Coupled Parabolic and Hyperbolic Equations Modeling Age-Dependent Epidemic Dynamics with Nonlinear Diffusion*

(Dedicated to Professor Avner Friedman at his 60th birthday)

by

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Abstract. This paper considers a system of coupled second order parabolic and first order hyperbolic equations arising from the age-dependent diffusion population dynamics with an infectious disease. The diffusion is assumed to be nonlinear which leads the parabolic equation to be degenerate. A notion of weak solutions is introduced. Under mild conditions, we have proved the global existence of weak solutions. The result is further improved for one dimensional case.

Keywords. population model, nonlinear diffusion, coupled parabolic and hyperbolic equations.

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§1. Introduction.

In this paper, we study the population dynamics of a single species with an infectious disease. The population is divided into two groups, one of which is called susceptibles (who are possible to catch the disease) and the other is called infectives (who can infect the disease). We consider the problem in the whole space $\mathbb{R}^n$ (in practice, $n \leq 3$) and whole time interval $[0,\infty)$. By $(x,t) \in \mathbb{R}^n \times [0,\infty)$ we mean at location x and at time t. We let $a \in [0,\infty)$ be the age of the infectives from catching the disease. Next, we let

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$\rho(x,t,a)$ be the age distribution of the infectives. Roughly speaking, this is the number of infectives who have caught the disease for time length a (time units, say days or hours) at location x and at time t. Thus, the density of the infectives at $(x,t) \in \mathbb{R}^n \times [0,\infty)$ is

$$u(x,t) \equiv \int_0^\infty \rho(x,t,a)da. \tag{1.1}$$

The density of the susceptibles at (x,t) is denoted by $v(x,t)$. In this paper, we do not consider the birth and the natural death. Such a mixed situation will be considered in our future publications. We let $\lambda(x,t,a)$, $\beta(x,t,a)$ and $\gamma(x,t,a)$ be the death rate for the infectives due to the disease, the recover rate and infection rate, at (x,t,a), respectively. If there is no diffusion with respect to the space variable $x \in \mathbb{R}^n$, then, all the functions are independent of x and we have the following equations for ρ and v: ($\mathbb{R}^+ = (0,\infty)$)

$$\begin{cases} \rho_t + \rho_a = -\lambda(t,a)\rho - \beta(t,a)\rho, \qquad (t,a) \in \mathbb{R}^+ \times \mathbb{R}^+, \\ v_t = -(\int_0^\infty \gamma(t,a)\rho(t,a)da)v + \int_0^\infty \beta(t,a)\rho(t,a)da, \qquad t \in \mathbb{R}^+, \\ \rho\,\big|_{t=0} = \rho_0(a), \qquad a \in \mathbb{R}^+, \\ \rho\,\big|_{a=0} = (\int_0^\infty \gamma(t,a)\rho(t,a)da)v(t), \qquad t \in \mathbb{R}^+, \\ v\,\big|_{t=0} = v_0. \end{cases} \tag{1.2}$$

See [13] and [2,7,15] for the relevant details. Now, we are interested in the case that the population is also subject to space diffusion. We assume that the diffusion is due to the overcrowding. From the model introduced in [6] (see [9–11] also), the diffusion velocity can be taken to be $-\nabla(u+v)$. Here, we notice that $u+v$ is the density of total population. It is clear that in order to take this diffusion effect into account, we need to add terms $\nabla \cdot [\rho\nabla(u+v)]$ and $\nabla \cdot [v\nabla(u+v)]$ into the first two equations in (1.2), respectively. This can be justified by the conservation of the population in both of the two groups (infectives and susceptibles). Hence, we end up with the following system:

$$\rho_t + \rho_a = \nabla \cdot [\rho\nabla(u+v)] - \lambda(x,t,a)\rho - \beta(x,t,a)\rho, \qquad (x,t,a) \in \mathbb{R}^n \times \mathbb{R}^+ \times \mathbb{R}^+, \tag{1.3}$$

$$v_t = \nabla \cdot [v\nabla(u+v)] - (\int_0^\infty \gamma(x,t,a)\rho(x,t,a)da)v + \int_0^\infty b(x,t,a)\rho(x,t,a)da, \qquad (x,t) \in \mathbb{R}^n \times \mathbb{R}^+, \tag{1.4}$$

with the following initial conditions:

$$\rho \,\Big|_{t=0} = \rho_0(x,a), \qquad (x,a) \in \mathbb{R}^n \times \mathbb{R}^+, \tag{1.5}$$

$$\rho \,\Big|_{a=0} = \Big(\int_0^\infty \gamma(x,t,a)\rho(x,t,a)da\Big)v(x,t), \qquad (x,t) \in \mathbb{R}^n \times \mathbb{R}^+, \tag{1.6}$$

and

$$v \,\Big|_{t=0} = v_0(x), \qquad x \in \mathbb{R}^n. \tag{1.7}$$

Now, let us further simplify the model in order to catch the essence of it. To this end, we let λ, β and γ be independent of a. Then, the density u of the infectives defined in (1.1) and the density v of the susceptibles satisfy the following coupled system:

$$u_t = \nabla \cdot [u \nabla (u+v)] + \gamma(x,t)uv - [\lambda(x,t) + \beta(x,t)]u, \qquad (x,t) \in \mathbb{R}^n \times \mathbb{R}^+, \tag{1.8}$$

$$v_t = \nabla \cdot [v \nabla (u+v)] - \gamma(x,t)uv + \beta(x,t)u, \qquad (x,t) \in \mathbb{R}^n \times \mathbb{R}^+. \tag{1.9}$$

If we set

$$p = \begin{pmatrix} u \\ v \end{pmatrix}, \qquad A = \begin{pmatrix} u & u \\ v & v \end{pmatrix},$$

then, the equations (1.8)–(1.9) can be written as

$$p_t - A\Delta p = G(x,t,p,\nabla p), \tag{1.10}$$

with some nonlinear function G. This system is not parabolic since the matrix $(A + A^T)/2$ has a negative eigenvalue if $u \neq v$. To overcome this difficulty, we introduce a new variable $w = u + v$. This is nothing but the density of the total population. Then, system (1.8)–(1.9) can be transformed into the following form:

$$w_t = \nabla \cdot (w \nabla w) - \lambda u, \qquad (x,t) \in \mathbb{R}^n \times \mathbb{R}^+, \tag{1.11}$$

$$u_t = \nabla u \cdot \nabla w + u\Delta w + (\gamma w - \lambda - \beta)u - \gamma u^2, \qquad (x,t) \in \mathbb{R}^n \times \mathbb{R}^+. \tag{1.12}$$

with initial conditions

$$u\big|_{t=0} = u_0(x) \equiv \int_0^\infty \rho_0(x,a)da, \qquad x \in \mathbb{R}^n, \tag{1.13}$$

$$w\big|_{t=0} = w_0(x) \equiv u_0(x) + v_0(x), \qquad x \in \mathbb{R}^n. \tag{1.14}$$

For fixed u, (1.11) is an inhomogeneous porous medium equation and for fixed w, (1.12) is a first order nonlinear hyperbolic equation, which is not of conservation law type. Hence, (1.11)–(1.14) is a system of a second order degenerate parabolic equation coupled with a first order hyperbolic equation. For the porous medium equation, we know that, in general, there exists no classical solutions if the initial data is not strictly positive everywhere. Thus, we can not expect to have a classical solution for (1.11)–(1.14). In this paper, we shall study the weak solutions for this system. In section 2, we introduce the weak formulation of (1.11)–(1.14). Section 3 is devoted to the study of the approximate problems. The existence of weak solutions will be proved in section 4. In section 5, the age distribution function ρ for the infectives is recovered from (1.3). Finally, in section 6, the one dimensional case is discussed.

§2. A Weak Formulation.

In this section, we introduce a weak formulation of (1.11)–(1.14). Let us first introduce some notation. Let $\Omega = \mathbb{R}^n$. For any $T > 0$, we denote $\Omega_T = (0,T) \times \mathbb{R}^n$, $\bar{\Omega}_T^0 = [0,T) \times \mathbb{R}^n$ and $\bar{\Omega}_T = [0,T] \times \mathbb{R}^n$. For any integer $k \geq 0$, let $C^k(\Omega)$ be the set of all k-time continuously differentiable functions $u(x)$ defined on Ω with all partial derivatives up to order k (inclusively) being bounded. The norm of $C^k(\Omega)$ is denoted by $\|\cdot\|_{C^k(\Omega)}$. For $k \geq 0$ and $0 < \alpha < 1$, we denote $C^{k+\alpha}(\Omega)$ the Banach space of all functions $u(x)$ in $C^k(\Omega)$ with the k-th order partial derivatives being α-Hölder continuous. The norm of this space is denoted by $\|\cdot\|_{C^{k+\alpha}(\Omega)}$. We can define the spaces $C^k(\bar{\Omega}_T)$ and $C^{k+\alpha}(\bar{\Omega}_T)$ in a similar way, treating x and t equally.

Next, we need to introduce spaces which are suitable for parabolic problems. To this end, we define the parabolic distance as follows:

$$d((x,t),(x',t')) = (|x-x'|^2 + |t-t'|)^{1/2}, \qquad \forall (x,t),(x',t') \in \mathbb{R}^n \times \mathbb{R}^+. \tag{2.1}$$

For $0 < \alpha \leq 1$, we denote $C^{0,\alpha}(\bar{\Omega}_T)$ to be the space of all functions $u(x,t)$ in $C^0(\bar{\Omega}_T)$ for which

$$\|u\|_{C^{0,\alpha}(\bar{\Omega}_T)} \equiv \|u\|_{C^0(\bar{\Omega}_T)} + \sup_{(x,t),(x't')\in\bar{\Omega}_T,(x,t)\neq(x't')} \frac{|u(x,t)-u(x',t')|}{d((x,t),(x',t'))^\alpha} \tag{2.2}$$

is finite. We set $C^{0,0}(\bar{\Omega}_T) \equiv C^0(\bar{\Omega}_T)$ with the same norm as $C^0(\bar{\Omega}_T)$. For $0 \leq \alpha \leq 1$, we define the space $C^{1,\alpha}(\bar{\Omega}_T)$ to be the set of all $C(\bar{\Omega}_T)$ functions $u(x,t)$ in $C^0(\bar{\Omega}_T)$ having the first order partial derivatives in x with

$$\|u\|_{C^{1,\alpha}(\bar{\Omega}_T)} \equiv \|u\|_{C^{0,\alpha}(\bar{\Omega}_T)} + \sum_{i=1}^n \|u_{x_i}\|_{C^{0,\alpha}(\bar{\Omega}_T)}, \tag{2.3}$$

being finite and $C^{2,\alpha}(\bar{\Omega}_T)$ to be the set of all functions $u(x,t)$ in $C^1(\bar{\Omega}_T)$ having the second order partial derivatives in x for which

$$\|u\|_{C^{2,\alpha}(\bar{\Omega}_T)} \equiv \|u\|_{C^{1,\alpha}(\bar{\Omega}_T)} + \|u_t\|_{C^{0,\alpha}(\bar{\Omega}_T)} + \sum_{i,j=1}^n \|u_{x_i x_j}\|_{C^{0,\alpha}(\bar{\Omega}_T)} \tag{2.4}$$

is finite.

Suppose (w,u) is a classical solution of (1.11)–(1.14). Multiplying (1.11) by any $\xi \in C_0^\infty(\bar{\Omega}_T^0)$ and then integrating by parts, we obtain

$$\int_{\Omega_T} \{w\xi_t - \frac{1}{2}\nabla w^2 \cdot \nabla\xi - \lambda u\xi\} dxdt - \int_{\mathbb{R}^n} w_0\xi(0,x)dx = 0. \tag{2.5}$$

On the other hand, (1.12) can be written as

$$u_t = \nabla\cdot[u\nabla w] + (\gamma w - \lambda - \beta)u - \gamma u^2, \qquad (x,t)\in\mathbb{R}^n\times\mathbb{R}^+. \tag{2.6}$$

Thus, multiplying (2.6) by any $\eta \in C_0^\infty(\bar{\Omega}_T^0)$ and integrating by parts, we have

$$\int_{\Omega_T} \{u\eta_t - u\nabla w\cdot\nabla\eta + [(\gamma w - \lambda - \beta)u - \gamma u^2]\eta\} dxdt - \int_{\mathbb{R}^n} u_0\eta(0,x)dx = 0. \tag{2.7}$$

The above suggests us to introduce the following weak formulation for (1.11)–(1.14).

Definition 2.1. A triple of functions (w,u,χ) defined on Ω_T is called a weak solution of (1.11)–(1.14) in Ω_T if the following hold

$$w,u,\chi \in L^\infty(\Omega_T), \qquad \nabla w^2 \in L^2_{loc}(\Omega_T), \tag{2.8}$$

$$w \geq u \geq 0, \qquad w^2 \geq \chi \geq u^2 \geq 0, \tag{2.9}$$

and (w, u, χ) satisfy (2.5) and

$$\int_{\Omega_T} \{u\eta_t - u\nabla w \cdot \nabla\eta + [(\gamma w - \lambda - \beta)u - \gamma\chi]\eta\} dxdt - \int_{\mathbb{R}^n} u_0\eta(0, x)dx = 0, \tag{2.10}$$

for all $\xi, \eta \in C_0^\infty(\bar{\Omega}_T^0)$. If (w, u, χ) is defined on $\mathbb{R}^n \times (0, \infty)$ and is a weak solution in Ω_T for any $T > 0$, we call it a global weak solution of (1.11)–(1.14).

In the next sections, we will show that there exists a weak solution (w, u, χ) of (1.11)–(1.14). Here, we point out that the u^2 in (2.7) has been replaced by χ in (2.10). This is technically necessary because we will encounter the weak convergence of some approximate sequence u_ε of u. As is well-known that the weak limit χ of u_ε^2, if it ever exists, is not necessarily equal to u^2. From the above definition of weak solutions, it seems that the function χ is not well determined. However, the following result tells us that this χ can not be arbitrary.

Proposition 2.2. *Let (w, u, χ) and $(\widehat{w}, \widehat{u}, \widehat{\chi})$ be two weak solutions of (1.11)–(1.14) in the sense of Definition 2.1. Then, the following hold:*

(i) *If $w = \widehat{w}$, then $u = \widehat{u}$ and $\chi = \widehat{\chi}$.*

(ii) *If $u = \widehat{u}$, then $w = \widehat{w}$ and $\chi = \widehat{\chi}$.*

Proof. (i) Let $w = \widehat{w}$. By (2.5), we obtain $u = \widehat{u}$. Then, by (2.10), we have $\chi = \widehat{\chi}$. We can similarly prove (ii) by using (2.5) and (2.10). □

The above result implies that the function χ is uniquely determined by the pair (w, u). This, however, does not say that the weak solution is unique. In §4, we will give another alternative way of representing the function χ via the so-called Young's measure. In §6, we will further show that for one dimensional case, if the initial data $w_0(x)$ has a positive lower bound, then

$$\chi(x, t) = u(x, t)^2.$$

Hence, (w, u) actually satisfy (2.5) and (2.7) in this case.

To conclude this section, we make the following hypotheses, which will be assumed throughout this paper:

$$\lambda(x, t), \beta(x, t), \gamma(x, t) \in C^1(\mathbb{R}^n \times \mathbb{R}^+), \tag{2.11}$$

$$w_0, u_0 \in C^{2+\alpha}(\mathbb{R}^n), \tag{2.12}$$

$$0 \leq \lambda, \beta, \gamma \leq M, \qquad \text{for some constant } M, \tag{2.13}$$

$$w_0(x) \geq u_0(x) \geq 0, \qquad \forall x \in \mathbb{R}^n, \tag{2.14}$$

The first two assumptions above are for convenience, which can be relaxed. The rest of two are physically reasonable.

§3. Approximate Problems.

In this section, we study the following approximate problem of (1.11)–(1.14): Let $T > 0$,

$$w_t = \nabla \cdot (w \nabla w) - \lambda w + \lambda E_\delta(w - \varphi_\varepsilon * u), \qquad \text{in } \Omega_T, \tag{3.1}$$

$$u_t = \nabla u \cdot \nabla w + (\Delta w + \gamma w - \lambda - \beta) u - \gamma u^2, \qquad \text{in } \Omega_T, \tag{3.2}$$

$$w \Big|_{t=0} = w_0(x) + \varepsilon \equiv w_0^\varepsilon(x), \qquad x \in \mathbb{R}^n. \tag{3.3}$$

$$u \Big|_{t=0} = u_0(x) + \varepsilon \equiv u_0^\varepsilon(x), \qquad x \in \mathbb{R}^n, \tag{3.4}$$

In the above, $0 < \varepsilon, \delta \leq 1$; $E_\delta(s)$ is a nonnegative smooth function defined in $\mathbb{R}$ nondecreasingly converges to $E_0(s) \equiv \max\{s, 0\}$ as $\delta \downarrow 0$ and with $E'_\delta(s)$ being bounded independent of δ (this is possible since $E_0(s)$ is Lipschitz continuous); and φ_ε is an ε-mollifier:

$$\varphi_\varepsilon(x,t) = \frac{1}{\varepsilon^{n+1}} \varphi(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}),$$

with

$$\varphi \in C_0^\infty(\mathbb{R}^{n+1}), \quad \operatorname{supp} \varphi \subset \{(x,t) \in \mathbb{R}^{n+1} \;\big|\; |x|^2 + t^2 \leq 1, t \geq 0\},$$
$$\varphi(x,t) \geq 0, \quad \int_{\mathbb{R}^{n+1}} \varphi(x,t) dx dt = 1.$$

We call problem (3.1)–(3.4) the approximate problem of (1.11)–(1.14).

We have seen that (3.2) is the same as (1.12). However, (1.11) is changed to (3.1). The purpose of making this approximation is that if w is a solution of (3.1), then, $w \geq 0$. We do not a priori have such a property for a solution of (1.11).

The main result of this section is the following:

Theorem 3.1. *There exists a unique classical solution* $(w_{\varepsilon,\delta}, u_{\varepsilon,\delta})$ *of (3.1)–(3.4) for any* $T > 0$. *Moreover, this solution satisfies*

$$\varepsilon e^{-MT} \leq w_{\varepsilon,\delta}(x,t) \leq \|w_0\|_{C^0(\Omega)} + \varepsilon, \qquad (x,t) \in \bar{\Omega}_T, \tag{3.5}$$

$$0 < u_{\varepsilon,\delta}(x,t) \leq w_{\varepsilon,\delta}(x,t), \qquad \forall (x,t) \in \bar{\Omega}_T, \quad \forall T > 0. \tag{3.6}$$

To prove the above theorem, we begin with the study of equation (3.1). In what follows, we let $T_0 > 0$ be fixed and $0 < T \leq T_0$. The following lemma was proved in [11].

Lemma 3.2. *Let* $w \in C^0(\bar{\Omega}_T)$ *with* $D^2 w \in C^0(\bar{\Omega}_T)$ *and* $(\bar{x}, \bar{t}) \in \Omega_T$. *Then, there exists a unique solution* $\psi(t; \bar{x}, \bar{t})$ *of*

$$\begin{cases} \dfrac{d\psi}{dt} = -\nabla w(\psi, t), \qquad 0 < t < T, \\ \psi(\bar{t}; \bar{x}, \bar{t}) = \bar{x}. \end{cases} \tag{3.7}$$

Moreover, the solution $\psi(t; \bar{x}, \bar{t})$ *is differentiable with respect to all its arguments, and satisfies*

$$\frac{\partial \psi}{\partial \bar{t}} = D_{\bar{x}} \psi(t; \bar{x}, \bar{t}) \nabla w(\bar{x}, \bar{t}), \qquad t \in [0, T]. \tag{3.8}$$

$$\begin{cases} \dfrac{d}{dt} D_{\bar{x}} \psi(t; \bar{x}, \bar{t}) = -D^2 w(\psi(t; \bar{x}, \bar{t}), t) D_{\bar{x}} \psi(t; \bar{x}, \bar{t}), \\ D_{\bar{x}} \psi(\bar{t}; \bar{x}, \bar{t}) = I. \end{cases} \tag{3.9}$$

Consequently,

$$|D_{\bar{x}} \psi(t; \bar{x}, \bar{t})| \leq e^{T \|D^2 w\|_{C^0(\bar{\Omega}_T)}}, \qquad t \in [0, T]. \tag{3.10}$$

We refer to the solution ψ of (3.7) as the characteristics associated with w. The following result gives a representation of solutions to (3.2) and (3.4).

Lemma 3.3. *Let $w \in C^0(\bar{\Omega}_T)$ with $D^2 w \in C^0(\bar{\Omega}_T)$. Let $\psi(t; \bar{x}, \bar{t})$ be determined by (3.7). Let u be a classical solution of (3.2) and (3.4) in Ω_T. Then, u can be expressed as*

$$u(\bar{x}, \bar{t}) = \frac{u_0^{\varepsilon}(\psi(0; \bar{x}, \bar{t})) e^{\int_0^{\bar{t}} h(\psi(t; \bar{x}, \bar{t}), t) dt}}{1 + u_0^{\varepsilon}(\psi(0; \bar{x}, \bar{t})) \int_0^{\bar{t}} \gamma(\psi(t; \bar{x}, \bar{t}), t) e^{\int_0^t h(\psi(\tau; \bar{x}, \bar{t}), \tau) d\tau} dt}, \tag{3.11}$$

where $h(x,t) = \Delta w(x,t) + \gamma(x,t) w(x,t) - \lambda(x,t) - \beta(x,t)$.

Proof. We fix $(\bar{x}, \bar{t}) \in \Omega_T$ and set

$$u(t) = u(\psi(t; \bar{x}, \bar{t}), t).$$

Then, $u(t)$ satisfies the following:

$$\begin{cases} \dfrac{du(t)}{dt} = h(\psi(t; \bar{x}, \bar{t}), t) u(t) - \gamma u(t)^2, \\ \\ u(0) = u_0^{\varepsilon}(\psi(0; \bar{x}, \bar{t})) > 0. \end{cases} \tag{3.12}$$

It follows from (3.12) that $u(t) > 0$ for all $t \in [0, T]$. Thus, $q(t) = \frac{1}{u(t)}$ is well-defined and satisfies

$$\begin{cases} \dfrac{dq(t)}{dt} = -h q(t) + \gamma, \\ q(0) = \dfrac{1}{u(0)}. \end{cases} \tag{3.13}$$

Hence,

$$q(t) = e^{-\int_0^t h d\tau} q(0) + \int_0^t \gamma e^{-\int_s^t h d\tau} ds. \tag{3.14}$$

We thus obtain (3.11) by taking $t = \bar{t}$. $\square$

Note that u can always be defined by (3.11) provided $\Delta w \in C^0(\bar{\Omega}_T)$. However, if w has no more regularity, this function may not necessarily be a classical solution of (3.2).

Lemma 3.4. *Let $w \in C^0(\bar{\Omega}_T)$ with $\Delta w \in C^0(\bar{\Omega}_T)$. Let u be defined by (3.11). Then,*

$$\|u\|_{C^0(\bar{\Omega}_T)} \leq (1 + \|u_0\|_{C^0(\Omega)}) e^{T(\|\Delta w\|_{C^0(\bar{\Omega}_T)} + M \|w\|_{C^0(\bar{\Omega}_T)})}. \tag{3.15}$$

Proof. It is easy to see that (note (2.13))

$$e^{\int_0^{\bar t}(\Delta w+\gamma w-\lambda-\beta)dt} \le e^{T(\|\Delta w\|_{C^0(\bar\Omega_T)}+M\|w\|_{C^0(\bar\Omega_T)})}, \qquad \bar t \in [0,T]. \tag{3.16}$$

Hence, (3.15) follows from (3.11) (since $0<\varepsilon\le 1$). □

Lemma 3.5. *If $u \in C^0(\bar\Omega_T)$ and $u(x,t)\ge 0$ on $\bar\Omega_T$, then, (3.1) and (3.3) admits a unique solution $w \in C^{2,\alpha}(\bar\Omega_T)$, with $\alpha\in(0,1)$ only depending on $\varepsilon>0$. Moreover, this solution satisfies (3.5).*

Proof. For any $\sigma>0$, we let $G_\sigma(s)\in C^\infty$ be a function satisfying

$$G_\sigma(s)=\begin{cases} s, & s\ge\sigma,\\ \sigma/2, & s\le \sigma/2,\end{cases} \tag{3.17}$$

and $G_\sigma(s)\ge\sigma/2$. Since E_δ is C^1 with E'_δ bounded uniformly in $\delta>0$, by the standard results for the nonlinear parabolic equations ([12]), for any $u\in C^0(\bar\Omega_T)$, the Cauchy problem

$$\begin{cases} \bar w_t-\nabla\cdot[G_\sigma(\bar w)\nabla\bar w] = -\lambda\bar w+\lambda E_\delta(\bar w-\varphi_\varepsilon * u),\\ \bar w(0,x)=w_0^\varepsilon(x),\end{cases} \tag{3.18}$$

is uniquely solvable in $C^{2,\alpha}(\bar\Omega_T)$. On the other hand, since $E_\delta\ge 0$, we get

$$\bar w_t-\nabla\cdot[G_\sigma(\bar w)\nabla\bar w]+\lambda\bar w\ge 0. \tag{3.19}$$

By maximum principle, we thus obtain

$$\bar w(x,t)\ge\varepsilon e^{-MT}, \qquad \forall (x,t)\in\bar\Omega_T. \tag{3.20}$$

Therefore, if we choose $\sigma\le\varepsilon e^{-MT}$ at the beginning, then, $\bar w$ is actually the unique solution of (3.1) and (3.3). Now, we rewrite the equation (3.1) as the following:

$$\bar w_t-\nabla\cdot(\bar w\nabla\bar w)+c(x,t)\bar w=0, \tag{3.21}$$

where

$$c(x,t)=\lambda-\frac{\lambda E_\delta(\bar w-\varphi_\varepsilon * u)}{\bar w}. \tag{3.22}$$

By (3.20), we know that $\bar{w} > 0$. Thus, notice $\varphi_\varepsilon * u \geq 0$ and $E_\delta \uparrow E_0$, we have

$$0 \leq \frac{E_\delta(\bar{w} - \varphi_\varepsilon * u)}{\bar{w}} \leq \frac{E_0(\bar{w} - \varphi_\varepsilon * u)}{\bar{w}} \leq 1. \tag{3.23}$$

This yields that

$$0 \leq c(x,t) \leq M, \qquad (x,t) \in \bar{\Omega}_T. \tag{3.24}$$

Hence, by maximum principle, we obtain

$$\bar{w}(x,t) \leq \|w_0\|_{C^0(\Omega)} + \varepsilon. \tag{3.25}$$

Combining (3.20) and (3.25), we obtain (3.5). □

We now construct a mapping $\mathcal{A} : C^{2,0}(\bar{\Omega}_T) \to C^{2,\alpha}(\bar{\Omega}_T) \subset C^{2,0}(\bar{\Omega}_T)$ as follows: For any $w \in C^{2,0}(\bar{\Omega}_T)$, let $u \in C^0(\bar{\Omega}_T)$ be d efined by (3.11). We designate $\bar{w} = \mathcal{A}w$ as the unique solution of (3.1) and (3.3). From the above arguments $\mathcal{A}$ is well-defined and satisfies (3.5). We have the following lemma about this map.

Lemma 3.6. *There exists a $T_1 \in (0,T]$, such that the map $\mathcal{A} : C^{2,0}(\bar{\Omega}_{T_1}) \to C^{2,0}(\bar{\Omega}_{T_1})$ admits a fixed point $w \equiv w_{\varepsilon,\delta} \in C^{2,\alpha}(\bar{\Omega}_T)$.*

Proof. Consider the convex and closed set

$$B = \{w \in C^{2,0}(\bar{\Omega}_{T_1}) \mid \|w\|_{C^{2,0}(\bar{\Omega}_{T_1})} \leq K\},$$

where $0 < T_1 \leq T$ and $K > 0$ are undetermined. From (3.21) and (3.24), by maximum principle, the L^p-estimates and the Sobolev inequalities, we get

$$\|\bar{w}\|_{C^{1,\alpha}(\bar{\Omega}_{T_1})} \leq C(\varepsilon, T). \tag{3.26}$$

Here, the constant $C(\varepsilon,T)$ depending on ε due to (3.20). Then, applying the Schauder estimates, we derive (notice (3.15))

$$\begin{aligned} \|\bar{w}\|_{C^{2,\alpha}(\bar{\Omega}_{T_1})} &\leq C(\varepsilon,T)[1 + \|w_0\|_{C^{2+\alpha}(\Omega)} + \|\varphi_\varepsilon * u\|_{C^\alpha(\bar{\Omega}_{T_1})}] \\ &\leq C(\varepsilon,T)[1 + \|w_0\|_{C^{2+\alpha}(\Omega)} + \frac{C}{\varepsilon}\|u\|_{C^0(\bar{\Omega}_{T_1})}] \\ &\leq C(\varepsilon,T)[1 + \|w_0\|_{C^{2+\alpha}(\Omega)} + \frac{C}{\varepsilon}(1 + \|u_0\|_{C^0(\Omega)})e^{T_1K(1+M)}]. \end{aligned} \tag{3.27}$$

We now take

$$K = C(\varepsilon, T)[1 + \|w_0\|_{C^{2+\alpha}(\Omega)} + \frac{C}{\varepsilon} e(1 + \|u_0\|_{C^0(\Omega)})], \tag{3.28}$$
$$T_1 = \frac{1}{K(1+M)} \equiv T_1(\varepsilon, T, \|w_0\|_{C^{2+\alpha}(\Omega)}, \|u_0\|_{C^0(\Omega)}).$$

Then, from (3.27), we see that $\mathcal{A}$ maps B into itself. It is clear that this map is continuous and compact. By Schauder fixed point theorem, $\mathcal{A}$ possesses a fixed point . □

Lemma 3.7. *Let $w \equiv w_{\varepsilon,\delta}$ be a fixed point of the map $\mathcal{A}$ in $C^{2,\alpha}(\bar{\Omega}_{T_1})$ and let $u \equiv u_{\varepsilon,\delta}$ be defined by (3.11) with the $\psi \equiv \psi_{\varepsilon,\delta}$ being the characteristics associated with $w_{\varepsilon,\delta}$. Then, $u \in C^1(\bar{\Omega}_{T_1})$ and $(w, u) \equiv (w_{\varepsilon,\delta}, u_{\varepsilon,\delta})$ is a classical solution of (3.1)–(3.4) on Ω_{T_1} satisfying (3.5)–(3.6).*

Proof. To show (w, u) is a classical solution of (3.1)–(3.4) on Ω_{T_1}, we only need to verify (3.2). Since w satisfies (3.1), we get

$$\Delta w(\psi(t; \bar{x}, \bar{t}), t) = \frac{1}{w(\psi(t; \bar{x}, \bar{t}), t)} \frac{dw(\psi(t; \bar{x}, \bar{t}), t)}{dt} + c(\psi(t; \bar{x}, \bar{t}), t), \tag{3.29}$$

where $\psi(\cdot; \bar{x}, \bar{t})$ is the characteristics associated with $w \equiv w_{\varepsilon,\delta}$ (see (3.7)) and $c(\cdot)$ defined by (3.22), with $(\bar{w}, u) = (w, u) \equiv (w_{\varepsilon,\delta}, u_{\varepsilon,\delta})$. Hence, (still let $h(x,t) = \Delta w + \gamma w - \lambda - \beta$)

$$g(t; \bar{x}, \bar{t}) \equiv e^{\int_0^t h(\psi(\tau; \bar{x}, \bar{t}), \tau) d\tau} = e^{\int_0^t \frac{1}{w} \frac{dw}{dt} d\tau} \cdot e^{\int_0^t (c + \gamma w - \lambda - \beta) d\tau} \tag{3.30}$$
$$= \frac{w(\psi(t; \bar{x}, \bar{t}), t)}{w_0^\varepsilon(\psi(0; \bar{x}, \bar{t}))} e^{\int_0^t (c + \gamma w - \lambda - \beta) d\tau}.$$

Clearly, the right hand side of (3.30) belongs to C^1 (as a function of $(t, \bar{x}, \bar{t})$). By (3.11), the definition of u, and (3.30), we have

$$u(\bar{x}, \bar{t}) = \frac{u_0^\varepsilon(\psi(0; \bar{x}, \bar{t})) g(\bar{t}; \bar{x}, \bar{t})}{1 + u_0^\varepsilon(\psi(0; \bar{x}, \bar{t})) \int_0^{\bar{t}} \gamma(\psi(t; \bar{x}, \bar{t}) g(t; \bar{x}, \bar{t}) dt}. \tag{3.31}$$

Therefore, $u \in C^1$ (see (3.8)–(3.10)). Furthermore, by (3.8), we derive

$$\frac{\partial u_0^\varepsilon}{\partial \bar{t}}(\psi(0; \bar{x}, \bar{t})) = \nabla_x u_0^\varepsilon(\psi(0; \bar{x}, \bar{t})) D_{\bar{x}} \psi(0; \bar{x}, \bar{t}) \nabla_{\bar{x}} w(\bar{x}, \bar{t}) \tag{3.32}$$
$$= \nabla_{\bar{x}} u_0^\varepsilon(\psi(0; \bar{x}, \bar{t})) \cdot \nabla_{\bar{x}} w(\bar{x}, \bar{t}).$$

Analogously, we have

$$\frac{\partial g}{\partial \bar{t}}(t; \bar{x}, \bar{t}) = \nabla_{\bar{x}} g(t; \bar{x}, \bar{t}) \cdot \nabla_{\bar{x}} w(\bar{x}, \bar{t}), \tag{3.33}$$

and

$$\frac{\partial g}{\partial t}(t;\bar{x},\bar{t}) = h(\psi(t;\bar{x},\bar{t}),t)g(t;\bar{x},\bar{t}) \tag{3.34}$$
$$= (\Delta w + \gamma w - \lambda - \beta)g.$$

Then, by direct computation, we are able to show that u satisfies (3.2).

Next, from (3.11), $u > 0$ on Ω_{T_1}. Then, by Lemma 3.5, we know that (3.5) holds. Finally, we set $v = w - u$. Then, v solves

$$\begin{cases} v_t - \nabla v \cdot \nabla w - (\Delta w - \lambda - \gamma u)v = \beta u + \lambda E_\delta(w - \varphi_\varepsilon * u) \geq 0, \\ v \mid_{t=0} = w_0 - u_0 = v_0 \geq 0. \end{cases} \tag{3.35}$$

By integrating v along the characteristics (3.7), we obtain $v \geq 0$. Hence, $u \leq w$. This completes the proof. □

Now, we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. By Lemma 3.7, we know that there exists a $T_1 \leq T$, such that (3.1)–(3.4) has a classical solution on $\bar{\Omega}_{T_1}$. Let us show that the classical solution on Ω_{T_1} is unique. To this end, we suppose there are two solutions (w_1, u_1) and (w_2, u_2) in Ω_{T_1}. Let $\widetilde{u} = u_1 - u_2$ and $\widetilde{w} = w_2 - w_2$. Then,

$$\widetilde{w}_t - \frac{1}{2}(w_1 + w_2)\Delta\widetilde{w} = \nabla(w_1 + w_2) \cdot \nabla\widetilde{w} + a_1\widetilde{w} - a_2\varphi_\varepsilon * \widetilde{u}, \tag{3.36}$$

$$\widetilde{u}_t - \nabla\widetilde{u} \cdot \nabla w_1 = b_1\widetilde{u} + b_2, \tag{3.37}$$

$$\widetilde{w} \mid_{t=0} = 0, \tag{3.38}$$

$$\widetilde{u} \mid_{t=0} = 0, \tag{3.39}$$

where

$$\begin{aligned} a_1 &= \frac{1}{2}\Delta(w_1 + w_2) - \lambda + a_2, \\ a_2 &= \lambda \int_0^1 E'_\delta(\tau(\widetilde{w} - \varphi_\varepsilon * \widetilde{u}) + w_2 - \varphi_\varepsilon * u_2)d\tau, \\ b_1 &= \Delta w_1 + \gamma w_1 - \lambda - \beta - \gamma(u_1 + u_2), \\ b_2 &= \nabla u_2 \cdot \nabla\widetilde{w} + (\Delta\widetilde{w} + \gamma\widetilde{w})u_2. \end{aligned} \tag{3.40}$$

Since $w_1, w_2 \in C^{2,\alpha}(\bar{\Omega}_{T_1})$ and $u_1, u_2 \in C^1(\bar{\Omega}_{T_1})$, we have

$$
\begin{aligned}
&\|a_1\|_{C^{0,\alpha}(\bar{\Omega}_{T_1})}, \|a_2\|_{C^{0,\alpha}(\bar{\Omega}_{T_1})}, \|b_1\|_{C^{0,\alpha}(\bar{\Omega}_{T_1})} \leq C, \\
&\|b_2\|_{C^0} \leq C\|D^2\widetilde{w}\|_{C^0(\bar{\Omega}_{T_1})}.
\end{aligned} \tag{3.41}
$$

Integrating (3.37) along the characteristics corresponding to w_1, we get

$$
\widetilde{u}(\bar{x},\bar{t}) = \int_0^{\bar{t}} e^{\int_t^{\bar{t}} b_1(\psi(\tau;\bar{x},\bar{t}),\tau)d\tau} b_2\, dt. \tag{3.42}
$$

Hence, for any $\widetilde{T}_1 \leq T_1$, we have

$$
|\widetilde{u}(\bar{x},\bar{t})| \leq \bar{t}C\|b_2\|_{C^0(\bar{\Omega}_{\widetilde{T}_1})} \leq \bar{t}C\|D^2\widetilde{w}\|_{C^0(\bar{\Omega}_{\widetilde{T}_1})}, \qquad \forall \bar{t} \leq \widetilde{T}_1. \tag{3.43}
$$

On the other hand, by Schauder estimates, we get from (3.36) that

$$
\|\widetilde{w}\|_{C^{2+\alpha}(\bar{\Omega}_{\widetilde{T}_1})} \leq C\|\varphi_\varepsilon * \widetilde{u}\|_{C^\alpha(\bar{\Omega}_{\widetilde{T}_1})} \leq C_\varepsilon\|\widetilde{u}\|_{C^0(\bar{\Omega}_{\widetilde{T}_1})}. \tag{3.44}
$$

Substituting (3.44) into (3.43), we obtain that in $\Omega_{\widetilde{T}_1}$, $(\widetilde{T}_1 \leq T_1)$

$$
\|\widetilde{u}\|_{C^0(\bar{\Omega}_{\widetilde{T}_1})} \leq \widetilde{T}_1\widetilde{C}_\varepsilon\|\widetilde{u}\|_{C^0(\bar{\Omega}_{\widetilde{T}_1})}. \tag{3.45}
$$

Note that $\widetilde{C}_\varepsilon$ does not depend on $\widetilde{T}_1$. Hence, it follows that $\widetilde{u} = 0$ in $\bar{\Omega}_{\widetilde{T}_1}$ if $\widetilde{T}_1 \leq (2\widetilde{C}_\varepsilon)^{-1}$. Then, $\widetilde{w} = 0$ in the same region. Repeating this procedure, we can get $\widetilde{u} = \widetilde{w} = 0$ in Ω_{T_1}. This shows the Uniqueness of the local classical solutions of (3.1)–(3.4).

Next, we prove the existence of global classical solutions. To this end, we only need to derive some a priori estimate. Suppose (w,u) is a classical solution of (3.1)–(3.4) satisfying (3.5)–(3.6). Then, for any $\bar{T} \leq T$, similar to (3.27), we have

$$
\begin{aligned}
\|w\|_{C^{2,\alpha}(\bar{\Omega}_{\bar{T}})} &\leq C(\varepsilon,T)[1 + \|w_0\|_{C^{2+\alpha}(\Omega)} + \|\varphi_\varepsilon * u\|_{C^\alpha(\bar{\Omega}_{\bar{T}})}] \\
&\leq C(\varepsilon,T)[1 + \|w_0\|_{C^{2+\alpha}(\Omega)} + \frac{C}{\varepsilon}\|u\|_{C^0(\bar{\Omega}_{\bar{T}})}] \\
&\leq C(\varepsilon,T)[1 + \|w_0\|_{C^{2+\alpha}(\Omega)} + \frac{C}{\varepsilon}(\|w_0\|_{C^0(\Omega)} + \varepsilon)] \\
&\leq \bar{C}(\varepsilon,T)(1 + \|w_0\|_{C^{2+\alpha}(\Omega)}).
\end{aligned} \tag{3.46}
$$

In the above, we have used (3.5)–(3.6). Since the above estimate is uniform in $\bar{T} \in [0,T]$, we can repeat the procedure given in the proof of Lemma 3.6 to extend the solution (w,u)

to Ω_T. Since for any $T > 0$, there exists a unique classical solution (w, u) in Ω_T, the solution can be extend to $\Omega \times (0, \infty)$ to get a global solution. $\square$

§4. Existence of Weak Solutions.

In this section, we show the existence of a weak solution to (1.11)–(1.14). From the previous section, we know that for any $\delta, \varepsilon > 0$, there exists a unique classical solution $(w_{\delta,\varepsilon}, u_{\delta,\varepsilon})$ of the approximate problem (3.1)–(3.4). We first study the limit as $\delta \to 0$.

Lemma 4.1. *For any $\varepsilon > 0$, there exists a classical solution $(w_\varepsilon, u_\varepsilon)$ of*

$$w_t - \nabla \cdot [w \nabla w] = -\lambda w + \lambda E_0 (w - \varphi_\varepsilon * u), \tag{4.1}$$

and (3.2)–(3.4), such that

$$0 < u_\varepsilon \leq w_\varepsilon \leq \|w_0\|_{C^0(\Omega)}. \tag{4.2}$$

Proof. For fixed $T > 0$, from (3.5)–(3.6), we know that $w_{\delta,\varepsilon}$ and $u_{\delta,\varepsilon}$ are bounded uniformly with respect to $\delta > 0$. By the Schauder estimates, we have

$$\|w_{\varepsilon,\delta}\|_{C^{2,\alpha}(\bar{\Omega}_T)} \leq C(\varepsilon, T). \tag{4.3}$$

It follows from (3.11) that

$$\|u_{\varepsilon,\delta}\|_{C^1(\bar{\Omega}_T)} \leq C(\varepsilon, T). \tag{4.4}$$

Hence, we can select a subsequence, still denoted by itself, and $w_\varepsilon \in C^{2,\alpha}(\bar{\Omega}_T)$, u_ε being Lipschitz, such that for some $\mu \in (0, 1)$,

$$w_{\varepsilon,\delta} \to w_\varepsilon, \qquad \text{in } C^{2,\mu}(\bar{\Omega}_T), \tag{4.5}$$

$$u_{\varepsilon,\delta} \to u_\varepsilon, \qquad \text{in } C^{\mu}(\bar{\Omega}_T), \tag{4.6}$$

as $\delta \to 0$. By the diagonal argument, we can further select a subsequence and $w_\varepsilon \in C^{2,\alpha}(\mathbb{R}^+ \times \mathbb{R}^n)$ and u being Lipschitz in $\mathbb{R}^n \times \mathbb{R}^+$, such that (4.1) and (4.2) hold in Ω_T for any $T > 0$. It is clear that $(w_\varepsilon, u_\varepsilon)$ satisfies (4.1). Since the characteristics (3.7) depends continuously on $D^2 w_{\delta,\varepsilon}$, we can pass to the limits through (3.11). It follows that the limit u_ε has the same expression (3.11). From the proof of Lemma 3.7, $u_\varepsilon \in C^1$ and $(w_\varepsilon, u_\varepsilon)$ satisfies (3.2). finally, (4.2) follows from (3.5)–(3.6) and (3.11). $\square$

Next, we look at the variational forms of ε-approximate solution $(w_\varepsilon, u_\varepsilon)$. It is easy to see that for any $\xi, \eta \in C_0^\infty(\bar{\Omega}_T^0)$, $(w_\varepsilon, u_\varepsilon)$ satisfy the following:

$$\begin{aligned}(4.7)\qquad &\int_{\Omega_T} \{w_\varepsilon \xi_t - \frac{1}{2}\nabla w_\varepsilon^2 \cdot \nabla \xi - \lambda w_\varepsilon \xi + \lambda E_0(w_\varepsilon - \varphi_\varepsilon * u_\varepsilon)\xi\} dxdt \\ &\qquad = \int_{\mathbb{R}^n} (w_0(x) + \varepsilon)\xi(x,0)dx.\end{aligned}$$

$$\begin{aligned}(4.8)\qquad &\int_{\Omega_T} \{u_\varepsilon \eta_t - \frac{1}{2}\theta(w_\varepsilon, u_\varepsilon)\nabla w_\varepsilon^2 \cdot \nabla \eta + [(\gamma w_\varepsilon - \lambda - \beta)u_\varepsilon - \gamma u_\varepsilon^2]\eta\} dxdt \\ &\qquad = \int_{\mathbb{R}^n} (u_0(x) + \varepsilon)\eta(x,0)dx,\end{aligned}$$

where

$$(4.9)\qquad \theta(w,u) = \begin{cases} \dfrac{u}{w}, & w > 0, \\ 0, & w \le 0. \end{cases}$$

We call $(w_\varepsilon, u_\varepsilon)$ the ε-approximate solution.

In what follows, we assume that $\operatorname{supp} w_0$ is compact and

$$(4.10)\qquad w_0(x) \ge M_0 \text{ dist } (x, \Gamma_0), \qquad \text{for } x \in \mathbb{R}^n \text{ with } w_0(x) > 0,$$

where $M_0 > 0$ is a constant and $\Gamma_0 = \partial\{w_0 > 0\}$ is the boundary of the support of w_0.

Lemma 4.2. *Let $(w_\varepsilon, u_\varepsilon)$ be as in Lemma 4.1, and $\Omega_T(R) = B_R(0) \times (0,T)$, where $B_R(0)$ is the ball in $\mathbb{R}^n$ centered at the origin with radius R. Then, there exists a constant $C(R)$, depending only on R, such that*

$$(4.11)\qquad \int_{\Omega_T(R)} |\nabla w_\varepsilon^2|^2 dxdt \le C(R), \qquad \forall \varepsilon > 0.$$

Proof. In (4.7) and (4.8), we put $2T$ in the place of T and $w_\varepsilon^2\xi^2$ in the place of ξ with $\xi \in C^\infty(\mathbb{R}^n \times \mathbb{R}^+)$, $0 \le \xi \le 1$, $\xi = 1$ on $\Omega_T(R)$ and $\xi = 0$ outside of $\Omega_{2T}(2R)$. It follows that

$$\begin{aligned} \int_{\Omega_{2T}} |\nabla w_\varepsilon^2|^2\xi^2 \le \int_{\Omega_{2T}} w_\varepsilon^2|\nabla w_\varepsilon^2||\nabla\xi|^2 + \int_{\Omega_{2T}} 2[\lambda w_\varepsilon^3 + \lambda E_0(w_\varepsilon - \varphi_\varepsilon * u_\varepsilon)w_\varepsilon^2]\xi^2 \\ + |\int_{\Omega_{2T}} 4w_\varepsilon^2(w_\varepsilon)_t\xi^2| + |\int_{\Omega_{2T}} 4w_\varepsilon^2\xi\xi_t| + \int_\Omega (w_0+\varepsilon)^3\xi(x,0)^2. \end{aligned} \tag{4.12}$$

We have

$$\int_{\Omega_{2T}} w_\varepsilon^2(w_\varepsilon)_t\xi^2 = \int_{\Omega_{2T}} \frac{1}{3}(w_\varepsilon^3)_t\xi^2 = -\frac{2}{3}\int_{\Omega_{2T}} \xi\xi_t w_\varepsilon^3 - \frac{1}{3}\int_{\mathbb{R}^n}(w_0+\varepsilon)^3\xi(x,0).$$

Since w_ε and u_ε are uniformly bounded, (4.11) follows from (4.12). $\square$

Lemma 4.3. *Let $(w_\varepsilon, u_\varepsilon)$ be the ε-approximate solution. Then, there exist $w \in C^{0,\alpha}(\bar\Omega_T)$, $u \in L^\infty(\Omega_T)$, such that for any bounded open set $Q \subset \mathbb{R}^n \times \mathbb{R}^+$,*

$$u_\varepsilon \to u, \qquad \text{weakly in } L^2(Q), \tag{4.13}$$

$$\nabla w_\varepsilon^2 \to \nabla w^2, \qquad \text{weakly in } L^2(Q), \tag{4.14}$$

$$w_\varepsilon \to w, \qquad \text{strongly in } C^{0,\alpha}(Q). \tag{4.15}$$

Proof. For fixed $R > 0$ and $T > 0$, from Lemma 4.2 and the previous section, w_ε, ∇w_ε^2 and u_ε are bounded in $L^2(\Omega_T(R))$. Hence, we can select a sequence such that (4.13)–(4.14) hold in $\Omega_T(R)$. By [3,4], we know that there exists a subsequence such that (4.15) holds in $\Omega_T(R)$. Then, the assertion of the lemma follows from the diagonalization argument. $\square$

We now can pass to the limits in (4.7). The limit w is a weak solution of the porous medium equation

$$w_t - \nabla\cdot[w\nabla w] = -\lambda w + \lambda G, \tag{4.16}$$

where G is the weak limit of $E_0(w_\varepsilon - \varphi_\varepsilon * u_\varepsilon)$. Set

$$\Omega_T^+ = \{(x,t) \in \Omega_T : w(x,t) > 0\},$$

and Γ_T is the boundary of Ω_T^+. Since w is continuous, Ω_T^+ is open. Let $\Omega^t = \{x \in \mathbb{R}^n : w(x,t) > 0\}$. The following result gives some properties of Ω^t.

Lemma 4.4. *Let (w,u) be a limit of ε-approximate solution $(w_\varepsilon, u_\varepsilon)$ in the sense of (4.13)–(4.15). Then, the region Ω^t is increasing in t and the boundary Γ_T of Ω_T^+ has the $(n+1)$-dimensional Lebesgue measure zero.*

Proof. Since $0 \le E_0(w_\varepsilon - \varphi_\varepsilon * u_\varepsilon) \le w_\varepsilon$, we have

$$0 \le G \le w. \tag{4.17}$$

Let

$$g = -\lambda + \lambda \frac{G}{w}. \tag{4.18}$$

Then, g is bounded and w satisfies

$$w_t - \nabla \cdot (w \nabla w) = g w. \tag{4.19}$$

By Lemma 3.6 in [11], the assertion follows. □

Theorem 4.5. *There exists a weak solution (w,u,χ) of the problem (1.11)–(1.14). Moreover, $w \in C^{0,\alpha}(\bar{\Omega}_T)$ and $\nabla w \in C^{0,\alpha}_{loc}(\Omega_T^+)$.*

Proof. From lemma 4.3, we can select a subsequence such that (4.13)–(4.15) hold. For any $\xi \in C_0^\infty(\bar{\Omega}_T^0)$, we write

$$\begin{aligned} \int_{\Omega_T} \lambda E_0(w_\varepsilon - \varphi_\varepsilon * u_\varepsilon)\xi &= \int_{\Omega_T} \lambda (w_\varepsilon - \varphi_\varepsilon * u_\varepsilon)\xi \\ &+ \int_{\Omega_T} \lambda [E_0(w_\varepsilon - \varphi_\varepsilon * u_\varepsilon) - (w_\varepsilon - \varphi_\varepsilon * u_\varepsilon)]\xi. \end{aligned} \tag{4.20}$$

Since $u_\varepsilon \to u$ weakly in $L^2_{loc}(\Omega_T)$, it is clear that $\varphi_\varepsilon * u_\varepsilon \to u$ weakly in $L^2_{loc}(\Omega_T)$. Hence,

$$\int_{\Omega_T} \lambda (w_\varepsilon - \varphi_\varepsilon * u_\varepsilon)\xi \to \int_{\Omega_T} \lambda (w-u)\xi. \tag{4.21}$$

By the definition,

$$E_0(w_\varepsilon - \varphi_\varepsilon * u_\varepsilon) - (w_\varepsilon - \varphi_\varepsilon * u_\varepsilon) = \begin{cases} 0, & \text{if } w_\varepsilon - \varphi_\varepsilon * u_\varepsilon \ge 0, \\ -(w_\varepsilon - \varphi_\varepsilon * u_\varepsilon), & \text{if } w_\varepsilon - \varphi_\varepsilon * u_\varepsilon < 0. \end{cases} \tag{4.22}$$

On the other hand, by (4.2),

$$\varphi_\varepsilon * u_\varepsilon - w_\varepsilon \leq \varphi_\varepsilon * w_\varepsilon - w_\varepsilon. \tag{4.23}$$

It follows that for any $(x,t) \in \Omega_T$,

$$0 \leq E_0(w_\varepsilon - \varphi_\varepsilon * u_\varepsilon) - (w_\varepsilon - \varphi_\varepsilon * u_\varepsilon) \leq |\varphi_\varepsilon * w_\varepsilon - w_\varepsilon| \to 0, \tag{4.24}$$

since $w_\varepsilon \to w$ in $C^{0,\alpha}(\bar{\Omega}_T)$ locally. Hence,

$$\int_{\Omega_T} \lambda[E_0(w_\varepsilon - \varphi_\varepsilon * u_\varepsilon) - (w_\varepsilon - \varphi_\varepsilon * u_\varepsilon)]\xi \to 0. \tag{4.25}$$

This yields that

$$E_0(w_\varepsilon - \varphi_\varepsilon * u_\varepsilon) \to \lambda(w-u), \qquad \text{weakly in } L^2_{loc}(\Omega_T). \tag{4.26}$$

Thus, (2.5) follows by sending $\varepsilon \to 0$ in (4.7).

In Ω_T^+, $w > 0$. Applying the $C^{1,\alpha}$-estimates in (4.1), we find that in any bounded open set Q with $\bar{Q} \subset \Omega_T^+$, $\|w_\varepsilon\|_{C^{1,\alpha}(Q)}$ is uniformly bounded. Hence, after extracting a subsequence, $\nabla w_\varepsilon \to \nabla w$ everywhere in Ω_T^+. Now, take a point $(x_0,t_0) \notin \bar{\Omega}_T^+$. Let $B_{2r}(x_0,t_0) \subset \Omega_T \setminus \bar{\Omega}_T^+$ be a ball centered at (x_0,t_0), with $r > 0$ small enough. We choose a smooth ξ in (4.12) such that $\xi = 1$ in $B_r(x_0,t_0)$, $0 \leq \xi \leq 1$ and $\xi = 0$ outside of $B_{2r}(x_0,t_0)$. From (4.12), we get

$$\int_{B_r(x_0,t_0)} |\nabla w_\varepsilon^2|^2 \leq C \sup_{(x,t)\in B_{2r}(x_0,t_0)} |w_\varepsilon(x,t)| \to 0, \qquad \varepsilon \to .0 \tag{4.27}$$

Therefore,

$$\nabla w_\varepsilon^2 \to 0, \qquad \text{a.e. } (x,t) \in \Omega_T \setminus \bar{\Omega}_T^+. \tag{4.28}$$

By Lemma 4.4, we know that Γ_T has measure zero. Thus, $\nabla w_\varepsilon^2 \to \nabla w^2$ almost everywhere in Ω_T. Finally, since $0 \leq u_\varepsilon \leq w_\varepsilon$ and $w_\varepsilon \to w$, we have

$$\theta(w_\varepsilon, u_\varepsilon) \to \theta(w,u), \qquad \text{weakly in } L^2(\Omega_T). \tag{4.29}$$

Let χ be the weak limit of u_ε^2. We now pass to the limit in (4.8) to obtain (2.7). The proof is completed. □

To conclude this section, let us give another way of expressing χ, using the so-called Young's measure. The result is stated as follows.

Proposition 4.6. *There exists a parametrized probability measure $\nu(x,t,dr)$, supported on $[0,w(x,t)]$, such that*

$$u(x,t)=\int_0^{w(x,t)} r\nu(x,t,dr), \qquad \text{a.e. } (x,t)\in\Omega\times(0,\infty), \tag{4.30}$$

$$\chi(x,t)=\int_0^{w(x,t)} r^2\nu(x,t,dr), \qquad \text{a.e. } (x,t)\in\Omega\times(0,\infty). \tag{4.31}$$

Proof. We choose the sequence u_ε such that (4.13) holds. Then, similar to [14], we can find a parametrized probability measure $\nu(x,t,dr)$ supported on $[0,w(x,t)]$, such that

$$F(u_\varepsilon^2(x,t))\overset{w}{\rightharpoonup}\int_0^{w(x,t)} F(r)\nu(x,t,dr), \qquad \in L^2(Q), \tag{4.32}$$

for any continuous function F and any bounded open set $Q\subset\Omega\times(0,\infty)$. Thus, we obtain (4.30) and (4.31). $\square$

We see that the above result gives a relation between u and χ. As soon as the parametrized measure $\nu(x,t,dr)$ is determined, the functions u and χ are automatically determined.

§5. Existence of Weak Infective Age Distributions.

In the previous section, we have proved the existence of weak solutions (w,u,χ) of (1.11)–(1.14). In particular, we have found a density u of the infectives. From our original model, we need also to find an age distribution $\rho(x,t,a)$ of the infectives. The purpose of this section is to determine such a distribution.

For convenience, let us rewrite the equation (1.3) and the conditions (1.5)–(1.6) for ρ in the present case (i.e., λ,β and γ are independent of a, and $v=w-u$):

$$\rho_t+\rho_a=\nabla\cdot[\rho\nabla w]-\lambda(x,t)\rho-\beta(x,t)\rho, \qquad (x,t,a)\in\mathbb{R}^n\times\mathbb{R}^+\times\mathbb{R}^+, \tag{5.1}$$

$$\rho\,\big|_{t=0}=\rho_0(x,a), \qquad (x,a)\in\mathbb{R}^n\times\mathbb{R}^+, \tag{5.2}$$

$$\rho\Big|_{a=0} = \gamma(x,t)[u(x,t)w(x,t) - u(x,t)^2], \qquad (x,t) \in \mathbb{R}^n \times \mathbb{R}^+, \tag{5.3}$$

We observe that if all the data involved in (5.1)–(5.3) are smooth and ρ is a classical solution of (5.1)–(5.3), smooth up to the boundary $\{t=0\}\bigcup\{a=0\}$, then the following compatibility condition should satisfy.

$$\rho_0(x,0) = \gamma(x,0)[u_0(x)w_0(x) - u_0(x)^2] \equiv \gamma(x,0)v_0(x)\int_0^\infty \rho_0(x,a)da, \qquad x \in \mathbb{R}^n. \tag{5.4}$$

Here, we should note that $\rho_0(x,a)$ and $v_0(x)$ are given functions. Also, it is reasonable to assume that

$$\begin{aligned} &\rho_0 \in L^\infty(\mathbb{R}^n \times \mathbb{R}^+)\bigcap C^1(\mathbb{R}^n \times \mathbb{R}^+), \qquad \rho_0(x,a) \geq 0, \qquad (x,a) \in \mathbb{R}^n \times \mathbb{R}^+, \\ &\int_0^\infty \rho_0(x,a)da \equiv u_0(x) \in C^0(\mathbb{R}^n)\bigcap L^\infty(\mathbb{R}^n). \end{aligned} \tag{5.5}$$

In what follows, we will keep assumptions (5.4)–(5.5). On the other hand, if an age distribution $\rho(x,t,a)$ exists on Ω_T satisfying (5.1)–(5.3) and (1.1), then, for any $\zeta \in C_0^\infty(\bar{\Omega}_T^0 \times [0,\infty))$, we have

$$\begin{aligned} &\int_{\Omega_T \times [0,\infty)} \rho\{-\zeta_t - \zeta_a + \nabla w \cdot \nabla\zeta + (\lambda+\beta)\zeta\}dxdtda \\ &+ \int_{\Omega\times[0,\infty)} \rho_0\zeta(x,0,a)dxda + \int_{\Omega_T} \gamma(uw - u^2)\zeta(x,t,0)dxdt = 0. \end{aligned} \tag{5.6}$$

This suggests us to give to following definition (compare with Definition 2.1).

Definition 5.1. Let (w,u,χ) be a weak solution of (1.11)–(1.14) on Ω_T and ρ_0 are given satisfying (5.4)–(5.5). We call $\rho \in L^\infty(\Omega_T; \mathcal{M}(0,\infty))$ a weak infective age distribution associated with (w,u,χ) on $\Omega_T \times [0,\infty)$ if for any $\zeta \in C^\infty(\bar{\Omega}_T^0 \times [0,\infty))$,

$$\begin{aligned} &\int_{\Omega_T \times [0,\infty)} \{-\zeta_t - \zeta_a + \nabla w \cdot \nabla\zeta + (\lambda+\beta)\zeta\}\rho(x,t,da)dxdt \\ &+ \int_{\Omega\times[0,\infty)} \rho_0\zeta(x,0,a)dxda + \int_{\Omega_T} \gamma(uw - \chi)\zeta(x,t,0)dxdt = 0, \end{aligned} \tag{5.7}$$

moreover, $\rho(x,t;\cdot)$ is a nonnegative measure for almost all (x,t) and

$$\int_0^\infty \rho(x,t,da) = u(x,t), \qquad \text{a.e. } (x,t) \in \Omega_T. \tag{5.8}$$

If the above holds for all $T>0$, we simply call ρ a weak infective age distribution associated with (w,u,χ).

In the above, $\mathcal{M}(0,\infty)$ denotes the set of all finite Borel measures on $[0,\infty)$. Our main result of this section is the following:

Theorem 5.2. *Let ρ_0 satisfy (5.4)–(5.5) and (w_0,u_0) be as in previous sections. Then, there exists a weak solution (w,u,χ) of (1.11)–(1.14) for which there is an associated weak infective age distribution ρ.*

To prove the above result, we let $0<\varepsilon,\delta\le 1$ and let $(w_{\varepsilon,\delta},u_{\varepsilon,\delta})$ be the solution of (3.1)–(3.4). We consider the following system:

$$\text{(5.9)}\qquad \begin{cases} \rho_t+\rho_a=\nabla\rho\cdot\nabla w_{\varepsilon,\delta}+(\Delta w_{\varepsilon,\delta}-\lambda-\beta)\rho, \\ \qquad\qquad (x,t,a)\in\mathbb{R}^n\times\mathbb{R}^+\times\mathbb{R}^+, \\ \rho\,\big|_{t=0}=\rho_0^\varepsilon\equiv\rho_0+\varepsilon\gamma(w_0-u_0), \qquad (x,a)\in\mathbb{R}^n\times\mathbb{R}^+, \\ \rho\,\big|_{a=0}=\gamma[u_{\varepsilon,\delta}w_{\varepsilon,\delta}-u_{\varepsilon,\delta}^2]\equiv\theta^{\varepsilon,\delta}, \qquad (x,t)\in\mathbb{R}^n\times\mathbb{R}^+. \end{cases}$$

Note that

$$\text{(5.10)}\qquad \rho_0^\varepsilon(x,0)=\gamma(x,0)[u_0^\varepsilon(x)w_0^\varepsilon(x)-u_0^\varepsilon(x)^2]=\theta^{\varepsilon,\delta}(x,0), \qquad \forall x\in\mathbb{R}^n.$$

This compatibility condition is needed in order to obtain a classical solution of (5.9).

We have the following lemma.

Lemma 5.3. *There exists a unique classical solution $\rho_{\varepsilon,\delta}$ of (5.9), which is given by*

$$\text{(5.11)}\qquad \rho_{\varepsilon,\delta}(\bar x,\bar t,\bar a)=\begin{cases} \rho_0^\varepsilon(\psi(0;\bar x,\bar t),\bar a-\bar t)e^{\int_0^{\bar t}\ell(\psi(t;\bar x,\bar t),t)dt}, & \bar t\le\bar a, \\ \theta^{\varepsilon,\delta}(\psi(\bar t-\bar a;\bar x,\bar t),\bar t-\bar a)e^{\int_{\bar t-\bar a}^{\bar t}\ell(\psi(t;\bar x,\bar t),t)dt}, & \bar t\ge\bar a, \end{cases}$$

where $\psi(\cdot;\bar x,\bar t)\equiv\psi_{\varepsilon,\delta}(\cdot;\bar x,\bar t)$ is the characteristics associated with $w_{\varepsilon,\delta}$:

$$\text{(5.12)}\qquad \begin{cases} \dfrac{d\psi}{dt}=-\nabla w_{\varepsilon,\delta}(\psi,t), \\ \psi\,\big|_{t=\bar t}=\bar x, \end{cases}$$

and $\ell=\Delta w_{\varepsilon,\delta}-\lambda-\beta$. Moreover, this solution satisfies

$$\text{(5.13)}\qquad \int_0^\infty\rho_{\varepsilon,\delta}(x,t,a)da=u_{\varepsilon,\delta}(x,t), \qquad (x,t)\in\mathbb{R}^n\times\mathbb{R}^+.$$

Proof. Suppose ρ is a classical solution of (5.9). We let ψ be defined by (5.12) and set

$$\widetilde{\rho}(t) = \rho(\psi(t;\bar{x},\bar{t}), t, t-\bar{t}+\bar{a}), \tag{5.14}$$

Then, we see that

$$\frac{d}{dt}\widetilde{\rho}(t) = \rho_t + \rho_a - \nabla\rho\cdot\nabla w_{\varepsilon,\delta} = \ell\widetilde{\rho}(t); \quad \widetilde{\rho}(\bar{t}) = \rho(\bar{x},\bar{t},\bar{a}). \tag{5.15}$$

Hence, we can obtain that ρ has the form (5.11). This gives the uniqueness of classical solutions of (5.9). Next, we let $\rho_{\varepsilon,\delta}$ be given by (5.11). Then, similar to the proof of Lemma 3.7, we are able to show, with some lengthy but straightforward computation, that $\rho_{\varepsilon,\delta}$ is a classical solution of (5.9). Now, we compute

$$\begin{aligned}
\int_0^\infty \rho_{\varepsilon,\delta}(\bar{x},\bar{t},\bar{a})d\bar{a} &= \int_0^{\bar{t}} \theta^{\varepsilon,\delta}(\psi(\bar{t}-\bar{a};\bar{x},\bar{t}),\bar{t}-\bar{a})e^{\int_{\bar{t}-\bar{a}}^{\bar{t}} \ell(\psi(t;\bar{x},\bar{t}),t)dt}d\bar{a}\\
&\quad + e^{\int_0^{\bar{t}} \ell(\psi(t;\bar{x},\bar{t}),t)dt}\int_{\bar{t}}^\infty \rho_0^\varepsilon(\psi(0;\bar{x},\bar{t}),\bar{a}-\bar{t})d\bar{a}\\
&= e^{\int_0^{\bar{t}} \ell(\psi(t;\bar{x},\bar{t}),t)dt}\{\int_0^{\bar{t}} \theta^{\varepsilon,\delta}(\psi(\bar{t}-\bar{a};\bar{x},\bar{t}),\bar{t}-\bar{a})e^{-\int_0^{\bar{t}-\bar{a}} \ell(\psi(t;\bar{x},\bar{t}),t)dt}d\bar{a}\\
&\quad + \int_0^\infty \rho_0^\varepsilon(\psi(0;\bar{x},\bar{t}),\bar{a})d\bar{a}\}\\
&= e^{\int_0^{\bar{t}} \ell(\psi(t;\bar{x},\bar{t}),t)dt}u_0^\varepsilon(\psi(0;\bar{x},\bar{t})\\
&\quad + \int_0^{\bar{t}}(\gamma u_{\varepsilon,\delta}w_{\varepsilon,\delta}-\gamma u_{\varepsilon,\delta})(\psi(\tau;\bar{x},\bar{t}),\tau)e^{-\int_\tau^{\bar{t}} \ell(\psi(t;\bar{x},\bar{t}),t)dt}d\tau = u_{\varepsilon,\delta}(\bar{x},\bar{t}).
\end{aligned} \tag{5.16}$$

This complete the proof of the lemma. □

We note that (see (3.30))

$$e^{\int_0^{\bar{t}} \ell(\psi(t;\bar{x},\bar{t}),t)dt} = \frac{w_{\varepsilon,\delta}(\bar{x},\bar{t})}{w_0^\varepsilon(\psi(0;\bar{x},\bar{t}))}e^{\int_0^{\bar{t}}(c-\lambda-\beta)(\psi(t;\bar{x},\bar{t}),t)dt}, \tag{5.17}$$

$$e^{\int_{\bar{t}-\bar{a}}^{\bar{t}} \ell(\psi(t;\bar{x},\bar{t}),t)dt} = \frac{w_{\varepsilon,\delta}(\bar{x},\bar{t})}{w^{\varepsilon,\delta}(\psi(\bar{t}-\bar{a};\bar{x},\bar{t}),\bar{t}-\bar{a})}e^{\int_{\bar{t}-\bar{a}}^{\bar{t}}(c-\lambda-\beta)(\psi(t;\bar{x},\bar{t}),t)dt}, \tag{5.18}$$

where (see (3.22))

$$c-\lambda-\beta = -(\frac{\lambda E_\delta(w_{\varepsilon,\delta}-\varphi_\varepsilon * u_{\varepsilon,\delta})}{w_{\varepsilon,\delta}}+\beta) \le 0. \tag{5.19}$$

Thus, from (5.11), we obtain that for $\bar{t} \leq \bar{a}$ (see (3.5)–(3.6))

$$0 \leq \rho_{\varepsilon,\delta}(\bar{x}, \bar{t}, \bar{a}) \leq \frac{\|\rho_0\|_{C^0(\Omega)}}{\varepsilon} + M(\|w_0\|_{C^0(\Omega)} + \varepsilon)^2, \tag{5.20}$$

and for $\bar{t} \geq \bar{a}$,

$$0 \leq \rho_{\varepsilon,\delta}(\bar{x}, \bar{t}, \bar{a}) \leq M(\|w_0\|_{C^0(\Omega)} + \varepsilon)^2, \tag{5.21}$$

This proves the following

Corollary 5.4. *The solution $\rho_{\varepsilon,\delta}$ satisfies*

$$\begin{aligned} 0 \leq &\rho_{\varepsilon,\delta}(\bar{x}, \bar{t}, \bar{a}) \leq \frac{\|\rho_0\|_{C^0(\Omega)}}{\varepsilon} + M(\|w_0\|_{C^0(\Omega)} + \varepsilon)^2, \\ &\forall (\bar{x}, \bar{t}, \bar{a}) \in \mathbb{R}^n \times \mathbb{R}^+ \times \mathbb{R}^+. \end{aligned} \tag{5.22}$$

Now, we are ready to prove Theorem 5.2.

Proof of Theorem 5.2. First, we will let $\delta \to 0$. By (4.5)–(4.6), we know that

$$\psi_{\varepsilon,\delta}(t; \bar{x}, \bar{t}) \to \psi_\varepsilon(t; \bar{x}, \bar{t}), \qquad \delta \to 0, \tag{5.23}$$

uniformly for $(t, \bar{x}, \bar{t})$ in any compact sets, where ψ_ε is the characteristics associated with w_ε. Thus, by (5.11) and (5.17)–(5.18),

$$\rho_{\varepsilon,\delta}(\bar{x}, \bar{t}, \bar{a}) \to \rho_\varepsilon(\bar{x}, \bar{t}, \bar{a}), \qquad \delta \to 0, \tag{5.24}$$

uniformly for all $(\bar{x}, \bar{t}, \bar{a})$ in compact sets and

$$\rho_\varepsilon(\bar{x}, \bar{t}, \bar{a}) = \begin{cases} \rho_0^\varepsilon(\psi_\varepsilon(0; \bar{x}, \bar{t}), \bar{a} - \bar{t}) e^{\int_0^{\bar{t}} \ell_\varepsilon(\psi_\varepsilon(t; \bar{x}, \bar{t}), t) dt}, & \bar{t} \leq \bar{a}, \\ \theta^\varepsilon(\psi_\varepsilon(\bar{t} - \bar{a}; \bar{x}, \bar{t}), \bar{t} - \bar{a}) e^{\int_{\bar{t}-\bar{a}}^{\bar{t}} \ell_\varepsilon(\psi_\varepsilon(t; \bar{x}, \bar{t}), t) dt}, & \bar{t} \geq \bar{a}, \end{cases} \tag{5.25}$$

with $\theta^\varepsilon = \gamma(u_\varepsilon w_\varepsilon - u_\varepsilon^2)$ and $\ell_\varepsilon = \Delta w_\varepsilon - \lambda - \beta$. By (5.13) and (5.22), using Dominated Convergence Theorem, we have

$$\int_0^\infty \rho_\varepsilon(x, t, a) da = u_\varepsilon(x, t), \qquad \forall (x, t) \in \Omega_T. \tag{5.26}$$

On the other hand, using the variational form for $\rho_{\varepsilon,\delta}$, and sending $\delta \to 0$, we have

$$
\begin{aligned}
&\int_{\Omega_T\times[0,\infty)} \{-\zeta_t - \zeta_a + \nabla w_\varepsilon \cdot \nabla\zeta + (\lambda+\beta)\zeta\}\rho_\varepsilon dxdtda \\
&+ \int_{\Omega\times[0,\infty)} \rho_0\zeta(x,0,a)dxda + \int_{\Omega_T} \gamma(u_\varepsilon w_\varepsilon - u_\varepsilon^2)\zeta(x,t,0)dxdt = 0.
\end{aligned}
\tag{5.27}
$$

Next, we need to send $\varepsilon \to 0$. By (5.26) and (5.22), we see that for any $A > 0$, $\{\rho_\varepsilon\}_{\varepsilon>0}$ is bounded in $L^\infty(\Omega_T; L^1(0,A))$. Since

$$
L^\infty(\Omega_T; L^1(0,A)) \hookrightarrow L^2(\Omega_T; \mathcal{M}(0,A)) = L^2(\Omega_T; C([0,A]))^*,
\tag{5.28}
$$

we may let (using diagonalization argument)

$$
\rho_\varepsilon \to \rho, \qquad \text{weakly in } L^2(\Omega_T; \mathcal{M}(0,A)), \quad \forall A > 0.
\tag{5.29}
$$

Then, for any $\zeta \in C_0^\infty(\Omega_T \times [0,\infty))$, to obtain (5.27), we only need to prove the following:

$$
\int_{\Omega_T\times[0,\infty)} (\nabla w_\varepsilon \cdot \nabla\zeta)\rho_\varepsilon dxdtda \to \int_{\Omega_T\times[0,\infty)} (\nabla w \cdot \nabla\zeta)\rho(x,t,da)dxdt.
\tag{5.30}
$$

We note that

$$
\int_{\Omega_T} \Big(\int_0^\infty |(\nabla w_\varepsilon)\rho_\varepsilon| da\Big)^2 dxdt = \frac{1}{4}\int_{\Omega_T} |\nabla w_\varepsilon^2|^2 \frac{u_\varepsilon^2}{w_\varepsilon^2} dxdt \le C.
\tag{5.31}
$$

Thus, $\{(\nabla w_\varepsilon)\rho_\varepsilon\}$ is bounded in $L^2(\Omega_T; L^1(0,\infty))^n$ and we may let

$$
(\nabla w_\varepsilon)\rho_\varepsilon \to \omega, \qquad \text{weakly in } L^2(\Omega_T; \mathcal{M}(0,\infty))^n.
\tag{5.32}
$$

Now, for any compact set $Q \subset \Omega_T^+ \equiv \{(x,t) \in \Omega_T \bigm| w(x,t) > 0\}$, ∇w_ε converges to ∇w in $C^{1,\alpha}(Q)$; for any compact set $B \subset \Omega_T \setminus \bar\Omega_T^+$, ∇w_ε^2 converges to 0 in $L^2(B)$; and the boundary $\partial\Omega_T^+$ of Ω_T^+ is of zero measure (see Lemma 4.4). Hence, similar to the proof of Theorem 4.5, we can identify

$$
\omega(x,t,da) = (\nabla w(x,t))\rho(x,t,da), \qquad (x,t) \in \Omega_T.
\tag{5.33}
$$

Then, we can pass to the limit in (5.27) to obtain (5.7). Finally, by (5.26) and (5.29), we immediately obtain that for any $\xi \in C_0^\infty(\Omega_T)$,

$$
\begin{aligned}
&\int_{\Omega_T} u\xi dxdt = \lim_{\varepsilon\to 0}\int_{\Omega_T} u_\varepsilon \xi dxdt \\
&= \lim_{\varepsilon\to 0}\int_{\Omega_T\times[0,\infty)} \xi\rho_\varepsilon dxdtda = \int_{\Omega_T} \xi\Big[\int_0^\infty \rho(x,t,da)\Big]dxdt.
\end{aligned}
\tag{5.34}
$$

Hence, (5.8) follows. $\square$

Remark 5.5. If there exists a $\sigma > 0$, such that

$$w_0(x) \geq \sigma, \qquad \forall x \in \mathbb{R}^n. \tag{5.35}$$

Then, (5.22) can be replaced by

$$\begin{aligned} 0 \leq & \rho_{\varepsilon,\delta}(\bar{x},\bar{t},\bar{a}) \leq \frac{\|\rho_0\|_{C^0(\Omega)}}{\sigma} + M(\|w_0\|_{C^0(\Omega)} + \varepsilon)^2, \\ & \forall(\bar{x},\bar{t},\bar{a}) \in \mathbb{R}^n \times \mathbb{R}^+ \times \mathbb{R}^+. \end{aligned} \tag{5.36}$$

In this case, we see that the weak infective age distribution ρ is actually in $L^\infty(\mathbb{R}^n \times [0,\infty); L^1(0,\infty))$.

§6. One Dimensional Case.

In this section, we study the one dimensional case. We shall show that if (w,u,χ) is a weak solution of (1.11)–(1.14), then, near the points at which $w_0(x) > 0$, we have $\chi = u^2$. In particular, if for some $\eta > 0$, $w_0(x) \geq \eta$ for all $x \in \mathbb{R}$, then $\chi = u^2$ everywhere.

Let us start with the following lemma.

Lemma 6.1. *Let $(w_\varepsilon, u_\varepsilon)$ be the solution of (3.2)–(3.4) and (4.1). Then, for any $\eta > 0$ with $w_0(x) + \varepsilon \geq \eta$, we have*

$$\|u_\varepsilon\|_{C^1(\bar{\Omega}_T)} \leq C(\eta,T)(1 + \|\nabla w_\varepsilon\|_{C(\bar{\Omega}_T)})^2. \tag{6.1}$$

Proof. By Lemma 3.3, u_ε can be expressed by (3.11), where ψ is the characteristics (3.7) corresponding to w_ε, and $h = \Delta w_\varepsilon + \gamma w_\varepsilon - \lambda - \beta$. Since w_ε solves (4.1), we have

$$\begin{aligned} &\int_0^{\bar{t}} \Delta w_\varepsilon(\psi(t;\bar{x},\bar{t}),t)dt \\ &= \int_0^{\bar{t}} [\frac{1}{w_\varepsilon}\frac{d}{dt} w_\varepsilon(\psi(t;\bar{x},\bar{t}),t) + \lambda - \frac{\lambda E_0(w_\varepsilon - \varphi_\varepsilon * u_\varepsilon)}{w_\varepsilon}]dt \\ &= \log[\frac{w_\varepsilon(\bar{x},\bar{t})}{w_0^\varepsilon(\psi(0;\bar{x},\bar{t}))}] + \int_0^{\bar{t}} [\lambda - \lambda\frac{E_0(w_\varepsilon - \varphi_\varepsilon * u_\varepsilon)}{w_\varepsilon}]dt. \end{aligned} \tag{6.2}$$

On the other hand, $D_{\bar{x}}\psi(t;\bar{x},\bar{t})$ solves (3.9) (as a function of t). Hence, by computation analogous to (6.2), we get

$$D_{\bar{x}}\psi(t;\bar{x},\bar{t}) = e^{\int_t^{\bar{t}} w_{\varepsilon xx}(\psi(t;\bar{x},\bar{t}),t)dt} = \frac{w_\varepsilon(\bar{x},\bar{t})}{w_\varepsilon(\psi(t;\bar{x},\bar{t}),t)} e^{\int_0^{\bar{t}}[\lambda-\lambda\frac{E_0(w_\varepsilon-\varphi_\varepsilon * u_\varepsilon)}{w_\varepsilon}]dt}. \tag{6.3}$$

By maximum principle and (4.1), we have

$$w_\varepsilon(x,t) \geq \eta e^{-MT}, \qquad (x,t) \in \Omega_T.$$

Thus, from (6.3) and (3.8) we obtain

$$|D_{\bar{x}}\psi(t;\bar{x},\bar{t})| \leq C(\eta,T), \tag{6.4}$$

$$|D_{\bar{t}}\psi(t;\bar{x},\bar{t})| \leq C(\eta,T)\|\nabla w_\varepsilon\|_{C(\bar{\Omega}_T)}. \tag{6.5}$$

By directly taking derivatives in (3.11), we can get (6.1). □

Theorem 6.2. *Let* $\eta > 0$ *and*

$$w_0(x) \geq \eta, \qquad \forall x \in \mathbb{R}. \tag{6.6}$$

Let (w,u,χ) *be a weak solution of (1.11)–(1.14). Then,*

$$\chi(x,t) = u(x,t)^2, \qquad \forall (x,t) \in \mathbb{R} \times (0,\infty). \tag{6.7}$$

Proof. Due to (6.6), by the Schauder type estimates, we have that for any $T > 0$, there exists a constant $C(T)$, such that

$$\|w_\varepsilon\|_{C^{2,\alpha}(\bar{\Omega}_T)} \leq C(T)(1 + \|u_\varepsilon\|_{C^\alpha(\bar{\Omega}_T)}). \tag{6.8}$$

It follows from (6.1) that both $\|w_\varepsilon\|_{C^{2,\alpha}(\bar{\Omega}_T)}$ and $\|u_\varepsilon\|_{C^1(\bar{\Omega}_T)}$ are uniformly bounded. Since (u,χ) is the weak limit of $(u_\varepsilon, u_\varepsilon^2)$, we obtain (6.7). □

Next, we would like to consider the case that w_0 does not have the uniform positive lower bound. In this case, we have the following local result.

Theorem 6.3. *Let $x_0 \in \mathbb{R}$ be such that $w(x_0) > 0$. Then, there exists a neighborhood $B_1(x_0)$ of x_0 in $\mathbb{R}$ and a $T_1 > 0$, such that*

$$\chi(x,t) = u(x,t)^2, \qquad \forall (x,t) \in B_1(x_0) \times [0, T_1]. \tag{6.9}$$

Proof. From Lemma 4.4, we know that there exists a neighborhood $B(x_0)$ of x_0 and a $T > 0$ such that for some $\eta > 0$,

$$w(x,t) \geq \eta, \qquad (x,t) \in B(x_0) \times [0, T]. \tag{6.10}$$

Then, it follows that for $\varepsilon > 0$ small enough,

$$w_\varepsilon(x,t) \geq \frac{1}{2}\eta, \qquad (x,t) \in B(x_0) \times [0, T]. \tag{6.11}$$

Hence, by the interior $C^{1+\alpha}$-estimates, we have some constant $C > 0$, independent of $\varepsilon > 0$, such that

$$\|w_\varepsilon\|_{C^{1,\alpha}(\bar{B}(x_0)\times[0,T])} \leq C, \qquad \forall \varepsilon > 0. \tag{6.12}$$

Let ψ_ε be the characteristics corresponding to w_ε. By (3.7) and (6.12), we can find a small neighborhood $B_1(x_0) \times [0, T_1] \subset B(x_0) \times [0, T]$, such that for any $(\bar{x}, \bar{t}) \in B_1(x_0) \times [0, T_1]$, $\psi_\varepsilon(t; \bar{x}, \bar{t})$ always stay in $B(x_0)$ for $t \in [0, T_1]$. Therefore, we have

$$|D_{\bar{x}}\psi_\varepsilon(t; \bar{x}, \bar{t})| \leq C(\eta, T_1), \tag{6.13}$$

$$|D_{\bar{t}}\psi_\varepsilon(t; \bar{x}, \bar{t})| \leq C(\eta, T_1)\|\nabla w_\varepsilon\|_{C(\bar{B}(x_0)\times[0,T_1])}, \tag{6.14}$$

for all $t \in [0, t_1]$ and $(\bar{x}, \bar{t}) \in B_1(x_0) \times [0, T_1]$. By the same argument as in the proof of Theorem 6.2, we obtain (6.9). $\square$

Corollary 6.4. *Let $I \subset \mathbb{R}$ be a compact set such that*

$$w_0(x) > 0, \qquad \forall x \in I. \tag{6.15}$$

Then, there exists a neighborhood $\mathcal{O} \subset \mathbb{R} \times [0, \infty)$ of $I \times \{0\}$, such that

$$\chi(x,t) = u(x,t)^2, \qquad (x,t) \in \mathcal{O}. \tag{6.16}$$

The proof is immediate.

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