$\mathcal{H}_\infty$ Design of General Multirate Sampled-Data Control Systems*

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Abstract

Direct digital design of general multirate sampled-data systems is considered. To tackle causality constraints, a new and natural framework is proposed using nest operators and nest algebras. Based on this framework explicit solutions to the $\mathcal{H}_\infty$ and $\mathcal{H}_2$ multirate control problems are developed in the frequency domain.

Keywords: multirate systems, digital control, sampled-data systems, discrete systems, $\mathcal{H}_\infty$ optimization, $\mathcal{H}_2$ optimization, causality constraint, nest algebra.

1 Introduction

There are several reasons to use a multirate sampling scheme in digital control systems:

- In complex, multivariable control systems, often it is unrealistic, or sometimes impossible, to sample all physical signals uniformly at one single rate. In such situations, one is forced to use multirate sampling.
- In general one gets better performance if one can sample and hold faster. But faster A/D and D/A conversions mean higher cost in implementation. For signals with different bandwidths, better trade-offs between performance and implementation cost can be obtained using A/D and D/A converters at different rates.
- Multirate controllers are in general time-varying. Thus multirate control systems can achieve what single-rate systems cannot; for example, gain margin improvement [25, 16], simultaneous stabilization [25, 31] and decentralized control [2, 41, 36].

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- Multirate controllers are normally more complex than single-rate ones; but often they are periodic in a certain sense and hence can be implemented on microprocessors via difference equations with finitely many coefficients. Therefore, like single-rate controllers, multirate controllers do not violate the finite memory constraint in microprocessors.

The study of multirate systems began in late 1950's [26, 22, 23]; recent interests are reflected in the LQG/LQR designs [7, 1, 9, 29], the parametrization of all stabilizing controllers [27, 34], and the work in [30, 3, 19, 10, 36]. The controller parametrization in [27, 34] provides a basis for designing optimal multirate systems. However, the special structure due to causality in the feedthrough terms of lifted controllers presents a difficult constraint in design; treating this causality constraint is the new feature in multirate optimal design.

Causality constraints also arise in discrete-time periodic control [25], where after lifting, the feedthrough terms in controllers must be block lower-triangular. Explicit solutions were obtained for the one-block $\mathcal{H}_\infty$ problem [14, 17] and $\mathcal{H}_2$ problem [42]. However, these results do not generalize easily to multirate systems since most multirate designs leads to four-block problems, i.e., the transfer matrices in the associated model-matching problems are in general nonsquare.

Our work has been greatly influenced by the recent trend in sampled-data research, namely, direct digital design based on continuous-time performance specs. Related work on single-rate sampled-data design has been completed in $\mathcal{H}_2$ [8, 24, 6] and $\mathcal{H}_\infty$ [20, 40, 5, 38, 39, 37, 21] frameworks. In [33], we performed direct designs for a multirate system with a uniform sampling rate and a uniform hold rate and proposed effective ways to tackle the causality constraint. Though the setup in [33] captures the essential issue of causality (in a simplified form), it also limits the applicability of the results.

Our goal in this paper is to treat *general* multirate systems. In order to do so, a general framework for attacking causality constraints is developed; this is based on ideas from nest spaces and nest algebras. Under this framework, the results on causality [27, 34] become quite transparent; moreover, and more importantly, explicit solutions are obtained for direct multirate designs with $\mathcal{H}_\infty$ and $\mathcal{H}_2$ performance criteria.

Setup

To bring in the multirate sampled-data setup, we need to define precisely the two basic elements, the periodic sampler S_τ and the (zero-order) hold H_τ (the subscript denotes the period): S_τ maps a continuous signal to a discrete signal and is defined via

$$\psi = S_\tau y \Longleftrightarrow \psi(k) = y(k\tau).$$

H_τ maps discrete to continuous via

$$u = H_\tau v \Longleftrightarrow u(t) = v(k), \qquad k\tau \le t < (k+1)\tau.$$

Note that the sampler and hold are synchronized at $t = 0$. The signals may be vector-valued; in this case, for example, $\psi = S_\tau y$ simply means

$$\begin{bmatrix} \psi_1 \\ \vdots \\ \psi_m \end{bmatrix} = \begin{bmatrix} S_\tau & & \\ & \ddots & \\ & & S_\tau \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix},$$

which corresponds to grouping m samplers with the same rate together.

The general multirate sampled-data system is shown in Figure 1. We have used continuous arrows for continuous signals and dotted arrows for discrete signals. Here, G is the continuous-

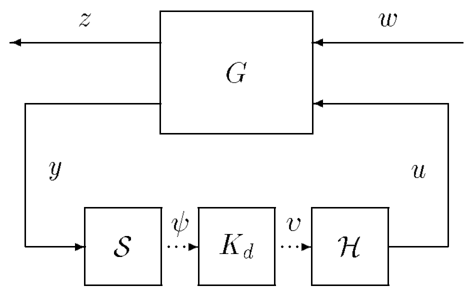

Figure 1: The general multirate sampled-data setup

time generalized plant with two inputs, the exogenous input w and the control input u, and two outputs, the signal z to be controlled and the measured signal y. $\mathcal{S}$ and $\mathcal{H}$ are multirate sampling and hold operators and are defined as follows:

$$\mathcal{S} = \begin{bmatrix} S_{m_1 h} & & \\ & \ddots & \\ & & S_{m_p h} \end{bmatrix}, \quad \mathcal{H} = \begin{bmatrix} H_{n_1 h} & & \\ & \ddots & \\ & & H_{n_q h} \end{bmatrix}.$$

These correspond to sampling p channels of y periodically with periods $m_i h$, $i = 1, \cdots, p$, respectively and holding q channels of v with periods $n_j h$, $j = 1, \cdots, q$ respectively. Here m_i and n_j are different integers and h is a real number referred as the *base period.* If we partition the signals accordingly

$$y = \begin{bmatrix} y_1 \\ \vdots \\ y_p \end{bmatrix}, \quad \psi = \begin{bmatrix} \psi_1 \\ \vdots \\ \psi_p \end{bmatrix}, \quad v = \begin{bmatrix} v_1 \\ \vdots \\ v_q \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ \vdots \\ u_q \end{bmatrix},$$

then

$$\begin{aligned} \psi_i(k) &= y_i(k m_i h), \quad i = 1, \cdots, p \\ u_j(t) &= v_j(k), \quad k n_j h \le t < (k+1) n_j h, \quad j = 1, \cdots, q. \end{aligned}$$

We shall allow each channel in y and u to be vector-valued as well; thus without loss of generality we can assume that no two m_i are equal and neither are two n_j. K_d is the discrete-time multirate controller, implemented via a microprocessor; it is synchronized with $\mathcal{S}$ and $\mathcal{H}$ in the sense that it inputs a value from the i-th channel at times $k(m_i h)$ and outputs a value to the j-th channel at $k(n_j h)$.

Figure 1 gives a compact way of describing multirate systems. It is clear that this model captures all multirate systems in which the rates are rationally related, i.e., the ratio of any two rates is rational. Note that any common factor among the m_i and n_j can be absorbed into h; thus we can assume without loss of generality that the greatest common factor among m_i and n_j is 1. With this condition, for any multirate system in which rates are rationally related, there exists a unique base period h and a unique set of integers m_i and n_j so that the system can be put into the framework of Figure 1.

Now we introduce a useful notation: Given an operator K and an operator matrix

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix},$$

the associated linear fractional transformation is denoted

$$\mathcal{F}(P, K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}.$$

Here we assume that the domains and co-domains of the operators are compatible and the inverse exists.

Throughout the paper, G is linear time-invariant (LTI) and finite-dimensional and K_d is linear; additional properties of K_d will be discussed in Section 3. The closed-loop map $w \mapsto z$ in Figure 1 is $\mathcal{F}(G, \mathcal{H}K_d\mathcal{S})$. We can now state our main synthesis problem: Design a K_d to give closed-loop stability (to be made precise in Section 4) and achieve the $\mathcal{H}_\infty$ performance requirement $\|\mathcal{F}(G, \mathcal{H}K_d\mathcal{S})\| < \gamma$, where $\gamma > 0$ is a pre-specified performance level and the norm is $\mathcal{L}_2$-induced. Note that the performance requirement is defined on the continuous-time map and so intersample behaviour is captured in the design spec. Such continuous-time specs are natural since sampled-data systems operate in a continuous-time environment, though controllers are digital. Necessary and sufficient conditions will be given under which this multirate $\mathcal{H}_\infty$ control problem is solvable; once solvable, an explicit solution will also be given.

Organization

The rest of the paper is organized as follows. In Section 2 we present some concepts and facts about nest operators and nest algebras. These have direct applications in subsequent sections.

Section 3 discusses desirable properties for multirate controllers; they are periodicity, causality, and finite-dimensionality. Causality constraints are defined using operators between appropriate nests. This provides a natural and transparent framework for studying causality constraints in multirate systems.

Section 4 deals with internal stability of the setup in Figure 1 and relates it to internal stability of some discrete-time system.

Section 5 contains the main contribution of this paper, namely, an explicit solution to the multirate $\mathcal{H}_\infty$ control problem. This is achieved by first reducing it to a constrained $\mathcal{H}_\infty$ model-matching problem and then solving the latter problem using results in Section 2. A frequency-domain approach is used consistently.

In Section 6 we briefly consider the $\mathcal{H}_2$-optimal design of general multirate systems. The techniques developed in this paper also yield an explicit solution to the $\mathcal{H}_2$ problem.

Finally, Section 6 contains some concluding remarks.

The notation is quite standard and will be defined when introduced. Throughout the paper, we choose to use λ-transforms instead of z-transforms, where $\lambda = z^{-1}$; in this case, discrete-time spaces such as $\mathcal{H}_2$ and $\mathcal{H}_\infty$ are defined on the open unit disk. Finally, if G is an LTI system, $\hat{G}$ denotes its transfer matrix.

2 Preliminaries

In this section we address some topics and computation on nests and nest algebras which are useful in the sequel. We shall restrict our attention to finite-dimensional spaces; more general treatment can be found in [4, 12].

Nests, Nest Operators, and Nest Algebras

Let $\mathcal{X}$ be a finite-dimensional space. A *nest* in $\mathcal{X}$, denoted $\{\mathcal{X}_i\}$, is a chain of subspaces in $\mathcal{X}$, including $\{0\}$ and $\mathcal{X}$, with the nonincreasing ordering:

$$\mathcal{X} = \mathcal{X}_0 \supseteq \mathcal{X}_1 \supseteq \cdots \supseteq \mathcal{X}_{n-1} \supseteq \mathcal{X}_n = \{0\}.$$

Let $\mathcal{X}$ and $\mathcal{Y}$ be both finite-dimensional inner-product spaces with nests $\{\mathcal{X}_i\}$ and $\{\mathcal{Y}_i\}$ respectively. Assume the two nests have the same number of subspaces, say, $n+1$ as above. A linear map $T : \mathcal{X} \to \mathcal{Y}$ is a *nest operator* if

$$T\mathcal{X}_i \subseteq \mathcal{Y}_i, \quad i = 0, 1, \cdots, n. \tag{1}$$

This gives $n+1$ relations; the first and the last are trivially satisfied. We shall allow repetitions in $\{\mathcal{X}_i\}$ and $\{\mathcal{Y}_i\}$. Thus redundancy may occur in (1) and in the results to follow. However, for computation one can eliminate this redundancy as follows: If $\mathcal{X}_i = \mathcal{X}_{i+1}$, the i-th relation, namely, $T\mathcal{X}_i \subseteq \mathcal{Y}_i$, is redundant since $\mathcal{Y}_i \supseteq \mathcal{Y}_{i+1}$ and therefore can be eliminated; similarly, if $\mathcal{Y}_i = \mathcal{Y}_{i+1}$, we eliminate the $(i+1)$-st relation. Let $\Pi_{\mathcal{X}_i} : \mathcal{X} \to \mathcal{X}_i$ and $\Pi_{\mathcal{Y}_i} : \mathcal{Y} \to \mathcal{Y}_i$ be orthogonal projections. Then the condition in (1) is equivalent to

$$(I - \Pi_{\mathcal{Y}_i})T\Pi_{\mathcal{X}_i} = 0, \quad i = 0, 1, \cdots, n.$$

The set of all such operators is denoted $\mathcal{N}(\{\mathcal{X}_i\}, \{\mathcal{Y}_i\})$ and abbreviated $\mathcal{N}(\{\mathcal{X}_i\})$ if $\{\mathcal{X}_i\} = \{\mathcal{Y}_i\}$. The following properties are straightforward to verify.

Lemma 1

(a) If $T_1 \in \mathcal{N}(\{\mathcal{X}_i\}, \{\mathcal{Y}_i\})$ and $T_2 \in \mathcal{N}(\{\mathcal{Y}_i\}, \{\mathcal{Z}_i\})$, then $T_2T_1 \in \mathcal{N}(\{\mathcal{X}_i\}, \{\mathcal{Z}_i\})$.

(b) $\mathcal{N}(\{\mathcal{X}_i\})$ forms an algebra, called nest algebra.

(c) If $T \in \mathcal{N}(\{\mathcal{X}_i\})$ and T is invertible, then $T^{-1} \in \mathcal{N}(\{\mathcal{X}_i\})$.

Factorization

It is a fact that every operator on $\mathcal{X}$ can be factored as the product of a unitary operator and an operator in $\mathcal{N}(\{\mathcal{X}_i\})$.

Lemma 2 *Let T be an operator on $\mathcal{X}$.*

(a) There exists a unitary operator U_1 on $\mathcal{X}$ and an operator R_1 in $\mathcal{N}(\{\mathcal{X}_i\})$ such that $T = U_1R_1$.

(b) There exists an operator R_2 in $\mathcal{N}(\{\mathcal{X}_i\})$ and a unitary operator U_2 on $\mathcal{X}$ such that $T = R_2U_2$.

Note that R_1 and R_2 are invertible if T is invertible. We shall give an elementary proof of this lemma, for it provides a way to compute such factorizations via the well-known QR factorization.

Proof of Lemma 2 We shall look at part (a); part (b) follows similarly. Since $\mathcal{X}_i \supseteq \mathcal{X}_{i+1}$, we write $(\mathcal{X}_{i+1})^{\perp}_{\mathcal{X}_i}$ as the orthogonal complement of $\mathcal{X}_{i+1}$ in $\mathcal{X}_i$. Decompose $\mathcal{X}$ into

$$\mathcal{X} = (\mathcal{X}_1)^{\perp}_{\mathcal{X}_0} \oplus (\mathcal{X}_2)^{\perp}_{\mathcal{X}_1} \oplus \cdots \oplus (\mathcal{X}_n)^{\perp}_{\mathcal{X}_{n-1}}.$$

It follows that under this decomposition any operator R belongs to $\mathcal{N}(\{\mathcal{X}_i\})$ iff its matrix is block lower-triangular, all the diagonal blocks being square. Thus it suffices to show that for any matrix T on $\mathcal{X}$ we can write $T = U_1R_1$ where U_1 is orthogonal and R_1 is block lower-triangular. This follows from a QR type of factorization for square matrices: $T = U_1R_1$ with U_1 orthogonal and R_1 lower-triangular; partition R_1 accordingly to get that R_1 is also block lower-triangular. **QED**

A Distance Problem

Finally, we look at a distance problem. Let $\mathcal{X}$ and $\mathcal{Y}$ be finite-dimensional inner-product spaces with nests $\{\mathcal{X}_i\}$ and $\{\mathcal{Y}_i\}$. Let T be an operator $\mathcal{X} \to \mathcal{Y}$. We want to find the distance (via induced norms) of T to $\mathcal{N}(\{\mathcal{X}_i\}, \{\mathcal{Y}_i\})$, abbreviated $\mathcal{N}$:

$$\operatorname{dist}(T, \mathcal{N}) := \inf_{Q \in \mathcal{N}} \|T - Q\|. \tag{2}$$

It is clear that

$$\operatorname{dist}(T, \mathcal{N}) \geq \max_i \|(I - \Pi_{\mathcal{Y}_i})T\Pi_{\mathcal{X}_i}\|.$$

Theorem 1

$$\operatorname{dist}(T, \mathcal{N}) = \max_i \|(I - \Pi_{\mathcal{Y}_i}) T \Pi_{\mathcal{X}_i}\|.$$

This is Corollary 9.2 in [12] specialized to operators on finite-dimensional spaces; it is an extension of a result in [32, 11] on norm-preserving dilation of operators, which is specialized to matrices below. We denote the Moore-Penrose generalized inverse of a matrix M by $M^\dagger$.

Lemma 3 *Assume that A, B, C are fixed matrices of appropriate dimensions. Then*

$$\inf_X \left\| \begin{bmatrix} C & A \\ X & B \end{bmatrix} \right\| = \max\{\| \begin{bmatrix} C & A \end{bmatrix} \|, \| \begin{bmatrix} A \\ B \end{bmatrix} \|\} := \alpha.$$

Moreover, a minimizing X is given by

$$X = -BA^*(\alpha I - AA^*)^\dagger C.$$

It will be of interest to us how to compute a Q to achieve the infimum in (2); this can be done by sequentially applying Lemma 3:

Step 1 Decompose the spaces $\mathcal{X}$ and $\mathcal{Y}$:

$$\begin{aligned} \mathcal{X} &= (\mathcal{X}_1)^\perp_{\mathcal{X}_0} \oplus (\mathcal{X}_2)^\perp_{\mathcal{X}_1} \oplus \cdots \oplus (\mathcal{X}_n)^\perp_{\mathcal{X}_{n-1}} \\ \mathcal{Y} &= (\mathcal{Y}_1)^\perp_{\mathcal{Y}_0} \oplus (\mathcal{Y}_2)^\perp_{\mathcal{Y}_1} \oplus \cdots \oplus (\mathcal{Y}_n)^\perp_{\mathcal{Y}_{n-1}}. \end{aligned}$$

We get matrix representations for T and Q:

$$T = \begin{bmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ T_{21} & T_{22} & \cdots & T_{2n} \\ \vdots & \vdots & & \vdots \\ T_{n1} & T_{n2} & \cdots & T_{nn} \end{bmatrix}, \quad Q = \begin{bmatrix} Q_{11} & 0 & \cdots & 0 \\ Q_{21} & Q_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ Q_{n1} & Q_{n2} & \cdots & Q_{nn} \end{bmatrix},$$

Q being block lower-triangular.

Step 2 Define $X_{ij} = T_{ij} - Q_{ij}$, $i \geq j$, and

$$P = \begin{bmatrix} X_{11} & T_{12} & \cdots & T_{1n} \\ X_{21} & X_{22} & \cdots & T_{2n} \\ \vdots & \vdots & & \vdots \\ X_{n1} & X_{n2} & \cdots & X_{nn} \end{bmatrix}.$$

The problem reduces to

$$\min_{X_{ij}} \|P\|,$$

where T_{ij}, $i < j$, are fixed. Minimizing X_{ij} can be selected as follows. First, set $X_{11}, X_{21}, \cdots, X_{n1}$ and $X_{n2}, \cdots, X_{nn}$ to zero. Second, choose X_{22} by Lemma 3 such that the norm of the matrix $(I - \Pi_{\mathcal{Y}_2})P\Pi_{\mathcal{X}_1}$ (obtained by retaining the first 2 block rows and the last $n-1$ block columns in P) is minimized:

$$\|(I - \Pi_{\mathcal{Y}_2})P\Pi_{\mathcal{X}_1}\| = \max\{\|(I - \Pi_{\mathcal{Y}_1})T\Pi_{\mathcal{X}_1}\|, \|(I - \Pi_{\mathcal{Y}_2})T\Pi_{\mathcal{X}_2}\|\}.$$

Fix this X_{22}. Third, choose $\begin{bmatrix} X_{32} & X_{33} \end{bmatrix}$ again by Lemma 3 to minimize

$$\|(I - \Pi_{\mathcal{Y}_3})P\Pi_{\mathcal{X}_2}\| = \max\{\|(I - \Pi_{\mathcal{Y}_2})T\Pi_{\mathcal{X}_1}\|, \|(I - \Pi_{\mathcal{Y}_3})T\Pi_{\mathcal{X}_3}\|\}.$$

In this way, we can find all X_{ij} such that

$$\min_{X_{ij}} \|P\| = \max_i \|(I - \Pi_{\mathcal{Y}_i})T\Pi_{\mathcal{X}_i}\|.$$

This procedure also gives a constructive proof of the theorem.

3 Multirate Systems

In this section we shall examine the multirate controller K_d in Figure 1 as a discrete-time linear operator. To be studied are three basic properties: periodicity, causality, and finite dimensionality.

Periodicity

The sampled-data controller $\mathcal{H}K_d\mathcal{S}$ is in general time-varying. However, the operation at each channel of $\mathcal{S}$ and $\mathcal{H}$ is periodic. Let

$$l = \mathrm{LCM}\,\{m_1, \cdots, m_p, n_1, \cdots, n_q\},$$

where LCM means least common multiple. Thus $\sigma := lh$ is the least common period for all sampling and hold channels, i.e., σ is the least time interval in which the sampling and hold schedule repeats itself. K_d can be chosen so that $\mathcal{H}K_d\mathcal{S}$ is σ-periodic in continuous-time. For this, we need a few definitions.

Let ℓ be the space of sequences, perhaps vector-valued, defined on the time set $\{0, 1, 2, \cdots\}$. Let U be the unit time delay on ℓ and U^* the unit time advance. Define the integers

$$\begin{aligned} \bar{m}_i &= \frac{l}{m_i}, \quad i = 1, 2, \cdots, p \\ \bar{n}_j &= \frac{l}{n_j}, \quad j = 1, 2, \cdots, q. \end{aligned}$$

We say K_d is (m_i, n_j)-*periodic* if

$$\begin{bmatrix} (U^*)^{\bar{n}_1} & & \\ & \ddots & \\ & & (U^*)^{\bar{n}_q} \end{bmatrix} K_d \begin{bmatrix} U^{\bar{m}_1} & & \\ & \ddots & \\ & & U^{\bar{m}_p} \end{bmatrix} = K_d.$$

This means shifting the i-th input channel by $\bar{m}_i$ time units ($\bar{m}_i m_i h = \sigma$) corresponds to shifting the j-th output channel by $\bar{n}_j$ units ($\bar{n}_j n_j h = \sigma$). Thus $\mathcal{H}K_d\mathcal{S}$ is σ-periodic in continuous time iff K_d is (m_i, n_j)-periodic. Since G is LTI, it follows that the sampled-data system in Figure 1 is σ-periodic if K_d is (m_i, n_j)-periodic. We shall refer σ as the *system period.*

Now we lift K_d to get an LTI system. For an integer $m > 0$, define the discrete lifting operator L_m via $\underline{v} = L_m v$:

$$\{v(0), v(1), \cdots\} \mapsto \left\{ \begin{bmatrix} v(0) \\ \vdots \\ v(m-1) \end{bmatrix}, \begin{bmatrix} v(m) \\ \vdots \\ v(2m-1) \end{bmatrix}, \cdots \right\}.$$

L_m maps ℓ to ℓ^m, the external direct sum of m copies of ℓ. If the underlying period for v is τ, then the underlying period for $\underline{v}$ is $m\tau$. Now extend the input and output spaces of K_d so that the underlying period is σ; this corresponds to lifting the controller K_d in the following way:

$$\underline{K_d} := \begin{bmatrix} L_{\bar{n}_1} & & \\ & \ddots & \\ & & L_{\bar{n}_q} \end{bmatrix} K_d \begin{bmatrix} L_{\bar{m}_1}^{-1} & & \\ & \ddots & \\ & & L_{\bar{m}_p}^{-1} \end{bmatrix}.$$

It is an easy matter to check, see, e.g., [28], that the lifted controller $\underline{K_d}$ is LTI iff K_d is (m_i, n_j)-periodic.

Causality

Figure 1 is a real-time system. So for K_d to be implementable, $\mathcal{H}K_d\mathcal{S}$ must be causal in continuous time. This implies that $\underline{K_d}$, as a single-rate system, must be causal; and moreover, the feedthrough term $\underline{D}$ in $\underline{K_d}$ must satisfy a certain constraint, that is, some blocks in $\underline{D}$ must be zero [27, 34]. Now let us characterize this constraint on $\underline{D}$ using nest operators.

Write $\underline{v} = \underline{K_d}\,\underline{\psi}$; then $\underline{v}(0) = \underline{D}\underline{\psi}(0)$, where by definitions

$$\begin{aligned} \underline{\psi}(0) &= \left(\begin{bmatrix} L_{\bar{m}_1} & & \\ & \ddots & \\ & & L_{\bar{m}_p} \end{bmatrix} \begin{bmatrix} \psi_1 \\ \vdots \\ \psi_p \end{bmatrix} \right)(0) \\ &= \begin{bmatrix} \psi_1(0)' & \cdots & \psi_1(\bar{m}_1 - 1)' & \cdots & \psi_p(0)' & \cdots & \psi_p(\bar{m}_p - 1)' \end{bmatrix}' \end{aligned}$$

Note that $\psi_i(k)$ is sampled at $t = km_i h$. Similarly,

$$\underline{v}(0) = \begin{bmatrix} v_1(0)' & \cdots & v_1(\bar{n}_1 - 1)' & \cdots & v_p(0)' & \cdots & v_p(\bar{n}_q - 1)' \end{bmatrix}'$$

and $v_j(k)$ occurs at $t = kn_j h$. Let Σ be the set of sampling or hold instants in the interval $[0, \sigma)$ (modulo the base period h), i.e.,

$$\Sigma := \left(\bigcup_i \{0, m_i, 2m_i, \cdots, l - m_i\} \right) \bigcup \left(\bigcup_j \{0, n_j, 2n_j, \cdots, l - n_j\} \right).$$

This is a finite set of, say, $n+1$ elements (not counting repetitions); order Σ increasingly ($\sigma_r < \sigma_{r+1}$):

$$\Sigma = \{\sigma_r : \ r = 0, 1, \cdots, n\}.$$

Let $\underline{\psi}(0)$ and $\underline{v}(0)$ live in the finite-dimensional spaces $\mathcal{X}$ and $\mathcal{Y}$ respectively. For $r = 0, 1, \cdots, n$, define

$$\begin{aligned} \mathcal{X}_r &= \operatorname{span}\{\underline{\psi}(0) : \ \psi_i(k) = 0 \text{ if } km_i < \sigma_r\} \\ \mathcal{Y}_r &= \operatorname{span}\{\underline{v}(0) : \ v_j(k) = 0 \text{ if } kn_j < \sigma_r\}. \end{aligned}$$

$\mathcal{X}_r$ and $\mathcal{Y}_r$ correspond to, respectively, the inputs and outputs occurred after and including time $\sigma_r h$. It is easily checked that $\{\mathcal{X}_r\}$ and $\{\mathcal{Y}_r\}$ are nests and that the causality condition on $\underline{D}$ (the output at time $\sigma_r h$ depends only on inputs up to $\sigma_r h$) is exactly

$$\underline{D}\mathcal{X}_r \subseteq \mathcal{Y}_r, \qquad r = 0, 1, \cdots, n.$$

Thus we define $\underline{D}$ to be (m_i, n_j)-*causal* if $\underline{D} \in \mathcal{N}(\{\mathcal{X}_r\}, \{\mathcal{Y}_r\})$. This is the same causality constraint in [27, 34] defined in terms of the elements of $\underline{D}$.

For later benefit, we define $\underline{D}$ to be (m_i, n_j)-*strictly causal* if

$$\underline{D}\mathcal{X}_r \subseteq \mathcal{Y}_{r+1}, \qquad r = 0, 1, \cdots, n-1.$$

This means that the output at time $\sigma_{r+1}h$ depends only on inputs up to time $\sigma_r h$.

The following lemma, which is straightforward to prove, justifies our use of terminology from a continuous-time viewpoint.

Lemma 4

(a) $\mathcal{H}K_d\mathcal{S}$ is causal in continuous time iff $\underline{K_d}$ is causal and $\underline{D}$ is (m_i, n_j)-causal.

(b) $\mathcal{H}K_d\mathcal{S}$ is strictly causal in continuous time iff $\underline{K_d}$ is causal and $\underline{D}$ is (m_i, n_j)-strictly causal.

Some conclusions on causality issues [27] are transparent under this new formulation.

Lemma 5

(a) If $\underline{D}_1$ is (m_i, p_k)-causal and $\underline{D}_2$ is (p_k, n_j)-causal, then $\underline{D}_2\underline{D}_1$ is (m_i, n_j)-causal; furthermore, if $\underline{D}_1$ or $\underline{D}_2$ is strictly causal, then $\underline{D}_2\underline{D}_1$ is also strictly causal.

(b) If $\underline{D}$ is (m_i, m_i)-causal and invertible, then $\underline{D}^{-1}$ is (m_i, m_i)-causal.

(c) If $\underline{D}$ is (m_i, m_i)-strictly causal, then $(I - \underline{D})^{-1}$ exists and is (m_i, m_i)-causal.

Proof Part (a) follows immediately from Lemma 4:

$$\begin{aligned}
&\underline{D}_1, \underline{D}_2 \ \text{ are causal} \\
&\Rightarrow \ \mathcal{H}\underline{D}_1\mathcal{S}, \mathcal{H}\underline{D}_2\mathcal{S} \ \text{ are causal in continuous time} \\
&\Rightarrow \ \mathcal{H}\underline{D}_2\underline{D}_1\mathcal{S} = \mathcal{H}\underline{D}_2\mathcal{S}\mathcal{H}\underline{D}_1\mathcal{S} \ \text{ is causal in continuous time} \\
&\Rightarrow \ \underline{D}_2\underline{D}_1 \ \text{ is causal.}
\end{aligned}$$

Part (a) also follows from Lemma 1 (a) by some renumbering of the indices. Part (b) follows directly from Lemma 1 (c). For part (c), note that under appropriate decomposition, $\underline{D}$ is strictly block lower-triangular; so $(I - \underline{D})^{-1}$ exists and is (m_i, m_i)-causal [part (b)]. **QED**

Let us define K_d to be (m_i, n_j)-*causal* if $\underline{K_d}$ is causal and $\underline{D}$ is (m_i, n_j)-causal.

Finite Dimensionality

We assume K_d is (m_i, n_j)-periodic and -causal. Then $\underline{K_d}$ is LTI and causal. To get finite-dimensional difference equations for K_d, we further assume $\underline{K_d}$ is finite-dimensional. Thus $\underline{K_d}$ has a state model

$$\underline{\hat{K}_d}(\lambda) = \left[\begin{array}{c|ccc} A & B_1 & \cdots & B_p \\ \hline C_1 & D_{11} & \cdots & D_{1p} \\ \vdots & \vdots & & \vdots \\ C_q & D_{q1} & \cdots & D_{qp} \end{array}\right].$$

Let the state for $\underline{K_d}$ be η. The corresponding equations for $\underline{K_d}$ ($\underline{v} = \underline{K_d}\,\underline{\psi}$) are

$$\begin{aligned}
\eta(k+1) &= A\eta(k) + \sum_{i=1}^{p} B_i \underline{\psi}_i(k) \\
\underline{v}_j(k) &= C_j\eta(k) + \sum_{i=1}^{p} D_{ji}\underline{\psi}_i(k), \quad j = 1, 2, \cdots, q.
\end{aligned}$$

Note that $\underline{\psi}_i = L_{\bar{m}_i}\psi_i$ and $\underline{v}_j = L_{\bar{n}_j} v_j$. Partitioning the matrices accordingly

$$\begin{aligned}
B_i &= \begin{bmatrix} (B_i)_0 & \cdots & (B_i)_{\bar{m}_i - 1} \end{bmatrix}, \\
C_j &= \begin{bmatrix} (C_j)_0 \\ \vdots \\ (C_j)_{\bar{n}_j - 1} \end{bmatrix}, \quad D_{ji} = \begin{bmatrix} (D_{ji})_{00} & \cdots & (D_{ji})_{0, \bar{m}_i - 1} \\ \vdots & & \vdots \\ (D_{ji})_{\bar{n}_j - 1, 0} & \cdots & (D_{ji})_{\bar{n}_j - 1, \bar{m}_i - 1} \end{bmatrix}
\end{aligned}$$

(certain blocks in D_{ji} must be zero for the causality constraint), we get the difference equations for K_d ($v = K_d\psi$):

$$\eta(k+1) = A\eta(k) + \sum_{i=1}^{p} \sum_{s=0}^{\bar{m}_i - 1} (B_i)_s \psi_i(k\bar{m}_i + s) \tag{3}$$

$$v_j(k\bar{n}_j + r) = (C_j)_r \eta(k) + \sum_{i=1}^{p} \sum_{s=0}^{\bar{m}_i - 1} (D_{ji})_{rs} \psi_i(k\bar{m}_i + s), \tag{4}$$

where the indices in (4) go as follows: $j = 1, 2, \cdots, q$ and $r = 0, 1, \cdots, \bar{n}_j - 1$. These are the equations for implementing K_d on microprocessors and they require only finite memory. Note that the state vector η for K_d is updated every system period σ.

In summary, in this paper we are interested in the class of multirate K_d which are (m_i, n_j)-periodic and -causal and finite-dimensional; this class is called the *admissible* class of K_d and can be modeled by difference equations (3-4) with $\underline{D}$ (m_i, n_j)-causal. The corresponding admissible class of $\underline{K_d}$ is characterized by LTI, causal, and finite-dimensional $\underline{K_d}$ with the same constraint on $\underline{D}$.

The causality constraint, namely, that $\underline{D}$ must be a nest operator, is the new feature in lifted multirate systems, which is the main concern in the subsequent designs. A seemingly easy way out would be to take $\underline{D} = 0$, which corresponds to delay the control input u by a system period σ. However, we would like to argue that this would result in serious performance degradation since for complex multirate systems, the system periods are usually appreciably large.

4 Internal Stability

In this section we look at stability of Figure 1. We assume the continuous G is LTI, causal, and finite-dimensional; partition G as follows:

$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix}.$$

G has a state model

$$\hat{G}(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right].$$

Let the plant state be x and the controller state be η (K_d is admissible). Note that the system in Figure 1 is σ-periodic. Define the continuous-time vector

$$x_{sd}(t) := \begin{bmatrix} x(t) \\ \eta(k) \end{bmatrix}, \qquad k\sigma \le t < (k+1)\sigma.$$

The (autonomous) system in Figure 1 is *internally stable*, or K_d *internally stabilizes* G, if for any initial value $x_{sd}(t_0)$, $0 \le t_0 < \sigma$, $x_{sd}(t) \to 0$ as $t \to \infty$.

In the definition, $w = 0$; so Figure 1 reduces to Figure 2, where we assume G_{22} has the same state as G. Moving $\mathcal{S}$ and $\mathcal{H}$ around the loop and defining $G_{22d} = \mathcal{S}G_{22}\mathcal{H}$, the multirate discretization of G_{22}, we arrive at a multirate discrete-time system. Now lift K_d as before and G_{22d} by

$$\underline{G_{22d}} = \begin{bmatrix} L_{\bar{m}_1} & & \\ & \ddots & \\ & & L_{\bar{m}_p} \end{bmatrix} G_{22d} \begin{bmatrix} L_{\bar{n}_1}^{-1} & & \\ & \ddots & \\ & & L_{\bar{n}_p}^{-1} \end{bmatrix}$$

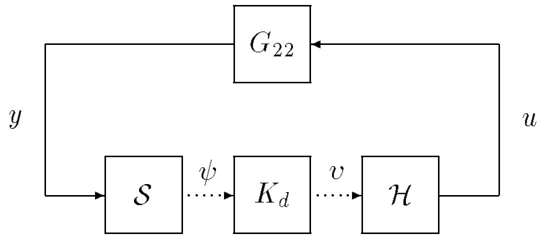

Figure 2: For stability analysis

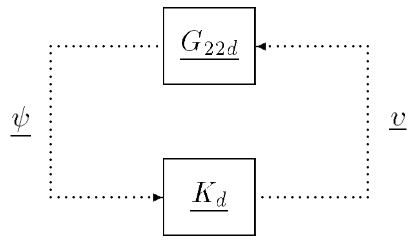

Figure 3: The lifted system for stability

to get the lifted system of Figure 3. Because G_{22} is LTI and strictly causal, G_{22d} is (n_j, m_i)-periodic and -strictly causal. Thus $\underline{G_{22d}}$ is LTI and causal with $\underline{D}_{22d}$ (n_j, m_i)-strictly causal. So Figure 3 gives an LTI discrete system. In fact, a state model for $\underline{G_{22d}}$ can be obtained [28]; its state being $\xi := S_\sigma x$, or $\xi(k) = x(k\sigma)$.

Let us see that Figure 3 is well-posed, i.e., the matrix $I - \underline{D}_{22d}\underline{D}$ is invertible, where $\underline{D}$ is the feedthrough term of $\underline{K_d}$. This follows from Lemma 5: $\underline{D}_{22d}\underline{D}$ is (m_i, m_i)-strictly causal [Lemma 5 (a)] and so $I - \underline{D}_{22d}\underline{D}$ is invertible [Lemma 5 (c)]. This also implies that the multirate system of Figure 1 is well-posed.

The system in Figure 3 is *internally stable*, or $\underline{K_d}$ *internally stabilizes* $\underline{G_{22d}}$ if for any initial states $\xi(0)$ and $\eta(0)$,

$$\begin{bmatrix} \xi(k) \\ \eta(k) \end{bmatrix} \to 0 \quad \text{as} \quad k \to \infty.$$

It is clear that Figure 3 is internally stable if Figure 1 is.

Theorem 2 *K_d internally stabilizes G iff $\underline{K_d}$ internally stabilizes $\underline{G_{22d}}$.*

Proof Suppose $\underline{K_d}$ internally stabilizes $\underline{G_{22d}}$. It suffices to show that $x(t) \to 0$ as $t \to \infty$. Internal stability of Figure 3 implies that $\underline{v}(k) \to 0$ as $k \to \infty$ in Figure 3 and hence $u(t) \to 0$ as $t \to \infty$ in Figure 2. Now since for $k\sigma \le t < (k+1)\sigma$,

$$x(t) = \mathrm{e}^{(t-k\sigma)A}\xi(k) + \int_{k\sigma}^{t} \mathrm{e}^{(t-\tau)A} B_2 u(\tau)\, d\tau,$$

it follows that $x(t) \to 0$ as $t \to \infty$. **QED**

Sufficient conditions for the internal stability to be achievable are that (A, B_2) and (C_2, A) are stabilizable and detectable respectively and that the system period σ is non-pathological in a certain sense, see, e.g., [16, 33].

5 $\mathcal{H}_\infty$-Optimal Control

With reference to Figure 1, we now study the main synthesis problem: Design an admissible K_d that internally stabilizes G and achieves the continuous-time $\mathcal{H}_\infty$ performance requirement $\|\mathcal{F}(G, \mathcal{H}K_d\mathcal{S})\| < \gamma$, where γ is pre-specified and positive. By proper scaling, we can always take $\gamma = 1$.

The general idea in the solution is to reduce the multirate sampled-data problem to a discrete $\mathcal{H}_\infty$ model-matching problem with the causality constraint and then solve the constrained problem explicitly using techniques presented in Section 2 on nest operators and nest algebras. A special case of the reduction process was reported in [33] where a uniform sampling rate and a uniform hold rate are assumed. The solution process is complex enough to require several distinct steps. Appropriate connections to some recent work on $\mathcal{H}_\infty$ control are made in each step.

We start with a state model for G:

$$\hat{G}(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & 0 & 0 \end{array}\right].$$

As we saw in the preceding section, the zero block in D_{22} guarantees the well-posedness of the closed-loop multirate system in Figure 1. For $\|\mathcal{F}(G, \mathcal{H}K_d\mathcal{S})\|$ to be finite, we must have $D_{21} = 0$. The zero block in D_{11} is for a technical simplification, as in the single-rate case [5, 21]. We shall assume that (A, B_2) is stabilizable and (C_2, A) is detectable.

$\mathcal{H}_\infty$ Discretization

The original problem is posed in continuous time; so the first step is to recast it as a discrete-time problem with time-varying control. The reduction is based on recent advances in $\mathcal{H}_\infty$ sampled-data control in the single-rate setting.

Introduce the *discrete sampling operator* $S_m : \ell \to \ell$ defined via

$$\psi = S_m\phi \Longleftrightarrow \psi(k) = \phi(km)$$

and the *discrete hold operator* $H_n : \ell \to \ell$ via

$$v = H_n\phi \Longleftrightarrow v(kn + r) = \phi(k), \quad r = 0, 1, \cdots, n-1.$$

It is easily checked that $S_{m_i h} = S_{m_i} S_h$ and $H_{n_j h} = H_h H_{n_j}$. So the multirate sampling and hold operators $\mathcal{S}$ and $\mathcal{H}$ can be factored as

$$\mathcal{S} = \begin{bmatrix} S_{m_1} & & \\ & \ddots & \\ & & S_{m_p} \end{bmatrix} S_h, \quad \mathcal{H} = H_h \begin{bmatrix} H_{n_1} & & \\ & \ddots & \\ & & H_{n_q} \end{bmatrix}.$$

Defining

$$K_{d1} = \begin{bmatrix} H_{n_1} & & \\ & \ddots & \\ & & H_{n_q} \end{bmatrix} K_d \begin{bmatrix} S_{m_1} & & \\ & \ddots & \\ & & S_{m_p} \end{bmatrix}, \tag{5}$$

we can view Figure 1 as a fictitious single-rate system but with a time-varying controller K_{d1} as in Figure 4. Now the results in, e.g., [5, 21] (there, discrete controllers are not required to be time-invariant), are applicable.

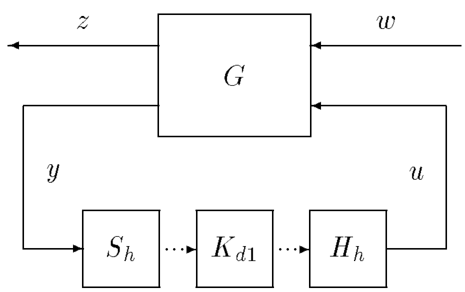

Figure 4: An equivalent single-rate system

Let $\underline{D}_{11h} : \mathcal{L}_2[0,h) \to \mathcal{L}_2[0,h)$ be defined by

$$(\underline{D}_{11h} w)(t) = C_1 \int_0^t \mathrm{e}^{(t-\tau)A} B_1 w(\tau)\, d\tau$$

and assume

$$\|\underline{D}_{11h}\| < 1.$$

Since $\underline{D}_{11h}$ is the restriction of $\mathcal{F}(G, \mathcal{H}K_d\mathcal{S})$ on $\mathcal{L}_2[0,h)$, this condition is necessary for $\|\mathcal{F}(G, \mathcal{H}K_d\mathcal{S})\| < 1$; how to verify this condition was studied in [5].

For the multirate sampled-data $\mathcal{H}_\infty$ problem, invoke the single-rate results to get the equivalent discrete system shown in Figure 5 and the equivalent discrete-time problem: Design K_{d1} to give internal stability and achieve $\|\mathcal{F}(G_d, K_{d1})\| < 1$, where the norm now is ℓ_2-induced. The $\mathcal{H}_\infty$ discretization G_d (for $\gamma = 1$) is LTI and causal and has a state model

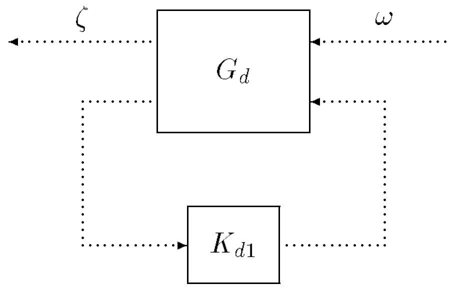

Figure 5: $\mathcal{H}_\infty$ discretized system

$$\hat{G}_d(\lambda) = \left[\begin{array}{c|cc} A_d & B_{1d} & B_{2d} \\ \hline C_{1d} & D_{11d} & D_{12d} \\ C_{2d} & 0 & 0 \end{array}\right].$$

The computation of the matrices in $\hat{G}_d$ is given in, e.g., [5, 21].

In this way, we arrive at an equivalent discrete $\mathcal{H}_\infty$ problem; however, K_{d1} is constrained to be of the form in (5) with K_d admissible.

Discrete Lifting

The system of Figure 5 is single-rate with the underlying period being h. The next step is to lift to get an LTI configuration with underlying period σ. Partition G_d as usual:

$$G_d = \begin{bmatrix} G_{11d} & G_{12d} \\ G_{21d} & G_{22d} \end{bmatrix}.$$

Define $\underline{K_d}$ as before and

$$\underline{G_d} = \begin{bmatrix} L_l & & & \\ & L_{\bar{m}_1} S_{m_1} & & \\ & & \ddots & \\ & & & L_{\bar{m}_p} S_{m_p} \end{bmatrix} G_d \begin{bmatrix} L_l^{-1} & & & \\ & H_{n_1} L_{\bar{n}_1}^{-1} & & \\ & & \ddots & \\ & & & H_{n_q} L_{\bar{n}_q}^{-1} \end{bmatrix}$$

to get the lifted system of Figure 6, where $\underline{\omega} = L_l \omega$ and $\underline{\zeta} = L_l \zeta$. Since G_d is LTI, causal, and finite-dimensional with G_{22d} strictly causal, it is an easy exercise to verify the following properties of $\underline{G_d}$.

Lemma 6 *$\underline{G_d}$ is LTI, causal, and finite-dimensional. Moreover, the feedthrough term $\underline{D}_{22d}$ of $\underline{G_{22d}}$ is (n_j, m_i)-strictly causal.*

In fact, a state model for $\underline{G_d}$ can be obtained using the lemma below.

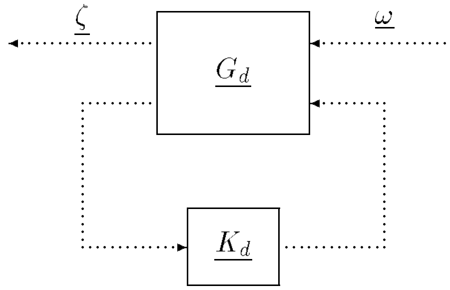

Figure 6: The lifted system

Let P be a discrete-time system with state ξ and transfer matrix

$$\hat{P}(\lambda) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right].$$

Let $m, n, \bar{m}, \bar{n}, l$ be positive integers such that

$$m\bar{m} = n\bar{n} = l.$$

Define

$$\underline{P} := L_{\bar{m}} S_m P H_n L_{\bar{n}}^{-1}$$

and the characteristic function on integers

$$\chi_{[p,q)}(r) = \begin{cases} 1, & p \leq r < q \\ 0, & \text{else.} \end{cases}$$

Lemma 7 *A state model for $\underline{P}$ is*

$$\underline{\hat{P}}(\lambda) = \left[\begin{array}{c|cccc} A^l & \sum_{r=0}^{n-1} A^{l-1-r}B & \sum_{r=n}^{2n-1} A^{l-1-r}B & \cdots & \sum_{r=l-n}^{l-1} A^{l-1-r}B \\ \hline C & D_{00} & D_{01} & \cdots & D_{0,\bar{n}-1} \\ CA^m & D_{10} & D_{11} & \cdots & D_{1,\bar{n}-1} \\ \vdots & \vdots & \vdots & & \vdots \\ CA^{l-m} & D_{\bar{m}_1,0} & D_{\bar{m}_1,1} & \cdots & D_{\bar{m}-1,\bar{n}-1} \end{array}\right],$$

where

$$D_{ij} = D\chi_{[jn,(j+1)n)}(im) + \sum_{r=jn}^{(j+1)n-1} CA^{im-1-r}B\chi_{[0,im)}(r).$$

The corresponding state vector is $\underline{\xi} = S_l \xi$.

The lemma can be proven by manipulating the input-output equations for P. Note that the transfer matrices for all blocks in $\underline{G_d}$ can be obtained from this lemma; for example, for the (1,1) block, namely, $\underline{G_{11d}} = L_l G_{11d} L_l^{-1}$, we take $m = n = 1$ and $\bar{m} = \bar{n} = l$ in the lemma to get

$$\underline{\hat{G}_{11d}}(\lambda) = \left[\begin{array}{c|cccc} A_d^l & A_d^{l-1}B_{1d} & A_d^{l-2}B_{1d} & \cdots & B_{1d} \\ \hline C_{1d} & D_{11d} & 0 & \cdots & 0 \\ C_{1d}A_d & C_{1d}B_{1d} & D_{11d} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ C_{1d}A_d^{l-1} & C_{1d}A_d^{l-2}B_{1d} & C_{1d}A_d^{l-3}B_{1d} & \cdots & D_{11d} \end{array}\right].$$

This realization is also given in, e.g., [16].

From the definitions of $\underline{K_d}$ and $\underline{G_d}$, we get after some algebra that the closed-loop map $\mathcal{F}(\underline{G_d}, \underline{K_d})$ in Figure 6 equals $L_l \mathcal{F}(G_d, K_{d1}) L_l^{-1}$. So $\|\mathcal{F}(\underline{G_d}, \underline{K_d})\| = \|\mathcal{F}(G_d, K_{d1})\|$ since L_l is norm-preserving. Thus the equivalent LTI problem is now: Design an admissible $\underline{K_d}$ that internally stabilizes $\underline{G_d}$ and achieves $\|\mathcal{F}(\underline{\hat{G}_d}, \underline{\hat{K}_d})\|_\infty < 1$. Notice that the feedthrough term $\underline{\hat{K}_d}(0)$ must be (m_i, n_j)-causal; so this is a constrained $\mathcal{H}_\infty$ control problem in discrete time.

Constrained Model-Matching Problem

Now we use the controller parametrization [27, 34] to reduce the problem further to a model-matching problem. In order to internally stabilize $\underline{G_d}$, it suffices to internally stabilize $\underline{G_{22d}}$.

Bring in a doubly-coprime factorization for $\underline{\hat{G}_{22d}}$:

$$\underline{\hat{G}_{22d}} = \hat{N}\hat{M}^{-1} = \hat{\tilde{M}}^{-1}\hat{\tilde{N}}$$

$$\begin{bmatrix} \hat{\tilde{X}} & -\hat{\tilde{Y}} \\ -\hat{\tilde{N}} & \hat{\tilde{M}} \end{bmatrix} \begin{bmatrix} \hat{M} & \hat{Y} \\ \hat{N} & \hat{X} \end{bmatrix} = I$$

with the conditions:

$$\hat{M}(0) = I, \quad \hat{\tilde{M}}(0) = I,$$
$$\hat{N}(0) = \hat{\tilde{N}}(0) = \underline{D}_{22d},$$
$$\hat{X} = I, \quad \hat{\tilde{X}} = I,$$
$$\hat{Y}(0) = \hat{\tilde{Y}}(0) = 0.$$

The standard procedure in [15] yields such a factorization. Since $\underline{D}_{22d}$ is (n_j, m_i)-strictly causal, it follows from [27, 34] that the set of admissible $\underline{K_d}$ that provide internal stability is parametrized by

$$\underline{\hat{K}_d} = (\hat{Y} - \hat{M}\hat{Q})(\hat{X} - \hat{N}\hat{Q})^{-1}, \quad \hat{Q} \in \mathcal{RH}_\infty, \quad \hat{Q}(0) \ \ (m_i, n_j)\text{-causal}.$$

Note that the causality constraint on $\underline{\hat{K}_d}$ translates exactly to the same constraint on $\hat{Q}(0)$. Now define the three $\mathcal{RH}_\infty$ matrices as in [15]

$$\begin{aligned}\hat{T}_1 &= \underline{\hat{G}_{11d}} + \underline{\hat{G}_{12d}}\hat{M}\hat{\tilde{Y}}\underline{\hat{G}_{21d}} \\ \hat{T}_2 &= \underline{\hat{G}_{12d}}\hat{M} \\ \hat{T}_3 &= \hat{\tilde{M}}\underline{\hat{G}_{21d}}\end{aligned}$$

to get the closed-loop transfer matrix of Figure 6

$$\mathcal{F}(\underline{\hat{G}_d}, \underline{\hat{K}_d}) = \hat{T}_1 - \hat{T}_2\hat{Q}\hat{T}_3.$$

Recall in Section 3 that $\hat{Q}(0)$ is (m_i, n_j)-causal iff $\hat{Q}(0) \in \mathcal{N}(\{\mathcal{X}_r\}, \{\mathcal{Y}_r\})$, abbreviated $\mathcal{N}$, where the nests $\{\mathcal{X}_r\}$ and $\{\mathcal{Y}_r\}$ were defined in Section 3. In this way we arrive at the constrained $\mathcal{H}_\infty$ model-matching problem: Find $\hat{Q} \in \mathcal{RH}_\infty$ with $\hat{Q}(0) \in \mathcal{N}$ such that

$$\|\hat{T}_1 - \hat{T}_2\hat{Q}\hat{T}_3\|_\infty < 1.$$

If such a $\hat{Q}$ exists, we say the multirate $\mathcal{H}_\infty$ problem is *solvable.*

An Explicit Solution

Let us focus on the constrained $\mathcal{H}_\infty$ model-matching problem. We write $\hat{T}^\sim(\lambda)$ for $\hat{T}(\lambda^{-1})'$. For regularity, we need the following assumption:

For every λ on the unit circle, $\hat{T}_2(\lambda)$ and $\hat{T}_3^\sim(\lambda)$ are both injective.

Under this assumption, perform an inner-outer factorization $\hat{T}_2 = \hat{T}_{2i}\hat{T}_{2o}$ and an co-inner-outer factorization $\hat{T}_3 = \hat{T}_{3co}\hat{T}_{3ci}$, where $\hat{T}_{2o}$ and $\hat{T}_{3co}$ are both invertible over $\mathcal{RH}_\infty$. Apply unitary transformations to $\hat{T}_1 - \hat{T}_2\hat{Q}\hat{T}_3$ and define

$$\hat{R} = \begin{bmatrix} \hat{R}_{11} & \hat{R}_{12} \\ \hat{R}_{21} & \hat{R}_{22} \end{bmatrix} := \begin{bmatrix} \hat{T}_{2i}^\sim \\ I - \hat{T}_{2i}\hat{T}_{2i}^\sim \end{bmatrix} \hat{T}_1 \begin{bmatrix} \hat{T}_{3ci}^\sim & I - \hat{T}_{3ci}^\sim\hat{T}_{3ci} \end{bmatrix}. \tag{6}$$

The constrained model-matching problem is equivalent to the following four-block problem of finding a $\hat{Q} \in \mathcal{RH}_\infty$ with $\hat{Q}(0) \in \mathcal{N}$ such that

$$\left\| \begin{bmatrix} \hat{R}_{11} - \hat{T}_{2o}\hat{Q}\hat{T}_{3co} & \hat{R}_{12} \\ \hat{R}_{21} & \hat{R}_{22} \end{bmatrix} \right\|_\infty < 1. \tag{7}$$

Dropping the causality constraint on $\hat{Q}(0)$ temporarily allows us to parametrize all $\hat{Q}$ in $\mathcal{RH}_\infty$ achieving (7). We know from [13] that there exists a $\hat{Q} \in \mathcal{RH}_\infty$ such that (7) holds iff

$$\left\| \begin{bmatrix} \Pi_{\mathcal{H}_2^\perp} & 0 \\ 0 & I \end{bmatrix} \hat{R}|_{\mathcal{H}_2 \oplus \mathcal{L}_2} \right\| < 1. \tag{8}$$

If (8) is satisfied, then a procedure in [18] allows us to find an $\mathcal{RH}_\infty$ matrix

$$\hat{K} = \begin{bmatrix} \hat{K}_{11} & \hat{K}_{12} \\ \hat{K}_{21} & \hat{K}_{22} \end{bmatrix}$$

with $\hat{K}_{12}^{-1}, \hat{K}_{21}^{-1} \in \mathcal{RH}_\infty$ and $\|\hat{K}_{22}\|_\infty < 1$ such that all $\hat{Q} \in \mathcal{RH}_\infty$ satisfying (7) are characterized by

$$\hat{Q} = \mathcal{F}(\hat{K}, \hat{Q}_1), \quad \hat{Q}_1 \in \mathcal{RH}_\infty, \quad \|\hat{Q}_1\|_\infty < 1. \tag{9}$$

We refer to [18] for the details of checking inequality (8) and the expression of $\hat{K}$. Hereafter, we shall assume that (8) is true. This is also necessary for the solvability of the multirate $\mathcal{H}_\infty$ problem.

In general $\hat{K}_{22}(0) \neq 0$, so $\hat{Q}(0)$ depends on $\hat{Q}_1(0)$ in a linear fractional manner. However, it is possible to simplify this relation by introducing another linear fractional transformation [33]:

$$\hat{Q}_1 = \mathcal{F}(V, \hat{Q}_2).$$

Here V, partitioned as usual, is a constant unitary matrix. It follows that the mapping $\hat{Q}_2 \mapsto \hat{Q}_1$ is bijective from the open unit ball of $\mathcal{RH}_\infty$ onto itself [35]. Thus all $\hat{Q}$ satisfying (7) can be re-parametrized by

$$\begin{aligned} \hat{Q} &= \mathcal{F}[\hat{K}, \mathcal{F}(V, \hat{Q}_2)] \\ &= \mathcal{F}(\hat{L}, \hat{Q}_2), \quad \hat{Q}_2 \in \mathcal{RH}_\infty, \quad \|\hat{Q}_2\|_\infty < 1, \end{aligned}$$

where $\hat{L}$, partitioned as usual, can be written in terms of $\hat{K}$ and V:

$$\hat{L} = \begin{bmatrix} \hat{K}_{11} + \hat{K}_{12} V_{11} (I - \hat{K}_{22} V_{11})^{-1} \hat{K}_{21} & \hat{K}_{12} (I - V_{11} \hat{K}_{22})^{-1} V_{12} \\ V_{21} (I - \hat{K}_{22} V_{11})^{-1} \hat{K}_{21} & V_{21} (I - \hat{K}_{22} V_{11})^{-1} \hat{K}_{22} V_{12} + V_{22} \end{bmatrix}.$$

For $\hat{L}_{22}(0) = 0$, we choose the unitary matrix V to be

$$V = \begin{bmatrix} \hat{K}_{22}'(0) & \left[I - \hat{K}_{22}'(0) \hat{K}_{22}(0) \right]^{1/2} \\ \left[I - \hat{K}_{22}(0) \hat{K}_{22}'(0) \right]^{1/2} & -\hat{K}_{22}(0) \end{bmatrix}.$$

It can be checked that $\hat{L}_{12}(0)$ and $\hat{L}_{21}(0)$ are still nonsingular.

To recap, the set of all $Q \in \mathcal{RH}_\infty$ achieving (7) is parametrized by

$$\hat{Q} = \mathcal{F}(\hat{L}, \hat{Q}_2), \quad \hat{Q}_2 \in \mathcal{RH}_\infty, \quad \|\hat{Q}_2\|_\infty < 1.$$

Here $\hat{L}$ has the desirable properties that $\hat{L}_{22}(0) = 0$, $\hat{L}_{12}(0)$ and $\hat{L}_{21}(0)$ are nonsingular. Thus

$$\hat{Q}(0) = \hat{L}_{11}(0) + \hat{L}_{12}(0) \hat{Q}_2(0) \hat{L}_{21}(0). \tag{10}$$

This is an affine function $\hat{Q}_2(0) \mapsto \hat{Q}(0)$.

Now we bring in the causality constraint on $\hat{Q}(0)$. Our goal is to find a $\hat{Q}_2 \in \mathcal{RH}_\infty$ with $\|\hat{Q}_2\|_\infty < 1$ such that $\hat{Q}(0)$ in (10) lies in $\mathcal{N}(\{\mathcal{X}_r\},\{\mathcal{Y}_r\})$. Since $\hat{Q}(0)$ depends only on $\hat{Q}_2(0)$ and in general $\|\hat{Q}_2\|_\infty \geq \|\hat{Q}_2(0)\|$, the equivalent problem is to find a constant matrix $\hat{Q}_2(0)$ with $\|\hat{Q}_2(0)\| < 1$ such that $\hat{Q}(0) \in \mathcal{N}$.

Now we use Lemma 2 to reduce the problem to a distance problem. Introduce matrix factorizations (Lemma 2)

$$L_{12}(0) = R_1 U_1, \quad L_{21}(0) = -U_2 R_2,$$

where R_1, R_2, U_1, U_2 are all invertible, U_1, U_2 are orthogonal, and R_1, R_2 belongs to the nest algebras $\mathcal{N}(\{\mathcal{Y}_r\}), \mathcal{N}(\{\mathcal{X}_r\})$ respectively.

The computation of such factorizations follow from the proof of Lemma 2: First, change coordinates in $\mathcal{X}$ and $\mathcal{Y}$ so that

$$\begin{aligned} \mathcal{X} &= (\mathcal{X}_1)^\perp_{\mathcal{X}_0} \oplus (\mathcal{X}_2)^\perp_{\mathcal{X}_1} \oplus \cdots \oplus (\mathcal{X}_n)^\perp_{\mathcal{X}_{n-1}} \\ \mathcal{Y} &= (\mathcal{Y}_1)^\perp_{\mathcal{Y}_0} \oplus (\mathcal{Y}_2)^\perp_{\mathcal{Y}_1} \oplus \cdots \oplus (\mathcal{Y}_n)^\perp_{\mathcal{Y}_{n-1}}. \end{aligned}$$

This corresponds to permute the components in $\mathcal{X}$ and $\mathcal{Y}$ according to the order of times when they occur. Second, do standard QR type factorizations to get the matrices under the new coordinates, see the proof of Lemma 2. Finally, change coordinates back via permutations to get the desired matrices.

Substitute the factorizations into (10) and pre- and post-multiply by R_1^{-1} and R_2^{-1} respectively to get

$$R_1^{-1}\hat{Q}(0)R_2^{-1} = R_1^{-1}\hat{L}_{11}(0)R_2^{-1} - U_1\hat{Q}_2(0)U_2.$$

Define

$$W := R_1^{-1}\hat{Q}(0)R_2^{-1}, \quad T := R_1^{-1}\hat{L}_{11}(0)R_2^{-1}, \quad P := U_1\hat{Q}_2(0)U_2.$$

It follows that $\hat{Q}(0) \in \mathcal{N}(\{\mathcal{X}_r\},\{\mathcal{Y}_r\})$ iff $W \in \mathcal{N}(\{\mathcal{X}_r\},\{\mathcal{Y}_r\})$ (Lemma 1) and $\|\hat{Q}_2(0)\| < 1$ iff $\|P\| < 1$. Therefore, we arrive at the following equivalent matrix problem: Given T, find P with $\|P\| < 1$ such that $W = T - P \in \mathcal{N}$; or equivalently, find $W \in \mathcal{N}$ such that $\|T - W\| < 1$. This can be solved via the distance problem studied in Theorem 1:

$$\mathrm{dist}\,(T, \mathcal{N}) = \max_r \{\|(I - \Pi_{\mathcal{Y}_r})T\Pi_{\mathcal{X}_r}\|\} =: \mu.$$

Let $W_{opt} \in \mathcal{N}$ achieves the distance, i.e., $\|T - W_{opt}\| = \mu$. The following result summarizes what we have derived.

Theorem 3 *The matrix problem is solvable, i.e., there exists a matrix P with $\|P\| < 1$ such that $T - P \in \mathcal{N}$, iff $\mu < 1$. Moreover, if $\mu < 1$, $P := T - W_{opt}$ solves the problem with $\|P\| = \mu$.*

How to compute μ and W_{opt} were discussed in the procedure given at the end of section 2: After a change of coordinates in $\mathcal{X}$ and $\mathcal{Y}$, which corresponds to permuting their components chronologically, μ can be found via computing the spectral norms of several matrices and

W_{opt} via sequentially applying Lemma 3. If $\mu < 1$, we can backtrack to get P (Theorem 3), $\hat{Q}_2(\lambda) = \hat{Q}_2(0)$, and finally $\hat{Q}$ which solves the multirate $\mathcal{H}_\infty$ control problem.

To summarize, let us list the solvability conditions for the multirate $\mathcal{H}_\infty$ control problem $\|\mathcal{F}(G, \mathcal{H}K_d\mathcal{S})\| < 1$:

(a) $\|\underline{D}_{11h}\| < 1$;

(b) $\| \begin{bmatrix} P_{\mathcal{H}_2^\perp} & 0 \\ 0 & I \end{bmatrix} \hat{R}|_{\mathcal{H}_2 \oplus \mathcal{L}_2} \| < 1$;

(c) $\mu < 1$.

Condition (a) was studied in detail in [5] and would usually be satisfied for a reasonable design since normally the base period h is much smaller than the system period σ. Condition (b) is the solvability condition for a standard four-block $\mathcal{H}_\infty$ problem, see, e.g., [18] for checking this condition. When conditions (a-b) hold, condition (c) amounts to computing the norms of several constant matrices.

Finally, we remark that multirate $\mathcal{H}_\infty$ controllers which are arbitrarily close to optimality can be computed based on the solvability conditions (a-c) (with proper scaling) and a standard bisection search.

6 $\mathcal{H}_2$-Optimal Control

In this section we use the techniques developed to solve explicitly a general multirate $\mathcal{H}_2$-optimal model-matching problem. The model-matching problem arises in multirate control problems from either a sampled-data point of view [33] or from a discrete-time LQG point of view [29].

The problem is as follows: Design a $\hat{Q} \in \mathcal{RH}_\infty$ with $\hat{Q}(0)$ being (m_i, n_j)-causal such that the $\mathcal{H}_2$-norm of the transfer matrix $\hat{T}_1 - \hat{T}_2\hat{Q}\hat{T}_3$ is minimized, i.e.,

$$\min_{\hat{Q} \in \mathcal{RH}_\infty, \hat{Q}(0) \in \mathcal{N}} \|\hat{T}_1 - \hat{T}_2\hat{Q}\hat{T}_3\|_2, \tag{11}$$

where $\hat{T}_i$ are all in $\mathcal{RH}_\infty$. Here we shall make the same regularity assumption as in Section 5:

For every λ on the unit circle, $\hat{T}_2(\lambda)$ and $\hat{T}_3^\sim(\lambda)$ are injective.

Bring in an inner-outer factorization $\hat{T}_2 = \hat{T}_{2i}\hat{T}_{2o}$ and a co-inner-outer factorization $\hat{T}_3 = \hat{T}_{3co}\hat{T}_{3ci}$ with $\hat{T}_{2o}$ and $\hat{T}_{3co}$ both invertible over $\mathcal{RH}_\infty$. Defining R as in (6), we get

$$\begin{aligned} \|\hat{T}_1 - \hat{T}_2\hat{Q}\hat{T}_3\|_2^2 &= \| \begin{bmatrix} \hat{T}_{2i}^\sim \\ I - \hat{T}_{2i}\hat{T}_{2i}^\sim \end{bmatrix} (\hat{T}_1 - \hat{T}_2\hat{Q}\hat{T}_3) \begin{bmatrix} \hat{T}_{3ci}^\sim & I - \hat{T}_{3ci}^\sim\hat{T}_{3ci} \end{bmatrix} \|_2^2 \\ & \quad \| \begin{bmatrix} \hat{R}_{11} - \hat{T}_{2o}\hat{Q}\hat{T}_{3co} & \hat{R}_{12} \\ \hat{R}_{21} & \hat{R}_{22} \end{bmatrix} \|_2^2 \\ &= \|\hat{R}_{11} - T_{2o}\hat{Q}\hat{T}_{3co}\|_2^2 + \|\hat{R}_{12}\|_2^2 + \|\hat{R}_{21}\|_2^2 + \|\hat{R}_{22}\|_2^2. \end{aligned}$$

The last three terms are independent of Q; so the problem in (11) reduces to minimizing only the first term. Writing

$$\hat{Q} = Q_0 + \lambda \hat{Q}_1, \quad Q_0 \in \mathcal{N}, \quad \hat{Q}_1 \in \mathcal{RH}_\infty,$$

we have

$$\begin{aligned}
&\inf_{\hat{Q} \in \mathcal{RH}_\infty, Q_0 \in \mathcal{N}} \|\hat{R}_{11} - \hat{T}_{2o}\hat{Q}\hat{T}_{3co}\|_2^2 \\
&= \inf_{Q_0 \in \mathcal{N}} \inf_{\hat{Q}_1 \in \mathcal{RH}_\infty} \|\hat{R}_{11} - \hat{T}_{2o}Q_0\hat{T}_{3co} - \lambda \hat{T}_{2o}\hat{Q}_1\hat{T}_{3co}\|_2^2 \\
&= \inf_{Q_0 \in \mathcal{N}} \inf_{\hat{Q}_1 \in \mathcal{RH}_\infty} \|\lambda^{-1}[\hat{R}_{11} - \hat{T}_{2o}Q_0\hat{T}_{3co}] - \hat{T}_{2o}\hat{Q}_1\hat{T}_{3co}\|_2^2 \\
&\geq \inf_{Q_0 \in \mathcal{N}} \|\Pi_{\mathcal{H}_2^\perp}\left\{\lambda^{-1}[\hat{R}_{11} - \hat{T}_{2o}Q_0\hat{T}_{3co}]\right\}\|_2^2 \\
&= \|\Pi_{\mathcal{H}_2^\perp}\hat{R}_{11}\|_2^2 + \inf_{Q_0 \in \mathcal{N}} \|R_{110} - \hat{T}_{2o}(0)Q_0\hat{T}_{3co}(0)\|_2^2,
\end{aligned}$$

where R_{110} is the constant term in $\hat{R}_{11}$. Note that the equality is achieved by setting

$$\hat{Q}_1 = \hat{T}_{2o}^{-1}\Pi_{\mathcal{H}_2}\left[\lambda^{-1}(\hat{R}_{11} - \hat{T}_{2o}Q_0\hat{T}_{3co})\right]\hat{T}_{3co}^{-1}. \tag{12}$$

Thus the optimal $\hat{Q}$ can be obtained in two steps: Solve the matrix 2-norm optimization

$$\inf_{Q_0 \in \mathcal{N}} \|R_{110} - \hat{T}_{2o}(0)Q_0\hat{T}_{3co}(0)\|_2^2$$

for Q_0 and then substitute Q_0 into (12) to get the optimal $\hat{Q}_1$.

The matrix 2-norm optimization problem can be solved via matrix factorization as well. Note that the matrix 2-norm is induced by the inner product:

$$\langle A, B\rangle := \text{trace }(A'B).$$

Thus the set of matrices in $\mathcal{N}$ can be regarded as a subspace in the space of matrices mapping $\mathcal{X}$ to $\mathcal{Y}$. So the orthogonal projections $\Pi_{\mathcal{N}}$ and $\Pi_{\mathcal{N}^\perp}$ are well-defined. In fact, $\Pi_{\mathcal{N}}$ amounts to simply retaining the blocks corresponding to the unconstrained blocks in $\mathcal{N}$ and zeroing the other blocks.

For square and nonsingular matrices $\hat{T}_{2o}(0)$ and $\hat{T}_{3co}(0)$, bring in useful factorizations (Lemma 2)

$$\hat{T}_{2o}(0) = U_2R_2, \quad \hat{T}_{3co}(0) = R_3U_3,$$

where U_2, R_2, U_3, R_3 are all square, U_2, U_3 are orthogonal, and R_2, R_3 are invertible and belong to the nest algebras $\mathcal{N}(\{\mathcal{Y}_r\})$ and $\mathcal{N}(\{\mathcal{X}_r\})$ respectively. Then

$$\begin{aligned}
&\min_{Q_0 \in \mathcal{N}} \|R_{110} - U_2R_2Q_0R_3U_3\|_2 \\
&= \min_{Q_0 \in \mathcal{N}} \|U_2'R_{110}U_3' - R_2Q_0R_3\|_2 \\
&\geq \|\Pi_{\mathcal{N}^\perp}[U_2'R_{110}U_3']\|_2
\end{aligned}$$

The inequality follows from the fact that $R_2 Q_0 R_3 \in \mathcal{N}$ iff $Q_0 \in \mathcal{N}$ (Lemma 1) and becomes equality if we select

$$Q_0 = R_2^{-1} \Pi_{\mathcal{N}} [U_2' R_{110} U_3'] R_3^{-1}.$$

This optimal Q_0 is in $\mathcal{N}$ also by Lemma 1.

On summarizing, we have derived the following result.

Theorem 4 *The optimal $\hat{Q}$ in (11) is given by*

$$\hat{Q}_{opt} = Q_{0,opt} + \lambda \hat{T}_{2o}^{-1} \Pi_{\mathcal{H}_2} \left[\lambda^{-1} (\hat{R}_{11} - \hat{T}_{2o} Q_{0,opt} \hat{T}_{3co}) \right] \hat{T}_{3co}^{-1},$$

where the optimal constant term $Q_{0,opt}$ is

$$Q_{0,opt} = R_2^{-1} \Pi_{\mathcal{N}} [U_2' R_{110} U_3'] R_3^{-1}.$$

7 Conclusions

In this paper we introduced a new framework based on nest operators for handling causality constraints in multirate systems. This framework allows us to develop explicit solutions to syntheses of general multirate control systems with $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance criteria.

The results in this paper are presented in an operator setting; for example, the solution techniques in Sections 5 and 6 are developed in the frequency domain. For computational efficiency, it would be useful to develop a time-domain approach via state-space methods; this would make a good project for future research.

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