Recent Results on Lyapunov-theoretic Techniques for Nonlinear Stability

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Abstract

This paper presents a Converse Lyapunov Function Theorem motivated by robust control analysis and design. Our result is based upon, but generalizes, various aspects of well-known classical theorems. In a unified and natural manner, it (1) includes arbitrary bounded disturbances acting on the system, (2) deals with global asymptotic stability, (3) results in smooth (infinitely differentiable) Lyapunov functions, and (4) applies to stability with respect to not necessarily compact invariant sets.

As a corollary of the obtained Converse Theorem, we show that the well-known Lyapunov sufficient condition for "input-to-state stability" is also necessary, settling positively an open question raised by several authors during the past few years.

1 Introduction

This work is motivated by problems of robust nonlinear stabilization. One of our main contributions is to provide a statement and proof of a Converse Lyapunov Function Theorem which is in a form particularly useful for the study of such feedback control analysis and design problems. We provide a single (and natural) unified result that:

1. applies to stability with respect to not necessarily compact invariant sets;
2. deals with global (as opposed to merely local) asymptotic stability;
3. results in smooth (infinitely differentiable) Lyapunov functions;
4. most importantly, applies to stability in the presence of bounded disturbances acting on the system.

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(This latter property is sometimes called "total stability" and it is equivalent to the stability of an associated differential inclusion.)

The interest in stability with respect to possibly non-compact sets is motivated by applications to areas such as output-control (one needs to stabilize with respect to the zero set of the output variables) and Luenberger-type observer design ("detectability" corresponds to stability with respect to the diagonal set $\{(x, x)\}$, as a subset of the composite state/observer system). Such applications and others are explored in [15], Chapter 5.

Smooth Lyapunov functions, as opposed to merely continuous or once-differentiable, are required in order to apply "backstepping" techniques in which a feedback law is built by successively taking directional derivatives of feedback laws obtained for a simplified system. (See for instance [10] for more on backstepping design.)

Finally, the effect of disturbances, and the study of associated Lyapunov functions, are topics of interest in robust control theory.

In practice, control systems are very often affected by noise, expressed for instance as perturbations on controls and errors on observations. Thus, it is desirable for a system not only to be stable, but also to display so-called "input/state" stability properties. Intuitively, this means that the behavior of the system should remain bounded when its inputs are bounded, and should tend to equilibrium when inputs tend to zero. These notions are closely related to the topic of stability under perturbations (total stability), studied in the classical dynamical systems literature.

In the late 1980s, one of the coauthors introduced a particular precise definition of *input/state stability*, and established a few basic results; see for instance [21], [23], and [22]. These results then were applied in different areas, including observer design and new small-gain theorems; see for instance [25], [27], [7], and [19].

One of the central tools used in [21] and the follow-up papers has always been the fact that a system is input/state stable if it admits an "ISS-Lyapunov function." This motivates checking the ISS property by investigating ISS-Lyapunov functions for the given system. In this work, we show that it is also necessary for a input/state stable system to admit an ISS-Lyapunov function. The proof is based on a reduction to a question about systems with disturbances and relies on our converse theorem.

1.1 Organization of Paper

The paper is organized as follows.

The next section provides the basic definitions and the statement of the main result. Actually, two versions are given, one that applies to global asymptotic stability with respect to arbitrary invariant sets, but assuming completeness of the system —that is, global existence of solutions for all inputs— and another version which does not assume completeness but only applies to the special case of compact invariant sets (in particular, to the usual case of global asymptotic stability with respect to equilibria).

Equivalent characterizations of stability by means of decay estimates have proved very useful in control theory –see e.g. [21]– and this is the subject of Section 3. Some technical facts about Lyapunov functions, including a result on the smoothing of such functions around an attracting set, are given in Section 4. After this, Section 5 establishes some basic facts about complete systems needed for the main result.

Section 6 contains the proof of the main result for the general case. Our proof is based upon, and follows to a great extent the outline of, the one given by Wilson in [30], who provided in the late 1960s a converse Lyapunov function theorem for local asymptotic stability with respect to closed sets. There are however some major differences with that work: we want a global rather than a local result, and several technical issues appear in that case; moreover, and most importantly, we have to deal with disturbances, which makes the careful analysis of uniform bounds of paramount importance. (In addition, even for the case of no disturbances and local stability, several critical steps in the proof are only sketched in [30], especially those concerning Lipschitz properties and smoothness around the attracting set; thus it seems useful to have an expository detailed and self-contained proof in the literature.) A needed technical result on smoothing functions, based closely also on [30], is placed in an Appendix for convenience. Section 7 deals with the compact case, essentially by reparameterization of trajectories.

An example, motivated by related work of Tsinias and Kalouptsidis, is given in Section 8 to show that the analogous theorems are false for unbounded disturbances.

Obviously in a topic such as this one, there are many connections to previous work. While it is likely that we have missed many relevant references, we discuss in Section 9 some relationships between our work and other results in the literature. Relations to work using "prolongations" are particularly important, and are the subject of some more detail in Section 10.

Section 11 presents the input to state stability results.

2 Definitions and Statements of Main Results

Consider the following system:

$$\dot{x}(t) = f(x(t), d(t)), \tag{1}$$

where for each $t \in \mathbb{R}$, $x(t) \in \mathbb{R}^n$ and $d(t) \in \mathcal{D}$, and where $\mathcal{D}$ is a compact subset of $\mathbb{R}^m$, for some positive integers n and m. The map $f : \mathbb{R}^n \times \mathcal{D} \to \mathbb{R}^n$ is assumed to satisfy the following two properties:

- f is continuous.
- f is locally Lipschitz on x uniformly on d, that is, for each compact subset K of $\mathbb{R}^n$ there is some constant c so that $|f(x, \mathbf{d}) - f(z, \mathbf{d})| \leq c\,|x - z|$ for all $x, z \in K$ and all $\mathbf{d} \in \mathcal{D}$, where $|\cdot|$ denotes the usual Euclidian norm.

Note that these properties are satisfied, for instance, if f extends to a continuously differentiable function on a neighborhood of $\mathbb{R}^n \times \mathcal{D}$.

Let $\mathcal{M}_\mathcal{D}$ be the set of all measurable functions from $\mathbb{R}$ to $\mathcal{D}$. We will call functions $d \in \mathcal{M}_\mathcal{D}$ *disturbances*. For each $d \in \mathcal{M}_\mathcal{D}$, we denote by $x(t, x_0, d)$ (and sometimes simply by $x(t)$ if there is no ambiguity from the context) the solution at time t of (1) with $x(0) = x_0$. This is defined on some maximal interval $(T^-_{x_0,d}, T^+_{x_0,d})$ with $-\infty \leq T^-_{x_0,d} < 0 < T^+_{x_0,d} \leq +\infty$.

Sometimes we will need to consider disturbances d that are functions defined only on some interval $I \subseteq \mathbb{R}$. In those cases, by abuse of notation, $x(t, x_0, d)$ will still be used, but only times $t \in I$ will be considered.

The system is said to be *forward complete* if $T^+_{x_0,d} = +\infty$ for all x_0 all $d \in \mathcal{M}_\mathcal{D}$. It is *backward complete* if $T^-_{x_0,d} = -\infty$ for all x_0 all $d \in \mathcal{M}_\mathcal{D}$, and it is *complete* if it is both forward and backward complete.

We say that a closed set $\mathcal{A}$ is an *invariant set* for (1) if

$$\forall x_0 \in \mathcal{A},\ \forall d \in \mathcal{M}_{\mathcal{D}},\ T^{+}_{x_0,d} = +\infty \quad \text{and} \quad x(t,\, x_0,\, d) \in \mathcal{A}, \quad \forall t \geq 0.$$

Remark 2.1 An equivalent formulation of invariance is in terms of the associated differential inclusion

$$\dot{x} \in F(x)\,, \tag{2}$$

where $F(x) = \{f(x,\, \mathbf{d}),\ \mathbf{d} \in \mathcal{D}\}$. The set $\mathcal{A}$ is invariant for (1) if and only if it is invariant with respect to (2) (see e.g. [1]). The notions of stability to be considered later can be rephrased in terms of (2) as well. □

We will use the following notation: for each nonempty subset $\mathcal{A}$ of $\mathbb{R}^n$, and each $\xi \in \mathbb{R}^n$, we denote

$$|\xi|_{\mathcal{A}} \stackrel{\text{def}}{=} d(\xi,\, \mathcal{A}) = \inf{}_{\eta\in\mathcal{A}} d(\xi,\, \eta)\,,$$

the common point-to-set distance, and $|\xi|_{\{0\}} = |\xi|$ is the usual norm.

Let $\mathcal{A} \subseteq \mathbb{R}^n$ be a closed, invariant set for (1). We emphasize that we do not require $\mathcal{A}$ to be compact. We will assume throughout this work that the following mild property holds:

$$\sup_{\xi\in\mathbb{R}^n} \{|\xi|_{\mathcal{A}}\} = \infty\,. \tag{3}$$

This is a minor technical assumption, satisfied in all examples of interest, which will greatly simplify our statements and proofs. (Of course, this property holds automatically whenever $\mathcal{A}$ is compact, and in particular in the important special case in which $\mathcal{A}$ reduces to an equilibrium point.)

Definition 2.2 System (1) is *(absolutely) uniformly globally asymptotically stable* (UGAS) *with respect to the closed invariant set* $\mathcal{A}$ if it is forward complete and the following two properties hold:

1. *Uniform Stability.* There exists a $\mathcal{K}_\infty$-function $\delta(\cdot)$ such that for any $\varepsilon \geq 0$,

$$|x(t,\, x_0,\, d)|_{\mathcal{A}} \,\leq\, \varepsilon \quad \text{for all } d \in \mathcal{M}_{\mathcal{D}}, \quad \text{whenever } |x_0|_{\mathcal{A}} \leq \delta(\varepsilon) \text{ and } t \geq 0\,. \tag{4}$$

2. *Uniform Attraction.* For any $r, \varepsilon > 0$, there is a $T > 0$, such that for every $d \in \mathcal{M}_{\mathcal{D}}$,

$$|x(t,\, x_0,\, d)|_{\mathcal{A}} \,<\, \varepsilon \tag{5}$$

whenever $|x_0|_{\mathcal{A}} < r$ and $t \geq T$. □

For the definitions of the standard comparison classes of $\mathcal{K}_\infty$ and $\mathcal{KL}$ functions, we refer the reader to the appendix.

Observe that when $\mathcal{A}$ is compact the forward completeness assumption is redundant, since in that case property (4) already implies that all solutions are bounded.

In the particular case in which the set $\mathcal{D}$ consists of just one point, the above definition reduces to the standard notion of set asymptotic stability of differential equations. (Note, however, that this definition differs from those in [3], and [30], which are not global.) If, in addition, $\mathcal{A}$ consists of just an equilibrium point x_0, this is the usual notion of global asymptotic stability for the solution $x(t) \equiv x_0$.

Remark 2.3 It is an easy exercise to verify that an equivalent definition results if one replaces $\mathcal{M}_{\mathcal{D}}$ by the subset of piecewise constant disturbances. $\square$

Remark 2.4 Note that the uniform stability condition is equivalent to: there is a $\mathcal{K}_\infty$-function φ so that

$$|x(t, x_0, d)|_{\mathcal{A}} \leq \varphi(|x_0|_{\mathcal{A}}), \quad \forall x_0, \ \forall t \geq 0, \quad \text{and } \forall d \in \mathcal{M}_{\mathcal{D}}\,.$$

(Just let $\varphi = \delta^{-1}$.) $\square$

The following characterization of the UGAS property will be extremely useful.

Proposition 2.5 The system (1) is UGAS with respect to a closed, invariant set $\mathcal{A} \subseteq \mathbb{R}^n$ if and only if it is forward complete and there exists a $\mathcal{KL}$-function β such that, given any initial state x_0, the solution $x(t, x_0, d)$ satisfies

$$|x(t, x_0, d)|_{\mathcal{A}} \leq \beta(|x_0|_{\mathcal{A}}, t)\,, \quad \text{any } t \geq 0\,, \tag{6}$$

for any $d \in \mathcal{M}_{\mathcal{D}}$.

Observe that when $\mathcal{A}$ is compact the forward completeness assumption is again redundant, since in that case property (6) implies that solutions are bounded.

Next we introduce Lyapunov functions with respect to sets. For any differentiable function $V : \mathbb{R}^n \longrightarrow \mathbb{R}$, we use the standard Lie derivative notation

$$L_{f_{\mathbf{d}}} V(\xi) \stackrel{\text{def}}{=} \frac{\partial V(\xi)}{\partial x} \cdot f_{\mathbf{d}}(\xi)\,,$$

where for each $\mathbf{d} \in \mathcal{D}$, $f_{\mathbf{d}}(\cdot)$ is the vector field defined by $f(\cdot, \mathbf{d})$. By "smooth" we always mean infinitely differentiable.

Definition 2.6 A *Lyapunov function* for the system (1) with respect to a nonempty, closed, invariant set $\mathcal{A} \subseteq \mathbb{R}^n$ is a function $V : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that V is smooth on $\mathbb{R}^n \backslash \mathcal{A}$ and satisfies

1. there exist two $\mathcal{K}_\infty$-functions α_1 and α_2 such that for any $\xi \in \mathbb{R}^n$,

$$\alpha_1(|\xi|_{\mathcal{A}}) \leq V(\xi) \leq \alpha_2(|\xi|_{\mathcal{A}})\,; \tag{7}$$

2. there exists a continuous, positive definite function α_3 such that for any $\xi \in \mathbb{R}^n \backslash \mathcal{A}$, and any $\mathbf{d} \in \mathcal{D}$,

$$L_{f_{\mathbf{d}}} V(\xi) \leq -\alpha_3(|\xi|_{\mathcal{A}})\,. \tag{8}$$

A *smooth* Lyapunov function is one which is smooth on all of $\mathbb{R}^n$. $\square$

Remark 2.7 Continuity of V on $\mathbb{R}^n \backslash \mathcal{A}$ and property 1. in the definition imply:

- V is continuous on all of $\mathbb{R}^n$;
- $V(x) = 0 \iff x \in \mathcal{A}$; and
- $V : \mathbb{R}^n \stackrel{\text{onto}}{\longrightarrow} \mathbb{R}_{\geq 0}$ (recall the assumption in equation (3)). $\square$

Our main results will be two converse Lyapunov theorems. The first one is for general closed invariant sets and assumes completeness of the system.

Theorem 1 *Assume that the system (1) is complete. Let $\mathcal{A} \subseteq \mathbb{R}^n$ be a nonempty, closed invariant subset for this system. Then, (1) is* UGAS *with respect to $\mathcal{A}$ if and only if there exists a smooth Lyapunov function V with respect to $\mathcal{A}$.*

The following result does not assume completeness but instead applies only to compact $\mathcal{A}$:

Theorem 2 *Let $\mathcal{A} \subseteq \mathbb{R}^n$ be a nonempty, compact invariant subset for the system (1). Then, (1) is* UGAS *with respect to $\mathcal{A}$ if and only if there exists a smooth Lyapunov function V with respect to $\mathcal{A}$.*

3 Some Preliminaries about UGAS

It will be useful to have a restatement of the second condition in tthe definition of UGAS stated in terms of uniform attraction times:

Lemma 3.1 The uniform attraction property defined in Definition 2.2 is equivalent to the following: There exists a family of mappings $\{T_r\}_{r>0}$ with

- for each fixed $r > 0$, $T_r : \mathbb{R}_{>0} \xrightarrow{\text{onto}} \mathbb{R}_{>0}$ is continuous and is strictly decreasing;
- for each fixed $\varepsilon > 0$, $T_r(\varepsilon)$ is (strictly) increasing as r increases and $\lim_{r\to\infty} T_r(\varepsilon) = \infty$;

such that, for each $d \in \mathcal{M}_{\mathcal{D}}$,

$$|x(t,\, x_0,\, d)|_{\mathcal{A}} < \varepsilon \text{ whenever } |x_0|_{\mathcal{A}} < r \text{ and } t \geq T_r(\varepsilon)\,. \tag{9}$$

Proof. Sufficiency is clear. Now we show the necessity part. For any $r, \varepsilon > 0$, let

$$A_{r,\varepsilon} \stackrel{\text{def}}{=} \{T \geq 0 :\ \forall\, |x_0|_{\mathcal{A}} < r,\ \forall t \geq T, \forall d \in \mathcal{M}_{\mathcal{D}},\ |x(t,\, x_0,\, d)|_{\mathcal{A}} < \varepsilon\} \subseteq \mathbb{R}_{\geq 0}\,. \tag{10}$$

Then from the assumptions, $A_{r,\varepsilon} \neq \emptyset$ for any $r, \varepsilon > 0$. Moreover,

$$A_{r,\varepsilon_1} \subseteq A_{r,\varepsilon_2}, \text{ if } \varepsilon_1 \leq \varepsilon_2, \text{ and } A_{r_2,\varepsilon} \subseteq A_{r_1,\varepsilon}, \text{ if } r_1 \leq r_2\,.$$

Now define $\bar{T}_r(\varepsilon) \stackrel{\text{def}}{=} \inf A_{r,\varepsilon}$. Then $\bar{T}_r(\varepsilon) < \infty$, for any $r, \varepsilon > 0$, and it satisfies

$$\bar{T}_r(\varepsilon_1) \geq \bar{T}_r(\varepsilon_2), \text{ if } \varepsilon_1 \leq \varepsilon_2, \text{ and } \bar{T}_{r_1}(\varepsilon) \leq \bar{T}_{r_2}(\varepsilon), \text{ if } r_1 \leq r_2.$$

So we can define for any $r, \varepsilon > 0$,

$$\tilde{T}_r(\varepsilon) \stackrel{\text{def}}{=} \frac{2}{\varepsilon}\int_{\varepsilon/2}^{\varepsilon} \bar{T}_r(s)\, ds\,. \tag{11}$$

Since $\bar{T}_r(\cdot)$ is decreasing, $\tilde{T}_r(\cdot)$ is well defined and is locally absolutely continuous. Also

$$\tilde{T}_r(\varepsilon) \geq \frac{2}{\varepsilon}\,\bar{T}_r(\varepsilon)\int_{\varepsilon/2}^{\varepsilon} ds = \bar{T}_r(\varepsilon)\,. \tag{12}$$

Furthermore,

$$\begin{aligned}\frac{d\tilde{T}_r(\varepsilon)}{d\varepsilon} &= -\frac{2}{\varepsilon^2}\int_{\varepsilon/2}^{\varepsilon}\bar{T}_r(s)\,ds + \frac{2}{\varepsilon}\left(\bar{T}_r(\varepsilon) - \frac{1}{2}\bar{T}_r(\frac{\varepsilon}{2})\right) \\ &= \frac{1}{\varepsilon}\left[\bar{T}_r(\varepsilon) - \frac{2}{\varepsilon}\int_{\varepsilon/2}^{\varepsilon}\bar{T}_r(s)\,ds\right] + \frac{1}{\varepsilon}\left[\bar{T}_r(\varepsilon) - \bar{T}_r\left(\frac{\varepsilon}{2}\right)\right] \\ &= \frac{1}{\varepsilon}\left[\bar{T}_r(\varepsilon) - \tilde{T}_r(\varepsilon)\right] + \frac{1}{\varepsilon}\left[\bar{T}_r(\varepsilon) - \bar{T}_r\left(\frac{\varepsilon}{2}\right)\right] \le 0, \quad \text{a.e.,} \end{aligned} \tag{13}$$

hence $\tilde{T}_r(\cdot)$ decreases (not necessarily strictly). Since $\bar{T}_{(\cdot)}(\varepsilon)$ increases, from the definition, $\tilde{T}_{(\cdot)}(\varepsilon)$ also increases. Finally, define

$$T_r(\varepsilon) \stackrel{\text{def}}{=} \tilde{T}_r(\varepsilon) + \frac{r}{\varepsilon} \tag{14}$$

Then it follows that

- for any fixed r, $T_r(\cdot)$ is continuous, maps $\mathbb{R}_{>0} \stackrel{\text{onto}}{\longrightarrow} \mathbb{R}_{>0}$, and is strictly decreasing;
- for any fixed ε, $T_r(\varepsilon)$ is increasing as r increases, and $\lim_{r\to\infty} T_r(\varepsilon) = \infty$.

So the only thing left to be shown is that T_r defined by (14) satisfies (9). To do this, pick any x_0 and t with $|x_0|_{\mathcal{A}} < r$ and $t \ge T_r(\varepsilon)$. Then

$$t \ge T_r(\varepsilon) > \tilde{T}_r(\varepsilon) \ge \bar{T}_r(\varepsilon)\,.$$

Hence, by the definition of $\bar{T}_r(\varepsilon)$, $|x(t, x_0, d)|_{\mathcal{A}} < \varepsilon$, as claimed. ∎

3.1 Proof of Characterization via Decay Estimate

We now provide a proof of Proposition 2.5.

[$\Longleftarrow$] Assume that there exists a $\mathcal{KL}$-function β such that (6) holds. Let

$$c_1 \stackrel{\text{def}}{=} \sup \beta(\cdot, 0) \le \infty\,,$$

and choose $\delta(\cdot)$ to be any $\mathcal{K}_\infty$-function with

$$\delta(\varepsilon) \le \bar{\beta}^{-1}(\varepsilon), \quad \text{any } 0 \le \varepsilon < c_1\,,$$

where $\bar{\beta}^{-1}$ denotes the inverse function of $\bar{\beta}(\cdot) \stackrel{\text{def}}{=} \beta(\cdot, 0)$. (If $c_1 = \infty$, we can simply choose $\delta(\varepsilon) \stackrel{\text{def}}{=} \bar{\beta}^{-1}(\varepsilon)$.) Clearly $\delta(\varepsilon)$ is the desired $\mathcal{K}_\infty$-function for the uniform stability property.

The uniform attraction property follows from the fact that for every fixed r, $\lim\limits_{t\to\infty} \beta(r, t) = 0$.

[$\Longrightarrow$] Assume that (1) is UGAS with respect to the closed set $\mathcal{A}$, and let δ be as in the definition. Let $\varphi(\cdot)$ be the $\mathcal{K}$-function $\delta^{-1}(\cdot)$. As mentioned in Remark 2.4, it follows that $|x(t, x_0, d))|_{\mathcal{A}} \le \varphi(|x_0|_{\mathcal{A}})$ for any $x_0 \in \mathbb{R}^n$, any $t \ge 0$, and any $d \in \mathcal{M}_{\mathcal{D}}$.

Let $\{T_r\}_{r\in(0,\infty)}$ be as in Lemma 3.1, and for each $r \in (0, \infty)$ denote $\psi_r \stackrel{\text{def}}{=} T_r^{-1}$. Then, for each $r \in (0, \infty)$, $\psi_r : \mathbb{R}_{>0} \longrightarrow \mathbb{R}_{>0}$ is again continuous, onto, and strictly decreasing. We also write $\psi_r(0) = +\infty$, which is consistent with that fact that

$$\lim_{t\to 0^+} \psi_r(t) = +\infty\,.$$

(Note: The property that $T_{(\cdot)}(t)$ increases to ∞ is not needed here.)

Claim: For any $|x_0|_{\mathcal{A}} < r$, any $t \geq 0$ and any $d \in \mathcal{M}_{\mathcal{D}}$, $|x(t, x_0, d)|_{\mathcal{A}} \leq \psi_r(t)$.

Proof: It follows from the definition of the maps T_r that, for any r, $\varepsilon > 0$, and for any $d \in \mathcal{M}_{\mathcal{D}}$,

$$|x_0|_{\mathcal{A}} < r, \;\; t \geq T_r(\varepsilon) \;\; \Longrightarrow \;\; |x(t, x_0, d)|_{\mathcal{A}} < \varepsilon \, .$$

As $t = T_r(\psi_r(t))$ if $t > 0$, we have, for any such x_0 and d,

$$|x(t, x_0, d)|_{\mathcal{A}} \; < \; \psi_r(t) \, , \; \forall t > 0 \, . \tag{15}$$

The claim follows by combining (15) and the fact that $\psi_r(0) = +\infty$.

Now for any $s \geq 0$ and $t \geq 0$, let

$$\bar{\psi}(s, t) \; \stackrel{\text{def}}{=} \; \min \left\{ \varphi(s), \inf{}_{r \in (s, \infty)} \psi_r(t) \right\} \, . \tag{16}$$

Because of the definition of φ and the above claim, we have, for each x_0, $d \in \mathcal{M}_{\mathcal{D}}$, and $t \geq 0$:

$$|x(t, x_0, d)|_{\mathcal{A}} \leq \bar{\psi}(|x_0|_{\mathcal{A}} \, , t) \, . \tag{17}$$

If $\bar{\psi}$ would be of class $\mathcal{KL}$, we would be done. This may not be the case, so we next majorize $\bar{\psi}$ by such a function.

By its definition, for any fixed t, $\bar{\psi}(\cdot, t)$ is an increasing function (not necessarily strictly). Also because for any fixed $r \in (0, \infty)$, $\psi_r(t)$ decreases to 0 (this follows from the fact that $\psi_r : \mathbb{R}_{>0} \stackrel{\text{onto}}{\longrightarrow} \mathbb{R}_{>0}$ is continuous and strictly decreasing), it follows that

$$\text{for any fixed } s, \;\; \bar{\psi}(s, t) \text{ decreases to 0 as } t \to \infty.$$

Next we construct a function $\tilde{\psi} : \mathbb{R}_{[0, \infty)} \times \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}$ with the following properties:

- for any fixed $t \geq 0$, $\tilde{\psi}(\cdot, t)$ is continuous and strictly increasing;
- for any fixed $s \geq 0$, $\tilde{\psi}(s, t)$ decreases to 0 as $t \to \infty$;
- $\tilde{\psi}(s, t) \geq \bar{\psi}(s, t)$.

Such a function $\tilde{\psi}$ always exists; for instance, it can be obtained as follows. Define first

$$\hat{\psi}(s, t) \; \stackrel{\text{def}}{=} \; \int_s^{s+1} \bar{\psi}(\varepsilon, t) \, d\varepsilon \, . \tag{18}$$

Then $\hat{\psi}(\cdot, t)$ is an absolutely continuous function on every compact subset of $\mathbb{R}_{\geq 0}$, and it satisfies

$$\hat{\psi}(s, t) \geq \bar{\psi}(s, t) \int_s^{s+1} d\varepsilon = \bar{\psi}(s, t) \, .$$

It follows that

$$\frac{\partial \hat{\psi}(s, t)}{\partial s} = \bar{\psi}(s + 1, t) - \bar{\psi}(s, t) \geq 0 \, , \text{ a.e.,}$$

and hence $\hat{\psi}(\cdot, t)$ is increasing. Also since for any fixed s, $\bar{\psi}(s, \cdot)$ decreases, so does $\hat{\psi}(s, \cdot)$. Note that

$$\bar{\psi}(s, t) \leq \bar{\psi}(s, 0) \; = \; \min \left\{ \inf{}_{r \in (s, \infty)} \psi_r(0), \varphi(s) \right\} \; = \; \varphi(s) \, ,$$

(recall that $\psi_r(0) = +\infty$), so by the Lebesgue dominated convergence theorem, for any fixed $s \geq 0$,

$$\lim_{t\to\infty} \hat{\psi}(s,\, t) = \int_s^{s+1} \lim_{t\to\infty} \bar{\psi}(\varepsilon,\, t)\, d\varepsilon = 0\,.$$

Now we see that the function $\hat{\psi}(s,\, t)$ satisfies all of the requirements for $\tilde{\psi}(s,\, t)$ except possibly for the strictly increasing property. We define $\tilde{\psi}$ as follows:

$$\tilde{\psi}(s,\, t) \stackrel{\text{def}}{=} \hat{\psi}(s,\, t) + \frac{s}{(s+1)(t+1)}\,.$$

Clearly it satisfies all the desired properties.

Finally, define

$$\beta(s,\, t) \stackrel{\text{def}}{=} \sqrt{\varphi(s)}\sqrt{\tilde{\psi}(s,\, t)}\,.$$

Then it follows that $\beta(s,\, t)$ is a $\mathcal{KL}$-function, and, for all $x_0,\, t,\, d$:

$$|x(t,\, x_0, d)|_{\mathcal{A}} \leq \sqrt{\varphi(|x_0|_{\mathcal{A}})}\sqrt{\bar{\psi}(|x_0|_{\mathcal{A}}\,,\, t)} \leq \beta(|x_0|_{\mathcal{A}}\,,\, t)\,,$$

which concludes the proof of the Proposition.

4 Some Preliminaries about Lyapunov Functions

In this section we provide some technical results about set Lyapunov functions. A lemma on differential inequalities is also given, for later reference.

Remark 4.1 One may assume in Definition 2.6 that all of $\alpha_1, \alpha_2, \alpha_3$ are smooth in $(0,\, +\infty)$ and of class $\mathcal{K}_\infty$. For α_1 and α_2, this is proved simply by finding two functions $\tilde{\alpha}_1, \tilde{\alpha}_2$ in $\mathcal{K}_\infty$, smooth in $(0,\, +\infty)$ so that

$$\tilde{\alpha}_1(s) \leq \alpha_1(s) \leq \alpha_2(s) \leq \tilde{\alpha}_2(s)\,, \text{ for all } s.$$

For α_3, a new Lyapunov function W and a function $\tilde{\alpha}_3$ which satisfies (8) with respect to W, but is smooth in $(0,\, +\infty)$ and of class $\mathcal{K}_\infty$, can be constructed as follows. First, pick $\tilde{\alpha}_3$ to be any $\mathcal{K}_\infty$-function, smooth in $(0,\, +\infty)$, such that

$$\tilde{\alpha}_3(s) \leq s\alpha_3(s)\,,\ \forall s \in [0,\, \alpha_1^{-1}(1)].$$

This is possible since α_3 is positive definite. Then let

$$\gamma : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}$$

be a $\mathcal{K}_\infty$-function, smooth in $(0,\, +\infty)$, such that

- $\gamma(r) \geq \alpha_1^{-1}(r)$ for all $r \in [0,\, 1]$;
- $\gamma(r) > \dfrac{\tilde{\alpha}_3(\alpha_1^{-1}(r))}{\alpha_3(\alpha_1^{-1}(r))}$ for all $r > 1$.

Now define $\beta(s) \stackrel{\text{def}}{=} \int_0^s \gamma(r)\,dr$. Note that β is a $\mathcal{K}_\infty$-function, smooth in $(0, +\infty)$. Let $W(\xi) \stackrel{\text{def}}{=} \beta(V(\xi))$. This is smooth on $\mathbb{R}^n \backslash \mathcal{A}$, and $\beta \circ \alpha_1$, $\beta \circ \alpha_2$ bound W as in equation (7). Moreover,

$$\beta'(V(\xi)) = \gamma(V(\xi)) \geq \gamma(\alpha_1(|\xi|_\mathcal{A}))\,,$$

so

$$L_{f_\mathbf{d}} W(\xi) = \beta'(V(\xi)) L_{f_\mathbf{d}} V(\xi) \leq -\gamma(\alpha_1(|\xi|_\mathcal{A}))\alpha_3(|\xi|_\mathcal{A})\,. \tag{19}$$

We claim that this is bounded by $-\tilde{\alpha}_3(|\xi|_\mathcal{A})$. Indeed, if $s \stackrel{\text{def}}{=} |\xi|_\mathcal{A} \leq \alpha_1^{-1}(1)$, then from the first item above and the definition of $\tilde{\alpha}_3$,

$$\gamma(\alpha_1(s)) \geq s \geq \frac{\tilde{\alpha}_3(s)}{\alpha_3(s)}\,;$$

if instead $s > \alpha_1^{-1}(1)$, then from the second item, also

$$\gamma(\alpha_1(s)) \geq \frac{\tilde{\alpha}_3(s)}{\alpha_3(s)}\,.$$

In either case, $\gamma(\alpha_1(s))\alpha_3(s) \geq \tilde{\alpha}_3(s)$, as desired. From now on, whenever necessary, we assume that α_1, α_2, α_3 are $\mathcal{K}_\infty$-functions, smooth in $(0, +\infty)$. $\square$

4.1 Smoothing of Lyapunov Functions

When dealing with control system design, one often needs to know that V can be taken to be globally smooth, rather than just smooth outside of $\mathcal{A}$.

Proposition 4.2 If there is a Lyapunov function for (1) with respect to $\mathcal{A}$, then there is also a smooth such Lyapunov function. $\square$

The proof relies on constructing a smooth function of the form $W = \beta \circ V$, where

$$\beta : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}$$

is built using a partition of unity.

Again let $\mathcal{A} \subseteq \mathbb{R}^n$ be nonempty and closed. For a multi-index $\varrho = (\varrho_1, \varrho_2, \ldots, \varrho_n)$, we use $|\varrho|$ to denote $\sum_{i=1}^n \varrho_i$. The following is a standard regularization result, but we have not been able to find it stated in the way needed here, so we include a proof.

Lemma 4.3 Assume that $V : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ is C^0, the restriction $V|_{\mathbb{R}^n \backslash \mathcal{A}}$ is C^∞, and also $V|_\mathcal{A} = 0$, $V|_{\mathbb{R}^n \backslash \mathcal{A}} > 0$. Then there exists a $\mathcal{K}_\infty$-function β, smooth on $(0, \infty)$ and so that $\beta^{(i)}(t) \to 0$ as $t \to 0^+$ for each $i = 0, 1, \ldots$ and having $\beta'(t) > 0$, $\forall t > 0$, such that

$$W \stackrel{\text{def}}{=} \beta \circ V$$

is a C^∞ function on all of $\mathbb{R}^n$.

Proof. Let $K_1, K_2, \ldots,$ be compact subsets of $\mathbb{R}^n$ such that $\mathcal{A} \subseteq \bigcup_{i=1}^\infty \operatorname{int}(K_i)$. For any $k \geq 1$, let

$$I_k \stackrel{\text{def}}{=} \left(\frac{1}{k+2}, \frac{1}{k}\right) \subseteq \mathbb{R}$$

and $I_0 \stackrel{\text{def}}{=} I_1$. Pick for any $k \geq 1$, a smooth (C^∞) function $\gamma_k : \mathbb{R}_{>0} \to [0, 1]$ satisfying

- $\gamma_k(t) = 0$ if $t \notin I_k$; and
- $\gamma_k(t) > 0$ if $t \in I_k$.

Define for any $k \geq 1$,

$$\mathcal{G}_k \stackrel{\text{def}}{=} \left\{ x \in \mathbb{R}^n : \; x \in \bigcup_{i=1}^{k} K_i, \; V(x) \in \operatorname{clos} I_k \right\}.$$

Then $\mathcal{G}_k$ is compact (because of compactness of the sets K_i and continuity of V). Observe that each derivative $\gamma_k^{(i)}$ has a compact support included in $\operatorname{clos} I_k$, so it is bounded. For each $k = 1, 2, \ldots$, let $c_k \in \mathbb{R}$ satisfy

1. $c_k \geq 1$;
2. $c_k \geq |(D^\varrho V)(x)|$ for any multi-index $|\varrho| \leq k$ and any $x \in \mathcal{G}_k$; and
3. $c_k \geq |\gamma_k^{(i)}(t)|$, for any $i \leq k$ and any $t \in \mathbb{R}_{>0}$.

Choose the sequence d_k to satisfy

$$0 < d_k < \frac{1}{2^k (k+1)! c_k^k}, \quad k = 1, 2, \ldots. \tag{20}$$

Let $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a C^∞ function such that $\alpha \equiv 0$ on $\left[0, \frac{1}{3}\right]$ and $\alpha \geq 1$ on $\left[\frac{1}{2}, \infty\right)$. Define $\gamma(0) \stackrel{\text{def}}{=} 0$ and

$$\gamma(t) \stackrel{\text{def}}{=} \sum_{k=1}^{\infty} d_k \gamma_k(t) + \alpha(t), \quad \forall t > 0. \tag{21}$$

Notice that for any $t \in (0, 1)$, if $k \stackrel{\text{def}}{=} \left\lfloor \frac{1}{t} \right\rfloor \geq 1$ denotes the largest integer $\leq \frac{1}{t}$, then $t \in I_{k-1}$, and

$$t \notin I_j \quad \text{if} \quad j \neq k, \, k-1.$$

Hence the sum in (21) at most consists of three terms (for $t \geq 1$ the sum is just $\gamma = \alpha$), and so γ is C^∞ at each $t \in (0, \infty)$.

Claim: For any $i \geq 0$, $\lim_{t \to 0^+} \gamma^{(i)}(t) = 0$.

Proof: Fix any $i \geq 0$. Given any $\varepsilon > 0$, let $k_0 \in \mathbb{Z}$ be such that $\varepsilon > \frac{1}{k_0} > 0$. Let

$$T \stackrel{\text{def}}{=} \min \left\{ \frac{1}{k_0}, \frac{1}{i+1}, \frac{1}{3} \right\}.$$

We will show that $t \in (0, T) \implies \left|\gamma^{(i)}(t)\right| < \varepsilon$. Indeed, as $0 < t < \min \left\{ \frac{1}{k_0}, \frac{1}{i+1}, \frac{1}{3} \right\}$, it follows that $k \stackrel{\text{def}}{=} \left\lfloor \frac{1}{t} \right\rfloor \geq \max\{i+1, k_0, 3\}$. So

$$\gamma^{(i)}(t) \leq d_{k-1} \gamma_{k-1}^{(i)}(t) + d_k \gamma_k^{(i)}(t),$$

and noticing that

$$i \leq k-1 < k \implies c_k \geq \left|\gamma_k^{(i)}(t)\right|, \quad c_{k-1} \geq \left|\gamma_{k-1}^{(i)}(t)\right|,$$

we have

$$\left|\gamma^{(i)}(t)\right| \leq d_{k-1}c_{k-1} + d_k c_k \leq \frac{1}{2k!} + \frac{1}{2(k+1)!} < \frac{1}{k!} < \frac{1}{k} \leq \frac{1}{k_0} < \varepsilon,$$

as wanted.

Note also that if $t \geq \frac{1}{2}$, then $\gamma(t) \geq \alpha(t) \geq 1 > 0$; and if $t \in \left(0, \frac{1}{2}\right)$, then $\gamma(t) \geq d_{k-1}\gamma_{k-1}(t) > 0$ with $k \stackrel{\mathrm{def}}{=} \left\lfloor \frac{1}{t} \right\rfloor \geq 2$, so the function

$$\beta(t) \stackrel{\mathrm{def}}{=} \int_0^t \gamma(s)\, ds \tag{22}$$

is also a $\mathcal{K}_\infty$-function, smooth on $(0,\infty)$. Furthermore, β satisfies $\beta^{(i)}(t) \to 0$ as $t \to 0^+$ for each $i = 0, 1, \ldots$.

Finally, we show that $W = \beta \circ V$ is C^∞. For this, is is it is enough to show that $D^{\varrho_0}W(x_n) \to 0$ as $x_n \to \bar{x} \in \partial\mathcal{A}$, for each multi-index ϱ_0 and each sequence $\{x_n\} \subseteq \mathbb{R}^n \backslash \mathcal{A}$ converging to a point $\bar{x}$ in the boundary of $\mathcal{A}$. (In general, see e.g. [4] (p. 52), if $\mathcal{A} \subseteq \mathbb{R}^n$ is closed and $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}$ satisfies that $\varphi|_{\mathcal{A}} = 0$, $\varphi|_{\mathbb{R}^n \backslash \mathcal{A}}$ is C^∞, and for each boundary point a of $\mathcal{A}$ and all multi-indices $\varrho = (\varrho_1, \varrho_2, \ldots, \varrho_n)$, it holds that $\lim_{\substack{x \to a \\ x \notin \mathcal{A}}} D^\varrho \varphi(x) = 0$, then φ is C^∞ on $\mathbb{R}^n$.)

Pick one such ϱ_0 and any sequence $\{x_n\}$ with $x_n \to \bar{x} \in \partial\mathcal{A}$. If $|\varrho_0| = 0$, one only needs to show that $W(x_n) \to 0$, which follows easily from the facts that $\beta \in \mathcal{K}_\infty$ and $V(x_n) \to 0$. So from now on, we can assume that $|\varrho_0| \stackrel{\mathrm{def}}{=} i \geq 1$. As $\mathcal{A} \subseteq \cup_{j=0}^\infty \mathrm{int}\, K_j$, $\bar{x} \in \mathrm{int}\, K_l$ for some l, and without loss of generality we may assume that there is some fixed l so that

$$x_n \in K_l, \quad \text{for all } n.$$

Pick any $\varepsilon > 0$. We will show that there exists some N such that

$$n > N \implies |D^{\varrho_0}W(x_n)| < \varepsilon.$$

Let $k \in \mathbb{Z}$ be so that

$$k > \max\left\{i, \log_2\left(\frac{1}{\varepsilon}\right), l\right\}$$

and let $T \in \left(0, \frac{1}{3}\right)$ be such that $T < \frac{1}{k+2}$. Observe that if $t < T$, then $t \notin I_1 \cup \cdots \cup I_k$.

As V is C^0 everywhere, $V = 0$ at $\mathcal{A}$, $V(x_n) \to V(\bar{x}) = 0$. So there exists N such that $V(x_n) < T$ whenever $n > N$. Fix an N like this. Then for any $n > N$,

$$\gamma_s^{(j)}(V(x_n)) = 0, \ \forall j, \ \forall s = 1, 2, \ldots, k,$$

(since γ_s vanishes outside I_s). Pick any $j \in \mathbb{N}$ with $j \leq i$, any $h \in \mathbb{N}$ with $h \leq i-1$, and $\varrho_1, \ldots, \varrho_h$ multi-indices such that $|\varrho_\mu| \leq i$, $\forall \mu = 1, \ldots, h$. Then for any $q \in \mathbb{N}$ with $q > k$, by the way we chose c_k,

$$\left|\gamma_q^{(j)}(V(x_n))\right| \leq c_q,$$

since $q > k > i \geq j$. Also, if $V(x_n) \in I_q$, then again by the properties of the sequence c_k,

$$|D^{\varrho_\mu} V(x_n)| \leq c_q \,,$$

(since $q > k > l$ and $x_n \in K_l$ imply $x_n \in K_1 \cup \cdots \cup K_q$, and $|\varrho_\mu| \leq i < k < q$). Therefore, for such q, if $V(x_n) \in I_q$,

$$\left|\gamma_q^{(j)}(V(x_n))\right| \, |D^{\varrho_1} V(x_n)| \cdots |D^{\varrho_h} V(x_n)| \;\leq\; c_q^{h+1} \;\leq\; c_q^i \;<\; c_q^q \,. \tag{23}$$

If instead it would be the case that $V(x_n) \notin I_q$, then $\gamma_q^{(j)}(V(x_n)) = 0$, and hence the inequality (23) still holds. Since

$$\gamma^{(j)}(V(x_n)) = \sum_{q=k+1}^{\infty} d_q \gamma_q^{(j)}(V(x_n)) \,,$$

we also have

$$\left|\gamma^{(j)}(V(x_n))\right| \, |D^{\varrho_1} V(x_n)| \cdots |D^{\varrho_h} V(x_n)| \;\leq\; \sum_{q=k+1}^{\infty} d_q c_q^q \;<\; \sum_{q=k+1}^{\infty} \frac{1}{2^q (q+1)!}$$

$$< \;\; \left(\sum_{q=k+1}^{\infty} \frac{1}{2^q}\right) \frac{1}{(k+1)!} = \frac{1}{2^k (k+1)!} \;<\; \frac{\varepsilon}{(k+1)!} \,. \tag{24}$$

Now observe that

$$(D^{\varrho_0} W)(x) = (D^{\varrho_0} (\beta \circ V))(x)$$

is a sum of $\leq i!$ terms (recall $0 < i = |\varrho_0|$), each of which is of the form

$$\beta^{(p)}(V(x)) \, (D^{\varrho_1} V)(x) \cdots (D^{\varrho_h} V)(x) \,,$$

where $0 < p \leq i$, $h \leq i - 1$, and each $|\varrho_\mu| \leq i$. Each

$$\beta^{(p)}(V(x)) = \gamma^{(j)}(V(x)) \,, \; j = p - 1 \leq i - 1 \,,$$

so (24) applies, and we conclude

$$|\,(D^{\varrho_0} W)(x_n)| \leq i! \frac{\varepsilon}{(k+1)!} < \varepsilon \,,$$

(since $k > i$.) ∎

Now let us return to the proof of the Proposition 4.2.

Proof of Proposition 4.2. Assume $\mathcal{A}$, V and $\alpha_1, \alpha_2, \alpha_3$ are as defined in Definition 2.6. Let β, W be as in Lemma 4.3. We show that W is a smooth Lyapunov function as required.

Let $\hat{\alpha}_i \stackrel{\text{def}}{=} \beta \circ \alpha_i, i = 1, 2$. These are again $\mathcal{K}_\infty$-functions, and they satisfy

$$\hat{\alpha}_1(|\xi|_{\mathcal{A}}) \leq W(\xi) \leq \hat{\alpha}_2(|\xi|_{\mathcal{A}}) \,.$$

We define, for $s > 0$:

$$\check{\beta}(s) \stackrel{\text{def}}{=} \min{}_{t \in [\alpha_1(s), \alpha_2(s)]} \beta'(t) \;>\; 0 \,.$$

Let also $\check{\beta}(0) \stackrel{\text{def}}{=} 0$. Define $\hat{\alpha}_3(s) \stackrel{\text{def}}{=} \check{\beta}(s)\alpha_3(s)$. Then $\hat{\alpha}_3$ is a continuous positive definite function. Also, for any $\xi \in \mathbb{R}^n \backslash \mathcal{A}$,

$$\begin{aligned} L_{f_{\mathbf{d}}} W(\xi) \;&=\; \beta'(V(\xi)) L_{f_{\mathbf{d}}} V(\xi) \;\leq\; -\beta'(V(\xi))\alpha_3(|\xi|_{\mathcal{A}}) \\ &\leq\; -\check{\beta}(|\xi|_{\mathcal{A}})\alpha_3(|\xi|_{\mathcal{A}}) \;=\; -\hat{\alpha}_3(|\xi|_{\mathcal{A}}) \,, \end{aligned}$$

which concludes the proof of the Proposition. ∎

4.2 A Useful Estimate

The following lemma establishes a useful comparison principle.

Lemma 4.4 For each continuous and positive definite function α, there exists a $\mathcal{KL}$-function $\beta_\alpha(s,t)$ with the following property: if $y(\cdot)$ is any (locally) absolutely continuous function defined for $t \geq 0$ and with $y(t) \geq 0$ for all t, and $y(\cdot)$ satisfies the differential inequality

$$\dot{y}(t) \leq -\alpha(y(t)), \quad \text{for almost all} \quad t \tag{25}$$

with $y(0) = y_0 \geq 0$, then it holds that

$$y(t) \leq \beta_\alpha(y_0, t)$$

for all $t \geq 0$.

Proof. Define for any $s > 0$, $\eta(s) \overset{\text{def}}{=} -\int_1^s \frac{dr}{\alpha(r)}$. This is a strictly decreasing differentiable function on $(0, \infty)$. Without loss of generality, we will assume that $\lim_{s\to 0^+} \eta(s) = +\infty$. If this were not the case, we could consider instead the following function:

$$\bar{\alpha}(s) \overset{\text{def}}{=} \begin{cases} \min\{s, \alpha(s)\}, & \text{if } 0 \leq s < 1, \\ \alpha(s), & \text{if } s \geq 1. \end{cases}$$

This function is again continuous, positive definite, satisfies $\bar{\alpha}(s) \leq \alpha(s)$ for any $s \geq 0$, and

$$\lim_{s\to 0^+} \int_s^1 \frac{dr}{\bar{\alpha}(r)} \;\geq\; \int_s^1 \frac{dr}{r} \;=\; +\infty\,.$$

Moreover, if $\dot{y}(t) \leq -\alpha(y(t))$ then also $\dot{y}(t) \leq -\bar{\alpha}(y(t))$, so $\beta_{\bar{\alpha}}$ could be used to bound solutions.

Let

$$0 < a \overset{\text{def}}{=} -\lim_{s\to+\infty} \eta(s)\,.$$

Then the range of η, and hence also the domain of η^{-1}, is the open interval $(-a, \infty)$. (We allow the possibility that $a = \infty$.) For $(s,t) \in \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$, define

$$\beta_\alpha(s,t) \overset{\text{def}}{=} \begin{cases} 0, & \text{if } s = 0, \\ \eta^{-1}(\eta(s) + t), & \text{if } s > 0. \end{cases}$$

We claim that for any $y(\cdot)$ satisfying the conditions in the Lemma,

$$y(t) \;\leq\; \beta_\alpha(y_0, t), \quad \text{for all } t \geq 0\,. \tag{26}$$

As $\dot{y}(t) \leq -\alpha(y(t))$, it follows that $y(t)$ is nonincreasing, and if $y(t_0) = 0$ for some $t_0 \geq 0$, then $y(t) \equiv 0$, $\forall t \geq t_0$. Without loss of generality, assume that $y_0 > 0$. Let

$$t_0 \overset{\text{def}}{=} \inf\{t : \; y(t) = 0\} \;\leq\; +\infty\,.$$

It is enough to show (26) holds for $t \in [0, t_0)$.

As η is strictly decreasing, we only need to show that $\eta(y(t)) \;\geq\; \eta(y_0) + t$, that is,

$$-\int_1^{y(t)} \frac{dr}{\alpha(r)} \;\geq\; -\int_1^{y_0} \frac{dr}{\alpha(r)} + t\,,$$

which is equivalent to

$$\int_{y(t)}^{y_0} \frac{dr}{\alpha(r)} \;\geq\; t\,. \tag{27}$$

From (25), one sees that

$$\int_0^t \frac{\dot{y}(\tau)}{\alpha(y(\tau))}\, d\tau \;\leq\; -\int_0^t d\tau \;=\; -t\,.$$

Changing variables in the integral, this gives (27).

It only remains to show that β_α is of class $\mathcal{KL}$. The function β_α is continuous since both η and η^{-1} are continuous in their domains, and $\lim_{r\to\infty} \eta^{-1}(r) = 0$. It is strictly increasing in s for each fixed t since since both η and η^{-1} are strictly decreasing. Finally, $\beta_\alpha(s,\,t) \to 0$ as $t \to \infty$ by construction. So β_α is a $\mathcal{KL}$-function. ∎

5 Some Properties of Complete Systems

We need to first establish some technical properties that hold for complete systems, and in particular a Lipschitz continuity fact.

For each $\xi \in \mathbb{R}^n$ and $T > 0$, let

$$\mathcal{R}^T(\xi) \;\stackrel{\text{def}}{=}\; \{\eta :\; \eta = x(T,\xi,d),\; d \in \mathcal{M}_{\mathcal{D}}\}\,.$$

This is the reachable set of (1) from ξ at time T. We use $\mathcal{R}^{\leq T}(\xi)$ to denote $\bigcup_{0\leq t\leq T} \mathcal{R}^t(\xi)$. If S is a subset of $\mathbb{R}^n$, we write

$$\mathcal{R}^T(S) \;\stackrel{\text{def}}{=}\; \bigcup_{\xi\in S} \mathcal{R}^T(\xi)\,,\quad \mathcal{R}^{\leq T}(S) \;\stackrel{\text{def}}{=}\; \bigcup_{\xi\in S} \mathcal{R}^{\leq T}(\xi)\,.$$

In what follows we use $\overline{S}$ to denote the closure of S for any subset S of $\mathbb{R}^n$.

Proposition 5.1 Assume that (1) is forward complete. Then for any compact subset K of $\mathbb{R}^n$ and any $T > 0$, the set $\overline{\mathcal{R}^{\leq T}(K)}$ is compact.

To prove Proposition 5.1, we first need to make a couple of technical observations.

Lemma 5.2 Let K be a compact subset of $\mathbb{R}^n$ and let $T > 0$. Then the set $\overline{\mathcal{R}^{\leq T}(K)}$ is compact if and only if $\overline{\mathcal{R}^{\leq T}(\xi)}$ is compact for each $\xi \in K$.

Proof. It is clear that the compactness of $\overline{\mathcal{R}^{\leq T}(K)}$ implies the compactness of $\overline{\mathcal{R}^{\leq T}(\xi)}$ for any $\xi \in K$.

Now assume, for $T > 0$ and a compact set K, that $\overline{\mathcal{R}^{\leq T}(\xi)}$ is compact for each $\xi \in K$. Pick any $\xi \in K$, and let $\mathcal{U} = \{\eta :\; d(\eta, \overline{\mathcal{R}^{\leq T}(\xi)}) < 1\}$. Then $\overline{\mathcal{U}}$ is compact. Let C be a Lipschitz constant for f with respect to x on $\overline{\mathcal{U}}$, and let $r = e^{-CT}$. For each $d \in \mathcal{M}_{\mathcal{D}}$ and each η with $|\eta - \xi| < r$, let $\tilde{t} = \inf\{t \geq 0 :\; |x(t,\,\eta,\,d) - x(t,\,\xi,\,d)| \geq 1\}$. Then, using Gronwall's Lemma, one can show that $\tilde{t} \geq T$, from which it follows that

$$\mathcal{R}^{\leq T}(\eta) \subseteq \overline{\mathcal{U}}, \qquad \forall |\eta - \xi| < r.$$

Thus, for each $\xi \in K$, there is a neighborhood $\mathcal{V}_\xi$ of ξ such that $\overline{\mathcal{R}^{\leq T}(\mathcal{V}_\xi)}$ is compact. By compactness of K, it follows that $\overline{\mathcal{R}^{\leq T}(K)}$ is compact. ∎

Lemma 5.3 For any subset S of $\mathbb{R}^n$ and any $T > 0$,

$$\mathcal{R}^T\left(\overline{S}\right) \subseteq \overline{\mathcal{R}^T(S)}, \quad \mathcal{R}^{\leq T}\left(\overline{S}\right) \subseteq \overline{\mathcal{R}^{\leq T}(S)}.$$

In particular, $\overline{\mathcal{R}^{\leq T}\left(\overline{S}\right)} = \overline{\mathcal{R}(S,\, T)}$.

Proof. The first conclusion follows from the continuity of solutions on initial states; see [24], Theorem 1. The second is immediate from there. ∎

We now return to the proof of Proposition 5.1. By Lemma 5.2, it is enough to show that $\overline{\mathcal{R}^{\leq T}(\xi)}$ is compact for each $\xi \in \mathbb{R}^n$ and each $T > 0$. Pick any $\xi_0 \in \mathbb{R}^n$, and let

$$\tau = \sup\{T \geq 0 : \ \overline{\mathcal{R}^{\leq T}(\xi_0)} \text{ is compact }\}.$$

Note that $\tau > 0$. This is because $|x(t,\, \xi_0,\, d) - \xi_0| \leq 1$ for any $0 \leq t < 1/M$ and any $d \in \mathcal{M}_{\mathcal{D}}$, where

$$M = \max\{|f(\xi,\, \mathbf{d})| : \ |\xi - \xi_0| \leq 1,\, \mathbf{d} \in \mathcal{D}\}.$$

We must show that $\tau = \infty$.

Assume that $\tau < \infty$. Using the same argument used above, one can show that if $\overline{\mathcal{R}^{\leq t}(\xi_0)}$ is compact for some $t > 0$ then there is some $\delta > 0$ such that $\overline{\mathcal{R}^{\leq (t+\delta)}(\xi_0)}$ is compact. From here it follows that $\overline{\mathcal{R}^{\leq \tau}(\xi_0)}$ is not compact. By definition, $\overline{\mathcal{R}^{\leq t}(\xi_0)}$ is compact for any $t < \tau$.

Let $\tau_1 = \tau/2$. Then there is some $\eta_1 \in \overline{\mathcal{R}^{\tau_1}(\xi_0)}$ such that $\overline{\mathcal{R}^{\leq (\tau-\tau_1)}(\eta_1)}$ is not compact; otherwise, by Lemma 5.2, $\overline{\mathcal{R}^{\leq (\tau-\tau_1)}\left(\overline{\mathcal{R}^{\tau_1}(\xi_0)}\right)}$ would be compact. This, in turn, would imply that $\overline{\mathcal{R}^{\leq \tau}(\xi_0)}$ is compact, since

$$\mathcal{R}^{\leq \tau}(\xi_0) \subseteq \mathcal{R}^{\leq \tau_1}(\xi_0) \bigcup \mathcal{R}^{\leq (\tau-\tau_1)}(\mathcal{R}^{\tau_1}(\xi_0)) \subseteq \overline{\mathcal{R}^{\leq \tau_1}(\xi_0)} \bigcup \overline{\mathcal{R}^{\leq (\tau-\tau_1)}\left(\overline{\mathcal{R}^{\tau_1}(\xi_0)}\right)}.$$

On the other hand, combining Lemma 5.3 with the fact that $\overline{\mathcal{R}^{\leq t}(\mathcal{R}^{\tau_1}(\xi_0))}$ is compact for any $0 \leq t < \tau - \tau_1$, one sees that $\overline{\mathcal{R}^{\leq t}(\eta_1)}$ is compact for any $0 \leq t < \tau - \tau_1$.

Since $\eta_1 \in \overline{\mathcal{R}^{\tau_1}(\xi_0)}$, there exists a sequence $\{z_n\} \to \eta_1$ with $z_n \in \mathcal{R}^{\tau_1}(\xi_0)$. Assume, for each n, that $z_n = x(\tau_1,\, \xi_0,\, d_n)$ for some $d_n \in \mathcal{M}_{\mathcal{D}}$. For each $d \in \mathcal{M}_{\mathcal{D}}$, and each $s \in \mathbb{R}$, we use d_s to denote the function defined by $d_s(t) = d(s+t)$. Then by uniqueness, one has that for each n, $x(s,\, z_n,\, (d_n)_{\tau_1}) \in K_1$ for any $-\tau_1 \leq s \leq 0$, where $K_1 = \overline{\mathcal{R}^{\leq \tau_1}(\xi_0)}$. We want to claim next that, by compactness of K_1 and Gronwall's Lemma,

$$|x(-\tau_1,\, \eta_1,\, (d_n)_{\tau_1}) - \xi_0| = |x(-\tau_1,\, \eta_1,\, (d_n)_{\tau_1}) - x(-\tau_1,\, z_n,\, (d_n)_{\tau_1})| \longrightarrow 0, \quad \text{as } \ n \to \infty.$$

The only potential problem is that the solution $x(-\tau_1, \eta_1, (d_n)_{\tau_1})$ may fail to exist a priori. However, it is possible to modify $f(x, \mathbf{d})$ outside a neighborhood of $K_1 \times \mathcal{D}$ so that it now has compact support and is hence globally bounded. The modified dynamics is complete. Now the above limit holds for the modified system, and *a fortriori* it also holds for the original system.

Choose n_0 such that

$$|x(-\tau_1,\, \eta_1,\, (d_{n_0})_{\tau_1}) - \xi_0| < 1/2. \tag{28}$$

Let $v_1 = d_{n_0}$, and let $\eta_0 = x(-\tau_1,\, \eta_1,\, (d_{n_0})_{\tau_1})$. Then, by continuity on initial conditions, there is a neighborhood $\mathcal{U}_1$ of η_1 contained in $B(\eta_1, 1)$ such that

$$|x(-\tau_1,\, \xi,\, (v_1)_{\tau_1}) - \eta_0| < 1/2, \quad \forall \xi \in \mathcal{U}_1, \tag{29}$$

where $B(\eta, r)$ denotes the open ball centered at η with radius r. Combining (28) and (29), one has

$$x(-\tau_1, \xi, (v_1)_{\tau_1}) \in \mathcal{U}_0, \quad \forall \xi \in \mathcal{U}_1,$$

where $\mathcal{U}_0 = B(\xi_0, 1)$.

Let $\tau_2 = \tau_1/2 = (\tau - \tau_1)/2$. Applying the above argument with ξ_0 replaced by η_1, τ replaced by $(\tau - \tau_1)$, and τ_1 replaced by τ_2, one shows that there exists some $\eta_2 \in \overline{\mathcal{R}^{\tau_2}(\eta_1)}$ such that $\overline{\mathcal{R}^{\leq t}(\eta_2)}$ is compact for any $0 \leq t < \tau - \sigma_2$, and $\overline{\mathcal{R}^{\leq(\tau-\sigma_2)}(\eta_2)}$ is not compact, where $\sigma_2 = \tau_1 + \tau_2$, and there exist some v_2 defined on $[0, \tau_2)$ and some neighborhood $\mathcal{U}_2$ of η_2 contained in $B(\eta_2, 1)$, such that

$$x(-\tau_2, \xi, (v_2)_{\tau_2}) \in \mathcal{U}_1, \quad \forall \xi \in \mathcal{U}_2.$$

By induction, one can get, for each $k \geq 1$ a point η_k, a neighborhood $\mathcal{U}_k$ of η_k contained in $B(\eta_k, 1)$, and a function v_k defined on $[0, \tau_k)$ (where $\tau_k = 2^{-k}\tau$) such that

- $\overline{\mathcal{R}^{\leq(\tau-\sigma_k)}(\eta_k)}$ is not compact, where $\sigma_k = \tau_1 + \tau_2 + \cdots + \tau_k = \tau(1 - 2^{-k}) \to \tau$;
- $x(-\tau_k, \xi, (v_k)_{\tau_k}) \in \mathcal{U}_{k-1}$, for any $\xi \in \mathcal{U}_k$.

Now define v on $[0, \tau)$ by concatenating all the v_k's. That is, $v(t) = v_k(t)$ for $t \in [\sigma_{k-1}, \sigma_k)$ (with $\sigma_0 \stackrel{\text{def}}{=} 0$). Then $v \in \mathcal{M}_{\mathcal{D}}$. For each k, let

$$\zeta_k = x(-\sigma_k, \eta_k, (v^k)_{\sigma_k}),$$

where v^k is the restriction of v to $[0, \sigma_k)$. By induction,

$$x\left(-(\sigma_k - \sigma_i), \eta_k, (v^k)_{\sigma_k}\right) \in \mathcal{U}_{k-i},$$

for each $0 \leq i \leq k$, from which it follows that $\zeta_k \in \mathcal{U}_0$ for each k. By compactness of $\overline{\mathcal{U}}_0$, there exists some subsequence of $\{\zeta_k\}$ converging to some point $\zeta_0 \in \mathbb{R}^n$. For ease of notation, we still use $\{\zeta_k\}$ to denote this convergent subsequence. Our aim is next to prove that the solution starting at ζ_0 and applying the measurable disturbance v does not exist for time τ, contradicting forward completeness.

First notice that for any compact set S, there exists some k such that $\eta_k \notin S$. Otherwise, assume that there exists some compact set S such that $\eta_k \in S$ for all k. Let $S_1 = \{\eta : d(\eta, S) \leq 1\}$. The compactness of S implies that there exists some $\delta > 0$ such that

$$\mathcal{R}^{\leq t}(\eta) \subseteq S_1$$

for any $\eta \in S$, and any $t \in [0, \delta]$. In particular, it implies that $\mathcal{R}^{\leq(\tau-\sigma_k)}(\eta_k) \subseteq S_1$ for k large enough so that $\tau - \sigma_k < \delta$. This contradicts the fact that $\overline{\mathcal{R}^{\leq(\tau-\sigma_k)}(\eta_k)}$ is not compact for each k.

Assume that $x(\tau, \zeta_0, v)$ is defined. By continuity on initial conditions, this would imply that $x(t, \zeta_k, v)$ is defined for all $t \leq \tau$ and for all k large enough, and it converges uniformly to $x(t, \zeta_0, v)$. Thus, $x(t, \zeta_k, v)$ remains in a compact for all $t \in [0, \tau]$ and all k. But

$$x(\sigma_k, \zeta_k, v) = x(\sigma_k, \zeta_k, v^k) = \eta_k,$$

contradicting what was just proved. So $x(\tau, \zeta_0, v)$ is not defined, which contradicts the forward completeness of the system. ∎

Remark 5.4 For $T>0$ and $\xi \in \mathbb{R}^n$, let

$$\mathcal{R}^{-T}(\xi) = \{\eta :\ \eta = x(-T, \xi, d),\ d \in \mathcal{M}_{\mathcal{D}}\}, \quad \text{and} \quad \mathcal{R}^{\geq -T}(\xi) = \bigcup_{t\in[-T,0]} \mathcal{R}^t(\xi)\,.$$

These are the reachable sets from ξ for the reversed system

$$\dot{x}(t) = -f(x(t), d(t))\,. \tag{30}$$

Similarly, one defines $\mathcal{R}^{-T}(S)$ and $\mathcal{R}^{\geq -T}(S)$ for subsets S of $\mathbb{R}^n$. If (1) is backward complete, that is, if (30) is forward complete, and applying Proposition 5.1 to (30), one concludes, for system (1), that $\overline{\mathcal{R}^{\geq -T}(K)}$ is compact for any $T>0$ and any compact subset K of $\mathbb{R}^n$. In particular, for systems that are (forward and backward) complete,

$$\overline{\mathcal{R}^{\geq -T}(K) \bigcup \mathcal{R}^{\leq T}(K)}$$

is compact for any compact set K and any $T>0$. □

Combining the above conclusion and Gronwall's Lemma, one has the following fact:

Proposition 5.5 Assume that (1) is complete. For any fixed $T>0$ and any compact $K \subseteq \mathbb{R}^n$, there is a constant $C>0$ (which only depends on the set K and T), such that for the trajectories $x(t, x_0, d)$ of the system (1),

$$|x(t,\xi,d) - x(t,\eta,d)| \leq C|\xi - \eta|$$

for any $\xi, \eta \in K$, any $|t| \leq T$, and any $d \in \mathcal{M}_{\mathcal{D}}$. □

6 Proof of the First Converse Lyapunov Theorem

Proof. [$\Longleftarrow$] Pick any $x_0 \in \mathbb{R}^n$ and any $d \in \mathcal{M}_{\mathcal{D}}$, and let $x(\cdot)$ be the corresponding trajectory. Then we have

$$\frac{dV(x(t))}{dt} \leq -\alpha_3(|x(t)|_{\mathcal{A}}) \leq -\alpha(V(x(t)))\,,\ \text{a.e. } t \geq 0\,,$$

where α is the $\mathcal{K}_\infty$-function defined by

$$\alpha(\cdot) \stackrel{\text{def}}{=} \alpha_3(\alpha_2^{-1}(\cdot))\,.$$

Now let β_α be the $\mathcal{KL}$-function as in Lemma 4.4 with respect to α, and define

$$\beta(s,t) \stackrel{\text{def}}{=} \alpha_1^{-1}(\beta_\alpha(\alpha_2(s), t))\,. \tag{31}$$

Then β is a $\mathcal{KL}$-function, since both α_1 and α_2 are $\mathcal{K}_\infty$-functions. By Lemma 4.4,

$$V(x(t)) \leq \beta_\alpha(V(x_0), t)\,, \quad \text{any } t \geq 0\,.$$

Hence

$$|x(t)|_{\mathcal{A}} \leq \beta(|x_0|_{\mathcal{A}}, t)\,,\ \text{any } t \geq 0\,.$$

Therefore the system (1) is UGAS with respect to $\mathcal{A}$, by Proposition 2.5.

[$\Longrightarrow$] We will show the existence of a not necessarily smooth Lyapunov function; then the existence of a smooth function will follow from Proposition 4.2. Assume that the system is UGAS with respect to the set $\mathcal{A}$. Let δ and T_r be as in Definition 2.2 and Lemma 3.1.

Define $g : \mathbb{R}^n \longrightarrow \mathbb{R}$ by

$$g(\xi) \stackrel{\text{def}}{=} \inf{}_{t\leq 0, d\in\mathcal{M}_\mathcal{D}} \{|x(t,\, \xi,\, d)|_\mathcal{A}\} \,. \tag{32}$$

Note that, by uniqueness of solutions, for each $t_0 > 0$ and each d, it holds that

$$x(t-t_0,\, x(t_0,\, \xi,\, d),\, d_{t_0}) \,=\, x(t,\, \xi, d)\,,$$

where d_{t_0} is defined by $d_{t_0}(t) = d(t+t_0)$. Pick any $d \in \mathcal{M}_\mathcal{D}$, $\xi \in \mathbb{R}^n$, and $t_1 > 0$. Let $\xi_1 = x(t_1,\, \xi,\, d)$. Then for any $t < 0$, and $v \in \mathcal{M}_\mathcal{D}$,

$$x(t,\, \xi, v) \,=\, x(t-t_1,\, \xi_1,\, v_{t_1}\#d_{t_1})\,,$$

where

$$v_{t_1}\#d_{t_1}(s) \,=\, \begin{cases} d(s+t_1), & \text{if } -t_1 \leq s \leq 0, \\ v(s+t_1), & \text{if } t-t_1 \leq t < -t_1. \end{cases}$$

Thus,

$$\begin{aligned} g(\xi) &= \inf{}_{t\leq 0, v\in\mathcal{M}_\mathcal{D}} |x(t,\, \xi,\, v)|_\mathcal{A} = \inf{}_{t\leq 0, d\in\mathcal{M}_\mathcal{D}} \left|x(t-t_1,\, \xi_1,\, v_{t_1}\#d_{t_1})\right|_\mathcal{A} \\ &= \inf{}_{\tau\leq -t_1, v\in\mathcal{M}_\mathcal{D}} \left|x(\tau,\, \xi_1,\, v_{t_1}\#d_{t_1})\right|_\mathcal{A} \geq \inf{}_{\tau\leq 0, v\in\mathcal{M}_\mathcal{D}} \left|x(\tau,\, \xi_1,\, v)\right|_\mathcal{A} \\ &= g(\xi_1)\,. \end{aligned}$$

This implies that

$$g(x(t,\, \xi,\, d)) \leq g(\xi), \quad \forall t > 0\,, \quad \forall d \in \mathcal{M}_\mathcal{D}\,. \tag{33}$$

Also one has

$$\delta(|\xi|_\mathcal{A}) \leq g(\xi) \leq |\xi|_\mathcal{A}\,. \tag{34}$$

The second half of (34) is obvious from $x(0,\, \xi,\, d) = \xi$. On the other hand, if the first half were not true, then there would be some $d \in \mathcal{M}_\mathcal{D}$ and some $t_0 \leq 0$ such that

$$\delta(|\xi|_\mathcal{A}) > |x(t_0,\, \xi,\, d)|_\mathcal{A}\,.$$

Pick any $0 < \varepsilon < |\xi|_\mathcal{A}$ so that $|x(t_0,\, \xi,\, d)|_\mathcal{A} < \delta(\varepsilon)$. By the uniform stability property, applied with $t = -t_0$ and $x_0 = x(t_0,\, \xi,\, d)$,

$$|\xi|_\mathcal{A} = |x(-t_0,\, x(t_0,\, \xi,\, d),\, d_{t_0})|_\mathcal{A} < |\xi|_\mathcal{A}\,,$$

which is a contradiction.

For any $0 < \varepsilon < r$, define $K_{\varepsilon,r} \stackrel{\text{def}}{=} \{\xi \in \mathbb{R}^n :\ \varepsilon \leq |\xi|_\mathcal{A} < r\}$.

Fact 1: For all ε and r with $0 < \varepsilon < r$, there exists $q_{\varepsilon,r} \leq 0$, such that:

$$\xi \in K_{\varepsilon,r},\ d \in \mathcal{M}_\mathcal{D}, \ \text{ and }\ t < q_{\varepsilon,r} \quad \Longrightarrow \quad |x(t,\, \xi,\, d)|_\mathcal{A} \geq r\,.$$

Proof: If the statement were not true, then there would exist ε, r with $0 < \varepsilon < r$ and three sequences $\{\xi_k\} \subseteq K_{\varepsilon,r}$, $\{t_k\} \subseteq \mathbb{R}$ and $d_k \in \mathcal{M}_\mathcal{D}$ with $\lim_{k\to\infty} t_k = -\infty$ such that for all k:

$$|x(t_k,\, \xi_k, d_k)|_\mathcal{A} < r\,.$$

Pick k large enough so that $-t_k > T_r(\varepsilon)$, then by the uniform attraction property,

$$|\xi_k|_{\mathcal{A}} = |x(-t_k,\, x(t_k,\, \xi_k,\, d_k),\, (d_k)_{t_k})|_{\mathcal{A}} < \varepsilon\,,$$

which is a contradiction. This proves the fact.

Therefore, for any $\xi \in K_{\varepsilon,r}$,

$$g(\xi) = \inf\{|x(t,\, \xi,\, d)|_{\mathcal{A}} :\; t \in [q_{\varepsilon,r},\, 0],\, d \in \mathcal{M}_{\mathcal{D}}\}.$$

Lemma 6.1 The function $g(\xi)$ is locally Lipschitz on $\mathbb{R}^n \backslash \mathcal{A}$, and continuous everywhere.

Proof. Fix any $\xi_0 \in \mathbb{R}^n\backslash\mathcal{A}$, and let $s = \dfrac{|\xi_0|_{\mathcal{A}}}{2}$. Let $\bar{B}(\xi_0,\, s)$ denote the closed ball centered at ξ_0 and with radius s. Then $\bar{B}(\xi_0,\, s) \subseteq K_{\sigma,r}$ for some $0 < \sigma < r$. Pick a constant C as in Proposition 5.5 with respect to this closed ball and $T = |q_{\sigma,r}|$. Pick any $\zeta, \eta \in \bar{B}(\xi_0,\, s)$. For any $\varepsilon > 0$, there exist some $d_{\eta,\varepsilon}$ and $t_{\eta,\varepsilon} \in [q_{\sigma,r},\, 0]$ such that $g(\eta) \geq |x(t_{\eta,\varepsilon},\, \eta,\, d_{\eta,\varepsilon})|_{\mathcal{A}} - \varepsilon$. Thus

$$g(\zeta) - g(\eta) \;\leq\; |x(t_{\eta,\varepsilon},\, \zeta,\, d_{\eta,\varepsilon})|_{\mathcal{A}} - |x(t_{\eta,\varepsilon},\, \eta,\, d_{\eta,\varepsilon})|_{\mathcal{A}} + \varepsilon \;\leq\; C|\zeta - \eta| + \varepsilon\,. \tag{35}$$

Note that (35) holds for all $\varepsilon > 0$, so it follows that

$$g(\zeta) - g(\eta) \;\leq\; C\,|\zeta - \eta|\,.$$

Similarly, $g(\eta) - g(\zeta) \leq C|\zeta - \eta|$. This proves that g is locally Lipschitz on $\mathbb{R}^n\backslash\mathcal{A}$.

Note that g is 0 on $\mathcal{A}$, and for $\xi \in \mathcal{A}$, $\eta \in \mathbb{R}^n$:

$$|g(\eta) - g(\xi)| = |g(\eta)| \leq |\eta|_{\mathcal{A}} \leq |\eta - \xi|\,,$$

thus g is globally continuous. (We are not claiming that g is locally Lipschitz on $\mathbb{R}^n$, though.) ∎

Now define $U :\, \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ by

$$U(\xi) \;\stackrel{\text{def}}{=}\; \sup_{t\geq 0, d\in\mathcal{M}_{\mathcal{D}}} \Big\{g(x(t,\, \xi,\, d))\, k(t)\Big\}\,, \tag{36}$$

where $k : R_{\geq 0} \longrightarrow \mathbb{R}_{>0}$ is any strictly increasing, smooth function that satisfies:

- there are two constants $0 < c_1 < c_2 < \infty$, such that $k(t) \in [c_1,\, c_2]$ for all $t \geq 0$;
- there is a bounded positive decreasing continuous function $\tau(\cdot)$, such that

$$k'(t) \geq \tau(t) \quad \text{for all} \;\; t \geq 0\,.$$

(For instance, $\dfrac{c_1 + c_2 t}{1 + t}$ is one example of such a function.) Observe that

$$U(\xi) \;\leq\; \sup_{t\geq 0}\Big(g(\xi)\, k(t)\Big) \leq\; c_2 g(\xi) \;\leq\; c_2\, |\xi|_{\mathcal{A}}\,, \tag{37}$$

and

$$U(\xi) \;\geq\; \sup_{d\in\mathcal{M}_{\mathcal{D}}} g(x(t,\, \xi,\, d))\, k(t)|_{t=0} \;\geq\; c_1 g(\xi) \;\geq\; c_1\delta(|\xi|_{\mathcal{A}})\,. \tag{38}$$

For any $\xi \in \mathbb{R}^n$, since

$$|x(t,\,\xi,\,d)|_{\mathcal{A}} \;\le\; \beta(|\xi|_{\mathcal{A}}\,,t),\;\; \forall d\,,\;\; \forall t \ge 0,$$

for some $\mathcal{KL}$-function β, and $0 \le g(x(t,\,\xi,\,d)) \le |x(t,\,\xi,\,d)|_{\mathcal{A}}$ for all $t \ge 0$, it follows that

$$\lim_{t\to+\infty} \sup_{d} g(x(t,\,\xi,\,d)) = 0\,.$$

Thus there exists some $\tau_\xi \in [0,\,\infty)$ such that

$$U(\xi) = \sup_{0\le t\le \tau_\xi, d\in\mathcal{M}_{\mathcal{D}}} g(x(t,\,\xi,\,d))\,k(t)\,.$$

In fact, we can get the following explicit bound.

Fact 2: For any $0 < |\xi|_{\mathcal{A}} < r$,

$$U(\xi) \;=\; \sup_{0\le t\le t_\xi, d\in\mathcal{M}_{\mathcal{D}}} g(x(t,\,\xi,\,d))k(t)\,,$$

where $t_\xi = T_r\left(\dfrac{c_1}{2c_2}\delta(|\xi|_{\mathcal{A}})\right)$.

Proof: If the statement is not true, then for any $\varepsilon > 0$, there exists some $t_\varepsilon > T_r\left(\frac{c_1}{2c_2}\delta(|\xi|_{\mathcal{A}})\right)$ and some d_ε such that

$$U(\xi) \;\le\; g(x(t_\varepsilon,\,\xi,\,d_\varepsilon))k(t_\varepsilon) \,+\, \varepsilon\,.$$

So we have

$$\begin{aligned}\delta(|\xi|_{\mathcal{A}}) \;&\le\; \frac{1}{c_1}\,U(\xi) \le \frac{1}{c_1}\,g(x(t_\varepsilon,\,\xi,\,d_\varepsilon))\,k(t_\varepsilon) \,+\, \frac{\varepsilon}{c_1}\\ &\le\; \frac{c_2}{c_1}\,g(x(t_\varepsilon,\,\xi,\,d_\varepsilon)) \,+\, \frac{\varepsilon}{c_1} \;\le\; \frac{c_2}{c_1}\,|x(t_\varepsilon,\,\xi,\,d_\varepsilon)|_{\mathcal{A}} \,+\, \frac{\varepsilon}{c_1} \;<\; \frac{\delta(|\xi|_{\mathcal{A}})}{2} + \frac{\varepsilon}{c_1}\,.\end{aligned}$$

Taking the limit as ε tends to 0 results in a contradiction.

For any compact set $K \subseteq \mathbb{R}^n\backslash\mathcal{A}$, let

$$t_K \;\stackrel{\mathrm{def}}{=}\; \max_{\xi\in K} t_\xi < \infty\,.$$

(Finiteness follows from Fact 2, as $K \subseteq \{\xi:\; 0 < |\xi|_{\mathcal{A}} < r\}$ for some $r > 0$.)

Lemma 6.2 The function $U(\cdot)$ defined by (36) is locally Lipschitz on $\mathbb{R}^n\backslash\mathcal{A}$, and continuous everywhere.

Proof. For $\xi_0 \notin \mathcal{A}$, pick up a compact neighborhood K_0 of ξ_0 so that $K_0 \cap \mathcal{A} = \emptyset$. By (38), one knows that

$$U(\xi) \;>\; r_0\,,\;\; \forall \xi \in K_0\,,$$

for some constant $r_0 > 0$. Let $r_1 = \dfrac{r_0}{2c_2}$ and let

$$K_1 = K_0 \bigcap \left\{\eta:\;\; |\eta - \xi_0| \le \frac{r_1}{4C}\right\}\,,$$

where C is such a constant that

$$|x(t,\,\xi,\,d) - x(t,\eta,\,d)| \;\le\; C\,|\xi - \eta|\,,\;\; \forall \xi,\eta \in K_0,\;\; 0 \le t \le t_{K_0},\;\; d \in \mathcal{M}_{\mathcal{D}}. \tag{39}$$

In the following we will show that there exists some $L > 0$ such that for any $\xi, \eta \in K_1$, it holds that

$$|U(\xi) - U(\eta)| \leq L\,|\xi - \eta| \,. \tag{40}$$

First of all, for any $\xi \in K_1$ and any $\varepsilon \in (0, r_0/2)$, there exists $t_{\xi,\varepsilon} \in [0, t_{K_0}]$ and $d_{\xi,\varepsilon} \in \mathcal{M}_{\mathcal{D}}$, such that

$$U(\xi) \leq g(x(t_{\xi,\varepsilon}, \xi, d_{\xi,\varepsilon}))k(t_{\xi,\varepsilon}) + \varepsilon \leq c_2\,|x(t_{\xi,\varepsilon}, \xi, d_{\xi,\varepsilon})|_{\mathcal{A}} + \varepsilon \,,$$

from which it follows that

$$|x(t_{\xi,\varepsilon}, \xi, d_{\xi,\varepsilon})|_{\mathcal{A}} \geq r_1 \,.$$

It follows from (39) that for any $\eta \in K_1$,

$$|x(t_{\xi,\varepsilon}, \eta, d_{\xi,\varepsilon})|_{\mathcal{A}} \geq |x(t_{\xi,\varepsilon}, \xi, d_{\xi,\varepsilon})|_{\mathcal{A}} - |x(t_{\xi,\varepsilon}, \xi, d_{\xi,\varepsilon}) - x(t_{\xi,\varepsilon}, \eta, d_{\xi,\varepsilon})| \geq \frac{r_1}{2} \,.$$

By Proposition 5.1 one knows that there exists some compact set K_2 such that

$$x(t, \xi, d) \in K_2, \quad \forall \xi \in K_1,\ \forall t \in [0, t_{K_1}],\ \text{and}\ \forall d \in \mathcal{M}_{\mathcal{D}} \,.$$

Again, applying Lemma 6.1 to the compact set $K_2 \bigcap \{\zeta :\ |\zeta|_{\mathcal{A}} \geq r_1/2\}$, one sees that

$$|g(x(t_{\xi,\varepsilon}, \xi, d_{\xi,\varepsilon})) - g(x(t_{\xi,\varepsilon}, \eta, d_{\xi,\varepsilon}))| \leq C_1\,|x(t_{\xi,\varepsilon}, \xi, d_{\xi,\varepsilon}) - x(t_{\xi,\varepsilon}, \eta, d_{\xi,\varepsilon})| \,,$$

for some $C_1 > 0$. Therefore, we have the following:

$$\begin{aligned} U(\xi) - U(\eta) &\leq g(x(t_{\xi,\varepsilon}, \xi, d_{\xi,\varepsilon}))k(t_{\xi,\varepsilon}) + \varepsilon - g(x(t_{\xi,\varepsilon}, \eta, d_{\xi,\varepsilon}))k(t_{\xi,\varepsilon}) \\ &\leq c_2\,|g(x(t_{\xi,\varepsilon}, \xi, d_{\xi,\varepsilon})) - g(x(t_{\xi,\varepsilon}, \eta, d_{\xi,\varepsilon}))| + \varepsilon \\ &\leq C_1 c_2\,|x(t_{\xi,\varepsilon}, \xi, d_{\xi,\varepsilon}) - x(t_{\xi,\varepsilon}, \eta, d_{\xi,\varepsilon})| + \varepsilon \\ &\leq L\,|\xi - \eta| + \varepsilon \,, \end{aligned}$$

for some constant L that only depends on the compact set K_1. Note that the above holds for any $\varepsilon \in (0, r_0/2)$, thus,

$$U(\xi) - U(\eta) \leq L\,|\xi - \eta| \,, \quad \forall \xi, \eta \in K_1 \,.$$

By symmetry, one proves (40).

To prove the continuity of U on $\mathbb{R}^n$, note that for any $\xi \in \mathcal{A}$, it holds that $U(\xi) = 0$, and so for all $\eta \in \mathbb{R}^n$:

$$|U(\xi) - U(\eta)| = U(\eta) \leq c_2\,|\eta|_{\mathcal{A}} \leq c_2\,|\xi - \eta| \,.$$

The proof of Lemma 6.2 is thus concluded. ∎

We next start proving that U decreases along trajectories. Now pick any $\xi \notin \mathcal{A}$. Let $h_0 > 0$ be such that

$$|x(t, \xi, \mathbf{d})|_{\mathcal{A}} \geq \frac{|\xi|_{\mathcal{A}}}{2}, \quad \forall \mathbf{d} \in \mathcal{D},\ \forall t \in [0, h_0] \,,$$

where $\mathbf{d}$ denotes the constant function $d(t) \equiv \mathbf{d}$. Such an h_0 exists by continuity. Pick any $h \in [0, h_0]$. For each $\mathbf{d} \in \mathcal{D}$, let $\eta_{\mathbf{d}} = x(h, \xi, \mathbf{d})$. For any $\varepsilon > 0$, there exist some $t_{\mathbf{d},\varepsilon}$ and $d_{\mathbf{d},\varepsilon} \in \mathcal{M}_{\mathcal{D}}$ such that

$$\begin{aligned} U(\eta_{\mathbf{d}}) &\leq g(x(t, \eta_{\mathbf{d}}, d_{\mathbf{d},\varepsilon}))k(t_{\mathbf{d},\varepsilon}) + \varepsilon \\ &= g(x(t_{\mathbf{d},\varepsilon} + h, \xi, \tilde{d}_{\mathbf{d},\varepsilon}))k(t_{\mathbf{d},\varepsilon} + h)\left(1 - \frac{k(t_{\mathbf{d},\varepsilon} + h) - k(t_{\mathbf{d},\varepsilon})}{k(t_{\mathbf{d},\varepsilon} + h)}\right) + \varepsilon \\ &\leq U(\xi)\left(1 - \frac{k(t_{\mathbf{d},\varepsilon} + h) - k(t_{\mathbf{d},\varepsilon})}{c_2}\right) + \varepsilon \,, \end{aligned} \tag{41}$$

where $\tilde{d}_{\mathbf{d},\varepsilon}$ is the concatenation of $\mathbf{d}$ and $d_{\mathbf{d},\varepsilon}$. Still for these ξ and h, and for any $r > |\xi|_{\mathcal{A}}$, define

$$T^r_{\xi,h} \stackrel{\text{def}}{=} \max_{0\leq \bar{t}\leq h,\mathbf{d}\in\mathcal{D}} T_r\left(\frac{c_1}{2c_2}\delta(|x(\bar{t},\,\xi,\,\mathbf{d})|_{\mathcal{A}})\right). \tag{42}$$

Claim: $t_{\mathbf{d},\varepsilon} + h \leq T^r_{\xi,h}$, for all $\mathbf{d}\in\mathcal{D}$ and for all $\varepsilon \in \left(0,\,\frac{c_1}{2}\delta\left(\frac{|\xi|_{\mathcal{A}}}{2}\right)\right)$.

Proof: If this were not true, then there would exist some $\tilde{\mathbf{d}}$ and some $\tilde{\varepsilon} \in \left(0,\,\frac{c_1}{2}\delta\left(\frac{|\xi|_{\mathcal{A}}}{2}\right)\right)$ such that $t_{\tilde{\mathbf{d}},\tilde{\varepsilon}} + h > T^r_{\xi,h}$, and hence in particular, for $\bar{t} = h$ and $\mathbf{d} = \tilde{\mathbf{d}}$ it holds that

$$t_{\tilde{\mathbf{d}},\tilde{\varepsilon}} + h > T_r\left(\frac{c_1}{2c_2}\delta(\left|\eta_{\tilde{\mathbf{d}}}\right|_{\mathcal{A}})\right),$$

which implies that

$$\left|x(t_{\tilde{\mathbf{d}},\tilde{\varepsilon}},\,\eta_{\tilde{\mathbf{d}}},\,d_{\tilde{\mathbf{d}},\tilde{\varepsilon}})\right|_{\mathcal{A}} = \left|x(t_{\tilde{\mathbf{d}},\tilde{\varepsilon}} + h,\,\xi,\,v)\right|_{\mathcal{A}} < \frac{c_1}{2c_2}\delta(\left|\eta_{\tilde{\mathbf{d}}}\right|_{\mathcal{A}}),$$

where v is the concatenated function defined by

$$v(t) = \begin{cases} \tilde{\mathbf{d}}, & \text{if } 0 \leq t \leq h, \\ d_{\tilde{\mathbf{d}},\tilde{\varepsilon}}(t-h), & \text{if } t > h. \end{cases}$$

Using (38), one has

$$\begin{aligned}\delta\left(\left|\eta_{\tilde{\mathbf{d}}}\right|_{\mathcal{A}}\right) &\leq \frac{1}{c_1}U(\eta_{\tilde{\mathbf{d}}}) \leq \frac{1}{c_1}g(x(t_{\tilde{\mathbf{d}},\tilde{\varepsilon}},\,\eta_{\tilde{\mathbf{d}}},\,d_{\tilde{\mathbf{d}},\tilde{\varepsilon}}))k(t_{\tilde{\mathbf{d}},\tilde{\varepsilon}}) + \frac{\tilde{\varepsilon}}{c_1} \\ &\leq \frac{c_2}{c_1}\left|x(t_{\tilde{\mathbf{d}},\tilde{\varepsilon}},\,\eta_{\tilde{\mathbf{d}}},\,d_{\tilde{\mathbf{d}},\tilde{\varepsilon}})\right|_{\mathcal{A}} + \frac{\tilde{\varepsilon}}{c_1} < \frac{1}{2}\delta\left(\left|\eta_{\tilde{\mathbf{d}}}\right|_{\mathcal{A}}\right) + \frac{\tilde{\varepsilon}}{c_1},\end{aligned}$$

which is a contradiction, since $\tilde{\varepsilon} < \frac{c_1}{2}\delta\left(\frac{|\xi|_{\mathcal{A}}}{2}\right) \leq \frac{c_1\delta\left(\left|\eta_{\tilde{\mathbf{d}}}\right|_{\mathcal{A}}\right)}{2}$. This proves the claim.

From (41), we have for any $\mathbf{d}\in\mathcal{D}$ and for any $\varepsilon > 0$ small enough,

$$U(x(h,\,\xi,\,\mathbf{d})) - U(\xi) \leq -U(\xi)\frac{\left(k(t_{\mathbf{d},\varepsilon} + h) - k(t_{\mathbf{d},\varepsilon})\right)}{c_2} + \varepsilon = -\frac{U(\xi)}{c_2}\,k'(t_{\mathbf{d},\varepsilon} + \theta h)h + \varepsilon,$$

where θ is some number in $(0,\,1)$. Hence, by the assumptions made on the function k, we have

$$U(x(h,\,\xi,\,\mathbf{d})) - U(\xi) \leq -\frac{U(\xi)}{c_2}\tau(t_{\mathbf{d},\varepsilon} + \theta h)h + \varepsilon \leq -\frac{U(\xi)}{c_2}\tau(T^r_{\xi,h})\,h + \varepsilon.$$

Again, since ε can be chosen arbitrarily small, we have

$$U(x(h,\,\xi,\,\mathbf{d})) - U(\xi) \leq -\frac{U(\xi)}{c_2}\tau(T^r_{\xi,h})h, \quad \forall \mathbf{d}\in\mathcal{D}.$$

Thus we showed that for any $\mathbf{d}$ and any $h > 0$ small enough,

$$\frac{U(x(h,\,\xi,\,\mathbf{d})) - U(\xi)}{h} \leq -\frac{U(\xi)}{c_2}\tau(T^r_{\xi,h}).$$

Since U is locally Lipschitz on $\mathbb{R}^n \backslash \mathcal{A}$, it is differentiable almost everywhere in $\mathbb{R}^n \backslash \mathcal{A}$, and hence for any $\mathbf{d} \in \mathcal{D}$ and for any $r > |\xi|_{\mathcal{A}}$,

$$\begin{aligned}
L_{f_{\mathbf{d}}} U(\xi) &= \lim_{h \to 0^+} \frac{U(x(h, \xi, \mathbf{d})) - U(\xi)}{h} \le - \lim_{h \to 0^+} \frac{U(\xi)}{c_2} \tau(T^r_{\xi, h}) \\
&= -\frac{U(\xi)}{c_2} \tau\left(\lim_{h \to 0^+} T^r_{\xi, h}\right) = -\frac{U(\xi)}{c_2} \tau\left(T_r\left(\frac{c_1}{2c_2} \delta(|\xi|_{\mathcal{A}})\right)\right) \\
&\le -\frac{c_1 \delta(|\xi|_{\mathcal{A}})}{c_2} \tau\left(T_r\left(\frac{c_1}{2c_2} \delta(|\xi|_{\mathcal{A}})\right)\right) \qquad (43) \\
&= -\bar{\alpha}_r(|\xi|_{\mathcal{A}}), \text{ a.e. }, \qquad (44)
\end{aligned}$$

where

$$\bar{\alpha}_r(s) = \frac{c_1 \delta(s)}{c_2} \tau\left(T_r\left(\frac{c_1}{2c_2} \delta(s)\right)\right).$$

Now define the function $\bar{\alpha}$ by

$$\bar{\alpha}(s) = \sup_{r > s} \bar{\alpha}_r(s).$$

Note that $\bar{\alpha}_r(0) = 0$ for any $r > 0$, so $\bar{\alpha}(0) = 0$. Also, applying to $r = 2s$, we have

$$\bar{\alpha}(s) \ge \frac{c_1 \delta(s)}{c_2} \tau\left(T_{2s}\left(\frac{c_1}{2c_2} \delta(s)\right)\right) > 0$$

for all $s > 0$. Notice that (44) holds for any $r > |\xi|_{\mathcal{A}}$, so it follows that for every $\mathbf{d} \in \mathcal{D}$, $L_{f_{\mathbf{d}}} U(\xi) \le -\bar{\alpha}(|\xi|_{\mathcal{A}})$ for almost all $\xi \in \mathbb{R}^n \backslash \mathcal{A}$. Now let

$$\breve{\alpha}(s) = \frac{c_1 \delta(s)}{c_2} \int_{2s}^{2s+1} \tau\left(T_r\left(\frac{c_1}{2c_2} \delta(s)\right)\right) dr,$$

for $s > 0$, and let $\breve{\alpha}(0) = 0$. Then $\breve{\alpha}$ is continuous on $[0, \infty)$ (the continuity at $s = 0$ is because τ is bounded and $\delta(0) = 0$), and for $s > 0$, it holds that

$$0 < \breve{\alpha}(s) \le \frac{c_1 \delta(s)}{c_2} \tau\left(T_{2s}\left(\frac{c_1}{2c_2} \delta(s)\right)\right)$$

because of the monotonicity properties of T and τ. Furthermore,

$$L_{f_{\mathbf{d}}} U(\xi) \le -\bar{\alpha}(|\xi|_{\mathcal{A}}) \le -\breve{\alpha}(|\xi|_{\mathcal{A}}),$$

for almost all $\xi \in \mathbb{R}^n \setminus \mathcal{A}$.

By Theorem 4 provided in the appendix, there exists a C^∞ function $V : \mathbb{R}^n \backslash \mathcal{A} \longrightarrow \mathbb{R}_{\ge 0}$ such that for almost all $\xi \in \mathbb{R}^n \backslash \mathcal{A}$,

$$|V(\xi) - U(\xi)| < \frac{U(\xi)}{2} \text{ and } L_{f_{\mathbf{d}}} V(\xi) \le -\frac{1}{2} \breve{\alpha}(|\xi|_{\mathcal{A}}), \ \forall \mathbf{d} \in \mathcal{D}.$$

Extend V to $\mathbb{R}^n$ by letting $V|_{\mathcal{A}} = 0$ and again denote the extension by V. Note that V is continuous on $\mathbb{R}^n$. So V is a Lyapunov function, as desired, with $\alpha_1(s) = \frac{c_1}{2} \delta(s)$, $\alpha_2(s) = \frac{3c_2}{2} s$ and $\alpha_3(s) = \frac{1}{2} \breve{\alpha}(s)$. ∎

7 Proof of the Second Converse Lyapunov Theorem

We need a couple of Lemmas. The first one is trivial, so we omit its proof.

Lemma 7.1 Let $f : \mathbb{R}^n \times \mathcal{D} \longrightarrow \mathbb{R}^n$ be continuous, where $\mathcal{D}$ is a compact subset of $\mathbb{R}^l$. Then there exists a smooth function $a_f : \mathbb{R}^n \longrightarrow \mathbb{R}$, with $a_f(x) \geq 1$ everywhere, such that $|f(x, \mathbf{d})| \leq a_f(x)$ for all x and all $\mathbf{d}$. □

Now for any given system

$$\Sigma : \quad \dot{x} = f(x, \mathbf{d}),$$

not necessarily complete, consider the following system:

$$\Sigma_b : \quad \dot{x} = \frac{1}{a_f(x)} f(x, \mathbf{d}).$$

Note that the system Σ_b is complete since $\dfrac{|f(x, \mathbf{d})|}{a_f(x)} \leq 1$ for all $x, \mathbf{d}$. We let $x_b(\cdot, x_0, d)$ denote the trajectory of Σ_b corresponding to the initial state x_0 and the disturbance d. The following result is a simple consequence of the fact that the trajectories of Σ are the same as those of Σ_b up to a rescaling of time. We provide the details to show clearly that the uniformity conditions are not violated.

Lemma 7.2 Assume that $\mathcal{A}$ is a compact set. Suppose that system Σ is UGAS with respect to $\mathcal{A}$. Then, system Σ_b is UGAS with respect to $\mathcal{A}$.

Proof. Pick a disturbance $d \in \mathcal{M}_{\mathcal{D}}$ and an initial state $x_0 \in \mathbb{R}^n$. Let $\gamma_b(t)$ denote $x_b(t, x_0, d)$. Let $\tau_{\gamma_b}(t)$ denote the solution for $t \geq 0$ of the following initial value problem:

$$\dot{\tau} = a_f(\gamma_b(\tau)), \quad \tau(0) = 0. \tag{45}$$

Since a_f is smooth, and γ_b is Lipschitz, $a_f \circ \gamma_b$ is locally Lipschitz as well. It follows that a unique $\tau_{\gamma_b}(t)$ is at least defined in some interval $[0, \bar{t}\,)$. Note that τ_{γ_b} is strictly increasing, so $\bar{t} < +\infty$ would imply $\lim_{t \to \bar{t}^-} \tau_{\gamma_b}(t) = +\infty$.

Claim: For every trajectory γ_b of Σ_b, $\tau_{\gamma_b}(t)$ is defined for all $t \geq 0$.

Proof: If the claim is not true, then there exist some trajectory γ_b of Σ_b and some $t_1 > 0$ such that $\lim_{t \to t_1^-} \tau_{\gamma_b}(t) = \infty$. Now for $t \in [0, t_1)$, one has:

$$\frac{d}{dt}\gamma_b(\tau_{\gamma_b}(t)) = \frac{1}{a_f(\gamma_b(\tau_{\gamma_b}(t)))} f\Big(\gamma_b(\tau_{\gamma_b}(t)), d(\tau_{\gamma_b}(t))\Big) \frac{d}{dt}\tau_{\gamma_b}(t) = f\Big(\gamma_b(\tau_{\gamma_b}(t)), d(\tau_{\gamma_b}(t))\Big). \tag{46}$$

Thus $\gamma_b(\tau_{\gamma_b}(t))$ is a solution of Σ on $[0, t_1)$. By the stability of Σ, it follows that

$$|\gamma_b(\tau_{\gamma_b}(t))|_{\mathcal{A}} < \delta^{-1}(|x_0|_{\mathcal{A}}), \quad t \in [0, t_1),$$

where $x_0 = \gamma_b(0)$, and δ is the function for Σ as defined in Definition 2.2. (c.f. Remark 2.4.) Let $c = \delta^{-1}(|x_0|_{\mathcal{A}})$, and let $M = \sup_{|\xi|_{\mathcal{A}} \leq c} a_f(\xi)$. ($M$ is finite because the set $\{\xi : |\xi|_{\mathcal{A}} \leq c\}$ is a compact set.) From here one sees that $|\tau_{\gamma_b}(t)| \leq M t_1$ for any $t \in [0, t_1)$. This is a contradiction. Thus $\tau_{\gamma_b}(t)$ is defined for all $t \geq 0$. This proves the claim.

Since $a_f(s) \geq 1$ and, for every trajectory γ_b of Σ_b, $\tau_{\gamma_b}(0) = 0$, it follows that $\tau_{\gamma_b}(\cdot) \in \mathcal{K}_\infty$ for each trajectory γ_b of Σ_b. From (46), one also sees that if $\gamma_b(t)$ is a trajectory of Σ_b, then $\gamma_b(\tau_{\gamma_b}(t))$ is a trajectory of Σ, and furthermore,

$$|\gamma_b(\tau_{\gamma_b}(s))|_{\mathcal{A}} < \varepsilon \quad \forall s \geq 0\,, \text{ if } |\gamma_b(0)|_{\mathcal{A}} \leq \delta(\varepsilon)\,.$$

It follows that

$$|\gamma_b(t)|_{\mathcal{A}} = \left|\gamma_b(\tau_{\gamma_b}(\tau_{\gamma_b}^{-1}(t)))\right|_{\mathcal{A}} < \varepsilon, \quad \forall t \geq 0, \text{ whenever } |\gamma_b(0)|_{\mathcal{A}} \leq \delta(\varepsilon)\,.$$

This shows that condition (1) of Definition 2.2 holds for Σ_b, with the same function δ.

Fix any $r, \varepsilon > 0$. Pick any x_0 with $|x_0|_{\mathcal{A}} < r$ and any $d \in \mathcal{M}_{\mathcal{D}}$. Again let $\gamma_b(t)$ denote the corresponding trajectory of Σ_b. Then

$$|\gamma_b(t)|_{\mathcal{A}} = \left|\gamma_b(\tau_{\gamma_b}(\tau_{\gamma_b}^{-1}(t)))\right|_{\mathcal{A}} < \delta^{-1}(r), \quad \forall t \geq 0\,.$$

Let

$$L = \sup\{a_f(\xi) : \ |\xi|_{\mathcal{A}} \leq \delta^{-1}(r)\}.$$

Then one sees that $|\dot{\tau}(t)| \leq L$, which implies that $\tau_{\gamma_b}(t) \leq Lt$ for all $t \geq 0$. Note that for the given $r, \varepsilon > 0$, by the UGAS property for Σ, there exists $T > 0$ such that for every $d \in \mathcal{M}_{\mathcal{D}}$,

$$|\gamma_b(\tau_{\gamma_b}(s))|_{\mathcal{A}} < \varepsilon$$

whenever $|\gamma_b(0)|_{\mathcal{A}} < r$ and $s \geq T$. This implies that

$$|\gamma_b(t)|_{\mathcal{A}} < \varepsilon$$

whenever $|\gamma_b(0)|_{\mathcal{A}} < r$ and $t \geq \tau_{\gamma_b}(T)$. Combining this with the fact that $\tau_{\gamma_b}(t) \leq Lt$, one proves that for any $d \in \mathcal{M}_{\mathcal{D}}$, it holds that

$$|\gamma_b(t)|_{\mathcal{A}} < \varepsilon$$

whenever $|\gamma_b(0)|_{\mathcal{A}} < r$ and $t \geq LT$. Hence we conclude that Σ_b is UGAS. ∎

In Lemma 7.2, the assumption that $\mathcal{A}$ is compact is crucial. Without this assumption, the conclusion may fail as the following example shows.

Example 7.3 Consider the following system Σ:

$$\dot{x} = (1 + y^2)\tanh x, \quad \dot{y} = y^4\,. \tag{47}$$

(Here f is independent of d.) Let $\mathcal{A} = \{(x, y) : \ x = 0\}$. Clearly the system is UGAS with respect to $\mathcal{A}$. For this system, a natural choice of a_f is $2 + y^4$. Thus, the corresponding Σ_b is as follows:

$$\dot{x} = (\tanh x)\frac{1 + y^2}{2 + y^4}, \quad \dot{y} = \frac{y^4}{2 + y^4}.$$

However, the system Σ_b is not UGAS with respect to $\mathcal{A}$. This can be seen as follows. Assume that Σ_b is UGAS. Then for $\varepsilon = 1/2$, there exists some $T > 0$ such that for any solution $(x(t), y(t))$ of Σ_b with $x(0) = 1$, it holds that

$$|x(t)| < \frac{1}{2}, \quad \forall t \geq T\,. \tag{48}$$

Since $\dfrac{1+y^2}{2+y^4} \to 0$ as $y \to \infty$, it follows that there exists some $y_0 > 0$ such that

$$\left|\frac{1+y^2}{2+y^4}\right| < \frac{1}{3T}, \quad \forall y \geq y_0 \,.$$

Now consider the trajectory $(x(t),\, y(t))$ of Σ_b with $x(0) = 1, y(0) = y_0$, where y_0 is as above. Clearly $y(t) \geq y_0$ for all $t \geq 0$, and thus,

$$\dot{x} = (\tanh x)\frac{1+y^2}{2+y^4} \leq (\tanh x)\,\frac{1}{3T} \,\leq\, \frac{1}{3T}\,,$$

which implies that

$$|x(T)| \,\geq\, 1 - \frac{1}{3T}\,T = \frac{2}{3}\,.$$

This contradicts (48). From here one sees that Σ_b is not UGAS with respect to $\mathcal{A}$. $\square$

We now prove Theorem 2.

The proof of the sufficiency part is the same as in the proof of Theorem 1. Observe that the fact that $V(\xi)$ is nonincreasing along trajectories implies, by compactness of $\mathcal{A}$, that trajectories are bounded, so $x(t)$ is defined for all $t \geq 0$. We now prove necessity.

Let a_f be a function for f as in Lemma 7.1, and let Σ_b be the corresponding system. Then by Lemma 7.2, one knows that the system Σ_b is UGAS. Applying Theorem 1 to the complete system Σ_b, one knows that there exists a smooth Lyapunov function V for Σ_b such that

$$\alpha_1(|\xi|_{\mathcal{A}}) \,\leq V(\xi)\, \leq \alpha_2(|\xi|_{\mathcal{A}})\,, \quad \forall \xi \in \mathbb{R}^n\,,$$

and

$$L_{\tilde{f}_{\mathbf{d}}} V(\xi) \,\leq\, -\alpha_3(|\xi|_{\mathcal{A}})\,, \quad \forall \xi \notin \mathcal{A}\,,\ \forall \mathbf{d} \in \mathcal{D}\,,$$

for some $\mathcal{K}_\infty$-functions α_1, α_2 and some positive definite function α_3, where

$$\tilde{f}_{\mathbf{d}}(\xi) \,=\, \frac{f(\xi,\mathbf{d})}{a_f(\xi)}\,.$$

Since $a_f(\xi) \geq 1$ everywhere, it follows that

$$L_{f_{\mathbf{d}}} V(\xi) \,\leq\, -\alpha_3(|\xi|_{\mathcal{A}}), \quad \forall \xi \notin \mathcal{A},\ \forall \mathbf{d} \in \mathcal{D}\,.$$

Thus, one concludes that V is also a Lyapunov function of Σ.

8 An Example

In general, for a noncompact disturbance value set $\mathcal{D}$, the converse Lyapunov theorem will fail, even if the vector fields $f(\xi,\mathbf{d})$ are locally Lipschitz uniformly on $\mathbf{d}$ on any compact subset of $\mathcal{D}$ (for instance, if f is smooth everywhere). To illustrate this fact, consider the common case of systems affine in controls:

$$\dot{x} \,=\, f(x) + g(x)\mathbf{d},$$

where for simplicity we consider only the unconstrained single-input case, that is, $\mathcal{D} = \mathbb{R}$. Assume that there would exist a Lyapunov function V for this system in the sense of definition 2.6. Then, calculating Lie derivatives, we have that, in particular,

$$L_f V(\xi) + \mathbf{d} L_g V(\xi) < 0, \quad \forall \xi \neq 0, \quad \forall \mathbf{d} \in \mathbb{R},$$

which implies that

$$L_g V(\xi) \; = \; 0, \quad \forall \xi \neq 0.$$

Thus V must be constant along all the trajectories of the differential equation

$$\dot{x} = g(x).$$

In general, such a property will contradict the properness or the positive definiteness of V, unless the vector field g is very special. As a way to construct counterexamples, consider the following property of a vector field g, which is motivated by the prolongation ideas in [26].

Consider the closure $W(\xi_0)$ of the trajectory through ξ_0 with respect to the vector field g. Note that if $\xi_1 \in W(\xi_0)$, then the fact that V is constant on trajectories, coupled with continuity of V, implies that $V(\xi_1) = V(\xi_0)$. Now assume that there is a chain $\xi_0, \xi_1, \xi_2, \ldots$ so that for each $i = 1, 2, \ldots, \xi_i \in W(\xi_{i-1})$. Then we conclude that $V(\xi_i) = V(\xi_0)$ for all i. If the sequence $\{\xi_i\}$ converges to zero (and $\xi_0 \neq 0$) or diverges to infinity, we contradict positive definiteness or properness of V respectively. For an example, take the following two dimensional system, which was used in [8] to show essentially the same fact.

Let $\mathfrak{S}$ be the spiral that describes the solution of the differential equation

$$\dot{x} = -x - y, \quad \dot{y} = x - y,$$

passing through the point $(1, 0)$. Explicitly, $\mathfrak{S}$ can be parameterized as $x = e^{-t}\cos t, \; y = e^{-t}\sin t, \; -\infty < t < \infty$. In polar coordinates, the spiral is given by $r = e^{-\theta}, \; -\infty < \theta < \infty$. Let $a(x, y)$ be any nonnegative smooth function which is zero exactly on the closure of the spiral $\mathfrak{S}$ (that is, $\mathfrak{S}$ plus the origin). (Such a function always exists since any closed subset of Euclidean space can be described as the zero set of a smooth function; see for instance [6]. Now consider the system

$$\begin{aligned} \dot{x} &= -x - y + x a(x, y)\mathbf{d}, \\ \dot{y} &= x - y + y a(x, y)\mathbf{d}. \end{aligned} \tag{49}$$

Note that the system is smooth everywhere. Let $\mathcal{D} = \mathbb{R}$, and let $\mathcal{A}$ be the origin. In polar coordinates, the system (49) on $\mathbb{R}^2 \backslash \{0\}$ satisfies the equations

$$\dot{r} = -r + r a(r\cos\theta, \, r\sin\theta)\mathbf{d}, \quad \dot{\theta} = 1. \tag{50}$$

(This can be seen as a system on $\mathbb{R}_{>0} \times S^1$.) In polar coordinates, then, the trajectory passing through $(r, \theta) = (1, 0)$ is precisely the spiral $r = e^{-\theta}$, for any $d \in \mathcal{M}_{\mathcal{D}}$. Pick any trajectory $(r(t), \theta(t))$ with $(r(0), \theta(0)) = (r_0, \theta_0)$, where $\theta_0 \in [0, 2\pi)$. Then there exists some integer $k \geq 0$ such that $r_0 < e^{-\theta_0 + 2k\pi}$.

Claim: It holds that

$$r(t) \; \leq \; e^{-\theta_0 + 2k\pi - t} \; \leq \; e^{2k\pi - t}, \quad \forall t \geq 0\,. \tag{51}$$

Assume that (51) is not true. Then there exists some $t_1 > 0$ such that

$$r(t_1) = e^{-\theta_0 + 2k\pi - t_1}.$$

Note that we also have $\theta(t_1) = \theta_0 + t_1$. Now let $(\bar{r}(t), \bar{\theta}(t)) = (e^{-\theta_0 + 2k\pi - t}, \theta_0 - 2k\pi + t)$. Then $(\bar{r}(t), \bar{\theta}(t))$ is a trajectory of the system, and furthermore, $(\bar{r}(0), \bar{\theta}(0))$ and $(r(0), \theta_0)$ are different points since $\bar{r}(0) \neq r(0)$. However, the points $(r(t_1), \theta(t_1))$ and $(\bar{r}(t_1), \bar{\theta}(t_1))$ are the same point on the xy-plane. This violates the uniqueness of solutions. Therefore, (51) holds for $t \geq 0$.

Note that in the above discussion, one can always choose $k = r_0 + 1$. It then follows from (51) that for any trajectory of the system with $r(0) = r_0$, it holds that

$$r(t) \leq e^{2(r_0+1)\pi - t}, \quad \forall t \geq 0, \quad \forall d. \tag{52}$$

Thus we conclude that the system is UGAS.

However, this system fails to admit a Lyapunov function. In this example, the vector field g is $(xa(x, y), ya(x, y))$. Consider the sequence of points in the xy-plane $\{\xi_k\}$ with $\xi_k = (e^{2k\pi}, 0)$ for $k \geq 0$. Note that for each $k \geq 1$,

$$\xi_k \in W(\xi_{k-1}^j),$$

where $\xi_k^j = \left(e^{2k\pi} + \dfrac{1}{j}, 0\right)$. Therefore, $V(\xi_k) = V(\xi_{k-1}^j)$ for any j and any k. This implies that

$$V(\xi_k) = V(\xi_0), \quad \forall k \geq 1,$$

contradicting the properness of V. This shows that it is impossible for the system to have a Lyapunov function.

It is worthwhile to note that by the same argument, one sees that not only there is no smooth Lyapunov function for the system, but also there is not even a Lyapunov function which is merely continuous (in the sense that V is not even smooth away from $\mathcal{A}$, and the Lie derivative condition is replaced by a condition asking that V should decrease along trajectories).

In [16], a simple example is given illustrating that uniform global asymptotic stability with respect merely to *constant* disturbances is also not sufficient to guarantee the existence of Lyapunov functions.

9 Relations to Other Work

The study of smooth Converse Lyapunov Theorems has a long history. In the special case of stability with respect to equilibria, and for systems without disturbances, the first complete work was that done in the early 1950s by Massera and Kurzweil; see for instance the papers [17] and [12]. (Although more general because we deal with set stability and disturbances, there is one important aspect in which our results are weaker than some of this classical work, especially that of Kurzweil: we assume enough regularity on the original system, so that there are unique solutions and there is continuous dependence. We do so because lack of regularity is not an issue in the main applications in which we are interested. Of course, the proofs become much simpler under regularity assumptions.) In the late 1960s, Wilson, in [30], extended the Massera and Kurzweil results to a converse Lyapunov function theorem for local asymptotic stability with respect to closed sets. As explained earlier, our proof is modeled along the lines of that paper. See also the textbooks [31] and [11] for many of these classical results.

Nondifferentiable Lyapunov functions have been studied in many papers and textbooks. Among these we may mention the classic book [3] by Bhatia and Szegö, as well as Zubov's work (see for instance [32]) which study in detail continuous Lyapunov function characterizations for global asymptotic stability with respect to arbitrary closed invariant sets. Also, in [28] and [26], and related work, the authors obtained the existence of continuous Lyapunov functions for systems which are stable, uniformly on disturbances and with respect to compact sets, assuming various additional conditions involving prolongations of dynamical systems. (The next section provides some more details on the prolongation approach.) Many results on converse Lyapunov functions with respect to sets can also be found in the many books and articles by Lakshmikantham and several coauthors. For instance, in [13], Theorem 3.4.1, a Massera-type proof is provided of a general converse theorem on local asymptotic stability with ¡respect to two $\mathcal{K}$-functions, that provides a Lipschitz Lyapunov function. As the authors point out, their theorem immediately provides a set-stability result (when using distance to the set as one of the comparison functions). It is also possible to reformulate stability for systems with disturbances in terms of differential inclusions, as explained earlier; see e.g. [1] and [2]. The first of these books employs Lyapunov functions in sufficiency characterizations of viability properties (not the same as stability with respect to all solutions), while the second one (see Chapter 6, and especially Section 4) shows various converse theorems that result in nondifferentiable Lyapunov functions, connecting their existence with the solution of optimal control problems.

The questions addressed in this paper are related to studies of "total stability," which typically ask about the preservation of stability when considering a new system $\dot{x} = f(x) + R(x,t)$, where $R(x,t)$ is a perturbation. (Sometimes the original system may be allowed to be time-varying, that is, it has equations $\dot{x} = f(x,t)$; in that case, its stability can in turn be interpreted in terms of stability of the set $\{x = 0\}$ for the extended system $\dot{x} = f(x,z)$, $\dot{z} = 1$.) In [14], Lefschetz discussed stability with respect to equilibria under perturbations (referred by the author as quasi-stability). In [11] and [31], one can find such studies, and relationships to the special case of $\dot{x} = f(x) + d(t)$, with results proved regarding stability under integrable disturbances (not arbitrary bounded ones). Under suitable technical conditions, systems with perturbations can also be treated as general dynamical systems in the sense studied in [20] and [32], where stability with respect to closed sets was also investigated.

Finally, let us mention the recent work [18], which proved results analogous to those in this paper for the special case of linear differential inclusions, resulting in homogeneous "quasi-quadratic" Lyapunov functions.

10 Relations to Stability of Prolongations

In [8, 9, 26, 28, 29], the authors considered various notions of stability for systems of the type (1) (with $\mathcal{D}$ not necessarily compact). These properties are defined in terms of the "prolongations" of the original system. The above papers investigated the relationships between such stability notions and the existence of continuous, not necessarily smooth, Lyapunov functions. In this section, we briefly discuss relations between UGAS stability and the notions considered in those papers, with the purpose of clarifying relations to this related previous work. For the more details on the definitions and elementary properties of prolongation maps and the corresponding stability concepts, we refer the reader to the papers mentioned above.

We start with some abstract definitions. Let $F : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to 2^{\mathbb{R}^n}$, $(\xi, t) \mapsto F(\xi, t) \subseteq \mathbb{R}^n$ be any map from $\mathbb{R}^n \times \mathbb{R}_{\geq 0}$ to the set of subsets of $\mathbb{R}^n$. Associated to F, one defines $\mathfrak{D}F$ and $\mathfrak{J}F$ by

$$\begin{aligned}
\mathfrak{D}F(\xi, t) &= \Big\{\eta \in \mathbb{R}^n : \ \text{there exist sequences } \ \xi_n, \, \eta_n \in \mathbb{R}^n, \ \text{ and } \ t_n \geq 0 \\
&\qquad \text{with } \ \xi_n \to \xi, \, \eta_n \to \eta, \ t_n \to t, \ \eta \in F(\xi_n, t_n)\Big\}, \\
\mathfrak{J}F(\xi, t) &= \Big\{\eta \in \mathbb{R}^n : \ \text{there exist } \ t_1, t_2, \ldots, t_k \geq 0 \ \text{ with} \\
&\qquad \sum_{i=1}^{k} t_i = t, \ \text{ such that } \ \eta \in F\Big(F(\cdots F(F(\xi_n, t_1), t_2)\cdots, t_{k-1}), t_k\Big)\Big\},
\end{aligned}$$

where $F(S, t) \stackrel{\text{def}}{=} \bigcup_{\xi \in S} F(\xi, t)$ for any subset S of $\mathbb{R}^n$.

The map F is called *cluster* if $\mathfrak{D}F = F$, and F is called *transitive* if $\mathfrak{J}F = F$.

For any system (1), consider the reachable set $\mathcal{R}^t(\xi)$ defined in section 5, seen now as a set-valued map. The prolongation map Γ associated with (1) is then defined by letting $\Gamma(\xi, t)$ be the smallest set containing $\mathcal{R}^t(\xi)$ such that Γ is both transitive and cluster. For further discussion regarding the definition of the map Γ, we refer the reader to [26] and the other papers mentioned above.

For subsets A and B of $\mathbb{R}^n$, we denote the usual distance between the two sets by $d(A, B) = \inf \{d(\xi, \eta) : \ \xi \in A, \ \eta \in B\}$. We say that a system (1) is T-stable (we use here the "T" for the name of the author of [26]) with respect to a closed, invariant set $\mathcal{A}$ if the following two properties hold:

- There exists a $\mathcal{K}_\infty$-function $\delta(\cdot)$ such that for any $\varepsilon > 0$,

$$d(\Gamma(\xi, t), \mathcal{A}) < \varepsilon, \qquad \text{whenever} \ \ |\xi|_{\mathcal{A}} \leq \delta(\varepsilon), \ \text{ and } \ t \geq 0\,;$$

- For any $r, \varepsilon > 0$, there is a $T > 0$ such that

$$d\,(\Gamma(\xi, t), \mathcal{A}) < \varepsilon, \qquad \text{whenever} \ \ |\xi|_{\mathcal{A}} < r, \ \text{ and } t \geq T.$$

Note that this is the same as what is called "global absolute asymptotic stability" (global A.A.S) in [26] for the special case when $\mathcal{A}$ is compact. Clearly, if a system is T-stable, then it is UGAS. It was shown in [26], under some extra technical assumptions, but without the compactness of $\mathcal{D}$, that global A.A.S implies the existence of a continuous, not necessarily smooth, Lyapunov function. (meaning that V is globally merely continuous; the condition $L_{f_{\mathbf{d}}} V(\xi) \leq -\alpha_3(|\xi|_{\mathcal{A}})$ is replaced by a condition that V should decrease along trajectories).

We will show next that, at least when $\mathcal{D}$ is compact, UGAS implies (and is therefore equivalent to) T-stability. So in what follows in this section, we assume that $\mathcal{D}$ is compact, and also that all systems involved are forward complete. We first need the following fact.

Lemma 10.1 For system (1), $\Gamma(\xi, t) = \overline{\mathcal{R}^t(\xi)}$ for any $\xi \in \mathbb{R}^n$ and any $t \geq 0$.

Proof. First note that the cluster property of Γ implies that $\Gamma(\xi, t)$ is closed for each $\xi \in \mathbb{R}^n$ and each $t \geq 0$. Thus it is enough to show that the map $\mathfrak{R} : \ (\xi, t) \ \mapsto \ \overline{\mathcal{R}^t(\xi)}$ is cluster and transitive.

Take $\xi_0 \in \mathbb{R}^n$ and $\tau > 0$. (The case when $t = 0$ is trivial.) Pick $\eta_0 \in \mathfrak{D}\mathfrak{R}(\xi_0, \tau)$. Then, by definition, there exist sequences $\{\xi_n\}$, $\{\eta_n\}$ and $\{t_n\}$ with $t_n \geq 0$ such that $\xi_n \to \xi_0$, $\eta_n \to \eta_0$, $t_n \to \tau$ and $\eta_n \in \overline{\mathcal{R}^{t_n}(\xi_n)}$.

Note then that for each n, there exists d_n such that

$$|\eta_n - x(t_n, \xi_n, d_n)| < \frac{1}{n}.$$

Let $\zeta_n = x(t_n, \xi_n, d_n)$. Then $\zeta_n \in \mathcal{R}^{t_n}(\xi_n)$ and $\zeta_n \to \eta_0$. Let K_0 be a compact set such that $\xi_n \in K_0$ for each n, and let $T > 0$ be such that $t_n \leq T$ for any n. Then by Proposition 5.1, there exists a compact set K_1 such that $\mathcal{R}(K_0, T) \subseteq K_1$. Let L be a Lipschitz constant for f with respect to states in K_1. Then it follows from Gronwall's Lemma that, for n large enough so that $|\xi_n - \xi_0| < e^{-LT}$, it holds that

$$|x(t, \xi_0, d_n) - x(t, \xi_n, d_n)| \leq |\xi_0 - \xi_n| e^{LT},$$

for any $0 \leq t \leq T$. Let $\kappa_n = x(\tau, \xi_0, d_n)$. Then

$$\begin{aligned} |\kappa_n - \zeta_n| &= |x(\tau, \xi_0, d_n) - x(t_n, \xi_n, d_n)| \\ &\leq |x(\tau, \xi_0, d_n) - x(\tau, \xi_n, d_n)| + |x(\tau, \xi_n, d_n) - x(t_n, \xi_n, d_n)| \\ &\leq |\xi_0 - \xi_n| e^{\tau L} + M|\tau - t_n| \end{aligned}$$

where $M = \max\{|f(\xi, \mathbf{d})|,\ d(\xi, K_1) \leq 1,\ \mathbf{d} \in \mathcal{D}\}$. It then follows that $\kappa_n \in \mathcal{R}^\tau(\xi_0)$ for each n and $\kappa_n \to \eta_0$. Thus, we conclude that $\eta_0 \in \overline{\mathcal{R}^\tau(\xi_0)}$. Hence we showed that $\mathfrak{D}\overline{\mathcal{R}^\tau(\xi_0)} = \overline{\mathcal{R}^\tau(\xi_0)}$ for any $\tau > 0$ and any $\xi_0 \in \mathbb{R}^n$, that is, the map $\mathfrak{R}$ is cluster.

To show the transitivity of $\mathfrak{R}$, first note that, by induction, it is enough to show that

$$\mathfrak{R}(\mathfrak{R}(\xi, t_1), t_2) \subseteq \mathfrak{R}(\xi, t_1 + t_2) \tag{53}$$

for any $\xi \in \mathbb{R}^n$ and any $t_1, t_2 \geq 0$.

Applying Lemma 5.3 to $S = R^{t_1}(\xi)$ together with the fact that

$$\mathcal{R}^{t_2}\left(\mathcal{R}^{t_1}(\xi)\right) = \mathcal{R}^{t_1+t_2}(\xi),$$

one immediately gets (53). ∎

Rewritting the definition of UGAS in terms of reachable sets, one has that a system (1) is UGAS if and only if the following properties hold:

- There exists a $\mathcal{K}_\infty$-function $\delta(\cdot)$ such that for any $\varepsilon > 0$,

$$d\left(\mathcal{R}^t(\xi), \mathcal{A}\right) < \varepsilon, \quad \text{whenever} \quad |\xi|_{\mathcal{A}} \leq \delta(\varepsilon), \quad \text{and} \quad t \geq 0;$$

- For any $r, \varepsilon > 0$, there is a $T > 0$ such that

$$d\left(\mathcal{R}^t(\xi), \mathcal{A}\right) < \varepsilon, \quad \text{whenever} \quad |\xi|_{\mathcal{A}} < r, \quad \text{and } t \geq T.$$

The following conclusion then follows immediately from the continuity of the function $\xi \mapsto d(\xi, \mathcal{A})$ and Lemma 10.1:

Proposition 10.2 For compact $\mathcal{D}$, a system (1) is UGAS with respect to $\mathcal{A}$ if and only if it is T-stable. ∎

Remark 10.3 In the special case when $\mathcal{A}$ is compact, a UGAS system is always forward complete. Thus in that case Proposition 10.2 is still true without completeness. □

Remark 10.4 The compactness condition on $\mathcal{D}$ is essential. Without the compactness of $\mathcal{D}$, Proposition 10.2 is in general not true. For instance, the system defined by (50) in section 8 is UGAS with respect to the origin $(0, 0)$. However the system is not T-stable, since $\Gamma(0, t) = \mathbb{R}^2$ for any $t > 0$. Note that for this example, $\overline{R^t(0, t)} = \{0\}$ for any $t > 0$ which is different from $\Gamma(0, t)$. The inconsistency with the conclusion of Lemma 10.1 is caused by the noncompactness of $\mathcal{D}$. □

11 Input to State Stability

In this section we introduce the property of input/state stability.

Consider the following general nonlinear system:

$$\dot{x} = f(x, u)\,. \tag{54}$$

Here $f : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is continuously differentiable and satisfies $f(0, 0) = 0$.

Controls or *inputs* are measurable locally essentially bounded functions $u : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}^m$. The set of all such functions, endowed with the (essential) supremum norm $\|u\| = \sup\{|u(t)|, t \geq 0\} \leq \infty$, is denoted by L_∞^m. (Everywhere, $|\cdot|$ denotes the usual Euclidian norm.) For each $\xi \in \mathbb{R}^n$ and each $u \in L_\infty^m$, we denote by $x(t, \xi, u)$ the trajectory of the system (54) with initial state $x(0) = \xi$ and the input u. This is defined on some maximal interval $[0, T_{\xi,u})$, with $T_{\xi,u} \leq +\infty$.

Definition 11.1 The system (54) is *(globally) input/state stable (ISS)* if there exist a $\mathcal{KL}$-function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}$, and a $\mathcal{K}$-function γ, such that, for each input $u \in L_\infty^m$ and each $\xi \in \mathbb{R}^n$, it holds that

$$|x(t, \xi, u)| \;\leq\; \beta(|\xi|, t) + \gamma(\|u\|) \tag{55}$$

for each $t \geq 0$. □

Note that, by causality, the same definition would result if one would replace $\|u\|$ by $\|u_t\|$ in (55), where u_t is the truncation of u at t; i.e., $u_t(s) = u(s)$ if $s < t$, and $u(s) = 0$ if $s \geq t$.

The definition is intended as a nonlinear generalization of the bound $|x(t)| \leq |\xi|\, e^{-\alpha t} + c\|u\|$ which holds for linear systems $\dot{x} = Ax + Bu$ when the matrix A is asymptotically stable. Using this definition, it was proved in [21] and [23] that a system can be stabilized by a smooth feedback if and only if it is feedback equivalent to an ISS system. Thus the definition appears to be natural, and the further characterizations given later confirm this fact.

Definition 11.2 A smooth function $V : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ is called an *ISS-Lyapunov function* for system (54) if there exist $\mathcal{K}_\infty$-functions α_1, α_2, and $\mathcal{K}$-functions α_3 and χ, such that

$$\alpha_1(|\xi|) \;\leq\; V(\xi) \;\leq\; \alpha_2(|\xi|) \tag{56}$$

for any $\xi \in \mathbb{R}^n$ and

$$\nabla V(\xi) \cdot f(\xi, \mu) \leq -\alpha_3(|\xi|) \tag{57}$$

for any $\xi \in \mathbb{R}^n$ and any $\mu \in \mathbb{R}^m$ so that $|\xi| \;\geq\; \chi(|\mu|)$. □

Observe that if V is an ISS-Lyapunov function for (54), then V is a Lyapunov function, in the usual sense, for the autonomous system $\dot{x} = f(x, 0)$ obtained when no controls are applied.

Note that the first inequality in Equation (56) states that V is positive definite (because $\alpha_1(r)$ is nonzero for $r \neq 0$) and proper, that is, "radially unbounded" (because $\alpha_1(r)$ increases to infinity). The second inequality in Equation (56) is redundant, since the existence of a function α_2 is an immediate consequence of continuity of V, but it is useful to have the function α_2 explicitely for proofs, and when dealing with more general stability considerations (such as stability with respect to invariant sets, or time-varying systems). Finally, observe that Equation (57) states that the derivative $\dot{V}$ along trajectories is negative definite for large enough x, given any control magnitude.

In [21] as well as in the literature since the publication of that paper, the fact that a system is ISS has always been established by showing that there is an ISS-Lyapunov function. The main result of this paper will be that the converse holds as well:

The system (54) is ISS if and only if it admits an ISS-Lyapunov function.

There turns out to be also an interesting connection between ISS stability and robust stability.

By a *feedback law* for a system (54) we will mean any (at least measurable) function $k : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ for which the differential equation corresponding to using k as a feedback,

$$\dot{x} = f(x, k(t, x)), \tag{58}$$

is well-posed; that is, for each initial state $x(0)$ there is an absolutely continuous solution, defined at least for small times, and any two such solutions coincide on their interval of existence. (For instance this will happen if $k(t, \xi)$ is continuous in ξ and measurable locally essentially bounded in t, and is locally Lipschitz in ξ uniformly with respect to t on finite intervals.)

Let ρ be any $\mathcal{K}_\infty$ function. A feedback law will be said to *bounded by* ρ if for each $\xi \in \mathbb{R}^n$

$$|k(t, \xi)| \leq \rho(|\xi|)$$

holds for almost all (recall that k is assumed to be only measurable) $t \in \mathbb{R}_{\geq 0}$.

We will say that the system (54) is *robustly stable* if there exist a $\mathcal{K}_\infty$ function ρ (called a *stability margin*) and a $\mathcal{KL}$ function β such that, for every feedback law bounded by ρ it holds that

$$|x(t)| \leq \beta(|x(0)|, t) \quad \forall t \geq 0, \tag{59}$$

for every solution of the corresponding system (58).

Remark 11.3 Note that this is not a notion of local stability with respect to "small" perturbations. It is a global notion, and perturbations can be arbitrarily large (since the function ρ is in class $\mathcal{K}_\infty$). In some sense, this is analogous to exponential stability for linear systems, where a perturbation of the spectrum preserves global asymptotic stability. □

The main result in this context will be:

The system (54) is ISS if and only if it is robustly stable.

After introducing several other notions, we will prove a Theorem that will imply the above two claims as well as the equivalence with several other control-systems properties.

Remark 11.4 Besides its characterization in terms of equivalences, one may ask about the intuitive meaning of the ISS definition. Definition 11.1 implies in particular the following two standard properties:

- with $u \equiv 0$, system
$$\dot{x} = f(x,\, u)$$
is GAS (globally asymptotically stable);
- with bounded u, the trajectory stays within a bounded distance — namely, $\beta(|\xi|,\, 0) + \gamma(\|u\|)$ — from the origin ("BIBS" or bounded-input bounded-state stability).

One may ask if these two properties together imply that the system is ISS. The answer is negative. Indeed, consider the following system on $\mathbb{R}^2$:

$$\dot{x} \;=\; x\left[\left(\sin\frac{\pi y}{2}\right)^2 - 1\right], \quad \dot{y} \;=\; -y + u\,,$$

where the control u takes values in $\mathbb{R}$. It can be shown that the system is GAS when $u \equiv 0$ by taking the Lyapunov function $V \stackrel{\text{def}}{=} (x^2 + y^2)/2$. The system is also BIBS: For any input u, it can be easily shown that $|y(t)| \le e^{-t}\,|y_0| + \|u\|$ and $|x(t)| \le |x_0|$.

On the other hand, this system is not ISS: Otherwise, fix the control $u \equiv 1$. The ISS property would imply that for any $x_0 \in \mathbb{R}$, there exists a time T_{x_0}, such that, the trajectory $(x(t),\, y(t))$ with the initial value $(x(0),\, y(0)) = (x_0,\, 1)$ satisfies:

$$|x(t)| + |y(t)| < 1 + \gamma(1)\,, \;\; \text{if } t \ge T_{x_0}\,, \tag{60}$$

for some $\mathcal{K}$-function γ. Consider the solution with the initial value $(2 + \gamma(1),\, 1)$. Then since

$$y(t) \equiv 1\,, \;\; \text{and} \;\; x(t) \equiv 2 + \gamma(1)\,,$$

these functions cannot possibly satisfy (60) for any t. The contradiction implies that the system is not ISS. □

11.1 Restatements of the Definition of ISS

In this section, we introduce various notions and establish their equivalence with input/state stability.

Remark 11.5 An equivalent form of decay estimation for ISS systems can be given as follows: the system (54) is ISS if and only if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that

$$|x(t,\, \xi,\, u)| \;\le\; \max\left\{\beta(|\xi|\,,\, t),\, \gamma(\|u\|)\right\} \quad \forall \xi \in \mathbb{R}^n,\; \forall u,\; \forall t \ge 0\,. \tag{61}$$

Proof. Clearly, if (61) holds for (54), then (54) is ISS.

Assume now that (54) is ISS. Then there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that (55) holds. Let $\tilde{\beta} = 2\beta,\; \tilde{\gamma} = 2\tilde{\gamma}$. Then

$$|x(t,\, \xi,\, u)| \;\le\; \max\left\{\tilde{\beta}(|\xi|\,,\, t),\, \tilde{\gamma}(\|u\|)\right\}\,,$$

for all $\xi \in \mathbb{R}^n$, all inputs u, and all $t \ge 0$. ■

The next lemma relates ISS to the "input/output stability" property introduced in [21]:

Lemma 11.6 A system (54) is ISS if and only if there exist $\mathcal{KL}$ functions β_0, β_1 and a $\mathcal{K}$-function γ such that, for any $\xi \in \mathbb{R}^n$ and any input u, it holds that

$$|x(t,\,\xi,\,u)| \le \beta_0(|\xi|\,,\,t) + \beta_1(\|u_T\|,\,t-T) + \gamma(\|u^T\|) \tag{62}$$

for any $0 \le T \le t$, where u^T is defined by

$$u^T(\tau) = \begin{cases} 0 & \text{if } 0 \le \tau \le T; \\ u(\tau) & \text{if } \tau > T. \end{cases}$$

Proof. Clearly, (62) implies (55). We now assume that there exist a $\mathcal{KL}$-function β and a $\mathcal{K}$ function γ such that (55) holds. Then one has the following:

$$\begin{aligned} |x(t,\,\xi,\,u)| &\le \beta\Big(\,|x(T,\,\xi,\,u)|\,,t-T\Big) \,+\, \gamma(\|u^T\|) \\ &\le \beta\Big(\beta(\,|\xi|\,,T) \,+\, \gamma(\|u_T\|),\; t-T\Big) + \gamma(\|u^T\|). \end{aligned}$$

for any $\xi \in \mathbb{R}^n$, any u and any $0 \le T \le t$. Since $\beta(r+s,\tau) \le \beta(2r,\,\tau) + \beta(2s,\,\tau)$ holds for any $r, s, \tau \ge 0$, it follows that

$$|x(t,\,\xi,\,u)| \,\le \beta\Big(2\beta(\,|\xi|\,,T),\,t-T\Big) \,+\, \beta\Big(2\gamma(\|u_T\|),\,t-T\Big) \,+\, \gamma(\|u^T\|)\,. \tag{63}$$

Now define β_0 by letting

$$\beta_0(s,t) = \max\left\{\beta(2\beta(s,\tau),\,t-\tau):\; 0 \le \tau \le t\right\}\,.$$

Then $\beta_0(0,t) = 0$ for any $t \ge 0$; and for each fixed t, $\beta_0(s,t)$ is strictly increasing in s and continuous. Thus, $\beta_0(\cdot,t) \in \mathcal{K}$ for each $t \ge 0$. It is not hard to see that for each fixed $s \ge 0$, $\beta_0(s,t)$ is decreasing in t. To show that $\beta_0(s,t) \to 0$ as $t \to \infty$ for each fixed s, just notice that

$$\beta(2\beta(s,T),\,t-T) \,\le\, \max\Big\{\beta(2\beta(s,\,t/2),\,0),\; \beta(2\beta(s,\,0),\,t/2)\Big\}$$

holds for any $s \ge 0$, $0 \le T \le t$, and both of the functions in the parentheses are $\mathcal{KL}$-functions.

Let β_1 be defined by $\beta_1(s,\,t) = \beta(2\gamma(s),\,t)$. Then $\beta_1 \in \mathcal{KL}$, and it follows from (63) that

$$|x(t,\,\xi,\,u)| \,\le \beta_0(\,|\xi|\,,t) \,+\, \beta_1(\|u_T\|,\,t-T) \,+\, \gamma(\|u^T\|)\,,$$

holds for any $\xi \in \mathbb{R}^n$, any u, and any $0 \le T \le t$. ∎

The ISS property can also be described without using class $\mathcal{KL}$ functions, in a manner analogous to the standard definition of global asymptotic stability for systems with no controls:

Lemma 11.7 The system (54) is ISS if and only if the following two properties hold:

1. For each $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|x(t,\,\xi,\,u)| \,\le\, \varepsilon \;\; \forall t \ge 0, \tag{64}$$

   for all inputs u and initial states ξ with $|\xi| \le \delta$ and $\|u\| \le \delta$.

2. There exists a $\mathcal{K}$-function γ such that, for any $r, \varepsilon > 0$, there is a $T > 0$ so that for every input u:
$$|x(t,\,\xi,\,u)| \,\le\, \varepsilon + \gamma(\|u\|) \tag{65}$$
whenever $|\xi| \le r$ and $t \ge T$.

This will be proved below.

Remark 11.8 In Lemma 11.7, Property 1 can be replaced by:

1$'$. There exists a $\mathcal{K}_\infty$-function $\delta(\cdot)$ and a $\mathcal{K}$-function $\tilde{\gamma}$ such that
$$|x(t,\,\xi,\,u)| \,\le\, \varepsilon + \tilde{\gamma}(\|u\|) \quad \forall t \ge 0, \forall u, \text{ whenever } |\xi| \le \delta(\varepsilon) \text{ and } t \ge 0\,. \tag{66}$$

Clearly Property 1$'$ implies Property 1. Next we show that Property 1 implies Property 1$'$. Pick $r, s > 0$, and consider the trajectories $x(t,\xi,u)$ with $|\xi| \le s$ and $\|u\| \le r$. Applying Property 2 with $\varepsilon = 1$, one knows that there exists a $T > 0$ such that
$$|x(t,\,\xi,\,u)| \le 1 + \gamma(r) \tag{67}$$
for all $t \ge T$. Note that Property 2 also implies that the system is forward complete, that is, $T_{\xi,u} = \infty$ for all $\xi \in \mathbb{R}^n$ and all u. By Proposition 5.1, there is an $L > 0$ such that $|x(t,\,\xi,\,u)| \le L$ for all $0 \le t \le T$, all $|\xi| \le s$ and all $\|u\| \le r$. Combining this with (67), one concludes that
$$|x(t,\,\xi,\,u)| \le C, \quad \forall t \ge 0,\ |\xi| \le s,\ \|u\| \le r, \tag{68}$$
where $C = 1 + L + \gamma(r)$. Now let
$$\varphi(r) = \inf\,\{C \ge 0 :\ |x(t,\,\xi,\,u)| \le C,\ \forall t \ge 0,\ \forall\,|\xi| \le r,\ \forall \|u\| \le r\}\,.$$
Note that $\varphi(r) < \infty$ for each $r \ge 0$ because of (68). Clearly, φ is nondecreasing, and
$$|x(t,\,\xi,\,u)| \le \varphi(|\xi|) + \varphi(\|u\|)$$
for all ξ, all u, and all $t \ge 0$. Also, it follows from Property 1 that $\varphi(r) \to 0$ as $r \to 0$. Let
$$\tilde{\varphi}(r) = \frac{1}{r}\int_r^{2r} \varphi(s)\,ds, \quad \text{for } r \ge 0,$$
and let $\tilde{\varphi}(0) = 0$. Then $\tilde{\varphi}$ is continuous, and $\varphi(r) \le \tilde{\varphi}(r)$. Now we define
$$\bar{\varphi}(r) = r + \max_{0 \le s \le r} \tilde{\varphi}(s).$$
Then $\bar{\varphi}$ is of $\mathcal{K}_\infty$ class, and
$$|x(t,\,\xi,\,u)| \le \bar{\varphi}(|\xi|) + \bar{\varphi}(\|u\|)$$
holds for all $t \ge 0$, all $\xi \in \mathbb{R}^n$ and all inputs u. Let $\delta(r) = \bar{\varphi}^{-1}(r)$; then $\delta(\cdot)$ is a $\mathcal{K}_\infty$-function. Clearly, with such a choice of δ and $\tilde{\gamma} = \bar{\varphi}$, it holds that
$$|x(t,\,\xi,\,u)| \le \varepsilon + \tilde{\gamma}(\|u\|)\,,$$
for all $|\xi| \le \delta(\varepsilon)$, all u, and all $t \ge 0$. $\square$

To prove Lemma 11.7, we also need the following conclusion:

Lemma 11.9 Property 2 in Lemma 11.7 is equivalent to the following: There exist a $\mathcal{K}$ function γ and a family of mappings $\{T_r\}_{r>0}$ with the properties

- for each fixed $r > 0$, $T_r : \mathbb{R}_{>0} \xrightarrow{\text{onto}} \mathbb{R}_{>0}$ is continuous and is strictly decreasing, and in particular, $\lim_{s\to+\infty} T_r(s) = 0$;
- for each fixed $\varepsilon > 0$, $T_r(\varepsilon)$ is (strictly) increasing as r increases and $\lim_{r\to\infty} T_r(\varepsilon) = \infty$;

such that, for each input u,

$$|x(t,\,\xi,\,d)| \;\le\; \varepsilon + \gamma(\|u\|) \;\text{ whenever }\; |\xi| \le r \text{ and } t \ge T_r(\varepsilon)\,. \tag{69}$$

Proof. Sufficiency is clear. Now we show the necessity part. This is very similar to the proof of Lemma 3.1, so only a scketch is given. Let γ be given as in Property 2 of the ISS definition. For each $r, \varepsilon > 0$, let

$$\tilde{T}_{r,\varepsilon} = \inf\{\tau :\; |x(t,\,\xi,\,u)| \le \varepsilon + \gamma(\|u\|),\; \forall t \ge \tau,\; \forall\, |\xi| \le r\}.$$

Note then that $\tilde{T}_{r,\varepsilon} < \infty$ for any $r, \varepsilon > 0$, and

$$\tilde{T}_{r_1,\varepsilon} \le \tilde{T}_{r_2,\varepsilon} \;\text{ if } r_1 \le r_2, \;\text{ and } \tilde{T}_{r,\varepsilon_1} \ge \tilde{T}_{r,\varepsilon_2} \;\text{ if } \varepsilon_1 \le \varepsilon_2\,.$$

Also, Property 1' implies that, for every fixed r, $\tilde{T}_r(s) \to 0$ as $s \to \infty$. Now, for each $r > 0$, let

$$\bar{T}_r(s) = \frac{1}{2s}\int_{s/2}^{s} \tilde{T}_{r,\sigma}\, d\sigma\,.$$

Then, for each fixed $r > 0$, $\bar{T}_r$ is a continuous function, and $\bar{T}_r(s) \ge \tilde{T}_r(s)$. Finally, for each $r > 0$, we let

$$T_r(s) = \frac{r}{s} + \sup_{\sigma \ge s} \bar{T}_r(\sigma).$$

One can easily check that the family $\{T_r\}_{r>0}$ satisfies all conditions in the lemma. ∎

We now return to prove Lemma 11.7.

Proof. [$\Longrightarrow$] Assume that system (54) is ISS; that is, there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that (55) holds. Then

$$|x(t,\,\xi,\,u)| \;\le\; \beta(|\xi|\,,0) + \gamma(\|u\|)$$

holds for all $t \ge 0$, all ξ, and all u. Let $\bar{\beta}(r) = \beta(r, 0) + r$ and let $\delta(r) = \bar{\beta}^{-1}(r)$. Then $\delta(\cdot) \in \mathcal{K}_\infty$, and with such a choice of δ, (66) holds with $\tilde{\gamma} = \gamma$. Property 2 in Lemma 11.7 holds because for each fixed $s \ge 0$, $\lim_{t\to\infty} \beta(s,t) = 0$.

[$\Longleftarrow$] Assume now that conditions in Lemma 11.7 hold. Now let δ be as in Property $1'$, and without loss of generality, we assume that $\tilde{\gamma}$ in Property 1' and γ in Property 2 are the same function, denoted by γ. Let $\varphi(\cdot)$ be the $\mathcal{K}_\infty$-function δ^{-1}. Then it holds that

$$|x(t,\,\xi,\,u)| \le \varphi(|\xi|) + \gamma(\|u\|),\;\; \forall t \ge 0\,. \tag{70}$$

Let $\{T_r\}_{r>0}$ be as in Lemma 11.9, and for each $r > 0$, let $\psi_r(s) = T_r^{-1}(s)$ for $s > 0$, and for $s = 0$, we also denote $\psi_r(0) = \infty$. Note then that ψ_r is continuous on $(0, \infty)$ and $\lim_{s\to 0} \psi_r(s) = \infty$ for each $r > 0$. Since

$$|\xi| \le r,\ t \ge T_r(\varepsilon) \ \Rightarrow\ |x(t, \xi, u)| \le \varepsilon + \gamma(\|u\|),$$

and $t = T_r(\psi_r(t))$ for $t > 0$, it follows from the above, applied in particular for $t = T_r(\varepsilon)$, that for $t > 0$:

$$|x(t, \xi, u)| \le \psi_r(t) + \gamma(\|u\|) \tag{71}$$

for any u and any $|\xi| \le r$. This formula also holds for $t = 0$ by the definition $\psi_r(0) = \infty$.

Now for any $s \ge 0$ and $t \ge 0$, let

$$\bar{\psi}(s,t) = \min\,\{\inf{}_{r\ge s}\psi_r(t),\ \varphi(s)\}\ .$$

Then by (70) and (71), one has

$$|x(t, \xi, u)| \le \bar{\psi}(|\xi|\,, t) + \gamma(\|u\|). \tag{72}$$

Pick any function $\tilde{\psi} : \mathbb{R}_{\ge 0} \times \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ with the following property:

- for any fixed $t \ge 0$, $\tilde{\psi}(\cdot, t)$ is continuous and strictly increasing;
- for any fixed $s \ge 0$, $\tilde{\psi}(s, t)$ decreases to 0 as $t \to \infty$;
- $\tilde{\psi}(s, t) \ge \bar{\psi}(s, t)$.

It is an easy exercise to show that such a majorizing function $\tilde{\psi}$ for $\bar{\psi}(s, t)$ always exists; for details, we refer the reader to similar step in the proof of Proposition 2.5. Finally, we let

$$\beta(s,t) = \sqrt{\varphi(s)}\sqrt{\tilde{\psi}(s,t)}\,.$$

Then β is of class $\mathcal{KL}$, and moreover,

$$\beta(s,t) \ge \min\,\{\varphi(s),\ \tilde{\psi}(s,t)\} \tag{73}$$

for all $s \ge 0$, $t \ge 0$. Combining (72) and (73), one concludes that

$$|x(t, \xi, u)| \le \beta(|\xi|\,, t) + \gamma(\|u\|)$$

for all $\xi \in \mathbb{R}^n$, all $t \ge 0$ and all inputs u. ∎

11.2 Robustness with Respect to Feedback

Lemma 11.10 If there exists an ISS-Lyapunov function for system (54), then the system is robustly stable.

Proof. Assume that V is an ISS-Lyapunov function. Let α_i $(i = 1, 2, 3)$ and χ be as in Definition 11.2. Without loss of generality, we assume that $\chi \in \mathcal{K}_\infty$ (otherwise, replace $\chi(r)$ by $\chi(r) + r$). Let $\rho(r) = \chi^{-1}(r)$; then $\rho \in \mathcal{K}_\infty$ as well, and

$$DV(\xi)f(\xi,\ \mu) \le -\alpha_3(|\xi|) \le -\alpha_3 \circ \alpha_2^{-1}(V(\xi))$$

whenever $\mu \leq \rho(|\xi|)$, which implies that

$$DV(\xi)f(\xi,\, k(\xi,t)) \leq -\hat{\alpha}_3(V(\xi))$$

for any feedback k bounded by ρ and for almost all t, where $\hat{\alpha}_3 = \alpha_3 \circ \alpha_2^{-1}$. This implies that for any such feedback, and for every solution $x(t)$ of the corresponding system (58), it holds that

$$\frac{d}{dt}V(x(t)) \;\leq\; -\hat{\alpha}_3(V(x(t)))$$

for almost all t. A simple comparison principle, Lemma 4.4, implies that there exists some $\mathcal{KL}$-function $\tilde{\beta}$, which depends only on $\hat{\alpha}_3$ and not on the particular k being used, such that

$$V(x(t)) \;\leq\; \tilde{\beta}\Big(V(x(0)),\, t\Big), \quad \forall t \geq 0,$$

for every solution $x(t)$ of (58) and feedback bounded by ρ, from which it follows that there exists some $\mathcal{KL}$-function β such that (59) holds for every solution of (58) whenever k is a feedback of the form considered. ∎

A particular case of the above setup is as follows. Fix any smooth function φ. Then, for each any $d(t) \in \mathcal{M}_{\mathcal{D}}$ = the set of all measurable functions from $\mathbb{R}$ to $\mathcal{D} = [-1,\, 1]^m$, the function $k(t,\xi) = d(t)\varphi(\xi)$ is an admissible feedback law. We view the system

$$\dot{x}(t) \;=\; f(x(t),\, d(t)\varphi(x(t))) \;=\; g(x(t), d(t)) \tag{74}$$

as a "system with disturbances" $d(t)$. For such systems, there is a natural definition of uniform asymptotically stability (UGAS); see Definition 2.2. This definition is exactly the same as saying that an estimate like the one in Equation (59) must hold, for every solution, when using any $d(t) \in \mathcal{M}_{\mathcal{D}}$, where β does not depend on the particular d.

We say that system (54) is *weakly robustly stable* if there exists a smooth function φ which is positive definite and proper or "radially unbounded" (that is, for some $\mathcal{K}_\infty$-function ψ, $\psi(|\xi|) \leq \varphi(\xi)$ for all ξ), so that the corresponding system (74) is UGAS.

Clearly, if a system (54) is robustly stable, then it is also weakly robustly stable: Given ρ, pick any smooth positive definite proper function φ with $\varphi(\xi) \leq \rho(|\xi|)$ (such functions exist, by a routine argument based on taking a scalar smooth function that approaches the origin very fast and is majorized by ρ). Now $d(t)\varphi(\xi)$ is just a particular type of feedback bounded by ρ. Consequently, if there exists an ISS-Lyapunov function for system (54), then (54) is weakly robustly stable. The following two lemmas provide the key connections between ISS, robust stability, and existence of ISS-Lyapunov functions.

Lemma 11.11 If system (54) is ISS, then it is also weakly robustly stable.

Proof. Assume that (54) is ISS. Then by Remark 11.5, there exist some $\mathcal{KL}$-function β and some $\mathcal{K}$-function γ such that

$$|x(t,\, \xi,\, u)| \;\leq\; \max\{\beta(|\xi|\,,\, t),\; \gamma(\|u\|)\} \tag{75}$$

for any ξ and any u. Let $\alpha(r) = \beta(r, 0)$. Then α is a $\mathcal{K}$-function. Without loss of generality, one can always assume that $\alpha(r) > r$ for all $r > 0$ (otherwise, one can replace $\alpha(r)$ by

$\max\{\alpha(r), 3r/2\}$) and thus α is $\mathcal{K}_\infty$. Similarly, one can assume that γ is also $\mathcal{K}_\infty$. It follows that α^{-1} is $\mathcal{K}_\infty$ and $\alpha^{-1}(r) < r$ for all $r > 0$. Now let $\sigma(r)$ be a $\mathcal{K}_\infty$ function satisfying

$$\sigma(r) < \gamma^{-1}\left(\frac{1}{4}\alpha^{-1}(r)\right)$$

for all $r > 0$. For instance, one can simply let $\sigma(r)$ be $\frac{1}{2}\gamma^{-1}\left(\frac{1}{4}\alpha^{-1}(r)\right)$. Let $\varphi(\xi)$ be any smooth, proper, and positive definite function $\mathbb{R}^n$ such that

$$\varphi(\xi) \leq \sigma(|\xi|)\,.$$

(It is an easy exercise to show existence of such functions.) Let ψ be any class $\mathcal{K}_\infty$ function such that $\varphi(\xi) \geq \psi(|\xi|)$ for all ξ.

Now for the fixed function φ, consider system (74). In what follows we show that with the φ chosen above, system (74) is UGAS. Let $x_\varphi(t, \xi, d)$ denote the solution of (74) with initial state ξ and disturbance d. To prove the desired conclusion, we first show that

$$\gamma\Big(|d(t)\varphi(x_\varphi(t, \xi, d))|\Big) \leq \frac{|\xi|}{2} \quad \text{a.e. } t \geq 0 \tag{76}$$

for any $\xi \in \mathbb{R}^n$ and any $d \in \mathcal{M}_{\mathcal{D}}$. For this it is enough to show, because of the monotonicity of γ, that

$$\gamma(\varphi(x_\varphi(t, \xi, d))) \leq \frac{|\xi|}{2} \quad \forall t \geq 0\,. \tag{77}$$

Pick any $\xi \in \mathbb{R}^n$ and $d \in \mathcal{M}_{\mathcal{D}}$, and use simply $x(t)$ to denote $x_\varphi(t, \xi, d)$. Notice then that

$$\gamma(\varphi(x(t))) \leq \frac{|\xi|}{4}$$

for all t small enough, since

$$\gamma(\varphi(x(0))) \leq \gamma(\sigma(|\xi|)) < \frac{1}{4}\alpha^{-1}(|\xi|) \leq \frac{|\xi|}{4}\,.$$

Now let

$$t_1 = \inf\left\{t > 0 :\ \gamma(\varphi(x(t))) > \frac{|\xi|}{2}\right\}\,.$$

Assume that $t_1 < \infty$. Then (77) holds for all $t \in [0, t_1)$, from which it follows that

$$\gamma(|d(t)\varphi(x(t))|) \leq \frac{1}{2}\alpha(|\xi|), \quad \text{a.e. } t \in [0, t_1)\,.$$

By (75), one sees that

$$|x(t)| \leq \beta(|\xi|, 0) \leq \alpha(|\xi|)$$

for all $0 \leq t \leq t_1$, which, in turn, implies that

$$\gamma(|\varphi(x(t_1))|) \leq \gamma(\sigma(|x(t_1)|)) \leq \frac{\alpha^{-1}(|x(t_1)|)}{4} \leq \frac{|\xi|}{4}\,.$$

This contradicts the definition of t_1 (by continuity, from the definition it must hold that $\gamma(|\varphi(x(t_1))|) \geq \frac{|\xi|}{2}$). Thus $t_1 = \infty$, and (77) is proved.

Claim: for each $r > 0$ there is some $T_r \geq 0$ so that

$$t \geq T_r,\ |\xi| \leq r \quad \Rightarrow \quad |x_\varphi(t,\,\xi,\,d)| \leq \frac{r}{2}\,. \tag{78}$$

To establish this claim, note that from (75) and (76) it follows that

$$|x_\varphi(t,\,\xi,\,d)| \leq \max\left\{\beta(|\xi|\,,\,t),\ \frac{|\xi|}{2}\right\}$$

for all ξ. On the other hand, since $\beta \in \mathcal{KL}$, for each $r > 0$ there exists $T_r > 0$ such that $\beta(r,\,t) < r/2$ for all $t \geq T_r$. This T_r satisfies the requirements of the claim.

Now pick any $\varepsilon > 0$. Let k be a positive integer such that $2^{-k}r < \varepsilon$. Let $r_1 = r$ and and $r_i = r_{i-1}/2$ for $i \geq 2$, and let

$$\tau = T_{r_1} + T_{r_2} + \cdots + T_{r_k}\,.$$

Then for $t \geq \tau$:

$$|x_\varphi(t,\,\xi,\,d)| \;\leq\; \frac{r}{2^k} \;<\; \varepsilon$$

for all $|\xi| \leq r$, all $d \in \mathcal{M}_\mathcal{D}$ and all $t \geq \tau$. This shows that the origin is a uniform attractor for system (74).

To show the uniform stability for the system, notice that (75) and (76) imply that

$$|x_\varphi(t,\,\xi,\,d)| \leq \beta(|\xi|\,,\,0)$$

for all $t \geq 0$, all $\xi \in \mathbb{R}^n$, and all $d \in \mathcal{M}_\mathcal{D}$. We conclude that system (74) is UGAS. ∎

Lemma 11.12 If system (54) is weakly robustly stable, then there exists an ISS-Lyapunov function for the system.

Proof. This is an easy consequence of the Converse Lyapunov Theorem for systems with bounded disturbances, Theorem 1 The details of the proof are as follows.

Assume that system (54) is weakly robustly stable. Then there exist a smooth function $\varphi:\ \mathbb{R}^n \to \mathbb{R}$, such that $\varphi(\xi) \geq \psi(|\xi|)$ for some $\mathcal{K}_\infty$-function ψ, for which the system (74) is UGAS. It then follows from Theorem 1 that there exists a uniform smooth Lyapunov function V for the system (74), that is, a smooth V so that, for some positive definite functions $\mathcal{K}_\infty$-functions α_i, $i = 1, 2, 3$, the following inequalities hold:

$$\alpha_1(|\xi|) \leq V(\xi) \leq \alpha_2(|\xi|) \quad \forall \xi \in \mathbb{R}^n$$

and

$$\nabla V(\xi)\cdot f(\xi,\,\mathbf{d}\,\varphi(\xi)) \leq -\alpha_3(|\xi|) \;\; \forall \xi \in \mathbb{R}^n, \quad \forall\,|\mathbf{d}| \leq 1\,. \tag{79}$$

Note that from (79), it follows that

$$\nabla V(\xi)\cdot f(\xi,\,v) \;\leq\; -\alpha_3(|\xi|)\,, \tag{80}$$

whenever $|v| \leq \varphi(\xi)$. Since $\psi(|\xi|) \leq \varphi(\xi)$, (80) holds whenever $|v| \leq \psi(|\xi|)$. Let $\chi(r) = \psi^{-1}(r)$. Clearly χ is a function as required for Definition 11.2, and thus, V is an ISS-Lyapunov function for (54). ∎

Summarizing all the above, we have our main result:

Theorem 3 *The following properties are equivalent for any system:*

1. *It is ISS.*
2. *It admits an ISS-Lyapunov function.*
3. *It is robustly stable.*
4. *There exist $\mathcal{KL}$-functions β_0, β_1 and a $\mathcal{K}$-function γ so that (62) holds.*
5. *There exists a $\mathcal{K}$-function γ, such that for any $\varepsilon > 0$, (64) and (65) hold for properly chosen δ and T.*
6. *It is weakly robustly stable.*

Proof. 1. $\Longleftrightarrow$ 4. See Lemma 11.6.

1. $\Longleftrightarrow$ 5. See Lemma 11.7.

1. $\Longrightarrow$ 6. See Lemma 11.11.

6. $\Longrightarrow$ 2. See Lemma 11.12.

2. $\Longrightarrow$ 1. See [21].

2. $\Longrightarrow$ 3. See Lemma 11.10.

3. $\Longrightarrow$ 6. Clear.

∎

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Appendix

A Some Basic Definitions

In this section we recall some standard concepts from stability theory.

A function $\gamma : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}$ is:

- a $\mathcal{K}$*-function* if it is continuous, strictly increasing and $\gamma(0) = 0$;
- a $\mathcal{K}_\infty$*-function* if it is a $\mathcal{K}$-function and also $\gamma(s) \to \infty$ as $s \to \infty$;
- a *positive definite* function if $\gamma(s) > 0$ for all $s > 0$, and $\gamma(0) = 0$.

A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}$ is a $\mathcal{KL}$*-function* if:

- for each fixed $t \geq 0$ the function $\beta(\cdot, t)$ is a $\mathcal{K}$-function, and
- for each fixed $s \geq 0$ it is decreasing to zero as $t \to \infty$.

Note that we are not requiring β to be continuous in both variables simultaneously; however it will turn out in our results that this stronger property will usually hold.

B Smooth Approximations of Locally Lipschitz Functions

In the proof of the converse Lyapunov theorem, we used a parameterized version of an approximation theorem given in [30]. For convenience of reference, and to make this work self-contained and expository, we next provide the needed variation of the theorem and its proof. (Several details, missing in the proof in [30], have been included as well.)

Theorem 4 *Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$, and let $\mathcal{D}$ be a compact subset of $\mathbb{R}^l$, and assume given:*

- *a locally Lipschitz function $\Phi : \mathcal{O} \longrightarrow \mathbb{R}$;*
- *a continuous map $f : \mathbb{R}^n \times \mathcal{D} \longrightarrow \mathbb{R}^n$, $(x, \mathbf{d}) \mapsto f(x, \mathbf{d})$ which is locally Lipschitz on x uniformly on $\mathbf{d}$;*
- *a continuous function $\alpha : \mathcal{O} \longrightarrow \mathbb{R}$ and continuous functions $\mu, \nu : \mathcal{O} \longrightarrow \mathbb{R}_{>0}$*

such that for each $\mathbf{d} \in \mathcal{D}$,

$$L_{f_{\mathbf{d}}}\Phi(\xi) \leq \alpha(\xi)\,, \quad \text{a.e. } \xi \in \mathcal{O}\,, \tag{81}$$

where $f_{\mathbf{d}}$ is the vector field defined by $f_{\mathbf{d}}(\cdot) = f(\cdot, \mathbf{d})$, (Recall that $\nabla\Phi$ is defined a.e., since Φ is locally Lipschitz by Rademacher's Theorem, see e.g. [5], page 216). Then there exists a smooth function $\Psi : \mathcal{O} \longrightarrow \mathbb{R}$ such that

$$|\Phi(\xi) - \Psi(\xi)| < \mu(\xi)\,, \quad \forall\, \xi \in \mathcal{O}$$

and for each $\mathbf{d} \in \mathcal{D}$,

$$L_{f_{\mathbf{d}}}\Psi(\xi) \leq \alpha(\xi) + \nu(\xi)\,, \quad \forall\, \xi \in \mathcal{O}\,.$$

To prove the theorem, we need first some easy facts about regularization. Let $\psi : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a smooth nonnegative function which vanishes outside of the unit disk and satisfies

$$\int_{\mathbb{R}^n} \psi(s)\, ds = 1\,.$$

For any measurable, locally essentially bounded function $\Phi : \mathcal{O} \longrightarrow \mathbb{R}$ and $0 < \sigma \leq 1$, define the function Φ_σ by convolution with $\dfrac{1}{\sigma^n}\,\psi\left(\dfrac{s}{\sigma}\right)$, that is:

$$\Phi_\sigma(\xi) \stackrel{\text{def}}{=} \int_{\mathbb{R}^n} \Phi(\xi + \sigma s)\psi(s)\, ds\,. \tag{82}$$

We think of this function as defined only for those ξ so that $\xi + \sigma s \in \mathcal{O}$ for all $|s| \leq 1$. Note that the integral is finite, as the integrand is essentially bounded and of compact support. The following observation is a standard approximation exercise, so we omit its proof.

Lemma B.1 For each compact subset K of $\mathcal{O}$, there exists some $\sigma_0 > 0$ such that Φ_σ is defined on K, and smooth there, for all $\sigma < \sigma_0$. Moreover, if Φ is continuous, then Φ_σ approaches Φ uniformly on K, as σ tends to 0. □

Now assume that Φ is a locally Lipschitz function. Then, for each $\mathbf{d} \in \mathcal{D}$, $L_{f_\mathbf{d}}\Phi$ is defined almost everywhere, and furthermore, on any compact subset $K \subseteq \mathcal{O}$,

$$\left|L_{f_\mathbf{d}}\Phi(\xi)\right| \le k\,|f(\xi,\,\mathbf{d})|\,, \quad \text{a.e. } \xi \in K\,,\ \forall \mathbf{d} \in \mathcal{D}\,,$$

where k is a Lipschitz constant for Φ on K. Therefore, for each $\mathbf{d}$, (omitting from now on the $\mathbb{R}^n$ in integrals)

$$(L_{f_\mathbf{d}}\Phi)_\sigma(\xi) = \int (L_{f_\mathbf{d}}\Phi)(\xi + \sigma s)\psi(s)\,ds$$

is well defined as long as $\xi + \sigma s \in \mathcal{O}$ for all $|s| \le 1$. Applying Lemma B.1 to $(L_{f_\mathbf{d}}\Phi)_\sigma$, this is smooth for any $\sigma > 0$ small.

Suppose that for all $\mathbf{d} \in \mathcal{D}$,

$$L_{f_\mathbf{d}}\Phi(\xi) \le \alpha(\xi)\,, \quad \text{a.e. } \xi \in \mathcal{O}\,, \tag{83}$$

for some continuous function α. Pick any compact subset $K \subseteq \mathcal{O}$. On this set K, we have

$$\begin{aligned}(L_{f_\mathbf{d}}\Phi)_\sigma(\xi) &= \int (L_{f_\mathbf{d}}\Phi)(\xi+\sigma s)\psi(s)\,ds \le \int \alpha(\xi+\sigma s)\psi(s)\,ds \\ &\le \alpha(\xi) + \max_{|s|\le 1,\,\xi\in K} |\alpha(\xi+\sigma s) - \alpha(\xi)|\,.\end{aligned}$$

From here we get the following conclusion:

Lemma B.2 For any compact subset K of $\mathcal{O}$, $(L_{f_\mathbf{d}}\Phi)_\sigma$ is a C^∞ function defined on K for all σ small enough, and, if (83) holds for all $\mathbf{d} \in \mathcal{D}$ and all $\xi \in \mathcal{O}$, then for any $\varepsilon > 0$ given, there exists some $\sigma_0 > 0$ such that

$$(L_{f_\mathbf{d}}\Phi)_\sigma(\xi) \le \alpha(\xi) + \varepsilon$$

for all $\sigma \le \sigma_0$, all $\mathbf{d} \in \mathcal{D}$, and all $\xi \in K$. □

The following lemma illustrates the relationship between $L_{f_\mathbf{d}}(\Phi_\sigma)$ and $(L_{f_\mathbf{d}}\Phi)_\sigma$.

Lemma B.3 On any compact subset K of $\mathcal{O}$,

$$\sup_{\mathbf{d}\in\mathcal{D},\xi\in K}\left|L_{f_\mathbf{d}}(\Phi_\sigma)(\xi) - (L_{f_\mathbf{d}}\Phi)_\sigma(\xi)\right| \to 0$$

as σ tends to 0.

Proof. For each $\xi \in \mathcal{O}$, we use $\varphi(t,\,\xi,\,\mathbf{d})$ to denote the solution of the differential equation:

$$\dot{x} = f(x,\,\mathbf{d})$$

with the initial condition $\varphi(0,\,\xi,\,\mathbf{d}) = \xi$. It follows from the assumptions on f and compactness of K and $\mathcal{D}$ that there exist some compact neighborhood V of K and some $\tau_1 > 0$ and $\sigma_0 > 0$ such that $\varphi(t,\,\xi+\sigma s,\,\mathbf{d}) \in V$ for all $\xi \in K$, $|s| \le 1$, $\sigma \le \sigma_0$, $\mathbf{d} \in \mathcal{D}$ and $|t| \le \tau_1$.

For the Lipschitz function Φ, we have, for all $\xi, \mathbf{d}$ and $\sigma \le \sigma_0$:

$$\begin{aligned}L_{f_\mathbf{d}}(\Phi_\sigma)(\xi) &= \left.\frac{d}{dt}\right|_{t=0} \Phi_\sigma(\varphi(t,\,\xi,\,\mathbf{d})) = \left.\frac{d}{dt}\right|_{t=0} \int \Phi(\varphi(t,\,\xi,\,\mathbf{d}) + \sigma s)\psi(s)\,ds \\ &= \lim_{t\to 0}\frac{1}{t}\int (\Phi(\varphi(t,\,\xi,\,\mathbf{d}) + \sigma s) - \Phi(\xi+\sigma s))\psi(s)\,ds\,,\end{aligned}$$

and

$$(L_{f_{\mathbf{d}}}\Phi)_\sigma(\xi) = \int L_{f_{\mathbf{d}}}\Phi(\xi+\sigma s)\psi(s)\,ds \tag{84}$$

$$= \int \frac{d}{dt}\Big|_{t=0} \Phi(\varphi(t,\,\xi+\sigma s,\,\mathbf{d}))\psi(s)\,ds \tag{85}$$

$$= \lim_{t\to 0}\frac{1}{t}\int \left[\Phi(\varphi(t,\,\xi+\sigma s,\,\mathbf{d})) - \Phi(\xi+\sigma s)\right]\psi(s)\,ds\,. \tag{86}$$

Notice that the integrand in (84) equals that in (85) almost everywhere on s (for each fixed ξ and σ) and that (86) follows from (85) because of the Lebesgue Dominated Convergence Theorem and the following fact:

$$\begin{aligned}&\frac{1}{|t|}\left|\Phi(\varphi(t,\,\xi+\sigma s,\,\mathbf{d})) - \Phi(\xi+\sigma s)\right|\psi(s)\\ &\leq \frac{k}{|t|}\left|\varphi(t,\,\xi+\sigma s,\,\mathbf{d}) - (\xi+\sigma s)\right|\psi(s) \;\leq\; kC\psi(s)\,,\quad \forall t\in[-\tau_1,\,\tau_1]\,,\end{aligned}$$

where $C \stackrel{\text{def}}{=} \max_{\xi\in V,\mathbf{d}\in\mathcal{D}} |f(\xi,\mathbf{d})|$ and k is a Lipschitz constant for Φ on V.

Now one sees that

$$L_{f_{\mathbf{d}}}(\Phi_\sigma)(\xi) - (L_{f_{\mathbf{d}}}\Phi)_\sigma(\xi) = \lim_{t\to 0}\frac{1}{t}\int[\Phi(\varphi(t,\,\xi,\mathbf{d})+\sigma s) - \Phi(\varphi(t,\,\xi+\sigma s,\,\mathbf{d}))]\psi(s)\,ds\,.$$

Thus it is enough to show that for any $\varepsilon>0$, there exist some $\delta>0$ and $\tau^*>0$ such that the above integral is bounded by ε for all $\mathbf{d}\in\mathcal{D}$, $\xi\in K$, $|t|<\tau^*$ and $\sigma<\delta$. This is basically a standard argument on continuous dependence on initial conditions, but we provide the details. For $0\leq\tau\leq\tau_1$, let

$$\gamma(\tau) \stackrel{\text{def}}{=} \sup\left\{|f(\varphi(t,\,\zeta,\,\mathbf{d}),\,\mathbf{d}) - f(\zeta,\,\mathbf{d})| :\; |t|\leq\tau,\;\zeta\in V,\;\mathbf{d}\in\mathcal{D}\right\}\,.$$

Then $\gamma(0)=0$, and γ is nondecreasing and continuous at $t=0$, because

$$|f(\varphi(t,\,\zeta,\,\mathbf{d}),\,\mathbf{d}) - f(\zeta,\,\mathbf{d})| \;\leq\; C_3\,|\varphi(t,\,\zeta,\,\mathbf{d}) - \zeta| \;\leq\; C_3C_4\,|t|\,,$$

where C_3 is a (uniform) Lipschitz constant for f on V_1, C_4 is an upper bound for $|f(\xi,\,\mathbf{d})|$ on V_1, and V_1 is some compact neighborhood of V such that $\varphi(t,\,\zeta,\,\mathbf{d})\in V_1$ for any $\zeta\in V$, $\mathbf{d}\in\mathcal{D}$ and $|t|\leq\tau_1$. For any $\zeta\in V$, $\mathbf{d}\in\mathcal{D}$ and $|t|\leq\tau_1$,

$$|\varphi(t,\,\zeta,\,\mathbf{d}) - (\zeta+tf(\zeta,\,\mathbf{d}))| \leq \int_0^{|t|}\gamma(\tau)\,d\tau \;\leq\; |t|\,\gamma(|t|)\,.$$

Now for $\xi\in K$, we have

$$\begin{aligned}&|\Phi(\varphi(t,\,\xi,\,\mathbf{d})+\sigma s) - \Phi(\varphi(t,\,\xi+\sigma s),\,\mathbf{d})| \;\leq\; k\,|\varphi(t,\,\xi,\,\mathbf{d})+\sigma s - \varphi(t,\,\xi+\sigma s,\,\mathbf{d})|\\ &\quad\leq\; k\,|\xi+\sigma s+tf(\xi,\,\mathbf{d}) - (\xi+\sigma s+tf(\xi+\sigma s,\,\mathbf{d}))|\\ &\qquad\quad +k\,|\varphi(t,\,\xi,\,\mathbf{d}) - (\xi+tf(\xi,\,\mathbf{d}))| \;+\; k\,|\varphi(t,\,\xi+\sigma s,\mathbf{d}) - (\xi+\sigma s+tf(\xi+\sigma s,\,\mathbf{d}))|\\ &\quad\leq\; k\,|t|\,|f(\xi,\mathbf{d}) - f(\xi+\sigma s,\,\mathbf{d})| + 2k\,|t|\,\gamma(|t|)\,.\end{aligned} \tag{87}$$

Finally, for $\varepsilon > 0$, let δ and τ^* be such that

$$\gamma(\tau) < \frac{\varepsilon}{3k} \quad \text{and} \quad |f(\xi, \mathbf{d}) - f(\xi + \sigma s, \mathbf{d})| < \frac{\varepsilon}{3k}$$

for any $\xi \in K$, $\mathbf{d} \in \mathcal{D}$, $|s| \leq 1$, $\sigma < \delta$ and $|t| < \tau^*$. It then follows from (87) that

$$\frac{1}{|t|} \int [\Phi(\varphi(t, \xi, \mathbf{d}) + \sigma s) - \Phi(\varphi(t, \xi + \sigma s, \mathbf{d}))] \psi(s)\, ds < \int \varepsilon \psi(s)\, ds = \varepsilon$$

for any $\xi \in K$, $\mathbf{d} \in \mathcal{D}$, $|t| < \tau^*$ and $\sigma < \delta$, which implies

$$\left| L_{f_{\mathbf{d}}}(\Phi_\sigma)(\xi) - (L_{f_{\mathbf{d}}}\Phi)_\sigma(\xi) \right| < \varepsilon$$

for any $\sigma < \sigma_0$, $\mathbf{d} \in \mathcal{D}$ and $\xi \in K$. ∎

Combining the previous three lemmas, we obtain the following conclusion:

Lemma B.4 Let K be a compact subset of $\mathcal{O}$. Then for any given $\varepsilon > 0$, there exists some smooth function Ψ defined on K such that

$$|\Psi(\xi) - \Phi(\xi)| < \varepsilon \quad \text{and} \quad L_{f_{\mathbf{d}}}\Psi(\xi) \leq \alpha(\xi) + \varepsilon$$

for all $\xi \in K$, $\mathbf{d} \in \mathcal{D}$. □

Now we are ready to complete the proof of Theorem (4). For the open subset $\mathcal{O}$ of $\mathbb{R}^n$, let $\{\mathcal{U}_i\}$ be a locally finite countable cover of $\mathcal{O}$ with $\bar{\mathcal{U}}_i$ compact and $\bar{\mathcal{U}}_i \subseteq \mathcal{O}$. Let $\{\beta_i\}$ be a partition of unity on $\mathcal{O}$ subordinate to $\{\mathcal{U}_i\}$. For any given positive functions $\mu(\cdot)$ and $\nu(\cdot)$, let

$$\varepsilon_i \stackrel{\text{def}}{=} \min\left\{\inf_{\xi \in \mathcal{U}_i} \mu(\xi),\ \inf_{\xi \in \mathcal{U}_i} \nu(\xi)\right\}.$$

For each i, it follows from Lemma B.4 that there exists some smooth function Ψ_i defined on $\bar{\mathcal{U}}_i$ such that

$$|\Phi(\xi) - \Psi_i(\xi)| < \frac{\varepsilon_i}{2^{i+1}(1+\tau_i)} \quad \text{and} \quad L_{f_{\mathbf{d}}}\Psi_i(\xi) \leq \alpha(\xi) + \frac{\varepsilon_i}{2}$$

on $\bar{\mathcal{U}}_i$, where $\tau_i \stackrel{\text{def}}{=} \max\{\left|L_{f_{\mathbf{d}}}\beta_i(\xi)\right| :\ \xi \in \bar{\mathcal{U}}_i,\ \mathbf{d} \in \mathcal{D}\}$. We define $\Psi = \sum_i \beta_i \Psi_i$. Clearly Ψ is a smooth function defined on $\mathcal{O}$, and

$$\begin{aligned} |\Psi(\xi) - \Phi(\xi)| &\leq \sum_{j \in \mathcal{J}_\xi} \beta_j(\xi)\, |\Psi_j(\xi) - \Phi(\xi)| \\ &< \max_{j \in \mathcal{J}_\xi} \varepsilon_j \leq \mu(\xi), \end{aligned}$$

where $\mathcal{J}_\xi \stackrel{\text{def}}{=} \left\{ j :\ \xi \in \mathcal{U}_j \right\}$.

For $L_{f_{\mathbf{d}}}\Psi$, one has

$$\begin{aligned} L_{f_{\mathbf{d}}}\Psi(\xi) &= L_{f_{\mathbf{d}}}\Phi(\xi) + L_{f_{\mathbf{d}}}(\sum_i \beta_i(\Psi_i - \Phi))(\xi) \\ &= L_{f_{\mathbf{d}}}\Phi(\xi) + \sum (L_{f_{\mathbf{d}}}\beta_i)(\Psi_i - \Phi)(\xi) + \sum \beta_i(L_{f_{\mathbf{d}}}\Psi_i(\xi) - L_{f_{\mathbf{d}}}\Phi(\xi)) \\ &= \sum_{j \in \mathcal{J}_\xi} (L_{f_{\mathbf{d}}}\beta_j)(\Psi_j - \Phi)(\xi) + \sum_{j \in \mathcal{J}_\xi} \beta_j L_{f_{\mathbf{d}}}\Psi_j(\xi) \end{aligned}$$

$$
\begin{aligned}
&< \sum_{j \in \mathcal{J}_\xi} \frac{\varepsilon_j}{2^{j+1}} + \sum_{j \in \mathcal{J}_\xi} \beta_j(\xi) \left(\alpha(\xi) + \frac{\varepsilon_i}{2} \right) \\
&\leq \frac{1}{2} \max_{j \in \mathcal{J}_\xi} \{\varepsilon_j\} + \alpha(\xi) + \frac{1}{2} \max_{j \in \mathcal{J}_\xi} \{\varepsilon_j\} \\
&\leq \alpha(\xi) + \nu(\xi)\,.
\end{aligned}
$$

We conclude that Ψ is the desired function. ∎