Some Stability Concepts and Their Applications in Optimal Control Problems*

Frédéric Bonnans
INRIA, Domaine de Voluceau, BP 105
Rocquencourt, 78153 Le Chesnay, France
and
Eduardo Casas
Dpto. de Matemática Aplicada y Ciencias de la Computación
E.T.S.I. de Caminos – Universidad de Cantabria
39071 Santander, Spain

Abstract

In this work we are concerned with state-constrained optimal control problems. Our aim is to derive the optimality conditions and to prove the convergence of the numerical approximations. To deal with these questions, whose difficulty is motivated by the presence of the state constraints, we consider some concepts of stability of the optimal cost functional with respect to small perturbations of the set of feasible states. While weak and strong stability on the right allow us to derive optimality conditions in a non-qualified and qualified form respectively, the weak stability on the left is the key to prove the convergence of the numerical approximations.

1 Introduction

In this paper we consider an optimal control problem with pointwise state constraints governed by a semilinear elliptic partial differential equation. We study the influence of some stability properties in the derivation of the optimality conditions satisfied by the optimal control, obtained through the penalization of the state constraints, and in the convergence of the numerical approximations of the optimal control problem. Since the ideas are quite general, this study can be

*This work was completed while the second author was visiting IMA, University of Minnesota, USA. Moreover the research of this author was partially supported by IMA and Dirección General de Investigación Científica y Técnica (Madrid)

extended to other state-constrained control problems, for instance to the control of evolution partial differential equations or ordinary differential equations.

It is well known that optimal control problems become more difficult to be studied when some state constraints are present. Under the hypothesis of non-empty interior of the feasible state set, it is possible to derive some optimality conditions in a non-qualified form (or also called Fritz John type conditions). The Slater condition has been widely used to obtain these conditions in a qualified form; see [2], [3], [4], [5], [13]. In these papers the calmness notion introduced by Rockafellar and Clarke, see [16], was considered as an alternative to Slater condition. This condition has been also used by the second author to prove the convergence of the numerical approximation of optimal control problems; see [11], [12].

While the Slater condition is a stability hypothesis that looks at the optima l control in connection with the state constraints, here we will consider some stabi lity conditions that point out the dependence of the optimal cost functional with respect to small perturbations of the set of feasible states. We will distinguish between weak and strong stability and stability on the left and on the right. The strong stability on the right is equivalent, in a local sense, to the existence of a exact penalty function, which can be used to derive the optimality conditions in a qualified form. This concept has been used under the name of calmness, Burke [9]. The weak stability on the right allows to utilize the classical differentiable non-exact penalty function to derive these optimality conditions in a non-qualified form; see [1], [8], [7] for the application of these ideas to the derivation of the Pontryagin's principle for state-constrained control problems. The stability on the left has shown to be useful in the analysis of the convergence of the numerical approximation of state-constrained optimal control problems; see [14].

The plan of the paper is as follows: in the next section we formulate the optimal control problem and prove the existence of a solution; in Section 3 the definitions of the stability concepts are given and the stability of the control problem is investigated; in Section 4 the optimality conditions are derived and finally the convergence of the numerical approximations is stated in Section 5.

2 Setting of the control problem

Let Ω be an open bounded convex subset of R^n $(n \leq 3)$ with boundary Γ. Let us consider the following boundary value problem:

$$Ay = -\sum_{i,j=1}^{n} \partial_{x_j} \left(a_{ij}(x)\partial_{x_i} y(x)\right) + a_0(x)y(x),$$

with $a_{ij} \in C^{0,1}(\overline{\Omega})$ and $a_0 \in L^\infty(\Omega)$ satisfying:

$$\begin{cases} \exists m > 0 \text{ such that } \displaystyle\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq m|\xi|^2 \quad \forall \xi \in R^n \text{ and } \forall x \in \Omega, \\ \\ a_0(x) \geq 0 \quad \text{a. e. } x \in \Omega. \end{cases}$$

Let $\phi : R \longrightarrow R$ be a increasing monotone function of class C^1, with $\phi(0) = 0$. Given $u \in L^2(\Omega)$, then the Dirichlet problem

$$\begin{cases} Ay + \phi(y) = u & \text{in } \Omega \\ \qquad\quad y = 0 & \text{on } \Gamma \end{cases} \tag{1}$$

has a unique solution $y_u \in H_0^1(\Omega) \cap H^2(\Omega)$. The regularity of y_u can be obtained as follows: first we deduce the boundedness of y_u by adapting the techniques of Stampacchia [21] to the nonlinear case or some other method, see for example Bonnans and Casas [6], and then the $H^2(\Omega)$-regularity is consequence of Grisvard's regularity results [20]. Furthermore the following estimate can be proved:

$$\|y_u\|_{H^2(\Omega)} \leq C(1 + \|u\|_{L^2(\Omega)}), \tag{2}$$

where C is independent of u.

Now the control problem is formulated

$$(P_\delta) \begin{cases} \text{minimize } \; J(u) = \displaystyle\int_\Omega L(x, y_u(x), u(x))dx \\ \\ \text{subject to } u \in K \;\text{ and }\; g(y_u(x)) \leq \delta \quad \forall x \in \overline{\Omega}, \end{cases}$$

where $g : R \longrightarrow R$ is continuous, $L : \Omega \times R^2 \longrightarrow R$ is a Carathéodory function, δ is a fixed real number and K is a bounded convex weakly-$\star$ closed nonempty subset of $L^\infty(\Omega)$. We will make the following additional assumptions about L:

H1) For every $(x, y) \in \Omega \times R$, $L(x, y, \cdot) : R \longrightarrow R$ is a convex function and for every $x \in \Omega$ $L(x, \cdot, \cdot) : R^2 \longrightarrow R$ is a function of class C^1.

H2) For each $M > 0$ there exists a function $\psi_M \in L^1(\Omega)$ such that for all $x \in \Omega$, $|y| \leq M$ and $|u| \leq M$ the following inequality holds

$$\left|\frac{\partial L}{\partial u}(x, y, u)\right| + \left|\frac{\partial L}{\partial y}(x, y, u)\right| + |L(x, y, u)| \leq \psi_M(x).$$

Under these hypotheses we have the following theorem of existence of solution:

Theorem 1 *There exists a number $\delta_0 \in R$ such that (P_δ) has at least one solution for every $\delta \geq \delta_0$, whereas (P_δ) has no feasible control for $\delta < \delta_0$.*

Proof. Using the continuity of the inclusion $H^2(\Omega) \subset C(\overline{\Omega})$ and the boundedness of K in $L^\infty(\Omega)$, we deduce from (2) the existence of a constant $C_1 > 0$ such that

$$\|y_u\|_{L^\infty(\Omega)} \leq C_1 \quad \forall u \in K.$$

Let M and m be the supremum and infimum respectively of g on $[-C_1, +C_1]$. Then it is obvious that (P_δ) has no feasible control for $\delta < m$, and however all elements of K are feasible for $\delta \geq M$. Let δ_0 be the infimum of the values δ for which (P_δ) has feasible controls. Then $m \leq \delta_0 \leq M$ and (P_δ) has no feasible control for $\delta < \delta_0$. Let us prove that there exists at least one feasible control for (P_{δ_0}). Let $\{\delta_j\}$ be a decreasing sequence converging to δ_0 and $\{u_j\} \subset K$ a sequence of controls such that u_j is feasible for (P_{δ_j}). Since K is bounded, we can extract a subsequence, denoted in the same way, converging weakly-$\star$ in $L^\infty(\Omega)$ towards an element $u_0 \in K$. Then the compactness of the inclusion $H^2(\Omega) \subset C(\overline{\Omega})$ implies the uniform convergence of $\{y_{u_j}\}$ to y_{u_0} and therefore

$$g(y_{u_0}(x)) = \lim_{j\to\infty} g(y_{u_j}(x)) \leq \lim_{j\to\infty} \delta_j = \delta_0 \quad \forall x \in \overline{\Omega},$$

which proves that u_0 is a feasible control for (P_{δ_0}).

To conclude the proof we must establish the existence of an optimal control of (P_δ) for all $\delta \geq \delta_0$. Let $\{u_k\} \subset K$ be a minimizing sequence of (P_δ). We can extract a subsequence, denoted again in the same manner, that converges weakly-$\star$ in $L^\infty(\Omega)$ to an element $u_\delta \in K$. Arguing as above, it can be proved that $g(y_{u_\delta}(x)) \leq \delta$ for every $x \in \overline{\Omega}$. Then u_δ is a feasible control for (P_δ) and moreover

$$J(u_\delta) \leq \liminf_{k\to\infty} J(u_k) = \inf(P_\delta),$$

where the first inequality is obtained with the help of hypotheses **H1)** and **H2)** (see Cesari [15] or Ekeland and Temam [18]) and the last equality follows from the definition of minimizing sequence. Consequently u_δ is a solution of (P_δ). ■

3 Some stability concepts associated to (P_δ)

Now we give some stability definitions previously introduced by Bonnans and Casas in [8]:

Definition 1 *We will say that problem (P_δ) is weakly stable on the right if*

$$\lim_{\delta' \searrow \delta} \inf(P_{\delta'}) = \inf(P_\delta), \tag{3}$$

and weakly stable on the left if

$$\lim_{\delta' \nearrow \delta} \inf(P_{\delta'}) = \inf(P_\delta). \tag{4}$$

(P_δ) is said strongly stable on the right (resp. left) if there exist $\epsilon > 0$ and $r > 0$ such that:

$$\inf(P_\delta) - \inf(P_{\delta'}) \leq r(\delta' - \delta) \quad \forall \delta' \in [\delta, \delta + \epsilon], \tag{5}$$

respectively

$$\inf(P_{\delta'}) - \inf(P_\delta) \leq r(\delta - \delta') \quad \forall \delta' \in [\delta - \epsilon, \delta]. \tag{6}$$

If (P_δ) is weakly (resp. strongly) stable on the left and on the right, it will be called weakly (resp. strongly) stable.

The first result to state is that almost all problems are stable

Proposition 1 *Let δ_0 be as in Theorem 1. Then for every $\delta \geq \delta_0$, except at most a countable number of them (resp. a set of zero measure), the problem (P_δ) is weakly (resp. strongly) stable.*

Proof. If we define $\varphi : [\delta_0, +\infty) \longrightarrow R$ by $\varphi(\delta) = \inf(P_\delta)$, then φ is a decreasing monotone function and therefore φ is continuous (resp. differentiable) at each point except at most a countable number of them (resp. a set of zero measure). Finally it is obvious that the continuity (resp. differentiability) of φ at δ implies the weak (resp. strong) stability of (P_δ). ■

By using the maximum principle for elliptic partial differential equations, Casas proved in [14] the following result:

Proposition 2 *Let us assume that $g : [a, b] \longrightarrow R$ is a strictly increasing monotone function, where a and b are defined by the expressions*

$$a = \inf\{y_u(x) : x \in \overline{\Omega}, u \in K\} \quad \text{and} \quad b = \sup\{y_u(x) : x \in \overline{\Omega}, u \in K\}.$$

If $\delta > \delta_0$, u_δ is a solution of (P_δ) and there exists $u_0 \in K$, $u_0 \neq u_\delta$, such that $u_0(x) \leq u_\delta(x)$ for each $x \in \overline{\Omega}$, then (P_δ) is weakly stable on the left.

Next we will study sufficient conditions for strong stability on the left of (P_δ). First we prove that the Slater condition implies strong stability on the left. Let us remark how the classical Slater condition is written for (P_δ) at the optimal control u_δ. If g is of class C^1, then the Slater hypothesis at u_δ is enounced as follows: there exists an element $u_0 \in K$ such that if y_δ is the state associated to u_δ and $z_0 \in H_0^1(\Omega) \cap H^2(\Omega)$ is the solution of

$$\left\{ \begin{array}{l} Az_0 + \phi'(y_\delta) z_0 = u_0 - u_\delta \ \text{ in } \Omega \\ z_0 = 0 \ \text{ on } \Gamma, \end{array} \right. \tag{7}$$

then

$$g(y_\delta(x)) + g'(y_\delta(x)) z_0(x) < \delta \quad \forall x \in \overline{\Omega}. \tag{8}$$

Proposition 3 *Let us suppose that g is of class C^1. If the Slater condition (8) is satisfied at u_δ, then (P_δ) is strongly stable on the left.*

Proof. Let $u_\lambda = u_\delta + \lambda(u_0 - u_\delta)$. Then we deduce from (8)

$$\lim_{\lambda \searrow 0} \left[g(y_{u_\delta}) + \frac{g(y_{u_\lambda}) - g(y_{u_\delta})}{\lambda} \right] = g(y_{u_\delta}) + g'(y_{u_\delta}) z_0 < \delta.$$

Since the previous convergence is uniform, then there exist two numbers $\lambda_0 > 0$ and $\theta \in (0, \delta)$ such that

$$g(y_{u_\delta}) + \frac{g(y_{u_\lambda}) - g(y_{u_\delta})}{\lambda} \le \theta < \delta \quad \forall \lambda \le \lambda_0.$$

Let us take $\epsilon > 0$ such that

$$\lambda = \frac{\delta - \delta'}{\delta - \theta} \le \lambda_0 \quad \forall \delta' \in [\delta - \epsilon, \delta). \tag{9}$$

Then for every $\delta' \in [\delta - \epsilon, \delta)$ we have

$$\sup_{x \in \Omega} g(y_{u_\lambda}(x)) = \sup_{x \in \Omega} \left[g(y_{u_\delta}(x)) + \lambda \frac{g(y_{u_\lambda}(x)) - g(y_{u_\delta}(x))}{\lambda} \right] \le$$

$$\lambda \sup_{x \in \Omega} \left[g(y_{u_\delta}(x)) + \frac{g(y_{u_\lambda}(x)) - g(y_{u_\delta}(x))}{\lambda} \right] + (1 - \lambda) \sup_{x \in \Omega} g(y_{u_\delta}(x)) \le$$

$$\lambda\theta + (1 - \lambda)\delta = \delta',$$

which implies that u_λ is feasible for $(P_{\delta'})$.

Finally, denoting by

$$c = \sup_{\lambda \in (0, \lambda_0]} \frac{J(u_\lambda) - J(u_\delta)}{\lambda} < +\infty \quad \text{and} \quad r = \frac{c}{\delta - \theta},$$

we get with (9)

$$\inf(P_{\delta'}) \le J(u_\lambda) = J(u_\delta) + \lambda \frac{J(u_\lambda) - J(u_\delta)}{\lambda} \le$$

$$J(u_\delta) + c\lambda = \inf(P_\delta) + c \frac{\delta - \delta'}{\delta - \theta} = \inf(P_\delta) + r(\delta - \delta') \quad \forall \delta' \in [\delta - \epsilon, \delta),$$

which concludes the proof. ∎

Now we consider the case of a convex nondifferentiable function g, which corresponds to the classical case $g(t) = |t|$. In this context the Slater condition is formulated as follows:

$$g(y_\delta(x) + z_0(x)) < \delta \quad \forall x \in \overline{\Omega}, \tag{10}$$

where z_0 is the solution of (7). Then we have the following result:

Proposition 4 *If g is convex and Slater condition (10) is satisfied at u_δ, then (P_δ) is strongly stable on the left.*

Proof. This proof follows the same steps than the previous one. We define u_λ in the same way and from (10) and the continuity of g we deduce the existence of $\theta \in (0, \delta)$ and $\lambda_0 > 0$ satisfying

$$g\left(y_{u_\delta} + \frac{y_{u_\lambda} - y_{u_\delta}}{\lambda}\right) \leq \theta < \delta \quad \forall \lambda \leq \lambda_0.$$

Now taking λ as in (9) and using the convexity of g we obtain

$$\sup_{x \in \Omega} g(y_{u_\lambda}(x)) = \sup_{x \in \Omega} g\left(y_{u_\delta}(x) + \lambda \frac{y_{u_\lambda}(x) - y_{u_\delta}(x)}{\lambda}\right) \leq$$

$$\lambda \sup_{x \in \Omega} g\left(y_{u_\delta}(x) + \frac{y_{u_\lambda}(x) - y_{u_\delta}(x)}{\lambda}\right) + (1 - \lambda) \sup_{x \in \Omega} g(y_{u_\delta}(x)) \leq \delta'.$$

The proof finishes as that of Proposition 3. ■

Corollary 1 *If ϕ is linear and g is convex, then (P_δ) is strongly stable on the left for every $\delta > \delta_0$.*

Proof. If ϕ is linear, then the Slater condition (10) becomes: there exists $u_0 \in K$ such that $g(y_{u_0}(x)) < \delta$ for every $x \in \overline{\Omega}$. Therefore it is enough to take u_0 as any feasible control for (P_{δ_0}) and apply the previous proposition. ■

So far we have studied the stability on the left. Concerning to the stability on the right of (P_δ), we have the following result:

Proposition 5 *For every $\delta \geq \delta_0$ the problem (P_δ) is weakly stable on the right.*

Proof. Let u_δ be a solution of (P_δ). Since K is bounded, we deduce the existence of a sequence $\{\delta_j\}_{j=1}^\infty$ such that $\delta_j \searrow \delta$ when $j \to \infty$ and

$$\lim_{j \to \infty} u_{\delta_j} = \overline{u} \quad \text{weakly-}\star \text{ in } L^\infty(\Omega)$$

for some $\overline{u} \in K$. If y_j and $\overline{y}$ are the states associated to u_{δ_j} and $\overline{u}$ respectively, we have that $y_j \to \overline{y}$ uniformly in $\overline{\Omega}$. Therefore $\overline{u}$ is a feasible control for (P_δ). Now, using the convexity of L with respect to the third variable and the feasibility of u_δ for every $(P_{\delta'})$, with $\delta' > \delta$, we get

$$\inf(P_\delta) \leq J(\overline{u}) \leq \liminf_{j \to \infty} J(u_{\delta_j}) = \lim_{\delta' \searrow \delta} \inf(P_{\delta'}) \leq J(u_\delta) = \inf(P_\delta),$$

which proves (3). ■

4 Optimality conditions

The aim of this section is to prove the following theorem

Theorem 2 *Let $\overline{u} \in K$ be a solution of (P_δ) and $\overline{y} \in H^2(\Omega) \cap H_0^1(\Omega)$ the associated state. Then there exist a real number $\overline{\alpha} \geq 0$, a Borel measure $\overline{\mu} \in M(\Omega)$ and an element $\overline{p} \in W_0^{1,s}(\Omega)$, for all $s \in [1, n/(n-1))$, satisfying*

$$\overline{\alpha} + \|\overline{\mu}\|_{M(\Omega)} > 0, \tag{11}$$

$$\begin{cases} A^\star \overline{p} + \phi'(\overline{y})\overline{p} = \overline{\alpha}\dfrac{\partial L}{\partial y}(x, \overline{y}(x), \overline{u}(x)) + g'(\overline{y}(x))\overline{\mu} \ \textit{in } \Omega, \\ \overline{p} = 0 \ \textit{on } \Gamma, \end{cases} \tag{12}$$

$$\int_\Omega (z(x) - g(\overline{y}(x)))d\overline{\mu}(x) \leq 0 \quad \forall z \in C_0(\Omega) \textit{ with } z(x) \leq \delta \ \forall x \in \Omega, \tag{13}$$

and

$$\int_\Omega \left(\overline{p}(x) + \overline{\alpha}\frac{\partial L}{\partial u}(x, \overline{y}(x), \overline{u}(x))\right)(u(x) - \overline{u}(x))dx \geq 0 \quad \forall u \in K. \tag{14}$$

Moreover, if (P_δ) is strongly stable on the right, then (12)–(14) is satisfied with $\overline{\alpha} = 1$.

Remark 1 *From (13) follows that $\overline{\mu}$ is a positive measure whose support is included in the set*

$$F_\delta = \{x \in \Omega : g(\overline{y}(x)) = \delta\}.$$

Some qualitative properties of the optimal control can be deduced from the previous optimality system. In particular, it can be stated in many situations that the optimal control is a continuous function in Ω F_δ, i.e. the singularities of $\overline{u}$ are located at the points where the state constraint is active; see for example [10].

The proof of Theorem 2 is divided into several steps. First for every $\gamma > 0$ we introduce a penalized control problem:

$$(Q_\gamma) \begin{cases} \min J_\gamma(u) = \displaystyle\int_\Omega \left[L(x, y_u(x), u(x)) + \frac{1}{2\gamma}((g(y(x)) - \delta)^+)^2\right] dx \\ u \in K. \end{cases}$$

The first issue to remark is the following

Proposition 6 *The following identity holds*

$$\lim_{\gamma \searrow 0} \inf(Q_\gamma) = \inf(P_\delta).$$

Proof. Let $\{u_\gamma\}$ be a family of γ-solutions of problems (Q_γ) and $\{y_\gamma\}$ the associated states:

$$J(u_\gamma) \leq \inf(Q_\gamma) + \gamma.$$

From the definition of (Q_γ) it follows that $(g(y_\gamma(x)) - \delta)^+ \longrightarrow 0$ in $L^2(\Omega)$, which, together with (2) and the compactness of the inclusion $H_0^1(\Omega) \cap H^2(\Omega) \subset C_0(\Omega)$, implies the convergence $(g(y_\gamma(x)) - \delta)^+ \longrightarrow 0$ in $C_0(\Omega)$. Therefore

$$\delta_\gamma = \|(g(y_\gamma(x)) - \delta)^+\|_{L^\infty(\Omega)} + \delta \longrightarrow \delta \text{ if } \gamma \searrow 0.$$

As u_γ is feasible pair for (P_{δ_γ}) we deduce that

$$\inf(P_{\delta_\gamma}) \leq J(y_\gamma, u_\gamma) \leq \inf(Q_\gamma) + \gamma.$$

Then, using the weak stability of (P_δ) on the right, Proposition 5, we obtain

$$\inf(P_\delta) = \lim_{\gamma \searrow 0} \inf(P_{\delta_\gamma}) \leq \lim_{\gamma \searrow 0} \{\inf(Q_\gamma) + \gamma\} = \lim_{\gamma \searrow 0} \inf(Q_\gamma) \leq \inf(P_\delta),$$

with the last inequality due to the fact that (y_u, u) is feasible for (Q_γ) whenever it is feasible for (P_δ), with the same cost. ■

Let us remark that the convexity assumption stated in **H1)** can be replaced by the weak stability of (P_δ) on the right and the proof of the previous proposition is still valid. In fact the same remark remains true for Theorem 2.

To prove the first part of Theorem 2 we will use the next result, called Ekeland's variational principle:

Theorem 3 (Ekeland [17]) *Let (E, d) be a complete metric space, $F : E \longrightarrow R \cup \{+\infty\}$ a lower semicontinuous function and let $e_\epsilon \in E$ satisfy*

$$F(e_\epsilon) \leq \inf_{e \in E} F(e) + \epsilon^2.$$

Then there exists an element $\overline{e}_\epsilon \in E$ such that

$$F(\overline{e}_\epsilon) \leq F(e_\epsilon), \quad d(\overline{e}_\epsilon, e_\epsilon) \leq \epsilon$$

and

$$F(\overline{e}_\epsilon) \leq F(e) + \epsilon d(e, \overline{e}_\epsilon) \quad \forall e \in E.$$

Proof of Theorem 2: First Part. Thanks to Proposition 6 we have that $\overline{u}$ is a ϵ_γ^2-solution of (Q_γ), i.e.

$$J_\gamma(\overline{u}) = J(\overline{u}) = \inf(P_\delta) \leq \inf(Q_\gamma) + \epsilon_\gamma^2,$$

with $\epsilon_\gamma \searrow 0$ when $\gamma \longrightarrow 0$. Applying Theorem 3, with $E = K$, $F = J_\gamma$ and $d(u, v) = \|u - v\|_{L^\infty(\Omega)}$, we obtain the existence of a control $u_\gamma \in K$ satisfying

$$\left\{ \begin{array}{l} J_\gamma(u_\gamma) \leq J_\gamma(\overline{u}), \quad \|u_\gamma - \overline{u}\|_{L^\infty(\Omega)} \leq \epsilon_\gamma \quad \text{and} \\ \\ J_\gamma(u_\gamma) \leq J_\gamma(u) + \epsilon_\gamma \|u_\gamma - u\|_{L^\infty(\Omega)} \quad \forall u \in K. \end{array} \right. \tag{15}$$

Since J_γ is Gâteaux differentiable and $\varphi(u) = \|u_\gamma - u\|_{L^\infty(\Omega)}$ is a convex function we obtain (for instance, see [5, Lemma 2])

$$J'_\gamma(u_\gamma)\cdot(u-u_\gamma) + \epsilon_\gamma \|u_\gamma - u\|_{L^\infty(\Omega)} \geq 0 \quad \forall u \in K. \tag{16}$$

Now it is easy to verify that

$$J'_\gamma(u_\gamma)\cdot(u-u_\gamma) = \int_\Omega \left(p_\gamma(x) + \frac{\partial L}{\partial y}(x, y_\gamma(x), u_\gamma(x)) \right) dx, \tag{17}$$

where y_γ is the state associated with u_γ and p_γ is the solution of

$$\left\{ \begin{array}{l} A^\star p_\gamma + \phi'(y_\gamma)p_\gamma = \dfrac{\partial L}{\partial y}(\cdot, y_\gamma, u_\gamma) + \dfrac{1}{\gamma}(g(y_\gamma)-\delta)^+ g'(y_\gamma) \text{ in } \Omega, \\ p_\gamma = 0 \text{ on } \Gamma. \end{array} \right. \tag{18}$$

Remark that $\dfrac{\partial L}{\partial y}(\cdot, y_\gamma, u_\gamma) \in L^1(\Omega)$ thanks to assumption **H2)**. Therefore $p_\gamma \in W_0^{1,s}(\Omega)$ for all $s \in [1, n/(n-1))$; see [21] and [10].

Defining

$$\overline{\alpha}_\gamma = \left(1 + \| \frac{1}{\gamma}(g(y_\gamma)-\delta)^+ \|_{M(\Omega)} \right)^{-1}$$

$$\overline{\mu}_\gamma = \frac{\overline{\alpha}_\gamma}{\gamma}(g(y_\gamma)-\delta)^+, \quad \text{and} \quad \overline{p}_\gamma = \overline{\alpha}_\gamma p_\gamma,$$

we obtain

$$\overline{\alpha}_\gamma + \|\overline{\mu}_\gamma\|_{M(\Omega)} = 1, \tag{19}$$

$$\int_\Omega (z(x) - g(y_\gamma(x)))\overline{\mu}_\gamma(x) \leq 0 \quad \forall z \in C_0(\Omega) \text{ with } z(x) \leq \delta \ \forall x \in \Omega, \tag{20}$$

with (18)

$$\left\{ \begin{array}{l} A^\star \overline{p}_\gamma + \phi'(y_\gamma)p_\gamma = \overline{\alpha}_\gamma \dfrac{\partial L}{\partial y}(\cdot, y_\gamma, u_\gamma) + g'(y_\gamma)\overline{\mu}_\gamma \quad \text{in } \Omega, \\ \overline{p}_\gamma = 0 \quad \text{on } \Gamma, \end{array} \right. \tag{21}$$

and with (16) and (17)

$$\int_\Omega \left(\overline{p}_\gamma(x) + \overline{\alpha}_\gamma \frac{\partial L}{\partial y}(x, (y_\gamma(x), u_\gamma(x)) \right) (u(x) - u_\gamma(x)) dx +$$

$$\epsilon_\gamma \|u_\gamma - u\|_{L^\infty(\Omega)} \geq 0 \quad \forall u \in K. \tag{22}$$

Now we must pass to the limit in (19)–(21) to deduce (11)–(14). From (15) we have that $u_\gamma \to \overline{u}$ in $L^\infty(\Omega)$, hence $y_\gamma \to \overline{y}$ in $H_0^1(\Omega) \cap H^2(\Omega)$. On the other hand, from the definition of $\overline{\mu}_\gamma$ follows that $\{\overline{\mu}_\gamma\}_{\gamma>0}$ is bounded in $M(\Omega)$, so we extract a subsequence weakly-$\star$ convergent to an element $\overline{\mu} \in M(\Omega)$. Using the assumption **H2)** we obtain that the right hand side of (21) converges in $M(\Omega)$ for a subsequence to the the right hand side of (12). Then the corresponding subsequence of $\{\overline{p}_\gamma\}_{\gamma>0}$ converges weakly in $W_0^{1,s}(\Omega)$ to an element $\overline{p}$ satisfying (12). To obtain (11) it is enough to remember (19) and remark that

$$\overline{\alpha} + \|\overline{\mu}\|_{M(\Omega)} = \overline{\alpha} + \frac{1}{\delta} < \overline{\mu}, g(\overline{y}) >=$$

$$\lim_{\gamma \to 0} \left(\overline{\alpha}_\gamma + \frac{1}{\delta} < \overline{\mu}_\gamma, g(y_\gamma) > \right) = \lim_{\gamma \to 0} \left(\overline{\alpha}_\gamma + \|\overline{\mu}_\gamma\|_{M(\Omega)} \right) = 1.$$

Finally, it is easy to pass to the limit in (19), (21) and (22). ■

The rest of the section is devoted to the proof of the optimality conditions (12)–(14), with $\overline{\alpha} = 1$, under the assumption of strong stability of (P_δ) on the right. Instead of the classical differentiable penalization considered above, now we will analyze the exact penalization of the state constraints.

Proposition 7 *If $r > 0$ satisfies (5), then $\overline{u}$ is a local solution in (K, d) of the penalized control problem*

$$\left\{ \begin{array}{l} min\ J_r(u) = \displaystyle\int_\Omega L(x, y_u(x), u(x))dx + r\|(g(y_u) - \delta)^+\|_{L^\infty(\Omega)} \\ \\ u \in K. \end{array} \right.$$

Proof. From (5) it follows

$$\inf(P_\delta) = \inf \left\{ J(u) + r(\delta' - \delta) : u \in K,\ g(y_u(x)) \le \delta',\ \delta' \in [\delta, \delta + \epsilon] \right\}.$$

Minimizing first with respect to δ' for fixed u we find

$$\inf(P_\delta) = \inf \left\{ J(u) + r\|(g(y_u) - \delta)^+\|_\infty : u \in K,\ g(y_u(x)) \le \delta + \epsilon \right\}.$$

Since the mapping $u \in L^\infty(\Omega) \longrightarrow y_u \in C_0(\Omega)$ is continuous, we deduce the existence of a ball $B_\lambda(\overline{u})$, $\lambda > 0$, such that

$$\|g(y_u)\|_{L^\infty(\Omega)} < \delta + \epsilon \quad \forall u \in B_\lambda(\overline{u}),$$

which, together with the previous identity, proves that $\overline{u}$ is a local solution of the penalized control problem. ■

Let us take $\lambda > 0$ as in the proof of the previous proposition and $r > 0$ verifying (5). We introduce the problem

$$(Q_r) \left\{ \begin{array}{l} \min\ J_r(u) = \displaystyle\int_\Omega L(x, y_u(x), u(x))dx + r\|(g(y_u) - \delta)^+)\|_{L^\infty(\Omega)} \\ u \in B_\lambda(\overline{u}). \end{array} \right.$$

Then $\overline{u}$ is a solution of this problem. We passed from a state-constrained control problem to another control problem without state constraints. The difficulty in this new problem is that the penalization term is not differentiable. To overcome this difficulty we define

$$J_{r,q}(u) = \int_\Omega L(x, y_u(x), u(x))dx + r\left(q^{-q} + \int_\Omega [(g(y_u(x)) - \delta)^+]^q)dx\right)^{1/q}$$

and

$$(Q_{r,q})\begin{cases} \min J_{r,q}(u) \\ u \in B_\lambda(\overline{u}), \end{cases}$$

with $q > 1$. Note that $(Q_{r,q})$ has a differentiable cost and moreover it represents an approximation of (Q_r):

Proposition 8 *The following identity holds*

$$\inf(Q_r) = \lim_{q\to\infty} \inf(Q_{r,q}).$$

Proof. From the convergence $\|z\|_{L^q(\Omega)} \to \|z\|_{L^\infty(\Omega)}$ for every $z \in L^\infty(\Omega)$ and the inequalities

$$\|(g(y_u) - \delta)^+\|_{L^q(\Omega)} \leq \left(q^{-q} + \int_\Omega [(g(y_u(x)) - \delta)^+]^q)dx\right)^{1/q} \leq$$

$$\frac{1}{q} + \|(g(y_u) - \delta)^+\|_{L^q(\Omega)}$$

we deduce that $J_{r,q}(u) \to J_r(u)$ when $q \to +\infty$. Therefore for every $u \in B_\lambda(\overline{u})$

$$\limsup_{q\to+\infty} \inf(Q_{r,q}) \leq \limsup_{q\to+\infty} J_{r,q}(u) = J_r(u),$$

hence

$$\limsup_{q\to+\infty} \inf(Q_{r,q}) \leq \inf(Q_r). \tag{23}$$

Now we prove the converse inequality. Let $\{u_q\}_{q\geq 1} \subset B_\lambda(\overline{u})$ be a sequence such that

$$J_{r,q}(u_q) \leq \inf(Q_{r,q}) + \frac{1}{q}.$$

Then using (23) we obtain

$$\liminf_{q\to\infty} J_{r,q}(u_q) \leq \limsup_{q\to\infty} J_{r,q}(u_q) = \limsup_{q\to\infty} \inf(Q_{r,q}) \leq$$

$$\inf(Q_r) \leq \liminf_{q\to\infty} J_r(u_q).$$

We end the proof by checking that

$$\liminf_{q\to\infty} J_r(u_q) \le \liminf_{q\to\infty} J_{r,q}(u_q).$$

Let $\{q_k\}_{k=1}^{\infty}$ a sequence converging to $+\infty$ such that

$$\liminf_{q\to\infty} J_r(u_q) = \lim_{k\to\infty} J_r(u_{q_k}).$$

Let $\{y_{q_k}\}_{k=1}^{\infty}$ the states associated to $\{y_{u_k}\}_{k=1}^{\infty}$. Since $\{y_{q_k}\}_{k=1}^{\infty}$ is bounded in $H_0^1(\Omega)\cap H^2(\Omega)$ we can extract a subsequence, denoted in the same way, such that $y_{q_k} \to \tilde{y}$ in $C_0(\Omega)$. Let $z = (g(\tilde{y}) - \delta)^+$ and let us assume that $\tilde{y}$ attains its maximum in $x_0 \in \overline{\Omega}$. Then for every $\epsilon > 0$ there exist $\rho_\epsilon > 0$ and $k_\epsilon \in N$ such that

$$|(g(y_{q_k}(x) - \delta)^+| \ge \|z\|_{L^\infty(\Omega)} - \epsilon, \quad \forall x \in \overline{\Omega},\ \|x - x_0\| \le \rho_\epsilon \text{ and } \forall k \ge k_\epsilon.$$

From this we deduce that

$$\left(\|z\|_{L^\infty(\Omega)} - \epsilon\right)[m(B_{\rho_\epsilon}(x_0))]^{1/q_k} \le \|(g(y_{q_k}(x)) - \delta)^+\|_{L^{q_k}(\Omega)} \le$$

$$\|(g(y_{q_k}(x)) - \delta)^+\|_{L^\infty(\Omega)}[m(\Omega)]^{1/q_k},$$

hence

$$\|z\|_{L^\infty(\Omega)} - \epsilon \le \lim_{k\to\infty} \|(g(y_{q_k}(x) - \delta)^+\|_{L^{q_k}(\Omega)} \le \|z\|_{L^\infty(\Omega)}.$$

Since $\epsilon > 0$ is arbitrary, we have

$$\lim_{k\to\infty} \|(g(y_{q_k}(x)) - \delta)^+\|_{L^{q_k}(\Omega)} = \|z\|_{L^\infty(\Omega)}.$$

Finally

$$\liminf_{q\to\infty} J_r(u_q) = \liminf_{k\to\infty} J(u_{q_k}) + \|z\|_{L^\infty(\Omega)} =$$

$$\liminf_{k\to\infty} J_{r,q_k}(u_{q_k}) \le \liminf_{q\to\infty} J_{r,q}(u_q). \blacksquare$$

Proof of Theorem 2: Second Part. Thanks to propositions 7 and 8 we deduce that $\overline{u}$ is a ϵ_q^2-solution of $(Q_{r,q})$, with $\epsilon_q \searrow 0$ when $q \to \infty$. Then Theorem 3 claims the existence of a control $u_q \in K$ satisfying

$$\begin{cases} J_{r,q}(u_q) \le J_{r,q}(\overline{u}), \quad \|u_q - \overline{u}\|_{L^\infty(\Omega)} \le \epsilon_q \quad \text{and} \\ J_{r,q}(u_q) \le J_{r,q}(u) + \epsilon_q\|u_q - u\|_{L^\infty(\Omega)} \quad \forall u \in K. \end{cases} \tag{24}$$

Using again [5, Lemma 2], we obtain

$$J'_{q,r}(u_q)\cdot(u - u_q) + \epsilon_q\|u_q - u\|_{L^\infty(\Omega)} \ge 0 \quad \forall u \in K, \tag{25}$$

with

$$J'_{r,q}(u_q)\cdot(u-u_q)=\int_\Omega\left(p_q(x)+\frac{\partial L}{\partial y}(x,y_q(x),u_q(x))\right)dx, \tag{26}$$

where y_q is the state associated with u_q and p_q is the solution of

$$\left\{\begin{array}{l} A^\star p_q+\phi'(y_q)p_q=\dfrac{\partial L}{\partial y}(\cdot,y_q,u_q)+\mu_q g'(y_q) \text{ in } \Omega,\\ p_q=0 \text{ on } \Gamma \end{array}\right. \tag{27}$$

and

$$\mu_q=r\left(q^{-q}+\int_\Omega[(g(y_q(x))-\delta)^+]^q dx\right)^{1/q-1}[(g(y_q)-\delta)^+]^{q-1}.$$

From here we deduce

$$\|\mu_q\|_{M(\Omega)}=\|\mu_q\|_{L^1(\Omega)}\le r\left(\int_\Omega[(g(y_q(x))^+]^q dx\right)^{1/q-1}\int_\Omega[(g(y_q(x))^+]^{q-1}dx=$$

$$r\|(g(y_q)-\delta)^+\|^{1-q}_{L^q(\Omega)}\|(g(y_q)-\delta)^+\|^{q-1}_{L^{q-1}(\Omega)}.$$

Applying Hölder's inequality with exponents $q/(q-1)$ and q it follows

$$\|z\|_{L^{q-1}(\Omega)}\le m(\Omega)^{1/q}\|z\|_{L^q(\Omega)}\quad \forall z\in L^q(\Omega),$$

which together with the previous relation leads to

$$\|\mu_q\|_{M(\Omega)}\le m(\Omega)^{1/q}r\|(g(y_q)-\delta)^+\|_{L^q(\Omega)}\le$$

$$m(\Omega)^{2/q}r\|(g(y_q)-\delta)^+\|_\infty\le M<+\infty\quad\forall q>1.$$

Now we can pass to the limit as in the proof of the first part of the theorem, with the help of (24)–(27), and obtain (12)–(14) with $\overline{\alpha}=1$. ■

5 Finite element approximations of (P_δ)

The aim of this section is to give a convergence result of the numerical approximations of (P_δ). Here the convexity assumption stated in **H1)** is essential, however the differentiability hypothesis on L is not necessary. The presentation of this section follows the steps of [14]. Let $\{\mathcal{T}_h\}_{h>0}$ be a family of triangulations in $\overline{\Omega}$. We associate two parameters with each element $T\in\mathcal{T}_h$: $\rho(T)$ and $\sigma(T)$, $\rho(T)$ denotes the diameter of the set T and $\sigma(T)$ is the diameter of the largest ball contained in T and it is supposed that $h=\max_{T\in\mathcal{T}_h}\rho(T)$. We make the following regularity hypotheses of the triangulation:

- There exist two positive numbers ρ and σ such that:

$$\frac{\rho(T)}{\sigma(T)} \leq \sigma \quad \text{and} \quad \frac{h}{\rho(T)} \leq \rho \quad \forall T \in \mathcal{T}_h \ \text{ and } \ \forall h > 0.$$

- Let us take $\overline{\Omega}_h = \cup_{T \in \mathcal{T}_h} T$, Ω_h its interior and Γ_h its boundary. Then we assume that $\overline{\Omega}_h$ is convex and the vertices of $\mathcal{T}_h$ placed on the boundary Γ_h are points of Γ.

To every boundary triangle T of $\mathcal{T}_h$ we associate another triangle $\tilde{T} \subset \overline{\Omega}$ with two interior sides to Ω coincident with two sides of T and the third side is the curvilinear arc of Γ limited by the other two sides. We denote by $\tilde{\mathcal{T}}_h$ the family formed by these boundary triangles with a curvilinear side and the interior triangles to Ω of $\mathcal{T}_h$, so $\overline{\Omega} = \cup_{T \in \tilde{\mathcal{T}}_h} T$.

Now let us consider the spaces

$$U_h = \{u_h \in L^\infty(\Omega) : u_h|_T \ \text{ is constant } \ \forall T \in \tilde{\mathcal{T}}_h\}$$

and

$$V_h = \{y_h \in C(\overline{\Omega}) : y_h|_T \in \mathcal{P}_1 \ \forall T \in \mathcal{T}_h \text{ and } \ y_h(x) = 0 \ \ \forall x \in \overline{\Omega} \backslash \Omega_h\},$$

where $\mathcal{P}_1$ is the space of the polynomials of degree less than or equal to 1. It is obvious that $V_h \subset H_0^1(\Omega)$. For each $u_h \in U_h$ we denote by $y_h(u_h)$ the unique element of V_h that satisfies:

$$\sum_{i,j=1}^n \int_{\Omega_h} a_{ij}(x) \partial_{x_i} y_h(x) \partial_{x_j} z_h(x) dx + \int_{\Omega_h} a_0(x) y_h(x) z_h(x) dx +$$

$$\int_{\Omega_h} \phi(y_h) z_h dx = \int_{\Omega_h} u_h z_h dx \ \ \forall z_h \in V_h.$$

For each $h > 0$ let K_h be a nonempty convex bounded closed subset of U_h in such a way that $\{K_h\}_{h>0}$ constitutes an internal approximation of K in the following sense:

1. $K_h \subset K$ for each $h > 0$.
2. For every $u \in K$ there exists a family $\{u_h\}_{h>0}$, with $u_h \in K_h$, verifying that $u_h(x) \to u(x)$ for almost every point $x \in \Omega$.

See for example Glowinski et al. [19, Vol. 1] for these questions.

Finally we state the finite dimensional optimal control problem:

$$(P_{\delta h}) \begin{cases} \text{minimize } \ J_h(u_h) = \displaystyle\int_{\Omega_h} L(x, y_h(u_h), u_h) dx \\ \\ \text{subject to } u_h \in K_h \text{ and } g(y_h(u_h)(x_j)) \leq \delta \ \forall j \in I_h, \end{cases}$$

where $\{x_j\}_{j=1}^{n(h)}$ is the set of vertices of $\mathcal{T}_h$ and I_h is the index set corresponding to interior vertices. Note that $y_h(u_h)$ is zero at the boundary vertices.

The following theorem states the existence of a solution for $(P_{\delta h})$.

Theorem 4 *For every $\delta > \delta_0$ there exists $h_\delta > 0$ such that $(P_{\delta h})$ has at least one solution $\overline{u}_h$ for all $h \leq h_\delta$.*

Proof. Since K_h is a compact set and J_h is continuous, the existence of a solution of $(P_{\delta h})$ will be assured if we prove that the set of feasible controls is nonempty. Let $u_0 \in K$ be a feasible control for (P_{δ_0}) and let us take $u_{0h} \in K_h$ such that $u_{0h}(x) \to u_0(x)$ for almost every point $x \in \Omega$ as $h \to 0$. Then $y_h(u_{0h}) \to y_{u_0}$ uniformly in $\overline{\Omega}$. Since $g(y_{u_0}(x)) \leq \delta_0$ for every $x \in \overline{\Omega}$, from the uniform convergence and the inequality $\delta > \delta_0$ we deduce the existence of $h_\delta > 0$ such that the inequality $g(y_h(u_{0h})(x)) \leq \delta$ holds for all $x \in \overline{\Omega}$ and each $h \leq h_\delta$. Therefore u_{0h} is a feasible control for $(P_{\delta h})$, which completes the proof. ∎

Remark 2 *From the previous proof it follows easily that $h_\delta \geq h_{\delta'}$ if $\delta > \delta'$. On the other hand, we do not know if problems $(P_{\delta_0 h})$ have feasible controls. Obviously the argument used in the proof is valid only if $\delta > \delta_0$.*

Finally we prove the convergence result of the numerical approximation.

Theorem 5 *Let us assume that (P_δ) is a weakly stable problem on the left and let $h_\delta > 0$ be as in Theorem 4. Given a family of controls $\{\overline{u}_h\}_{h<h_\delta}$, $\overline{u}_h$ being a solution of $(P_{\delta h})$, then there exist subsequences $\{\overline{u}_{h_k}\}_{k\in N}$, with $h_k \to 0$ as $k \to \infty$, and elements $\overline{u} \in K$ such that $\overline{u}_{h_k} \to \overline{u}$ in the weak-$\star$ topology of $L^\infty(\Omega)$. Each one of these limit points is a solution of (P_δ). Moreover we have*

$$\lim_{h\to 0} J_h(\overline{u}_h) = \inf(P_\delta).$$

Proof. Let $\overline{y}_h$ be the state associated to $\overline{u}_h$. Since $\{\overline{u}_h\}_{h\leq h_\delta} \subset K$ and K is bounded in $L^\infty(\Omega)$, we can extract a subsequence $\{\overline{u}_{h_k}\}$ such that $h_k \to 0$ and $\overline{u}_{h_k} \to \overline{u}$ weakly-$\star$ in $L^\infty(\Omega)$ for some element $\overline{u} \in K$. Now we prove that $\overline{u}$ is a solution of (P_δ). Let $\overline{y}$ the state associated to $\overline{u}$. Since $\overline{y}_{h_k} \to \overline{y}$ uniformly in $\overline{\Omega}$ and $g(\overline{y}_{h_k}(x_j)) \leq \delta$ for every $j \in I_h$, it follows that $g(\overline{y}(x)) \leq \delta$ and therefore $\overline{u}$ is a feasible control for the problem (P_δ).

Let $\delta' \in [\delta_0+\epsilon, \delta)$, with $0 < \epsilon < \delta - \delta_0$ fixed, and let $u_{\delta'}$ be a solution of $(P_{\delta'})$. For each h let us take $u_{\delta' h} \in K_h$ such that $u_{\delta' h}(x) \to u_{\delta'}(x)$ for almost every point of Ω. From the uniform convergence $y_h(u_{\delta' h}) \to y_{u_{\delta'}}$ and the inequality $g(y_{u_{\delta'}}(x)) \leq \delta' < \delta$ for all $x \in \overline{\Omega}$ we deduce the existence of $h_{\delta'} \geq h_{\delta_0+\epsilon}$ such that $g(y_h(u_{\delta' h})(x)) \leq \delta\ \forall x \in \overline{\Omega}$ and $\forall h \leq h_{\delta'}$. Hence $u_{\delta' h}$ is a feasible control for $(P_{\delta h})$ always that $h \leq h_{\delta'}$. Then, given $k_0 > 0$ such that $h_k \leq h_{\delta_0+\epsilon}\ \ \forall k \geq k_0$, we obtain that $J_{h_k}(\overline{u}_{h_k}) \leq J_{h_k}(u_{\delta' h_k})$ for every $k > k_0$. Using the convexity of L with respect to the third variable we get

$$J(\overline{u}) \leq \liminf_{k\to\infty} J_{h_k}(\overline{u}_{h_k}) \leq \liminf_{k\to\infty} J_{h_k}(u_{\delta' h_k}) = J(u_{\delta'}) = \inf(P_{\delta'}).$$

Finally, the feasibility of $\overline{u}$ for (P_δ) and the stability condition (4) enables us to conclude that

$$\inf(P_\delta) \leq J(\overline{u}) \leq \lim_{\delta' \nearrow \delta} \inf(P_{\delta'}) = \inf(P_\delta),$$

which proves that $\overline{u}$ is a solution of (P_δ). The rest of theorem is immediate. ∎

Remark 3 *We can imagine that nonstable problems (P_δ) have isolated solutions which are far from the solutions of problems $(P_{\delta'})$, with $\delta' < \delta$. Obviously these type of problems are not easy to solve numerically. To deal with these cases we must increase δ when the numerical approximations are carried out, by taking $\delta + \epsilon_h$, $\epsilon_h \to 0$ as $h \to 0$. In order to find out the region containing the solution of (P_δ) we must choose ϵ_h large enough. However, if ϵ_h is too large, then the approximation $\overline{u}_h$ could be too far from the solution of (P_δ). This equilibrium is possible, though sometimes difficult, because the set of numbers δ corresponding to nonstable problems is numerable. Anyway to enlarge δ in the approximations is not a good idea for stable problems because this procedure becomes slower the convergence.*

References

[1] J.F. Bonnans. Pontryagin's principle for the optimal control of semilinear elliptic systems with state constraints. In *30th IEEE Conference on Control and Decision*, pages 1976–1979, Brighton, England, 1991.

[2] J.F. Bonnans and E. Casas. Contrôle de systèmes non linéaires comportant des contraintes distribuées sur l'état. Technical Report 300, INRIA Rocquencourt, May 1984.

[3] J.F. Bonnans and E. Casas. On the choice of the function spaces for some state-constrained control problems. *J. on Numerical Functional Analysis & Optimization*, 7(4):333–348, 1985.

[4] J.F. Bonnans and E. Casas. Contrôle de systèmes elliptiques semilinéaires comportant des contraintes sur l'état. In H. Brezis and J.L. Lions, editors, *Nonlinear Partial Differential Equations and Their Applications. Collège de France Seminar*, volume 8, pages 69–86. Longman Scientific & Technical, New York, 1988.

[5] J.F. Bonnans and E. Casas. Optimal control of semilinear multistate systems with state constraints. *SIAM J. on Control & Optim.*, 27(2):446–455, 1989.

[6] J.F. Bonnans and E. Casas. Un principe de Pontryagine pour le contrôle des systèmes elliptiques. *J. Differential Equations*, 90(2):288–303, 1991.

[7] J.F. Bonnans and E. Casas. A boundary Pontryagin's principle for the optimal control of state-constrained elliptic equations. *International Series of Numerical Mathematics*, 107:241–249, 1992.

[8] J.F. Bonnans and E. Casas. An extension of Pontryagin's principle for state-constrained optimal control of semilinear elliptic equations and variational inequalities. *To appear*, 1992.

[9] J.V. Burke. An exact penalization viewpoint of constrained optimization. *SIAM J. on Control & Optim.*, 29(4):968–998, 1991.

[10] E. Casas. Control of an elliptic problem with pointwise state constraints. *SIAM J. on Control & Optim.*, 24(6):1309–1318, 1986.

[11] E. Casas. Optimality conditions and numerical approximations for some optimal design problems. *Control and Cybernetics*, 19(3–4):73–91, 1990.

[12] E. Casas. Optimal control in coefficients with state constraints. *Applied Mathematics and Optimization*, 26:21–37, 1992.

[13] E. Casas. Boundary control of semilinear elliptic equations with pointwise state constraints. *SIAM J. on Control & Optim.*, To appear.

[14] E. Casas. Finite element approximations for some state-constrained optimal control problems. In P. Borne and S.G. Tzafestas, editors, *Mathematics of the Analysis and Design of Process Control.* Elsevier, To appear.

[15] L. Cesari. *Optimization. Theory and Applications.* Springer-Verlag, New York-Heidelberg-Berlin, 1983.

[16] F.H. Clarke. A new approach to Lagrange multipliers. *Math. Op. Res.*, 1(2):165–174, 1976.

[17] I. Ekeland. Nonconvex minimization problems. *Bull. Amer. Math. Soc.*, 1(3):76–91, 1979.

[18] I. Ekeland and R. Temam. *Analyse Convexe et Problèmes Variationnels.* Dunod-Gauthier Villars, Paris, 1974.

[19] R. Glowinski, J.L. Lions, and R. Trémolières. *Analyse Numérique des Inéquations Variationelles.* Dunod, Paris, 1976.

[20] P. Grisvard. *Elliptic Problems in Nonsmooth Domains.* Pitman, Boston-London-Melbourne, 1985.

[21] G. Stampacchia. Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus. *Ann. Inst. Fourier (Grenoble)*, 15:189–258, 1965.