ANOMALOUS EXPONENTS IN NONLINEAR DIFFUSION

BY

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ABSTRACT

We present a technique to prove existence and analytic dependence with respect to the relevant perturbation parameters of the anomalous exponents appearing often in problems of continuum mechanics. Such exponents are related to the existence of suitable self-similar solutions. The basic ingredients of the technique are the existence of analytic families of curves, matching by means of the Implicit Function Theorem, and continuation in the relevant parameter.

We demonstrate the method by investigating in full detail two quite different situations which involve diffusive phenomena and are formulated in terms of nonlinear parabolic equations: they are the source-type solutions of the Barenblatt equation of elasto-plastic filtration and the focusing solutions of the porous medium equation.

The method is not limited to these two problems and has quite general scope. We describe several additional problems involving anomalous exponents where this technique can be applied, and we outline the results of the cases that have been successfully treated.

The results give a rigorous foundation to perturbative methods popular in the physical literature for some of these problems.

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Introduction

A. This paper is concerned with the properties of some self-similarity exponents appearing in the description of the behaviour of a number of nonlinear equations in different asymptotic situations. It is well-known after the work of Barenblatt [B1] that the intermediate asymptotics of many interesting phenomena can be described in terms of self-similar solutions of the form

$$u(x,t) = t^{-\alpha} f(\eta) \qquad \text{with} \qquad \eta = x\, t^{-\beta}\,. \tag{1.1}$$

The similarity exponents α and β are of primary physical importance since α represents the rate of decay of the magnitude u, while β is the rate of spread (or contraction if $\beta < 0$) of the space distribution as time goes on. Often, these exponents are easily determined from the equation and some conservation law. This is the so-called self-similarity of the first kind. However, it sometimes occurs that the exponents are not easily ascertained, being nonlinear eigenvalues of some related elliptic problems. In that case it is said that we face *self-similarity of the second kind*, and also that the exponents are *anomalous*. Of course, the profile f is also an important object of the investigation.

The occurrence and properties of anomalous exponents is still not well understood. It is often the case that the equation contains a parameter, say ε, and the anomalous exponent is a function of ε. Moreover, for some distinguished value, say $\varepsilon = 0$, the self-similarity is of the first kind (normal self-similarity). Then, we can consider the generic anomalous behaviour for $\varepsilon \neq 0$ as a perturbation of the normal case, $\varepsilon = 0$. The description of this transition is an important subject of study. Perturbative techniques are often used. The justification of such methods relies on establishing sufficient smoothness for the solution profiles and the exponents in that transition. It is precisely the question of regularity of the anomalous exponents and profiles that is addressed in this paper by means of technique based on matching, which uses the Implicit Function Theorem and a continuation argument. We shall refer to our technique as the *IFT Method* due to the prominent role played by the Implicit Function Theorem. Contrary to what happens with perturbative techniques the investigation will not be reduced to the transition from normal to anomalous self-similarity, but will describe the phenomenon of second-kind self-similarity in the whole parameter range of existence.

Barenblatt proposed in [B1; Introduction, page 7] the following example to explain the nature of the idea of similarity of the second kind. Let us modify somewhat the classical problem of heat conduction so that the specific heat of the medium is equal to one constant if the medium is heated, and to another constant if it is cooled. For this modified medium the equation of heat conduction becomes

$$\begin{aligned} u_t &= \kappa_1 \Delta u \qquad \text{when} \quad u_t > 0 \\ u_t &= \kappa_2 \Delta u \qquad \text{when} \quad u_t < 0, \end{aligned} \tag{1.2}$$

where κ_1 and κ_2 are positive constants. Such a heat conduction model applies in materials in which pores appear. It also occurs in filtration theory when an elastic fluid flows through an elasto-plastic porous medium, under the assumption that the porous medium is irreversibly deformable, see [BER]. The equation is known in the literature as the *Barenblatt equation of elasto-plastic mode of filtration.* It is also interesting to note that (1.2) can be viewed as an example of the parabolic *Bellman equations* studied in the theory of Dynamic Programming. Thus, for $\kappa_1 > \kappa_2$ (1.2) can be written as

$$u_t = \max\{\kappa_1 \Delta u, \kappa_2 \Delta u\}, \tag{1.3}$$

while for $\kappa_1 < \kappa_2$ we have to replace *max* by *min*. See [EL] for recent work on such equations.

It is remarkable that the behaviour of the solutions to equation (1.2) depends crucially on the quotient κ_2/κ_1, which is an essential parameter in the equation. Let us write $\kappa_2/\kappa_1 = 1 + \varepsilon$. Then, for $\varepsilon = 0$ we recover the classical heat equation and the large-time behaviour of all nonnegative solutions with, say, compactly supported initial data is described by the well-known fundamental solution, which is a self-similar solution of the form (1.1) with $\alpha = N/2$ and $\beta = 1/2$.

However, for $\varepsilon \neq 0$ Barenblatt and collaborators, [BK], [Ka], [BS], showed that the existence of source-type self-similar solutions leads to an anomalous exponent α. The existence, uniqueness and properties of such solutions was established in [KPV], where it is shown that such self-similar solutions of the second kind give the large-time behaviour of a general class of solutions.

In this paper we show that the anomalous exponent α is in fact an analytic function of the parameter ε in the interval $(-1, \infty)$ (where we are dealing with a parabolic problem). The method is in itself a complete proof of the existence of such *solution branch* $u = u_\varepsilon(x,t)$, $\varepsilon \in (-1, \infty)$. Since for $\varepsilon = 0$ the solution is explicit we can exactly calculate the derivative $\partial\alpha/\partial\varepsilon$ for this value of ε.

Actually, starting from a self-similar solution of the first kind is in no way essential; what we need as a starting point is a self-similar solution whose exponents and profile are explicitly known. Our second main example demonstrates that point. We consider a very different problem, arisen in the theory of flows through porous media. The equation under consideration is

$$u_t = \Delta u^m, \quad m > 1, \tag{1.4}$$

where u stands for the density of the fluid. The equation appears as a model in many other contexts, like plasma physics, where u stands for a temperature, cf. [ZR]. For a survey on the mathematical state of the art cf. [A2]. In this equation the large-time behaviour of nonnegative solutions with finite mass is described by a self-similar solution of the first kind, [B], [FK]. An interesting anomalous exponent appears when we want to describe the phenomenon of *hole-filling*, whereby a solution is initially supported in the complement of a ball (the hole) and the support spreads with time to fill this hole. The model for this *focusing behaviour* is given by a one-parameter family of self-similar solutions with anomalous exponent, so-called Graveleau solutions, [AG]. In this case it is convenient for thechnical reasons to write the self-similar solution in the form

$$U^{m-1} = \frac{r^2}{T-t}\phi(\xi) \quad \text{with} \quad \xi = (T-t)r^{-\alpha} \ge 0\,, \tag{1.5}$$

where T is the focusing time, and α is the anomaluos exponent. It is interesting to notice that $2-\alpha$ is the Hölder-continuity exponent of the solution at the focusing point (and at the same time the maximal regularity of that solution). As in the first example, this family of self-similar solutions describes the generic situation; indeed, it has been recently shown, [AA], that every radially symmetric solution of the porous medium equation whose initial data are supported in an annulus behaves as time approaches the focusing time precisely as one these focusing self-similar solutions.

B. Let us describe in some more detail the two problems, our technique and the results. Consider first equation (1.2). If we put $\gamma = 0$, it is well-known that the most important properties of the solutions to this equation can be described in terms of the fundamental solution

$$U(x,t) = \frac{1}{(4\pi t^N)^{1/2}}\exp(-\frac{x^2}{4t})\,. \tag{1.6}$$

It was discovered in [Ka], [BS] (for dimension $N=1$) that formula (1.5) can be continuously extended to a nonnegative self-similar solution of the form (1.1) for equation (1.2) for $\varepsilon \neq 0$. This solution continues to satisfy the initial condition $U(x,0) = 0$ for $x \neq 0$, but, contrary to (1.4), the singularity of the function $U(x,t)$ at $(x,t) = (0,0)$ is never of the form of a Dirac mass if $\varepsilon \neq 0$. This is a direct consequence of the value of the similarity exponents. Indeed, though β is still $1/2$ (otherwise (1.1) would not be a solution of equation (1.2)), the exponent α varies monotonically with ε: $\alpha > 1/2$ for $\varepsilon > 0$ and $\alpha < 1/2$ for $\varepsilon < 0$. The anomaly of the similarity decay rate is related to the failure of the mass conservation law, $\int u(x,t)\,dx =$constant, which clearly cannot hold for nontrivial solution of (1.2) with $\varepsilon \neq 0$.

More recently, Goldenfeld and collaborators [GMOL] used a *renormalization group* technique together with a perturbative approach to calculate the variation of α with ε near $\varepsilon = 0$. A detailed account of the main ideas and applicability of this approach can be found in the recent book by Goldenfeld [G], see especially Chapter 10. As to rigorous analytical studies, the existence of the nonnegative self-similar solution of (1.2) with exponential decay in space was established in [KPV] and the properties of the corresponding anomalous exponent investigated in all space dimensions $N \ge 1$. It was also shown there that such a solution describes the asymptotic behaviour of all nonnegative solutions of equation (1.2) posed in $Q = \mathbf{R}^N \times (0,\infty)$ whose initial data decay fast as $x \to \infty$. In this way one of main properties the fundamental solution (1.4) is extended to the case $\varepsilon \neq 0$.

Let us now give a more detailed account of the four steps involved in the IFT Method.

(i) The first step consists of writing the ODE satisfied by the profile function f and separating the curve $\xi = f(\eta)$ into two branches, each corresponding to a sign of Δu.

Consequently, each branch is a representation of a self-similar caloric function and as such analytic in the variables and parameters.

(ii) Simultaneously, we observe that for $\varepsilon = 0$ the solution to this problem is known; indeed, it is given by the explicit solution (1.4).

(iii) The value of α for $\varepsilon \neq 0$ is then determined in a unique way so that the union of the two branches is a solution of the whole equation. Everything is thus reduced to a suitable *matching* problem, which is solved locally near a known solution by means of the Implicit Function Theorem, once we appropriate determinant of partial derivatives is shown to be nonzero (in other words, a *transversality condition* holds at the meeting of the two branches). In particular, for $\varepsilon = 0$, the computations become explicit, and this allows us to find the formula for the derivative $\partial\alpha/\partial\varepsilon$

$$\frac{d\alpha}{d\varepsilon}(0) = \frac{(N/2)^{N/2}}{e^{N/2}\,\Gamma(N/2)}\,. \tag{1.7}$$

This formula was obtained for dimension $N = 1$ in [GMOL] by a completely different approach!

(iv) The construction of a solution for all ε in the pertinent range, i.e., a *maximal solution branch* $\alpha = \alpha(\varepsilon)$, proceeds via a continuation argument, starting in the way explained in (iii) at the explicit solution (1.4) and using a priori estimates to ensure the continuation of the branch.

These results are proved in Section 2. We remark that they allow for a rigorous justification of the results of [GMOL]. Notice that the existence of a solution for ε far away from 0 is not covered by such approach. The above method is in itself a complete proof of the existence of the self-similar solution with anomalous exponent, different from that of [KPV]. Moreover, we prove that the profile f depends analytically on ε away from the matching point.

C. We turn now our attention to the focusing solution of the porous medium equation. The *Graveleau solution* is written in [AG] in the form (1.5), where the anomalous exponent $\alpha \in (1,2)$ for $N \geq 2$, while the exponent is 1 for $N = 1$. The value of α controls the optimal regularity of U. In fact, at the focusing time U is given by

$$U(x,T) = c\,|x|^{\frac{2-\alpha}{m-1}}\,, \tag{1.8}$$

with $c > 0$ a constant. This shows that whenever $\alpha > 1$ the gradient of the pressure function $v = \frac{m}{m-1}\,u^{m-1}$ is singular at the origin. Lipschitz-continuity of the pressure is known to be the optimal regularity of the nonnegative solutions of the PME in one space dimension [A], and is also true in several dimensions for large times [CVW].

The four-step method outlined above works also in the present problem, even with surprising formal parallels. We can construct a solution u with a hole which is filled in finite time, and also show analytic dependence of α and ϕ with respect to the parameters $N > 0$ and $m > 1$ (Theorem 3). It is worth mentioning that continuation is done for fixed m with respect to the *dimension* N taken as a real quantity (since we are looking for radially symmetric solutions we may allow ourselves noninteger dimensions!). Again in this way we obtain a complete proof of the existence of the focusing solutions wholly independent of [AG] (apart from the use of some a priori estimates taken from [AG] that we need in the continuation step). As in the previous example, the calculation is explicit at the starting solution and we can explicitly calculate the derivative of α with respect to N for $N = 1$, $m > 1$ (Theorem 4).

D. The above project was discussed with G.I. Barenblatt during his visit to the United States in 1991, and at the time the two applications above were worked out and the results announced in the Workshop held at the IMA on Degenerate Diffusions, [V]. It was apparent that the method could be applied to a number of similar situations in very different applied contexts. During the last two years a number of problems have been studied by several authors by the IFT Method. Section 4 will devoted to a detailed review of some of the progress already achieved, and Section 5 lists some new directions.

2. The elasto-plastic equation

By suitably scaling the time or space variable we can eliminate one of the parameters in equation (1.2) and write it as

$$u_t = \begin{cases} \Delta u & \text{if} \quad \Delta u > 0 \\ (1+\varepsilon)\Delta u & \text{if} \quad \Delta u < 0\,, \end{cases} \tag{2.1}$$

with $\varepsilon \in (-1, \infty)$. In can also be written in the equivalent form

$$u_t + \gamma |u_t| = \Delta u\,, \quad 0 < |\gamma| < 1\,. \tag{2.2}$$

The parameters γ and ε are related by

$$\varepsilon = \frac{2\gamma}{1-\gamma}\,, \quad \gamma = \frac{\varepsilon}{2+\varepsilon}\,. \tag{2.3}$$

(To complete the equivalence it is also necessary to rescale the time). In a more compact way we write (2.2) as $F(u_t) = \Delta u$, where

$$F(s) = s + \gamma|s| = \begin{cases} (1+\gamma)s & \text{for} \quad s \geq 0 \\ (1-\gamma)s & \text{for} \quad s < 0\,. \end{cases} \tag{2.4}$$

We will consider nonnegative solutions defined in $Q = \{(x,t) : x \in \mathbf{R}^N, t > 0\}$. We have already mentioned that we are interested in nonnegative, source-type, self-similar solutions $U = U(x,t)$ of the form (1.1). In this section we will choose the notation (2.2) so that the driving parameter is γ. The results are readily expressed in terms of ε.

Satisfying the equation forces the second similarity exponent, β, to be $1/2$. The problem is then to determine the decay exponent α and the profile f in such a way that U be a nontrivial, nonnegative solution of (2.2) with initial data

$$u(x,0) = 0 \quad \text{for} \quad x \neq 0\,. \tag{2.5}$$

The profile f satisfies the equation

$$f'' + \frac{N-1}{\eta} f' = F(-(\frac{1}{2}\eta f' + \alpha f)) \qquad \text{for} \quad \eta > 0\,. \tag{2.6}$$

In view of the regularity of $u(x,t)$ at the origin for $t > 0$, we must have $f'(0) = 0$ and, if $x \neq 0$, we require in view of (2.5),

$$\lim_{t \downarrow 0} u(x,t) = |x|^{-2\alpha} \lim_{\eta \to \infty} \eta^{2\alpha} f(\eta) = 0. \tag{2.7}$$

Finally, because the differential equation and the conditions at the origin and infinity are not affected by a multiplicative constant, there will be no loss of generality if we set $f(0) = 1$. Thus, we shall be investigating the problem

$$\text{(I)} \quad \begin{cases} (2.8) \quad f'' + \frac{N-1}{\eta} f' = F(-(\frac{1}{2}\eta f' + \alpha f)), \quad f > 0, \qquad 0 < \eta < \infty, \\ (2.9) \quad f(0) = 1, \quad f'(0) = 0, \\ (2.10) \quad \eta^{2\alpha} f(\eta) \to 0 \quad \text{as} \quad \eta \to \infty\,. \end{cases}$$

In [KPV] the existence of such solutions with an anomalous exponent was proved. The result is

[KPV, Theorems 2.1 and 2.3]. *For any $\gamma \in (-1,1)$ there exists a unique $\alpha(\gamma) > 0$ such that Problem (I) has a solution. The exponent α is a strictly increasing and continuous function of γ, and covers the interval $([N-2]_+/2, \infty)$.*

The reader should bear in mind in checking the results of [KPV] that the decay exponent is written in that work as $\alpha/2$, so all the values of α are doubled with respect to ours. Our methods give the following result.

Theorem 1. *There exists a unique analytic function $\alpha = \alpha(\gamma, N)$ defined for $-1 < \gamma < 1$ and $N > 0$ such that Problem (I) admits a solution. Moreover, we have $\partial\alpha/\partial\gamma > 0$.*

Proof: **A)** We begin by recalling the change of variables

$$t = \frac{1}{4}\eta^2 \quad \text{and} \quad y(t) = \frac{-\eta f'}{f}\,, \tag{2.11}$$

which proved useful in [KPV]. Thus, in terms of y and t equation (2.6) becomes

$$y' = H(t, y; \alpha) \equiv \kappa(2\alpha - y) + \frac{1}{2t}\{y^2 - (N-2)y\}, \tag{2.12}$$

where

$$\kappa = \begin{cases} \kappa_1 = 1 - \gamma & \text{if} \quad 0 < y < 2\alpha \\ \kappa_2 = 1 + \gamma & \text{if} \quad y > 2\alpha. \end{cases} \tag{2.13}$$

The initial conditions (2.9) give $y(0) = 0$. It then follows from the equation that

$$y'(0) = \frac{4(1-\gamma)\alpha}{N}. \tag{2.14}$$

The solutions we look for will be confined to the region $y > 0$, $t > 0$. An analysis of (2.12) shows that there are only three possible evolutions in forward time for a solution of (2.12). The solution we look for (with fast, i.e. exponential decay in f) corresponds here to solutions such that $y(t)$ exists and is positive for all $t > 0$ and $y(t)/\sqrt{t} \to \infty$ as $t \to \infty$. There exists only one such solution curve. There can also be solutions changing sign in the (f,η)-plane which correspond here to solutions blowing up in finite time. Finally, solutions can be bounded in time and then they tend to 2α as $t \to \infty$. Another important observation about equation (2.12) is that $\partial H/\partial \alpha > 0$. This immediately implies that the solutions $y = y(t;\alpha)$ of (2.12) with $y(0) = 0$ depend monotonically on α. All this is explained in detail in [KPV] where it is also been observed that since

$$H(t, 2\alpha; \alpha) = \frac{2\alpha}{t}\left(\alpha - \frac{N-2}{2}\right), \tag{2.15}$$

the existence of a solution to our problem implies that $\alpha > 0$, $\alpha > (N-2)/2$. We will discuss these bounds below. Then the solution crosses the line $y = 2\alpha$ only once because H is always positive on the line $y = 2\alpha$ (so that it crosses upwards). The equation $y = 2\alpha$ is equivalent to saying that $(r^{N-1}f')' = 0$, or in terms of u that $\Delta u = 0$. Thus the sign of Δu changes only once.

B) We now pose the problem in our terms. As long as the solutions of (2.8) do not change 'concavity', i.e. the sign of Δu remains the same, we are just dealing with self-similar solutions of the heat equation. Now, given α and γ there exists precisely one solution f_1 of the equation

$$f'' + \frac{N-1}{\eta} f' + (1-\gamma)(\frac{1}{2}\eta f' + \alpha f)) = 0, \quad f > 0, \tag{2.16}$$

defined for some $\eta > 0$ with initial conditions (2.9), and precisely one solution f_2 (up to a constant) of

$$f'' + \frac{N-1}{\eta} f' + (1+\gamma)(\frac{1}{2}\eta f' + \alpha f)) = 0, \quad f > 0, \tag{2.17}$$

defined for all large η and which decays exponentially as $\eta \to \infty$. Moreover, these solutions are analytic functions of the variable η and the parameters α and γ. Our method consists in finding an α and a point η_*, both depending on γ and N, such that f_1 and f_2 meet at η_* and the following contact takes place:

(i) the values of f_i and f_i' coincide;

(ii) the concavity of f_1, i.e. $(r^{N-1}f_1')'$, is negative in $(0,\eta_*)$ and that of f_2 positive in (η_*,∞). Both quantities vanish at η_*.

All of this is much simpler to view in terms of the variables y, t. It amounts to using equation (2.12) and first shoot from the origin to obtain a solution y_1 until it reaches the line $y=2\alpha$ at a point $t=t_*(\gamma,N)$. The other branch, namely the solution of (2.17) with exponential decay near infinity, becomes the solution y_2 of (2.12) which is defined and positive for all large t and such that $y(t)\to\infty$ as $t\to\infty$. If y_2 crosses the line $y=2\alpha$ at the same point t_* as y_1, then both branches combine into a solution of our problem.

C) Our construction method is based on continuation in the parameter γ starting from the known solution of the heat equation. Indeed, for $\gamma=0$ the solution is just the fundamental solution, given by

$$\alpha = N/2, \quad \text{and} \quad f(\eta)=\exp(-\frac{x^2}{4t}), \tag{2.18}$$

which in the (y,t)-plane reads $y=2t$. The crossing of the $\{y=N\}$-line takes place at $t_*(0,N)=N/2$. As we now move the values continuously of γ and α, we see that the intersection point of each of the branches y_1 (where $y<2\alpha$) and y_2 (where $y>2\alpha$) with the line $y=2\alpha$ will move continuously. Observe that $H(t,2\alpha;\alpha)>0$ implies that the crossing is transversal. Coincidence of the branches, hence a solution of our problem, means that a certain point t_* we have the *matching conditions*

$$F_1 \equiv y_1(t_*;\gamma,\alpha)-2\alpha=0 \tag{2.19}$$

$$F_2 \equiv y_2(t_*;\gamma,\alpha)-y_1(t_*;\gamma,\alpha)=0\,. \tag{2.20}$$

Since both y_1 and y_2 are smooth functions of their arguments we can attempt to apply the Implicit Function Theorem in a neighbourhood of the values $t_*=N/2$, $\gamma=0$, $\alpha=N/2$ to obtain solutions for γ close to 0. Once the solution function is known locally we can continue it in γ if we make sure that the Implicit Function Theorem is valid in a neighbourhood of any solution.

The application of the Implicit Function Theorem depends on the value of the Jacobian determinant

$$J=\frac{\partial(F_1,F_2)}{\partial(t_*,\alpha)}=\frac{\partial y_1}{\partial t}\frac{\partial(y_1-y_2)}{\partial\alpha}-\frac{\partial(y_1-y_2)}{\partial t}(\frac{\partial y_1}{\partial\alpha}-1) \tag{2.21}$$

not being zero. This determinant has to be evaluated at the intersection point of the two branches of a solution, i.e. for given γ and α and for $t=t_*$, $y_i=2\alpha$. On the line $y=2\alpha$ formula (2.12) prescribes the same slope for both branches, hence $\partial(y_1-y_2)/\partial t=0$. Since $\partial y_1/\partial t=H(t_*,2\alpha;\alpha)$, J reduces to

$$J=\frac{\alpha(2\alpha+2-N)}{t_*}\left.\left(\frac{\partial(y_1-y_2)}{\partial\alpha}\right)\right|_{t=t_*}. \tag{2.22}$$

So, as long as $\alpha > 0$, $\alpha > (N-2)/2$, we only have to prove that the latter factor does not vanish. Let us compute the derivatives $\partial y_i/\partial\alpha$ at the point $t = t_*, y_i = 2\alpha$. Differentiating (2.12) with respect to the parameter α we get for $z_i = \partial y_i/\partial\alpha$ the equation

$$\frac{dz_i}{dt} = \Phi_i(t) z_i + 2\kappa_i\,, \quad \Phi_i(t) = \frac{2y_i(t) + 2 - N}{2t} - \kappa_i\,. \tag{2.23}$$

This is a linear equation that can be integrated explicitly to give

$$z_i(t) = C\Psi_i(t) + 2\kappa_i \Psi_i(t) \int^t \frac{ds}{\Psi_i(s)}\,, \tag{2.24}$$

where

$$\Psi_i(t) = \exp(\int^t \Phi_i(s) ds), \tag{2.25}$$

and the constant C has to be determined from the initial conditions. Let us examine z_1 in the interval $0 \le t \le t_*$. In view of (2.25), (2.23) and (2.14) we have for $t \approx 0$

$$\Psi_1(t) \approx t^{\frac{2-N}{2}} e^{(\frac{4\alpha}{N} - 1)\kappa_1 t}\,.$$

It is then easy to see that the only solution of (2.23) starting from (0,0) with a finite slope is given by formula (2.24) with $C = 0$ and integration in the last term from 0 to t, i.e.,

$$z_1(t) = 2\kappa_1 \Psi_1(t) \int_0^t \frac{ds}{\Psi_1(s)} > 0\,. \tag{2.26}$$

A similar analysis at infinity leads to the formula

$$z_2(t) = 2\kappa_2 \Psi_2(t) \int_\infty^t \frac{ds}{\Psi_2(s)} < 0\,. \tag{2.27}$$

It follows from these formulas that the factor we are looking for, $z_1(t_*) - z_2(t_*)$, is nonzero as desired (more precisely, it is positive).

The application of the IFT gives for $\partial\alpha/\partial\gamma$ the formula

$$\frac{\partial\alpha}{\partial\gamma} = -\frac{\partial(y_1 - y_2)/\partial\gamma}{\partial(y_1 - y_2)/\partial\alpha}, \tag{2.28}$$

with both derivatives computed at $t = t_*$, $y = 2\alpha$. Curiously, the numerator turns out to be very easy to compute for general γ. In fact, we can eliminate the explicit appearance of γ from equation (2.12) by making the change of time $\tau = \kappa t$, which transforms (2.12) into

$$\frac{dy}{d\tau} = (2\alpha - y) + \frac{1}{2\tau}\{y^2 - (N-2)y\}. \tag{2.29}$$

It follows that $y(t;\alpha,\kappa)=y(\kappa t;\alpha,1)$. Hence,

$$\frac{\partial y_i}{\partial\gamma}(t;\alpha,\kappa)=t\,\frac{\partial y_i}{\partial t}(\kappa t;\alpha,1)\,\frac{\partial\kappa_i}{\partial\gamma}\,, \tag{2.30}$$

so that

$$\frac{\partial y_1}{\partial\gamma}=-ty'(t)\quad\text{and}\quad\frac{\partial y_2}{\partial\gamma}=ty'(t)\,.$$

This means that the numerator of (2.28) becomes, in view of (2.12),

$$A=-2t_*y'(t_*)=-2\alpha(2\alpha+2-N)\,. \tag{2.31}$$

We see that (2.28) gives a positive value for the derivative $\partial\alpha/\partial\gamma$.

Therefore, starting from the known solution for $\gamma=0$ and $\alpha=N/2$ we can apply the IFT to obtain a *local solution branch*, $\{\alpha=\alpha(\gamma),\ y=y(t,\gamma,\alpha(\gamma))\}$, defined in an interval $\gamma\in I_0$ around $\gamma=0$, and α is a strictly monotone and analytic function of γ. Such solution is uniquely determined in its interval of existence.

D) Let us now discuss the question of *global existence*, i.e., the continuation of the solution to the maximal parameter interval $(-1,1)$. By the uniqueness just mentioned we can consider the maximal γ-interval $I\subset(-1,1)$ to which the solution can be extended in such a way that $\alpha(\gamma)>0$, $\alpha(\gamma)>(N-2)/2$, and $y(t,\gamma,\alpha(\gamma))$ is a solution of our problem for every $\gamma\in I$. By the IFT such an interval is open.

We can also prove that this interval is closed in $(-1,1)$, hence it must be the whole $(-1,1)$. For this we have to examine what happens when taking limits. Indeed, if we have a sequence of solutions y_n of our problem with corresponding values of the parameters $\gamma_n\to\tilde\gamma$ and $\alpha_n\to\tilde\alpha$, as long as $\tilde\gamma\in(-1,1)$ and $0<\tilde\alpha<\infty$, then there exists a function $\tilde y=\lim y_n$ which is a solution of (2.12) with initial value $\tilde y(0)=0$. Moreover, it follows from (2.15) that under the additional condition $\tilde\alpha>(N-2)/2$ such a solution crosses the line $y=2\alpha$ transversally, so that we can apply the IFT Method to continue the solution branch. Summing up, if we make sure that $\tilde\alpha$ is not 0, ∞ or $(N-2)/2$, continuation applies at any endpoint $\tilde\gamma$ interior to $(-1,1)$, and we conclude that the maximal interval is $(-1,1)$.

The above estimates on α have been obtained in [KPV] as a part of the construction of the solution. Verifying them directly for a limit $\tilde\alpha$ of our solution branch is an easy exercise about the trajectories of (2.12). Let us for instance show why a limit $\tilde\alpha$ cannot be $(N-2)/2$. If this is so, then the equation becomes

$$y'=\frac{1}{2t}(N-2-y)(2t\kappa-y)\,.$$

It is then clear from the flow portrait that $y=2(1-\gamma)t$ is a supersolution for the first branch, while $y=2(1+\gamma)t$ is a subsolution of the second. It follows that there is no possible intersection point. Moreover, such intersection is impossible by continuity for all sufficiently close parameters, γ and α.

In view if this the argument showing that α does not go to 0 has to be checked only for $N = 1$. Inspection of the flow portrait shows that as $\alpha \to 0$ the first branch collapses into a part of the axis $y = 0$ and the second branch takes on the whole y-range $0 < y < \infty$. It is also clear from the picture that for $1 + \gamma > 0$ the crossing time for the second branch must go to 0. However, this is not the case for the first branch; a simple supersolution argument shows that for small α there is a positive lower bound for the crossing time of this branch of the form $O(1/(1-\gamma))$, independent of α. Contradiction ensues. The proof that α does not diverge is easier usinf the fact that the slope $H \to \infty$ when $\alpha \to \infty$ for fixed $t > 0$, $y > 0$ and $\gamma \in (-1, 1)$, and also $y'(0) \to \infty$. ∎

Let us turn now to the actual computation of the variation of α and f with γ. We obtain the following result.

Theorem 2. *At $\gamma = 0$ we have*

$$\frac{\partial \alpha}{\partial \gamma}(0) = \frac{2(N/2)^{N/2}}{e^{N/2}\,\Gamma(N/2)}\,. \tag{2.32}$$

Proof: We use formula (2.28). Its numerator is given in (2.31). The computation of the denominator of (2.28) is done in part C) of the proof of Theorem 1. For the particular value $\gamma = 0$, when $\alpha = N/2$, $\kappa_i = 1$, $t_* = N/2$ and $y_i(t) = 2t$ the formulas become explicit. Indeed,

$$z_1(t_*) - z_2(t_*) = 2\Psi(t_*) \int_0^\infty \frac{ds}{\Psi(s)}\,, \tag{2.33}$$

where $\Psi(t) = t^{\frac{2-N}{2}}\, e^t$. It follows that

$$z_1(t_*) - z_2(t_*) = 2(N/2)^{\frac{2-N}{2}} e^{N/2} \Gamma(N/2)\,. \tag{2.34}$$

This gives finally formula (2.32). ∎

Remarks: 1) For $N = 1$, $\varepsilon = \gamma = 0$, we get the formula $\partial\alpha/\partial\varepsilon = 1/\sqrt{2e\pi}$ obtained by Goldenfeld and collaborators in [GMOL]. On the other hand, notice that expression (2.32) gives for large N the limit

$$\frac{\partial \alpha}{\partial \gamma}(0, N) \approx \frac{2}{\sqrt{\pi N}}\,.$$

2) We can perform a perturbation analysis for the profile f, or equivalently for y. Thus, we consider the function

$$z(t, \gamma) = \frac{\partial y}{\partial \gamma}\,, \tag{2.35}$$

where $y = y(t, \gamma, \alpha(\gamma))$, i.e. we have used Theorem 1 to calculate α as a function of γ so as to obtain a one-parameter family of solutions to our problem with parameter $\gamma \in (-1, 1)$. Then z satisfies with respect to t the ordinary differential equation

$$z' = \frac{\partial H}{\partial y} z + \frac{\partial H}{\partial \gamma} + \frac{\partial H}{\partial \alpha} . \alpha'(\gamma) .$$

Working out the details, and substituting the particular values $\gamma = 0$, $\alpha = N/2$, $y = 2t$, $\kappa = 1$, we get

(2.36)
$$z' = (1 + \frac{2-N}{2t}) z + 2\alpha'(0) \pm (2t - N) ,$$

with the $\pm$ taking different signs in the two intervals $0 \leq t \leq N/2$, $N/2 \leq t < \infty$. Looking at the initial conditions we find that $z(0) = 0$ and $z'(0) = -2 + 4\alpha'(0)/N$ (see formula (2.14)), and then z is given in the first interval by

$$z(t) = 2t^{\frac{2-N}{2}} e^t \int_0^t (\alpha'(0) + s - \frac{N}{2}) s^{\frac{N-2}{2}} e^{-s} ds ,$$

with the corresponding sign modification in the integrand for the interval $t \geq N/2$.

3) It is also possible to compute higher derivatives of the function $\alpha(\gamma)$ at $\gamma = 0$ and thus obtain the Taylor series of the function.

3. The Graveleau solutions for the porous medium equation

We take the porous medium equation

(3.1)
$$u_t = \Delta(u^m) , \qquad m > 1 ,$$

and look for a self-similar solution with a hole. It is most convenient to replace the density variable u by the pressure

(3.2)
$$v = \frac{m}{m-1} u^{m-1}$$

and write thus the equation as

(3.3)
$$v_t = (m-1) v \Delta v + |\nabla v|^2$$

with quadratic nonlinearities. Following [AG] we will take as origin of time the moment where the hole disappears, consider the self-similar solution $v = v(r, t)$ for $t < 0$ and $r = |x| \geq 0$, and write it in the form

(3.4)
$$v(x, t) = \frac{r^2}{-t} \phi(\xi) \quad \text{with} \quad \xi = (-t) r^{-\alpha} \geq 0 .$$

The problem consists in determining a constant $\alpha > 0$ (the anomalous exponent), and a function $\phi \geq 0$ (the profile) such that v is a solution of (3.3) and ϕ vanishes for $\xi \geq R$ and is positive for $0 < \xi < R$. In the case of one space dimension $N = 1$, there is an explicit solution

$$v(x,t) = (t + |x|)^+ \,, \tag{3.5}$$

which corresponds to the values $\alpha = 1$, $R = 1$ and

$$\phi = \xi - \xi^2 \,. \tag{3.6}$$

This solution has the nice property of not depending on m. The existence of such α, ϕ and R for $N > 1$ is proved in [AG]. Moreover, it so happens that $\alpha \in (1,2)$ for $N > 1$. This has a great interest for the regularity of solutions.

Our approach here will be to construct a solution branch in terms of a convenient parameter, starting from the known solution. We have two parameters to choose, namely the dimension N and the polytropical exponent m. We will work with N. Since we are dealing with radial symmetry there is no formal problem in treating non-integer dimensions. This is precisely what we shall do!

We first summarize the results we use from [AG].

[AG], Theorem. *There exists a unique function $\alpha = \alpha(N,m)$ defined for $N > 0$ and $m > 1$ such that equation (3.3) admits a nonnegative solution of the form (3.4) with a hole: $\phi(\xi) = 0$ if $\xi \geq R(N,m) > 0$. Moreover,*

$$\frac{2((m-1)N+2)}{(m-1)(N+2)+4} < \alpha(N,m) < 2 \,, \tag{3.7}$$

so that $\alpha(N,m) \to 2$ as $N \to \infty$.

Estimate (3.7) gives a very precise information on the range of variation of α. In fact, in our construction we will only need to know that α is a priori bounded away from 0 and infinity. Our result is the following

Theorem 3. *The anomalous exponent $\alpha = \alpha(N,m)$ is defined and analytic in the domain $\{N > 0, m > 1\}$. The profile ϕ is also an analytic function of N and m.*

PROOF: **A)** We begin by writing the equation for ϕ in terms of $\xi > 0$:

$$\frac{1}{\xi^2}\phi - \frac{1}{\xi}\phi' = \frac{A}{\xi^2}\phi^2 - \frac{\alpha}{\xi}(B - \mu\alpha)\phi\phi' + \mu\alpha^2\phi\phi'' + \alpha^2(\phi')^2, \tag{3.8}$$

with

$$\mu = m - 1, \quad A = 2(\mu N + 2) \quad \text{and} \quad B = \mu(N+2) + 4 \,. \tag{3.9}$$

We then transform this equation into an autonomous dynamical system by introducing the variable

$$\theta(\xi) = -\alpha\xi\phi'(\xi)\,. \tag{3.10}$$

We get the system

$$\begin{cases} \text{(a)} & \alpha\xi\phi' = -\theta \\ \text{(b)} & \mu\alpha\phi\xi\theta' = -(\phi + (\theta/\alpha) - \theta^2 - A\phi^2 - B\theta\phi). \end{cases} \tag{3.11}$$

After a suitable reparametrization of the form

$$\frac{d\tau}{d\xi} = \frac{1}{\alpha\xi\phi},$$

the system is regularized into a quadratic system and we find that it has three stationary points, namely $P_1 = (0,0)$, $P_2 = (0, 1/\alpha)$ and $P_3 = (1/A, 0)$. The corresponding equation in the phase-plane (ϕ, θ) is

$$\frac{d\theta}{d\phi} = H(\phi, \theta; N, \mu, \alpha) \equiv \frac{1}{\mu\phi\theta}(\phi + (\theta/\alpha) - \theta^2 - A\phi^2 - B\theta\phi)\,. \tag{3.12}$$

Only the half-plane $\phi \geq 0$ will be taken into account. We remark that the vector field H is monotone decreasing in α since

$$\frac{\partial H}{\partial \alpha} = -\frac{1}{\mu\alpha^2\phi}\,. \tag{3.13}$$

The situation is not so easy with respect to N. In fact,

$$\frac{\partial H}{\partial N} = -\frac{2\phi + \theta}{\theta}\,. \tag{3.14}$$

It is shown in [AG] that $\theta > -\alpha\phi$ and $\alpha < 2$, therefore the numerator $2\phi + \theta$ in (3.14) is always positive and the sign of the derivative is opposite to that of θ.

It has been proved in [AG] that the solution we are looking for corresponds to a trajectory joining the stationary points P_1 and P_2. The trajectory starts at $\xi = 0$ from $P_1 = (0,0)$, along the center-unstable manifold with slope $-\alpha$, and travels with increasing ϕ and first decreasing and then increasing θ until it reaches the ϕ-axis at a point $Q = (l, 0)$ at a value ξ_1. The point Q is located to the right of P_3, i.e. $l > 1/A$. The trajectory then enters the quadrant $\{\phi > 0, \theta > 0\}$ and travels monotonically in ϕ and ξ towards the θ-axis, which is attained at P_2 at a time ξ_2. It enters P_2 through the stable manifold which has slope $(\alpha - B)/m$.

B) Our idea is to shoot, for given N, m and α, from both end-points P_1 and P_2 (there is only one possibility, namely along the respective center-unstable and stable manifolds) until the branches reach the ϕ-axis. If they meet there we have obtained a solution. This is where the value of α comes into play.

To start with, we know that such branches are given when $N = 1$ by the parametric representation

$$\phi = \xi - \xi^2\,, \quad \theta = -\xi + 2\xi^2\,, \tag{3.15}$$

with corresponding $\alpha = 1$, $Q = (1/4, 0)$, $\xi_1 = 1/2$ and $\xi_2 = 1$.

Let $\Gamma_1 = \{(\phi, \theta_1(\phi)) : 0 \le \phi \le l_1\}$ be the lower branch starting from P_1 and ending at $Q_1 = (l_1, 0)$ and let $\Gamma_2 = \{(\phi, \theta_2(\phi)) : 0 \le \phi \le l_2\}$ be the upper branch, joining P_2 with $Q_2 = (l_2, 0)$. We have $\theta_1(\phi) \le 0$, $\theta_2(\phi) \ge 0$. The *matching condition* reads simply

$$l_1(N, \alpha, \mu) = l_2(N, \alpha, \mu)\,. \tag{3.16}$$

Both branches Γ_1 and Γ_2 are given by analytic functions of ϕ, N, α and m. The analyticity is well-known for Γ_2 since it is the stable manifold of a hyperbolic rest point, cf. [GH], [Pk]. On the other hand, since Γ_1 is the center-unstable manifold of a degenerate rest point (saddle-node), its analyticity is not so straightforward. Write (3.12) in the form

$$\mu\phi\theta\frac{d\theta}{d\phi} = \phi + (\theta/\alpha) - \theta^2 - A\phi^2 - B\theta\phi\,,$$

where A and B are polynomials in μ, N, as given by (3.9). This equation has a unique formal solution of the form

$$\theta(\phi; \alpha, N, \mu) = \sum_{k=1}^{\infty} \sigma_k(\alpha, N, \mu)\phi^k\,,$$

where $\sigma_1(\alpha, N, \mu) = -\alpha$ and the remaining σ_k's are polynomials in their arguments. By the results of Sibuya [S] (see also [RS]) there exists an actual solution, $\Theta(\phi; \alpha, N, \mu)$, which is analytic in a domain

$$\begin{aligned}\mathcal{D} = \{(\phi, \alpha, N, \mu) : &|\arg(\phi)| < \delta_1, |\arg(\alpha)| < \delta_2, |\arg(\mu)| < \delta_4,\\ &0 < |\phi| < M_1, 0 < |\alpha| < M_2, 0 < |N| < M_3, 0 < |\mu| < M_4\}\,,\end{aligned}$$

and which has the property that for each positive integer P there exists a positive constant $K(P)$ such that

$$|\Theta(\phi; \alpha, N, \mu) - \sum_{k=0}^{P} \sigma_k(\alpha, N, \mu)\phi^k| \le K(P)\phi^{P+1}$$

in $\mathcal{D}$. Here the δ_j are small, while M_2, M_3 and M_4 can be arbitrarily large provided that M_1 be sufficiently small.

Let us continue the proof. In order to apply the IFT Method we need one more fact, namely the *transversality condition* at the matching point

$$\frac{\partial(l_1 - l_2)}{\partial\alpha} \neq 0\,, \tag{3.17}$$

the derivative being computed at a solution orbit joining P_1 to P_2. That will be proved in **C**. Granted these facts, we construct a function $\alpha = \alpha(N, m)$ defined for $N > 0$ and $m > 1$ such that a solution of our problem exists for the every parameter triple (N, m, α). Since the construction will be done for fixed m, we may safely omit all reference to m in the argument to follow.

We know the solution for $N = 1$, $\alpha = 1$. Thanks to the Implicit Function Theorem, equation (3.16) can be solved in a neighbourhood of the starting values $(N = 1, \alpha = 1)$ and we will obtain an analytic function $\alpha = \alpha(N)$, defined for $N \approx 1$ and such that (3.16) holds, thus producing a solution of our original problem. In particular, we will have

$$\frac{d\alpha}{dN} = -\frac{\partial l/\partial N}{\partial l/\partial\alpha}\,, \quad l = l_1 - l_2\,. \tag{3.18}$$

Now we consider the maximal subinterval I of the half-line $L = (0, \infty)$ in which we can define a solution $\alpha = \alpha(N)$. Let us prove that $I = L$. Clearly, I is a well-defined open interval since by the Implicit Function Theorem, equation (3.18) and the a priori estimate for α, a solution with parameters (N_*, α_*) can be continued in a unique way to a neighbourhood of N_*. But we can also prove that it is a closed interval. For that it is sufficient to know that α remains always bounded away from 0 and infinity, a consequence of the a priori estimate (3.7). Indeed, if this is so and $N_* = \sup(I)$ then we have solutions for a sequence (N_n, α_n) with $N_n \to N_*$. By our assumption we also have along some subsequence $a_n \to \alpha_*$ with $0 < \alpha_* < \infty$. It is then clear taking limits in (3.12) that our problem admits a solution with exponent α_* for $N = N_*$. The same happens at $\inf(I)$. This ends the proof that the solution can be continued to all the half-line L.

C) Let us now make sure that our claim (3.17) is valid. Computing derivatives of l_i, $i = 1$ or 2, with respect to N or α directly is not convenient because of the singular behaviour at $\theta = 0$. However, we can compute them indirectly by computing the derivatives of θ_i^2 at the endpoints l_i. This is done as follows. ¿From equation (3.12) we can conclude that near an intersection with the ϕ-axis at a point $(l, 0)$ different from P_3 we have

$$\theta\, d\theta \approx \frac{1 - A\,l}{\mu} d\phi\,, \tag{3.19}$$

from which it follows that for our branches

$$\frac{\partial l_i}{\partial N}(N_0, \alpha_0) = \frac{\mu}{2(Al_i^0 - 1)}\frac{\partial\theta^2}{\partial N}(l_i^0; N_0, \alpha_0)\,, \tag{3.20}$$

where $l_i^0 = l_i(N_0, \alpha_0)$, and similarly

$$\frac{\partial l_i}{\partial\alpha}(N_0, \alpha_0) = \frac{\mu}{2(Al_i^0 - 1)}\frac{\partial\theta^2}{\partial\alpha}(l_i^0; N_0, \alpha_0)\,. \tag{3.21}$$

D) We now proceed with the computation of $\partial\theta_i^2/\partial\alpha$ at the common endpoint $\phi = l$ of two meeting branches. We introduce the functions

$$z_i(\phi) = \frac{\partial\theta_i^2(\phi; N, \alpha)}{\partial\alpha} \tag{3.22}$$

defined in the interval $I = \{0 < \phi < l\}$. It is easy to see that, as a function of ϕ, z_i satisfies the linear ODE

$$\begin{aligned} \frac{dz_i}{d\phi} &= 2\frac{\partial\theta}{\partial\alpha}H + 2\theta(\frac{\partial H}{\partial\alpha} + \frac{\partial H}{\partial\theta}\frac{\partial\theta}{\partial\alpha}) \\ &= z(\frac{H}{\theta} + \frac{\partial H}{\partial\theta}) + 2\theta\,\frac{\partial H}{\partial\alpha}\,. \end{aligned} \tag{3.23}$$

Working out the details we get

$$\frac{dz_i}{d\phi} = F_i(\phi)z_i + G_i(\phi) \tag{3.24}$$

with

$$F_i(\phi) = \frac{(1/\alpha) - B\phi - 2\theta_i}{\mu\phi\theta_i}\,, \quad G_i(\phi) = \frac{-2\theta_i}{\mu\alpha^2\phi}\,. \tag{3.25}$$

Moreover, since $\theta_1(0) = 0$ we get the initial condition for z_1: $z_1(0) = 0$. On the other hand, $\theta_2(0) = 1/\alpha$, hence $z_2(0) = -2/\alpha^3$. The inspection of the dependence of H with respect to α done before showed that $\partial H/\partial\alpha < 0$ for $\phi > 0$. Therefore, $\partial\theta_i/\partial\alpha < 0$ and, consequently $z_1 \geq 0$ since $\theta_1 < 0$, while $z_2 \leq 0$ for $\theta_2 > 0$. This can also be seen from the fact that G has the same sign as $-\theta$. Since $G \neq 0$ in $0 < \phi < l$ we conclude that strict inequality, $z_1 > 0 > z_2$, holds inside the interval I. But, can the solutions z_i be continued up to $\phi = l$ and are the limit values different from 0? The answer is yes, and it all depends on two facts: the sign of G and the fact that F is integrable near $\phi = l$ (since $\theta \approx c(l-\phi)^{1/2}$ there, see (3.19)). In fact, we can write (3.24) as

$$(z_i\,\exp(-\int^\phi F_i(s)ds))' = G_i\,\exp(-\int^\phi F_i(s)ds)\,. \tag{3.26}$$

We immediately see that the function $\tilde z_i = z_i\,\exp(-\int^\phi F_i(s)ds)$ is monotone, hence it has a limit. Moreover, $\tilde z_1$ is positive and increasing, hence the limit is positive. Since G_i is bounded near $\phi = l$ (actually, it goes to zero), the limit is finite. In conclusion, we have proved that there exists $a_1 > 0$ such that

$$\lim_{\phi\to l} z_1(\phi) = a_1\,.$$

In the same way we prove that there exists $a_2 < 0$ such that

$$\lim_{\phi\to l} z_2(\phi) = a_2\,.$$

In this way, recalling (3.21) we get with $[z] = a_1 - a_2$, i.e. the jump between the two branches at $\phi = l$

$$\frac{\partial l}{\partial \alpha} = \frac{\mu}{2(Al-1)}[z] > 0\,. \tag{3.27}$$

E) The computation of the derivative with respect to N proceeds in a similar way. We introduce the functions

$$w_i(\phi) = \frac{\partial \theta_i^2(\phi; N, \alpha)}{\partial N}\,, \tag{3.28}$$

defined in the interval $I = \{0 < \phi < l\}$. As a function of ϕ, w_i satisfies the linear ODE

$$\frac{dw_i}{d\phi} = w_i\left(\frac{H}{\theta} + \frac{\partial H}{\partial \theta}\right) + 2\theta\,\frac{\partial H}{\partial N} = F_i(\phi)w_i + K_i(\phi)\ , \tag{3.29}$$

where F_i is as in (3.25) and K_i is given by

$$K_i(\phi) = -2(2\phi + \theta_i)\,. \tag{3.30}$$

Again K_i is a bounded function and F_i is integrable, hence we obtain a limit for the functions w_i as $\phi \to l$

$$\lim_{\phi \to l} w_i(\phi) = b_i\,.$$

As in (3.27) we conclude that if we set $[w] = b_1 - b_2$

$$\frac{\partial l}{\partial N} = \frac{\mu}{2(Al-1)}[w]\,. \tag{3.31}$$

This gives for the derivative of α with respect to N

$$\frac{d\alpha}{dN} = -\frac{[w]}{[z]}\,. \tag{3.32}$$

Our proof of existence and analytic dependence is finished. ▮.

An interesting question is the monotonicity of the function $\alpha = \alpha(N, m)$. Numerical experiments done by I.G. Kevrekidis [Ks] in Princeton suggest that α is monotonically increasing in N (and also in m for $N \geq 1$). We can try to use formula (3.32) to prove that $\partial\alpha/\partial N > 0$. We know that $[z] > 0$. However, the sign of $[w]$ is not so easy. If we look at the branch w_2 the signs are clear: $w_2(0) = 0$ and $K_2 < 0$ in I , so that b_2 is negative. As for w_1, we can show that K_1 is also negative, hence w_1 is negative and $b_1 < 0$. But, is $b_2 - b_1 = -[w]$ positive? We have been unable to settle that question in general. However, we will prove next that this is true for $N \approx 1$.

Theorem 4. *Let $N = 1$ and $m > 1$. Then, $\alpha(1, m) = 1$ and*

$$\frac{\partial \alpha}{\partial N}(1, m) = 2 \int_0^\infty \frac{e^{-t} dt}{(1 + \mu t)^{\frac{1}{\mu}}} - 1\,, \tag{3.33}$$

where $\mu = m - 1$. Thus, the derivative is positive, increasing in m and its range is the interval $(0, 1)$. In particular, for $N = 1$, $m \approx 1$ we have $\partial\alpha/\partial N = \frac{1}{4}(m - 1) + o(m - 1)$.

Remark: The integral in (3.33) can be computed in terms of incomplete Gamma functions as

$$\int_0^\infty \frac{e^{-t} dt}{(1 + \mu t)^{\frac{1}{\mu}}} = \mu^{-\frac{1}{\mu}} e^{\frac{1}{\mu}} \Gamma(1 - \frac{1}{\mu}, \frac{1}{\mu})\,. \tag{3.34}$$

Such functions can be found for example in Mathematica [W].

Proof: It is just a question of computing explicitly the formulas for $[z]$ and $[w]$.
(i) *General formulas.* The actual computation of $[z]$ and $[w]$ is best done after changing the independent variable ϕ in (3.24), (3.29) into ξ using (3.11.a). We first do this job for general N and m. We get

$$\frac{dz}{d\xi} = f(\xi) z + g(\xi)\,, \tag{3.35.a}$$

$$\frac{dw}{d\xi} = f(\xi) w + k(\xi)\,, \tag{3.35.b}$$

with

$$f(\xi) = \phi'(\xi) F(\phi(\xi), \theta(\xi)) = \frac{B\phi + 2\theta - (1/\alpha)}{\alpha\mu\phi\xi} \tag{3.36.a}$$

$$g(\xi) = \phi'(\xi) G(\phi(\xi), \theta(\xi)) = \frac{2\theta^2}{\alpha^3\mu\phi\xi} \tag{3.36.b}$$

$$k(\xi) = \phi'(\xi) K(\phi(\xi), \theta(\xi)) = \frac{2\theta(2\phi + \theta)}{\alpha\xi}. \tag{3.36.c}$$

Here we do not need to label the branches because they are represented by disjoint and adjoining ξ-intervals. The first branch from $\xi = 0$ to $\xi = \xi_1$ (where ϕ takes a maximum), and the second branch from $\xi = \xi_1$ to $\xi = \xi_e$, where ϕ is again zero. Observe that near $\xi = 0$ we have $\phi \approx c\xi$ and $\theta \approx -\alpha c\xi$, hence

$$f(\xi) \approx -\frac{1}{\mu\alpha^2 c\xi^2}, \quad g(\xi) \approx \frac{2c}{\mu\alpha}, \qquad k(\xi) \approx -2(2 - \alpha)c^2\xi,$$

while near the endpoint $\xi = \xi_e$ we have and $\theta = 1/\alpha$, $\phi = 0$ and $\phi' = -1/(\alpha^2\xi_e)$, so that

$$f(\xi) \approx \frac{1/\mu}{\xi_e - \xi}, \quad g(\xi) \approx \frac{2/(\mu\alpha^3)}{\xi_e - \xi}, \qquad k(\xi) \approx \frac{2}{\alpha^3\xi_e}.$$

Finally, notice that there is no singularity at the intermediate point ξ_1. Let us introduce for convenience the function

$$h(\xi) \equiv \exp(-\int_c^{\xi} f(s)ds). \tag{3.37}$$

for some $c \in (0, \xi_e)$. The best choice is $c = \xi_1$. We see that h is positive in $(0, \xi_e)$, it goes to zero at $\xi = 0$ and $\xi = \xi_e$. In fact, it behaves like $O(\exp(-c/\xi))$ at $\xi = 0$, and like $(\xi_e - \xi)^{1/\mu}$ for $\xi \approx \xi_e$.

With these observations we proceed with the integration of (3.35.a). Taking initial condition $z(0) = 0$ we get

$$z_1(\xi) = \frac{1}{h(\xi)} \int_0^{\xi} g(s)h(s)ds \tag{3.38.a}$$

which corresponds to the first branch. For the second branch we work out the variation of parameters formula and put $C = 0$ to obtain

$$z_2(\xi) = \frac{1}{h(\xi)} \int_{\xi_e}^{\xi} g(s)h(s)ds\,, \tag{3.38.b}$$

which satisfies the end value $z_2(\xi_e) = -2/\alpha^3$, as can be verified by L'Hôpital's rule

$$\lim_{\xi \to \xi_e} z_2(\xi) = -\lim_{\xi \to \xi_e} \frac{g(\xi)}{f(\xi)} = -\frac{2}{\alpha^3}.$$

This means that

$$[z] = \frac{1}{h(\xi_1)} \int_0^{\xi_e} g(s)h(s)ds. \tag{3.39}$$

Choosing $c = \xi_1$ in (3.40) we get the simpler form

$$[z] = \int_0^{\xi_e} g(s)h(s)ds. \tag{3.40}$$

It is crucial that *this quantity does not vanish.* Actually, it is positive since g and h are.

As for the computation of $[w]$ we get from (3.35.b) the formulas

$$w_1(\xi) = \frac{1}{h(\xi)} \int_0^{\xi} k(s)h(s)ds \tag{3.41.a}$$

corresponding to $w_1(0) = 0$, and

$$w_2(\xi) = \frac{1}{h(\xi)} \int_{\xi_e}^{\xi} k(s)h(s)ds, \tag{3.41.b}$$

corresponding to $w(\xi_e) = 0$. Hence

$$[w] = \frac{1}{h(\xi_1)} \int_0^{\xi_e} k(s)h(s)ds. \tag{3.42}$$

and the denominator disappears if $c = \xi_1$. Therefore,

$$\frac{\partial \alpha}{\partial N} = -\frac{\int_0^{\xi_e} k(s)h(s)ds}{\int_0^{\xi_e} g(s)h(s)ds}. \qquad \blacksquare \tag{3.43}$$

(ii) *Computing at* $N = 1$. When $N = 1$ we know that $\alpha = 1$, and we also have an explicit formula for the solution from which our branches start. Using the parametric representation (3.15) for the branches, i.e.

$$\phi = \xi - \xi^2, \quad \theta = -\xi + 2\xi^2,$$

we get $\xi_e = 1$, $\xi_1 = 1/2$ and

$$\begin{aligned} f(\xi) &= \frac{-3\mu\xi^2 + (3\mu + 2)\xi - 1}{\mu\xi^2(1-\xi)} \\ &= \frac{3 + 1/\mu}{\xi} - \frac{1}{\mu\xi^2} + \frac{1}{\mu(1-\xi)}, \end{aligned} \tag{3.44.a}$$

and

$$g(\xi) = (1 - 2\xi)G(\phi(\xi)) = \frac{2(1-2\xi)^2}{\mu(1-\xi)}, \tag{3.44.b}$$

$$k(\xi) = (1 - 2\xi)K(\phi(\xi)) = -2\xi(1 - 2\xi). \tag{3.44.c}$$

The first branch corresponds here to $0 \leq \xi \leq 1/2$ and the second to $1/2 \leq \xi \leq 1$. Now the above formulas become explicit:

$$z_1(\xi) = \frac{2}{\mu}\xi^{3+\frac{1}{\mu}}(1-\xi)^{-\frac{1}{\mu}} e^{\frac{1}{\mu\xi}} \int_0^{\xi} \frac{(1-2s)^2}{(1-s)^{1-\frac{1}{\mu}}} s^{-3-\frac{1}{\mu}} e^{-\frac{1}{\mu s}}\, ds, \tag{3.45.a}$$

$$z_2(\xi) = \frac{2}{\mu}\xi^{3+\frac{1}{\mu}}(1-\xi)^{-\frac{1}{\mu}} e^{\frac{1}{\mu\xi}} \int_1^{\xi} \frac{(1-2s)^2}{(1-s)^{1-\frac{1}{\mu}}} s^{-3-\frac{1}{\mu}} e^{-\frac{1}{\mu s}}\, ds, \tag{3.45.b}$$

and

$$w_1(\xi) = -2\xi^{3+\frac{1}{\mu}}(1-\xi)^{-\frac{1}{\mu}} e^{\frac{1}{\mu\xi}} \int_0^{\xi} (1-2s)s^{-2-\frac{1}{\mu}}(1-s)^{\frac{1}{\mu}} e^{-\frac{1}{\mu s}}\, ds, \tag{3.46.a}$$

$$w_2(\xi) = -2\xi^{3+\frac{1}{\mu}}(1-\xi)^{-\frac{1}{\mu}} e^{\frac{1}{\mu\xi}} \int_1^{\xi} (1-2s)s^{-2-\frac{1}{\mu}}(1-s)^{\frac{1}{\mu}} e^{-\frac{1}{\mu s}}\, ds. \tag{3.46.b}$$

Putting $\xi = 1/2$ we get

$$(3.47)\qquad [z] = \frac{e^{\frac{2}{\mu}}}{4\mu}\int_0^1 \frac{(1-2\xi)^2}{(1-\xi)^{1-\frac{1}{\mu}}}\xi^{-3-\frac{1}{\mu}}e^{-\frac{1}{\mu\xi}}\,d\xi = \frac{e^{\frac{2}{\mu}}}{4\mu}\int_1^\infty \frac{(s-2)^2}{(s-1)^{1-\frac{1}{\mu}}}e^{-\frac{s}{\mu}}\,ds > 0\,.$$

In fact, putting $s = t+1$, the integral can be computed in terms of Γ functions. We get

$$(3.48)\qquad [z] = \frac{1}{4}\mu^{\frac{1}{\mu}}e^{\frac{1}{\mu}}\Gamma(\frac{1}{\mu})\,.$$

Observe that this quantity is positive. On the other hand,

$$(3.49)\qquad -[w] = \frac{e^{\frac{2}{\mu}}}{4}\int_0^1 \frac{1-2\xi}{\xi^{2+\frac{1}{\mu}}}(1-\xi)^{\frac{1}{\mu}}e^{-\frac{1}{\mu\xi}}\,d\xi = \frac{e^{\frac{2}{\mu}}}{4}\int_1^\infty (s-2)(s-1)^{\frac{1}{\mu}}e^{-\frac{s}{\mu}}\,\frac{ds}{s}\,.$$

Now the integral is not so easy to evaluate.

Lemma 3.3. *We have*

$$(3.50)\qquad \int_1^\infty (s-2)(s-1)^{\frac{1}{\mu}}e^{-\frac{s}{\mu}}\,\frac{ds}{s} = \mu^{\frac{1}{\mu}}e^{-\frac{1}{\mu}}\Gamma(\frac{1}{\mu})\left\{2\int_0^\infty \frac{e^{-t}dt}{(1+\mu t)^{\frac{1}{\mu}}} - 1\right\}.$$

Accepting for a moment this result, we get

$$(3.51)\qquad -[w] = \frac{1}{4}\mu^{\frac{1}{\mu}}e^{\frac{1}{\mu}}\Gamma(\frac{1}{\mu})\left\{2\int_0^\infty \frac{e^{-t}dt}{(1+\mu t)^{\frac{1}{\mu}}} - 1\right\}.$$

Finally, we use formula (3.32) and immediately get (3.33).

Proof of Lemma 3.3: Let us put $\lambda = 1/\mu$. We introduce the function

$$(3.52)\qquad I(r,\lambda) = \int_1^\infty (s-2)(s-1)^r e^{-\lambda s}\,\frac{ds}{s}\,.$$

This integral is well-defined for r and $\lambda > 0$. Differentiating with respect to λ we get

$$\frac{\partial I}{\partial \lambda} = -\int (s-2)(s-1)^r e^{-\lambda s}\,ds\,,$$

which can be worked out in terms of Γ functions to give

$$\frac{\partial I}{\partial \lambda} = e^{-\lambda}\left(\frac{\Gamma(r+1)}{\lambda^{r+1}} - \frac{\Gamma(r+2)}{\lambda^{r+2}}\right)\,.$$

Integrating this formula from in λ from λ to ∞, and taking into account that $I(r,\infty) = 0$, we arrive at

$$(3.53)\qquad I(r,\lambda) = \Gamma(r+2)\int_\lambda^\infty \frac{e^{-s}}{s^{r+2}}ds - \Gamma(r+1)\int_\lambda^\infty \frac{e^{-s}}{s^{r+1}}ds.$$

We are interested in $I(\lambda, \lambda)$. We use (3.53), $\Gamma(r+2) = (r+1)\Gamma(r+1)$, put $s = t + \lambda$ and integrate by parts to obtain

$$I(r, \lambda) = \frac{e^{-\lambda}\Gamma(r+1)}{\lambda^{r+1}} \left\{ 1 - 2\int_0^\infty \frac{e^{-t}dt}{(1+\frac{1}{\lambda}t)^{r+1}} dt \right\}.$$

Now we put $r = \lambda$ and integrate by parts a second time to obtain (3.50). ∎

Let us now analyze the behaviour of the function

$$f(\mu) = 2\int_0^\infty \frac{e^{-t}dt}{(1+\mu t)^{\frac{1}{\mu}}} - 1\,, \tag{3.54}$$

which gives the derivative $\partial\alpha/\partial N$ for $N = 1$. The derivative is positive. This can be shown by recalling the fact that for every $n > 0$ $(1+n)^{1/n} < e^n$. Applying this to the integral in (3.52) with $n = \mu t$ we get

$$\int_0^\infty \frac{e^{-t}dt}{(1+\mu t)^{\frac{1}{\mu}}} > \int_0^\infty e^{-2t}dt = \frac{1}{2}.$$

Moreover, it is clear that as $\mu \to 0$ we have

$$f(\mu) \to 2\int_0^\infty e^{-2t}dt - 1 = 0,$$

while for $\mu \to \infty$ we have with $s = \mu t = t/\varepsilon$

$$f(\mu) = 2\int_0^\infty \frac{e^{-\varepsilon s}\varepsilon ds}{(1+s)^{\frac{1}{\mu}}} - 1 \approx 2\int_0^\infty e^{-\varepsilon s}\varepsilon ds - 1 \to 1.$$

The fact that f is increasing can be verified immediately by differentiation

$$f'(\mu) = \frac{2}{\mu^2}\int_0^\infty \frac{e^{-t}}{(1+\mu t)^{\frac{1}{\mu}+1}}\{(1+\mu t)\log(1+\mu t) - \mu t\}dt.$$

Note that the integrand is positive. In particular, when when $m \to 1$ we get

$$\lim_{\mu\to 0} f'(\mu) = \int_0^\infty e^{-2t}t^2 dt = \frac{1}{4}.$$

This ends the proof of Theorem 4. ∎

Regarding the expression for the integral in $f(\mu)$ given in (3.34), let us recall that the Incomplete Gamma Function is defined as

$$\Gamma(a, z) = \int_z^\infty s^{a-1}e^{-s}ds.$$

We can write

$$\int_0^\infty \frac{e^{-t}dt}{(1+\mu t)^{\frac{1}{\mu}}} = \mu^{-\frac{1}{\mu}}e^{\frac{1}{\mu}}\int_{\frac{1}{\mu}}^\infty s^{-\frac{1}{\mu}}e^{-s}ds,$$

by making the change of variables $t + (1/\mu) = s$. (3.34) follows.

4. Further application of the IFT Method

We will now devote some space to present several problems of continuum mechanics to which the method has been successfully applied.

(1) A first example is provided by the *travelling wave with minimal speed* for the *porous-Fisher equation*,

$$u_t = (u^m)_{xx} + u(1-u), \tag{4.1}$$

a nonlinear reaction-diffusion variant of the famous KPP model which is described in [A3]. Here we want to find a travelling wave

$$u = u(x-ct)\,, \tag{4.2}$$

joining the level $u = 1$ at $x = -\infty$ with $u = 0$ for all large x. The speed c and profile of the wave have to be determined as functions of m. The application of the IFT Method with $\xi = x - ct$, $q(\xi) = u^{m-1}$ and $p(\xi) = q'(\xi)$ gives a system of ODE's. We start from a known explicit solution for $m = 2$, $c = 1$, given by

$$p = \frac{1}{2}(q-1)\,. \tag{4.3}$$

Working out the details we get the precise value

$$\left.\frac{\partial c}{\partial m}\right|_{m=2} = -\frac{7}{24}. \tag{4.4}$$

Work on this problem is being done by E. Durand at the University of Minnesota, [D].

(2) Let us now consider the fundamental solutions of the *modified porous medium equation*,

$$u_t + |\gamma| u_t = \Delta u^m\,. \quad -1 < \gamma < 1, m > 1\,. \tag{4.5}$$

[B2] derives this equation for the motion of a groundwater mound through a porous medium which overlies an impermeable horizontal stratum. Here u is the pressure, $m = 2$ and $\gamma > 0$. Observe that this equation combines the two main models discussed in the Introduction. Here we can take as the driving parameter γ and start from the standard Barenblatt solutions for $\gamma = 0$, $m \neq 1$. The IFT Method analysis is due to Hulshof and Vazquez (and MapleV), cf. [HV]. A renormalization analysis has been performed by Chen, Goldenfeld and Oono, [CGO].

(3) A model for the propagation of turbulent bursts in the ocean is, according to [B3], the following nonlocal evolution equation

$$u_t = l(t)(u^{3/2})_{xx} - \kappa\frac{u^{3/2}}{l(t)}, \tag{4.6}$$

called the *turbulent burst equation*, [B3], [BGL]. Of interest are solutions having compact support in space for all $t > 0$; the data are radially symmetric and $l(t)$ is the radius of the support of the turbulent density $u(x,t)$. We look for source-type solutions, the driving parameter is κ and the starting solution is again the Barenblatt solution for $\kappa = 0$. References to work on this problem are [HP], [KV]; the IFTM analysis is worked out in [QV]. It is worth noting that though the physical problem was posed for $\kappa > 0$, the mathematical analysis produces a *global branch* in the parameter interval $\kappa_* < \kappa < \infty$, $\kappa_* < 0$, thus including a reaction range with $\kappa < 0$. It ends on the left with an asymptote for a finite $\kappa = \kappa_*$, which marks the transition to blow-up regimes. The analysis is actually done for general diffusion exponent $m > 1$ (instead of just 3/2), and dimension $N \geq 1$.

(4) The k-ε model of turbulence. This is a system of two equations for the turbulent energy density and the turbulent energy dissipation rate, [Ko], [B3], [BDK],

$$(4.7)\qquad \begin{aligned} k_t &= \alpha\left(\frac{k^2}{\varepsilon}k_x\right)_x - \varepsilon \\ \varepsilon_t &= \beta\left(\frac{k^2}{\varepsilon}\varepsilon_x\right)_x - \gamma\frac{\varepsilon^2}{k}, \end{aligned}$$

where α, β and γ are positive parameters, $\gamma > 3/2$. This is probably to date the most difficult example of application of our technique, because, being a system, it involves the study of a four-dimensional phase plane. The detailed description is contained in a very recent paper by Hulshof [H]. Starting point is the self-similar solution obtained by Barenblatt and collaborators [BGL] for parameter values $\alpha = \beta$. Self-similar solutions are now constructed for a certain range of $\alpha \neq \beta$. The analysis performed is only local around the solution for $\alpha = \beta$. The question of constructing a global branch is open and seems to be a very difficult problem.

(5) The *dipole solutions* for the elasto-plastic model (1.2) in one space dimension. Here we want to find the self-similar solutions which describe the large-time behaviour of all solutions with antisymmetric and integrable initial data, positive in $0 < x < \infty$. This is equivalent to solving the Dirichlet problem in a quadrant $\{(x,t) : x > 0, t > 0\}$ with $u = 0$ on the lateral boundary $x = 0$, $t > 0$. For $\gamma = 0$ the asymptotic behaviour is given by the derivative in x of the fundamental solution of the heat equation, which is a self-similar solution with initial data $u(x,0) = \delta'(x)$ (a dipole). We then have similarity of the first kind, corresponding to the conservation of the first moment.

For $\gamma \neq 0$ we have an anomalous exponent. The application of the IFT Method to this example turns out to be a variation of Section 2. Let us briefly review the details for the reader's convenience. With the notation of Section 2 the explicit dipole solution for $\gamma = 0$ is given by

$$(4.8)\qquad u(x,t) = xt^{-3/2}\exp(-|x|^2/4t),$$

which has the self-similar form (1.1) with $\alpha = 1$, $\beta = 1/2$ and $f(\eta) = \eta\exp(-\eta^2/4)$. With the notations (2.12) this corresponds to

$$(4.9)\qquad y = 2t - 1\,.$$

The solution we are looking for when $\gamma \neq 0$ is given in the (y,t)-plane by a trajectory of the same equation as in Section 2, namely,

$$y' = \kappa(2\alpha - y) + \frac{1}{2t}(y + y^2)\,, \tag{4.10}$$

starting from the point $(0,-1)$ and such that $y \to +\infty$ as t goes to infinity. Matching is done again at the level $y = 2\alpha$. We still get formulas (2.26), (2.27), (2.28) and (2.31). In particular, at $\gamma = 0$, $\alpha = 1$, we get $t_* = 3/2$. Then, using (2.31) we get $A = -6$, while the denominator of (2.28) amounts to

$$2t_*^{-1/2} e^{t_*} \int_0^\infty t^{1/2} e^{-t} dt = 2(2e^3/3)^{1/2} \Gamma(3/2)\,,$$

so that

$$\left.\frac{\partial \alpha}{\partial \gamma}\right|_{\gamma=0} = \sqrt{\frac{54}{\pi e^3}} = 0.92508\,. \tag{4.11}$$

Final comments

Summing up, in all cases discussed in the paper a family of equations depending on a continuous parameter is studied and a solution is known for a particular parameter value. The problem is then cast as the problem of matching two regular solution branches. This is done by the Implicit Function Theorem, starting from the known solution. A global solution is obtained by continuation thanks to suitable a priori estimates.

The above list is by no means exhaustive. It rather reflects the interests and knowledge of the group of researchers involved in this area. Some further applications come easily to mind. For instance, we can consider the problem of focusing behaviour for other equations with finite speed of propagation, such as

$$u_t = \nabla \cdot (|\nabla u|^{p-2} \nabla u), \quad p > 2, \tag{5.1}$$

the so-called p-Laplacian evolution equation. Another related problem is the study of the fundamental solutions of the equation

$$u_t = |u_{xx}|^{m-1} u_{xx}, \quad 0 < m < \infty, \tag{5.2}$$

which appears in nonlinear elasticity. Such solution exhibits (for $m \neq 1$) an anomalous exponent which has been studied by Bernis, Hulshof and Vazquez in [BHV]. A further example is the diffusion-convection model studied by renormalization techniques by Ginzburg, Entov and Teodorov, [GET],

$$(mc + a(c))_t + v(t)c_t = Dc_{xx}, \tag{5.3}$$

with $a(c)$ taking different constant values for $c_t > 0$ and $c_t < 0$. D is a constant. Thus, this model is related to (1.2).

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