Maximum Principle for State-Constrained Optimal Control Problems Governed by Quasilinear Elliptic Equations*

by

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Abstract. In this paper, the authors study an optimal control problem for quasilinear elliptic PDEs with pointwise state constraints. Weak and strong optimality conditions of Pontryagin maximum principle type are derived. In proving these results, we penalized the state constraints and respectively use the Ekeland variational principle and an exact penalization method.

Keywords. quasilinear elliptic equations, optimal control, Pontryagin's principle, state constraints.

AMS(MOS) subject classifications. 49K20, 35J65, 35J85

§1. Introduction.

In this paper, our aim is to prove Pontryagin's principle for pointwise state-constrained optimal control problems governed by very general quasilinear elliptic equations. The control is distributed and takes values in a bounded subset, not necessarily convex, of some Euclidean space. The cost functional is Lagrange type.

Standard results of optimal control problems for linear elliptic equations with convex control set and convex functional can be found in [16]. In [1,5], the results were extended to linear or semilinear equations with state constraints. In the framework of semilinear elliptic equations, the Pontryagin type principle was first proved in [2] for problems without

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state constraints; later, in [3,4] and [21], different approaches were used to deal with the state-constrained case. In this paper, we improve the techniques of [3,4] and [21] so that the extension of the results to the quasilinear equations is possible. We remove the weak stability assumption made in [4] for the weak version of Pontryagin principle (see §4). We also obtain the strong version of Pontryagin principle, which was not carried out in [21]. As in [3,4], to prove this strong principle, we assume a stability condition for optimal cost functional with respect to small perturbations of the feasible state set. This leads an exact penalization of the state constraint. The penalty functional used here is different from that in [3,4], which allows us to shorten the proof.

Let us mention some other papers related to the present one. In [6], optimal control of quasilinear elliptic equations without state constraints was considered; and for the evolution case of finite and infinite dimensions, see [10,13,18], and the references cited therein.

This paper is organized as follows. In §2, optimal control problem is formulated and the state equation is studied. §3 is devoted to the derivation of the variation along given feasible pairs, which is needed to deal with the case of a not necessarily convex control set. The approach followed in this section is based on the method used in [12]. In §§4 and 5, we obtain the weak and strong Pontryagin maximum principles.

§2. Formulation of the Problem.

This section is devoted to a formulation of the control problem which will be studied in this paper. Our state equation is as follows:

$$
\begin{cases} -\nabla \cdot a(x, \nabla y(x)) = f(x, y(x), u(x)), & \text{in } \Omega, \\ y \big|_{\partial\Omega} = 0. \end{cases} \tag{2.1}
$$

In what follows, we always assume that Ω is a bounded region in $\mathbb{R}^n$ with a $C^{1,\gamma}$ boundary $\partial\Omega$, for some $\gamma > 0$ and U is a bounded measurable set in some Euclidean space. We use $|\cdot|$ as the norm of vectors in Euclidean spaces or of matrices, which can be identified from the context. Also, we let $\langle \cdot, \cdot \rangle$ be the inner products or duality in possibly different spaces. For any measurable set $S \subset \mathbb{R}^n$, we use $|S|$ to denote the Lebesgue measure of the set S.

We make the following assumptions.

(A1) The function $a : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous. For each $x \in \bar{\Omega}$, $a(x,\cdot)$ is differentiable and $a_\zeta(\cdot,\cdot)$ is continuous (we use ζ as the dummy argument for ∇y). Moreover there exist constants $\alpha > 1$, $0 < \sigma \le 1$, $\Lambda \ge \lambda > 0$ and $\kappa > 0$, such that for all $x, \widehat{x} \in \Omega$, $\zeta, \xi \in \mathbb{R}^n$,

$$\lambda(\kappa + |\zeta|)^{\alpha-2}|\xi|^2 \le \langle\, a_\zeta(x,\zeta)\xi, \xi \,\rangle \le \Lambda(\kappa + |\zeta|)^{\alpha-2}|\xi|^2, \tag{2.2}$$

$$|a(x,\zeta) - a(\widehat{x},\zeta)| \le \Lambda(1 + |\zeta|)^{\alpha-1}|x - \widehat{x}|^\sigma. \tag{2.3}$$

(A2) The function $f : \Omega \times \mathbb{R} \times U \to \mathbb{R}$ has the following properties: $f(\cdot,y,u)$ is measurable on Ω, $f(x,\cdot,u)$ is in $C^1(\mathbb{R})$ with $f(x,\cdot,\cdot)$ and $f_y(x,\cdot,\cdot)$ being continuous on $\mathbb{R} \times U$. Moreover,

$$f_y(x,y,u) \le 0, \qquad \forall (x,y,u) \in \Omega \times \mathbb{R} \times U, \tag{2.4}$$

and for any $R > 0$, there exists an $M_R > 0$, such that

$$|f(x,y,u)| + |f_y(x,y,u)| \le M_R, \qquad \forall (x,u) \in \Omega \times U, |y| \le R. \tag{2.5}$$

Next, we set

$$\mathcal{U} = \{u : \Omega \to U \;\big|\; u \quad \text{is measurable}\;\}.$$

Any element $u \in \mathcal{U}$ is referred to as a control. In what follows, we will denote by $C_0(\Omega)$ the set of all continuous function on $\bar{\Omega}$ which vanish on $\partial\Omega$ and by $C^{1,\beta}(\bar{\Omega})$ the set of all continuously differentiable functions on $\bar{\Omega}$ for which the first order partial derivatives are Hölder continuous with the exponent $\beta \in (0,1)$. Now, we state the following basic result.

Proposition 2.1. *Let (A1)–(A2) hold. Then, for any $u \in \mathcal{U}$, there exists a unique $y \equiv y(\cdot\,;u) \in C^{1,\beta}(\bar{\Omega}) \bigcap C_0(\Omega)$ solving (2.1) for some $\beta \in (0, \min\{\sigma,\gamma\})$. Furthermore, there exists a constant $C > 0$, independent of $u \in \mathcal{U}$, such that*

$$\|y(\cdot\,;u)\|_{C^{1,\beta}(\bar{\Omega})} \le C, \qquad \forall u \in \mathcal{U}. \tag{2.6}$$

Sketch of the Proof. First of all, we truncate f: For any $m > 0$, let

$$f_m(x,y,u) = \begin{cases} f(x,y,u), & \text{if } |y| \le m, \\ f(x,-m,u), & \text{if } y < -m, \\ f(x,m,u), & \text{if } y > m. \end{cases} \tag{2.7}$$

Then, we consider the following truncated problem:

$$\begin{cases} -\nabla\cdot a(x,\nabla y(x)) = f_m(x,y(x),u(x)), & \text{in } \Omega, \\ y\mid_{\partial\Omega} = 0. \end{cases} \tag{2.8}$$

By [15], we know that (2.8) admits a unique solution $y_m \in W_0^{1,\alpha}(\Omega)$. Then, as in [19], we are able to show that there exists a constant $C > 0$ independent of m and $u \in \mathcal{U}$, such that

$$\|y_m(\cdot\,;u)\|_{L^\infty(\Omega)} \le C, \qquad \forall m > 0, u \in \mathcal{U}. \tag{2.9}$$

Consequently, for $m > C$, we obtain that $y_m = y$ is a solution of (2.1). Thus, by [14], we obtain that in fact this y is in $C^{1,\beta}(\bar\Omega)$, for some $\beta \in (0, \min\{\sigma,\gamma\})$ and the estimate (2.6) holds. Finally, the uniqueness follows immediately from the coercivity of the operator (see (2.2) and (2.4)). $\square$

In what follows, any pair $(y,u) \in (C^{1,\beta}(\bar\Omega)\bigcap C_0(\Omega)) \times \mathcal{U}$ satisfying (2.1) is called a feasible pair and we refer to the corresponding y and u as feasible state and control, respectively. Clearly, under (A1)–(A2), $\mathcal{U}$ coincides with the set of all feasible controls and for each feasible control $u \in \mathcal{U}$ there corresponds a unique feasible state. Now, we let $f^0 : \Omega \times \mathbb{R} \times U \to \mathbb{R}$ be a given function. We make the following assumption on this function:

(A3) The function $f^0(\cdot,y,u)$ is measurable on Ω, $f^0(x,\cdot,u)$ is in $C^1(\mathbb{R})$ with $f^0(x,\cdot,\cdot)$ and $f^0_y(x,\cdot,\cdot)$ being continuous on $\mathbb{R}\times U$. Furthermore, for any $R > 0$, there exists a function $\varphi_R \in L^1(\Omega)$, such that

$$|f^0(x,y,u)| + |f^0_y(x,y,u)| \le \varphi_R(x), \qquad \forall (x,u) \in \Omega\times U, \quad |y| \le R. \tag{2.10}$$

It is easy to see that under (A1)–(A3), for any $u \in \mathcal{U}$, the following functional is well-defined:

$$J(u) = \int_\Omega f^0(x,y(x),u(x))dx. \tag{2.11}$$

This functional is referred to as the cost functional. Next, we introduce another map $g : \bar\Omega \times \mathbb{R} \to \mathbb{R}$. We assume the following:

(A4) The map g is continuous, $g_y(\cdot,\cdot)$ exists and is also continuous on $\bar\Omega\times\mathbb{R}$. Moreover we assume that

$$g(x,0)=0, \qquad \forall x\in\bar\Omega. \tag{2.12}$$

The assumption (2.12) can be relaxed; see Remark 4.2.

¿From above, we know that under (A1)–(A2), for any $u\in\mathcal{U}$, the corresponding feasible state y is in $C^{1,\beta}(\bar\Omega)$. Thus, we may talk about the state constraint of form

$$g(x,y(x))\le\delta, \qquad \forall x\in\bar\Omega, \tag{2.13}$$

where $\delta>0$ is given. Of course, for any given $u\in\mathcal{U}$, the corresponding state y does not necessarily satisfy the constraint (2.13). We refer to any feasible pair (y,u) satisfying (2.13) as an admissible pair and the corresponding y and u as admissible state and control, respectively. We denote the set of all admissible controls by $\mathcal{U}_\delta$, indicating the dependence on δ by the subscript. Now, our optimal control problem can be stated as follows:

Problem (P$_\delta$). Under (A1)–(A4), find a control $\bar u\in\mathcal{U}_\delta$, such that

$$J(\bar u)=\inf_{u\in\mathcal{U}_\delta}J(u). \tag{2.14}$$

Any admissible control $\bar u$ satisfying (2.14) is called an optimal control, the corresponding state $\bar y$ is called an optimal state and the pair $(\bar y,\bar u)$ is referred to as an optimal pair.

§3. Variation along Given Feasible Pairs.

In deriving necessary conditions for optimal pairs, one needs to make certain perturbations for the control and the corresponding variations of the state and the cost functional need to be determined. This section is devoted to such a determination. We note that since the control domain is not necessarily convex, the perturbation of the control is restricted to be of "spike" type. This causes the computation somewhat technical. Our basic idea here is taken from [12] and [13,21].

For any feasible pair (y,u), we define

$$
(3.1) \qquad \begin{cases} a_{ij}(x) = a_{i,\zeta_j}(x,\nabla y(x)), \qquad 1 \le i,j \le n, \\ a_0(x) = -f_y(x,y(x),u(x)), \\ c(x) = f^0_y(x,y(x),u(x)), \end{cases}
$$

and given $u \in \mathcal{U}$,

$$
(3.2) \qquad \begin{cases} h(x) = f(x,y(x),v(x)) - f(x,y(x),u(x)), \\ h^0(x) = f^0(x,y(x),v(x)) - f^0(x,y(x),u(x)). \end{cases}
$$

Set

$$
(3.3) \qquad Az(x) \equiv - \sum_{i,j=1}^{n} \partial_{x_j}\big(a_{ij}(x)\partial_{x_i} z(x)\big) + a_0(x)z(x).
$$

Since $y \in C^{1,\beta}(\bar{\Omega})$ with an estimate (2.6), by (2.2) and (2.4), we see that the following hold: For some constants $\Lambda_0 \ge \lambda_0 > 0$, independent of $u \in \mathcal{U}$,

$$
(3.4) \qquad \lambda_0|\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \le \Lambda_0|\xi|^2, \qquad \forall \xi \in \mathbb{R}^n, x \in \Omega,
$$

$$
(3.5) \qquad a_0(x) \ge 0, \qquad \text{a.e. } x \in \Omega.
$$

Moreover, each a_{ij} is in $C(\bar{\Omega})$ and the modulus of continuity for a_{ij} is uniform in $u \in \mathcal{U}$. Now, we consider the following problem

$$
(3.6) \qquad \begin{cases} Az(x) = h(x), \qquad \text{in } \Omega, \\ z\,\big|_{\partial\Omega} = 0. \end{cases}
$$

Clearly, since $h \in L^\infty(\Omega)$, this problem admits a unique solution $z \in W^{1,p}_0(\Omega) \bigcap C_0(\Omega)$ for every $p > 1$; see for instance [17].

Our main result of this section is the following.

Theorem 3.1. *Let (y,u) be a given feasible pair and $v \in \mathcal{U}$ be fixed. Then, for any $\rho \in (0,1)$, there exists a measurable set $E_\rho \subset \Omega$, with property*

$$
(3.7) \qquad |E_\rho| = \rho|\Omega|,
$$

such that if we define u_ρ by

$$u_\rho(x) = \begin{cases} u(x), & \text{if } x \in \Omega \setminus E_\rho, \\ v(x), & \text{if } x \in E_\rho, \end{cases} \tag{3.8}$$

and we let y_ρ be the state corresponding to u_ρ, then the following hold:

$$\begin{cases} y_\rho = y + \rho z + r_\rho, \\ \lim_{\rho \to 0} \frac{1}{\rho} \|r_\rho\|_{W^{1,p}(\Omega)} = 0; \end{cases} \tag{3.9}$$

and

$$\begin{cases} J(u_\rho) = J(u) + \rho z^0 + r_\rho^0, \\ \lim_{\rho \to 0} \frac{1}{\rho} |r_\rho^0| = 0; \end{cases} \tag{3.10}$$

where z is the solution of (3.6), p is any number in $[1,\infty)$ and z^0 is given by

$$z^0 = \int_\Omega [f_y^0(x,y(x),u(x))z(x) + f^0(x,y(x),v(x)) - f^0(x,y(x),u(x))]dx. \tag{3.11}$$

To prove the above result, we need some lemmas. First, we recall the so-called Ekeland distance. For any $u, v \in \mathcal{U}$, we let

$$d(u,v) = |\{x \in \Omega \bigm| u(x) \neq v(x)\}|. \tag{3.12}$$

It is standard that $(\mathcal{U}, d(\cdot,\cdot))$ is a complete metric space (see [9]). Our first lemma is concerning the continuity of the state $y(\cdot\,; u)$ with respect to the control u.

Lemma 3.2. *Let $u, \widehat{u} \in \mathcal{U}$ and $y, \widehat{y}$ be the corresponding states. Then,*

$$\|y - \widehat{y}\|_{W^{1,p}(\Omega)} \le \begin{cases} C_p d(u,\widehat{u})^{\frac{n+p}{np}}, & \text{if } p > \frac{n}{n-1}, \\ C_{p,q} d(u,\widehat{u})^{1/q}, \quad \forall q > 1, & \textit{if } p = \frac{n}{n-1}, \\ C_p d(u,\widehat{u}), & \text{if } 1 \le p < \frac{n}{n-1}, \end{cases} \tag{3.13}$$

with the constants C_p and $C_{p,q}$ being independent of u and $\widehat{u}$.

Proof. We denote

$$\begin{cases} \widetilde{a}_{ij}(x) = \int_0^1 a_{i,\zeta_j}(x, \nabla y(x) + \tau \nabla(\widehat{y}(x) - y(x)))d\tau, & 1 \le i,j \le n, \\ \widetilde{a}_0(x) = -\int_0^1 f_y(x, y(x) + \tau(\widehat{y}(x) - y(x)), \widehat{u}(x))d\tau. \end{cases} \tag{3.14}$$

Then, we define

$$\widetilde{A}z(x) \equiv -\sum_{i,j=1}^{n} \partial_{x_j}\big(\widetilde{a}_{ij}(x)\partial_{x_i} z(x)\big) + \widetilde{a}_0(x)z(x). \tag{3.15}$$

¿From (2.1), we see that $\widehat{y} - y$ satisfies the following:

$$\begin{cases} \widetilde{A}(\widehat{y}(x) - y(x)) = f(x, y(x), \widehat{u}(x)) - f(x, y(x), u(x)), & \text{in } \Omega, \\ (\widehat{y} - y)\big|_{\partial\Omega} = 0. \end{cases} \tag{3.16}$$

By the L^p estimate for the divergence form elliptic equations (see [17]), we obtain

$$\|\widehat{y} - y\|_{W^{1,p}(\Omega)} \le C\|f(\cdot, y, \widehat{u}) - f(\cdot, y, u)\|_{W^{-1,p}(\Omega)}. \tag{3.17}$$

By Sobolev embedding, we have

$$\begin{cases} L^{\frac{np}{n+p}}(\Omega) \hookrightarrow W^{-1,p}(\Omega), & \text{for } p > \dfrac{n}{n-1}, \\ L^q(\Omega) \hookrightarrow W^{-1,p}(\Omega), & \text{for } p = \dfrac{n}{n-1}, \quad \forall q > 1, \\ L^1(\Omega) \hookrightarrow W^{-1,p}(\Omega), & \text{for } 1 \le p < \dfrac{n}{n-1}. \end{cases} \tag{3.18}$$

This together with (3.17) and (2.5) gives (3.13). In the above, we should note that $\widehat{y}$ and y are bounded in $C^{1,\beta}(\bar{\Omega})$ and the constant in L^p estimate only depends on the modulus of continuity of the leading coefficients, the ellipticity constant, the bounds of the coefficients and the domain. Thus, the constant appeared in (3.13) is independent of controls u and $\widehat{u}$. □

Our next lemma is essential in this paper.

Lemma 3.3. *Let $p > n$, $b^0 \in L^1(\Omega)$ and $b \in L^p(\Omega)$. For any $\rho \in (0,1)$, let*

$$\mathcal{E}_\rho = \{E \subset \Omega \mid E \text{ is measurable with } |E| = \rho|\Omega|\}. \tag{3.19}$$

Then,

$$\inf_{E \in \mathcal{E}_\rho} \{|\int_\Omega (1 - \frac{1}{\rho}\chi_E(x))b^0(x)dx| + \|(1 - \frac{1}{\rho}\chi_E)b\|_{W^{-1,p}(\Omega)}\} = 0. \tag{3.20}$$

Proof. We let Γ be the kernel for the Newtonian potential:

$$\Gamma(x) = \begin{cases} \frac{1}{n(2-n)\omega_n}|x|^{2-n}, & n \ge 3, \\ \frac{1}{2\pi}\log|x|, & n = 2, \end{cases} \tag{3.21}$$

with ω_n being the volume of the unit ball in $\mathbb{R}^n$. Then, we let $E \in \mathcal{E}_\rho$ be undetermined and set

$$V^\rho(x) = \int_\Omega \Gamma(x-\xi)(1 - \frac{1}{\rho}\chi_E(\xi))b(\xi)d\xi, \qquad x \in \Omega. \tag{3.22}$$

We know that (see [11])

$$\Delta V^\rho(x) = (1 - \frac{1}{\rho}\chi_E(x))b(x), \qquad x \in \Omega. \tag{3.23}$$

Then, for any $\varphi \in W_0^{1,p'}(\Omega)$, ($p' = \frac{p}{p-1} < \frac{n}{n-1}$), we have

$$\begin{aligned} &\int_\Omega (1 - \frac{1}{\rho}\chi_E(x))b(x)\varphi(x)dx = \int_\Omega \Delta V^\rho(x)\varphi(x)dx \\ &= -\int_\Omega \nabla V^\rho(x)\cdot\nabla\varphi(x)dx \le \|\nabla V^\rho\|_{L^p(\Omega)}\|\varphi\|_{W^{1,p'}(\Omega)}. \end{aligned} \tag{3.24}$$

Thus,

$$\|(1 - \frac{1}{\rho}\chi_E)b\|_{W^{-1,p}(\Omega)} \le \|\nabla V^\rho\|_{L^p(\Omega)}. \tag{3.25}$$

Next, we estimate $\|\nabla V^\rho\|_{L^p(\Omega)}$. To this end, we denote

$$\theta(x,\xi) = \nabla_x\Gamma(x-\xi)b(\xi), \qquad x,\xi \in \bar{\Omega}. \tag{3.26}$$

Then, we have

$$\begin{aligned} &\lim_{x'\to x}\int_\Omega |\theta(x',\xi) - \theta(x,\xi)|d\xi \\ &\le \lim_{x'\to x}(\int_\Omega |\nabla_x\Gamma(x'-\xi) - \nabla_x\Gamma(x-\xi)|^{p'}d\xi)^{1/p'}\|b\|_{L^p(\Omega)} = 0. \end{aligned} \tag{3.27}$$

Hence, for any $\varepsilon > 0$, there exists a finite set $\{x^k, 1 \le k \le k_\varepsilon\} \subset \Omega$, such that for any $x \in \bar{\Omega}$, there exists an x^k, with the property that

$$\int_\Omega |\theta(x,\xi) - \theta(x^k,\xi)|d\xi < \varepsilon. \tag{3.28}$$

Next, we set

$$\Theta(\xi) = \begin{pmatrix} b^0(\xi) \\ \theta(x^1,\xi) \\ \theta(x^2,\xi) \\ \vdots \\ \theta(x^{k_\varepsilon},\xi) \end{pmatrix}, \qquad \xi \in \Omega. \tag{3.29}$$

Clearly, $\Theta \in L^1(\Omega; \mathbb{R}^{nk_\varepsilon+1})$. We can find a simple function

$$\widetilde{\Theta}(\xi) = \sum_{i=1}^{\ell} \Theta_i \chi_{F_i}(\xi), \qquad \xi \in \Omega, \Theta_i \in \mathbb{R}^{nk_\varepsilon+1}, \tag{3.30}$$

with the F_i's being mutually disjoint and $\Omega = \bigcup_{i=1}^{\ell} F_i$, such that

$$\int_\Omega |\Theta(\xi) - \widetilde{\Theta}(\xi)| d\xi < \varepsilon. \tag{3.31}$$

Then, we take $E_\rho^i \subset F_i$, such that

$$|E_\rho^i| = \rho |F_i|, \qquad 1 \le i \le \ell, \tag{3.32}$$

and we define

$$E_\rho = \bigcup_{i=1}^{\ell} E_\rho^i. \tag{3.33}$$

Clearly, $E_\rho \in \mathcal{E}_\rho$. Also, by the above construction, we have

$$\int_\Omega (1 - \frac{1}{\rho} \chi_{E_\rho}(\xi)) \widetilde{\Theta}(\xi) d\xi = 0. \tag{3.34}$$

Now, we take $E = E_\rho$ in (3.22). By Lemma 3.4, which is going to be proved below, we know that

$$\nabla V^\rho(x) = \int_\Omega \nabla_x \Gamma(x-\xi)(1 - \frac{1}{\rho}\chi_E(\xi)) b(\xi) d\xi = \int_\Omega (1 - \frac{1}{\rho}\chi_E(\xi)) \theta(x,\xi) d\xi. \tag{3.35}$$

Thus, for any $x \in \bar{\Omega}$, we let x^k satisfy (3.28). It follows that

$$\begin{aligned} |\nabla V^\rho(x)| &= |\int_\Omega (1 - \frac{1}{\rho}\chi_{E_\rho}(\xi))\theta(x,\xi) d\xi| \\ &\le |\int_\Omega (1 - \frac{1}{\rho}\chi_{E_\rho}(\xi))(\theta(x,\xi) - \theta(x^k,\xi)) d\xi| + |\int_\Omega (1 - \frac{1}{\rho}\chi_{E_\rho}(\xi))\theta(x^k,\xi) d\xi| \\ &\le (1 + \frac{1}{\rho})\varepsilon + (1 + \frac{1}{\rho}) \int_\Omega |\Theta(\xi) - \widetilde{\Theta}(\xi)| d\xi + |\int_\Omega (1 - \frac{1}{\rho}\chi_{E_\rho}(\xi)) \widetilde{\Theta}(\xi) d\xi| \\ &\le 2(1 + \frac{1}{\rho})\varepsilon. \end{aligned} \tag{3.36}$$

Furthermore, recalling the definition of $\Theta(\xi)$ and E_ρ, we also have

$$|\int_\Omega (1 - \frac{1}{\rho}\chi_{E_\rho}(\xi)) b_0(\xi) d\xi| < (1 + \frac{1}{\rho})\varepsilon. \tag{3.37}$$

Combining (3.25), (3.36) and (3.37), we obtain (3.20) since $\varepsilon > 0$ is arbitrary. $\square$

In the above proof, we have used the following result.

Lemma 3.4. *Let w be the Newtonian potential given by*

$$w(x) = \int_\Omega \Gamma(x-\xi) f(\xi) d\xi, \qquad x \in \Omega, \tag{3.38}$$

with Γ being given by (3.21) and $f \in L^p(\Omega)$. Then, for the case $p > n/2$, there exists a constant C depending only on the domain Ω, n and p, such that

$$\|w\|_{W^{2,p}(\Omega)} \le C\|f\|_{L^p(\Omega)}. \tag{3.39}$$

Furthermore, if $p > n$, then

$$\nabla w(x) = \int_\Omega \nabla_x \Gamma(x-\xi) f(\xi) d\xi, \qquad x \in \Omega. \tag{3.40}$$

Proof. First of all, by [11, p.230], for any $f \in L^p(\Omega)$, with $1 < p < \infty$, the Newtonian potential w defined by (3.38) is in $W^{2,p}(\Omega)$, and satisfies

$$\Delta w(x) = f(x), \qquad x \in \Omega, \tag{3.41}$$

and

$$\|D^2 w\|_{L^p(\Omega)} \le C\|f\|_{L^p(\Omega)}, \tag{3.42}$$

with C depends only on n and p. On the other hand, we know that there exists a constant $C_1 > 0$, such that

$$\|w\|_{W^{2,p}(\Omega)} \le C_1[\|w\|_{L^p(\Omega)} + \|D^2 w\|_{L^p(\Omega)}]. \tag{3.43}$$

Thus, by (3.42), for the case $p > n/2$, to get estimate (3.39), it suffices to estimate $\|w\|_{L^p(\Omega)}$. Since Ω is bounded, we may find $r > 0$ such that

$$\bar\Omega \subset B_r(x) \equiv \{\xi \in \mathbb{R}^n \mid |\xi - x| \le r\}, \qquad \forall x \in \bar\Omega. \tag{3.44}$$

Then, we have

$$
(3.45)\qquad
\begin{aligned}
\|w\|_{L^p(\Omega)} &= (\int_\Omega |\int_\Omega \Gamma(x-\xi)f(\xi)d\xi|^p dx)^{1/p} \\
&\le \{\int_\Omega (\int_{B_r(x)} |\Gamma(x-\xi)|^{p'} d\xi)^{p/p'} dx\}^{1/p} \|f\|_{L^p(\Omega)} \\
&= \|\Gamma\|_{L^{p'}(B_r(0))} |\Omega|^{1/p} \|f\|_{L^p(\Omega)} \le C_2 \|f\|_{L^p(\Omega)}.
\end{aligned}
$$

Here, we have used the fact that $p > n/2$ implies $\Gamma \in L^{p'}(B_r(0))$. Thus, (3.39) follows. Finally, for the case $p > n$, we let $\{f_k\}_{k=1}^\infty \subset \mathcal{D}(\Omega)$ be a sequence converging to f in $L^p(\Omega)$ and let

$$
(3.46)\qquad w_k(x) = \int_\Omega \Gamma(x-\xi) f_k(\xi) d\xi, \qquad x \in \Omega.
$$

¿From (3.39), we see that $w_k \to w$ strongly in $W^{2,p}(\Omega)$. On the other hand, from [11], we know that

$$
(3.47)\qquad \partial_{x_i} w_k(x) = \int_\Omega \partial_{x_i}\Gamma(x-\xi) f_k(\xi) d\xi, \qquad x \in \Omega.
$$

Since $p > n$, $\partial_{x_i}\Gamma \in L^{p'}(B_r(0))$ for all $r > 0$ (note $p' < \frac{n}{n-1}$). Therefore, we pass to the limits in (3.46) to get the desired result. $\square$

By using the result of [20], we can actually show that the results of Lemma 3.4 hold for any $1 < p < \infty$. Now, we are ready to prove our Theorem 3.1.

Proof of Theorem 3.1. It is enough to show the theorem for $p > n$. For any $\rho \in (0,1)$, by Lemma 3.3, we can find an $E_\rho \in \mathcal{E}_\rho$, such that

$$
(3.48)\qquad |\int_\Omega (1 - \frac{1}{\rho}\chi_{E_\rho}(x)) h^0(x) dx| + \|(1 - \frac{1}{\rho}\chi_{E_\rho}) h\|_{W^{-1,p}(\Omega)} \le \rho,
$$

where h^0 and h are given by (3.2). Let u_ρ be defined by (3.8) and let y_ρ be the corresponding state. Let us set

$$
(3.49)\qquad z_\rho(x) = \frac{y_\rho(x) - y(x)}{\rho}, \qquad x \in \Omega.
$$

Then, z_ρ satisfies the following

$$
(3.50)\qquad
\begin{cases}
-\sum_{i,j=1}^n \partial_{x_j}(a_{ij}^\rho(x)\partial_{x_i} z_\rho(x)) + a_0^\rho(x) z_\rho(x) = \frac{1}{\rho}\chi_{E_\rho}(x) h(x), \\
z_\rho \big|_{\partial\Omega} = 0,
\end{cases}
$$

where

$$
(3.51) \qquad \begin{cases} a_{ij}^{\rho}(x) = \displaystyle\int_0^1 a_{i,\zeta_j}(x, \nabla y(x) + \tau \nabla (y_\rho(x) - y(x)))d\tau, \qquad 1 \le i, j \le n, \\ a_0^{\rho}(x) = -\displaystyle\int_0^1 f_y(x, y(x) + \tau(y_\rho(x) - y(x)), u_\rho(x))d\tau. \end{cases}
$$

Clearly, by (2.6), Lemma 3.2 and (A1)–(A2), we see that

$$
(3.52) \qquad \begin{cases} a_{ij}^{\rho}(x) \to a_{ij}(x) \equiv a_{i,\zeta_j}(x, \nabla y(x)), \qquad \text{in } C(\bar{\Omega}), \\ a_0^{\rho}(x) \to a_0(x) \equiv -f_y(x, y(x), u(x)), \qquad \text{in } L^p(\Omega), \end{cases}
$$

By recalling z, the solution of (3.6), we have the following:

$$
(3.53) \qquad \begin{cases} -\displaystyle\sum_{i,j=1}^{n} \partial_{x_j}(a_{ij}^{\rho}(x)\partial_{x_i}(z_\rho(x) - z(x))) + a_0^{\rho}(x)(z_\rho(x) - z(x)) \\ \qquad = \displaystyle\sum_{i,j=1}^{n} \partial_{x_j}((a_{ij}^{\rho}(x) - a_{ij}(x))\partial_{x_i} z(x)) \\ \qquad\qquad - (a_0^{\rho}(x) - a_0(x))z(x) - (1 - \dfrac{1}{\rho})\chi_{E_\rho}(x))h(x), \\ (z_\rho - z)\big|_{\partial\Omega} = 0. \end{cases}
$$

By the result of [17] (see remark below), we have

$$
(3.54) \qquad \begin{aligned} \frac{\|r_\rho\|_{W^{1,p}(\Omega)}}{\rho} &= \|z_\rho - z\|_{W^{1,p}(\Omega)} \\ &\le C\{\sum_{i,j}^{n} \|(a_{ij}^{\rho} - a_{ij})\partial_{x_i} z\|_{L^p(\Omega)} + \|(a_0^{\rho} - a_0)z\|_{W^{-1,p}(\Omega)} \\ &\qquad + \|(1 - \frac{1}{\rho}\chi_{E_\rho})h\|_{W^{-1,p}(\Omega)}\} \\ &\le C\{\sum_{i,j}^{n} \|a_{ij}^{\rho} - a_{ij}\|_{L^\infty(\Omega)}\|z\|_{W^{1,p}(\Omega)} + \|(a_0^{\rho} - a_0)z\|_{L^p(\Omega)} + \rho\} = o(1). \end{aligned}
$$

This proves (3.9), for $p > n$. Finally, we define z^0 as in (3.11) and let

$$
r_\rho^0 = \frac{1}{\rho}(J(u_\rho) - J(u) - z^0).
$$

Then, using (3.48) and (3.52), we have

(3.55)
$$\begin{aligned}\frac{1}{\rho}|r_\rho^0| &= |\frac{J(u_\rho)-J(u)}{\rho} - z^0| \\ &\leq |\int_\Omega [\int_0^1 f_y^0(x,y(x)+\tau(y_\rho(x)-y(x)),u_\rho(x))d\tau z_\rho(x) - f_y^0(x,y(x),u(x))z(x)]dx| \\ &\qquad + |\int_\Omega (1-\frac{1}{\rho}\chi_{E_\rho}(x))h^0(x)dx| = o(1).\end{aligned}$$

This proves (3.10). □

Remark 3.5. In [17] the $W^{1,p}$–regularity used in (3.54) was proved for $p > n/(n-1)$, if $n > 2$; and for $p \geq 2$ if $n = 2$. In particular this result is true for all $p \geq 2$. Then by a duality argument we can conclude that the results remains for all $p \in (1,+\infty)$.

§4. Weak Pontryagin Maximum Principle.

In this section, we present a Pontryagin type maximum principle for optimal controls of our Problem (P_δ). We denote by $\mathcal{M}(\Omega)$ the space of all real Borel measures in Ω. Our main result of this section is the following:

Theorem 4.1. *Let (A1)–(A4) hold. Let $(\bar y,\bar u)$ be an optimal pair. Then, there exist a* $\psi^0 \leq 0$, a $\psi \in W_0^{1,p'}(\Omega)$ *with* $p' < n/(n-1)$ *and a* $\mu \in \mathcal{M}(\Omega)$, *such that*

$$|\psi^0| + \|\mu\|_{M(\Omega)} > 0 \tag{4.1}$$

$$\begin{cases} -\displaystyle\sum_{i,j=1}^n \partial_{x_i}(a_{i,\zeta_j}(x,\nabla\bar y(x))\partial_{x_j}\psi(x)) = f_y(x,\bar y(x),\bar u(x))\psi(x) \\ \qquad\qquad + \psi^0 f_y^0(x,\bar y(x),\bar u(x)) + g_y(x,\bar y(x))\mu, \qquad \text{in } \Omega, \\ \psi\mid_{\partial\Omega} = 0. \end{cases} \tag{4.2}$$

$$\int_\Omega (\eta(x) - g(x,\bar y(x))d\mu(x) \geq 0, \qquad \forall \eta \in C_0(\Omega),\ \textit{with } \eta(x) \leq \delta, \quad \forall x \in \bar\Omega. \tag{4.3}$$

$$H(x,\bar y(x),\bar u(x),\psi^0,\psi(x)) = \max_{v\in U} H(x,\bar y(x),v,\psi^0,\psi(x)), \qquad \text{a.e. } x \in \Omega, \tag{4.4}$$

where the Hamiltonian H is given by

$$H(x,y,u,\psi^0,\psi) = \psi^0 f^0(x,y,u) + \psi f(x,y,u), \qquad \forall (x,y,u,\psi^0,\psi) \in \Omega \times \mathbb{R} \times U \times \mathbb{R} \times \mathbb{R}. \tag{4.5}$$

Before proving the above theorem, let us give some preliminaries. Since $C_0(\Omega)$ is a separable Banach space, by [8, p.167], we know that there exists a norm, denoted by $|\cdot|_0$, which is equivalent to the norm $\|\cdot\|_{C_0(\Omega)}$, such that the dual of $(C_0(\Omega), |\cdot|_0)$ is strictly convex. It is clear that any element $\mu \in (C_0(\Omega), |\cdot|_0)^*$ can still be identified with an element of $\mathcal{M}(\Omega)$, such that

$$\langle \mu, \eta \rangle = \int_\Omega \eta(x) d\mu(x), \qquad \forall \eta \in C_0(\Omega). \tag{4.6}$$

In the rest of this section, whenever we write $C_0(\Omega)$, the norm of it is always taken to be the above $|\cdot|_0$, the dual space of it is still identified to be $\mathcal{M}(\Omega)$ and the corresponding dual norm is denoted by $|\cdot|_*$. Now, we define

$$Q = \{\eta \in C_0(\Omega) \bigm| \eta(x) \le \delta, \quad \forall x \in \bar{\Omega}\}. \tag{4.7}$$

Clearly, Q is a convex, closed and with a nonempty interior in $C_0(\Omega)$. Let

$$d_Q(\eta) = \inf_{\widetilde{\eta} \in Q} |\eta - \widetilde{\eta}|_0, \qquad \forall \eta \in C_0(\Omega). \tag{4.8}$$

Then, $d_Q : C_0(\Omega) \to \mathbb{R}$ is convex and Lipschitz continuous (with the Lipschitz constant being 1). From [7], we know that the Clarke's generalized gradient, denoted by ∂d_Q, which coincides with the subdifferential in the sense of the convex analysis in this case, is convex and weak*-compact. Therefore, given $\xi \in \partial d_Q(\eta)$, we have that

$$\langle \xi, \widetilde{\eta} - \eta \rangle + d_Q(\eta) \le d_Q(\widetilde{\eta}), \qquad \forall \widetilde{\eta} \in C_0(\Omega).$$

¿From this relation, it is easy to deduce that $|\xi|_* \le 1$, the identity $|\xi|_* = 1$ being true whenever $\eta \notin Q$; see [13]. Since $(\mathcal{M}(\Omega), |\cdot|_*)$ is strictly convex, then $\partial d_Q(\eta)$ is a singleton for every $\eta \notin Q$. Furthermore, $d_Q : C_0(\Omega) \to \mathbb{R}$ is Gâteaux differentiable at every point $\eta \notin Q$ and $\{\nabla d_Q(\eta)\} = \partial d_Q(\eta)$, hence

$$|\nabla d_Q(\eta)|_* = 1, \qquad \forall \eta \notin Q. \tag{4.9}$$

Now, we are ready to give a proof of our Theorem 4.1.

Proof of Theorem 4.1. Let $(\bar y, \bar u)$ be an optimal pair. For any $u \in \mathcal{U}$, let $y(\cdot\,; u)$ be the corresponding state, emphasizing the dependence of it on the control. For any $\varepsilon > 0$, we define

$$J_\varepsilon(u) = \{[(J(u) - J(\bar u) + \varepsilon)^+]^2 + d_Q(g(\cdot, y(\cdot\,; u)))^2\}^{1/2}. \tag{4.10}$$

Clearly, this functional is continuous on the (complete) metric space $(\mathcal{U}, d)$. Also, we have

$$J_\varepsilon(u) > 0, \qquad \forall u \in \mathcal{U}, \tag{4.11}$$

$$J_\varepsilon(\bar u) = \varepsilon \le \inf_{\mathcal{U}} J_\varepsilon(u) + \varepsilon. \tag{4.12}$$

Hence, by Ekeland's variational principle ([7,9]), we can find a $u^\varepsilon \in \mathcal{U}$, such that

$$d(\bar u, u^\varepsilon) \le \sqrt{\varepsilon}, \tag{4.13}$$

$$J_\varepsilon(u^\varepsilon) \le J_\varepsilon(\bar u), \tag{4.14}$$

$$J_\varepsilon(\widehat u) - J_\varepsilon(u^\varepsilon) \ge -\sqrt{\varepsilon}\, d(\widehat u, u^\varepsilon), \qquad \forall \widehat u \in \mathcal{U}. \tag{4.15}$$

We let $v \in \mathcal{U}$ and $\varepsilon > 0$ be fixed and let

$$y^\varepsilon = y(\cdot\,; u^\varepsilon).$$

Set

$$\begin{cases} a_{ij}^\varepsilon(x) = a_{i,\zeta_j}(x, \nabla y^\varepsilon(x)), \qquad 1 \le i, j \le n \\ a_0^\varepsilon(x) = -f_y(x, y^\varepsilon(x), u^\varepsilon(x)), \end{cases} \tag{4.16}$$

and

$$\begin{cases} h^\varepsilon(x) = f(x, y^\varepsilon(x), v(x)) - f(x, y^\varepsilon(x), u^\varepsilon(x)), \\ h^{0,\varepsilon}(x) = f^0(x, y^\varepsilon(x), v(x)) - f^0(x, y^\varepsilon(x), u^\varepsilon(x)). \end{cases} \tag{4.17}$$

Let A^ε be the elliptic differential operator with the coefficients given by (4.16):

$$A^\varepsilon z(x) \equiv -\sum_{i,j=1}^{n} \partial_{x_j}\left(a_{ij}^\varepsilon(x)\partial_{x_i} z(x)\right) + a_0^\varepsilon(x) z(x). \tag{4.18}$$

Then, (3.4)–(3.5) hold for this A^ε and the leading coefficients $a_{ij}^\varepsilon(x)$ are uniformly continuous in $\bar{\Omega}$ independent of ε. We consider the following problem

$$\begin{cases} A^\varepsilon z^\varepsilon(x) = h^\varepsilon(x), & \text{in } \Omega, \\ z^\varepsilon \big|_{\partial\Omega} = 0. \end{cases} \tag{4.19}$$

We know that the above problem admits a unique solution $z^\varepsilon \in W_0^{1,p}(\Omega)$. for all $p > 1$. By Theorem 3.1, we know that for any $\rho \in (0,1)$, there exists an $E_\rho^\varepsilon \subset \Omega$, with the property $|E_\rho^\varepsilon| = \rho|\Omega|$, such that if we define

$$u_\rho^\varepsilon(x) = \begin{cases} u^\varepsilon(x), & \text{if } x \in \Omega \setminus E_\rho^\varepsilon, \\ v(x), & \text{if } x \in E_\rho^\varepsilon, \end{cases} \tag{4.20}$$

and let $y_\rho^\varepsilon = y(\cdot\,; u_\rho^\varepsilon)$ be the corresponding state, then

$$\begin{cases} y_\rho^\varepsilon = y^\varepsilon + \rho z^\varepsilon + r_\rho^\varepsilon, \\ J(u_\rho^\varepsilon) = J(u^\varepsilon) + \rho z^{0,\varepsilon} + r_\rho^{0,\varepsilon}, \end{cases} \tag{4.21}$$

where z^ε is the solution of (4.19),

$$z^{0,\varepsilon} = \int_\Omega [f_y^0(x, y^\varepsilon(x), u^\varepsilon(x)) z^\varepsilon(x) + h^{0,\varepsilon}(x)]dx, \tag{4.22}$$

and for any $p \in [1,\infty)$,

$$\lim_{\rho\to 0} \frac{1}{\rho}\|r_\rho^\varepsilon\|_{W^{1,p}(\Omega)} = \lim_{\rho\to 0}\frac{1}{\rho}|r_\rho^{0,\varepsilon}| = 0. \tag{4.23}$$

Now, we take $\widehat{u} = u_\rho^\varepsilon$ in (4.15). Then, it follows that

$$\begin{aligned} -\sqrt{\varepsilon}|\Omega| &\le \frac{J_\varepsilon(u_\rho^\varepsilon) - J_\varepsilon(u^\varepsilon)}{\rho} \\ &= \frac{1}{J_\varepsilon(u_\rho^\varepsilon) + J_\varepsilon(u^\varepsilon)}\Big\{\frac{[(J(u_\rho^\varepsilon) - J(\bar{u}) + \varepsilon)^+]^2 - [(J(u^\varepsilon) - J(\bar{u}) + \varepsilon)^+]^2}{\rho} \\ &\qquad + \frac{d_Q(g(\cdot, y_\rho^\varepsilon))^2 - d_Q(g(\cdot, y^\varepsilon))^2}{\rho}\Big\} \\ &\to \frac{(J(u^\varepsilon) - J(\bar{u}) + \varepsilon)^+}{J_\varepsilon(u^\varepsilon)} z^{0,\varepsilon} \\ &\qquad + \langle\, \frac{d_Q(g(\cdot, y^\varepsilon))\xi_\varepsilon}{J_\varepsilon(u^\varepsilon)}, g_y(\cdot, y^\varepsilon) z^\varepsilon\,\rangle, \qquad (\rho \to 0), \end{aligned} \tag{4.24}$$

where

$$
\xi_\varepsilon = \begin{cases} \nabla d_Q(g(\cdot,y^\varepsilon)), & \text{if } g(\cdot,y^\varepsilon)\notin Q, \\ 0, & \text{if } g(\cdot,y^\varepsilon)\in Q. \end{cases} \tag{4.25}
$$

Next, we define $(\varphi^{0,\varepsilon},\varphi^\varepsilon)\in[0,1]\times\mathcal{M}(\Omega)$ as follows:

$$
\begin{cases} \varphi^{0,\varepsilon} = \dfrac{(J(u^\varepsilon)-J(\bar u)+\varepsilon)^+}{J_\varepsilon(u^\varepsilon)}, \\ \varphi^\varepsilon = \dfrac{d_Q(g(\cdot,y^\varepsilon))\xi_\varepsilon}{J_\varepsilon(u^\varepsilon)}. \end{cases} \tag{4.26}
$$

Then we see that (4.24) can be written as

$$
-\sqrt{\varepsilon}|\Omega| \le \varphi^{0,\varepsilon}z^{0,\varepsilon} + \int_\Omega g_y(x,y^\varepsilon(x))z^\varepsilon(x)d\varphi^\varepsilon(x), \tag{4.27}
$$

and from (4.9) and (4.10), we have

$$
|\varphi^{0,\varepsilon}|^2 + |\varphi^\varepsilon|_*^2 = 1. \tag{4.28}
$$

On the other hand, by the definition of subdifferential, we have

$$
\begin{aligned} \langle\, \varphi^\varepsilon, \eta - g(\cdot,y^\varepsilon)\,\rangle = \int_\Omega (\eta(x) - g(x,y^\varepsilon(x))d\varphi^\varepsilon(x) \le 0, \\ \forall \eta\in C_0(\Omega), \quad \eta(x)\le\delta, \quad \forall x\in\Omega. \end{aligned} \tag{4.29}
$$

Next, by (4.13) and Lemma 3.2, we have

$$
\|y^\varepsilon - \bar y\|_{W^{1,p}(\Omega)} \to 0, \qquad (\varepsilon\to 0). \tag{4.30}
$$

Thus, (4.29) implies

$$
\begin{aligned} \int_\Omega (\eta - g(\cdot,\bar y))d\varphi^\varepsilon(x) \le |g(\cdot,y^\varepsilon) - g(\cdot,\bar y)|_0 \equiv \sigma_\varepsilon \to 0, \quad (\varepsilon\to 0) \\ \forall\eta\in C_0(\Omega), \text{with } \eta(x)\le\delta, \quad \forall x\in\Omega. \end{aligned} \tag{4.31}
$$

By extracting some subsequence, still denoted by itself, one has

$$
\varphi^{0,\varepsilon}\to\varphi^0, \qquad \varphi^\varepsilon \overset{*}{\rightharpoonup} \varphi. \tag{4.32}
$$

Let $C > 0$ satisfy that $\| \cdot \|_{C_0(\Omega)} \leq C | \cdot |_0$. Let us fix an element $\eta_0 \in C_0(\Omega)$ and a real positive number r verifying that $\eta_0(x) < \delta - r$, $\forall x \in \bar{\Omega}$. Taking $\eta(x) = \eta_0(x) + \widehat{\eta}(x)$, with $|\widehat{\eta}|_0 \leq r/C$, in (4.31), we obtain

$$
\int_\Omega \widehat{\eta}(x) d\varphi^\varepsilon(x) \leq \int_\Omega (g(x, \bar{y}(x)) - \eta_0(x)) d\varphi^\varepsilon(x) + \sigma_\varepsilon, \qquad \forall |\widehat{\eta}|_0 \leq r/C. \tag{4.33}
$$

Taking the supremum in the left hand term, it follows that

$$
\frac{r}{C} |\varphi^\varepsilon|_* \leq \int_\Omega (g(x, \bar{y}(x)) - \eta_0(x)) d\varphi^\varepsilon(x) + \sigma_\varepsilon.
$$

Then, by (4.28), (4.31) and (4.32), we obtain

$$
(\frac{C}{r} \int_\Omega (g(x, \bar{y}(x)) - \eta_0(x)) d\varphi(x))^2 + |\varphi^0|^2 \geq \lim_{\varepsilon \to 0} [|\varphi^\varepsilon|_*^2 + |\varphi^{0,\varepsilon}|^2] = 1. \tag{4.34}
$$

On the other hand, from (4.30) we have

$$
\begin{cases} z^\varepsilon \to z, & \text{in } W^{1,p}(\Omega), \\ z^{0,\varepsilon} \to z^0, & \end{cases} \qquad (\varepsilon \to 0), \tag{4.35}
$$

where z is the solution of the following variational system:

$$
\begin{cases} - \sum_{i,j=1}^n \partial_{x_j} (a_{i,\zeta_j}(x, \nabla \bar{y}(x)) \partial_{x_i} z(x)) = f_y(x, \bar{y}(x), \bar{u}(x)) z(x) \\ \qquad\qquad + f(x, \bar{y}(x), v(x)) - f(x, \bar{y}(x), \bar{u}(x)), \qquad \text{in } \Omega, \\ z \big|_{\partial\Omega} = 0. \end{cases} \tag{4.36}
$$

and

$$
\begin{aligned} z^0 &= \int_\Omega \{ f_y^0(x, \bar{y}(x), \bar{u}(x)) z(x) dx \\ &\quad + \int_\Omega [f^0(x, \bar{y}(x), v(x)) - f^0(x, \bar{y}(x), \bar{u}(x))] dx. \end{aligned} \tag{4.37}
$$

We note that the solution z of (4.36) and the quantity z^0 defined by (4.37) depend on the choice of $v \in \mathcal{U}$. Thus, we denote them by $z(\cdot, v)$ and $z^0(v)$. respectively. Then, taking limits in (4.27), we obtain

$$
\varphi^0 z^0(v) + \langle \varphi, g_y(\cdot, \bar{y}) z(\cdot; v) \rangle \geq 0, \qquad \forall v \in \mathcal{U}. \tag{4.38}
$$

Now, we let

$$\psi^0 = -\varphi^0 \in [-1,0], \qquad \mu = -\varphi. \tag{4.39}$$

Then, (4.1) follows from (4.34). Also, we obtain (4.3) by taking limits in (4.31) (along the above-mentioned subsequence). Furthermore,

$$\psi^0 z^0(v) + \langle\, \mu, g_y(\cdot,\bar y) z(\cdot;v)\,\rangle \le 0, \qquad \forall v \in \mathcal{U}. \tag{4.40}$$

Since $\mu \in \mathcal{M}(\Omega) \subset W^{-1,p'}(\Omega)$, we know ([17]) that (4.2) admits a unique solution $\psi \in W_0^{1,p'}(\Omega)$. By some direct computation, we can reduce (4.40) to the following:

$$\begin{aligned}
&\int_\Omega \{\psi^0[f^0(x,\bar y(x),\bar u(x)) - f^0(x,\bar y(x),v(x))] \\
&\qquad + \psi(x)[f(x,\bar y(x),\bar u(x)) - f(x,\bar y(x),v(x))]\}dx \\
&\equiv \int_\Omega [H(x,\bar y(x),\bar u(x),\psi^0,\psi(x)) - H(x,\bar y(x),v(x),\psi^0,\psi(x))]dx \ge 0, \\
&\qquad\qquad\qquad \forall v \in \mathcal{U}.
\end{aligned} \tag{4.41}$$

We take a countable dense subset $\{u_k\}_{k\ge1} \subset U$. For each u_k, there exists a measurable set $\Omega_k \in \Omega$ with $|\Omega_k| = |\Omega|$, such that Ω_k consists of all Lebesgue points of the function $H(x,\bar y(x),\bar u(x),\psi^0,\psi(x)) - H(x,\bar y(x),u_k,\psi^0,\psi(x))$. Then, for any $x_0 \in \Omega_k$ and any small enough $r > 0$ (with $B_r(x_0) \equiv \{x \in \mathbb{R}^n \mid |x - x_0| \le r\} \subset \Omega$), we take v in (4.41) by

$$v(x) = \begin{cases} \bar u(x), & \text{if } x \in \Omega \setminus B_r(x_0), \\ u_k, & \text{if } x \in B_r(x_0). \end{cases} \tag{4.42}$$

Then, (4.41) reads

$$\int_{B_r(x_0)} [H(x,\bar y(x),\bar u(x),\psi^0,\psi(x)) - H(x,\bar y(x),u_k,\psi^0,\psi(x))]dx \ge 0, \qquad \forall r > 0. \tag{4.43}$$

Dividing by $|B_r(x_0)|$ and sending $r \to 0$, we obtain

$$\begin{aligned}
&H(x_0,\bar y(x_0),\bar u(x_0),\psi^0,\psi(x_0)) \ge H(x_0,\bar y(x_0),u_k,\psi^0,\psi(x_0)), \\
&\qquad\qquad\qquad \forall x_0 \in \Omega_k, k \ge 1.
\end{aligned} \tag{4.44}$$

Then, by the continuity of $f^0(x,y,u)$ and $f(x,y,u)$ in u and the density of the countable set $\{u_k\}_{k\ge1}$, we obtain (4.4). □

Remark 4.2. We note that from (4.3), it follows that μ is a nonpositive measure and

$$\operatorname{supp}\mu \subset \{x \in \Omega \mid g(x, \bar{y}(x)) = \delta\}. \tag{4.45}$$

Also let us mention that assumption (2.12) can be removed. If we assume that

$$g(x, 0) < \delta, \qquad \forall x \in \partial\Omega, \tag{4.46}$$

the state constraint is inactive on the boundary of Ω. So (4.45) is still true and Theorem 4.1 remains also true. Finally in the case where

$$\sup_{x \in \partial\Omega} g(x, 0) = \delta, \tag{4.47}$$

then the state constraint may be active on the boundary and consequently the support of the Lagrange multiplier μ may intersect the boundary of Ω. In this case the adjoint state equation should be modified.

Remark 4.3. Similar to [4], we may relax the continuity of the functions f and f^0 in the control variable u. One of such interesting cases is that the functions f and f^0 are given by

$$\begin{aligned} f(x, y, u) &= f_1(x, y, u) + f_2(x, u), \\ f^0(x, y, u) &= f_1^0(x, y, u) + f_2^0(x, u), \end{aligned} \tag{4.48}$$

with f_1 and f_1^0 satisfy (A2)–(A3), and f_2 and f_2^0 are merely measurable and bounded. Our result remains true for such a case. Some other cases are also possible. We omit the details here (see [4]).

§5. Strong Pontryagin Maximum Principle.

In this section we are going to prove that Theorem 4.1 holds with $\psi^0 = -1$ if Problem (P_δ) is stable in a certain sense that we make it precise in the following definition.

Definition 5.1. We say that problem (P_δ) is strongly stable on the right if there exist $\varepsilon > 0$ and $C > 0$ such that

$$\inf(P_\delta) - \inf(P_{\delta'}) \leq C(\delta' - \delta), \qquad \forall \delta' \in [\delta, \delta + \varepsilon]. \tag{5.1}$$

This concept was used by Bonnans and Casas [4] to derive the Pontryagin Principle in a qualified form, that is to say the same conditions (4.2)–(4.4) but with the parameter $\psi^0 = -1$. Here we will prove that the result stated in [4] for semilinear elliptic equations still holds for quasilinear elliptic equations. The approach we used here is simpler than that given in [4]. The key in the proof of this principle is that it is possible to make an exact penalization of the state constraint provided the control problem is strongly stable on the right. Therefore the first question to consider is whether this assumption is satisfied frequently or not. Fortunately most of problems are stable, more precisely

Proposition 5.2. *Let us denote by δ_0 a real number such that (P_{δ_0}) has at least one admissible control. Then for every $\delta \geq \delta_0$, except at most a set of zero Lebesgue measure, the problem (P_δ) is strongly stable on the right.*

See [4] for the proof of this proposition. Now we can state the main result of this section.

Theorem 5.3. *Under assumptions (A1)–(A4) and provided that (P_δ) is strongly stable on the right, then there exist a $\psi \in W_0^{1,p'}(\Omega)$, with $p' < n/(n-1)$, and $\mu \in \mathcal{M}(\Omega)$ such that (4.2)–(4.4) hold with $\psi^0 = -1$.*

Before proving this theorem we need some lemmas. Let Q and d_Q be defined as in the previous section.

Lemma 5.4. *There exists a $q > 0$ such that $\bar{u}$ is a solution in $(\mathcal{U}, d)$ of the penalized problem*

$$\min_{u \in \mathcal{U}} J_q(u) = J(u) + q d_Q(g(\cdot, y(\cdot; u))). \tag{5.2}$$

Proof. Suppose the contrary. Then for each $k > 0$, there exists a $u^k \in \mathcal{U}$, such that

$$J(u^k) + k d_Q(g(\cdot, y^k)) < J(\bar{u}), \qquad \forall k > 0, \tag{5.3}$$

where y^k is the feasible state corresponding to u^k. Then, we see that

$$0 < d_Q(g(\cdot, y^k)) \to 0, \qquad k \to \infty. \tag{5.4}$$

Since each $g(\cdot;y^k) \notin Q$, we have

$$\delta_k = \max_{x\in\Omega} g(x, y^k(x)) > \delta, \qquad \forall k > 0, \tag{5.5}$$

and by (5.4),

$$\lim_{k\to\infty} \delta_k = \delta. \tag{5.6}$$

Then, by the strong stability, we have some constant $C > 0$, such that

$$\inf(P_\delta) - \inf(P_{\delta_k}) \leq C(\delta_k - \delta), \qquad \forall k > 0. \tag{5.7}$$

However, this together (5.3) yields the following:

$$\begin{aligned} C(\delta_k - \delta) &\geq J(\bar{u}) - J(u^k) > k d_Q(g(\cdot;y^k)) \\ &\geq kC'\|(g(\cdot;y^k) - \delta)^+\|_{C_0(\Omega)} = kC'(\delta_k - \delta), \qquad \forall k > 0. \end{aligned} \tag{5.8}$$

This is a contradiction. $\square$

Since J_q is not Gâteaux differentiable at $\bar{u}$ we need to modify slightly this functional.

Lemma 5.5. *Let $\varepsilon > 0$ and consider the problem*

$$(P_{\delta,\varepsilon}) \qquad \min_{u\in\mathcal{U}} J_{q,\varepsilon}(u) = J(u) + q\{d_Q(g(\cdot;y(\cdot;u)))^2 + \varepsilon^2\}^{1/2}. \tag{5.9}$$

Then the following identity holds

$$\lim_{\varepsilon\to 0} \inf(P_{\delta,\varepsilon}) = \inf_{u\in\mathcal{U}} J_q(u). \tag{5.10}$$

Proof. It is an immediate consequence of the inequality

$$J_q(u) \leq J_{q,\varepsilon}(u) \leq J_q(u) + q\epsilon. \tag{5.11}$$

$\square$

Now, we present a proof of our main result of this section.

Proof of Theorem 5.1. Lemmas 5.4 and 5.5 imply that $\bar{u}$ is a σ_ε^2-solution of $(P_{\delta,\varepsilon})$, with $\sigma_\varepsilon \searrow 0$ as $\varepsilon \searrow 0$, that is to say:

$$J_{q,\varepsilon}(\bar{u}) \leq \inf(P_{\delta,\varepsilon}) + \sigma_\varepsilon^2. \tag{5.12}$$

Then we can apply again the Ekeland's variational principle and deduce the existence of an element $u^\varepsilon \in \mathcal{U}$, such that

$$d(\bar{u}, u^\varepsilon) \leq \sigma_\varepsilon, \qquad J_{q,\varepsilon}(u^\varepsilon) \leq J_{q,\varepsilon}(\bar{u}), \tag{5.13}$$

and

$$J_{q,\varepsilon}(u) - J_{q,\varepsilon}(u^\varepsilon) \geq -\sigma_\varepsilon d(u, u^\varepsilon), \qquad \forall u \in \mathcal{U}. \tag{5.14}$$

Now we can argue as in the proof of Theorem 4.1 and replace (4.27) by

$$\begin{aligned} -\sigma_\varepsilon|\Omega| &\leq \lim_{\rho\to 0} \frac{J_{q,\varepsilon}(u^\varepsilon_\rho) - J_{q,\varepsilon}(u^\varepsilon)}{\rho} \\ &= z^{0,\varepsilon} + \langle\, \varphi^\varepsilon, g_y(\cdot, y^\varepsilon) z_\varepsilon \,\rangle, \end{aligned} \tag{5.15}$$

where the element $\varphi^\varepsilon \in \mathcal{M}(\Omega)$ is given by

$$\varphi^\varepsilon = \begin{cases} q\dfrac{d_Q(g(\cdot, y^\varepsilon))}{\{d_Q(g(\cdot, y^\varepsilon))^2 + \varepsilon^2\}^{1/2}} \nabla d_Q(g(\cdot, y^\varepsilon)), & \text{if } g(\cdot, y^\varepsilon) \notin Q, \\ 0, \quad \text{if } g(\cdot, y^\varepsilon) \in Q. \end{cases} \tag{5.16}$$

Therefore, we have $|\varphi^\varepsilon|_* \leq q$. Now we can take a subsequence that converges weakly* to an element $\varphi \in \mathcal{M}(\Omega)$. The rest is as in the proof of Theorem 4.1, taking $\varphi^{0,\varepsilon} = 1$. □

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