MARTIN CAPACITY FOR MARKOV CHAINS AND RANDOM WALKS IN VARYING DIMENSIONS

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1 Introduction

Kakutani (1944) discovered that a compact set $\Lambda \subseteq \mathbb{R}^d$ is hit with positive probability by a d-dimensional Brownian motion ($d \geq 3$) if and only if Λ has positive Newtonian capacity. When capacity criteria were transferred to the discrete setting (by Ito and Mckean (1960) and Lamperti (1963)) it was in the form of a "Wiener Test" (c.f. Corollary 2.4). This kind of summability condition is quite effective in deciding whether a given subset of a lattice is hit infinitely often by a random walk, but does not yield estimates of the probability of ever hitting the set. Such estimates are obtained from the discrete analogue of the following.

Proposition 1.1 *Let $\{B_d(t)\}$ denote standard d-dimensional Brownian motion with $B_d(0) = \mathbf{0}$ and $d \geq 3$. Let $\Lambda \in \mathbb{R}^d$ be any compact set. Then*

$$\frac{1}{2}\text{Cap}_K(\Lambda) \leq \mathbf{P}[\exists t > 0 : B_d(t) \in \Lambda] \leq \text{Cap}_K(\Lambda) \tag{1}$$

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where

$$K(x,y) = \frac{||y||^{d-2}}{||x-y||^{d-2}}$$

for $x, y \in \mathbb{R}^d$*. Here* $||x-y||$ *is the Euclidean distance and*

$$\mathrm{Cap}_K(\Lambda) = \left[\inf_{\mu(\Lambda)=1} \int_\Lambda \int_\Lambda K(x,y)\, d\mu(x)\, d\mu(y) \right]^{-1}.$$

Remarks:

1. More detailed definitions will be given later.

2. The constant $1/2$ in (1) is sharp.

The classical criterion $\mathbf{P}[\exists t > 0 \,:\, B_d(t) \in \Lambda] > 0 \Leftrightarrow \mathrm{Cap}_G(\Lambda) > 0$, where $G(x,y) = ||x-y||^{2-d}$, is clearly contained in Proposition 1.1 ; passing from the Green kernel $G(x,y)$ to the Martin kernel $K(x,y) = G(x,y)/G(0,y)$ yields sharper estimates which are then useful in a discrete setting. By applying such estimates to an appropriate space-time Markov chain, we obtain criteria for a set A of integers to contain infinitely many times of return of a random walk to the origin (Corollary 2.5); in particular, an infinite expected number of returns is not sufficient (Examples 1 and 2 below); this is well-known. Lamperti's Wiener Test and a theorem of Lyons about percolation on trees are also obtained as corollaries.

Our initial motivation for these criteria was understanding *random walks in varying dimension.* Let F_2 and F_3 be two distributions with mean zero and finite variance on the lattice $\mathbf{Z}^3$, where F_2 is supported on a plane (but not on any line) and $\mathrm{supp}(F_3)$ is not contained in any plane. Given an increasing sequence of positive integers $\{a_n\}$ we consider the inhomogeneous random walk $\{S_k\}$ whose independent increments $S_k - S_{k-1}$ have distribution F_3 if $k \in \{a_n\}$ and distribution F_2 otherwise. Theorem 5.4 shows that the process $\{S_k\}$ is *recurrent* if

$$a_n \approx \exp(\exp(n^{1/2}))$$

but *transient* if $a_n \approx \exp(\exp(n^\theta))$ for any $\theta \in (0, 1/2)$. Here *recurrence* means that the number of k for which $S_k = \mathbf{0}$ is almost surely infinite, and *transience* means that this number is almost surely finite. (these alternatives are exhaustive, cf. Lemma 5.1.) An easy calculation shows that the expected number of visits to the origin by $\{S_k\}$ is infinite when $\theta < 1/2$ as well as when $\theta = 1/2$.

We also consider variants in other dimensions. For instance, there exists a recurrent random walk which interlaces two-dimensional, four-dimensional and six-dimensional steps (but the four-dimensional steps are indispensible here; see Corollary 5.3). Conversely, there is a transient process obtained by alternating blocks of one-dimensional and two-dimensional random walk steps (Proposition 6.1). Durrett, Kesten and Lawler (1991) analyze a random walk in one dimension that interlaces several increment distributions all having mean zero. In that setting, distributions without second moments are necessary in order to obtain transience. Another inhomogeneous model was analyzed by D. Scott (1990).

The rest of this paper is organized as follows. Martin capacity for Markov chains is the focus of Section 2. Several examples are given, including an interesting relation between simple random walk in three dimensions and the time-space chain arising from simple random walk in the plane. Section 3 shows how to derive Lyons' percolation Theorem from the general capacity estimate for Markov chains (Theorem 2.2.) In Section 4 we give the easy proofs of Proposition 1.1 and related results concerning Brownian motion. Random walks in varying dimension are analyzed in Section 5. This section is written so it can be read independently of the rest of the paper. However, it is connected to the previous sections both in the methods of proof and in that in both settings the number of returns to the origin can be almost surely finite but have infinite expectation. Finally, Section 6 contains the examples of transient walks which interlace one-dimensional and two-dimensional steps.

2 Polarity for Markov chains

First we recall some potential theory notions.

Definition 2.1 *Let Λ be a set and $\mathcal{B}$ a σ-field of subsets of Λ. Given a measurable function $F : \Lambda \times \Lambda \to [0, \infty]$ and a finite measure μ on $(\Lambda, \mathcal{B})$, the F-energy of μ is*

$$I_F(\mu) = \int_\Lambda \int_\Lambda F(x, y)\, d\mu(x)\, d\mu(y).$$

The capacity of Λ in the kernel F is

$$\mathrm{Cap}_F(\Lambda) = \left[\inf_\mu I_F(\mu)\right]^{-1}$$

where the infimum is over probability measures μ on $(\Lambda, \mathcal{B})$ and by convention, $\infty^{-1} = 0$.

If Λ is contained in Euclidean space, we always take $\mathcal{B}$ to be the Borel σ-field; if Λ is countable, we take $\mathcal{B}$ to be the σ-field of all subsets. When Λ is countable we also define the *asymptotic capacity* of Λ in the kernel F:

$$\mathrm{Cap}_F^{(\infty)}(\Lambda) = \inf_{\Lambda_0 \text{ finite}} \mathrm{Cap}_F(\Lambda \setminus \Lambda_0). \tag{2}$$

Let $\{p(x, y) : x, y \in Y\}$ be transition probabilities on the countable set Y, i.e. $\sum_y p(x, y) = 1$ for every $x \in Y$. Let $\rho \in Y$ be a distinguished starting state and let $\{X_n : n \geq 0\}$ be a Markov chain with $\mathbf{P}[X_{n+1} = y \,|\, X_n = x] = p(x, y)$.

Define the Green function

$$G(x, y) = \sum_{n=1}^{\infty} p^{(n)}(x, y) = \sum_{n=1}^{\infty} \mathbf{P}_x[X_n = y]$$

where $p^{(n)}(x, y)$ are the n-step transition probabilities and $\mathbf{P}_x$ is the law of the chain $\{X_n : n \geq 0\}$ when $X_0 = x$. We assume that the Markov chain $\{X_n\}$ is transient or equivalently that $G(x, y) < \infty$ for all $x, y \in Y$, and want to estimate the probability that a sample path $\{X_n\}$ hits a set $\Lambda \subseteq Y$.

Theorem 2.2 *Let $\{X_n\}$ be a transient Markov chain on the countable state space Y with initial state ρ and transition probabilities $p(x, y)$. For any subset Λ of Y we have*

$$\frac{1}{2}\mathrm{Cap}_K(\Lambda) \leq \mathbf{P}_\rho[\exists n \geq 0 : X_n \in \Lambda] \leq \mathrm{Cap}_K(\Lambda) \tag{3}$$

and

$$\frac{1}{2}\mathrm{Cap}_K^{(\infty)}(\Lambda) \leq \mathbf{P}_\rho[X_n \in \Lambda \textit{ infinitely often }] \leq \mathrm{Cap}_K^{(\infty)}(\Lambda) \tag{4}$$

where K is the Martin kernel

$$K(x,y) = \frac{G(x,y)}{G(\rho,y)} \tag{5}$$

defined using the intial state ρ.

Remarks:

1. The Martin kernel $K(x,y)$ can obviously be replaced by the symmetric kernel $\frac{1}{2}(K(x,y) + K(y,x))$ without affecting the energy of measures or the capacity of sets.
2. If the Markov chain starts according to an initial measure π on the state space, rather than from a fixed initial state, the theorem may be applied by adding an abstract initial state ρ with transition probabilities $p(\rho,y) = \pi(y)$ for $y \in Y$.

Proof: (i) The right hand inequality in (3) follows from an entrance time decomposition. Let τ be the first hitting time of Λ and let ν be the (possibly defective) hitting measure $\nu(x) = \mathbf{P}_\rho[X_\tau = x]$ for $x \in \Lambda$. Then

$$\nu(\Lambda) = \mathbf{P}[\exists n \geq 0 : X_n \in \Lambda] \tag{6}$$

and

$$\int G(x,y)\, d\nu(x) = \sum_{x\in\Lambda} \mathbf{P}_\rho[X_\tau = x] G(x,y) = G(\rho,y).$$

Therefore $\int K(x,y)\, d\nu(x) = 1$ for every $y \in \Lambda$. Consequently

$$I_F\left(\frac{\nu}{\nu(\Lambda)}\right) = \nu(\Lambda)^{-2} I_F(\nu) = \nu(\Lambda)^{-1},$$

so that $\mathrm{Cap}_K(\Lambda) \geq \nu(\Lambda)$. By (6), this proves half of (3).

To establish the left hand inequality in (3) we use the second moment method. Given a probability measure μ on Λ, consider the random variable

$$Z = \int_\Lambda G(\rho,y)^{-1} \sum_{n=1}^{\infty} \mathbf{1}_{\{X_n = y\}}\, d\mu(y).$$

By Tonelli and the definition of G,

$$\mathbf{E}_\rho Z = 1. \tag{7}$$

Now we bound the second moment:

$$\begin{aligned} \mathbf{E}_\rho Z^2 &= \mathbf{E}_\rho \int_\Lambda \int_\Lambda G(\rho,y)^{-1} G(\rho,x)^{-1} \sum_{m,n=0}^{\infty} \mathbf{1}_{\{X_m=x, X_n=y\}} \, d\mu(x)\, d\mu(y) \\ &\leq 2\mathbf{E}_\rho \int_\Lambda \int_\Lambda G(\rho,y)^{-1} G(\rho,x)^{-1} \sum_{0\leq m\leq n<\infty} \mathbf{1}_{\{X_m=x, X_n=y\}} \, d\mu(x)\, d\mu(y). \end{aligned}$$

For each m we have

$$\mathbf{E}_\rho \sum_{n=m}^{\infty} \mathbf{1}_{\{X_m=x, X_n=y\}} = \mathbf{P}_\rho[X_m = x] G(x,y).$$

Summing this over all $m \geq 0$ yields $G(\rho,x)G(x,y)$ and therefore

$$\mathbf{E}_\rho Z^2 \leq 2 \int_\Lambda \int_\Lambda G(\rho,y)^{-1} G(x,y) \, d\mu(x)\, d\mu(y) = 2 I_K(\mu).$$

By Cauchy-Schwartz and (7),

$$\mathbf{P}_\rho[\exists n \geq 0 : X_n \in \Lambda] \geq \mathbf{P}_\rho[Z > 0] \geq \frac{(\mathbf{E}_\rho Z)^2}{\mathbf{E}_\rho Z^2} \geq \frac{1}{2 I_K(\mu)}.$$

Since the left hand side does not depend on μ we conclude that

$$\mathbf{P}_\rho[\exists n \geq 0 : X_n \in \Lambda] \geq \frac{1}{2} \mathrm{Cap}_K(\Lambda)$$

as claimed. $\square$

To infer (4) from (3) observe that since $\{X_n\}$ is a transient chain, almost surely every state is visited only finitely often and therefore

$$\{X_n \in \Lambda \text{ infinitely often }\} = \bigcap_{\Lambda_0 \text{ finite}} \{\exists n \geq 0 : X_n \in \Lambda \setminus \Lambda_0\} \quad \text{a.s.}$$

Applying (3) and the definition (2) of asymptotic capacity yields (4). $\square$

The remainder of this section is devoted to deriving some consequences of Theorem 2.2. The first involves the notion of *intersection-equivalence*. Say that two random subsets W_1 and W_2

of a countable space are intersection-equivalent (or that their laws are intersection-equivalent) if for every subset A of the space, $\mathbf{P}[W_1 \cap A \neq \emptyset]$ and $\mathbf{P}[W_2 \cap A \neq \emptyset]$ are bounded by constant multiples of each other. It is easy to see that if W_1 and W_2 are intersection-equivalent then $\mathbf{P}[|W_1 \cap A| = \infty]$ and $\mathbf{P}[|W_2 \cap A| = \infty]$ are also bounded by the same constant multiples of each other. An immediate corollary of Theorem 2.2 is the following, one instance of which is given in Corollary 2.6.

Corollary 2.3 *Suppose the Green's functions for two Markov chains on the same state space are bounded by constant multiples of each other. Then their ranges are intersection-equivalent.*

Lamperti (1963) gave an alternative criterion for $\{X_n\}$ to visit the set Λ infinitely often. Fix $b > 1$. With the notations of Theorem 2.2, denote $Y(n) = \{x \in Y : b^{-n-1} < G(\rho, x) \leq b^{-n}\}$.

Corollary 2.4 (Lamperti's Wiener Test) *Assume that the set $\{x \in Y : G(\rho, x) > 1\}$ is finite and that for some constant C and all $x \in Y(m)$ and $y \in Y(m+n)$ we have*

$$G(x, y) < Cb^{-(m+n)}, \tag{8}$$

provided that m and n are sufficiently large. Then

$$\mathbf{P}[X_n \in \Lambda \text{ infinitely often}] > 0 \Leftrightarrow \sum_{n=1}^{\infty} b^{-n}\mathrm{Cap}_G(\Lambda \cap Y(n)) = \infty. \tag{9}$$

Remark: Clearly $\sum_{n=1}^{\infty} b^{-n}\mathrm{Cap}_G(\Lambda \cap Y(n)) = \infty$ if and only if $\sum_n \mathrm{Cap}_K(\Lambda \cap Y(m)) = \infty$. The equivalence then follows from a version of the Borel-Cantelli lemma proved in Lamperti's paper (a better proof is in Kochen and Stone (1964)). This corollary is useful in many cases; however the condition (8) excludes some natural transient chains such as simple random walk on a binary tree.

Next, we deduce a criterion for a recurrent Markov chain to visit its initial state infinitely often within a prescribed time set.

Corollary 2.5 *Let $\{X_n\}$ be a recurrent Markov chain on the countable state space Y with initial state $X_0 = \rho$ and transition probabilities $p(x, y)$. For nonnegative integers $m \le n$ denote*

$$\tilde{G}(m, n) = \mathbf{P}[X_n = \rho \mid X_m = \rho] = p^{(n-m)}(\rho, \rho)$$

and

$$\tilde{K}(m, n) = \frac{\tilde{G}(m, n)}{\tilde{G}(0, n)}.$$

Then for any set of times $A \subseteq \mathbf{Z}^+$:

$$\frac{1}{2}\mathrm{Cap}_{\tilde{K}}(A) \le \mathbf{P}[\exists n \in A : X_n = \rho] \le \mathrm{Cap}_{\tilde{K}}(A) \tag{10}$$

and

$$\frac{1}{2}\mathrm{Cap}^{(\infty)}_{\tilde{K}}(A) \le \mathbf{P}[\sum_{n=A} \mathbf{1}_{\{X_n=\rho\}} = \infty] \le \mathrm{Cap}^{(\infty)}_{\tilde{K}}(A). \tag{11}$$

Proof: Consider the space-time chain $\{(X_n, n) : n \ge 0\}$ on the state space $Y \times \mathbf{Z}^+$. This chain is obviously transient; let G denote its Green function. Since $G((\rho, m), (\rho, n)) = \tilde{G}(m, n)$ for $m \le n$, applying Theorem 2.2 with $\Lambda = \{\rho\} \times A$ shows that (10) and (11) follow respectively from (3) and (4). $\square$

Example 1: Random walk on $\mathbf{Z}$. Let S_n be the partial sums of mean-zero, finite variance, IID integer random variables. By the local limit theorem (c.f. Spitzer 1964),

$$\tilde{G}(0, n) = \mathbf{P}[S_n = 0] \approx cn^{-1/2}$$

provided that the summands $S_n - S_{n-1}$ are aperiodic. Therefore

$$\mathbf{P}[\sum_{n\in A} \mathbf{1}_{\{S_n=0\}} = \infty] > 0 \Leftrightarrow \mathrm{Cap}^{(\infty)}_F(A) > 0, \tag{12}$$

with $F(m, n) = (n^{1/2}/(m-n)^{1/2})\mathbf{1}_{\{m<n\}}$. By the Hewitt-Savage zero-one law, the event in (12) must have probability zero or one. Consider the special case in which A consists of separated blocks of integers:

$$A = \bigcup_{n=1}^{\infty} [2^n, 2^n + L_n]. \tag{13}$$

An easy calculation (or Lamperti's criterion) shows that in this case $S_n = 0$ for infinitely many $n \in A$ with probability one if and only if $\sum_n L_n^{1/2} 2^{-n/2} = \infty$. On the other hand, the expected number of returns $\sum_{n \in A} \mathbf{P}[S_n = 0]$ is infinite if and only if $\sum_n L_n 2^{-n/2} = \infty$. Thus an infinite expected number of returns in a time set does not suffice for almost sure return in the time set. When the walk is periodic, i.e.

$$r = \gcd\{n : \mathbf{P}[S_n = 0] > 0\} > 1,$$

the same criterion holds as long as A is contained in $r\mathbf{Z}^+$. Similar examples may be found in Ruzsa and Székely (1982) and Lawler (1991).

In some cases, the criterion of Corollary 2.5 can be turned around and used to estimate asymptotic capacity. For instance, if $\{S'_n\}$ is an independent random walk with the same distribution as $\{S_n\}$ and A is the random set $A = \{n : S'_n = 0\}$, then the positivity of $\mathrm{Cap}_F^{(\infty)}(A)$ follows from the recurrence of the planar random walk $\{(S_n, S'_n)\}$. This easily implies that the "discrete Hausdorff dimension" of A (in the sense of Barlow and Taylor (1992)) is almost surely $1/2$; detailed estimates of the Hausdorff measure of A were obtained in Khoshnevisan (1993).

Example 2: Random walk in $\mathbf{Z}^2$. Now we assume that S_n are partial sums of mean-zero, finite variance IID random variables in $\mathbf{Z}^2$. Denote $r = \gcd\{n : \mathbf{P}[S_n = 0] > 0\} > 1$ and let $A \subseteq r\mathbf{Z}$. Again, $\mathbf{P}[S_n = 0$ for infinitely many $n \in A]$ is zero or one and it is one if and only if $\mathrm{Cap}_F^{(\infty)}(A) > 0$ where $F(m,n) = \frac{m}{1+|m-n|}$. This follows from the local limit theorem (c.f. Spitzer 1964) which ensures that

$$\tilde{G}(0, rn) = \mathbf{P}[S_{rn} = 0] \approx cn^{-1} \text{ as } n \to \infty.$$

For instance, if A consists of disjoint blocks

$$A = \bigcup_n [2^n, 2^n + L_n]$$

then $\mathrm{Cap}_F^{(\infty)}(A) > 0$ if and only if $\sum_n 2^{-n} L_n / \log L_n = \infty$. The expected number of returns to zero is infinite if and only if $\sum 2^{-n} L_n = \infty$.

Comparing the kernel F with the Martin kernel for simple random walk on $\mathbf{Z}^3$ leads to the next corollary.

Corollary 2.6 *Let $\{S_n^{(d)}\}$ be a simple random walk on the d-dimensional lattice and let A be a set of positive even integers. Denote $A^{(3)} = \{(0,0,k) : k \in A\}$. Then the ratio*

$$\frac{\mathbf{P}[S_{2n}^{(2)} = 0 \textit{ for some } n \in A]}{\mathbf{P}[S_n^{(3)} \in A^{(3)} \textit{ infinitely often}]} \tag{14}$$

is bounded above and below by positive constants. Consequently,

$$\mathbf{P}[S_{2n}^{(2)} = 0 \textit{ for infinitely many } n \in A] == \mathbf{P}[S_n^{(3)} \in A^{(3)} \textit{ infinitely often}] \tag{15}$$

Note that both sides of 15 take only the values 0 or 1. Corollary 2.6 follows from the asymptotics $G(0,x) \sim c/|x|$ as $|x| \to \infty$ for the random walk $S_n^{(3)}$ (cf. Spitzer (1964)) and from Example 2 above. Erdös (1961) and Mckean (1961) showed that for $A = \{2p : p \text{ prime}\}$, the left-hand side of (15) is 1. To see why Corollary 2.6 is surprising, observe that the space-time chain $\{(S_n^{(2)}, n)\}$ travels to infinity faster than $S_n^{(3)}$, yet by Corollary 2.6, the same subsets of even lattice points on the positive z-axis are hit infinitely often by the two processes.

Example 3: Riesz-type kernels. The analogues of the Riesz kernels in the discrete setting are the kernels

$$F_\alpha(x,y) = \frac{||y||^\alpha}{1 + ||x-y||^\alpha}$$

on $\mathbf{Z}^d$, where $||\cdot||$ is any norm. We write $\text{Cap}_\alpha^{(\infty)}$ for $\text{Cap}_{F_\alpha}^{(\infty)}$. By Theorem 2.2, the asymptotics for the Green function, and the Hewitt-Savage law, simple random walk on $\mathbf{Z}^d$ visits a set $\Lambda \subseteq \mathbf{Z}^d$ i.o. a.s. if and only if $\text{Cap}_{d-2}^{(\infty)}(\Lambda) > 0$. More generally, if a random walk $\{S_n\}$ on the d-dimensional lattice has a Green function satisfying $G(0,x) \sim c|x|^{\alpha-d}$ as $|x| \to \infty$, then Theorem 2.2 implies that $S_n \in \Lambda$ for infinitely many n a.s. iff $\text{Cap}_{d-\alpha}^{(\infty)} > 0$ These asymptotics for the Green function are known to hold for many increment distributions in the domain of attraction of an α-stable distribution. (cf. Williamson (1968) for some sufficient conditions.)

Given a set of digits $D \subseteq \{0,1,\ldots,b-1\}$ containing zero, consider "the integer Cantor set"

$$\Lambda(D,b) = \{\sum_{n=0}^{N} a_n b^n : a_n \in D \text{ for all } n\,,\, N \geq 0\}.$$

A trite calculation shows that $\mathrm{Cap}_\alpha^{(\infty)}(\Lambda(D,b)) > 0$ if and only if $|D| \geq b^\alpha$. This motivates defining the dimension of $\Lambda \subseteq \mathbf{Z}^d$ by

$$dim(\Lambda) = \inf\{\alpha : \mathrm{Cap}_\alpha^{(\infty)}(\Lambda) = 0\}. \tag{16}$$

Corollary 8.4 in Barlow and Taylor (1991) shows that this definition is equivalent to the one in that paper.

As remarked before, a random walk on $\mathbf{Z}^d$ (or any abelian group) will visit a set infinitely often with probability 0 or 1 (by the Hewitt Savage zero-one law). Easy examples show that this fails for random walk on a free group. More generally the following "folklore" lemma holds.

Proposition 2.7 *Let μ be a probability measure whose support generates a group Y and let $\{S_n\}$ be the random walk with step distribution $S_n S_{n-1}^{-1} \sim \mu$. Then the probability* $\mathbf{P}[S_n \in \Lambda$ *infinitely often*$]$ *takes only the values zero and one as Λ ranges over subsets of Y, if and only if every bounded μ-harmonic function is constant. (Recall that $h : Y \to \mathbb{R}$ is μ-harmonic if $h(x) = \int_Y h(yx)\, d\mu(y)$ for all $x \in Y$.)*

Remark: When all bounded harmonic functions are constant, one says that the *Poisson boundary* of (Y, μ) is trivial; see Kaimanovich and Vershik (1982) for background.

Proof: Given a set $\Lambda \subseteq Y$, the function $h(x) = \mathbf{P}[S_n x \in \Lambda \text{ infinitely often}]$ is bounded and μ-harmonic. The Markov property and the martingale convergence theorem imply that

$$h(S_m) = \mathbf{P}\left[\{S_k : k \geq 0\} \text{ visits } \Lambda \text{ i.o. } \mid S_1, S_2, \ldots, S_m\right] \to \mathbf{1}_{\{S_k \text{ visits } \Lambda \text{ i.o.}\}}$$

as $m \to \infty$. Thus if all bounded harmonic functions are constant, the zero-one law holds. Conversely, assume the zero-one law holds and let h be a bounded μ-harmonic function. For $\alpha \in \mathbb{R}$, let $\Lambda_\alpha = \{y \in Y : h(y) < \alpha\}$. If $\mathbf{P}[S_n x \text{ visits } \Lambda_\alpha \text{ i.o.}] = 0$ then we consider the stopping time $\tau = \min\{n : h(S_n x) \geq \alpha\}$ and obtain $h(x) = h(S_0 x) = \mathbf{E} h(S_\tau x) \geq \alpha$. Similarly, if $\mathbf{P}[S_n x \text{ visits } \Lambda_\alpha \text{ i.o.}] = 1$ then $h(x) \leq \alpha$. Since the support of μ generates Y,

$$\mathbf{P}[S_n \text{ visits } \Lambda_\alpha \text{ i.o.}] = 0 \Leftrightarrow \mathbf{P}[S_n x \text{ visits } \Lambda_\alpha \text{ i.o.}] = 0$$

and it follows that $h(x) = h(e)$ for all $x \in Y$. $\square$

3 Independent percolation on trees

Theorem 2.2 yields a short proof of a fundamental result of R. Lyons concerning *percolation on trees.*

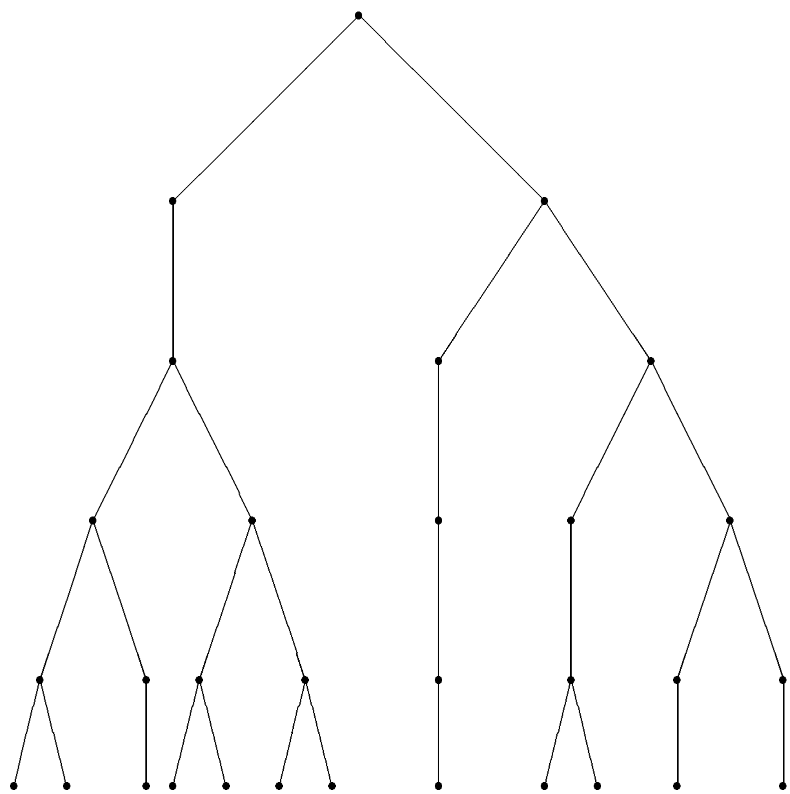

Figure 1: a tree

Notation: Let T be a finite, rooted tree. Vertices of degree one in T (apart from the root ρ) are called *leaves*, and the set of leaves is the *boundary* ∂T of T. The set of edges on the path connecting the root to a leaf x is denoted $\texttt{Path}(x)$.

Independent percolation on T is defined as follows. To each edge e of T, a parameter p_e in $[0, 1]$ is attached, and e is removed with probability $1 - p_e$, retained with probability p_e, with independence between edges. Say that a leaf x *survives the percolation* if all of $\mathtt{Path}(x)$ is retained, and say that the tree boundary ∂T survives if some leaf of T survives.

Theorem 3.1 (Lyons (1992)) *With the notation above, define a kernel F on ∂T by $F(x,y) = \prod\{p_e^{-1} : e \in \mathtt{Path}(x) \bigcap \mathtt{Path}(y)\}$ for $x \neq y$ and $F(x,x) = 2\prod\{p_e^{-1} : e \in \mathtt{Path}(x)\}$. Then*

$$\mathrm{Cap}_F(\partial T) \leq \mathbf{P}[\partial T \textit{ survives the percolation}] \leq 2\mathrm{Cap}_F(\partial T)$$

(The kernel F differs from the kernel used in Lyons (1992) on the diagonal, but this difference is unimportant in all applications).

Proof: Embed T in the plane. The random set of $r \geq 0$ leaves that survive the percolation may be enumerated from left to right as $V_1, V_2, \ldots, V_r$. The key observation is that The random sequence $\rho, V_1, V_2, \ldots, V_r, \Delta, \Delta, \ldots$ is a Markov chain on the state space $\partial T \bigcup \{\rho, \Delta\}$ (where ρ is the root and Δ is a formal absorbing cemetery).

Indeed, *given* that $V_k = x$, all the edges on $\mathtt{Path}(x)$ are retained, so that survival of leafs to the right of x is determined by the edges strictly to the right of $\mathtt{Path}(x)$, and is thus conditionally independent of $V_1, \ldots, V_{k-1}$. This verifies the Markov property, so Theorem 2.2 may be applied.

The transition probabilities for the Markov chain above are complicated, but it is easy to write down the Green kernel. Clearly, $G(\rho, y) = \mathbf{P}[y \text{ survives the percolation}] = \prod_{e \in \mathtt{Path}(y)} p_e$. Also, when x is to the left of y, $G(x,y)$ is just the probability that the range of the Markov chain contains y given that it contains x, which is just the probability of y surviving given that x survives, so

$$G(x,y) = \prod_{e \in \mathtt{Path}(y) \setminus \mathtt{Path}(x)} p_e$$

and hence

$$\frac{G(x,y)}{G(\rho,y)} = \prod_{e \in \mathtt{Path}(x) \cap \mathtt{Path}(y)} p_e^{-1}$$

and Lyons' Theorem follows from Theorem 2.2. $\square$

4 Martin capacity and Brownian motion

Proof of Proposition 1.1: To bound the hitting probability of Λ from above, consider the stopping time $\tau = \min\{t > 0 : B_d(t) \in \Lambda\}$. The distribution of $B_d(\tau)$ on the event $\tau < \infty$ is a possibly defective distribution ν satisfying

$$\nu(\Lambda) = \mathbf{P}[\tau < \infty] = \mathbf{P}[\exists t > 0 : B_d(t) \in \Lambda]. \tag{17}$$

Now recall the standard formula, valid when $\epsilon < ||y||$:

$$\mathbf{P}[\exists t > 0 : ||B_d(t) - y|| < \epsilon] = \frac{\epsilon^{d-2}}{||y||^{d-2}}. \tag{18}$$

By a first entrance decomposition, the probability in (18) is at least

$$\mathbf{P}[||B_d(\tau) - y|| > \epsilon \text{ and } \exists t > 0 : ||B_d(t) - y|| < \epsilon] \;=\; \int_{x:||x-y||>\epsilon} \frac{\epsilon^{d-2}}{||x - y||^{d-2}}\, d\nu(x).$$

Letting ϵ go to zero we obtain

$$\int_\Lambda \frac{d\nu(x)}{||x - y||^{d-2}} \leq \frac{1}{||y||^{d-2}},$$

i.e. $\int_\Lambda K(x,y)\, d\nu(x) \leq 1$ for all $y \in \Lambda$. Therefore $I_K(\nu) \leq \nu(\Lambda)$ and thus

$$\mathrm{Cap}_K(\Lambda) \geq [I_K(\nu/\nu(\Lambda))]^{-1} \geq \nu(\Lambda),$$

which by (17) yields the upper bound on the hitting probability of Λ.

To obtain a lower bound for this probability, a second moment estimate is used. For $\epsilon > 0$ and $y \in \mathbb{R}^d$ let $D(y, \epsilon)$ denote the Euclidean ball of radius ϵ about y and let $h_\epsilon(||y||)$ denote the probability that a Brownian path will hit this ball:

$$h_\epsilon(r) = \min\{1, \left(\frac{\epsilon}{r}\right)^{d-2}\}. \tag{19}$$

Define $h_\epsilon(r) = 1$ for $r < 0$. Given a probability measure μ on Λ, and $\epsilon > 0$, consider the random variable

$$Z_\epsilon = \int_\Lambda \mathbf{1}_{\{\exists t>0: B_d(t) \in D(y,\epsilon)\}} h_\epsilon(||y||)^{-1}\, d\mu(y).$$

Clearly $\mathbf{E}Z_\epsilon = 1$. We compute the second moment of Z_ϵ in order to apply Cauchy-Schwartz as in the proof of Theorem 2.2.

By symmetry,

$$\begin{aligned}
\mathbf{E}Z_\epsilon^2 &= 2\mathbf{E}\int_\Lambda\int_\Lambda \mathbf{1}_{\{\exists t>0: B_d(t)\in D(x,\epsilon) \text{ and } \exists s>t: B_d(s)\in D(y,\epsilon)\}} \frac{d\mu(x)d\mu(y)}{h_\epsilon(||x||)h_\epsilon(||y||)} \\
&\leq 2\mathbf{E}\int_\Lambda\int_\Lambda \mathbf{1}_{\{\exists t>0: B_d(t)\in D(x,\epsilon)\}} \frac{h_\epsilon(||y-x||-\epsilon)}{h_\epsilon(||x||)h_\epsilon(||y||)}\, d\mu(x)\, d\mu(y) \\
&= 2\int_\Lambda\int_\Lambda \frac{h_\epsilon(||y-x||-\epsilon)}{h_\epsilon(||y||)}\, d\mu(x)\, d\mu(y).
\end{aligned}$$

Since the last integrand is bounded by $2K(x,y)$ if $y \notin D(0,\epsilon)$ and by 1 if $y \in D(0,\epsilon)$, we get

$$\begin{aligned}
&\mathbf{E}Z_\epsilon^2 \leq 4\int\int \mathbf{1}_{\{||x-y||\leq 2\epsilon\}} K(x,y)\, d\mu(x)\, d\mu(y) + 2\mu(D(0,\epsilon)) \\
&+2\int\int \mathbf{1}_{\{||x-y||>2\epsilon\}} \left(\frac{||y||}{||y-x||+1}\right)^{d-2} d\mu(x)\, d\mu(y).
\end{aligned}$$

The first two terms drop out as $\epsilon \to 0$ (by dominated convergence) leaving

$$\lim_{\epsilon\downarrow 0} \mathbf{E}Z_\epsilon^2 \leq 2I_K(\mu) \tag{20}$$

provided $\mu(\{\mathbf{0}\}) = 0$. Clearly the hitting probability $\mathbf{P}[\exists t > 0, y \in \Lambda : B_d(t) \in D(y,\epsilon)]$ is at least

$$\mathbf{P}[Z_\epsilon > 0] \geq \frac{(\mathbf{E}Z_\epsilon)^2}{\mathbf{E}Z_\epsilon^2} = (\mathbf{E}Z_\epsilon^2)^{-1}$$

Transience of Brownian motion implies that if the Brownian path visits every ϵ-neighborhood of the compact set Λ then it almost surely intersects Λ itself. Therefore, by (20):

$$\mathbf{P}[\exists t > 0 : B_d(t) \in \Lambda] \geq \lim_{\epsilon\downarrow 0}(\mathbf{E}Z_\epsilon^2)^{-1} \geq \frac{1}{2I_K(\mu)}.$$

Since this is true for all probability measures μ on Λ, we get the desired conclusion:

$$\mathbf{P}[\exists t > 0 : B_d(t) \in \Lambda] \geq \frac{1}{2}\mathrm{Cap}_K(\Lambda). \tag{21}$$

$\square$

Remark: To see that the constant 1/2 in (21) cannot be increased, consider the spherical shell

$$\Lambda_R = \{x \in \mathbb{R}^d : 1 \leq ||x|| \leq R\};$$

it is easy to check that $\lim_{R\to\infty} \mathrm{Cap}_K(\Lambda_R) = 2$.

Next, we pass from the local to the global behavior of Brownian paths. Barlow and Taylor (1991) noted that for $d \geq 2$ the set of nearest neighbor lattice points to a Brownian path in $\mathbb{R}^d$ is a subset of $\mathbf{Z}^d$ with dimension 2, using their definition of dimension which is equivalent to (16). This is a property of the path near infinity; another such property is given by

Proposition 4.1 *Let $B_d(t)$ denote d-dimensional Brownian motion. Let $\Lambda \subseteq \mathbb{R}^d$ with $d \geq 3$ and let Λ_1 be the cubical fattening of Λ defined by*

$$\Lambda_1 = \{x \in \mathbb{R}^d : \exists y \in \Lambda \;\; s.t. \;\; ||y - x||_\infty \leq 1\}.$$

Then a necessary and sufficient condition for the almost sure existence of times $t_j \uparrow \infty$ at which $B_d(t_j) \in \Lambda_1$ is that $\mathrm{Cap}_{d-2}^{(\infty)}(\Lambda_1 \cap \mathbf{Z}^d) > 0$.

The proof is very similar to the proof of Theorem 2.2 and is omitted.

5 Random walk in varying dimension

To give meaning to the terms "recurrent" and "transient", we prove a "folklore" lemma which implies a 0-1 law for recurrence of RWVD.

Lemma 5.1 *Let $\{F_j : 1 \leq j \leq l\}$ be distributions on the abelian group Y and let*

$$(n(1), n(2), \ldots) \in \{1, 2, \ldots, l\}^{\mathbf{Z}^+}$$

be any sequence in which each value $1, \ldots, l$ occurs infinitely often. Let $\{X_k\}$ be independent random variables with distributions $F_{n(k)}$. Then any tail event for the sequence of partial sums $S_N = \sum_{k=1}^N X_k$ has probability 0 or 1.

Proof: If $l = 1$, this is a consequence of the Hewitt-Savage 0-1 law. If $l > 1$, assume for induction that the result is true for smaller values of l and let $\mathcal{F}_{l-1}$ denote the σ-field generated by

$$\{X_k : n(k) \leq l - 1\}. \tag{22}$$

Conditional on $\mathcal{F}_{l-1}$, the event B is exchangeable in the remaining variables $\{X_k : n(k) = l\}$; since these variables are identically distributed, the Hewitt-Savage 0-1 law shows that $\mathbf{P}[B \,|\, \mathcal{F}_{l-1}] \in \{0, 1\}$ almost surely. The set $\tilde{B} := \{\mathbf{P}[B \,|\, \mathcal{F}_{l-1}] = 1\}$ is $\mathcal{F}_{l-1}$-measurable, and it is a tail event for the partial sums of the variables in (22). By induction, $\mathbf{P}[\tilde{B}] \in \{0, 1\}$, which shows that $\mathbf{P}[B] \in \{0, 1\}$. $\square$

Next, recall the random walk in varying dimension considered in the introduction: a process $\{S_k\}$ in $\mathbf{Z}^3$ with independent increments $S_k - S_{k-1}$ distributed according to a truly 3-dimensional distribution F_3 if $k \in \{a_n : n \geq 1\}$, and according to the projection F_2 of F_3 to the x-y plane if $k \notin \{a_n\}$. We assume that:

$$F_3 \text{ makes the three coordinates independent, and} \tag{23}$$

$$F_3 \text{ has mean zero and finite variance.}$$

We first state an easy qualitative proposition which is sharpened in Theorem 5.4 below.

Proposition 5.2 *If $\{a_n\}$ grow sufficiently fast, then the RWVD in 2 and 3 dimensions is recurrent.*

Proof: Denote by π_z projection to the z-axis and by π_{xy} the projection map to the x-y plane. Since $\{\pi_{xy}(S_n)\}$ is a recurrent planar random walk, we may select a_n inductively to satisfy

$$P[\exists k \in (a_n, a_{n+1}] : \pi_{xy}(S_k) = 0] \geq 1/2. \tag{24}$$

The process $\{\pi_z(S_{a_n})\}$ is a recurrent one-dimensional random walk, so there is almost surely a random infinite sequence $N(1), N(2), \ldots$ for which $\pi_z(S_{a_{N(j)}}) = 0$ for all $j \geq 1$. Independence of $\{\pi_{xy}(S_n)\}$ and $\{\pi_z(S_n)\}$ implies that the set of j for which there exists a $k \in (a_{N(j)}, a_{N(j)+1}]$ such that $S_{a_k} = 0$ stochastically dominates the random set of positive integers gotten by including each one independently with probability 1/2. In particular, there are almost surely infinitely many such j, and for each j there is some $k \in (a_{N(j)}, a_{N(j)+1}]$ with $S_k = 0$, proving recurrence. $\Box$

The argument above is quite general and extends in an obvious way to the product of two recurrent Markov chains. Iterating this argument yields the next corollary.

Corollary 5.3 *If* $d_1 < d_2 < \cdots < d_N$ *and*

$$\max\{d_{j+1} - d_j : 1 \le j \le N-1\} \le 2, \tag{25}$$

then there exists a recurrent process $\{S_n\}$ *with independent increments which interlaces infinitely many* d_j*-dimensional steps for each* j*. More precisely,* $S_{k+1} - S_k$ *has a truly* $D(k)$*-dimensional distribution for each* k*, and the sequence* $\{D(k)\}$ *takes on only the values* $d_1, \ldots, d_N$*, each one infinitely often. If (25) is violated then (clearly) any such process* $\{S_n\}$ *must be transient.*

Next, we give the quantitative version, Theorem 5.4, of Proposition 5.2. This will be proved in detail. We also state similar theorems for RWVD in 2 and 4 dimensions and RWVD in 1 and 3 dimensions and give the necessary modifications to the proof of Theorem 5.4. Define

$$\phi(n) = \frac{\log(a_{n+1}/a_n)}{\log a_{n+1}}; \tag{26}$$

$$\phi_1(n) = \sqrt{\frac{a_{n+1} - a_n}{a_{n+1}}}. \tag{27}$$

Theorem 5.4 *For the "*$\mathbf{Z}^2$ *in* $\mathbf{Z}^3$*" random walk in varying dimension* $\{S_n\}$ *considered in Proposition 5.2, we have:*

(*i*) *If* $\sum_n n^{-1/2}\phi(n) < \infty$ *then* $\{S_n\}$ *is transient.*

(*ii*) *If* $\sum_n n^{-1/2}\phi(n) = \infty$ *and the sequence* $\{\phi(n)\}$ *is nonincreasing, then* $\{S_n\}$ *is recurrent.*

Remarks:

1. In particular, S_n is recurrent for $a_n = \exp(e^{n^{1/2}})$ and transient for $a_n = \exp(e^{n^\alpha})$ when $\alpha > 1/2$.

2. The monotonicity assumption in (ii) is far from necessary, and may be weakened in several ways. If ϕ is bounded below, S_n is recurrent and the proof is easier. If

$$\sup_{m>n} \phi(m)/\phi(n) < \infty, \tag{28}$$

then S_n is still recurrent when $\sum_n n^{-1/2}\phi(n) = \infty$. On the other hand, the hypothesis may not be discarded completely. To see this, let $A \subseteq \{1, 2, 3, \ldots\}$ be a set of times such that a simple random walk $\{Y_n\}$ on $\mathbf{Z}^1$ will have $Y_n = 0$ for only finitely many $n \in A$ almost surely, even though $\sum_{n\in A} \mathbf{P}[Y_n = 0] = \infty$ (c.f. Example 1). Define the sequence $\{a_n\}$ by $a_{n+1} = 2a_n - 1$ if $n \notin A$ and $a_{n+1} = a_n^2$ if $n \in A$. For $n \in A$, $\phi(n) = 1/2$, so the sum in (ii) is infinite by the assumption $\sum_{n\in A} \mathbf{P}[Y_n = 0] = \infty$. But with probability one, the S_{a_n} is in the x-y plane for only finitely many $n \in A$, while by Lemma 5.9, $\{S_k\}$ visits the origin finitely often in time intervals $[a_n, a_{n+1}]$ for $n \notin A$.

3. To see the connection to Theorem 2.2, let W be any subset of the positive integers and define

$$\psi(W) = \bigcup \{ [a_n, a_{n+1} - 1] \, : \, n \in W\}.$$

If W_1 is the set of times a one-dimensional random walk is at the origin and W_2 is the set of times an independent two-dimensional random walk is at the origin, then $\psi(W_1)$ intersects W_2 infinitely often if and only if the RWVD is recurrent. Exact capacity criteria are available for which sets intersect W_2 infinitely often as well as which gauges give W_1 positive capacity, but the complication introduced by the map ψ makes it easier to use the second moment method directly.

Theorem 5.5 *For the "$\mathbf{Z}^2$ in $\mathbf{Z}^4$" random walk in varying dimension,* *(i) If $\sum_n n^{-1}\phi(n) < \infty$ then $\{S_n\}$ is transient.*

(ii) If $\sum_n n^{-1}\phi(n) = \infty$ and the sequence $\{\phi(n)\}$ is nonincreasing, then $\{S_n\}$ is recurrent.

Theorem 5.6 *For the "$\mathbf{Z}^1$ in $\mathbf{Z}^3$" random walk in varying dimension,*

(i) If $\sum_n n^{-1}\phi_1(n) < \infty$ then $\{S_n\}$ is transient.

(ii) If $\sum_n n^{-1}\phi_1(n) = \infty$ and the sequence $\{\phi_1(n)\}$ is nonincreasing, then $\{S_n\}$ is recurrent.

The proofs begin with some elementary estimates on the probability of returning to the origin in a specified time interval.

Lemma 5.7 *Let $\{S_n\}$ be the partial sums of an aperiodic random walk on the one-dimensional integer lattice with mean zero and finite variance. Then there exist constants c_1 and c_2 depending only on the distribution of the increments, such that for sufficiently large integers $0 < a < b$,*

$$c_1\sqrt{\frac{b-a}{b}} \leq \mathbf{P}[S_n = 0 \text{ for some } a \leq n < b] \leq c_2\sqrt{\frac{b-a}{b}}. \tag{29}$$

Lemma 5.8 *Let $\{S_n\}$ be the partial sums of an aperiodic random walk on the two-dimensional integer lattice with mean zero and finite variance. Then there exist constants c_1 and c_2 depending only on the distribution of the increments, such that for sufficiently large integers $0 < a < b$,*

$$c_1\frac{\log(b/a)}{\log b} \leq \mathbf{P}[S_n = 0 \text{ for some } a \leq n < b], \tag{30}$$

and, in the case that $b > 2a$,

$$\mathbf{P}[S_n = 0 \text{ for some } a \leq n < b] \leq c_2\frac{\log(b/a)}{\log b}. \tag{31}$$

Proof of Lemma 5.7: The Local Central Limit Theorem (c.f. Spitzer 1964) gives

$$\mathbf{P}[S_n = 0] = \frac{c}{\sqrt{n}}(1 + o(1)) \tag{32}$$

for some constant c as $k \to \infty$. Write G for the event that $S_n = 0$ for some $k \in [a, b-1]$. Then

$$\mathbf{P}[G] = \frac{\mathbf{E}\#\{k : a \le k < b \text{ and } S_k = 0\}}{\mathbf{E}\left(\#\{k : a \le k < b \text{ and } S_k = 0\} \mid G\right)} .$$

The numerator is $(c + o(1))(\sqrt{b} - \sqrt{a})$ as $a \to \infty$ according to the Local CLT. To get an upper bound on the denominator, let $T = \min\{a \le k < b : S_k = 0\}$ be the (possibly infinite) hitting time and condition on T to get

$$\begin{aligned}
& \mathbf{E}\left(\#\{k : a \le k < b \text{ and } S_k = 0\} \mid G\right) \\
\le \quad & \sup_{a \le t < b} \mathbf{E}\left(\#\{k : a \le k < b \text{ and } S_k = 0\} \mid T = t\right) \\
= \quad & \sup_{0 \le t < b-a} \mathbf{E}\#\{k : 0 \le k < b - a \text{ and } S_k = 0\} \\
= \quad & (c + o(1))\sqrt{b-a}
\end{aligned}$$

as $b - a \to \infty$. Using Taylor's Theorem on the numerator now gives

$$\mathbf{P}[G] \ge \frac{(c + o(1))(b-a)/(2\sqrt{b})}{(c + o(1))\sqrt{b-a}} \ge c_1 \sqrt{\frac{b-a}{b}}$$

for some constant c_1 that takes into account small values of a and $b - a$.

To prove the second inequality, recompute

$$\mathbf{P}[G] = \frac{\mathbf{E}\#\{k : a \le k < 2b - a \text{ and } S_k = 0\}}{\mathbf{E}\left(\#\{k : a \le k < 2b - a \text{ and } S_k = 0\} \mid G\right)} .$$

The numerator is now $(c + o(1))(\sqrt{2b-a} - \sqrt{a})$ which, by Taylor's Theorem is at most $(2c + o(1))(b-a)/\sqrt{a}$. The denominator is at least

$$\inf_{a \le t < b} \mathbf{E}\left(\#\{k : a \le k < 2b - a \text{ and } S_k = 0\} \mid T = t\right) \ge (c + o(1))\sqrt{b-a}.$$

Taking the quotient proves the second inequality in the case $b \leq 2a$; the case $b > 2a$ is trivial. $\square$

Proof of Lemma 5.8: The Local CLT now gives

$$\mathbf{P}[S_n = 0] = \frac{c}{n}(1 + o(1)).$$

Defining G and T as before, it again follows that

$$\mathbf{P}[G] = \frac{\mathbf{E}\#\{k : a \leq k < b \text{ and } S_k = 0\}}{\mathbf{E}\left(\#\{k : a \leq k < b \text{ and } S_k = 0\} \mid G\right)} .$$

Using the Local CLT and conditioning on T as before, shows this to be at least

$$(1 + o(1))\frac{\log b - \log a}{\log(b - a)},$$

which proves (30). On the other hand, using the alternate expression

$$\mathbf{P}[G] = \frac{\mathbf{E}\#\{k : a \leq k < 2b - a \text{ and } S_k = 0\}}{\mathbf{E}\left(\#\{k : a \leq k < 2b - a \text{ and } S_k = 0\} \mid G\right)}$$

gives

$$\mathbf{P}[G] \leq (1 + o(1))\frac{\log(2b - a) - \log a}{\log(b - a)}$$

which is at most $c_2 \log(b/a)/\log b$ as long as $b > 2a$, proving (31). $\square$

Proof of Theorem 5.4: The second moment method will be used, as in the proof of Theorem 2.2. It is possible to get a good second moment estimate on the number of intervals $[a_n, a_{n+1} - 1]$ that contain a return to zero, but only after throwing out some of them. We must first prove:

Lemma 5.9 *The number of k for which $S_k = 0$ and $a_n \leq k < a_{n+1}$ for some n satisfying $a_{n+1} < 2a_n$ is almost surely finite.*

Proof: Let $m(1), m(2), \ldots$ enumerate the integers m for which $[2^{m-1}, 2^{m+1} - 1]$ contains some a_n. It suffices to show that finitely many intervals of the form $[2^{m(j)-1}, 2^{m(j)+1} - 1]$ contain values of k for which $S_k = 0$, since these cover all intervals of the form $[a_n, a_{n+1} - 1]$ satisfying $a_{n+1} < 2a_k$.

Fix j and let $n(j)$ denote the least n such that $n \in [2^{m(j)-1}, 2^{m(j)+1} - 1]$. By the independence of the coordinates of $\{S_k\}$, and by the Local CLT in one and two dimensions, one sees that for each $k \in [2^{m(j)-1}, 2^{m(j)+1} - 1]$, the probability of $S_k = 0$ is at most $c/(k\sqrt{n(j)})$. Summing this over all k in the interval gives

$$\mathbf{P}[S_k = 0 \text{ for some } 2^{m(j)-1} \leq k < 2^{m(j)+1}] \leq \frac{c}{\sqrt{n(j)}} .$$

Another way to get an upper bound on this is to see that the probability of this event is at most the product of the probability that the walk returns to the x-y plane during the interval with the probability that it returns to the z-axis during the interval. Lemmas 5.7 and 5.8 applied to the intervals $[n(j), n(j+1) - 1]$ and $[2^{m(j)-1}, 2^{m(j)+1} - 1]$ respectively show this product to be at most

$$c\sqrt{\frac{n(j+1) - n(j)}{n(j+1)}}\,\frac{1}{m(j)}.$$

Since $m(j) \geq j$, these two upper bounds may be written as

$$\mathbf{P}[S_k = 0 \text{ for some } 2^{m(j)-1} \leq k < 2^{m(j)+1}] \leq c \min\left(\frac{1}{\sqrt{n(j)}}\,,\,\sqrt{\frac{n(j+1) - n(j)}{n(j+1)}}\,\frac{1}{m(j)}\right).$$

Lemma 5.10 with $b_j = n(j+1) - n(j)$ now shows that these probabilities are summable in j, and Borel-Cantelli finishes the proof. For continuity's sake, the lemma (which is a fact about deterministic integer sequences) is given at the end of the section. $\square$

Proof of Theorem 5.4 (continued): Let $I_n = 1$ if $a_{n+1} \geq 2a_n$ and $S_k = 0$ for some $k \in [a_n, a_{n+1} - 1]$, and let $I_n = 0$ otherwise. Part (i) of the theorem is just Borel-Cantelli: the hypothesis in (i) and the estimate (31) in the case $b > 2a$ together imply that $\mathbf{E}I_n \leq n^{-1/2}\phi(n)$ is summable. Thus the random walk visits zero finitely often in intervals $[a_n, a_{n+1} - 1]$ for which $a_{n+1} \geq 2a_n$; this, together with Lemma 5.9, proves (i).

To prove (ii), it suffices, by the 0-1 law (Lemma 5.1), to show that the probability of S_k returning to the origin infinitely often is at least some $c > 0$. This follows from the two propositions: $\sum_{n=1}^\infty \mathbf{E}I_n = \infty$, and $\mathbf{E}(\sum_{n=1}^M I_n)^2 \leq c(\sum_{n=1}^M \mathbf{E}I_n)^2$ (c.f. something like Kechen-Stone).

Seeing that $\sum_{n=1}^\infty \mathbf{E}I_n = \infty$ is easy, since $\sum_{n=1}^\infty n^{-1/2}\phi(n)$ is assumed to be infinite; the difference between the two sums is

$$\sum_{n=1}^\infty \frac{\phi(n)}{n^{1/2}} \mathbf{1}_{a_{n+1}<2a_n}.$$

Letting $m(j)$ enumerate those j such that $2^{m(j)-1} \leq a_n < a_{n+1} < 2^{m(j)+1}$ for some n, the difference comes out to at most

$$\sum_{j=1}^\infty \frac{\log 4}{j^{1/2}\log(2^{m(j)-1})}$$

which is summable since $m(j) \geq j$. For use below, let C_0 denote this finite sum.

For the second moment computation, take M large enough so that $\sum_{n=1}^M \mathbf{E}I_n \geq 1$, so that $(\sum_{n=1}^M \mathbf{E}I_n)^2$ is greater than $\sum_{n=1}^M \mathbf{E}I_n$. The expected square may be expanded into terms $\mathbf{E}I_j I_r$, for $r \geq j$, which we now bound. Of course, $\mathbf{E}I_n I_r = 0$ if $a_{n+1} < 2a_n$ or $a_{r+1} < 2a_r$. Assume now that $\mathbf{E}I_n I_r > 0$ and that r and n are not consecutive among numbers k with $\mathbf{E}I_k > 0$. Then

$$\begin{aligned}\mathbf{E}I_n I_r &= (\mathbf{E}I_n)\mathbf{E}(I_r \mid I_n = 1) \\ &\leq c\frac{\phi(n)}{n^{1/2}} \max_{t<a_{n+1}} \mathbf{E}(I_r \mid S_t = 0).\end{aligned}$$

The inequality (31) may be used with $b = a_{r+1} - t$ and $a = a_r - t$ to see that

$$\mathbf{E}I_n I_r \leq c \max_{t<a_{n+1}} \frac{\phi(n)}{n^{1/2}} \frac{\log(a_{r+1}-t) - \log(a_r - t)}{\sqrt{r-n}\log(a_{r+1}-t)}.$$

As t increases, the numerator of the last term increases and the denominator decreases, and since $t < a_{n+1} < a_r/2$, this yields

$$\begin{aligned}\mathbf{E}I_n I_r &\leq c\frac{\phi(n)}{n^{1/2}} \frac{\log(a_{r+1}-a_r/2) - \log(a_r - a_r/2)}{\sqrt{r-n}\log(a_{r+1}-a_r/2)} \\ &\leq c'\frac{\phi(n)}{n^{1/2}} \frac{\log a_{r+1} - \log a_r}{\log a_{r+1}}\end{aligned}$$

$$
\begin{aligned}
&= c' \frac{\phi(n)}{n^{1/2}} \frac{\phi(r)}{(r-n)^{1/2}} \\
&\leq c' \frac{\phi(n)}{n^{1/2}} \frac{\phi(r-n)}{(r-n)^{1/2}},
\end{aligned}
$$

since $\phi(n)$ is assumed to be monotone decreasing.

Using the bound $\mathbf{E} I_n I_r \leq \mathbf{E} I_n$ for consecutive or identical nonzero terms, now gives

$$
\begin{aligned}
\mathbf{E}(\sum_{n=1}^{M} I_n)^2 &\leq 3c' \sum_{n=1}^{M} n^{-1/2}\phi(n) + 2c' \sum_{n=1}^{M} n^{-1/2}\phi(n)(\sum_{j=1}^{M} j^{-1/2}\phi(j)) \\
&\leq 5c'(C_0 + \sum_{n=1}^{M} \mathbf{E} I_n)^2.
\end{aligned}
$$

The second moment is thus bounded by a constant multiple of the square of the first moment, completing the proof of the theorem. □

Proofs of Theorems 5.5 and 5.6: To prove Theorem 5.5, it suffices everywhere to change $n^{-1/2}\phi(n)$ to $n^{-1}\phi(n)$. The analogue of Lemma 5.9 goes through without alteration, since $n^{-1} < n^{-1/2}$, and the second moment estimate is easily completed. To prove Theorem 5.6, one does not use a version of Lemma 5.9 (which would not, in fact be true). Instead, a decomposition into cases where $a_r < 2a_n$ and $a_r \geq 2a_n$ occurs inside the second moment estimate. Writing I_n for the indicator function of the existence of a $k \in [a_n, a_{n+1} - 1]$ for which $S_k = 0$, one has

$$
c_1 \frac{\phi_1(n)}{n} \leq \mathbf{E} I_n \leq c_2 \frac{\phi_1(n)}{n}
$$

for all n as a consequence of Lemma 5.7. Then

$$
\mathbf{E}(\sum_{n=1}^{M} I_n)^2 = \sum_{n=1}^{M} \mathbf{E} I_n + 2 \sum_{M \geq r, a_r \geq 2a_n} \mathbf{E} I_n I_r + \sum_{M \geq r, a_r < 2a_n} \mathbf{E} I_n I_r. \tag{33}
$$

Bound the middle term of (33) in the same manner as before:

$$
\begin{aligned}
\sum_{a_r \geq 2a_n} \mathbf{E} I_n I_r &\leq c \sum_{a_r \geq 2a_n} \frac{\phi_1(n)}{n} \frac{\sqrt{(a_{r+1} - a_r)/(a_{r+1} - a_{n+1})}}{r-n} \\
&\leq 2c \sum_{a_r \geq 2a_n} \frac{\phi_1(n)}{n} \frac{\phi_1(r)}{r-n}.
\end{aligned}
$$

Monotonicity of $\phi_1(n)$ allows us to replace $\phi_1(r)$ with $\phi_1(r-n)$, after which the middle term of (33) is at most $2c(\sum_{n=1}^M \mathbf{E}I_n)^2$.

Finally, to bound the last term of (33), write

$$\sum_{n=1}^M \mathbf{E}I_n I_r \leq c \sum_{n=1}^M \frac{\phi_1(n)}{n} \left(\sum_{a_n < a_r < 2a_n} \frac{c}{r-n} \sqrt{\frac{a_{r+1}-a_r}{a_{r+1}-a_{n_1}}} \right).$$

Since $\phi_1(r)^2 = \frac{a_{r+1}-a_r}{a_{r+1}}$ is nonincreasing, and since $a_{r+1} < 2a_n$ for all but possibly the last element of the inner sum, it follows that each $a_{n+k+1}-a_{n+k}$ is at least half of each $a_{n+k'+1}-a_{n+k'}$ when $k < k'$, except possibly for the last one. Thus

$$\frac{a_{n+k+1}-a_n}{a_{n+k+1}-a_{n+1}} \leq \frac{2}{k}$$

except when k is maximal, and hence the inner sum is bounded:

$$\sum_{a_n < a_r < 2a_n} \frac{1}{r-n} \sqrt{\frac{a_{r+1}-a_r}{a_{r+1}-a_{n_1}}} \leq 1 + \sum_{k=1}^{\infty} \frac{c}{k^{3/2}} < c'.$$

Thus the last term of (33) is at most a constant multiple of $\sum_{n=1}^M \mathbf{E}I_n$, and the second moment bound is complete. □

Having finished the proofs of Theorems 5.4, 5.5 and 5.6, it remains to prove the lemma that was used in the first proof.

Lemma 5.10 *Let $b_1, b_2, \ldots$ be any sequence of positive integers and let $B_n = \sum_{k=1}^n b_k$ be the partial sums. Then*

$$\sum_{n=1}^{\infty} \min \left(\frac{1}{\sqrt{B_{n-1}}}, \frac{1}{n}\sqrt{\frac{b_n}{B_n}} \right) < \infty.$$

Proof: Breaking down the terms according to whether $B_{n-1} \geq n^3$ gives

$$\sum_{n=1}^{\infty} \min \left(\frac{1}{\sqrt{B_{n-1}}}, \frac{1}{n}\sqrt{\frac{b_n}{B_n}} \right)$$

$$\leq \sum_{n=1}^{\infty}\left(\mathbf{1}_{B_{n-1}\geq n^3}\frac{1}{\sqrt{B_n}}+\mathbf{1}_{B_{n-1}<n^3}\frac{1}{n}\sqrt{\frac{b_n}{B_n}}\right)$$

$$\leq C+\sum_{n=1}^{\infty}\mathbf{1}_{B_{n-1}<n^3}\frac{1}{n}\sqrt{\frac{b_n}{B_n}}.$$

To show that the second term is finite, we estimate the sum over intervals $[M, 2M]$. Let $\delta_n = \sqrt{b_n/B_n}$, and, assuming the sum to be nonzero, let $T = \max\{j \leq 2M : B_j \leq (2M)^3\}$. Then

$$\sum_{n=M}^{2M}\frac{1}{n}\sqrt{\frac{b_n}{B_n}}\mathbf{1}_{B_{n-1}<n^3}$$

$$\leq \frac{1}{M}\sum_{n=M}^{2M}\delta_n\mathbf{1}_{B_{n-1}<(2M)^3}$$

$$= \frac{1}{M}\sum_{n=M}^{T+1}\delta_n.$$

Bince $1+\delta_n^2 = B_{n+1}/B_n$ and each $\delta_n < 1$, we have

$$\prod_{n=M}^{T}(1+\delta_n^2) \leq 2B_T/B_M \leq 16M^3.$$

Taking logs gives

$$\sum_{n=M}^{T}\delta_n^2 \leq \sum_{n=M}^{T}(\log 2)^{-1}\log(1+\delta_n^2) \leq c\log M$$

for $c > 3/\log 2$ and large M. By Cauchy-Schwartz,

$$\sum_{n=M}^{T+1}\delta_n \leq 1+\sqrt{M}\sqrt{\sum_{n=M}^{T}\delta_n^2} \leq c\sqrt{M\log M}.$$

Thus

$$\sum_{n=M}^{2M}\mathbf{1}_{B_{n-1}<n^3}\frac{1}{n}\sqrt{\frac{b_n}{B_n}} \leq c\sqrt{\frac{\log M}{M}}.$$

This is summable as M varies over powers of 2, which proves the lemma. $\square$

6 A transient inhomogenous random walk with fair bounded steps in one and two dimensions

The technical details of this example are similar to those encountered in the $\mathbf{Z}^1$ in $\mathbf{Z}^3$ and $\mathbf{Z}^2$ in $\mathbf{Z}^3$ random walks. Rather than produce a third similar analysis, we only state sufficient conditions for a $\mathbf{Z}^1$ in $\mathbf{Z}^2$ walk to be transient. The schema for the calculations, as before, is

$$\mathbf{P}[S_n = \hat{0} \text{ for some } n \in [A, B]] = \frac{\mathbf{E}\#\{n \in [A, B] : S_n = \hat{0}\}}{\mathbf{E}(\#\{n \in [A, B] : S_n = \hat{0}\} \mid \text{ at least one such } n)}.$$

Let $\{a_n\}$ and $\{b_n\}$ be sequences of positive integers and let $\{S_n\}$ be a random walk in $\mathbf{Z}^2$ which does a_1 steps of a random walk uniform over all 4 diagonal neighbors, then b_1 steps of a horizontal simple random walk, then a_2 diagonal steps, then b_2 horizontal steps and so on.

Proposition 6.1 *If the sequences $\{a_n\}$ and $\{b_n\}$ satisfy conditions (34) and (35) below then the random walk $\{S_n\}$ is transient.*

Proof: Denote the partial sums of the sequences $\{a_n\}$ and $\{b_n\}$ by

$$A_n = \sum_{j=1}^{n} a_j \text{ and } B_n = \sum_{j=1}^{n} b_j$$

and let (X_n, Y_n) denote the coordinates of S_n. If $S_n = \hat{0}$ for $A_j + B_{j-1} \leq n \leq A_j + B_j$ then $Y_{A_j+B_{j-1}}$ must be zero, and X_n must be zero for some $A_j + B_{j-1} \leq n \leq A_j + B_j$. X_n and Y_n are independent, and $\mathbf{P}[Y_{A_j+B_j} = 0] \approx C(A_j + B_j)^{-1/2}$. The probability that $X_n = 0$ for some n in this range may be esitmated by $C[\sqrt{A_j + B_j} - \sqrt{A_j + B_{j-1}}]/\sqrt{b_j}$. Thus the summability condition

$$\frac{1}{\sqrt{A_n}} \frac{\sqrt{A_n + B_n} - \sqrt{A_n + B_{n-1}}}{\sqrt{b_n}} < \infty \tag{34}$$

rules out infinitely many returns in intervals $A_j + B_{j-1} \leq n \leq A_j + B_j$.

The only other way $\{S_n\}$ can return to $\hat{0}$ infintely often is if it does so during infinitely many intervals $A_j + B_j < n < A_{j+1} + B_j$. The analogous calculation yields the condition

$$\sum_{n=1}^{\infty} \frac{\sum_{j=1}^{a_n} (B_n + A_n + j)^{-1/2} (A_n + j)^{-1/2}}{\log a_n} < \infty \tag{35}$$

which is sufficient to ensure finitely many returns during intervals $A_j + B_j < n < A_{j+1} + B_j$, and finishes the proof. $\square$

To see a concrete example, with $\{a_n\}$ and $\{b_n\}$ growing almost as slowly as is allowed, let $a_n = (\log n)^{2+\epsilon}$ and $b_n = (\log n)^{4+\epsilon}$ for some $\epsilon > 0$.

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