{i+2,2} = 1.$$

STEP 2. We substract off the first terms of the formal expansion of the solution. However, the reduction of [8] described in the introduction is not appropriate, since we need the differential part to remain *symmetric.*

Let us transform (9) further by letting

$$\left\{ \begin{array}{rcllll} u & = & \ln(2/T^2) & + & u^{(0)} & + \; u^{(1)}T + vT^2 \\ u_0 & = & -(2/T) & + & u^{(1)} & + \; v_0 T \\ u_i & = & & & u_i^{(0)} & + \; v_i T, \end{array} \right. \tag{10}$$

where $\vec{v} = (v, v_0, v_i)$ is the new unknown, and

$$e^{u^{(0)}} = \gamma; \qquad u^{(1)}\gamma + \Delta\psi = 0; \qquad u_i^{(0)} = \partial_i u^{(0)}.$$

After a calculation similar to the ones in [8], system (9) becomes a Fuchsian system, which is symmetric-hyperbolic for $T \neq 0$:

$$Q[T\partial_T + A]\vec{v} = \tilde{\varphi}(X) + TA^i\partial_i\vec{v} + TF(X,\vec{v}) \tag{11}$$

where

$$\tilde{\varphi} = \begin{pmatrix} 0 \\ -2R(X) \\ \partial_i u^{(1)} \end{pmatrix}; \qquad F = \begin{pmatrix} 0 \\ b_0 \\ 0 \end{pmatrix},$$

$R$ denoting the scalar curvature of $\Sigma$ for the metric induced by Minkowski space (see [8] for its expression), and

$$A(x) = \begin{pmatrix} 2 & -1 & \\ -2 & 1 & \\ & & I_n \end{pmatrix}.$$

This matrix has eigenvalues 0, 3 and 1, with multiplicities $1, 1, n$. The function $b_0$ is given by

$$b_0(X,T,\vec{v}) = -v_0\Delta\psi + \gamma(2u^{(1)}v + Tv^2) + \gamma h(T, u^{(1)} + Tv),$$

where $h(T,z) = z^3 \int_0^1 (1-\sigma)^2 \exp[\sigma T z]\, d\sigma$.

REMARK: The principal part of (11) is simply equal to the principal part of (9) multiplied by $T$.

STEP 3. (Introduction of logarithms.)

If the curvature condition holds, $\tilde{\varphi}$ belongs to the range of $A$ and we need not go any further.

If the curvature condition fails, (11) has no continuous solution; logarithms are therefore required, and we need a more complicated formula: let $t_0 = T$, $t_1 = T \ln T$ and introduce a new unknown $\vec{w} = (w, w_0, w_i)$ by the formulae

$$\left\{ \begin{array}{lllllll} u & = & \ln(2/t_0^2) & + & u^{(0)} & + & u^{(1)} t_0 + u^{(2)} t_0 t_1 + w(t_0, t_1, X) t_0^2 \\ u_0 & = & -(2/t_0) & + & u^{(1)} & + & (t_0 + 2t_1) u^{(2)} + w_0 t_0 \\ u_i & = & & & u_i^{(0)} & + & u_i^{(1)} t_0 + w_i t_0, \end{array} \right. \tag{12}$$

where

$$u^{(2)} = -\frac{2R}{3\gamma}.$$

In other words, we are defining $\vec{w}$ by

$$\left\{ \begin{array}{lllll} v & = & w & + & u^{(2)} t_1 / t_0 \\ v_0 & = & w_0 & + & u^{(2)} (1 + 2t_1/t_0) \\ v_i & = & w_i & + & u_i^{(1)} \end{array} \right.$$

Equation (11) now takes the form

$$Q(N+A)\vec{w} = t_0 A^i \partial_i \vec{w} + t_0 g_0(X, t_0, t_1, \vec{w}) + t_1 g_1(X, t_0, t_1, \vec{w}), \tag{13}$$

where $Q$, $A$, $A^i$ are as before, and

$$g_0 = \begin{pmatrix} 0 \\ -(w_0 + u^{(2)})\Delta\psi + \Delta u^{(1)} - \sum_i 2\psi_i \partial_i u^{(2)} \\ \qquad + \gamma[h(u^{(1)} + t_0 w + t_1 u^{(2)}) + (2u^{(1)} + t_0 w + t_1 u^{(2)}) w] \\ \partial_i u^{(2)} \end{pmatrix}$$

and

$$g_1 = \begin{pmatrix} 0 \\ \gamma(2u^{(1)} + t_0 w + t_1 u^{(2)}) u^{(2)} - 2u^{(2)} \Delta\psi - 4 \sum_i \psi_i u_i^{(2)} \\ -2u_i^{(1)} \end{pmatrix}.$$

The desired singular solution is then obtained via

$$v(X,T) = w(T, T\ln T, X) + u^{(2)}(X)\ln T,$$

combined with the first line of (10).

This ends the proof of the reduction of (7) to a Fuchsian equation. We summarize the results in the following theorem:

**Theorem 1** *There are symmetric matrices $Q$ and $A^j$, and a constant matrix $A$ as well as a function $f$ such that if $t = (t_0, t_1)$, and*

$$Q(N+A)\vec{w} = t_0 A^j \partial_j \vec{w} + t \cdot f(t, X, \vec{w}), \tag{14}$$

*then the first component $w$ of $\vec{w}$ generates, via (11), a singular solution $u$ of (7) which blows up for $T=0$, provided that $\{w_i\}$ are, for $T=Y=0$, the components of the gradient of a function.*

*Furthermore, if $\psi \in H^r(\mathbf{R}^n)$, we have $Q$ and $A^j$ in $H^{r-1}_{\rm loc}$, while $f$ maps $H^{r-1}$ to $H^{r-4}$ smoothly, if $r > n/2+4$.*

Remarks. 1) The rationale in both Step 2 and Step 3 is to ensure that the new unknown appears *at the same order* in all of the equations, so that the symmetry of the principal part remains; this requires it to enter *at different orders* in the expansion of the old unknown. The application of this remark to the other nonlinearities considered in [10] will be given elsewhere.

2) The coefficients of (13) being in $H^{r-4}_{\rm loc}$ (see the expressions of $A^i$, $Q$, $g_0$ and $g_1$), we are led to seek solutions such that $\vec{w}(0,0,X) \in H^{r-4} \cap \ker(A)$. Note that it doesn't help to assume better regularity on $w$ if it is not matched by a corresponding increase in the regularity assumption on $\psi$. One could make use of the fact that $A^i$ is much more regular that $f$, but we will refrain from it, for the sake of simplicity.

3) Note that we really only need to construct $\vec{w}$ near $t=0$, for as soon as this has been done, we may revert to system (11) which is a *bona fide* symmetric hyperbolic system.

4) It is possible to reduce the generalized Fuchsian system to a Fuchsian system by considering $g(\sigma, t, X) = \vec{w}(\sigma^M t)$, which solves

$$Q(\sigma\partial_\sigma + A)g = (\sigma^M t)_0 A^j \partial_j \vec{w} + (\sigma^M t)\cdot f(t, x, \vec{w}).$$

We must solve for $g$ upto $\sigma = 1$. In this formulation, $t$ is treated as a "space" variable. One should of course truncate the equation for large $t$ if one wishes to use energy integrals; this is harmless since we are only interested in small values of $t$.

## 3. Construction of solutions in $H^s$.

We prove an existence theorem for (11).

**Theorem 2** *For $r > n/2 + 5$, one can find infinitely many solutions of (11) of the form (12), where $\vec{w}(T, T\ln T, X)$ is continuous in $T$, with values in $H^{r-4}$, for $T$ small.*

Remarks: 1) The solution is determined by the choice of the first component of $\vec{w}$ for $T = 0$. It should be taken in $H^{s-4}$. The rest of the initial data are determined by the conditions that $w_i$ should equal $\partial_i w$, and that $\vec{w}$ should belong to the range of $A$.

2) It will follow from the proofs that the function $\vec{w}$ is smooth in $(T, Y, X)$ if $\psi$ and the data are.

3) We let from now on $s = r - 4$, and all coefficients and solutions will be estimated in $H^s$. Note that $s > n/2 + 1$.

The method of proof consists in solving the initial-value problem for (13).

§3.1 introduces a system which is slightly more general than (13), and which is expected to encompass equations generated by nonlinear equations with nonlinearities other than the one in (1). The main point is the introduction of a new scalar product related to $A$.

§3.2 treats the case of bounded nonlinearities (i.e., no derivative terms on the r.h.s.).

§3.3 proves energy-type estimates.

§3.4 introduces an approximate equation which can be solved using §3.2, and shows that the estimates of §3.3 apply to it.

§3.5 proves estimates of the time derivatives of the solutions of the approximate equation.

§3.6 presents the conclusion of the argument.

### 3.1. Formulation.

We wish to solve

$$Q(N+A)u = \sum_{k=0}^{l} t_k(B_k + f_k(t,u)) := t \cdot (\mathcal{B}u + f), \tag{15}$$

where the $B_k = \sum_j A_{jk}\partial_j$ are first-order differential operators, $t = (t_0, \ldots, t_l)$, and $u$ is vector-valued. We also define $N$ by

$$N = \sum_{i,j=0}^{l} m_{ij} t_j \partial/\partial t_i,$$

$M = (m_{ij})$ being a constant matrix with real, positive eigenvalues. The system obtained in Theorem 1 is clearly of this form. This greater generality is required to encompass the blow-up problem for equations more complicated than (1). The assumptions on the coefficients and the nonlinearity as spelled out below follow closely the needs of the blow-up problem, and are therefore certainly not optimal.

We seek solutions for which $u$ is, for $t = 0$, a prescribed element of the kernel of $A$. By re-defining $u$, we may, and will, assume that

$$u(0) = 0.$$

We will denote by $(u, v)$ both the Euclidean scalar product on $\mathbf{R}^{n+2}$ and the associated $L^2$ scalar product.

Our assumptions on $Q$, $A$, $M$, $\mathcal{B}$ and $f$ are as follows:

(H1) $A$ is constant, while multiplication by $Q$, $Q^{-1}$ and $A_{jk}$ are bounded operators in $H^s$; all the eigenvalues of $A$ have nonnegative real parts.

(H2) $f$ is $C^\infty$ function in $u$ and defines a map from $\mathbf{R}^{l+1} \times H^s$ to $H^s$; furthermore, $f \equiv 0$ if $\|u\|_{L^\infty}$ or $|t|$ is large enough.

(H3) There is a positive-definite matrix-valued function $V$, which commutes with $Q$ and $A_{jk}$, and such that $(u, VQAu) \geq 0$, and $(u, VQu)$ is equivalent to the $L^2$ norm. In addition, $VA_{jk} = A_{jk}$, and multiplication by $V$ is a bounded operator in $H^s$.

(H4) The eigenvalues of $M$ are real and positive, and $M + M^T$ is positive-definite ($M^T$ being the transpose of $M$).

REMARKS: 1) It is *not* convenient to divide the equation through by $Q$ at once, since this may destroy any symmetry properties of $\mathcal{B}$.

2) For (13), we may take $V = \mathrm{diag}(2\gamma, 1, I_n)$ and $s = r - 4$. The introduction of $V$ should not be confused with the change of scalar product commonly encountered in the theory of symmetric systems: it is due here to the fact that $(Au, u)$ changes sign, even though all the eigenvalues of $A$ are nonnegative. It is therefore a special feature of the Fuchsian set-up. Without it, we would not be able to derive an energy estimate by multiplying the equation by $u$. The growth condition in (H2) can be satisfied by truncating $g_0$ and $g_1$ for large $\vec{w}$; since we are only interested in the immediate vicinity of $T = Y = 0$, where $\vec{w}$ remains small, this truncation is reasonable. The second part of (H4) is required only to prove the estimates on the time derivatives of the solution, not the solution itself. To ensure it in our case, one may replace $t_k$ by $\varepsilon^{-k}t_k$, which has the effect of replacing $m_{jk}$ by $\varepsilon^{j-k}m_{jk}$. Since $M$ is here lower triangular, we may arrange so that the off-diagonal elements become arbitrarily small, while the diagonal elements remain the same. Since the latter are positive, it follows that $M + M^T$ can be assumed to be positive-definite. This argument works for all the operators $M$ arising in [8], and more generally, as soon as $M$ is triangular with positive eigenvalues.

The other hypotheses are easy to check.

3) It is important that the equation should contain $Q(N+A)$ rather than $QN + A$, because it is easy to see that one can find $Q$ diagonal and positive-definite and $A$ with eigenvalues in the right half-plane, for which $Q^{-1}A$ has some eigenvalues with *negative* real parts. (For such a case, there would be non-trivial solutions of $(QN + A)u = 0$ with zero initial data, and there would be no hope of a reasonable existence-uniqueness theorem.)

It is convenient to measure the size of $t$ in terms of a norm invariant under the characteristic flow of $M$; such a norm is given by the following lemma, which is proved in the appendix:

**Lemma 1** *There is a function $\delta(t)$ such that*

1. *$\delta$ is continuous for all $t$, and $C^\infty$ for $t \neq 0$,*
2. *$\delta(\sigma^M t)$ increases from 0 to $\delta(t)$ as $\sigma$ increases from 0 to 1.*
3. *$\delta(\sigma^M t) \leq \delta(t)\sigma^m(1 - \theta \ln \sigma)^q$ for some positive $m$ and $q$, and some $\theta \in (0, 1)$.*

We also note that one can convert the equation into an integral equation since $\lim_{\sigma\to 0}\sigma^A u(\sigma^M t)=0$:

$$u(t)=\int_0^1 \sigma^A Q^{-1}(\mathcal{B}u(\sigma^M t)+f(\sigma^M t,u(\sigma^M t)))\frac{d\sigma}{\sigma}, \tag{16}$$

provided that this integral converges. We will seek continuous solutions to this integral equation (the solutions will usually have additional regularity properties). They will be defined on sets of the form $\{\delta(t)\le\delta_0\}$, rather than $\{|t|\le\delta_0\}$, because the latter do not form a basis of neighborhoods of the origin which remains invariant under the characteristic flow of $M$.

**3.2. Case of bounded $\mathcal{B}$.**

We consider here the case when $\mathcal{B}$ is a bounded operator. In that case, we may simply include it in $f$ and assume, as we will, that $\mathcal{B}=0$. We may similarly assume that $Q=I$.

We must therefore solve

$$u(t)=T[u], \tag{17}$$

where

$$T[u]:=\int_0^1 \sigma^A\sigma^M t\cdot f(\sigma^M t,u(\sigma^M t))\frac{d\sigma}{\sigma}.$$

We let $c=\int_0^1|\sigma^A|\sigma^{m-1}(1-\theta\ln\sigma)^q\,d\sigma$. Since $s>n/2$, $f$ is globally Lipschitz on $L^2$ or $H^s$, uniformly in $t$, and we call $L$ an upper bound for these Lipschitz constants. The assumptions imply that $T$ is contractive on $C(\delta(t)\le\delta_0;H^s)$ provided that $\delta_0<1/(Lc-1)$ ($\delta_0<\infty$ if $Lc\ge 1$). The existence of continuous solutions with values in $H^s$ follows immediately.

For higher $t$-derivatives, one considers the functions $\{u_k\}$ defined by $u_0=0$, $u_{k+1}=T[u_k]$. We then observe that the differential $T'_u$ of $T$ with respect to $u$ at $u=u_k$ is contractive on $C(\delta(t)\le\delta_0;H^s)$, so that the sequence of first-order derivatives $\{\nabla_t u_k\}$ satisfies a recurrence relation

$$\nabla_t u_{k+1}=T'_u\nabla_t u_k+\int_0^1\sigma^A[\nabla_t(\sigma^M t)\cdot f(\sigma^M t,u_k(\sigma^M t))+\sigma^M t\cdot\partial f/\partial t]\frac{d\sigma}{\sigma},$$

which has the form

$$\nabla_t u_{k+1}=T'_u\nabla_t u_k+\varphi[u_k].$$

Since we already know that $\{u_k\}$ converges at an exponential rate in $C(\delta(t) \leq \delta_0; H^s)$, we have, all norms being taken in this space,

$$\|\nabla_t u_{k+1} - \nabla_t u_k\| \leq \varepsilon \|\nabla_t u_k - \nabla_t u_{k-1}\| + C\alpha^k,$$

for some $\varepsilon$ and $\alpha$ in (0,1). It follows by induction that one can choose $a$ and $\kappa$ such that $\|\nabla_t u_k - \nabla_t u_{k-1}\| \leq a(\kappa\alpha)^k$, while $0 < \kappa\alpha < 1$. The existence of $t$-derivatives and their continuity follows.

For higher derivatives, an iteration of the same argument proves their existence and continuity, QED.

Note that because of the boundedness of the r.h.s., there is no loss of regularity due to successive time differentiations.

**3.3. Estimates.**

We consider solutions of

$$Q(N + A)u = t \cdot (Bu + f(u)),$$

where $Q$, $N$, $f$, $A$ are as in (H1)-(H4), but $B$ is subject instead to the conditions of the next theorem.

We assume $u(0) = 0$.

We prove $L^2$ and $H^s$ *a priori* bounds. They will be applied to regularized equations, where $B$ will be a smooth approximation to $\mathcal{B}$.

We define $S = (1 - \Delta)^{s/2}$. An operator $P$ is said to be bounded above if $(Pu, u) \leq C(u, u)$. It is equivalent to require $P + P^*$ to be bounded above.

**Theorem 3** *(1) If $VB + (VB)^* = C_1$ is bounded above, one has $\|u(t)\|_{L^2} \leq C\delta(t)$ for small $t$;*

*(2) If $SQ^{-1}BS^{-1} - Q^{-1}B = C_2$ is bounded above, then $\|u(t)\|_{H^s} \leq C\delta(t)$ for small $t$.*

*Further, the constants in these estimates depend only on $Q$, $N$, $f$, $A$, and the bounds on $C_1$ and $C_2$.*

*Proof:* (1) We have

$$N(u, VQu) + (u, VQAu) = t \cdot (u, VBu + Vf(u)).$$

Now, $(u, VQAu) \geq 0$ and $(u, VBu) = (u, C_1 u)/2 \leq C(u, VQu)$. Therefore the quantity, $e(\sigma) = (u(\sigma^M t), VQu(\sigma^M t))$, which is equivalent to the square of the $L^2$ norm of $u(\sigma^M t)$, satisfies

$$\sigma\partial_\sigma e \leq C\delta(t)\sigma^\beta[1 + e],$$

where $\beta > -1$. We also have $e(0) = 0$. Integrating, we find $\ln(1 + e(1)) \leq C\delta(t)$, from which the result follows, for small $t$.

(2) Let $v = Su$. We are interested in an $L^2$ estimate on $v$. Now, since $A$ is constant, $v$ solves

$$(N + A)v = t \cdot (SQ^{-1}BS^{-1}v + SQ^{-1}f(S^{-1}v)).$$

Using the assumption, and multiplying by $Q$, we find

$$Q(N + A)v = t \cdot (Bv + QC_2v + QSQ^{-1}f(S^{-1}v)).$$

Since the nonlinear term is bounded and sublinear on $L^2$, we may apply the procedure of (1) to derive an $L^2$ estimate of $v$, QED.

**3.4. Approximate equation.**

The strategy consists in approximating $\mathcal{B}$ by bounded operators. We use the Yosida regularization

$$B_{i\lambda} = \lambda(\lambda - B_i)^{-1}B_i,$$

and let $\mathcal{B}_\lambda = (B_{0\lambda}, \ldots, B_{l\lambda})$.

Since $s > n/2 + 1$ and $\mathcal{B}$ is a first-order operator with coefficients in $H^s$, we know that $\mathcal{B} + \mathcal{B}^*$ is bounded on $L^2$, and that therefore $\mathcal{B}_\lambda$ exists for $\lambda$ real and large enough.

We consider the approximate equation

$$Q(N + A)u_\lambda = t \cdot (\mathcal{B}_\lambda u_\lambda + f(u_\lambda)), \tag{18}$$

with $u_\lambda(0) = 0$. The parameter $\lambda$ is large and positive, and will eventually tend to infinity.

The existence and differentiability of $H^s$ solutions to this equation follows from §3.2. We establish here *a priori* estimates which will be used in §3.5.

We proceed therefore to check the assumptions of Theorem 3 for $\mathcal{B}_\lambda$, taking care that the operators $C_1$ and $C_2$ be bounded above uniformly in $\lambda$.

First of all, since $V$ and $A_{jk}$ commute, we have

$$(V\mathcal{B} + (V\mathcal{B})^*)_k = -\sum_j \partial_j(VA_{jk}),$$

which is a bounded function if $s > n/2 + 1$.

To obtain information on $\mathcal{B}_\lambda$, we prove the following lemmas; throughout the rest of §3.4, we will write $B$ for any of the $B_i$'s, and let $R_\lambda = (\lambda - B)^{-1}$, $B_\lambda = \lambda B R_\lambda$.

**Lemma 2** *Assume $|\lambda R_\lambda| \le C$ for $\lambda > \lambda_0$, and $(Bu, u) \le C(u,u)$ for $u$ in the domain of $B$. Then $(\lambda R_\lambda Bu, u) \le C'(u,u)$ where $C'$ is independent of $\lambda > \lambda_0$.*

*Proof:* We write

$$\begin{aligned}\lambda R_\lambda B + (\lambda R_\lambda B)^* &= \lambda B R_\lambda + \lambda R_\lambda^* B^* \\ &= \lambda R_\lambda^*(B + B^* - 2B^*B/\lambda)\lambda R_\lambda,\end{aligned}$$

so that, since $(R_\lambda^* B^* B R_\lambda u, u) = (BR_\lambda u, BR_\lambda u) \ge 0$,

$$2(\lambda R_\lambda Bu, u) \le ((B+B^*)\lambda R_\lambda u, \lambda R_\lambda u) \le C(u,u),$$

which is the desired result.

**Lemma 3** *$VB_\lambda + (VB_\lambda)^*$ is bounded above on $L^2$.*

*Proof:* We know that $VB = B$. We show that $VB_\lambda = B_\lambda$, which, using Lemma 2, will give the desired result.

Now if $B_\lambda x = y$, we have $Bx = (\lambda - B)y$, and therefore $Bx = VBx = \lambda Vy - VBy = \lambda Vy - \lambda y + Bx$, or $\lambda(y - Vy) = 0$. Since we are interested only in positive values of $\lambda$, the result follows.

**Lemma 4** *There are bounded operators $C$ and $C'$ such that*

$$SQ^{-1}BS^{-1} = Q^{-1}B + C,$$

*and*

$$SBS^{-1} = B + C'.$$

*Proof:* Both $B$ and $Q^{-1}B$ are sums of terms of the form $a_j(x)\partial_j$, with $a_j \in H^s$ or $a_j$ constant. By a well-known theorem [6, 8, 9, 14],

$$\|[S, a_j]f\|_{L^2} \le C(\|a_j\|_{\mathrm{Lip}}\|f\|_{H^{s-1}} + \|a_j\|_{H^s}\|f\|_{L^\infty}).$$

In particular, for any $u$,

$$\|[S, a_j]\partial_j S^{-1}u\|_{L^2} \le C\|u\|_{L^2}.$$

Now

$$S(a_j\partial_j)S^{-1} - a_j\partial_j = [S, a_j]\partial_j S^{-1},$$

and we just saw that the r.h.s. is a bounded operator if $a_j \in H^s$, the case of constant $a_j$ being trivial. The lemma is therefore proved.

**Lemma 5** *$SQ^{-1}B_\lambda S^{-1}$ is bounded on $L^2$.*

*Proof:* Using Lemma 4 have

$$\begin{aligned}
SQ^{-1}B_\lambda S^{-1} &= \lambda SQ^{-1}B(\lambda - B)^{-1}S^{-1} \\
&= \lambda SQ^{-1}BS^{-1}S(\lambda - B)^{-1}S^{-1} \\
&= \lambda(Q^{-1}B + C)S(\lambda - B)^{-1}S^{-1}.
\end{aligned}$$

Now (cf. [11, pp.123 and 125),

$$S(\lambda - B)^{-1}S^{-1} = (\lambda - SBS^{-1})^{-1} = (\lambda - B - C')^{-1},$$

and since

$$(\lambda - B - C')^{-1} = R_\lambda(I - C'R_\lambda)^{-1},$$

we conclude that

$$\begin{aligned}
SQ^{-1}B_\lambda S^{-1} &= (Q^{-1}B + C)\lambda R_\lambda(I - C'R_\lambda)^{-1} \\
&= Q^{-1}B_\lambda + C\lambda R_\lambda(I - C'R_\lambda)^{-1} \\
&\qquad - Q^{-1}B\lambda R_\lambda C'R_\lambda(I - C'R_\lambda)^{-1}.
\end{aligned}$$

Since $C$, $C'$, and$\lambda R_\lambda$ are bounded, it follows that $C\lambda R_\lambda(I - C'R_\lambda)^{-1}$ and is bounded for $\lambda$ large enough.

As for $Q^{-1}B\lambda R_\lambda C'R_\lambda(I - C'R_\lambda)^{-1}$, it is bounded as well because

$$B\lambda R_\lambda C'R_\lambda = (\lambda R_\lambda - 1)C'\lambda R_\lambda$$

is.

This concludes the proof of Lemma 5.

By application of Theorem 3, we find that the approximate solutions satisfy

$$\|u_\lambda(t)\|_{H^s} \leq C\delta(t).$$

We also recall for later use the following classical result:

**Lemma 6** *If $B+B^*$ is bounded, then $\lambda R_\lambda$ and $BR_\lambda = \lambda R_\lambda - 1$ are uniformly bounded for $\lambda$ large enough.*

### 3.5. Estimating the time derivatives.

We prove there energy-type estimates on the time derivatives $z_\lambda := \nabla_t u_\lambda$. They form a vector of length $(l+1)(n+2)$. In this paragraph only we use the following notation: for any matrix or operator, such as $A$, we write $A'$ for the matrix made up with $(l+1)$ diagonal blocks equal to $A$:

$$A'z = (Az_0, \ldots, Az_l).$$

$Q'$ and $\mathcal{B}'_\lambda$ are related to $Q$ and $\mathcal{B}_\lambda$ in a similar way.

Differentiating the equation satisfied by $u_\lambda$, we find that

$$Q'(N+\tilde{A})z_\lambda = t \cdot [\mathcal{B}'_\lambda z_\lambda + f_u z_\lambda + \nabla_t f] + \mathcal{B}_\lambda u_\lambda + f(u_\lambda), \tag{19}$$

where $\tilde{A}$ is the block matrix

$$\tilde{A} = \begin{pmatrix} A + m_{00} & m_{10}I & \ldots \\ m_{01}I & A + m_{11}I & \ldots \\ \ldots & \ldots & \ldots \end{pmatrix}.$$

**Lemma 7** *The eigenvalues of $\tilde{A}$ are the numbers of the form $\lambda_A + \lambda_M$ where $\lambda_A$ and $\lambda_M$ are eigenvalues of $A$ and $M$ respectively.*

*Proof:* Assume that $P^{-1}MP$ is in upper-triangular Jordan form. Let $P = ((p_{ij}))$ and

$$\tilde{P} = \begin{pmatrix} p_{00}I & p_{10}I & \ldots \\ p_{01}I & p_{11}I & \ldots \\ \ldots & \ldots & \ldots \end{pmatrix}.$$

Then $\tilde{P}^{-1}\tilde{A}\tilde{P}$ is block-diagonal with blocks of the form

$$\begin{pmatrix} A+\lambda_M & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & A+\lambda_M \end{pmatrix}.$$

The lemma follows.

In particular, $\tilde{A}$ is invertible, and we may define uniquely $z_\lambda(0)$, which is bounded in $H^s$ (set $t=0$ in the equation, and use $u_\lambda(0) = 0$.). Substracting it from $z_\lambda$ and calling again the difference $z_\lambda$ for convenience, we end up with a system of the form (13), but with coefficients bounded in $H^{s-1}$: indeed, since we already have $\|u_\lambda(t)\|_{H^s} \le C\delta(t)$, we know that

$$\|\mathcal{B}_\lambda u_\lambda + f(u_\lambda) - f(0)\|_{H^s} \le C\delta(t)$$

as well, and we are therefore assured that the r.h.s. of the equation for $z_\lambda$ is $O(t)$. This will play the role of (H2) in the sequel.

To prove energy estimates on $z_\lambda$, we need to check the rest of (H1)–(H4) for (19). Only (H3) requires a separate argument. We claim that we may satisfy (H3) by using the the (positive-definite) matrix $V'$. Let $((\tilde{A}_{qr}))$ be the block decomposition of $\tilde{A}$:

$$\tilde{A}_{qr} = A + m_{rq}I.$$

Recall also that $(u, VQAu) \ge 0$ for any $u$. We must estimate from below

$$\begin{aligned}
(V'Q'\tilde{A}z, z) &= \sum_{q,r}(z_q, VQ\tilde{A}_{qr}z_r) \\
&= \sum_{q,r}(z_q, VQAz_q + VQm_{rq}z_r) \\
&\ge \sum_{q,r}\frac{1}{2}[m_{rq} + m_{rq}]((VQ)^{1/2}z_q, (VQ)^{1/2}z_r).
\end{aligned}$$

The result follows, since $M + M^T$ is positive-definite.

Applying the procedure of §3.3, we find that

$$\|\nabla_t u_\lambda\|_{H^{s-1}} \le C\delta(t)$$

uniformly in $\lambda$.

**3.6. End of proof.**

Recall that by translation of $u$, we have assumed $u_0 = 0$. Its regularity is therefore incorporated with the regularity assumptions on $f$.

We must pass to the limit $\lambda \to \infty$.

From the bounds of $u_\lambda$ in $C(\delta(t) \le \delta_0; H^s) \cap C^1(\delta(t) \le \delta_0; H^{s-1})$, we obtain the existence of a solution in $C(\delta(t) \le \delta_0; H^{s-1})$, by application of Ascoli's theorem. The *a priori* estimate in $H^s$ also implies that the solution is continuous at $t = 0$, in the $H^s$ topology, and is uniformly bounded in $H^s$.

Let us now turn to systems (11) and (13).

We now have a solution $w$ of (13) defined for small $(t_0, t_1)$, which means that we know that $w(X, T, T \ln T)$ in $H^s$ is continuous in $T$, with values in $H^{s-1}$, and which is bounded, and tends to zero as $T \to 0$, in the $H^s$ topology. Now, we may substitute back $t_0 = T$ and $t_1 = T \ln T$ into (13), which gives a symmetric-hyperbolic system for $T \neq 0$, similar to (11). For such a system, it is well-known that solutions which startin $H^s$ are continuous in time, with values in $H^s$, and therefore $w(X, T, T \ln T)$ is continuous at every $T \neq 0$, with values in $H^s$. Combining this with the estimates neat $t = 0$, we conclude that the solution is continuous for all $T$ with values in $H^s$. From the definition of $w$, it follows that $v$ has the same continuity properties. This completes the proof of Theorem 3.

Remark: In case all coefficients are $C^\infty$, one could derive $H^s$ estimates on higher-order derivatives, for all $s$, thereby proving the existence (and uniqueness, in the natural way) of a solution of class $C^\infty$ in $(t_0, \ldots, t_l, X)$. The domain of existence does not shrink with increasing $s$, by a standard continuation argument ($C^1$ bounds are sufficient for continuation). Finally, we note that in the case of (11) or (13), the matrices $A_{jk}$ are actually of class $H^{s+3}$, which might be used to gain one derivative when the curvature condition holds. We refrain however from pursuing such technicalities.

## 4. Regularity questions.

We analyze in more detail the situation for the one-dimensional model problem

$$u_t - u_x = \frac{u - u_0}{t}; \quad u(0) = u_0, \tag{20}$$

where $u_0 \in H^s$ is given. We prove that this problem can be reduced to a Fuchsian system to which the results of §3 apply, but that the solutions with optimal regularity are not obtained by taking the initial value for this system to be in $H^s$. The correct result is surprisingly complicated, and we propose an explanation in §4.4.

Our argument is in three steps, summarized in the following theorems:

First, as we show in §4.1 and §4.2,

**Theorem 4** *There are infinitely many solutions which are $H^s$ with respect to $s$. Furthermore, if $u_0$ is analytic,*

$$u = u_0 + \sum_{j\geq 1}(a_j(x) + b_j(x)\ln t)t^j \tag{21}$$

*where $a_1$ is arbitrary, and the other coefficients can be found inductively; in particular, $b_1 = u_{0x} := \partial_x u_0$.*

Next, we interpret the coefficient $a_1$ as an initial value for the Fuchsian system

$$\begin{cases} t\lambda_t + \mu & = & t\lambda_x + u_{0x}, \\ t\mu_t & = & t\mu_x \qquad . \end{cases} \tag{22}$$

More precisely,

**Theorem 5** *If $(\lambda, \mu)$ is a solution of (22) with*

$$\lambda(x,0) = a_1(x) \quad \text{and} \quad \mu(x,0) = u_{0x},$$

*then $u = u_0 + t(\lambda + \mu \ln t)$ solves (20).*

(see §4.3).

Now, system (22) has r.h.s. and initial conditions in $H^{s-1}$, and it would seem impossible to hope that the solution be better that $H^{s-1}$ for $t \neq 0$. *There is however a combination of $\lambda$ and $\mu$ which is in $H^s$ for every $t$:*

**Theorem 6** *For any $u_0 \in H^s$, (22) has infinitely many solutions such that $t\lambda + \mu t \ln t$ is in $H^s$ for every $t$. Such solutions are precisely the solutions of the initial-value problem for (22) with*

$$\lambda(x,0) - k\left(\frac{\partial}{\partial x}\right)u_0 \in H^s \quad \text{and} \quad \mu(x,0) = u_{0x},$$

*where $k(i\xi) = i\xi(\gamma + \ln(i\xi))$, $\gamma$ being Euler's constant.*

If $s$ is large enough, $u = u_0 + t(\lambda + \mu \ln t)$ would admit formal expansions of the form (21), truncated after a finite number of terms, with coefficients determined by the choice of $a_1$; however, one should *not* take $a_1 \in C^\infty$ to obtain maximum smoothness in $u$; rather, it should be chosen in the class $k(\partial/\partial x)u_0 + H^s$.

The proof of these statements follows.

**4.1. The general solution.**

We prove Th. 4.

Let us Fourier transform in $x$. We find

$$t(\hat{u}_t - i\xi\hat{u}) = \hat{u} - \hat{u}_0, \tag{23}$$

or

$$(t^{-1}e^{-i\xi t}\hat{u})_t = -t^{-2}e^{-i\xi t}\hat{u}_0. \tag{24}$$

The general solution therefore has the form

$$\hat{u}(\xi,t) = te^{i\xi t}\hat{\alpha}(\xi) + \left(t\int_t^\infty s^{-2}e^{i\xi(t-s)}\,ds\right)\hat{u}_0(\xi) \tag{25}$$

for $t > 0$, where $\alpha(\xi)$ is arbitrary. This can be rewritten

$$\hat{u} = (t\alpha(x+t))\hat{} + g(t)\hat{u}_0(\xi) \tag{26}$$

where

$$g(t,\xi) = \int_0^\infty \frac{t}{(t+\tau)^2}e^{-i\xi\tau}\,d\tau = \int_1^\infty \sigma^{-2}e^{i\xi t(1-\sigma)}\,d\sigma. \tag{27}$$

From the second form of $g$, it is apparent that our problem has infinitely many solutions that are in $H^s$ for each $t > 0$, if $u_0 \in H^s$; more precisely,

$$||u(t)||_s \leq ||u_0||_s + t||\alpha||_s. \tag{28}$$

This proves the first part of Th. 4.

For the second, we let $v = (u - u_0)/t$ so that $v$, too, solves a Fuchsian equation:

$$t(v_t - v_x) = u_{0x}. \tag{29}$$

where $u_{0x} = \partial_x u_0$. This equation has no solution of class $C^1$ unless $u_0$ is constant. On the other hand, from the study of the analytic case, that if $u_0$ is analytic, then $tv$ is an analytic function of $x$, $t$ and $t\ln t$. It is easy to show by substitution that (29) has infinitely many *formal* solutions of the form

$$\sum_{j\geq 1}\frac{1}{t}(a_j + b_j\ln t)t^j.$$

The coefficients $a_j$ and $b_j$ can be found recursively if $u_0$ and $a_1$ are given. One finds in particular $b_1 = u_{0x}$. The convergence follows from [8, Part II].

Theorem 4 is proved.

### 4.2. Associated Fuchsian system.

We now introduce a Fuchsian system which implies (29). To this end, let $v = \lambda(x,t) + \mu(x,t)\ln t$. The equation for $v$ becomes

$$t(\lambda_t - \lambda_x) + t\ln t(\mu_t - \mu_x) + \mu = u_{0x}.$$

Let us therefore consider the Fuchsian system

$$\begin{cases} t\lambda_t + \mu & = \quad t\lambda_x + u_{0x}, \\ t\mu_t & = \quad t\mu_x. \end{cases} \tag{30}$$

Theorem 5 is now checked by a straightforward calculation.

The issue has been reduced to describing the regularity of the solution of (22) with

$$\lambda(x,0) = a_1(x) \quad \text{and} \quad \mu(x,0) = u_{0x}.$$

### 4.3. Optimal regularity.

We show that the solution will not have the optimal regularity, namely $H^s$ in the space variable, if $a_1$ is $C^\infty$; one must instead take $a \in H^{s-2}$ of a very particular form, as given in Theorem 6.

To show this, we first evaluate $g(t)$ more precisely. The Laplace transform of $(t+\tau)^{-2}$ with respect ot $\tau$ is, for $t > 0$,

$$p \mapsto \frac{1}{t} - pe^{pt}E_1(pt),$$

where $E_1(z) = -\gamma - \ln z - \sum_{n\geq 1}(-1)^n z^n/(n.n!)$ is the exponential-integral function (also equal to $\int_z^\infty t^{-1}e^{-t}\,dt$; $\gamma$ is Euler's constant, and we are taking $|\arg z| < \pi$ with the principal determination of the logarithm). Therefore, we find

$$g(t) = 1 + i\xi t e^{i\xi t}(\gamma + \ln(i\xi t) + \sum_{n\geq 1}\frac{(-i\xi t)^n}{n.n!}). \tag{31}$$

Since $t > 0$, we may expand this as

$$g(t) = 1 + t\{i\xi\ln t + i\xi(\gamma + \ln(i\xi)) + O(t\ln t)\}.$$

This provides an expression for the most general solution in $H^s$.

Let us now assume $u = u_0 + t(\lambda + \mu \ln t)$, where $(\lambda, \mu)$ solve (22) with initial data $(a, u_{0x})$. From the equation for $\mu$, we find

$$\mu(x,t) = u_{0x}(x+t).$$

Therefore $\hat{\mu} = i\xi \hat{u}_0 e^{-it\xi} = i\xi\hat{u}_0(1+O(t))$ as $t \to 0$ for fixed $\xi$.

We can now compute $a_1$ for this solution (27): as $t \to 0$,

$$\frac{\hat{u} - \hat{u}_0}{t} = e^{i\xi t}\hat{\alpha}(\xi) + \frac{g(t)-1}{t}\hat{u}_0 = i\xi\hat{u}_0 \ln t + \hat{\alpha} + k(i\xi)\hat{u}_0 + o(1),$$

where $k(i\xi) = i\xi(\gamma + \ln(i\xi))$. Observe that $k$ is not a classical symbol. Using the expansion of $\hat{\mu}$, we compute $\hat{\lambda}$ and conclude that $\hat{a}_1 = \hat{\alpha} + k(i\xi)\hat{u}_0$.

Since $\alpha$ must be in $H^s$ for estimate (28) to hold, we see that $a_1$ cannot be in $H^s$. Rather,

$$a_1 - k(\partial/\partial x)u_0 \in H^s.$$

This is the conclusion of Th. 6.

**4.4. Why the formal procedure fails.**

The restriction on the arbitrary function occurring in the general solution is missed by the formal calculation in powers of $t$ and $t \ln t$. Let us show that this is natural, this restriction being of a "global" nature (similar to matching conditions.)

Set $\rho = \xi t$. We find

$$\rho(\hat{u}_\rho - i\hat{u}) = \hat{u} - \hat{u}_0(\xi).$$

All solutions of this equation are bounded near $\rho = 0$. They depend on one parameter (here, $a_1$ essentially). Any two solutions differ by a multiple of $\rho e^{i\rho}$; thus, at most one solution can remain bounded for all $\rho$.

Let us precisely require that

$$|\hat{u}| \leq C|\hat{u}_0|$$

for all $\rho$. This singles out the solution

$$\int_1^\infty \sigma^{-2} e^{i\rho(1-\sigma)}\, d\sigma\, u_0(\xi).$$

If we expand this solution in the form (21), there is no reason why the coefficient $a_1$ should turn out to be zero. In fact, is it easy to see that this coefficient contains the expression $k(\partial_x)u_0$ of Th. 6.

## 5. Remarks on power nonlinearities.

We briefly include here a few further results on the case

$$\Box u = u^p$$

where $p$ is still greater than 1, but is not assumed to be rational as in [3]. We show that a formal solution can also be constructed in this case, for arbitrary blow-up surfaces, with one arbitrary function; however, the solution is now analytic in $X, T$, and $T^{1/(p-1)}$. This will bring into play an operator $N$ of a slightly more general form than that used in the exponential case, thereby justifying the generality of the present set-up. We also include brief remarks on the global behavior of the blow-up surface in low dimensions.

Remark: If we write $1/(p-1) = 1 + \varepsilon$, we may think of $T^{1/(p-1)}$ as

$$\sum_{k \geq 0} T(\varepsilon \ln T)^k / \, k!,$$

so that the solution may be said, loosely speaking, to involve "infinitely many logarithms," as opposed to finitely many when $p$ is rational. The formal solution represents of course the Taylor expansion of the solution of a Fuchsian equation, as usual.

### 5.1. Formal series.

We consider solutions of

$$\Box u = c_p u^p,$$

where $c_p = 2(p+1)/(p-1)^2$.We seek again solutions which are singular for $t = \psi(x)$. We are therefore led to the equation

$$\gamma u_{TT} - \Delta u + 2\psi^i \partial_i u_T + (\Delta u) u_T = c_p u^p.$$

Letting $Y = T^{1/(p-1)}$ and $u = v(T, Y, X)Y^{-2}$, we find, after some calculation, a generalized Fuchsian equation of the form

$$\gamma[(N - 2\mu)^2 - (N - 2\mu)]v + O(T) = c_p v^p,$$

where $N = T\partial_T + \mu Y \partial_Y$, $\mu = 1/(p-1)$. We have lumped as $O(T)$ all the terms which will not be essential in the sequel. For $T = Y = 0$, we find $v = v_0 := \gamma^\mu$. If we seek $v$ as a power series in $T$ and $Y$, we find that since each monomial $T^k Y^l$ is simply transformed by $N$ into $(k + l\mu)T^k Y^l$, we may find the coefficients of this series recursively provided that

$$(k - (l-2)\mu)^2 - (k - (l-2)\mu) - 2(\mu + 1)(2\mu + 1) \neq 0.$$

This condition fails precisely when $k + l\mu = 2 + 4\mu$ or $-1$. If $p$ is irrational, this corresponds to one term only for the expansion of $u$. The exact form of this series will be studied elsewhere.

**5.2. The asymptotic behavior of the blow-up surface.**

Let us consider (1) again, and limit ourselves to two space dimensions. We prove that the solution cannot be free of logarithms if the Cauchy data on $t = 0$ are regular and compactly supported.

Indeed, in that case, the blow-up surface must be spacelike, ruled (zero scalar curvature), and limited by $\{t = 0\}$. Furthermore, by finite speed of propagation, it is limited below by a translate of the light cone. This forces any generatrix to lie on the cone, thus meeting $\{t = 0\}$. This contradicts the fact that there are no singularities on the initial surface. Thus, the curvature condition, when combined with a support condition, actually sets *global* constraints on the blow-up surface.

## 6. Appendix.

We prove Lemma 1. The argument is similar to that of [10, p.1886].

Assume that a linear change of variables has been performed to put $M$

in block diagonal form, with blocks of the form

$$\begin{pmatrix} \lambda_r & & & & \\ 1 & \lambda_r & & & \\ & 2 & \ddots & & \\ & & \ddots & \ddots & \\ & & & l_r & \lambda_r \end{pmatrix}.$$

with $\lambda_r > 0$. The integer $l_r$ may be zero, in the case of a $1 \times 1$ block. (It is convenient not to take the off-diagonal elements equal to 1.) Let us label $(t_{r0}, t_{r1}, \ldots, t_{r,l_r})$ the components of $t$ corresponding to the $r$th block.

We define

$$\delta(t) = \sum_r \left( \sum_{k=0}^{l_r} \theta^k |t_{rk}| \right).$$

We prove that one can choose $\theta > 0$ so that $\delta(t)$ has the required properties.

First, by solving the equation $\sigma \partial t / \partial \sigma = Mt$, one shows that

$$(\sigma^M t)_{rk} = \sum_{j=0}^{k} t_{rj} \binom{k}{j} \sigma^{\lambda_r} (\ln \sigma)^{k-j}$$

for $k \leq l_r$.

We now compute, for $\sigma \in (0,1)$,

$$\begin{aligned}
\delta(\sigma^M t) &= \sum_{k,r} \theta^k |(\sigma^M t)_{rk}| \\
&\leq \sum_r \sum_{0 \leq j \leq k \leq l_r} \sigma^{\lambda_r} |t_{rj}| \theta^k \binom{k}{j} (\ln \sigma)^{k-j} \\
&\leq \sum_{0 \leq j \leq k} \sigma^{\lambda_r} \binom{k}{j} [-\theta \ln \sigma]^{k-j} \theta^j |t_{rj}| \\
&= \sum_{0 \leq j \leq k} \sigma^{\lambda_r} \binom{k}{j} [-\theta \ln \sigma]^{j} \theta^{k-j} |t_{r,k-j}| \\
&\leq \sum_r \sum_{j=0}^{l_r} \binom{l_r}{j} [-\theta \ln \sigma]^j \sum_{k \geq j} \theta^{k-j} |t_{r,k-j}| \sigma^{\lambda_r} \\
&\leq \sum_r \sigma^{\lambda_r} \delta(t) (1 - \theta \ln \sigma)^{l_r}.
\end{aligned}$$

(We used the fact that $\binom{k}{j} \le \binom{l}{j}$ if $k \le l$).

If we choose $m$ such that $0 < m \le \lambda_r$ for all $r$, and estimate $l_r$ by, say $q$, we find that

$$\delta(\sigma^M t) \le \delta(t)\sigma^m(1-\theta\ln\sigma)^q,$$

We finally choose $\theta \in (0, m/q)$, to ensure that $\sigma^m(1-\theta\ln\sigma)^q$ be increasing for $\sigma \in (0,1)$.

All the desired properties follow.

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