AVERAGED MOTION OF CHARGED PARTICLES UNDER THEIR SELF-INDUCED ELECTRIC FIELD

AVNER FRIEDMAN* AND CHAOCHENG HUANG†

Abstract. In this paper we consider the averaged equations for a large number of small balls of uniform mass and charge moving under the force of their self-electric field. These equations are

$$\Delta\varphi(x,t) = -P(x,t),\ d^2\psi/dt^2 = -\nabla\varphi(\psi,t)$$

subject to $\psi(x,0) = x,\ \psi_t(x,0) = \psi_1(x)$ where φ is the electric potential and P is the limit concentration of the small balls as their number increases to infinity (and their radius goes to zero). The evolution of P is given by

$$P(x,t) = P_0(\psi^{-1}(x,t))J(\psi^{-1})(x,t)$$

where P_0 is the initial concentration, ψ^{-1} is the inverse of the mapping $x \to \psi(x,t)$ and $J(\psi^{-1})$ is its Jacobian. It is proved that if the initial data are in $C^{1+\alpha}$ then there exists a unique local solution with ψ in $C^{1+\alpha}$. The solution can be extended globally in time as long as ψ and ψ^{-1} remain uniformly bounded in C^1. There are however smooth initial data for which a global solution does not exist. One of the main results of the paper is that if $|P_0 - \chi_{B_1}|$ and $|\psi_1(x) - b_0x|$ are small enough, where χ_{B_1} is the characteristic function of the unit ball and b_0 is a positive real number, then there exists a unique global solution.

§1. The averaged equations. Let Ω be a domain in $\mathbb{R}^3$ and consider disjoint balls $B_\varepsilon(x_i^\varepsilon(t))$ in Ω of radius ε and centers $x_i^\varepsilon(t)$ which vary in time t. We assume that each ball carries electric charge, uniformly distributed within the ball, with total charge equal to its volume times a fixed constant. We denote by $\varphi_\varepsilon(x,t)$ the electric potential in Ω; it is generated by the electric charges of the balls and by the boundary conditions that are imposed at $\partial\Omega$. Then, for all $t > 0$,

$$\Delta\varphi_\varepsilon = -\xi_\varepsilon \tag{1.1}$$

where ξ_ε is the characteristic function of the disjoint union

$$\bigcup_{i=1}^{N_\varepsilon} B_\varepsilon(x_i^\varepsilon(t))$$

where N_ε = number of balls.

The centers of the balls vary according to Newton's law

$$\mu\frac{d^2x_i^\varepsilon}{dt^2} = -\fint_{B_\varepsilon(x_i^\varepsilon(t))} \nabla\varphi_\varepsilon(x,t)dx \tag{1.2}$$

where μ is a positive constant and $\fint$ means the average. We also impose initial conditions

$$x_i^\varepsilon(0) = x_{io}^\varepsilon\ ,\ \frac{dx_i^\varepsilon(0)}{dt} = x_{i1}^\varepsilon \tag{1.3}$$

and write $x_{i1}^\varepsilon = \psi_1(x_{io}^\varepsilon)$.

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* Institute for Mathematics and its Applications, University of Minnesota, Minneapolis, Minnesota 55455,
† University of Minnesota, School of Mathematics, Minneapolis, Minnesota 55455.

We are interested in the limiting behavior of the distribution of the balls as $\varepsilon \to 0$. We assume that there exists a limiting distribution $P_0(x)$ of the initial positions of the balls as $\varepsilon \to 0$, i.e.,

$$\sum_{i=1}^{N_\varepsilon} \varepsilon^3 \xi(x_{io}^\varepsilon) \to \int_\Omega P_0(x)\xi(x)dx \qquad \forall\ \xi \in C(\overline{\Omega}) \ . \tag{1.4}$$

We further assume that a limiting distribution $P(x,t)$ exists for each time $t > 0$, that is,

$$\sum_{i=1}^{N_\varepsilon} \varepsilon^3 \xi(x_i^\varepsilon(t)) \to \int_\Omega P(x,t)\xi(x)dx \qquad \forall\ \xi \in C(\overline{\Omega}) \tag{1.5}$$

as $\varepsilon \to 0$. We wish to determine the evolution of $P(x,t)$.

Formally, $\Delta\varphi_\varepsilon = -\xi_\varepsilon$ will approach

$$\Delta\varphi = -P \ , \tag{1.6}$$

and φ satisfies the same boundary conditions as φ_ε. Let $y_i^\varepsilon(t)$ be the solution of

$$\mu\frac{d^2 y}{dt^2} = -\nabla\varphi(y,t) \tag{1.7}$$

with the initial conditions (1.3). Assuming that $D^2\varphi$ and $D^2\varphi_\varepsilon$ are uniformly bounded (independently of ε), one can verify that

$$|x_i^\varepsilon(t) - y_i^\varepsilon(t)| \to 0 \quad \text{if} \quad \varepsilon \to 0\ ,\ \forall\ t > 0\ ,$$

and therefore, for any $\xi \in C(\overline{\Omega})$,

$$\sum_{i=1}^{N_\varepsilon} \varepsilon^3 \xi(y_i^\varepsilon) \to \int_\Omega P(x,t)\xi(x)dx \ . \tag{1.8}$$

Denote by $\psi(x,t)$ the solution of

$$\begin{aligned} &\mu\frac{d^2\psi}{dt^2} = -\nabla\varphi(\psi,t)\ , \quad t > 0\ , \\ &\psi(0) = x\ ,\ \frac{d\psi(0)}{dt} = \psi_1(x)\ . \end{aligned} \tag{1.9}$$

Then

$$y_i^\varepsilon(t) = \psi(x_{io}^\varepsilon, t)$$

and, by (1.8), (1.4),

$$\begin{aligned} \int_\Omega P(x,t)\xi(x)dx \leftarrow \sum_{i=1}^{N_\varepsilon} \varepsilon^3\xi(y_i^\varepsilon(t)) = \sum_{i=1}^{N_\varepsilon} \varepsilon^3\xi(\psi(x_{io}^\varepsilon,t)) \\ \to \int_\Omega P_0(x)\xi(\psi(x,t))dx \end{aligned}$$

for any $\xi \in C(\overline{\Omega})$, as $\varepsilon \to 0$. By change of variables $\psi(x,t) = y$, the last integral takes the form

$$\int_{\Omega} P_0(\psi^{-1}(y,t))J(\psi^{-1})(y,t)\xi(y)dy$$

where $\psi^{-1}(x,t)$ is the inverse $\psi(x,t)$ (for t fixed) and J denotes the Jacobian of the mapping. We then conclude that

$$P(x,t) = P_0(\psi^{-1}(x,t))J(\psi^{-1})(x,t) \ . \tag{1.10}$$

Thus the limiting distribution $P(x,t)$ evolves according to the system (1.6), (1.9), (1.10), with some prescribed boundary conditions on φ. More generally we may include some hydrodynamic effects. Thus, we can include Stokes' drag by adding a friction terms in the ODE (1.9). The system then becomes:

$$\begin{gathered}
\Delta\varphi = -P \quad \text{in} \quad \Omega \ , \\
P(x,t) = P_0(\psi^{-1}(x,t))J(\psi^{-1})(x,t) \ , \\
\mu\frac{d^2\psi}{dt^2} + \lambda\frac{d\psi}{dt} = -\nabla\varphi(\psi,t) \ , \\
\psi(x,0) = x, \ \frac{d\psi(0)}{dt} = \psi_1(x)
\end{gathered} \tag{1.11}$$

with prescribed boundary conditions for φ on $\partial\Omega$.

In the formal derivation of (1.11) we excluded collision of balls; this implies that $0 \le P \le 1$. However, we shall henceforth admit any nonnegative P; this means that we allow "soft" collisions in the approximating model.

The system (1.11) may also be viewed as plasma with uniformly distributed mass and non-uniformly distributed electric charge $P(x,t)$. Equation (1.10) is then a conservation law for the charge density.

§**2. The main results.** The results of this paper will be established in n-dimensional space for any $n \ge 2$. For simplicity we shall consider here only the case

$$\Omega = \mathbb{R}^n \ , \quad \text{and} \quad \varphi(x,t) \to 0 \quad \text{if} \quad |x| \to \infty \ . \tag{2.1}$$

We assume that

$$\psi_1(x) \in C^{1+\alpha}(\mathbb{R}^n) \ , \tag{2.2}$$

$$\begin{aligned}
&\Omega_0 \text{ is a bounded domain in } \mathbb{R}^n \text{ given by} \\
&\Omega_0 = \{g_0(x) > 0\}, \ \partial\Omega_0 = \{g_0(x) = 0\}, \ g_0 \in C^{1+\alpha}(\mathbb{R}^n) \ , \\
&\nabla g_0(x) \neq 0 \quad \text{on} \quad \partial\Omega_0 \ ,
\end{aligned} \tag{2.3}$$

$$\begin{aligned}
&P_0(x) \ge 0 \quad \text{if} \quad x \in \overline{\Omega}_0, \ P_0(x) = 0 \quad \text{if} \quad x \in \mathbb{R}^n \backslash \overline{\Omega}_0 \ , \\
&\text{and} \quad P_0|_{\overline{\Omega}_0} \in C^\alpha(\overline{\Omega}_0) \ .
\end{aligned} \tag{2.4}$$

Note that if P_0 is a nonnegative $C^\alpha(\mathbb{R}^n)$ function with compact support, then (2.4) is satisfied for suitable Ω_0 as in (2.3). Another special case is

$$P_0 = \chi_{\Omega_0} \tag{2.5}$$

where Ω_0 satisfies (2.3).

Denote by Ω_t the image of Ω_0 under the mapping $\psi(\cdot,t)$, that is, $\Omega_t = \{\psi(x,t) : x \in \Omega_0\}$. Since $P(x,t) = 0$ if $x \notin \overline{\Omega}_t$, we shall consider the functions $\psi(x,t)$ only for $x \in \Omega_0$ and $\psi^{-1}(x,t)$ only for $x \in \Omega_t$. We shall denote by $|\psi(\cdot,t)|_\alpha$ (or briefly by $|\psi|_\alpha$) the α-Hölder coefficient of $\psi(x,t)$ in $x \in \Omega_0$, and by $|\psi^{-1}(\cdot,t)|_\alpha$ (or briefly by $|\psi^{-1}|_\alpha$) the α-Hölder coefficient of $\psi^{-1}(x,t)$ in $x \in \Omega_t$. We also write

$$|\psi(\cdot,t)|_{C^{1+\alpha}} = |\psi(\cdot,t)|_{L^\infty} + |\nabla\psi(\cdot,t)|_{L^\infty} + |\nabla\psi(\cdot,t)|_\alpha$$

where the gradient is taken with respect to x and $|\ |_\alpha$ refers to the α-Hölder coefficient in $x \in \Omega_0$; the L^∞-norms are taken for x in Ω_0. Similarly we define

$$|\psi^{-1}(\cdot,t)|_{C^{1+\alpha}} \quad \text{for} \quad x \in \Omega_t \ .$$

Theorem 2.1. *If (2.2)–(2.4) hold then the system (1.11), (2.1) has a unique solution for some time interval $0 \le t \le t_0$ $(t_0 > 0)$ such that*

$$\sup_{0\le t\le t_0} |\psi(\cdot,t)|_{C^{1+\alpha}} < \infty \ , \ \sup_{0\le t\le t_0} |\psi^{-1}(\cdot,t)|_{C^{1+\alpha}} < \infty \ . \tag{2.6}$$

Theorem 2.2. *Under the assumptions of Theorem 2.1, if for some $T > 0$ there exists a solution of (1.11), (2.1) for all $t < T$ such that*

$$|\nabla\psi(\cdot,t)|_{L^\infty} \le C, \ |\nabla\psi^{-1}(\cdot,t)|_{L^\infty} \le C \qquad (C < \infty) \tag{2.7}$$

for all $0 \le t < T$, then the solution is unique and can be extended to $0 \le t \le T + \delta$ for some $\delta > 0$ depending only on C and the initial data, and (2.6) holds with $t_0 = T + \delta$.

Thus the á priori estimate (2.7) is sufficient for global existence. Theorems 2.1 and 2.2 are proved in §5 and §6 respectively; auxiliary estimates are established in §§3–4. In §7 it is shown that the estimate (2.7) is not always satisfied and, in fact, for some smooth radially symmetric initial data, the Jacobian $J(\psi^{-1})$ may blow up in finite time.

The remaining part of the paper is devoted to the case where the data are nearly radially symmetric:

$$\begin{aligned} &P_0 = \chi_{B_1} + \varepsilon P_{02} \quad (B_1 = \{x \in \mathbb{R}^n ;\ |x| < 1\}) \ , \\ &\psi_t(x,0) = b_0 x + \varepsilon b_1(x), \ b_0 > 0 \end{aligned} \tag{2.8}$$

where ε is positive and small. If $\lambda = 0,\ \varepsilon = 0$ then there exists a unique global solution, and ψ has the form $\psi(x,t) = f(t)x$.

Theorem 2.3. *If $P_{02} \in C^\alpha(\mathbb{R}^n)$, supp$P_{02} \subset B_1, b_1 \in C^{1+\alpha}(\mathbb{R}^n)$ and $\lambda = 0$, then for any ε sufficiently small there exists a unique global solution of (1.11), (2.1) with initial data $P_0, \psi_t(x,0)$ given by (2.8).*

The proof, given for simplicity only for the case $\mu = 1, n = 3$, is presented in §§8–13.

We conclude this section with some remarks on the case $\mu = 0$, $\lambda = 1$. In this case,

$$\frac{d\psi}{dt} = -\nabla\varphi(\psi, t)\ ,\ \ \psi(x,0) = x\ .$$

It is well known [3; p.25] that

$$J(\psi) = \exp\int_0^t \ \text{trace}\ \left(-\frac{\partial^2}{\partial x_i \partial x_j}\varphi\right)(\psi, t)dt\ ,$$

so that

$$J(\psi) = \exp\int_0^t [-\Delta\varphi(\psi, t)]dt = \exp\int_0^t P(\psi, t)dt$$

$$= \exp\int_0^t P_0(x)J(\psi)^{-1}(x, t)dt$$

by (1.11). It follows that

$$\frac{d}{dt}J(\psi) = J(\psi)P_0(x)J(\psi)^{-1} = P_0(x)$$

and thus

$$J(\psi) = 1 + tP_0(x)\ .$$

The system (1.11) then becomes

$$\begin{aligned} &\Delta\varphi = -P_0(\psi^{-1})(1 + tP_0(\psi^{-1}))^{-1}\ , \\ &\frac{d\psi}{dt} = -\nabla\varphi(\psi, t)\ ,\ \ \psi(x,0) = x\ . \end{aligned} \tag{2.9}$$

If P_0 is given by (2.5), then

$$\Delta\varphi = -\frac{1}{1+t} \quad \text{in} \quad \Omega_t\ .$$

Specializing to $n = 2$ we can then write

$$\begin{aligned} &\nabla\varphi = -\int_{\Omega_t} \frac{x-y}{|x-y|^2}\ \frac{1}{1+t}\ dy\ , \\ &\frac{d\psi}{dt} = -\nabla\varphi(\psi, t)\ ,\ \ \psi(x,0) = x\ . \end{aligned} \tag{2.10}$$

This problem is nearly identical to the vortex patch problem for the 2-d Euler equation; in the latter case

$$\begin{aligned} &\nabla\varphi = -\int_{\Omega_t} \frac{x-y}{|x-y|^2}\ dy\ , \\ &\frac{d\psi}{dt} = -\nabla^\perp\varphi(\psi, t),\ \ \psi(x,0) = x \end{aligned} \tag{2.11}$$

where $\nabla^\perp\varphi = (-\varphi_y, \varphi_x)$. Global existence and uniqueness of $C^{1+\alpha}$ solution for the vortex patch problem was established by Chemin [2]. A simpler proof was recently given by Bertozzi and Constantin [1]; see [1] [2] for further references regarding weak solutions.

Our proofs of Theorems 2.1–2.3 use potential theoretic estimates established by Friedman and Velázquez [4] in a work on electrophotography, as well as some estimates from [1].

§**3. Auxiliary Lemmas.** Let Ω be a bounded domain in $\mathbb{R}^n$ given by

$$\Omega = \{g(x) > 0)\} \ , \ \partial\Omega = \{g(x) = 0\} \ , \quad \text{where} \tag{3.1}$$
$$g \in C^{1+\alpha}(\mathbb{R}^n) \ , \ \inf_{\partial\Omega} |\nabla g| > 0 \ .$$

We denote by $|\ |_{L^\infty(D)}$ the $L^\infty(D)$-norm and by $|\ |_{C^\alpha(D)}$ the α-Hölder coefficient in D. Set

$$\delta = \left(\frac{\inf\limits_{\partial\Omega} |\nabla g|}{2|\nabla g|_\alpha} \right)^{1/\alpha} \tag{3.2}$$

where $|\nabla g|_\alpha = |\nabla g|_{C^\alpha(\mathbb{R}^n)}$.

Introduce the function

$$W(x) = \int\limits_\Omega \frac{x-y}{|x-y|^n}\, dy \ , \ x \in \Omega \ . \tag{3.3}$$

We denote by $d(\Omega)$ the diameter of Ω and by M any number $\geq \max\{1, d(\Omega)\}$.

Lemma 3.1. *There exists a constant C depending only on n, α and M such that*

$$|\nabla W|_{L^\infty(\Omega)} \leq C(2 + |\log\delta|) \ , \tag{3.4}$$

$$|\nabla W|_{C^\alpha(\Omega)} \leq \frac{C}{\delta^\alpha}(2 + |\log\delta|) \ . \tag{3.5}$$

Proof. The estimate (3.4) for $n = 2$ is precisely Proposition 1 in [1]; its proof depends on the Geometric Lemma in [1]. Both the lemma and the proposition easily extend to $n > 2$.

To prove (3.5), we shall first estimate the Hölder coefficient of ∇W on $\partial\Omega$. Take any point $x^0 \in \partial\Omega$ and choose the coordinates such that $x^0 = 0$ and

$$g_{x_i}(x^0) = 0 \quad \text{if} \quad i = 1, \ldots, n-1 \ .$$

Then, for any $x \in B_\delta(x^0) \cap \partial\Omega$,

$$g_{x_n}(x) \geq g_{x_n}(0) - |x|^\alpha |\nabla g|_\alpha > \frac{1}{2} \inf_{\partial\Omega} |\nabla g| \ . \tag{3.6}$$

Write $x' = (x_1, \ldots, x_{n-1})$; then $x = (x', x_n)$. From (3.6) it follows, by the implicit function theorem, that there exists a unique function $x_n = f(x')$ such that

$$\partial\Omega \cap B_\delta(x^0) \quad \text{is given by} \ \ x_n = f(x') \ ,$$

i.e.,

$$g(x', f(x')) = 0 \ ,$$

and

$$f \in C^{1+\alpha} \ , \ f(0) = 0 \ , \ |(x', f(x')| \leq \delta \ .$$

Clearly

$$f_{x_i}(x') = -\frac{g_{x_i}(x', f(x'))}{g_{x_n}(x', f(x'))} \ .$$

Therefore

$$(3.7)\qquad |f_{x_i}(x')| \le \frac{|g_{x_i}|_\alpha |(x', f(x'))|^\alpha}{\inf\limits_{\partial\Omega} |\nabla g|} \le \frac{|\nabla g|_\alpha}{\inf\limits_{\partial\Omega} |\nabla g|}\delta^\alpha \le 1 \ .$$

Also

$$|f_{x_i}(x') - f_{x_i}(\overline{x}')| \le \frac{|g_{x_i}(x', f(x')) - g_{x_i}(\overline{x}', f(\overline{x}'))|}{|g_{x_n}(x', f(x'))|}$$
$$+ \frac{|g_{x_i}(\overline{x}', f(\overline{x}'))|}{|g_{x_n}(x', f(x'))||g_{x_n}(\overline{x}', f(\overline{x}'))|} \ |g_{x_n}(x', f(x')) - g_{x_n}(\overline{x}', f(\overline{x}'))| \ ,$$

and using (3.7) we easily conclude that

$$(3.8)\qquad |f_{x_i}(x') - f_{x_i}(\overline{x}')| \le \frac{4|\nabla g|_\alpha}{\inf\limits_{\partial\Omega} |\nabla g|} \ |x' - \overline{x}'|^\alpha = \frac{|x' - \overline{x}|^\alpha}{\delta^\alpha} \ .$$

If $n = 2$ then by [4; Lemma 3.1] the tangential derivative of W along $\partial\Omega$ is in $C^{1+\alpha}$, and

$$(3.9)\qquad |\frac{d}{dx_1} \ W(x_1, f(x_1)) - \frac{d}{dx_1} \ W(\overline{x}_1, f(\overline{x}_1))| \le |x_1 - \overline{x}_1|^\alpha \frac{C(1 + |\log\delta|)}{\delta^\alpha} \ ;$$

here (3.7) and (3.8) have been used. It follows that

$$(3.10)\ |W(y_1, f(y_1)) - [W(x_1, f(x_1)) - \frac{d}{dx_1} W(x_1, f(x_1)) \cdot (y_1 - x_1)]| \le |x-y|^{1+\alpha} \frac{C(1+|\log\delta|)}{\delta^\alpha}.$$

This estimate is valid uniformly for $x, y \in \partial\Omega$ provided $|x - y| < \delta$ (with a parametrization f which depends on x); if $|x - y| > \delta$ then, by (3.4), (3.10) is again true if C is large enough.

The proof of (3.9) can be extended to the case $n > 2$; see Remark 3.1 below. Hence (3.10) can also be extended to $n > 2$. Each component of W is harmonic function (since $\Delta \int |x - y|^{2-n} dy =$ const. if $n > 2$ and $\Delta \displaystyle\int \log|x - y| =$ const. if $n = 2$). Therefore we can apply $C^{1+\alpha}$ estimates (Theorem 3.1 of Widman [6]) to deduce the assertion (3.5).

Remark 3.1. Lemma 12.1 (in §12) provides a proof of (3.9) for $n = 3$ in the more complicated case where Ω is replaced by $\psi(B_1)$. If in that proof we replace the balls $B_\eta(y_0), B_{\widetilde{\eta}}(\widetilde{y}_0)$ by half spaces $\Pi_\eta(x_0)$, $\Pi_\eta(\widetilde{x}_0)$ tangent to $\partial\psi(B_1)$, then we obtain a proof of (3.9) for $n = 3$, which is the extension of the proof of [4; Lemma 3.1] for $n = 2$; the proof for $n > 3$ is the same.

Set

$$(3.11)\qquad K(x) = (K_{ij}(x)),\ K_{ij}(x) = \frac{\partial^2}{\partial x_i \partial x_j} \ \frac{1}{|x|^{n-2}} = \frac{\sigma_{ij}}{|x|^n}$$

and consider

$$G(x) = \int\limits_{\mathbb{R}^n} K(x - y)(h(x) - h(y))w(y)dy \ .$$

We shall need later on the following lemma due to Bertozzi and Constantin [1; p. 26]:

Lemma 3.2. *If $h \in C^\alpha(\mathbb{R}^n)$ and $w \in L^\infty(\mathbb{R}^n)$ then*

$$|G|_{C^\alpha(\mathbb{R}^n)} \le C|h|_{C^\alpha(\mathbb{R}^n)} \left[|K * w|_{L^\infty(\mathbb{R}^n)} + |w|_{L^\infty(\mathbb{R}^n)}\right]$$

where C is a constant depending only on n, α.

§**4. A priori estimates.** For simplicity we shall prove Theorems 2.1, 2.2 only in case $\lambda = 0$; the proof for $\lambda \neq 0$ is similar.

We can solve

$$\Delta\varphi = -P \quad \text{in } \mathbb{R}^n, \quad \varphi = 0 \text{ at infinity}$$

by

$$\varphi(x,t) = \frac{1}{(n-2)\omega_n} \int_{\mathbb{R}^n} P(y) \frac{1}{|x-y|^{n-2}}\, dy \tag{4.1}$$

if $n > 2$, and then

$$\nabla\varphi(x,t) = -\gamma_n \int_{\mathbb{R}^n} P(y) \frac{x-y}{|x-y|^n}\, dy \qquad \left(\gamma_n = \frac{1}{\omega_n}\right) \tag{4.2}$$

where ω_n is the surface area of the unit sphere in $\mathbb{R}^n$; for $n = 2$, (4.1) is replaced by

$$\varphi(x,y) = \gamma_2 \int_{\mathbb{R}^2} P(y) \log \frac{1}{|x-y|}\, dy \qquad \left(\gamma_2 = \frac{1}{\omega_2} = \frac{1}{2\pi}\right) \tag{4.1$'$}$$

but (4.2) remains the same.

Setting

$$\Omega_t = \psi(\Omega_0, t) = \{\psi(x,t)\ ;\ x \in \Omega_0\} \tag{4.3}$$

we can rewrite (4.2) in the form

$$\begin{aligned} \nabla\varphi(x,t) \ &= \ -\gamma_n \int_{\Omega_t} P_0(\psi^{-1}(y,t) J(\psi^{-1})(y,t) \frac{x-y}{|x-y|^n}\, dy \\ &= \ -\gamma_n \int_{\Omega_0} P_0(z) \frac{x-\psi(z,t)}{|x-\psi(z,t)|^n}\, dz\ . \end{aligned} \tag{4.4}$$

The original problem (1.11), (2.1) then reduces to

$$\mu \frac{\partial^2 \psi(x,t)}{\partial t^2} = \gamma_n \int_{\Omega_0} P_0(z) \frac{\psi(x,t) - \psi(z,t)}{|\psi(x,t)-\psi(z,t)|^n}\, dz\ , \tag{4.5}$$

$$\psi(x,0) = x, \quad \psi_t(x,0) = \psi_1(x)\ . \tag{4.6}$$

In this section we assume that there exists a solution to (4.5), (4.6) with

$$\psi \in C^{1+\alpha}(\overline{\Omega}_0) \quad \text{and } J(\psi) \neq 0 \text{ for } 0 \le t < T \text{ for some } T \ge 0\ .$$

Then $\psi^{-1}(x,t)$ exists and it belongs to $C^{1+\alpha}(\overline{\Omega}_t)$. We shall derive $C^{1+\alpha}$ estimates which will be used in the next section to prove existence and uniqueness for (4.5), (4.6).

Set

$$V(x,t) = \int_{\Omega_t} P_0(\psi^{-1}(y,t))J(\psi^{-1})(y,t)\frac{x-y}{|x-y|^n}\, dy \ . \tag{4.7}$$

For any $x^0 \in \Omega_t$ choose $\rho > 0$ small enough so that $B_\rho(x^0) \subset \Omega_t$. For any $x \in B_\rho(x^0)$ we write

$$\begin{aligned} V(x,t) = & \int_{\Omega_t \setminus B_\rho(x^0)} P_0(\psi^{-1}(y,t))J(\psi^{-1})(y,t)\frac{x-y}{|x-y|^n}\, dy \\ & + \int_{B_\rho(x^0)} P_0(\psi^{-1}(y,t))J(\psi^{-1})(y,t)\frac{x-y}{|x-y|^n}\, dy \ . \end{aligned}$$

Then

$$\begin{aligned} \nabla_x V(x,t) = & \int_{\Omega_t \setminus B_\rho(x^0)} P_0(\psi^{-1}(y,t))J(\psi^{-1})(y,t)\frac{\sigma(x-y)}{|x-y|^n}\, dy \\ & + \int_{B_\rho(x^0)} \left[P_0(\psi^{-1}(y,t))J(\psi^{-1})(y,t)) - P_0(\psi^{-1}(x,t))J(\psi^{-1})(x,t)\right] \frac{\sigma(x-y)}{|x-y|^n}\, dy \\ & - \int_{\partial B_\rho(x^0)} \left[\frac{x-y}{|x-y|^n} \otimes \nu\right] P_0(\psi^{-1}(x,t))J(\psi^{-1})(x,t)\, dS_y \end{aligned}$$

where $\sigma = (\sigma_{ij})$, σ_{ij} as in (3.11), and ν is the outward normal. Taking $x = x^0$ and letting $\rho \to 0$, we get

$$\begin{aligned} \nabla_x V(x^0,t) = \lim\nolimits_{\rho\to 0} & \int_{\Omega_t \setminus B_\rho(x^0)} P_0(\psi^{-1}(y,t))J(\psi^{-1})(y,t)\frac{\sigma(x^0-y)}{|x^0-y|^n}\, dy \\ & + \omega_n P_0(\psi^{-1}(x^0,t))J(\psi^{-1})(x^0,t)I \end{aligned}$$

where I is the identity matrix. Therefore, by (4.4), (4.7), for any $x \in \Omega_0$,

$$\begin{aligned} -\frac{1}{\gamma_n}\, & \nabla(\nabla\varphi(\psi(x,t),t) = \nabla(V(\psi(x,t),t)) = \nabla V(\psi(x,t),t)\nabla\psi(x,t) \\ & = \lim_{\rho\to 0} \int_{\Omega_t \setminus B_\rho(\psi(x,t))} P_0(\psi^{-1}(y,t))J(\psi^{-1})(y,t)\frac{\sigma(\psi(x,t)-y)}{|\psi(x,t)-y|^n}\, dy \nabla\psi(x,t) \\ & + \omega_n P_0(x)J(\psi(x,t))^{-1}\nabla\psi(x,t)\ ; \end{aligned} \tag{4.8}$$

here we used the relation $J(\psi^{-1}(\psi(x,t))\cdot J(\psi(x,t)) = 1$. Changing variables, $y = \psi(z,t)$, and setting

$$\widetilde{B}_\rho(\psi(x,t)) = \psi^{-1}(B_\rho(x),t)\ ,$$

we obtain from the last formula,

$$
\begin{aligned}
(4.9)\qquad -\frac{1}{\gamma_n}\nabla(\nabla\varphi(\psi(x,t),t)) &= \lim_{\rho\to 0}\int\limits_{\Omega_0\setminus\widetilde{B}_\rho(\psi(x,t))} P_0(z)\frac{\sigma(\psi(x,t)-\psi(z,t))}{|\psi(x,t)-\psi(z,t)|^n}\,dz\nabla\psi(x,t)\\
&\quad+\omega_n P_0(x)J(\psi(x,t))^{-1}\nabla\psi(x,t)\equiv A\nabla\psi+B\nabla\psi\ .
\end{aligned}
$$

We shall denote the diameter of Ω_t by $d(\Omega_t)$ and denote by M any positive constant such that

$$
(4.10)\qquad \max\{1,d(\Omega_t)\}\le M\ .
$$

LEMMA 4.1. *There exists a constant C depending only on n,α and M such that, for all $t>0$,*

$$
\begin{aligned}
(4.11)\qquad |\nabla(\nabla\varphi(\psi,t))|_{L^\infty} &\le C(1+|P_0|_{L^\infty})|\nabla\psi|_{L^\infty}\Big[1\\
&\quad+|J(\psi^{-1})|_{L^\infty}\log\Big(1+|P_0|_\alpha+|\nabla\psi|_{L^\infty}+|\nabla\psi^{-1}|_{L^\infty}+|\nabla\psi|_\alpha\Big)\Big].
\end{aligned}
$$

Proof. In view of (4.9) it suffices to estimate the term A, which can be written as $A=\lim\limits_{\rho\to 0}A_\rho$ where

$$
\begin{aligned}
(4.12)\qquad A_\rho &= \int\limits_{\Omega_0\setminus\widetilde{B}_\rho(\psi(x,t))}[P_0(z)-P_0(x)]\frac{\sigma(\psi(x,t)-\psi(z,t))}{|\psi(x,t)-\psi(z,t)|^n}\,dz\\
&\quad+P_0(x)\int\limits_{\Omega_0\setminus\widetilde{B}_\rho(\psi(x,t))}\frac{\sigma(\psi(x,t)-\psi(z,t))}{|\psi(x,t)-\psi(z,t)|^n}\,dz\equiv K_1+P_0(x)K_2\ .
\end{aligned}
$$

Clearly

$$
\begin{aligned}
(4.13)\qquad |K_1| &\le \int\limits_{\Omega_t}|P_0(\psi^{-1}(y,t))-P_0(x)|\frac{C}{|\psi(x,t)-y|^n}\ |J(\psi^{-1})|_{L^\infty}\ dy\\
&= C|J(\psi^{-1})|_{L^\infty}\left[\ \int\limits_{\Omega_t\setminus B_{\rho_0}(\psi(x,t))}+\int\limits_{B_{\rho_0}(\psi(x,t))}\right]\\
&= C|J(\psi^{-1})|_{L^\infty}(K_{11}+K_{12})\quad(\text{for any }\rho_0>0)\ ,
\end{aligned}
$$

and

$$
\begin{aligned}
(4.14)\qquad K_{11} &\le C|P_0|_{L^\infty}\int\limits_{\rho_0}^{d(\Omega_t)}\frac{1}{r^n}\ r^{n-1}dr = C|P_0|_{L^\infty}|\log\rho_0|\ ,\\
K_{12} &\le C\int\limits_{B_{\rho_0}(\psi(x,t))}|P_0(\psi^{-1})|_\alpha\frac{1}{|\psi(x,t)-y|^{n-\alpha}}\ \le C|P_0(\psi^{-1})|_\alpha\rho_0^\alpha\ .
\end{aligned}
$$

Choosing

$$\rho_0^\alpha = \frac{1}{1 + |P_0(\psi^{-1})|_\alpha}$$

and using the inequality

$$|P_0(\psi^{-1})|_\alpha \le |P_0|_\alpha |\nabla\psi^{-1}|_{L^\infty}^\alpha \ ,$$

we find that

$$|K_1| \le C(1 + |P_0|_{L^\infty})|J(\psi^{-1})|_{L^\infty}(1 + \log(1 + |P_0|_\alpha |\nabla\psi^{-1}|_{L^\infty})) \ . \tag{4.15}$$

To estimate K_2 we introduce a small parameter $\rho_1 > \rho$ to be determined later on, and write $K_2 \equiv K_2(\rho)$ in the form

$$\begin{aligned} K_2 &= K_2(\rho_1) + I_1 \quad \text{where} \\ I_1 &= \int\limits_{\Omega_0 \cap (\widetilde{B}_{\rho_1} \setminus \widetilde{B}_\rho)} \frac{\sigma(\psi(x,t) - \psi(z,t))}{|\psi(x,t) - \psi(z,t)|^n}\, dz \ ; \end{aligned} \tag{4.16}$$

here $\widetilde{B}_r = \widetilde{B}_r(\psi(x,t))$. By change of variables,

$$\begin{aligned} |K_2(\rho_1)| &\le \int\limits_{\Omega_0 \setminus \widetilde{B}_{\rho_1}} \frac{C}{|\psi(x,t) - \psi(z,t)|^n}\, dz \\ &\le \int\limits_{\Omega_t \setminus B_{\rho_1}(\psi(x,t))} \frac{C}{|\psi(x,t) - y|^n} |J(\psi^{-1})(y,t)| dy \\ &\le |J(\psi^{-1}|_{L^\infty} \int\limits_{\rho_1}^{d(\Omega_t)} \frac{C}{r^n} r^{n-1} dr \le C|J(\psi^{-1})|_{L^\infty} |\log \rho_1| \ , \end{aligned} \tag{4.17}$$

and

$$I_1 = \int\limits_{\Omega_t \cap (B_{\rho_1} \setminus B_\rho)} \frac{\sigma(\psi(x,t)) - y)}{|\psi(x,t) - y|^n}\, J(\psi^{-1})(y,t) dy$$

where $B_r = B_r(\psi(x,t))$.

Write

$$\begin{aligned} I_1 \ &= \int\limits_{\Omega_t \cap (B_{\rho_1} \setminus B_\rho)} \frac{\sigma(\psi(x,t) - y)}{|\psi(x,t) - y|^n} [J(\psi^{-1})(y,t) - J(\psi^{-1})(\psi(x,t),t)] dy \\ &+ \int\limits_{\Omega_t \cap (B_{\rho_1} \setminus B_\rho)} \frac{\sigma(\psi(x,t) - y)}{|\psi(x,t) - y|^n}\, dy J(\psi^{-1})(\psi(x,t),t) \equiv I_{11} + I_{12} \ . \end{aligned} \tag{4.18}$$

Then

$$|I_{11}| \le C|J(\psi^{-1})|_\alpha \int\limits_{B_{\rho_1}\setminus B_\rho} \frac{dy}{|\psi(x,t)-y|^{n-\alpha}} \le C|J(\psi^{-1})|_\alpha \rho_1^\alpha \ . \tag{4.19}$$

Next, if $\rho < d$ where $d = \text{dist}\ (\psi(x,t), \partial\Omega_t)$, then

$$I_{12} = \int\limits_{\Omega_t\cap(B_{\rho_1}\setminus B_d)} \frac{\sigma(\psi(x,t)-y)}{|\psi(x,t)-y|^n}\ dy J(\psi^{-1})(\psi(x,t),t) \tag{4.20}$$

since

$$\int\limits_{\Omega_t\cap(B_d\setminus B_\rho)} \frac{\sigma(\psi(x,t)-y)}{|\psi(x,t)-y|^n}\ dy = 0$$

(The spherical average of σ is zero).

We can represent $\partial\Omega_t$ in the form

$$\partial\Omega_t = \{g(x,t) = 0\}$$

where

$$g(x,t) = g_0(\psi^{-1}(x,t)),\ g_0 \text{ as in (2.3).} \tag{4.21}$$

From the proof of Proposition 1 in [1; p. 23], if $n = 2$ then I_{12}, defined by (4.20), can be estimated in the form

$$|I_{12}| \le C\left(1 + |\log \frac{\delta}{d(\Omega_t)}|\right) |J(\psi^{-1})|_{L^\infty} \tag{4.22}$$

where

$$\delta^\alpha = \frac{\inf\limits_{\partial\Omega_t} |\nabla g|}{|\nabla g|_\alpha}\ . \tag{4.23}$$

The proof is based on the geometric lemma in [1; p. 23] (or on Lemma 10.1 in §10), and it easily extends to $n > 2$.

From (4.21) we get

$$\nabla g = \nabla\psi^{-1}\nabla g_0(\psi^{-1})\ ,\ \nabla\psi\nabla g = \nabla g_0\ ,$$

so that

$$\inf_{\partial\Omega_t} |\nabla g| \ge |\nabla\psi|_{L^\infty}^{-1} \inf_{\partial\Omega_0} |\nabla g_0| \ge C|\nabla\psi|_{L^\infty}^{-1}\ ,\ C > 0\ .$$

Also

$$\begin{aligned}|\nabla g|_\alpha &\le |\nabla\psi^{-1}|_{L^\infty}|\nabla g_0(\psi^{-1})|_\alpha + |\nabla\psi^{-1}|_\alpha|\nabla g_0|_{L^\infty}\\ &\le C(|\nabla\psi^{-1}|_{L^\infty}^{1+\alpha} + |\nabla\psi^{-1}|_\alpha)\ .\end{aligned}$$

Using these estimates we can obtain a lower bound on the δ in (4.23), namely,

$$\frac{1}{\delta^\alpha} \le C(|\nabla\psi^{-1}|_{L^\infty}^{1+\alpha} + |\nabla\psi^{-1}|_\alpha)|\nabla\psi|_{L^\infty} \ . \tag{4.24}$$

We then conclude from (4.22) that

$$|I_{12}| \le C\log(2 + |\nabla\psi|_{L^\infty} + |\nabla\psi^{-1}|_{L^\infty} + |\nabla\psi^{-1}|_\alpha)|J(\psi^{-1})|_{L^\infty} \ .$$

Combining this with (4.19) and choosing

$$\rho_1 = \left(|\nabla\psi^{-1}|_{L^\infty}^{1+\alpha}|\nabla\psi|_\alpha\right)^{-1/\alpha}$$

we deduce from (4.18) and from (4.16), (4.17) that

$$|K_2| \le C\left[|J(\psi^{-1})|_{L^\infty} + \frac{|J(\psi^{-1})|_\alpha}{|\nabla\psi^{-1}|^{1+\alpha}|\nabla\psi|_\alpha}\right]\log\left[2 + |\nabla\psi|_{L^\infty} + |\nabla\psi^{-1}|_{L^\infty} + |\nabla\psi^{-1}|_\alpha\right] \ . \tag{4.25}$$

If we expand the determinant $J(\psi^{-1})$ as sums of product of its elements, we find that

$$|J(\psi^{-1})|_{L^\infty} \le C|\nabla\psi^{-1}|_{L^\infty}^n \tag{4.26}$$

and

$$|J(\psi^{-1})|_\alpha \le C|\nabla\psi^{-1}|_{L^\infty}^{n-1}|\nabla\psi^{-1}|_\alpha \ .$$

Further, dropping t,

$$\frac{|\nabla\psi^{-1}(\psi(x)) - \nabla\psi^{-1}(\psi(y))|}{|\psi(x)-\psi(y)|^\alpha} = \frac{|\nabla\psi(x)^{-1} - \nabla\psi(y)^{-1}|}{|\psi(x)-\psi(y)|^\alpha}$$
$$= \frac{|\nabla\psi(x)^{-1}|\ |\nabla\psi(y) - \nabla\psi(x)|\ |\nabla\psi(y)^{-1}|}{|\psi(x)-\psi(y)|^\alpha}$$

so that

$$|\nabla\psi^{-1}|_{C^\alpha(\Omega_t)} \le |\nabla\psi|_\alpha\ |\nabla\psi^{-1}|_{L^\infty}^{2+\alpha} \ . \tag{4.27}$$

It follows that

$$|J(\psi^{-1})|_\alpha \le C|\nabla\psi^{-1}|_{L^\infty}^{n+1+\alpha}|\nabla\psi|_\alpha \ . \tag{4.28}$$

Using this and (4.26) in (4.25) we obtain

$$|K_2| \le C|\nabla\psi^{-1}|_{L^\infty}^n \log[2 + |\nabla\psi|_{L^\infty} + |\nabla\psi^{-1}|_{L^\infty} + |\nabla\psi|_\alpha] \ . \tag{4.29}$$

Recalling (4.15), we find that $A\nabla\psi = \lim_{\rho\to 0}(A_\rho\nabla\psi)$ (where A_ρ is defined in (4.12)) is bounded by the right-hand side of (4.11). Since the same is true for $B\nabla\psi$ (defined in (4.9)), the proof of (4.11) is complete.

The next lemma provides C_α estimates on $\nabla(\nabla\varphi(\psi(x,t),t)$. We shall use the notation

$$\Psi_1 = |\nabla\psi|_{L^\infty},\ \ \Psi_2 = |\nabla\psi^{-1}|_{L^\infty},\ \ \Psi = \max\{\Psi_1,\Psi_2\} \ . \tag{4.30}$$

Lemma 4.2 *There exists a constant C depending only on n, α and M such that*

$$
\begin{aligned}
|\nabla(\nabla\varphi(\psi(x,t),t))|_\alpha \le C(1+|P_0|_{L^\infty}+|P_0|_\alpha)\Psi^{n+5}(1+|\nabla\psi|_\alpha) \\
\times \log(2+|P_0|_\alpha+\Psi+|\nabla\psi|_\alpha)\ .
\end{aligned} \tag{4.31}
$$

Proof. We shall use (4.9). By (4.24), (4.26),

$$|B\nabla\psi|_\alpha \le C(|P_0|_{L^\infty}+|P_0|_\alpha)\Psi^{n+3}(1+|\nabla\psi|_\alpha)\ . \tag{4.32}$$

Next

$$|A\nabla\psi|_\alpha \le |A|_{L^\infty}|\nabla\psi|_\alpha + |A|_\alpha|\nabla\psi|_{L^\infty} \tag{4.33}$$

and $|A|_{L^\infty}$ is bounded by the right-hand side of (4.11), so that

$$|A|_{L^\infty}|\nabla\psi|_\alpha \le C(1+|P|_{L^\infty})\Psi^n|\nabla\psi|_\alpha \log[2+|P_0|_\alpha+\Psi+|\nabla\psi|_\alpha]\ . \tag{4.34}$$

We write A in the form (4.12) with $\rho=0$ and proceed to estimate $|K_1|_\alpha$. Since K_1 is a function of $\psi(x,t)$, we can write $K_1 = V(\psi(x,t))$ where

$$
\begin{aligned}
V(x) &= \int_{\Omega_0}[P_0(z)-P_0(\psi^{-1}(x,t))]\frac{\sigma(x-\psi(z,t))}{|x-\psi(z,t)|^n}\,dz \\
&= \int_{\Omega_t}[P_0(\psi^{-1}(y,t))-P_0(\psi^{-1}(x,t))]\frac{\sigma(x-y)}{|x-y|^n}\,J(\psi^{-1})(y,t)dy\ .
\end{aligned}
$$

By Lemma 3.2,

$$|V|_\alpha \le C|P_0(\psi^{-1}(x,t))|_\alpha[|J(\psi^{-1})|_{L^\infty}+|K*J(\psi^{-1})|_{L^\infty}]\ .$$

Observe that $(K*J(\psi^{-1}))(\psi(x,t))$ is equal (after change of variables in the integral) to K_2 as defined in (4.12) with $\rho=0$. Therefore it can be estimated by (4.25). Using also (4.26), 4.27) we conclude that

$$|K_1|_\alpha = |V(\psi(x,t))|_\alpha \le C|P_0|_\alpha\Psi_1^\alpha\Psi_2^{n+\alpha}\log[2+\Psi+|\nabla\psi|_\alpha]\ . \tag{4.35}$$

Next we need to estimate P_0K_2. Since K_2 is a function of $\psi(x,t)$, we can write $K_2 = U(\psi(x,t))$ where

$$
\begin{aligned}
U(\xi) &= \int_{\Omega_t}\frac{\sigma(\xi-y)}{|\xi-y|^n}\,J(\psi^{-1})(y,t)dy \\
&= \int_{\Omega_t}\frac{\sigma(\xi-y)}{|\xi-y|^n}[J(\psi^{-1})(y,t)-J(\psi^{-1})(\xi,t)]dy \\
&\quad + J(\psi^{-1})(\xi,t)\int_{\Omega_t}\frac{\sigma(\xi-y)}{|\xi-y|^n}\,dy \equiv H_1+H_2\ .
\end{aligned}
$$

By Lemma 3.2

$$|H_1|_\alpha \le C|J(\psi^{-1})|_\alpha\ .$$

By Lemma 3.1

$$|H_2|_\alpha \le |J(\psi^{-1})|_{L^\infty}\frac{C}{\delta^\alpha}(2+|\log\delta|)+|J(\psi^{-1})|_\alpha C(2+|\log\delta|)\ ,$$

and δ is bounded below by (4.24). Using (4.26)–(4.28) and (4.24), we easily deduce for $K_2 = U(\psi(x,t))$ the bound

$$\begin{aligned} |K_2|_\alpha &\le (|H_1|_\alpha + |H_2|_\alpha)|\nabla\psi|^\alpha_{L^\infty} \\ &\le C\Psi_1^\alpha\Psi_2^{n+1+\alpha}[1+(1+\Psi_2|\nabla\psi|_\alpha)\Psi_1\log(2+\Psi+|\nabla\psi|_\alpha)]. \end{aligned} \tag{4.36}$$

Recalling also (4.25), we find that

$$|P_0K_2|_\alpha \le C(|P_0|_{L^\infty}+|P_0|_\alpha)\Psi^{n+1}(1+|\nabla\psi|_\alpha)\log[2+\Psi+|\nabla\psi|_\alpha]\ . \tag{4.37}$$

From (4.35), (4.37) and (4.12) (with $\rho = 0$) we see that $|A|_\alpha$ is bounded by the right-hand side of (4.37). Combining this fact and (4.34) we find that the right-hand side of (4.33) is also bounded by the right-hand side of (4.37). Together with (4.32) and (4.9) the assertion (4.31) then follows.

Remark 4.1. From (4.29), (4.35) and (4.36) we obtain a more accurate Hölder estimate for the function A defined in (4.9) (and given also in the form (4.12) with $\rho = 0$):

$$\begin{aligned} |A|_\alpha \le C\Psi_2^n[&|P_0|_{L^\infty}\Psi_1^\alpha\Psi_2^{1+\alpha} + (|P_0|_\alpha\Psi_1^\alpha\Psi_2^\alpha + |P_0|_\alpha) \\ &+|P_0|_{L^\infty}\Psi_1^{1+\alpha}\Psi_2^{1+\alpha} + |P_0|_{L^\infty}\Psi_1^{1+\alpha}\Psi_2^{2+\alpha}|\nabla\psi|_\alpha\log(2+\Psi+|\nabla\psi|_\alpha)]\ . \end{aligned} \tag{4.38}$$

This estimate will be used in the proof of Theorem 2.3.

§**5. Proof of Theorem 2.1.** In this section we prove Theorem 2.1. In order to apply the proof step-by-step so as to get a global solution, we shall consider more general initial conditions than (4.6), namely,

$$\psi(x,0)=\psi_0(x),\quad \psi_t(x,0)=\psi_1(x) \tag{5.1}$$

where ψ_0, ψ_1 belong to $C^{1+\alpha}(\overline{\Omega}_0)$. We can rewrite (4.5), (5.1) in the form

$$\psi(x,t)=\psi_0(x)+t\psi_1(x)+\mathcal{A}\psi \equiv \mathcal{B}\psi \tag{5.2}$$

where

$$\mathcal{A}\psi = \int_0^t\int_0^s\int_{\Omega_0}\frac{\gamma_n}{\mu}P_0(z)\frac{\psi(x,\tau)-\psi(z,\tau)}{|\psi(x,\tau)-\psi(z,\tau)|^n}\,dzd\tau ds\ . \tag{5.3}$$

By Lemmas 4.1, 4.2,

$$|\nabla_x\mathcal{A}\psi|_{L^\infty} \le Ct^2(1+|P_0|_{L^\infty})\Psi^{n+1}\log[2+|P_0|_\alpha+\Psi+|\nabla\psi|_\alpha]\ , \tag{5.4}$$

$$|\nabla_x \mathcal{A}\psi|_\alpha \le Ct^2(1+|P_0|_{L^\infty}+|P_0|_\alpha)\Psi^{n+5}(1+|\nabla\psi|_\alpha) \times \log[2+|P_0|_\alpha+\Psi+|\nabla\psi|_\alpha] \ . \tag{5.5}$$

THEOREM 5.1. *Suppose*

$$|\nabla\psi_0| \le a,\ |\nabla\psi_0|_\alpha \le a,\ |\nabla\psi_1|_{L^\infty} \le a\ ,\ |\nabla\psi_1|_\alpha \le a \quad \textit{and} \quad |\nabla\psi_0^{-1}|_{L^\infty} \le b\ .$$

Then there exists a unique solution of (5.2) for $0 \le t \le t_0$, for some $t_0 > 0$ depending only on a and b.

Proof. For T, k_1, k_2 positive and $T \le 1$, define a set

$$\begin{aligned} K \equiv K(k_1,k_2,T) = \{\psi(x,t) \in C^1(\overline{\Omega}_0 \times [0,T])\ ,\\ \nabla_x\psi \in C_x^\alpha,\ |\nabla_x\psi|_{L^\infty} \le k_1, |\nabla_x\psi|_\alpha \le k_1\ ,\\ |D_t\psi| \le k_1,\ |\nabla_x\psi - \nabla_x\psi_0| \le k_2\}\ . \end{aligned}$$

Notice that $d(\Omega_t) \le d(\Omega_0)+k_1T$ and choose $M = 3+d(\Omega_0)+k_1$. Let $k_2 = \dfrac{1}{2b}$. Then

$$|\nabla_x\psi\nabla_x\psi_0^{-1} - I| \le |\nabla_x\psi_0^{-1}|k_2 < \frac{1}{2}\ .$$

Hence $|\nabla_x\psi^{-1}| \le 2b$.

From (5.4), (5.5) we deduce that

$$\begin{aligned} &|\nabla_x\mathcal{B}\psi|_{L^\infty} \le a(1+t)+Ct^2\ ,\\ &|\nabla_x\mathcal{B}\psi|_\alpha \le |\nabla\psi_0|_\alpha + t|\nabla\psi_1|_\alpha + Ct^2\ ,\\ &|D_t\mathcal{B}\psi|_{L^\infty} \le |\nabla\psi_1|_{L^\infty} + Ct\ , \quad \text{and}\\ &|\nabla_x\mathcal{B}\psi - \nabla\psi_0|_{L^\infty} \le t|\nabla\psi_1|_{L^\infty} + Ct^2 \end{aligned}$$

where C is a constant depending only on M, k_1, a and b. If $k_1 = 2a$ then for T small enough (depending only on a, b) $\mathcal{B}$ maps the set K into itself.

Next take any two functions $\psi, \overline{\psi}$ in K and set

$$\begin{aligned} &\delta(x,t) = \psi(x,t) - \overline{\psi}(x,t)\ ,\\ &\widetilde{\delta}(x,t) = (\mathcal{B}\psi)(x,t) - (\mathcal{B}\overline{\psi})(x,t)\ ,\\ &\delta(t) = |\delta(\cdot,t)|_{L^\infty}\ ,\ \widetilde{\delta}(t) = (\widetilde{\delta}(\cdot,t)|_{L^\infty}\ . \end{aligned}$$

We shall estimate $\widetilde{\delta}$ in terms of δ. Write

$$\begin{aligned} \widetilde{\delta}(x,t) &= \int_{\Omega_0} \frac{\gamma_n}{\mu}\ P_0(z)\left[\frac{\psi(x,t)-\psi(z,t)}{|\psi(x,t)-\psi(z,t)|^n} - \frac{\overline{\psi}(x,t)-\overline{\psi}(z,t)}{|\overline{\psi}(x,t)-\overline{\psi}(z,t)|^n}\right] dz \\ &= \int_{\Omega_0\cap B_{\rho_1}(x)} + \int_{\Omega_0\setminus B_{\rho_1}(x)} \equiv I_1 + I_2 \end{aligned} \tag{5.6}$$

where ρ_1 will be determined later on.

By the mean value theorem,

$$
\begin{aligned}
&|\nabla\psi^{-1}|_{L^\infty}|\psi(x,t)-\psi(z,t)| \geq |x-z| \ , \\
&|\nabla\overline{\psi}^{-1}|_{L^\infty}|\overline{\psi}(x,t)-\overline{\psi}(z,t)| \geq |x-z| \ .
\end{aligned} \tag{5.7}
$$

It follows that

$$
|I_1| \leq \int\limits_{\Omega_0\cap B_{\rho_1}(x)} \frac{C}{|x-z|^{n-1}}\ dz \leq C\rho_1 \ . \tag{5.8}
$$

Next,

$$
\begin{aligned}
\frac{\mu}{\gamma_n}\ I_2 = &\int\limits_{\Omega_0\setminus B_{\rho_1}(x)} \frac{\delta(x,t)-\delta(z,t)}{|\psi(x,t)-\psi(z,t)|^n}\ dz \\
&\int\limits_{\Omega_0\setminus B_{\rho_1}(x)} \frac{(\overline{\psi}(x,t)-\overline{\psi}(z,t))[|\overline{\psi}(x,t)-\overline{\psi}(z,t)|^n-|\psi(x,t)-\psi(z,t)|^n]}{|\psi(x,t)-\psi(z,t)|^n|\overline{\psi}(x,t)-\overline{\psi}(z,t)|^n}\ dz \\
&\equiv I_{21}+I_{22} \ .
\end{aligned}
$$

By (5.7),

$$
|I_{21}| \leq 2\delta(t) \int\limits_{\Omega_0\setminus B_{\rho_1}(x)} \frac{C}{|x-z|^n}\ dz \leq C\delta(t)(1+|\log\rho_1|)
$$

and

$$
|I_{22}| \leq \int\limits_{\Omega_0\setminus B_{\rho_1}(x)} \frac{C|\delta(x,t)-\delta(z,t)|}{|x-z|^n}\ dz \leq C\delta(t)(1+|\log\rho_1|) \ .
$$

Hence

$$
|I_2| \leq C\delta(t)(1+|\log\rho_1|) \ . \tag{5.9}
$$

Choosing $\rho_1=\delta(t)$ we conclude from (5.9), (5.8) and (5.6) that

$$
\tilde{\delta}(t) \leq C\int_0^t\int_0^s \delta(\tau)[1+|\log\delta(\tau)|]d\tau ds \ .
$$

Since $\delta(t)\leq Ct^2, |\log\delta(\tau)|>1$ if τ is small enough and, consequently,

$$
\tilde{\delta}(t) \leq C\int_0^t \delta(\tau)|\log\delta(\tau)|d\tau \ . \tag{5.10}
$$

Choose any $\psi^{(0)}$ in K and define

$$
\psi^{(j+1)} = \mathcal{B}\psi^{(j)} \ .
$$

Then the $\psi^{(j)}$ belong to K and, setting

$$\delta_{j+1}(t) = |\psi^{(j+1)}(\cdot,t) - \psi^{(j)}(\cdot,t)|_{L^\infty} ,$$

we have, by (5.10),

$$\delta_{j+1}(t) \le C \int_0^t \delta_j(s)|\log \delta_j(s)|ds .$$

By [4; §9] this implies that the full sequence $\psi^{(j)}$ is uniformly convergent to a fixed point of $\mathcal{B}$, which is of course a solution of (5.2). Uniqueness also follows from (5.10), as in [4; §9].

§**6. Proof of Theorem 2.2.** Recall that the solution ψ has the form

$$\psi(x,t) = \psi_0(x) + t\psi_1(x) + \int_0^t \int_0^s (\mathcal{M}\psi)(x,\tau)d\tau ds$$

where

$$(\mathcal{M}\psi)(x,t) = \int_\Omega P_0(z) \frac{\psi(x,t)-\psi(z,t)}{|\psi(x,t)-\psi(z,t)|^n} \, dz .$$

Introducing $\zeta = \dfrac{d\psi}{dt}$, we have

$$\tag{6.1} \frac{d}{dt}\begin{pmatrix} \nabla\zeta \\ \nabla\psi \end{pmatrix} = \begin{pmatrix} \nabla\mathcal{M}\psi \\ \nabla\zeta \end{pmatrix} , \qquad \begin{pmatrix} \nabla\zeta \\ \nabla\psi \end{pmatrix}_{t=0} = \begin{pmatrix} \nabla\psi_1 \\ \nabla\psi_0 \end{pmatrix} .$$

For any $x, \overline{x}$ in Ω_0,

$$\nabla\zeta(x,t) - \nabla\zeta(\overline{x},t) = \nabla\psi_1(x) - \nabla\psi_1(\overline{x})$$
$$+ \int_0^t [\nabla(\mathcal{M}\psi)(x,s) - \nabla(\mathcal{M}\psi)(\overline{x},s)]ds .$$

Hence, by Lemma 4.2 and (2.7)

$$|\nabla\zeta(x,t) - \nabla\zeta(\overline{x},t)| \le |\nabla\psi_1|_\alpha |x-\overline{x}|^\alpha$$
$$+ C\int_0^t C\log[2+|\nabla\psi|_\alpha][1+|\nabla\psi|_\alpha]|x-\overline{x}|^\alpha dt .$$

Hence

$$\tag{6.2} |\nabla\zeta|_\alpha \le C|\nabla\psi_1|_\alpha + C\int_0^t \log[2+|\nabla\psi|_\alpha][1+|\nabla\psi|_\alpha]dt .$$

From (6.1) we also have

$$\tag{6.3} |\nabla\psi|_\alpha \le C|\nabla\psi_0|_\alpha + C\int_0^t |\nabla\zeta|_\alpha dt .$$

Setting

$$G(t) = 2 + |\nabla\zeta|_\alpha + |\nabla\psi|_\alpha$$

we conclude that $G(t) > 2$ and

$$G(t) \le C_1 + C\int_0^t G(t)\log G(t)dt \equiv R(t)$$

where $C_1 = 1 + C(|\nabla\psi_0|_\alpha + |\nabla\psi_1|_\alpha)$. Since $G \le R$,

$$R' = CG\log G \le CR\log R\ .$$

It follows that

$$R\log\log R \le \log\log C_1 + Ct$$

and

$$G(t) \le R(t) \le \exp[\log C_1 e^{Ct}]\ .$$

This gives a priori bound on $|\nabla\psi|_\alpha$ and $|\nabla\psi_t|_\alpha$ for $0 < t < T$ in terms of $|\nabla\psi_0|_\alpha + |\nabla\psi_1|_\alpha$. One can also obtain (more easily) a priori bound on $|\nabla\psi|_{L^\infty}$ and $|\nabla\psi_t|_{L^\infty}$ for $0 < t < T$. We can then apply the proof of Theorem 5.1 in order to extend the solution from $t = T - \varepsilon$ (ε arbitrarily small) to $t = T + \delta$ for some $\delta > 0$ (independent of ε). This completes the proof of Theorem 2.2.

§**7. Radially symmetric solutions.** In this section we give a counterexample to global existence. We consider radial solutions, taking $n = 3$.

Setting

$$\psi(x,t) = \overline{\psi}(|x|,t)\frac{x}{|x|}$$

one can compute directly that

$$\psi^{-1}(x,t) = \overline{\psi}^{-1}(|x|,t)\frac{x}{|x|}$$

and

$$J(\psi^{-1})(r,t) = \frac{(\overline{\psi}^{-1}(r,t))^2}{r^2}\ \frac{d}{dr}\ \psi^{-1}(r,t)\ .$$

Writing $\varphi = \varphi(r,t), P = P(r,t)$ $(r = |x|)$, and taking $\psi_1(x) = \overline{\psi}_1(r)\frac{x}{|x|}$, the system (1.11) for $\mu = 1,\ \lambda = 0$ becomes

$$\begin{aligned}
&\varphi_{rr} + \frac{2}{r}\,\varphi_r = -P(r,t)\ ,\\
&P(r,t) = P_0(\overline{\psi}^{-1}(r,t))\frac{(\overline{\psi}^{-1})^2}{r^2}\left|\frac{d}{dr}\ \overline{\psi}^{-1}(r,t)\right|\ ,\\
&\frac{d^2\overline{\psi}(r,t)}{dt^2} = -\varphi_r(\overline{\psi},t)\ ,\\
&\overline{\psi}(r,0) = r, \frac{d\overline{\psi}(r,0)}{dt} = \overline{\psi}_1(r)\ .
\end{aligned}\tag{7.1}$$

Imposing the boundary condition $\varphi(r,t) \to 0$ if $r \to \infty$, we can solve the first equation:

$$\varphi_r(r,t) = -\frac{1}{r^2} \int_0^r s^2 P(s,t) ds$$
$$= -\frac{1}{r^2} \int_0^r s^2 P_0(\overline{\psi}^{-1}(s,t)) \left| \frac{d\overline{\psi}^{-1}(s,t)}{ds} \right| \left(\frac{\overline{\psi}^{-1}(s,t)}{s} \right)^2 ds \; .$$

Changing variables and noting, by symmetry, that $\overline{\psi}(0,t) = 0$, we get

$$\varphi_r(r,t) = -\frac{1}{r^2} \int_0^{\overline{\psi}^{-1}(r,t)} \rho^2 P_0(\rho) d\rho \; , \tag{7.2}$$

so that

$$\frac{d^2\overline{\psi}}{dt^2} = \frac{1}{\overline{\psi}^2(r,t)} \int_0^r \rho^2 P_0(\rho) d\rho \; . \tag{7.3}$$

Hence

$$\overline{\psi}(r,t) = r + t\overline{\psi}_1(r) + \int_0^t \int_0^s \frac{1}{\overline{\psi}^2(r,\tau)} \int_0^r \rho^2 P_0(\rho) d\rho d\tau ds. \tag{7.4}$$

It follows that

$$\overline{\psi}_r(r,t) = 1 + t\overline{\psi}_1'(r) + r^2 P_0(r) \int_0^t \int_0^s \frac{d\tau ds}{\overline{\psi}^2(r,\tau)} - \int_0^t \int_0^s \frac{2\overline{\psi}_r(r,\tau)}{\overline{\psi}^3(r,\tau)} \int_0^r \rho^2 P_0(\rho) d\rho d\tau ds \; . \tag{7.5}$$

Assume that $\overline{\psi}_1(r) \geq 0$ and $\overline{\psi}_1(0) = 0$. We see from (7.4) that $\overline{\psi}(r,t) \geq r$. Hence the right-hand side of (7.3) is $\leq C_1 r$, and from (7.4) we then get

$$r \leq \overline{\psi}(r,t) \leq C_1(1+t^2) r \; . \tag{7.6}$$

Consider now the question of global existence of solutions, and set

$$F = \overline{\psi}(r,t), \; b = \overline{\psi}_1(r), \; a(r) = \int_0^r \rho^2 P_0(\rho) d\rho \; .$$

Then $a \geq 0$, $b \geq 0$ and $a(0) = b(0) = 0$, and $F(r,t)$ satisfies:

$$\begin{aligned} F'' &= \frac{a(r)}{F^2(r,t)} \qquad \left(' = \frac{d}{dt} \right) \; , \\ F(r,0) &= r, \; F'(r,0) = b(r) \; . \end{aligned} \tag{7.7}$$

Since $\overline{\psi}_r(r,0) = 1 > 0$ and we are considering only solutions with $\overline{\psi}_r(r,t) \neq 0$ (i.e., $J(\psi^{-1}) \neq 0$), we need to impose the additional condition

$$F_r(r,t) > 0 \; . \tag{7.8}$$

From (7.6) we have the a priori estimates

$$r \le F(r,t) \le C_1(1+t^2)r \ . \tag{7.9}$$

Rewriting (7.7) in the form

$$F(r,t) = r + tb(r) + \int_0^t \int_0^s \frac{a(r)}{F(r,\tau)^2} \, d\tau ds \tag{7.10}$$

and using (7.9), one can easily establish global existence for (7.10), or (7.7). Thus in order to get global solution for (1.11), (2.1), it remains to verify the inequality (7.8).

For simplicity we shall specialize to

$$P_0 = \chi_{(0,1)}, \text{ so that } a(r) = \frac{r^3}{3} \ .$$

Theorem 7.1. *Let $P_0 = \chi_{(0,1)}$. Then (i) there exists a global solution to (1.11), (2.1) if either $b(r) \equiv 0$, or*

$$b(r) > 0, \ \ b(r)b'(r) + \frac{2}{3}\, r \ge 0 \qquad \forall\ 0 \le r \le 1\ ; \tag{7.11}$$

(ii) if $b(r) > 0$ for $0 < r \le 1$ but

$$b(r_0)b'(r_0) + \frac{2}{3}\, r_0 < 0 \quad \text{for some} \quad 0 < r_0 < 1\ , \tag{7.12}$$

then there does not exist a global solution to (1.11), (2.1).

Proof. To prove (i) we only need to verify (7.8). If $b(r) \equiv 0$ then $F = f(t)r$ where

$$f'' = \frac{1}{3f^2}\ ,\ f(0) = 1\ ,\ f'(0) = 0\ .$$

It is clear that $f(t)$ remains positive for all $t > 0$, so that $F_r = f > 0$.

To consider the case where $b(r) > 0$ for $0 < r \le 1$, we multiply both sides of the differential equation in (7.7) by F' and integrate. We obtain

$$F' = \left[b^2 + 2a\left(\frac{1}{r} - \frac{1}{F}\right)\right]^{1/2}\ ,$$

or

$$t = \int_r^{F(r,t)} \frac{dx}{\left[b^2 + 2a\left(\frac{1}{r} - \frac{1}{x}\right)\right]^{1/2}}\ . \tag{7.13}$$

Taking the derivative with respect to r and recalling that $a = r^3/3$, we easily find that

$$F_r(r,t) > 0 \quad (< 0) \quad \text{if and only if}$$

(7.14)

$$\frac{1}{b(r)} + \left(bb' + \frac{2}{3}\, r\right) \int_r^{F(r,t)} \frac{dx}{\left[b^2 + 2a\left(\frac{1}{r} - \frac{2}{x}\right)\right]^{3/2}} > 0 \quad (< 0)\ .$$

Consequently, under the assumptions of (7.11), $F_r > 0$.

Consider the case (ii). Then (7.14) is still valid. Also

$$\int_{r_0}^{F(r_0,t)} \frac{dx}{\left[b^2(r_0) + 2a(r_0)\left(\frac{1}{r_0} - \frac{1}{x}\right)\right]^{3/2}}$$

$$\geq \int_{r_0}^{F(r_0,t)} \frac{dx}{\left(b^2(r_0) + \frac{2}{3}\, r_0^2\right)\left[b^2(r_0) + 2a(r_0)\left(\frac{1}{r_0} - \frac{1}{x}\right)\right]^{1/2}}$$

$$= \frac{t}{b^2(r_0) + \frac{2}{3}\, r_0^2} \qquad \text{by (7.13).}$$

Recalling (7.12), it follows that at $r = r_0$

$$\frac{1}{b} + \left(bb' + \frac{r}{3}\right)\int_r^{F(r,t)} \frac{dx}{\left[b^2 + 2a\left(\frac{1}{r} - \frac{1}{x}\right)\right]^{3/2}}$$

$$\leq \frac{1}{b} + \left(bb' + \frac{r}{3}\right)\frac{t}{\left(b^2 + \frac{2}{3}\, r^2\right)} < 0$$

if t is sufficiently large. Hence (by (7.14)) $F_r(r_0, t) < 0$ if t is sufficiently large.

Remark 7.1. Consider the one dimensional case with $\Omega = \{-c < x < \infty\}$, $P_0 = \chi_{\Omega_0}$, $\Omega_0 = (a, b)$ where $-c < a < b < \infty$. Then

$$\begin{aligned}
&\varphi_{xx}(x,t) = -P(x,t) \quad \text{in} \quad \Omega\ , \\
&\psi_{tt}(x,t) = -\varphi_x(\psi(x,t),t) \quad \text{in} \quad \Omega_0\ , \\
&P(x,t) = P_0(\psi^{-1}(x,t))\psi_x^{-1}(x,t) \quad \text{in} \quad \Omega_t = \psi(\Omega_0, t)
\end{aligned}$$

and, assuming $\varphi_x(-c, t) = 0$, we find that

$$\varphi_x(x,t) = -\int_{\psi(a,t)}^{x} P(\xi,t)d\xi = -\int_a^{\psi^{-1}(x,t)} P_0(\eta)d\eta\ ,$$

so that

$$\psi(x,t) = x + t\psi_1(x) + \frac{t^2}{2}\int_a^x P_0(\eta)d\eta\ .$$

With whatever boundary condition we take at $x = \infty$, global solution (with $\psi_x \neq 0$) exists if and only if

$$\psi_x(x,t) = 1 + t\psi_1'(x) + \frac{t^2}{2}\, P_0(x) > 0\ ,$$

i.e., if and only if $\psi_1'(x) \geq -\sqrt{2}$.

§8. **Proof of Theorem 2.3.** For simplicity we give the proof in case $n=3,\ \mu=1$. Set $r=|x|$,

$$\chi_{(0,1)}(x)=\begin{cases}1 & \text{if } 0\le r<1\\ 0 & \text{if } r>1\ ,\end{cases}$$

and

$$P_{01}=\chi_{(0,1)}(r)\ . \tag{8.1}$$

Let $f(t)$ be the solution of

$$f''(t)=\frac{1}{3f^2}\quad \text{for}\quad t>0,\ f(0)=1,\quad f'(0)=b_0\ge 0\ . \tag{8.2}$$

Denote by (φ_1,ψ_1,P_1) the solution to (1.11), (2.1) corresponding to the initial data (2.8) when $\varepsilon=0$, i.e.,

$$\begin{aligned}&\Delta\varphi_1=-P_1\ ,\\ &\frac{d^2\psi_1}{dt^2}=-\nabla\varphi_1(\psi_1,t)\ ,\\ &P_1=P_{01}(\psi_1^{-1})J(\psi_1^{-1})\ ,\end{aligned} \tag{8.3}$$

As seen from the calculations in §7, the solution exists globally, and it is given by

$$\psi_1(x,t)=f(t)x\ , \tag{8.4}$$

$$\frac{\partial}{\partial r}\,\varphi_1(r,t)=-\frac{1}{r^2}\int\limits_0^{r/f(t)}\rho^2P_{01}(\rho)d\rho\ . \tag{8.5}$$

It is easy to check that

$$\begin{aligned}&f(t)>1,\ 0<f'(t)\le c\quad \text{for all } t>0,\\ &f(t)=ct(1+o(1)),\ f'(t)=c(1+o(1))\quad \text{as}\quad t\to\infty\end{aligned} \tag{8.6}$$

where $c=\left(\frac{2}{3}+b_0^2\right)^{1/2}$, $o(1)\to 0$ as $t\to\infty$. By assumption,

$$P_0=P_{01}+\varepsilon P_{02} \tag{8.7}$$

where

$$\operatorname{supp}P_{02}\subset B_1\ ,\ P_{02}\in C^\alpha(\mathbb{R}^3)\ , \tag{8.8}$$

and

$$\frac{\partial}{\partial t}\,\psi(x,0)=b_0x+\varepsilon b_1(x)\ ,\ b_1\in C^{1+\alpha}(\mathbb{R}^3)\ . \tag{8.9}$$

We wish to solve (1.11), (2.1) by writing ψ in the form

$$\psi = \psi_1(x,t) + \varepsilon\psi_2(x,t) \ ; \tag{8.10}$$

ψ_2 depends, of course, on ε. Then ψ_2 must satisfy (cf. (4.5))

$$\varepsilon\frac{d^2\psi_2}{dt^2} = \frac{1}{4\pi}\int_{B_1}(P_{01}+\varepsilon P_{02})\frac{\psi(x,t)-\psi(z,t)}{|\psi(x,t)-\psi(z,t)|^3}\,dz - \frac{d^2\psi_1}{dt^2}\ ,$$

(8.11)

$$\psi_2(x,0) = 0\ ,\ \psi_{2,t}(x,0) = b_1(x)\ .$$

Lemma 8.1. *There exists a small constant* $\lambda \in \left(0,\frac{1}{2}\right)$ *such that for any solution of (1.11), (2.1) for* $0 \le t \le T$*, written in the form (8.10), if*

$$|\varepsilon\nabla\psi_2(\cdot,t)|_{L^\infty} \le \lambda f(t),\ |\varepsilon\nabla\psi_2(\cdot,t)|_\alpha \le \lambda f(t) \tag{8.12}$$

then

$$|\nabla\psi_2(\cdot,t)|_{L^\infty} \le Cf(t),\ |\nabla\psi_2(\cdot,t)|_\alpha \le Cf(t)\ , \tag{8.13}$$

provided ε *is small enough, where* C *is a constant independent of* ε *and* T*.*

Assuming that the lemma is true we can complete the proof of Theorem 2.3 as follows.

Proof of Theorem 2.3

From (8.4) and (8.10) we have the following identity:

$$\varepsilon\nabla\psi_2(x,t) = [\nabla\psi(x,t)-\nabla\psi(x,\tau)] - [f(t)-f(\tau)]I + \varepsilon\nabla\psi_2(x,\tau) \tag{8.14}$$

for any $t,\tau \ge 0$.

By Theorem 2.1 there exists a unique solution for $0 \le t \le \tilde{T}$ (for some $\tilde{T} > 0$) for any $0 < \varepsilon < 1$, and

$$|\nabla\psi(\cdot,t)|_{L^\infty} \le C,\ |\nabla\psi(\cdot,t)|_\alpha \le C\ ,\ |\nabla\psi^{-1}(\cdot,t)|_{L^\infty} \le C\ .$$

By the identity (8.14) with $\tau = 0$ we have (since $f(0) = 1$)

$$\varepsilon|\nabla\psi_2(\cdot,t)|_{L^\infty} \le |\nabla\psi(\cdot,t)-\nabla\psi(\cdot,0)|_{L^\infty} + |f(t)-1| + \varepsilon|\nabla b_1|_{L^\infty}\ , \tag{8.15}$$

$$\varepsilon|\nabla\psi_2(\cdot,\tau)|_\alpha \le |\nabla\psi(\cdot,t)-\nabla\psi(\cdot,0)|_\alpha + \varepsilon|\nabla b_1|_\alpha\ . \tag{8.16}$$

We can estimate the first terms on the right-hand sides of (8.15) and (8.16) by using the representation (5.2) of ψ and the estimates (5.4), (5.5), to obtain the inequalities (8.12) for $0 \le t \le \tilde{T}$, provided $\tilde{T}$ is small enough.

Now let T be any positive number such that (8.12) holds for $0 \le t \le T$. Then, by Lemma 8.1, (8.13) holds. Since

$$\nabla\psi^{-1}(\psi(x,t),t) = (\nabla\psi(x,t))^{-1} = \frac{1}{f}\left(I + \frac{\varepsilon\nabla\psi_2}{f}\right)^{-1}\ , \tag{8.17}$$

we have

$$|\nabla\psi^{-1}|_{L^\infty} \le \frac{2}{f(t)} \ . \tag{8.18}$$

Hence, by the proof of Theorem 2.2, the solution can be uniquely extended to an interval $t \le T + \delta$ for some $\delta > 0$ depending only on C, and

$$|\nabla\psi(\cdot,t)|_{L^\infty} \le 2Cf(t), |\nabla\psi(\cdot,t)|_\alpha \le 2Cf(t), |\nabla\psi^{-1}(\cdot,t)|_{L^\infty} \le 2Cf(t) \ .$$

for $T \le t \le T+\delta$. By the same arguments as before, we may choose $\delta(C)$ such that

$$|\nabla\psi(\cdot,t) - \nabla\psi(\cdot,T)|_{L^\infty} \le \frac{1}{2}\ \lambda f(t) \ ,$$

$$|\nabla\psi(\cdot,t) - \nabla\psi(\cdot,T)|_\alpha \le \frac{1}{2}\ \lambda f(t) \ ,$$

for $T \le t \le T+\delta$. Using (8.14), it follows that

$$|\varepsilon\nabla\psi_2(\cdot,t)|_{L^\infty} \le |f(t) - f(T)| + |\varepsilon\nabla\psi_2(\cdot,T)|_{L^\infty} + \frac{1}{2}\ \lambda f(t) \ ,$$

$$|\varepsilon\nabla\psi_2(\cdot,t)|_{L^\infty} \le |\varepsilon\nabla\psi_2(\cdot,T)|_\alpha + \frac{1}{2}\ \lambda f(t) \ ,$$

for $T \le t \le T+\delta$. By (8.6) $|f(t) - f(T)| \le |t-T|\ |f'| \le \frac{1}{4}\ \lambda f(t)$ if δ is small. Therefore, if we take $\delta = \delta(C)$ small, $\varepsilon \le \varepsilon_1(C)$ small such that $\varepsilon C < \frac{\lambda}{4}$, and use (8.13) for $t = T$, then (8.12) follows for $T \le t \le T+\delta$. Since $\delta = \delta(C)$ and $\varepsilon_1(C)$ were independent of T, a step-by-step argument establishes global existence and uniqueness, as well as the estimates in (8.13). This completes the proof of Theorem 2.3.

The remaining part of the paper is devoted to the proof of Lemma 8.1. Various positive constants independent of ε and T will be denoted by the same symbol C.

Remark 8.1. The proofs of Lemma 8.1 and Theorem 2.3 can be extended to the case where $\lambda \ne 0$ and to more general radially symmetric functions P_{01} and functions P_{02}.

§**9. Proof of Lemma 8.1.** We shall often suppress the variable t, writing $\psi(x)$ instead of $\psi(x,t)$, etc. We shall denote by C constants independent of ε and λ.

Observe that (8.12) implies that

$$|\nabla\psi| \le 2f(t) \ , \ |\nabla\psi|_\alpha = |\varepsilon\nabla\psi_2|_\alpha \le f(t) \ . \tag{9.1}$$

From (8.14) and

$$J(\psi^{-1}) = \det(\nabla\psi)^{-1}$$

we get

$$|J(\psi^{-1})|_{L^\infty} \le \frac{C}{f^3(t)} \ . \tag{9.2}$$

Recalling (4.28) we then also have (using (9.1))

$$|J(\psi^{-1})|_\alpha \le \frac{C|\nabla\psi|_\alpha}{f^{4+\alpha}} \le \frac{C|\varepsilon\nabla\psi_2|_\alpha}{f^{4+\alpha}} \le \frac{C}{f^{3+\alpha}} \ . \tag{9.3}$$

Note (cf. (4.5)) that

$$\frac{d^2\psi_1}{dt^2} = \frac{1}{4\pi}\int\limits_{B_1} P_{01}\frac{\psi_1(x,t)-\psi_1(z,t)}{|\psi_1(x,t)-\psi_1(z,t)|^3}\ dz$$

and (cf. (4.4), (4.5), (4.9))

$$\frac{d^2\nabla\psi_1}{dt^2} = \left[\frac{1}{4\pi}\int\limits_{B_1} P_{01}\frac{\sigma(\psi_1(x)-\psi_1(z))}{|\psi_1(x)-\psi_1(z)|^3}\ dz + P_{01}J(\psi_1^{-1})\right]\nabla\psi_1 \ .$$

Applying ∇ to (8.11) we can reduce the problem (1.11), (2.1) (as in (4.5), (4.6)) to

$$\begin{aligned}
\frac{d^2\nabla\psi_2}{dt^2} &= \frac{1}{4\pi\varepsilon}\Big[\int\limits_{B_1}\frac{\sigma(\psi(x)-\psi(z))}{|\psi(x)-\psi(z)|^3}dz - \int\limits_{B_1}\frac{\sigma(\psi_1(x)-\psi_1(z))}{|\psi_1(x)-\psi_1(z)|^3}dz\Big]\nabla\psi_1(x)\\
&+\frac{1}{4\pi}\int\limits_{B_1}(P_{01}+\varepsilon P_{02})\frac{\sigma(\psi(x)-\psi(z))}{|\psi(x)-\psi(z)|^3}dz\cdot\nabla\psi_2(x)\\
&+\frac{1}{4\pi}\int\limits_{B_1}P_{02}\frac{\sigma(\psi(x)-\psi(z))}{|\psi(x)-\psi(z)|^3}\ dz\cdot\nabla\psi_1(x)\\
&+\frac{1}{4\pi}\ J(\psi^{-1})[P_{02}\nabla\psi_1(x)+(P_{01}+\varepsilon P_{02})\nabla\psi_2(x)]\\
&-\frac{1}{\varepsilon}\ P_{01}\nabla\psi_1(x)[J(\psi_1^{-1})-J(\psi^{-1})]\\
&\equiv I_1+I_2+I_3+I_4+I_5\ ,
\end{aligned}\tag{9.4}$$

with

$$\nabla\psi_2(x,0)=0\ ,\ \nabla\psi_{2,t}(x,0)=\nabla b_1(x)\ . \tag{9.5}$$

Note that in the above and later, all the singular integrals are understood as the principle value (in the sense of (4.9)).

In the remaining part of this section we estimate I_2, I_3, I_4 and I_5; I_1 will be estimated in the subsequent sections.

Step 1. L^∞ estimates.

In the same way that we introduced $A\nabla\psi$ in (4.9) and then decomposed A_ρ as in (4.12), we write

$$\begin{aligned}
4\pi I_2 &= \lim_{\rho\to 0}\int\limits_{B_1\setminus\widetilde{B}_\rho(\psi(x))}(P_{01}+\varepsilon P_{02})\frac{\sigma(\psi(x)-\psi(z))}{|\psi(x)-\psi(z)|^3}\ dz\cdot\nabla\psi_2(x)\\
&= \lim_{\rho\to 0}(K_1+P_0K_2)\cdot\nabla\psi_2(x)
\end{aligned}\tag{9.6}$$

where

$$K_1 = \int\limits_{B_1 \backslash \widetilde{B}_\rho(\psi(x))} [P_0(z) - P_0(x)] \frac{\sigma(\psi(x) - \psi(z))}{|\psi(x) - \psi(z)|^3}\, dz\ ,$$

$$K_2 = \int\limits_{B_1 \backslash \widetilde{B}_\rho(\psi(x))} \frac{\sigma(\psi(x) - \psi(z))}{|\psi(x) - \psi(z)|^3}\, dz\ .$$

By (4.25), (4.26), (4.29) we obtain

$$|K_1 + P_0 K_2| \le C|\nabla\psi^{-1}|^3_{L^\infty} \log[2 + |\nabla\psi|_{L^\infty} + |\nabla\psi^{-1}|_\infty + |\nabla\psi|_\alpha]\ .$$

Using (8.15), (9.1), and recalling (9.6), we find that

$$|I_2|_{L^\infty} \le \frac{C}{f^3} \log(2+f)|\nabla\psi_2|_{L^\infty}\ . \tag{9.7}$$

Next

$$4\pi I_3 = \lim_{\rho\to 0} \int\limits_{B_1 \backslash \widetilde{B}_\rho(\psi(x))} P_{02} \frac{\sigma(\psi(x) - \psi(z))}{|\psi(x) - \psi(z)|^3}\, dz \cdot \nabla\psi_1(x) \tag{9.8}$$

can be treated in the same way as I_2. This leads to

$$\begin{aligned} |I_3|_{L^\infty} &\le C|\nabla\psi^{-1}|^3_{L^\infty} |\nabla\psi_1|_{L^\infty} \log[2 + |\nabla\psi|_{L^\infty}|\nabla\psi^{-1}|_{L^\infty} + |\nabla\psi|_\alpha] \\ &\le \frac{C}{f^2}\ \log(2+f)\ . \end{aligned} \tag{9.9}$$

By (9.1), (9.2) we deduce that

$$|I_4|_{L^\infty} \le \frac{C}{f^2} + \frac{C}{f^3}|\nabla\psi_2|_{L^\infty}\ . \tag{9.10}$$

Finally, to estimate I_5, consider

$$J(\psi_1^{-1}) - J(\psi^{-1}) = \frac{1}{f^3}\left[1 - \det\left(I + \frac{\varepsilon\nabla\psi_2}{f}\right)^{-1}\right]\ .$$

Writing $D = \nabla\psi_2/f$ and noting that $(I+\varepsilon D)^{-1} = I - \varepsilon D(I+\varepsilon D)^{-1}$, it follows that

$$\begin{aligned} |J(\psi_1^{-1}) - J(\psi^{-1})| &= \frac{1}{f^3}|1 - \det(I - \varepsilon D(I+\varepsilon D)^{-1})| \\ &\le \frac{C}{f^3}\ |\varepsilon D| \le \varepsilon\frac{C}{f^4}\ |\nabla\psi_2|\ . \end{aligned} \tag{9.11}$$

Hence

$$|I_5| \le \frac{C}{f^3}\ |\nabla\psi_2|_{L^\infty}\ . \tag{9.12}$$

Combining the estimates (9.7), (9.9), (9.10) and (9.12), we get

$$(9.13) \qquad |I_2 + I_3 + I_4 + I_5|_{L^\infty} \le \frac{C\log(2+f)}{f^2}\left(1 + \frac{|\nabla\psi_2|_{L^\infty}}{f}\right) .$$

Step 2. C_α estimates.
Writing $I_2 = A\nabla\psi_2$ we have

$$|I_2|_\alpha \le |A|_\alpha |\nabla\psi_2|_{L^\infty} + |A|_{L^\infty}|\nabla\psi_2|_\alpha .$$

By (9.7)

$$|A|_{L^\infty} \le \frac{C}{f^3}\ \log(2+f) ,$$

and by (4.38) and (8.15), (9.1)

$$(9.14) \qquad |A|_\alpha \le \frac{C}{f^3}\ \log(2+f) .$$

It follows that

$$(9.15) \qquad |I_2|_\alpha \le \frac{C}{f^3}\ \log(2+f)(|\nabla\psi_2|_{L^\infty} + |\nabla\psi_2|_\alpha) .$$

Similarly we write $I_3 = \widetilde{A}\nabla\psi_1$ so that

$$|I_3|_\alpha \le |\widetilde{A}|_\alpha |\nabla\psi_1|_{L^\infty} + |\widetilde{A}|_{L^\infty}|\nabla\psi_1|_\alpha .$$

$|\widetilde{A}|_\alpha$ can be estimated in the same way as $|A|_\alpha$ in (9.14), whereas $|\nabla\psi_1|_\alpha = |f(t)I|_\alpha = 0$. Hence

$$(9.16) \qquad |I_3|_\alpha \le \frac{C}{f^2}\log(2+f) .$$

From the definition of I_4 we have

$$\begin{aligned} |I_4|_\alpha \le\ & C|J(\psi^{-1})|_\alpha(|\nabla\psi_1|_{L^\infty} + |\nabla\psi_2|_{L^\infty}) \\ & + C|J(\psi^{-1})|_{L^\infty}(|\nabla\psi_1|_\alpha + |\nabla\psi_2|_\alpha + |\nabla\psi_1|_{L^\infty} + |\nabla\psi_2|_{L^\infty}) . \end{aligned}$$

Using (9.2), (9.3), it follows that

$$(9.17) \qquad |I_4|_\alpha \le \frac{C}{f^2} + \frac{C}{f^3}(|\nabla\psi_2|_{L^\infty} + |\nabla\psi_2|_\alpha) .$$

Finally, from the definition of I_5,

$$\begin{aligned} |I_5|_\alpha \le\ & \frac{1}{\varepsilon}\ |P_{01}\nabla\psi_1|_\alpha |J(\psi_1^{-1}) - J(\psi^{-1})|_{L^\infty} \\ & + \frac{1}{\varepsilon}\ |P_{01}\nabla\psi_1|_{L^\infty}|J(\psi_1^{-1}) - J(\psi^{-1})|_\alpha \equiv I_{51} + I_{52} \end{aligned}$$

and

$$I_{51} \leq \frac{C}{f^3}\ |\nabla\psi_2|_{L^\infty}$$

by (9.11). Since $J(\psi_1^{-1}) = 1/f^3$ is a function of t only,

$$I_{52} \leq \frac{Cf}{\varepsilon}\ |J(\psi^{-1})|_\alpha \leq \frac{C}{f^{3+\alpha}}|\nabla\psi_2|_\alpha$$

by (9.3). It follows that

$$|I_5| \leq \frac{C}{f^3}(|\nabla\psi_2|_{L^\infty} + |\nabla\psi_2|_\alpha)\ . \tag{9.18}$$

Combining this with (9.15)–(9.17), we conclude that

$$|I_2 + I_3 + I_4 + I_5|_\alpha \leq \frac{C\log(2+f)}{f^2}\left(1 + \frac{|\nabla\psi_2|_{L^\infty} + |\nabla\psi_2|_\alpha}{f}\right)\ . \tag{9.19}$$

§**10. Auxiliary results.** In this section we prove three lemmas that will be needed for estimating I_1.

LEMMA **10.1.** *Let $z = \xi_i(x,y)\quad (i=1,2)$ be $C^{1+\alpha}$ surfaces such that $\xi_1(x_0,y_0) = \xi_2(x_0,y_0)$, $\nabla\xi_1(x_0,y_0) = \nabla\xi_2(x_0,y_0)$ and let $B_r(x_0,y_0,u_0)$ be a ball of center (x_0,y_0,u_0), $u_0 \in \mathbb{R}$. Set*

$$\begin{aligned} T(\xi_1,\xi_2) = \{(x,y,z);\xi_1(x,y) \leq z \leq \xi_2(x,y) \quad &\textit{if} \quad \xi_1(x,y) \leq \xi_2(x,y)\ , \\ \textit{or} \quad \xi_2(x,y) \leq z \leq \xi_1(x,y) \quad &\textit{if} \quad \xi_2(x,y) < \xi_1(x,y)\} \end{aligned}$$

and $T_r = T(\xi_1,\xi_2) \cap B_r(x_0,y_0,u_0)$. Then

$$\int\limits_{T_r} \frac{dw}{|w - (x_0,y_0,u_0)|^3} \leq C|\nabla(\xi_1 - \xi_2)|_\alpha r^\alpha \tag{10.1}$$

where C is a universal constant.

Proof. By the mean value theorem,

$$\begin{aligned} |\xi_1(x,y) - \xi_2(x,y)| &= |(\xi_1(x,y) - \xi_2(x,y)) - (\xi_1(x_0,y_0) - \xi_2(x_0,y_0))| \\ &\leq |\nabla(\xi_1 - \xi_2)(\overline{x},\overline{y})|\ |(x - x_0, y - y_0)| \end{aligned}$$

where $(\overline{x},\overline{y})$ is a point in the segment connecting (x,y) to (x_0,y_0). Since $\nabla(\xi_1 - \xi_2)(x_0,y_0) = 0$, we get

$$|\xi_1(x,y) - \xi_2(x,y)| \leq |\nabla(\xi_1 - \xi_2)|_\alpha |(x - x_0, y - y_0)|^{1+\alpha}\ .$$

Therefore the left-hand side of (10.1) is bounded by

$$\begin{aligned} &\int\limits_{B_r(x_0,y_0)} dxdy \left| \int\limits_{\xi_1(x,y)}^{\xi_2(x,y)} \frac{dz}{|(x,y,z) - (x_0,y_0,z_0)|^3} \right| \\ &\leq \int\limits_{B_r(x_0,y_0)} \frac{|\nabla(\xi_1 - \xi_2)|_\alpha |(x,y) - (x_0,y_0)|^{1+\alpha}}{|(x,y) - (x_0,y_0)|^3}\ dxdy \leq C|\nabla(\xi_1 - \xi_2)|_\alpha r^\alpha\ . \end{aligned}$$

LEMMA 10.2. *Let* $z_0 = (0,0,-d)$, $0 \le d \le 1$ *and let*

$$S_1 = \{x^2+y^2+z^2=1\},\ S_1^\varepsilon = \{x^2+y^2+(z-\varepsilon)^2 = (1+\varepsilon)^2\}\ ,\ 0<\varepsilon<1\ .$$

Consider a ray

$$\ell : \ell(t) = z_0 + t(a,b,c),\ t \ge 0$$

with $a^2+b^2+c^2=1$, *and denote by* z_1 *and* z_2 *the intersections of* ℓ *with* S_1 *and* S_1^ε, *respectively. Then*

$$|z_1 - z_2| \le \varepsilon\left(c + \frac{5}{1-d^2+c^2d^2}\right)\ . \tag{10.2}$$

Proof. The parameters t_i for which $\ell(t_i) = z_i$ are given by

$$t_1 = (c^2d^2+1-d^2)^{1/2}\ ,$$
$$t_2 = [c^2d^2+1-d^2+\varepsilon(2c^2d+2-2d+\varepsilon c^2)]^{1/2} + cd + c\varepsilon\ .$$

We now directly compute that $|z_1-z_2| = |t_1-t_2|$ is bounded by the right-hand side of (10.2).

LEMMA 10.3. *The function*

$$\int_{B_r(0)} \frac{\sigma(y-z)}{|y-z|^3}\,dz \tag{10.3}$$

is constant in $B_r(0)$, *and the constant is independent of* r.

Proof. For any $y \in B_r(0)$ the gradient of

$$\int_{B_r(0)\setminus B_\rho(0)} \frac{dz}{|y-z|}$$

is identically zero in $B_\rho(0)$ (by Newton's theorem [5; p. 22]). On the other hand the gravitational field of uniform mass distribution in $B_\rho(0)$ in the exterior of $B_\rho(0)$ is the same as if the entire mass were concentrated at the origin:

$$\nabla \int_{B_\rho(0)} \frac{dz}{|y-z|} = \nabla\left(\frac{4\pi}{3}\ \frac{\rho^3}{|y|}\right)\ .$$

It follows that

$$\nabla \int_{B_r(0)} \frac{dz}{|y-z|} = cy \tag{10.4}$$

where c is a constant independent of r. Since the gradient of the left-hand side of (10.4) differs from the integral in (10.3) by $\dfrac{1}{4\pi}\ I$, the proof of the lemma is complete.

§**11. Estimating** $|I_1|_{L^\infty}$. We can write

$$(11.1)\quad 4\pi I_1 = \frac{f(t)}{\varepsilon}\left[\int\limits_{\psi(B_1)} \frac{\sigma(\psi(x)-y)}{|\psi(x)-y|^3}\, J(\psi^{-1})(y)dy - \int\limits_{\psi_1(B_1)} \frac{\sigma(\psi_1(x)-y)}{|\psi_1(x)-y|^3}\, J(\psi_1^{-1})(y)dy\right]$$

or

$$4\pi I_1 = \frac{f(t)}{\varepsilon}\int\limits_{\psi(B_1)} \frac{\sigma(\psi(x)-y)}{|\psi(x)-y|^3}[J(\psi^{-1})(y) - J(\psi^{-1})(\psi(x))]dy$$

$$(11.2)\qquad +\frac{f(t)}{\varepsilon}\, J(\psi^{-1})(\psi(x))\left[\int\limits_{\psi(B_1)} \frac{\sigma(\psi(x)-y)}{|\psi(x)-y|^3}\, dy - \int\limits_{\psi_1(B_1)} \frac{\sigma(\psi_1(x)-y)}{|\psi_1(x)-y|^3}\, dy\right]$$

$$+\frac{f(t)}{\varepsilon}\left[J(\psi^{-1})(\psi(x)) - J(\psi_1^{-1})(\psi_1(x))\right]\int\limits_{\psi_1(B_1)} \frac{\sigma(\psi_1(x)-y)}{|\psi_1(x)-y|^3}\, dy$$

$$\equiv I_{11} + I_{12} + I_{13}\ .$$

We shall use (11.2) to estimate $|I_1|_{L^\infty}$ in this section, and $|I_1|_\alpha$ in the next section. Writing

$$(11.3)\qquad I_{11} = \frac{f(t)}{\varepsilon}\left[\int\limits_{\psi(B_1)\setminus B_{\rho_1}(\psi(x))} + \int\limits_{\psi(B_1)\cap B_{\rho_1}(\psi(x))}\right] = \frac{f(t)}{\varepsilon}(K_{11}+K_{12})\ ,$$

we can proceed to estimate K_{11}, K_{12} as in (4.14). First,

$$|K_{11}| \le C|\log\rho_1|\sup_{x,y}|J(\psi^{-1})(y) - J(\psi^{-1})(\psi(x))|\ .$$

Since $J(\psi_1^{-1})$ is independent of the space variable,

$$|J(\psi^{-1})(y) - J(\psi^{-1})(\psi(x))| \le |J(\psi^{-1})(y) - J(\psi_1^{-1})(y)|$$
$$+|J(\psi^{-1})(\psi(x)) - J(\psi_1^{-1})(\psi(x))| \le \frac{\varepsilon C}{f^4}\,|\nabla\psi_2|$$

by (9.11). Hence

$$(11.4)\qquad |K_{11}| \le \frac{\varepsilon C}{f^4}\,|\nabla\psi_2|_{L^\infty}|\log\rho_1|\ .$$

Next,

$$|K_{12}| \le \int\limits_{B_{\rho_1}(\psi(x))} |J(\psi^{-1})|_\alpha \frac{dy}{|\psi(x)-y|^{3-\alpha}} \le C|J(\psi^{-1})|_\alpha\rho_1^\alpha\ ,$$

$$(11.5)\qquad \le \frac{C}{f^{3+\alpha}}\,|\nabla\psi_2|_\alpha\rho_1^\alpha$$

by (9.3). Taking $\rho_1 = f(t)$ we obtain

$$|I_{11}| \le \frac{C}{f^3}\ \log(2+f)(|\nabla\psi_2|_{L^\infty} + |\nabla\psi_2|_\alpha)\ . \tag{11.6}$$

Later on we shall need to use Lemma 3.1 with $\Omega = \psi(B_1(0))$. We can write $B_1(0) = \{x; g_0(x) < 0\}$ where $g_0(x) = 1 - |x|^2$. Then $\Omega = \{x; g(x) < 0\}$ where $g(y) = 1 - |\psi^{-1}(y)|^2$, and

$$\delta^\alpha = \frac{\inf\limits_{\partial\Omega} |\nabla g|}{|\nabla g|_\alpha}\ .$$

Since

$$y = \psi(z) = fz + \varepsilon\psi_2(\psi^{-1}(y))\ ,$$

we have

$$\psi^{-1}(y) = \frac{1}{f}[y - \varepsilon\psi_2(\psi^{-1}(y))]\ , \tag{11.7}$$

and, using (4.24) and (4.27), we then easily find that

$$\frac{1}{\delta^\alpha} \le \frac{C}{f^\alpha}\ . \tag{11.8}$$

Hence,

$$\begin{array}{l}\text{(3.4) in Lemma 3.1 holds for } \Omega_t = (\psi_1 + \varepsilon\psi_2)(B_1(0), t)\\ \text{with } \delta \text{ replaced by } f(t) \text{ and } C \text{ independent of } \varepsilon (\varepsilon < 1).\end{array} \tag{11.9}$$

Using (11.9) with $\varepsilon = 0$ and (9.11), we get

$$|I_{13}| \le \frac{C}{f^3}\ \log(2+f)|\nabla\psi_2|_{L^\infty}\ . \tag{11.10}$$

To estimate I_{12} at any point $x_0 \in \psi(B_1)$, let $\tilde{x}_0 \in \partial\psi(B_1)$ be a point on $\partial\psi(B_1)$ such that

$$|x_0 - \tilde{x}_0| = \operatorname{dist}(x, \partial\psi(B_1))\ .$$

Choose a point y_0 and $\eta > 0$ such that the ball $B_\eta(y_0)$ is tangent to $\partial\psi(B_1)$ at $\tilde{x}_0$. Clearly x_0 and y_0 lie on the inner normal to $\partial\psi(B_1)$ at $\tilde{x}_0$; η (or y_0) will be chosen later on.

For any $\rho_0 > 0$ we can write, by Lemma 10.3,

$$\begin{aligned}\varepsilon f J(\psi^{-1}) I_{12} &= \int\limits_{\psi(B_1)} \frac{\sigma(x_0 - z)}{|x_0 - z|^3}\ dz - \int\limits_{B_\eta(y_0)} \frac{\sigma(x_0 - z)}{|x_0 - z|^3}\ dz \\ &= \left[\ \int\limits_{\psi(B_1)\setminus B_{\rho_0}(x_0)} \frac{\sigma(x_0 - z)}{|x_0 - z|^3}\ dz - \int\limits_{B_\eta(y_0)\setminus B_{\rho_0}(x_0)} \frac{\sigma(x_0 - z)}{|x_0 - z|^3}\ dz\right] \\ &\quad + \left[\ \int\limits_{\psi(B_1)\cap B_{\rho_0}(x_0)} \frac{\sigma(x_0 - z)}{|x_0 - z|^3}\ dz - \int\limits_{B_\eta(y_0)\cap B_{\rho_0}(x_0)} \frac{\sigma(x_0 - z)}{|x_0 - z|^3}\ dz\right] \equiv J_1 + J_2\ .\end{aligned} \tag{11.11}$$

To estimate J_2 we note that

$$|J_2| \leq \int\limits_{[\psi(B_1)\Delta B_\eta(y_0)]\cap B_{\rho_0}(x_0)} \frac{|\sigma(x_0 - z)|}{|x_0 - z|^3}\, dz\ . \tag{11.12}$$

We now recall that $\psi(B_1) = \{g < 0\}$ where, by (11.7),

$$\begin{aligned} g(x) &= 1 - |\psi^{-1}(x)|^2 = 1 - \left|\frac{x}{f} - \frac{\varepsilon}{f}\ \psi_2(\psi^{-1}(x))\right|^2 \\ &= \frac{1}{f^2}\ \left[f^2 - |x - \varepsilon\psi_2(\psi^{-1}(x))|^2\right]\ . \end{aligned} \tag{11.13}$$

From (3.6) we know that if

$$\rho_0 \leq \delta \tag{11.14}$$

then $\partial\psi(B_1) \cap B_{\rho_0}(x_0)$ can be represented in the form

$$x_3 = \xi_1(x_1, x_2)\ .$$

where the coordinates are chosen such that $\nabla g(\tilde{x}_0) = (0, 0, \ell)$. On the other hand we can write

$$B_\eta(y_0) = \{-\tilde{g} < 0\} \quad \text{where} \quad \tilde{g}(x) = \frac{1}{f^2}[\eta^2 - |x - y_0|^2]\ ,$$

and represent $\partial B_\eta(y_0) \cap B_{\rho_0}(x_0)$ in the form

$$x_3 = \xi_2(x_1, x_2)\ .$$

We wish to apply Lemma 10.1 and we therefore need to ensure that $\nabla\xi_1 = \nabla\xi_2$ at $\tilde{x}_0$, i.e., that

$$\nabla g(\tilde{x}_0) = \nabla\tilde{g}(\tilde{x}_0)\ . \tag{11.15}$$

Since

$$\begin{cases} \nabla g(x) = -\dfrac{2}{f^2}\ x + \dfrac{\varepsilon}{f^2}\ \theta(x)\ , \\ \nabla\tilde{g}(x) = -\dfrac{2}{f^2}(x - y_0) \end{cases} \tag{11.16}$$

where

$$\theta(x) = 2\psi_2(\psi^{-1}(x)) + \nabla(\psi_2(\psi^{-1}(x))) \cdot (x - \varepsilon\psi_2(\psi^{-1}(x)))\ , \tag{11.17}$$

equation (11.15) becomes

$$y_0 = \varepsilon\theta(\tilde{x}_0)\ . \tag{11.18}$$

Making this choice for y_0 also determines η:

$$\eta = |\tilde{x}_0 - y| = |\tilde{x}_0 - \varepsilon\theta(\tilde{x}_0)|\ .$$

Since $g(\tilde{x}_0) = 0$, (11.13) gives

$$f = |\tilde{x}_0 - \varepsilon\psi_2(\psi^{-1}(\tilde{x}_0))|$$

so that

$$\begin{aligned} \eta &= f + [|\tilde{x}_0 - \varepsilon\theta(\tilde{x}_0)| - |\tilde{x}_0 - \varepsilon\psi_2(\psi^{-1}(\tilde{x}_0))|] \\ &= f + \varepsilon\widetilde{\theta}(\tilde{x}_0) \ , \end{aligned} \tag{11.19}$$

where

$$|\widetilde{\theta}(x)| \leq Cf(t) \tag{11.20}$$

by (8.12) (8.18).

We now apply Lemma 10.1 to get

$$|J_2| \leq C|\nabla(\xi_1 - \xi_2)|_\alpha \rho_0^\alpha \ , \tag{11.21}$$

and it remains to estimate $|\nabla(\xi_1 - \xi)|_\alpha$.

To do that we begin by noting that (11.16) and (11.18) imply that

$$\begin{cases} |\nabla g - \nabla\tilde{g}| \leq \dfrac{\varepsilon}{f^2}\ |\theta(x) - \theta(\tilde{x}_0)|_{L^\infty} \ , \\ \nabla g - \nabla\tilde{g}|_\alpha \leq \dfrac{\varepsilon}{f^2}\ |\theta|_\alpha \ . \end{cases} \tag{11.22}$$

Using (8.12), (8.18), (8.19) and the fact that diam $\psi(B_1) \leq Cf$ (which follows from (8.12)), we easily establish the estimates

$$|\theta(x) - \theta(\tilde{x}_0)|_{L^\infty} \leq C|\nabla\psi_2|_{L^\infty} \ , \tag{11.23}$$

$$|\theta|_\alpha \leq C\left(\frac{1}{f^\alpha}\ |\nabla\psi_2|_{L^\infty} + \frac{1}{f}\ |\nabla\psi_2|_\alpha\right) \ . \tag{11.24}$$

It follows that

$$|\nabla g - \nabla\tilde{g}| \leq \frac{\varepsilon C}{f^2}\ |\nabla\psi_2|_{L^\infty} \ , \tag{11.25}$$

$$|\nabla g - \nabla\tilde{g}|_\alpha \leq \frac{\varepsilon C}{f^2}\left(\frac{1}{f^\alpha}\ |\nabla\psi_2|_{L^\infty} + \frac{1}{f}\ |\nabla\psi_2|_\alpha\right) \ . \tag{11.26}$$

The function $\theta(x)$ in (11.7) is bounded by Cf (by (8.12), (8.19)). Therefore, from (3.6) and (11.16),

$$\frac{C}{f} \leq \frac{1}{2}\ \inf_{\partial\Omega_t} |\nabla g| \leq \inf_{\partial\Omega_t} |g_{x_3}| \leq |\nabla g|_{L^\infty} \leq \frac{C}{f} \ . \tag{11.27}$$

Next we derive lower bound on $|\nabla g|_\alpha$. By (11.16) and (11.24),

$$\begin{aligned} \frac{|\nabla g(x) - \nabla g(y)|}{|x-y|^\alpha} &\geq \frac{2}{f^2}\left[|x-y|^{1-\alpha} - \varepsilon\frac{|\theta(x) - \theta(y)|}{|x-y|^\alpha}\right] \\ &\geq \frac{2}{f^2}\left[|x-y|^{1-\alpha} - \varepsilon C\left(\frac{|\nabla\psi_2|_{L^\infty}}{f^\alpha} + \frac{|\nabla\psi_2|_\alpha}{f}\right)\right] \ . \end{aligned}$$

Since diam $\psi(B_1) \sim cf$, using (8.12) we obtain that

$$(11.28) \qquad |\nabla g|_\alpha = \sup_{x,y\in\Omega_t} \frac{|\nabla g(x) - \nabla g(y)|}{|x-y|^\alpha} \geq \frac{2}{f^2}\left[f^{1-\alpha} - f^{1-\alpha}\lambda C\right] \geq \frac{C}{f^{1+\alpha}} ,$$

provided we choose λ such that $\lambda C \leq \frac{1}{2}$. We can now fix this λ and proceed to estimate $|\nabla(\xi_1 - \xi_2)|_\alpha$ (on the right-hand side of (11.21)). The i-th component of $\nabla\xi_1 - \nabla\xi_2$ is

$$-\frac{g_{x_i} - \tilde{g}_{x_i}}{g_{x_3}} - \frac{\tilde{g}_{x_i}(\tilde{g}_{x_3} - g_{x_3})}{g_{x_3}\tilde{g}_{x_3}} .$$

Therefore

$$\begin{aligned}
|\nabla\xi_1 - \nabla\xi_2|_\alpha &\leq \frac{|\nabla g - \nabla\tilde{g}|_\alpha}{\inf|g_{x_3}|} + 2\frac{|\nabla g - \nabla\tilde{g}|_{L^\infty}|\nabla g|_\alpha}{\inf|g_{x_3}|^2} \\
&\quad + 2\frac{|\nabla\tilde{g}|_{L^\infty}|\nabla g - \nabla\tilde{g}|_{L^\infty}|\nabla g|_\alpha}{\inf|g_{x_3}|} \\
&\leq \varepsilon C\left[\frac{|\nabla\psi_2|_{L^\infty}}{f}\ \frac{1}{\delta^\alpha} + \frac{1}{\delta^\alpha}\ \frac{|\nabla g - \nabla\tilde{g}|_\alpha}{|\nabla g|_\alpha}\right]
\end{aligned}$$

where we have used (11.27), (11.26) and the definition of δ. Using (11.28) and (11.26) we conclude that

$$(11.29) \qquad |\nabla\xi_1 - \nabla\xi_2|_\alpha \leq \frac{\varepsilon C}{\delta^\alpha}\left(\frac{|\nabla\psi_2|_{L^\infty}}{f} + \frac{|\nabla\psi_2|_\alpha}{f^{2-\alpha}}\right) .$$

Substituting this into (11.21) and choosing $\rho_0 = \delta$ (cf. (11.14)), we then obtain

$$(11.30) \qquad |J_2| \leq \varepsilon C\left(\frac{|\nabla\psi_2|_{L^\infty}}{f} + \frac{|\nabla\psi_2|_\alpha}{f}\right) .$$

Estimating J_1 (defined in (11.11)) is easier. Observe that if we choose the origin to be at $\varepsilon\psi_1(0,t)$ then

$$B_{r_1} \subset \psi(B_1, t) \subset B_{r_2}$$

where

$$r_1 = f - \varepsilon|\nabla\psi_1|_{L^\infty} , \ r_2 = f + \varepsilon|\nabla\psi_1|_{L^\infty} .$$

Hence

$$|J_1| \leq \int\limits_{\psi(B_1)\Delta B_\eta(y_0))\setminus B_\delta(x_0)} \frac{C}{|x_0 - z|^2} \leq \int\limits_S dS \int\limits_{\tilde{r}_1}^{\tilde{r}_2} \frac{d\rho}{\rho}$$

(S = surface of the unit sphere) where

$$|\tilde{r}_2 - \tilde{r}_1| \leq \varepsilon C|\nabla\psi_2|_{L^\infty}$$

by Lemma 10.2. It follows that

$$|J_1| \le C\left|\log \frac{\tilde r_2}{\tilde r_1}\right| \le C\left|\frac{\tilde r_2 - \tilde r_1}{\tilde r_1}\right| \le \frac{\varepsilon C |\psi_2|_{L^\infty}}{\delta} \le \varepsilon C \frac{\nabla \psi_2|_{L^\infty}}{f} \ ,$$

where (11.8) was used. Combining this with (11.30) and (11.11) we get:

LEMMA 11.1. *If ε is sufficiently small, then*

$$|I_{12}| \le \frac{C}{f^3}(|\nabla\psi_2|_{L^\infty} + |\nabla\psi_2|_\alpha) \ . \tag{11.31}$$

Using this and (11.6), (11.10) in (11.1), we conclude that

$$|I_1|_{L^\infty} \le \frac{C\log(2+f)}{f^3}(|\nabla\psi_2|_{L^\infty} + |\nabla\psi_2|_\alpha) \ . \tag{11.32}$$

§**12. Estimating** $|I_1|_\alpha$. Since $J(\psi_1^{-1})$ is a constant, (9.3) gives

$$|J(\psi^{-1}) - J(\psi_1^{-1})|_\alpha \le Cf^3 \ |\varepsilon D|_\alpha = \frac{C}{f^3}\ \varepsilon \frac{|\nabla\psi_2|_\alpha}{f} \ .$$

Also, by Lemma 10.3, the integral in I_{13} is independent of the space variable and by (11.9), it is bounded by $C\log(2+f)$. Hence

$$|I_{13}| \le \frac{Cf\log(2+f)}{\varepsilon}\ \frac{C}{f^3}\ \varepsilon\frac{|\nabla\psi_2|_\alpha}{f} \le \frac{C\log(2+f)}{f^3}\ |\nabla\psi_2|_\alpha \ . \tag{12.1}$$

Next, by Lemma 3.2 and the first part of Lemma 3.1 (as formulated in (11.9)) and (11.8)),

$$\begin{aligned} |I_{11}|_\alpha &\le \frac{Cf}{\varepsilon}\ |J(\psi^{-1})|_\alpha \log[2 + |\nabla\psi|_{L^\infty} + |\nabla\psi^{-1}|_{L^\infty} + |\nabla\psi|_\alpha] \\ &\le \frac{C}{f^3}\ \log(2+f)|\nabla\psi_2|_\alpha \end{aligned} \tag{12.2}$$

where (9.3) was used in the last inequality.

To estimate $|I_{12}|_\alpha$ we introduce the function

$$w(x) = \int\limits_{\psi(B_1)} \frac{\sigma(x-y)}{|x-y|^3}\ dy - \int\limits_{\psi_1(B_1)} \frac{\sigma(\psi_1(x)-y)}{|\psi_1(x)-y|^3}\ dy \ . \tag{12.3}$$

Then by the definition of I_{12} (in (11.1)) and by (11.9),

$$|I_{12}|_\alpha \le \frac{Cf}{\varepsilon}\ |J(\psi^{-1})|_\alpha \log(2+f) + \frac{C}{\varepsilon f^2}\ |w(\psi(x))|_\alpha \ .$$

Using (9.3) and the estimate $|w(\psi(x))|_\alpha \le C|w|_\alpha |\nabla\psi|^\alpha_{L^\infty}$, we then get

$$|I_{12}|_\alpha \le \frac{C}{f^{2-\alpha}}\ \frac{1}{\varepsilon}\ |w|_\alpha + \frac{C}{f^{3+\alpha}}\ \log(2+f)|\nabla\psi_2|_\alpha \ . \tag{12.4}$$

LEMMA 12.1. *The following estimate holds:*

$$|w|_\alpha \le \frac{\varepsilon C}{f^{1+\alpha}}(|\nabla\psi_2|_{L^\infty} + |\nabla\psi_2|_\alpha) \ . \tag{12.5}$$

Assuming the lemma, we conclude that

$$|I_{12}|_\alpha \le \frac{\varepsilon C}{f^3}(|\nabla\psi_2|_{L^\infty} + |\nabla\psi_2|_\alpha)$$

and then, together with (12.2), (12.1),

$$|I_1|_\alpha \le \frac{C\log(2+f)}{f^3}(|\nabla\psi_2|_{L^\infty} + |\nabla\psi_2|_\alpha) \ . \tag{12.6}$$

Proof of Lemma 12.1. The proof is somewhat similar to the proof of (3.5); however, instead of integrating over Ω we integrate over $\psi(B_1)$. The main step consists in estimating

$$|w(x_0) - w(\tilde{x}_0)| \quad \text{for} \quad x_0, \tilde{x}_0 \quad \text{on} \quad \partial\psi(B_1), \ \tilde{x}_0 \in B_{\frac{\delta}{16}}(x_0) \ . \tag{12.7}$$

As in §11 we introduce two balls $B_\eta(y_0)$ and $B_{\tilde{\eta}}(\tilde{y}_0)$ such that $B_\eta(y_0)$ is tangent to $\partial\psi(B_1)$ at x_0 and $B_{\tilde{\eta}}(\tilde{y}_0)$ is tangent to $\partial\psi(B_1)$ at $\tilde{x}_0$, and

$$\begin{aligned} &y_0 = \varepsilon\theta(x_0), \ \tilde{y}_0 = \varepsilon\theta(\tilde{x}_0) \ , \\ &\eta = f + \varepsilon\tilde{\theta}(x_0), \ \tilde{\eta} = f + \varepsilon\tilde{\theta}(\tilde{x}_0) \ . \end{aligned} \tag{12.8}$$

We write

$$\begin{aligned} w(x_0) - w(\tilde{x}_0) &= \left[\int\limits_{\psi(B_1)} \frac{\sigma(x_0 - y)}{|x_0 - y|^3} - \int\limits_{B_\eta(y_0)} \frac{\sigma(x_0 - y)}{|x_0 - y|^3} \right] \\ &\quad - \left[\int\limits_{\psi(B_1)} \frac{\sigma(\tilde{x}_0 - y)}{|\tilde{x}_0 - y|^3} - \int\limits_{B_{\tilde{\eta}}(\tilde{y}_0)} \frac{\sigma(\tilde{x}_0 - y)}{|\tilde{x}_0 - y|^3} \right] \\ &= \int\limits_{\psi(B_1)\Delta B_\eta(y_0)} \frac{\sigma(x_0 - y)}{|x_0 - y|^3} - \int\limits_{\psi(B_1)\Delta B_{\tilde{\eta}}(\tilde{y}_0)} \frac{\sigma(\tilde{x}_0 - y)}{|\tilde{x}_0 - y|^3} \equiv I(x_0) - J(\tilde{x}_0) \ , \end{aligned} \tag{12.9}$$

and

$$\begin{aligned} I(x_0) &= \int\limits_{(\psi(B_1)\Delta B_\eta(y_0))\setminus B_\delta(x_0)} \frac{\sigma(x_0 - y)}{|x_0 - y|^3} + \int\limits_{(\psi(B_1)\Delta B_\eta(y_0))\cap B_\delta(x_0)} \frac{\sigma(x_0 - y)}{|x_0 - y|^3} \\ &\equiv w_1(x_0) + w_2(x_0) \ , \\ J(x_0) &= \int\limits_{(\psi(B_1)\Delta B_{\tilde{\eta}}(\tilde{y}_0))\setminus B_\delta(x_0)} \frac{\sigma(\tilde{x}_0 - y)}{|\tilde{x}_0 - y|^3} + \int\limits_{(\psi(B_1)\Delta B_{\tilde{\eta}}(\tilde{y}_0))\cap B_\delta(x_0)} \frac{\sigma(\tilde{x}_0 - y)}{|\tilde{x}_0 - y|^3} \\ &\equiv \tilde{w}_1(\tilde{x}_0) + \tilde{w}_2(\tilde{x}_0) \ , \end{aligned} \tag{12.10}$$

where the integral $\int\limits_{A\Delta B}$ is understood as $\int\limits_A - \int\limits_B$.

We can write

$$
\begin{aligned}
w_1(x_0) - \tilde{w}_1(\tilde{x}_0) &= \int\limits_{\Omega_0} \left[\frac{\sigma(x_0-y)}{|x_0-y|^3} - \frac{\sigma(\tilde{x}_0-y)}{|\tilde{x}_0-y|^3} \right] \\
&+ \int\limits_{\Omega_{01}} \frac{\sigma(x_0-y)}{|x_0-y|^3} - \int\limits_{\Omega_{02}} \frac{\sigma(\tilde{x}_0-y)}{|\tilde{x}_0-y|^3} \equiv m_1 + m_2 + m_3
\end{aligned}
\tag{12.11}
$$

where

$$
\begin{aligned}
\Omega_0 &= [(\psi(B_1)\Delta B_\eta(y_0))\backslash B_\delta(x_0)] \cap [(\psi(B_1)\Delta B_{\tilde{\eta}}(\tilde{y}_0))\backslash B_\delta(x_0)] \ , \\
\Omega_{01} &= [(\psi(B_1)\Delta B_\eta(y_0))\backslash B_\delta(x_0)] \backslash [(\psi(B_1)\Delta B_{\tilde{\eta}}(\tilde{y}_0))\backslash B_\delta(x_0)] \ , \\
\Omega_{02} &= [(\psi(B_1)\Delta B_{\tilde{\eta}}(\tilde{y}_0))\backslash B_\delta(x_0)] \backslash [(\psi(B_1)\Delta B_\eta(y_0))\backslash B_\delta(x_0)] \ .
\end{aligned}
$$

Note that

$$
\begin{aligned}
\Omega_{01} &\subset B_\eta(y_0)\Delta B_{\tilde{\eta}}(\tilde{y}_0)\backslash B_\delta(x_0) \ , \\
\Omega_{02} &\subset B_\eta(y_0)\Delta B_{\tilde{\eta}}(\tilde{y}_0)\backslash B_\delta(x_0) \ .
\end{aligned}
$$

From (11.24), (11.19) we get

$$
\begin{aligned}
&|y_0 - \tilde{y}_0| \le \varepsilon |\theta|_\alpha |x_0 - \tilde{x}_0|^\alpha \le \varepsilon C \left(\frac{|\nabla\psi_2|_{L^\infty}}{f^\alpha} + \frac{|\nabla\psi_2|_\alpha}{f} \right) |x_0 - \tilde{x}_0|^\alpha \ , \\
&|\eta - \tilde{\eta}| \le \varepsilon |\tilde{\theta}|_\alpha |x_0 - \tilde{x}_0| \le \varepsilon \left(|\theta|_\alpha + |\nabla\psi_2|_{L^\infty} |\nabla\psi^{-1}|_{L^\infty} \right) |x_0 - \tilde{x}_0|^\alpha \\
&\qquad \le \varepsilon C \left(\frac{|\nabla\psi_2|_{L^\infty}}{f^\alpha} + \frac{|\nabla\psi_2|_\alpha}{f} \right) |x_0 - \tilde{x}_0|^\alpha \ , \\
&\eta, \tilde{\eta} \le Cf \ .
\end{aligned}
\tag{12.12}
$$

It follows that

$$
|B_\eta(y_0)\Delta B_{\tilde{\eta}}(\tilde{y}_0)| \le Cf^2 |y_0 - \tilde{y}_0|^2 \le \varepsilon C f^2 \left(\frac{|\nabla\psi_2|_{L^\infty}}{f^\alpha} + \frac{|\nabla\psi_2|_\alpha}{f} \right) |x_0 - \tilde{x}_0|^\alpha \ .
$$

Hence

$$
\begin{aligned}
|m_2| + |m_3| &\le \frac{1}{\delta^3} \ |B_\eta(y_0)\Delta B_{\tilde{\eta}}(\tilde{y}_0)| \\
&\le \frac{\varepsilon C}{f^{1+\alpha}} (|\nabla\psi_2|_{L^\infty} + |\nabla\psi_2|_\alpha) |x_0 - \tilde{x}_0|^\alpha \ .
\end{aligned}
\tag{12.13}
$$

To estimate m_1 we use the mean value theorem to get

$$\left|\frac{\sigma(x_0-y)}{|x_0-y|^3}-\frac{\sigma(\widetilde{x}_0-y)}{|\widetilde{x}_0-y|^3}\right| = \int_0^t \frac{d}{dt}\frac{\sigma((x_0-y)t+(1-t)(\widetilde{x}_0-y))}{|(x_0-y)t+(1-t)(\widetilde{x}_0-y)|^3}\,dt$$

(12.14)

$$\leq |x_0-\widetilde{x}_0|\int_0^1 \frac{C}{|(x_0-\widetilde{x}_0)t+(\widetilde{x}_0-y)|^4} \leq C|x_0-\widetilde{x}_0|\frac{1}{\left||y-x_0|-\dfrac{\delta}{2}\right|^4}\ .$$

Hence

$$\begin{aligned} |m_1| &= \int_{\rho_1}^{\rho_2} C|x_0-\widetilde{x}_0|\frac{r^2dr}{\left|r-\dfrac{\delta}{4}\right|^4} \quad \text{(using polar coordinates centered at } x_0) \\ &\leq C|x_0-\widetilde{x}_0|\left(\frac{1}{\rho_1}-\frac{1}{\rho_2}\right) \qquad (\text{since} \quad \rho_1>\delta) \\ &\leq C|x_0-\widetilde{x}_0|\ \frac{\rho_2-\rho_1}{\rho_1\rho_2}\ , \end{aligned}$$

and

$$|\rho_2-\rho_1|\leq \varepsilon C\left(\frac{|\nabla\psi_2|_{L^\infty}}{f^\alpha}+\frac{|\nabla\psi_2|_\alpha}{f}\right)$$

by (12.12) and Lemma 10.2. We thus get

$$|m_1|\leq \frac{\varepsilon C}{\delta^2}\ |x_0-\widetilde{x}_0|\left(\frac{|\nabla\psi_2|_{L^\infty}}{f^\alpha}+\frac{|\nabla\psi_2|_\alpha}{f}\right)\ ,$$

and $|x_0-\widetilde{x}_0|\leq|x_0-\widetilde{x}_0|^\alpha\delta^{1-\alpha}$. Using this and (12.13) in (12.11), we find that

$$|w_1(x_0)-\widetilde{w}_1(\widetilde{x}_0)|\leq\frac{\varepsilon C}{f^{1+\alpha}}\left(|\nabla\psi_2|_{L^\infty}+|\nabla\psi_2|_\alpha\right)|x_0-\widetilde{x}_0|^\alpha\ . \tag{12.15}$$

We next estimate $w_2(x_0)-\widetilde{w}_2(\widetilde{x}_0)$, writing

$$w_2(x_0)=\int\limits_{(\psi(B_1)\Delta B_\eta(y_0))\cap B_\delta(x_0)\cap B_\rho(x_0)}\frac{\sigma(x_0-y)}{|x_0-y|^3}$$

(12.16)

$$+\int\limits_{(\psi(B_1)\Delta B_\eta(y_0))\cap B_\delta(x_0)\setminus B_\rho(x_0)}\frac{\sigma(x_0-y)}{|x_0-y|^3}\equiv w_{21}(x_0)+w_{22}(x_0)\ ,$$

and

$$\widetilde{w}_2(\widetilde{x}_0)=\int\limits_{(\psi(B_1)\Delta B_{\widetilde{\eta}}(\widetilde{y}_0))\cap B_\delta(x_0)\cap B_\rho(\widetilde{x}_0)}\frac{\sigma(\widetilde{x}_0-y)}{|\widetilde{x}_0-y|^3}$$

(12.17)

$$+\int\limits_{(\psi(B_1)\Delta B_{\widetilde{\eta}}(\widetilde{y}_0))\cap B_\delta(x_0)\setminus B_\rho(\widetilde{x}_0)}\frac{\sigma(\widetilde{x}_0-y)}{|\widetilde{x}_0-y|^3}\equiv \widetilde{w}_{21}(\widetilde{x}_0)+\widetilde{w}_{22}(\widetilde{x}_0)\ ,$$

where $\rho = 4|x_0 - \tilde{x}_0|$.

Using Lemma 10.1 and (11.29), (11.8), we find that

$$
\begin{aligned}
|w_{21}(x_0)| + |\tilde{w}_{21}(\tilde{x}_0)| &\le C|\nabla\xi_1 - \nabla\xi_2|\rho^\alpha \\
&\le \frac{\varepsilon C}{f^{1+\alpha}}(|\nabla\psi_2|_{L^\infty} + |\nabla\psi_2|_\alpha)|x_0 - \tilde{x}_0|^\alpha\ . \qquad (12.18)
\end{aligned}
$$

To estimate $w_{22}(x_0) - \tilde{w}_{22}(\tilde{x}_0)$ it is helpful to refer to Figure 1.

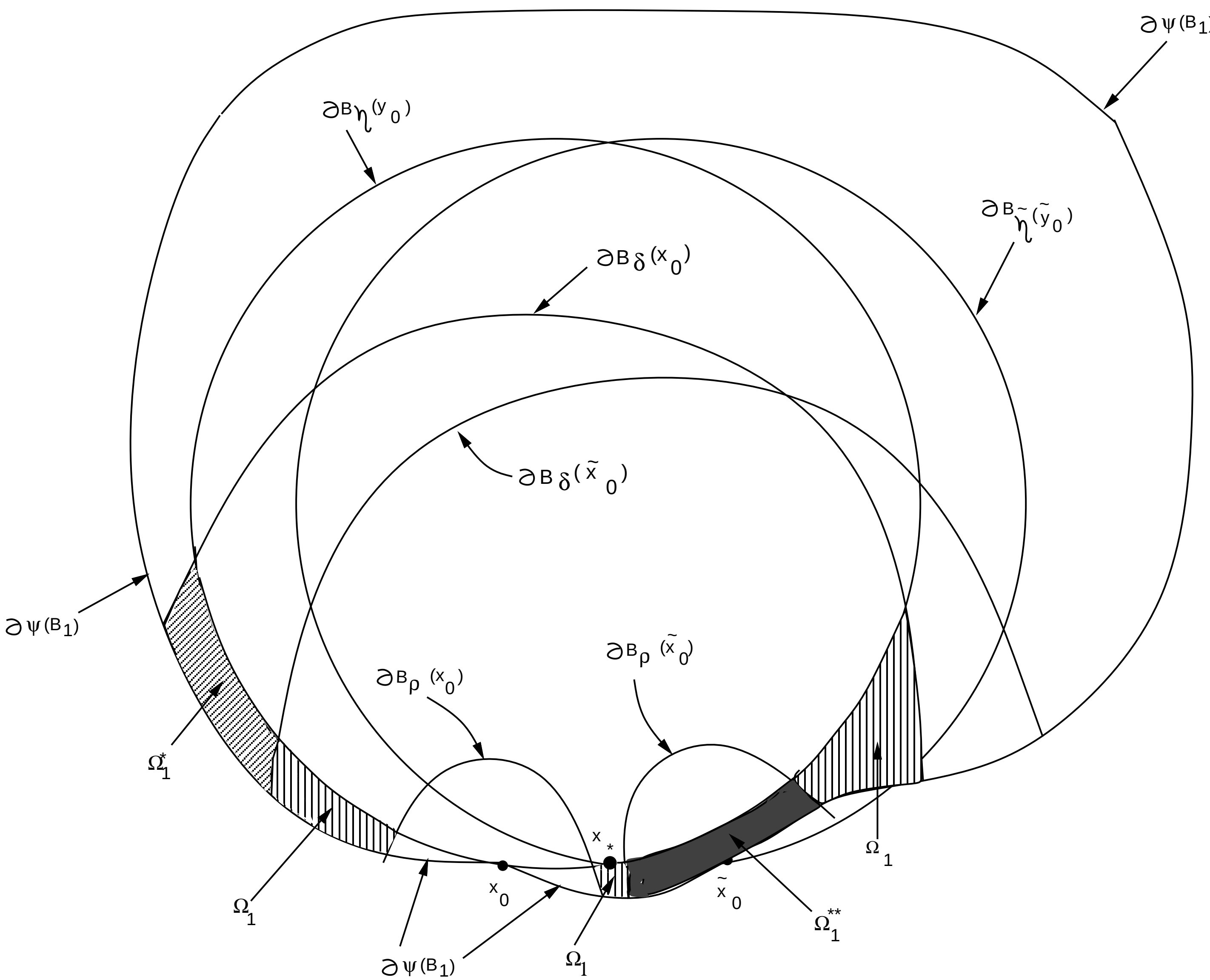

Figure 1

The set $(\psi(B_1)\Delta B_\eta(y_0)) \cap (B_\delta(x_0)\backslash B_\rho(x_0))$ is broken into three parts:

$$\Omega_1^{**} \text{ inside } B_\rho(\tilde{x}_0),\ \Omega_1^{*} \text{ outside } B_\delta(\tilde{x}_0), \text{ and } \Omega_1 \text{ in } B_\delta(\tilde{x}_0)\backslash B_\rho(\tilde{x}_0).$$

For Ω_1^{**} we use Lemma 10.1, (11.29) and the fact that $1/\delta \le 1/f$:

$$\Big| \int\limits_{\Omega_1^{**}} \frac{\sigma(x_0 - x)}{|x_0 - x|^3} \Big| \le |\nabla\xi_1 - \nabla\xi_2|_\alpha \rho^\alpha \le \frac{\varepsilon C}{f^{1+\alpha}} (|\nabla\psi_2|_{L^\infty} + |\nabla\psi_2|_\alpha) |x_0 - \widetilde{x}_0|^\alpha \ . \tag{12.19}$$

In Ω_1^* the integrand is $\le C/\delta^3$. Since the centers of the balls $B_\delta(x_0)$ and $B_\delta(\widetilde{x}_0)$ are distance $|x_0 - \widetilde{x}_0|$ apart, we get, by the proof of Lemma 10.1 and (11.29),

$$\begin{aligned} \Big| \int\limits_{\Omega_1^*} \frac{\sigma(x_0 - x)}{|x_0 - x|^3} \Big| &\le \int\limits_{\delta}^{\delta + |x_0 - \widetilde{x}_0|} \frac{C}{\delta^3} \, \delta^{2-\alpha} |\nabla\xi_1 - \nabla\xi_2|_\alpha \le \frac{|x_0 - \widetilde{x}_0|}{\delta^{1-\alpha}} \, |\nabla\xi_1 - \nabla\xi_2|_\alpha \\ &\le \frac{\varepsilon C}{f^{1+\alpha}} \, (|\nabla\psi_2|_{L^\infty} + |\nabla\psi_2|_\alpha) \, |x_0 - \widetilde{x}_0|^\alpha \end{aligned} \tag{12.20}$$

where $z = \xi_1$ represents $\partial\psi(B_1)$ and $z = \xi_2$ represents $\partial B_\eta(y_0)$.

Similarly, the set $(\psi(B_1) \Delta B_{\widetilde{\eta}}(\widetilde{y}_0)) \cap (B_\delta(x_0) \backslash B_\rho(\widetilde{x}_0))$ can be broken into three parts:

$$\Omega_2^{**} \text{ inside } B_\rho(x_0), \Omega_2^* \text{ outside } B_\delta(x_0), \text{ and } \Omega_2 \text{ in } B_\delta(x_0) \backslash B_\rho(x_0).$$

Estimates similar to (12.19), (12.20) hold for the corresponding integrals over Ω_2^* and Ω_2^{**}. Hence

$$|w_{22}(x_0) - \widetilde{w}_{22}(\widetilde{x}_0)| \le \frac{\varepsilon C}{f^{1+\alpha}} \, (|\nabla\psi_2|_{L^\infty} + |\nabla\psi_2|_\alpha) \, |x_0 - x_0|^\alpha + K \tag{12.21}$$

where

$$K = \Big| \int\limits_{\Omega_1} \frac{\sigma(x_0 - x)}{|x_0 - x|^3} - \int\limits_{\Omega_2} \frac{\sigma(\widetilde{x}_0 - x)}{|\widetilde{x}_0 - x|^3} \Big| \ ; \tag{12.22}$$

here, as mentioned following (12.10), $\int\limits_{\Omega_1}$ is understood as $\int\limits_{\Omega_{11}} - \int\limits_{\Omega_{12}}$, where

$$\Omega_{11} = (\psi(B_1) \backslash B_\eta(y_0)) \cap (B_\delta(x_0) \backslash B_\rho(x_0)) \cap (B_\delta(\widetilde{x}_0) \backslash B_\rho(\widetilde{x}_0)) \ ,$$

$$\Omega_{12} = (B_\eta(y_0) \backslash \psi(B_1)) \cap (B_\delta(x_0) \backslash B_\rho(x_0)) \cap (B_\delta(\widetilde{x}_0) \backslash B_\rho(\widetilde{x}_0)) \ ,$$

and $\int\limits_{\Omega_2} = \int\limits_{\Omega_{21}} - \int\limits_{\Omega_{22}}$, where

$$\begin{aligned} \Omega_{21} &= (\psi(B_1) \backslash B_{\widetilde{\eta}}(\widetilde{y}_0)) \cap (B_\delta(x_0) \backslash B_\rho(\widetilde{x}_0)) \cap (B_\delta(x_0) \backslash B_\rho(x_0)) \ , \\ \Omega_{22} &= (B_{\widetilde{\eta}}(\widetilde{y}_0) \backslash \psi(B_1)) \cap (B_\delta(x_0) \backslash B_\rho(\widetilde{x}_0)) \cap (B_\delta(x_0) \backslash B_\rho(x_0)) \ . \end{aligned}$$

Hence

$$K = \Big| \Bigg(\int\limits_{\Omega_{11}} \frac{\sigma(x_0 - x)}{|x_0 - x|^3} - \int\limits_{\Omega_{21}} \frac{\sigma(\widetilde{x}_0 - x)}{|\widetilde{x}_0 - x|^3} \Bigg) - \Bigg(\int\limits_{\Omega_{12}} \frac{\sigma(x_0 - x)}{|x_0 - x|^3} - \int\limits_{\Omega_{22}} \frac{\sigma(\widetilde{x}_0 - x)}{|\widetilde{x}_0 - x|^3} \Bigg) \Big| \ .$$

By the proof of Lemma 10.1, (11.29) and (12.14), for any subset $\tilde{\Omega} \subset \Omega_{11} \cup \Omega_{12} \cup \Omega_{21} \cup \Omega_{22}$, we have

$$\int\limits_{\tilde{\Omega}} \left| \frac{\sigma(x_0 - x)}{|x_0 - x|^3} - \frac{\sigma(\tilde{x}_0 - x)}{|\tilde{x}_0 - x|^3} \right| dx \le C|x_0 - \tilde{x}_0| \int\limits_{\tilde{\Omega}} \frac{dx}{\left(|x_0 - x| - \frac{\rho}{4}\right)^4}$$

$$\le \int\limits_{\rho/2}^{\delta} C|x_0 - \tilde{x}_0| \frac{|\nabla \xi_1 - \nabla \xi_2|_\alpha}{r^4} \; r^{2+\alpha} dr$$

(12.23)

$$\le C|x_0 - \tilde{x}_0| \; |\nabla \xi_1 - \nabla \xi_2|_\alpha \left(\frac{1}{\rho^{1-\alpha}} - \frac{1}{\delta^{1-\alpha}} \right)$$

$$\le \frac{\varepsilon C}{f^{1+\alpha}} (|\nabla \psi_2|_{L^\infty} + |\nabla \psi_2|_\alpha) |x_0 - \tilde{x}_0|^\alpha \; ,$$

since $\rho = 4|x_0 - \tilde{x}_0| < \dfrac{\delta}{4}$. Furthermore,

$$(\Omega_{11} \backslash \Omega_{21}) \cup (\Omega_{22} \backslash \Omega_{12}) = (B_{\tilde{\eta}}(\tilde{y}_0) \backslash B_\eta(y_0)) \cap B_\delta(x_0) \backslash \Sigma_1 \; ,$$

$$(\Omega_{21} \backslash \Omega_{11}) \cup (\Omega_{12} \backslash \Omega_{22}) = (B_\eta(y_0) \backslash B_{\tilde{\eta}}(\tilde{y}_0)) \cap B_\delta(x_0) \backslash \Sigma_2 \; ,$$

where Σ_1 and Σ_2 are contained in $(B_\rho(x_0) \cup B_\rho(\tilde{x}_0)) \cap [(\psi(B_1) \Delta B_{\tilde{\eta}}(y_0)) \cup (\psi(B_1) \Delta B_\eta(y_0))]$. Using (12.23) and the fact that the corresponding integrals over Σ_1 and Σ_2 are bounded as in (12.19), we find that

$$K \le \left| \int\limits_{(B_{\tilde{\eta}}(\tilde{y}_0) \backslash B_\eta(y_0)) \cap (B_\delta(x_0) \backslash B_\rho(\tilde{x}_0))} \frac{\sigma(\tilde{x}_0 - x)}{|\tilde{x}_0 - x|^3} - \int\limits_{(B_\eta(y_0) \backslash B_{\tilde{\eta}}(\tilde{y}_0)) \cap (B_\delta(x_0) \backslash B_\rho(x_0))} \frac{\sigma(x_0 - x)}{|x_0 - x|^3} \right|$$

(12.24)

$$+ \frac{\varepsilon C}{f^{1+\alpha}} (|\nabla \psi_2|_{L^\infty} + |\nabla \psi_2|_\alpha) |x_0 - \tilde{x}_0|^\alpha \; .$$

From the definitions of η and $\tilde{\eta}$ (see (11.19)) and the fact that $f \le \delta \le Cf$ (see (11.8) and (11.27), (11.28)), we have

$$|y_0 - \tilde{y}_0| \le \lambda C f, \; |\eta - \tilde{\eta}| \le \lambda C f, \; |\eta - f| \le \lambda C f \quad \text{and} \quad |\tilde{\eta} - f| \le \lambda C f$$

if λ is small. Hence for λ small such that $\lambda C \le \dfrac{1}{q}$ where $q \gg 1, \; B_\eta(y_0) \cap B_{\tilde{\eta}}(\tilde{y}_0) \ne \phi$ and

(12.25)

$$|y_0 - \tilde{y}_0| \le \frac{1}{q} \; f \le \eta \le \left(1 + \frac{1}{q}\right) f \; , \; |\eta - \tilde{\eta}| \le \frac{1}{q} \; f \; ,$$

$$\text{and} \quad \left(1 - \frac{1}{q}\right) f \le \tilde{\eta} \le \left(1 + \frac{1}{q}\right) f \; .$$

Consider first the case where

(12.26) $$x_0 \notin B_{\tilde{\eta}}(\tilde{y}_0) \quad \text{and} \quad \tilde{x}_0 \notin B_\eta(y_0) \; .$$

We shall choose a point x_* in $\partial B_{\tilde\eta}(\tilde y_0) \cap \partial B_\eta(y_0)$ and denote by $\pi(\tilde y_0)$ and $\pi(y_0)$ the tangent planes to $\partial B_{\tilde\eta}(\tilde y_0)$ and $\partial B_\eta(y_0)$ at x_*, respectively. Consider first the case $n = 2$. In this case $\partial B_{\tilde\eta}(\tilde y_0) \cap \partial B_\eta(y_0)$ consists of two points, and x_* is chosen to be the nearest one to x_0; see Figure 2.

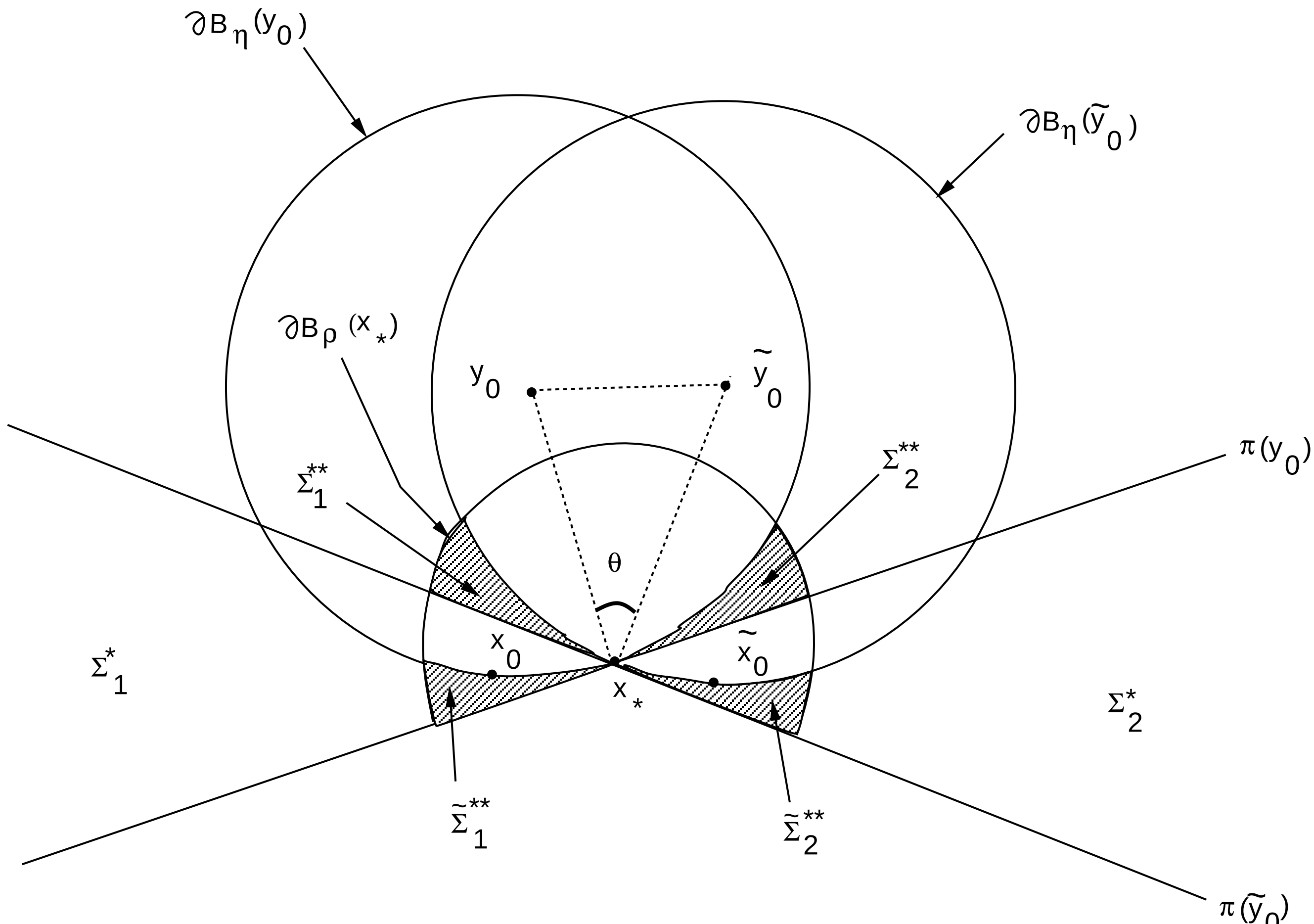

Figure 2

In the triangle $T(y_0, x_*, \tilde y_0)$ with vertices $(y_0, x_*, \tilde y_0)$ the angle θ at x_* goes to zero if $q \to \infty$. Hence in the triangle $T(x_0, x_*, \tilde x_0)$ with vertices $(x_0, x_*, \tilde x_0)$ the angle at x_* goes to π if $q \to \infty$, and, therefore,

$$|x_0 - x_*| \le |x_0 - \tilde x_0| \ , \ |\tilde x_0 - x_*| \le |x_0 - \tilde x_0| \tag{12.27}$$

if q is large (actually, $q \ge 8$ if $|x_0 - \tilde x_0|$ is small enough).

Consider next the 3-dimensional case, and introduce a plane $\tilde\pi$ passing through $x_0, \tilde x_0$ and y_0. Then $\partial B_\eta(y_0) \cap \tilde\pi$ is a circle of center y_0 and radius η. Denote by $\tilde y_1$ and $\tilde\eta_1$ the center and radius of the circle $\partial B_{\tilde\eta}(\tilde y_0) \cap \tilde\pi$. By (12.25),

$$|\tilde y_1 - y_0| \le \frac{1}{q}\ f \ , \ |\tilde\eta_1 - \eta| \le \frac{C}{q} \qquad (\tilde\eta_1^2 = \tilde\eta^2 - |\tilde y_0 - y_0|^2) \ .$$

We choose the point x_* in the intersection of the circles $\partial B_\eta(y_0) \cap \tilde\pi$ and $\partial B_{\tilde\eta}(\tilde y_0) \cap \tilde\pi$, near x_0, and then the estimates in (12.27) remain true (if q is large enough).

Similarly to (12.23) we can derive the estimates

$$\Big| \int\limits_{(B_{\widetilde{\eta}}(\widetilde{y}_0)\setminus B_\eta(y_0))\cap(B_\delta(x_0)\setminus B_\rho(x_*))} \left(\frac{\sigma(\widetilde{x}_0 - x)}{|\widetilde{x}_0 - x|^3} - \frac{\sigma(x_* - x)}{|x_* - x|^3}\right)\Big|$$

$$\leq \frac{\varepsilon C}{f^{1+\alpha}}(|\nabla\psi_2|_{L^\infty} + |\nabla\psi_2|_\alpha)|x_0 - \widetilde{x}_0|^\alpha$$

and

$$\Big| \int\limits_{(B_\eta(y_0)\setminus B_{\widetilde{\eta}}(\widetilde{y}_0))\cap(B_\delta(x_0)\setminus B_\rho(x_*))} \left(\frac{\sigma(x_0 - x)}{|x_0 - x|^3} - \frac{\sigma(x_* - x)}{|x_* - x|^3}\right)\Big|$$

$$\leq \frac{\varepsilon C}{f^{1+\alpha}}(|\nabla\psi_2|_{L^\infty} + |\nabla\psi_2|_\alpha)|x_0 - \widetilde{x}_0|^\alpha \ .$$

Using these estimates in (12.24), we obtain

$$K \leq K_1 + K_2 + \frac{\varepsilon C}{f^{1+\alpha}}(|\nabla\psi_2|_{L^\infty} + |\nabla\psi_2|_\alpha)|x_0 - \widetilde{x}_0|^\alpha \tag{12.28}$$

where

$$K_1 = \Big| \int\limits_{(B_{\widetilde{\eta}}(\widetilde{y}_0)\setminus B_\eta(y_0))\cap B_\delta(x_0)} \frac{\sigma(x_* - x)}{|x_* - x|^3} - \int\limits_{(B_\eta(y_0)\setminus B_{\widetilde{\eta}}(\widetilde{y}_0))\cap B_\delta(x_0)} \frac{\sigma(x_* - x)}{|x_* - x|^3}\Big| ,$$

$$K_2 = \Big| \int\limits_{(B_{\widetilde{\eta}}(\widetilde{y}_0)\setminus B_\eta(y_0))\cap B_\rho(x_*)} \frac{\sigma(x_* - x)}{|x_* - x|^3} - \int\limits_{(B_\eta(y_0)\setminus B_{\widetilde{\eta}}(y_0))\cap B_\rho(x_*)} \frac{\sigma(x_* - x)}{|x_* - x|^3}\Big| .$$

Since $x_* \in \overline{B_\eta(y_0)} \cap \overline{B_{\widetilde{\eta}}(\widetilde{y}_0)}$, by adding and subtracting the integral

$$\int\limits_{B_{\widetilde{\eta}}(\widetilde{y}_0)\cap B_\eta(y_0)} \frac{\sigma(x_* - x)}{|x_* - x|^3}$$

and using Lemma 10.3 we get

$$K_1 \leq \Big| \int\limits_{(B_{\widetilde{\eta}}(\widetilde{y}_0)\setminus B_\eta(y_0))\setminus B_\delta(x_0)} \frac{\sigma(x_* - x)}{|x_* - x|^3} - \int\limits_{(B_\eta(y_0)\setminus B_{\widetilde{\eta}}(\widetilde{y}_0))\setminus B_\delta(x_0)} \frac{\sigma(x_* - x)}{|x_* - x|^3}\Big|$$

$$\leq \frac{C}{\delta^3}\ |(B_{\widetilde{\eta}}(\widetilde{y}_0)\Delta B_\eta(y_0))\setminus B_\delta| \tag{12.29}$$

$$\leq \frac{\varepsilon C}{f}\ \left(\frac{|\nabla\psi_2|_{L^\infty}}{f^\alpha} + \frac{|\nabla\psi_2|_\alpha}{f}\right)|x_0 - \widetilde{x}_0|^\alpha$$

since, by (2.12), the "width" of $B_{\widetilde{\eta}}(\widetilde{y}_0)\Delta B_\eta(y_0)$ is bounded by

$$\varepsilon C\left(\frac{|\nabla\psi_2|_{L^\infty}}{f^\alpha} + \frac{|\nabla\psi_2|_\alpha}{f}\right)|x_0 - \widetilde{x}_0|^\alpha \ .$$

To estimate K_2 let Σ_1^* and Σ_2^* denote the two components of the set lying between the planes $\pi(y_0)$ and $\pi(\tilde{y}_0)$; see Figure 2.

By symmetry, we have

$$K_2 \le \Big| \int\limits_{\Sigma_1^{**}} \frac{\sigma(x_* - x)}{|x_* - x|^3} - \int\limits_{\tilde{\Sigma}_1^{**}} \frac{\sigma(x_* - x)}{|x_* - x|^3} \Big|$$

$$+ \Big| \int\limits_{\Sigma_2^{**}} \frac{\sigma(x_* - x)}{|x_* - x|^3} - \int\limits_{\tilde{\Sigma}_2^{**}} \frac{\sigma(x_* - x)}{|x_* - x|^3} \Big| = K_{21} + K_{22} \ ,$$

where

$$\Sigma_1^{**} = (B_\eta(y_0) \backslash B_{\tilde{\eta}}(\tilde{y}_0)) \cap B_\rho(x^*) \backslash \Sigma_1^* \ ,$$

$$\tilde{\Sigma}_1^{**} = (\Sigma_1^* \backslash (B_\eta(y_0) \backslash B_{\tilde{\eta}}(\tilde{y}_0))) \cap B_\rho(x^*) \ ,$$

$$\Sigma_2^{**} = (B_{\tilde{\eta}}(\tilde{y}_0) \backslash B_\eta(y_0)) \cap B_\rho(x^*) \backslash \Sigma_2^* \ ,$$

$$\tilde{\Sigma}_2^{**} = (\Sigma_2^* \backslash (B_\eta(y_0) \backslash B_{\tilde{\eta}}(\tilde{y}_0))) \cap B_\rho(x^*) \ .$$

Let A be a rotation about x_* such that $A\pi(\tilde{y}_0) = \pi(y_0)$. From (12.12) we deduce that

$$|A - I| \le \varepsilon C \left(\frac{|\nabla \psi_2|_{L^\infty}}{f^\alpha} + \frac{|\nabla \psi_2|_\alpha}{f} \right) |x_0 - \tilde{x}_0|^\alpha \frac{1}{f} \ , \tag{12.30}$$

where I is the identity matrix. By change of variables (recalling (3.11))

$$\int\limits_{\Sigma_1^{**}} \frac{\sigma(x_* - x)}{|x_* - x|^3} - \int\limits_{\tilde{\Sigma}_1^{**}} \frac{\sigma(x_* - x)}{|x_* - x|^3}$$

$$= A^{-1} \left(\int\limits_{A\Sigma_1^{**}} \frac{\sigma(x_* - y)}{|x_* - y|^3} \right) A - \int\limits_{\tilde{\Sigma}_1^{**}} \frac{\sigma(x_* - x)}{|x_* - x|^3} \ .$$

Observe that $A\Sigma_1^{**} \Delta \tilde{\Sigma}_1^{**}$ lies between two spheres that are tangent at x_*. Hence we can apply Lemma 10.1 in order to estimate the integral of $\sigma(x_* - x)/|x_* - x|^3$ over this set. Using also (12.30) we conclude that

$$|K_{21}| \le \frac{\varepsilon C}{f^{1+\alpha}} (|\nabla \psi_2|_{L^\infty} + |\nabla \psi_2|_\alpha) |x_0 - \tilde{x}_0|^\alpha \tag{12.31}$$

provided

$$\Big| \int\limits_{A\Sigma_1^*} \frac{\sigma(x_* - y)}{|x_* - y|^3} \Big| \le C \ . \tag{12.32}$$

The proof of (12.32) follows by noting that if we complete $A\Sigma_1^*$ into half a ball B^+ with its planer boundary on $\pi(\tilde{y}_0)$, then the integral of $\sigma(x_* - y)/|x_* - y|^3$ over B^+ is equal to half the integral over the complete ball, which is constant by Lemma 10.3. Hence it suffices to prove that (12.32) holds when $A\Sigma_1^*$ is replaced by $B^+\backslash A\Sigma_1^*$, and this follows by Lemma 3.1 (after we scale by f).

Having proved (12.31) we note that K_{22}, and then also K_2, can be estimated in the same way. Recalling (12.28), (12.29), we conclude that

$$K \le \frac{\varepsilon C}{f^{1+\alpha}}(|\nabla\psi_2|_{L^\infty} + |\nabla\psi_2|_\alpha)|x_0 - \tilde{x}_0|^\alpha \ . \tag{12.33}$$

So far we have assumed that (12.26) holds. If $x_0 \in B_{\tilde{\eta}}(\tilde{y}_0)$ (or if $\tilde{x}_0 \in B_\eta(y_0)$) then we can proceed as before, in fact more simply, by choosing $x_* = x_0$ (or $x_* = \tilde{x}_0$).

From (12.33) and (12.21) we obtain

$$|w_{22}(x_0) - \tilde{w}_{22}(\tilde{x}_0)| \le \frac{\varepsilon C}{f^{1+\alpha}}(|\nabla\psi_2|_{L^\infty} + |\nabla\psi_2|_\alpha)|x_0 - \tilde{x}_0|^\alpha \ .$$

Recalling (12.18) and (12.16), (12.17), we also get the same bound for $|w_2(x_0) - \tilde{w}_2(\tilde{x}_0)|$. Using (12.15) and recalling (12.9), (12.10), we get the same bound also for $|w(x_0) - \tilde{w}(\tilde{x}_0)|$, i.e.,

$$|w(x_0) - w(\tilde{x}_0)| \le \frac{\varepsilon C}{f^{1+\alpha}}(|\nabla\psi_2|_{L^\infty} + |\nabla\psi_2|_\alpha)|x_0 - \tilde{x}_0|^\alpha \tag{12.34}$$

provided $x_0, \tilde{x}_0 \in \partial\psi(B_1)$ and $|x_0 - \tilde{x}_0| < \dfrac{\delta}{16}$.

Observe that since $\psi_1(x) = f(t)x$, the second integral in (12.3) is a constant (by Lemma 10.3) and

$$w(x) = \int\limits_{\psi(B_1)} \frac{\sigma(x-y)}{|x-y|^3}\, dy - \int\limits_{\tilde{\psi}_1(B_1)} \frac{\sigma(x-y)}{|x-y|^3}\, dy$$

where $\tilde{\psi}_1(x) = \gamma x,\ \gamma = f(t) + \varepsilon|\psi_2|_{L^\infty} \le 2f(t)$. Since $\psi(B_1) \subset \tilde{\psi}_1(B_1)$, the function

$$Z(x) = \int\limits_{\psi(B_1)} \frac{dy}{|x-y|} - \int\limits_{\tilde{\psi}_1(B_1)} \frac{dy}{|x-y|}$$

is harmonic in $\psi(B_1)$. Consider its gradient $V = \nabla Z$, i.e.,

$$V(x) = -\int\limits_{\psi(B_1)} \frac{x-y}{|x-y|^3}\, dy + \int\limits_{\tilde{\psi}_1(B_1)} \frac{x-y}{|x-y|^3}\, dy \ .$$

Let $x_0 \in \partial\psi(B_1)$ and consider in a $\dfrac{\delta}{16}$-neighborhood of x_0 a representation $x_3 = h(x_1, x_2) = h(x')$ of $\partial\psi(B_1)$. Then

$$\frac{\partial}{\partial x_1}\, V(x', h(x')) = w(x)\cdot(1, 0, h_{x_1}) \qquad (x = (x', h(x'))) \ .$$

If $\tilde{x} = (\tilde{x}', h(\tilde{x}'))$ is another point in the $\dfrac{\delta}{16}$-neighborhood of x_0, then

$$\begin{aligned} \Big|\frac{\partial}{\partial x_1}\, V(x', h(x')) - \frac{\partial}{\partial x_1}\, V(\tilde{x}', h(\tilde{x}'))| &\le |w(x) - w(\tilde{x})||\nabla h| \\ &+ |w|_{L^\infty}|h_{x_1}(x') - h_{x_1}(\tilde{x}')| \ . \end{aligned} \tag{12.35}$$

Since $g(\psi^{-1})(x', h(x')) = 0$, we have

$$h_{x_1} = -\frac{g(\psi^{-1})_{x_1}}{g(\psi^{-1})_{x_3}} \ ,$$

and (cf. §11)

$$|h_{x_1}|_{L^\infty} \leq 1 \ , \ |h_{x_1}|_\alpha \leq \frac{C}{\delta^\alpha} \leq \frac{C}{f^\alpha} \ . \tag{12.36}$$

Recalling the definition of w and of I_{12} (in (11.2)) and using Lemma 11.1, we have

$$|w| \leq \frac{\varepsilon C}{f} \left(|\nabla \psi_2|_{L^\infty} + |\nabla \psi_2|_\alpha \right) \ . \tag{12.37}$$

Using also (12.34) and (12.36), we obtain from (12.35) the estimate

$$\left| \frac{\partial}{\partial x_1} V(x', h(x')) - \frac{\partial}{\partial x_1} V(\tilde{x}', h(\tilde{x}')) \right| \leq \frac{\varepsilon C}{f^{1+\alpha}} \left(|\nabla \psi_2|_{L^\infty} + |\nabla \psi_2|_\alpha \right) |x' - \tilde{x}'|^\alpha \tag{12.38}$$

provided $|x' - \tilde{x}'| \leq \dfrac{\delta}{16}$. A similar estimate holds for $\partial V / \partial x_2$.

We now rescale V by

$$\tilde{V}(x) = \frac{1}{f} \ V(fx) \ .$$

$\tilde{V}$ is then harmonic in $\dfrac{1}{f} \ \psi(B_1)$, and its boundary $\partial \left(\dfrac{1}{f} \ \psi(B_1) \right)$ can be represented locally by

$$x_n = \theta(x') = \frac{1}{f} \ h(fx') \ .$$

From (12.38) it follows that

$$\left| \frac{\partial}{\partial x_1} \ \tilde{V}(x', \theta(x')) - \frac{\partial}{\partial x_1} \ \tilde{V}(\tilde{x}', \theta(\tilde{x}')) \right| \leq \frac{\varepsilon C}{f} (|\nabla \psi_2|_{L^\infty} + |\nabla \psi_2|_\alpha) |x' - \tilde{x}'|^\alpha$$

for $|x' - \tilde{x}'| \leq \dfrac{\delta}{16f}$. The same estimate holds for $\dfrac{\partial \tilde{V}}{\partial \tilde{x}_2}$. Combining this estimate with (12.36) and the fact that $\delta / f \geq 1$, we conclude that the tangential derivatives of $\tilde{V}$ on the boundary of $\dfrac{1}{f} \ \psi(B_1)$ is α-Hölder continuous with the α-Hölder coefficient $M = \varepsilon C f^{-1} (|\nabla \psi_2|_{L^\infty} + |\nabla \psi_2|_\alpha)$. Note the $\dfrac{1}{f} \ \psi(B_1) \subset B_2$ and by (8.12) the $C^{1+\alpha}$ of $\dfrac{1}{f} \ \psi(B_1)$ is also uniformly bounded. We can therefore apply [6; Theorem 2.4 and its Remark] to $M^{-1} \tilde{V}(x)$ to obtain the $C^{1+\alpha}$ estimate

$$|\nabla (M^{-1} \tilde{V})|_\alpha \leq C$$

where C is independent of ε. By scaling back we conclude that for $x, \tilde{x} \in \psi(B_1)$,

$$\begin{aligned} |\nabla V(x) - \nabla V(\tilde{x})| &= \left| \nabla \tilde{V} \left(\frac{x}{f} \right) - \nabla \tilde{V} \left(\frac{\tilde{x}}{f} \right) \right| \\ &\leq MC \frac{|x - \tilde{x}|^\alpha}{f^\alpha} \ . \end{aligned}$$

It follows that

$$|\nabla V|_\alpha \leq \frac{\varepsilon C}{f^{1+\alpha}}(|\nabla \psi_2|_{L^\infty} + |\nabla \psi_2|_\alpha) \ ,$$

and this completes the proof of Lemma 12.1.

§13. Completion of the proof of Lemma 8.1. We shall need the following lemma.

LEMMA 13.1. *If*

$$u''(t) \geq h(t)u(t) \qquad (h(t) \geq 0)$$

for $t > 0$ *and* $u(0) > 0$, $u'(0) > 0$, *then* $u(t) > 0$ *for all* $t > 0$.

Indeed, otherwise there is a first $t_0 > 0$ such that $u(t_0) = 0$. For $0 < t < t_0$, $u'' > 0$ and therefore also $u' > 0$, so that $u(t_0) > u(0) > 0$, a contradiction.

Set

$$R(t) = |\nabla \psi(\cdot, t)|_{L^\infty} + |\nabla \psi(\cdot, t)|_\alpha \ .$$

From (9.4), (9.13), (11.32) and (9.19) and (12.6) we get

$$R(t) \leq a + bt + \int_0^t \int_0^s \frac{C \log(2+\tau)}{(1+\tau)^{1+\alpha}} \left(1 + \frac{R(\tau)}{1+\tau}\right) d\tau ds \ .$$

Introduce

$$F(t) = a + bt + \int_0^t \int_0^s \frac{C \log(2+\tau)}{(1+\tau)^{1+\alpha}} \left(1 + \frac{R(\tau)}{1+\tau}\right) d\tau ds \ . \tag{13.1}$$

Then $0 \leq R(t) \leq F(t)$ and F satisfies

$$F''(t) \leq \frac{C \log(2+t)}{(1+t)^{1+\alpha}} \left(1 + \frac{F(t)}{1+t}\right) \ , \ F(0) = a > 0, \ F'(0) = b \ . \tag{13.2}$$

Let

$$H'' = \frac{C \log(2+t)}{(1+t)^{1+\alpha}} \left(1 + \frac{H}{1+t}\right) \ , \quad t > 0 \ , \tag{13.3}$$

$$H(0) = a + 1 \ , \ H'(0) = |b| + 1 \ . \tag{13.4}$$

By Lemma 13.1,

$$F(t) \leq H(t) \ .$$

Clearly H and H' remain positive for all $t > 0$ and therefore, for any $T_0 > 0$,

$$0 \leq F(t) \leq H(T_0) \quad \text{if} \quad 0 < t < T_0 \ . \tag{13.5}$$

By (13.1) we then have

$$F''(t) \le \frac{C\log(2+t)}{(1+t)^{1+\alpha}}\left(1+\frac{H(T_0)}{1+t}\right) \equiv k(t) \ ,$$

and we easily deduce that

$$F'(t) \le B(T_0) \quad \text{if} \quad t \le T_0 \qquad \left(B(T_0) = b + \int_0^{T_0} k\right) \ . \tag{13.6}$$

We next introduce a function

$$G(t) = A[2(1+t) - (1+t)^\gamma] \quad \text{for} \quad t \ge T_0 \ , \tag{13.7}$$

where $0 < \gamma < 1$ and $A > 1$ are constants. Then

$$\begin{aligned} \frac{C\log(2+t)}{(1+t)^{1+\alpha}}\left[1+\frac{G}{1+t}\right] &\le \frac{C\log(2+t)}{(1+t)^{1+\alpha}}(1+2A) \\ &\le (1+t)^{\gamma-2}\left[\frac{C\log(2+t)}{(1+t)^{\alpha+\gamma-1}}(1+2A)\right] \end{aligned} \tag{13.8}$$

Choose $1-\alpha < \gamma$ and T_0 so large that

$$\frac{C\log(2+t)}{(1+t)^{\alpha+\gamma-1}} \le \frac{1}{4}\,\gamma(1-\gamma) \quad \text{if} \quad t \ge T_0 \ .$$

Then the right-hand side of (13.8) is

$$\le A\gamma(1-\gamma)(1+t)^{\gamma-2} = G''(t) \ .$$

Hence G is a supersolution to equation (13.3). Choosing T_0 and A such that also

$$G(T_0) = A[2(1+T_0) - (1+T_0)^\gamma] \ge A > H(T_0) \ ,$$

$$G'(T_0) = 2A - A\gamma(1+T_0)^{\gamma-1} \ge A > B(T_0) \ ,$$

and recalling (13.5), (13.6), we can apply Lemma 13.1 to $u = G - F$. We deduce that

$$R(t) \le F(t) \le G(t) \le 2A(1+t) \quad \text{if} \quad t \ge T_0 \ .$$

This together with (13.5) completes the proof of (8.13).

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