Rigidity of symplectic fillings, symplectic divisors and Dehn twist exact sequences

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Abstract

We present three different aspects of symplectic geometry in connection to complex geometry. Convex symplectic manifolds, symplectic divisors and Lagrangians are central objects to study on the symplectic side. The focus of the thesis is to establish relations of these symplectic objects to the corresponding complex analytic objects, namely *Stein fillings*, *divisors* and *coherent sheaves*, respectively.

Using pseudoholomorphic curve techniques and Gauge theoretic results, we systematically study obstructions to symplectic/Stein fillings of contact 3-manifolds arising from the rigidity of closed symplectic four-manifolds with non-positive Kodaira dimension. This perspective provides surprising consequences which, in particular, captures a new rigidity phenomenon for exact fillings of unit cotangent bundle of orientable surfaces and recovers many known results in a uniform way.

The most important source of Stein fillings comes from smoothing of a complex isolated singularities. This motivates us to study when a symplectic divisor admits a convex/concave neighborhood and we obtain a complete and very computable answer to this local behaviour of symplectic divisors. Globally speaking, symplectic divisors in a closed symplectic manifold that represent its first Chern class are of particular importance in mirror symmetry. Such a symplectic divisor, together with the closed symplectic manifold together is called a symplectic log Calabi-Yau surface. We obtain a complete classification of symplectic log Calabi-Yau surface up to isotopy of symplectic divisors.

Finally, we study algebraic properties of Fukaya category on the functor level and utilize Biran-Cornea's Lagrangian cobordism theory and Mau-Wehrheim-Woodward functor to provide a partial proof of Huybrechts-Thomas's conjecture.

Contents

Chapter 1

Introduction

A symplectic manifold (X^{2n}, ω) is a $2n$-dimensional smooth manifold X together with a closed, non-degenerate two form $\omega \in \Omega^2(X)$. A symplectic submanifold $C \subset (X, \omega)$ is a submanifold such that $\omega|_C$ is a symplectic. A Lagrangian submanifold L of a symplectic manifold (X, ω) is a submanifold such that $\omega|_L = 0$.

A Hamiltonian H of a symplectic manifold (X, ω) is a real-valued function. The associated Hamiltonian vector field V is the ω-dual of dH. A time-dependent Hamiltonian $H = (H_t)_{t\in[0,1]} : X \times [0,1] \to \mathbb{R}$ is a one parameter family of real-valued function. The associated time-dependent Hamiltonian vector field $V = (V_t)_{t\in[0,1]}$ is also called a Hamiltonian vector field. Hamiltonian flow ϕ_H^t is a flow generated by a Hamiltonian vector field of a Hamiltonian $H = (H_t)$. Since $\mathcal{L}_V \omega = 0$, Hamiltonian flow is a symplectomorphism (ie. diffeomorphism preserving ω), where $\mathcal{L}$ is the Lie derivative. An isotopy of Lagrangian that is generated by a Hamiltonian flow is called a Hamiltonian isotopy.

A contact manifold (Y^{2n-1}, ξ) is a $2n-1$-dimensional smooth manifold Y equipped with with a hyperplane distribution ξ (called the contact structure) such that for any $y \in Y$, there is a neighborhood $y \in U$ and a one-form $\alpha \in \Omega^1(U)$ satisfying $\ker(\alpha) = \xi$ and $\alpha \wedge (d\alpha)^{n-1}$ is non-vanishing. When ξ is co-orientable, there exists a globally well-defined $\alpha \in \Omega^1(Y)$ such that $\ker(\alpha) = \xi$ and $\alpha \wedge (d\alpha)^{n-1}$ is non-vanishing.

Symplectic manifolds with contact boundary, *union of codimension two symplectic submanifolds* and *Lagrangians up to Hamiltonian isotopy* constitute the three main aspects of the thesis. The focus of the thesis is to establish relations of these symplectic objects to the corresponding complex analytic objects, namely *Stein fillings*, *divisors* and *coherent sheaves*, respectively.

1.1 Symplectic manifolds with contact boundary - fillings and caps

Let (Y, ξ) be a closed co-oriented contact 3-manifold. A strong symplectic filling (N, ω_N) of (Y, ξ) is a symplectic manifold with boundary ∂N such that near ∂N, there is a locally defined Liouville vector field V (i.e. $d\iota_V \omega_N = \omega_N$) pointing **outward** along ∂N so that the induced contact structure $\xi_N = \ker(\iota_V \omega_N)|_N$ on ∂N makes $(\partial N, \xi_N)$ contactomorphic to (Y, ξ). When V is chosen, we denote the induced contact 1-form on $\partial N = Y$ as $\alpha_N = \iota_V \omega_N$. We also call a strong symplectic filling a symplectic filling, a filling or a convex symplectic manifold.

An exact symplectic filling (N, ω_N) of (Y, ξ) is a strong symplectic filling such that the Liouville vector field V can be extended to a globally defined vector field. A Stein filling is a special kind of exact symplectic filling and we refer readers to [81] for more details.

A strong symplectic capping (P, ω_P) of (Y, ξ) is defined similarly to strong symplectic filling. The only modification is that the locally defined Liouville vector field V is required to point **inward** instead of outward. The induced contact 1-form, which we call the **Liouville 1-form**, is denoted as α_P and (P, ω_P, α_P) forms a concave symplectic pair. A strong symplectic capping (P, ω_P) is also called as a symplectic cap, a cap or a concave symplectic manifold.

For a choice of a filling (N, ω_N) and a cap (P, ω_P) of (Y, ξ), we can glue them together to get a closed symplectic manifold $X = N \cup_Y P$ by inserting part of the symplectization of (Y, ξ) (see [28]). The symplectic form ω on X is not canonical and our convention is that $\omega|_P = \omega_P$ so $(N, \omega|_N)$ is obtained by attaching part of the symplectization of (Y, ξ) to $(N, \lambda_P \omega_N)$, for some $\lambda_P > 0$ small. The actual rescaling factor λ_P is not very important to our discussion so whenever a filling is glued with a cap, λ_P is chosen implicitly. These notations and conventions are used throughout.

Understanding symplectic fillings of a given contact 3-manifold (Y, ξ) is a very active research area. An ultimate goal is to classify all the Stein, exact or minimal strong symplectic fillings of a given contact manifold (Y, ξ). The first step towards this goal is to understand whether the given (Y, ξ) has finitely many or infinitely many fillings. Some families of contact 3-manifolds that admit finitely many Stein fillings are found ([25], [35], [64], [60], [85], [100], etc). For minimal strong fillings, Ohta, Ono and others have systematically investigated the links of isolated singularities ([77], [78], [79], [80], [11] etc), and established uniqueness/finiteness/infiniteness for different classes of singularities.

Instead of classifying completely all the fillings, one can ask whether topological quantities for fillings are bounded. It was conjectured by Stipsicz in [102] that all possible Euler characteristics and signatures of Stein fillings of a fixed (Y, ξ) are bounded. However, it was disproved by Baykur and Van Horn-Morris in [10]. Based on the work of many people ([5], [4], [10], [9], [21], [82] etc), we now know that many contact 3-manifolds have infinitely many Stein fillings up to diffeomorphism.

Even though Stipsicz's conjecture is not true in general, it is important to know for what contact 3-manifolds the boundedness does hold. Our focus is to address when the Betti numbers are bounded.

Definition 1.1.1. A contact 3-manifold (Y, ξ) is of **Stein** (resp. **exact**, **strong**) **Betti finite type** if there are only finitely many possible values of the tuple (b_1, b_2, b_3) of Betti numbers for all of its Stein (resp. exact, minimal strong) fillings.

Note also that this finiteness of Betti numbers guarantees the finiteness of e and σ. Planar contact 3-manifolds (i.e. contact 3-manifolds supported by open books of page genus zero) have Stein Betti finite type [45], but there are many other contact manifolds having this property too.

To study this question, we introduce three kinds of caps, namely Calabi-Yau caps, uniruled caps and adjunction caps. Calabi-Yau caps give surprising new restrictions to exact fillings and in particular, we apply it to study exact fillings of unit cotangent bundles. Uniruled and adjunction caps unify several known finiteness results into a single picture.

1.1.1 Calabi-Yau caps

Definition 1.1.2. A **Calabi-Yau cap** of a contact 3-manifold (Y, ξ) is a compact symplectic manifold (P, ω) which is a strong concave filling of (Y, ξ) such that $c_1(P)$ is torsion.

Theorem 1.1.3. *Suppose (Y, ξ) admits a Calabi-Yau cap (P, ω_P). Then (Y, ξ) is of exact Betti finite type.*

If, moreover, (P, ω_P) cannot be embedded in a uniruled manifold, then all exact fillings of (Y, ξ) have torsion first Chern class.

It is worthy to point out that in many situations, there are simple (topological) obstructions for a Calabi-Yau cap (P, ω_P) to be embedded in a uniruled manifold. On the other hand, the ingredients in the proof of Theorem 1.1.3 can be used to obtain the following surprising consequence.

Theorem 1.1.4. *Let Y be the unit cotangent bundle of a closed orientable surface Σ_g of genus g, equipped with the standard contact structure ξ_{std}. Then any exact filling of (Y, ξ_{std}) has the same integral homology and intersection form as $T^*\Sigma_g$ and has vanishing first Chern class.*

We remark that when $g = 0, 1$, (Y, ξ_{std}) has a unique exact filling up to symplectic deformation equivalence given by the disk cotangent bundle (See [41], [110] and [101]). However, there has been no good understandings of exact fillings for $g > 1$. In contrast, no exact/Stein filling of (Y, ξ_{std}) that is not diffeomorphic to $T^*\Sigma_g$ has been found. We heard from Chris Wendl the following conjecture.

Conjecture 1.1.5. *The diffeomorphism types of Stein/exact fillings of (Y, ξ_{std}) is unique and given by the disk cotangent bundle for any g.*

Therefore, Theorem 1.1.4 gives strong evidence to Conjecture 1.1.5. After we introduced Calabi-Yau caps, Sivek and Van Horn Morris in [99] were able to use it to obtain parts of Theorem 1.1.4 independently and derive the beautiful result that the Stein version of Conjecture 1.1.5 is true up to s-cobordism relative to boundary.

As an immediate application of Theorem 1.1.4, we obtain Corollary 1.1.6. This subtle difference between exact Betti finite type and strong Betti finite type is an interesting phenomenon that we do not find in the literature. It also implies that Theorem 1.1.3 cannot be strengthened to strong Betti finite type.

Corollary 1.1.6. *There exist infinitely many Stein fillable contact 3-manifolds such that each of them is of exact Betti finite type but not of strong Betti finite type.*

1.1.2 Uniruled/Adjunction caps

For any symplectic cap (symplectic concave filling) (P,ω) of (Y,ξ), there is Liouville vector field V defined near Y pointing inward along Y. The induced one form $\alpha = \iota_V\omega$ is a contact one form on ∂P such that $(\partial P, \ker(\alpha))$ is contactomorphic to (Y,ξ). For any choice of V, we call the induced one form α a **Liouville one form**. Given a Liouville one form α, $[(\omega,\alpha)]$ is a relative cohomology class in $H^2(P,\partial P,\mathbb{R})$.

Definition 1.1.7. A **uniruled cap** of a contact 3-manifold (Y,ξ) is a symplectic concave filling (P,ω) of (Y,ξ) such that $c_1(P)\cdot[(\omega,\alpha)] > 0$ for some Liouville one form α.

Since $[(\omega,\alpha)]$ is a relative class, $c_1(P)\cdot[(\omega,\alpha)]$ is well-defined and it is further explained in Subsection 2.1.2. We call a contact 3-manifold admitting a uniruled cap a **uniruled contact manifold**. A contact 3-manifold that is strong symplectic cobordant to a uniruled contact manifold is also uniruled (see Lemma 2.2.6). The class of uniruled contact manifolds is strictly larger than the planar class (note that (T^3,ξ_{std}) is non-planar but admits a uniruled cap).

If (Y,ξ) admits a uniruled cap, then we can derive restrictions on fillings that are strictly stronger than when (Y,ξ) admits a Calabi-Yau cap.

Theorem 1.1.8. *Suppose a contact 3-manifold (Y,ξ) admits a uniruled cap (P,ω_P). Then (Y,ξ) is of strong Betti finite type.*

Remark 1.1.9. We would like to point out that to the best of our knowledge, most previously known exact/strong Betti finite type contact 3-manifolds have a uniruled cap.

We also introduce another type of caps which we call **adjunction caps**. It is based on an observation that existence of a **smoothly** embedded surface in a closed symplectic manifold with sufficiently large self-intersection number relative to the genus implies that the symplectic manifold is uniruled (Proposition 2.2.3). Adjunction caps have similar properties as uniruled caps, including Theorem 1.1.8. Similarly, every planar contact 3-manifold also has an adjunction cap (see [29]) and we call a contact manifold admitting an adjunction cap an **adjunction contact manifold**.

1.2 Codimension two symplectic submanifolds - divisors

A *symplectic divisor* refers to a connected configuration of finitely many closed embedded symplectic surfaces $D = C_1 \cup \cdots \cup C_k$ in a symplectic 4 dimensional manifold (possibly with boundary or non-compact) (W, ω). D is further required to have the following properties: D has empty intersection with ∂W, no three C_i intersect at a point, and any intersection between two surfaces is transversal and positive. The orientation of each C_i is chosen to be positive with respect to ω. Since we are interested in the germ of a symplectic divisor, W is sometimes omitted in the writing and (D, ω), or simply D, is used to denote a symplectic divisor.

A closed regular neighborhood $P(D)$ of D is called a plumbing of D. The plumbings are well defined up to orientation preserving diffeomorphism, so we can introduce topological invariants of D using any of its plumbings. In particular, $b_2^{\pm}(D)$ is defined as $b_2^{\pm}$ of a plumbing. Similarly, we define the *boundary* of the divisor D, to be $\partial P(D)$.

A plumbing $P(D)$ of D is called a *concave (resp. convex) neigborhood* if $P(D)$ is a strong concave (resp. convex) filling of its boundary. A symplectic divisor D is called *concave* (resp. *convex*) if for any neighborhood N of D, there is a concave (resp. convex) neighborhood $P(D) \subset N$ for the divisor. Through out this thesis, all concave (resp. convex) fillings are symplectic strong concave (resp. strong convex) fillings and we simply call it cappings or concave fillings (resp. fillings or convex fillings).

Definition 1.2.1. Suppose that D is a concave (resp. convex) divisor. If a symplectic gluing ([28]) can be performed for a concave (resp. convex) neighborhood of D and a symplectic manifold Y with convex (resp. concave) boundary to obtain a closed symplecitc manifold, then we call D a **capping** (resp. **filling**) divisor. In both cases, we call D a **compactifying** divisor of Y.

1.2.1 Motivation

We provide some motivation from two typical families of examples in algebraic geometry together with some general symplectic compactification phenomena.

Suppose Y is a smooth affine algebraic variety over $\mathbb{C}$. Then Y can be compactified by a divisor D to a projective variety X. By Hironaka's resolution of singularities theorem, we could assume that X is smooth and D is a simple normal crossing divisor. In this case, Y is a Stein manifold and D has a concave neighborhood induced by a plurisubharmonic function on Y ([27]). Moreover, Y is symplectomorphic to the completion of a suitably chosen Stein domain $\overline{Y} \subset Y$ (See e.g. [71]). Therefore, compactifying Y by D in the algebro geometric situation is analogous to gluing $\overline{Y}$ with a concave neighborhood of D along their contact boundaries [28].

On the other hand, suppose we have a compact complex surface with an isolated normal singularity. We can resolve the isolated normal singularity and obtain a pair (W, D), where W is a smooth compact complex surface and D is a simple normal crossing resolution divisor. In this case, we can define a Kähler form near D such that D has a convex neighborhood $P(D)$. If the Kähler form can be extended to W, then the Kähler compactification of $W - D$ by D is analogous to gluing the symplectic manifold $W - Int(P(D))$ with $P(D)$ along their contact boundaries.

From the symplectic point of view, there are both flexibility and constraints for capping a symplectic 4 manifold Y with convex boundary. For flexibility, there are infinitely many ways to embed Y in closed symplectic 4-manifolds (Theorem 1.3 of [30]). This still holds even when Y has only weak convex boundary (See [26] and [29]). For constraints, it is well-known that Y does not have any exact capping by Stoke's theorem. From these perspectives, divisor cappings might provide a suitable capping model to study (See also [33] and [32]).

On the other hand, divisor fillings have been studied by several authors. For instance, it is known that they are the maximal fillings for the canonical contact structures on Lens spaces (See [60] and [12]).

In this setting, we want to answer the following question: Suppose D is a symplectic divisor. When is D also a convex/concave divisor?

1.2.2 Divisor Neighborhood

We recall some results from the literature for the filling side. It is proved in [34] that when the graph of a symplectic divisor is negative definite, it can always be perturbed to be a convex divisor.

Our main result in this direction is:

Theorem 1.2.2. *Let $D \subset (W, \omega_0)$ be a symplectic divisor. If the intersection form of D is not negative definite and ω_0 restricted to the boundary of D is exact, then ω_0 can be deformed through a family of symplectic forms ω_t on W keeping D symplectic and such that (D, ω_1) is a concave divisor.*

It is convenient to associate an augmented graph (Γ, a) to a symplectic divisor (D, ω), where Γ is the graph of D and a is the area vector for the embedded symplectic surfaces (See Section 3.2 for details). The intersection form of Γ is denoted by Q_Γ.

Definition 1.2.3. Suppose (Γ, a) is an augmented graph with k vertices. Then, we say that (Γ, a) satisfies the positive (resp. negative) **GS criterion** if there exists $z \in (0, \infty)^k$ (resp $(-\infty, 0]^k$) such that $Q_\Gamma z = a$.

A symplectic divisor is said to satisfy the positive (resp. negative) GS criterion if its associated augmented graph does.

One important ingredient for the proof of Theorem 1.2.2 is the following result.

Proposition 1.2.4. *Let (D, ω) be a symplectic divisor with ω-orthogonal intersections. Then, (D, ω) has a concave (resp. convex) neighborhood inside any regular neighborhood of D if (D, ω) satisfies the positive (resp. negative) GS criterion.*

The construction is essentially due to Gay and Stipsicz in [34], which we call the GS construction. We remark that GS criteria can be verified easily. They are conditions on wrapping numbers in disguise. Therefore, by a recent result of Mark McLean [72], Proposition 1.2.4 can be generalized to higher dimensions with GS criteria being replaced accordingly. Moreover, using techniques in [72], we establish the necessity of the GS criterion and answer the uniqueness question in [34].

Theorem 1.2.5. *Let $D \subset (W, \omega)$ be an ω-orthogonal symplectic divisor. If (D, ω) does not satisfy the positive (resp. negative) GS criterion. Then, there is a neighborhood N of D such that any plumbing $P(D) \subset N$ of D is not a concave (resp. convex) neighborhood.*

Theorem 1.2.6. *Let (D, ω_i) be ω_i-orthogonal symplectic divisors for $i = 0, 1$ such that both satisfy the positive (resp. negative) GS criterion. Then the concave (resp. convex) structures on the boundary of (D, ω_0) and (D, ω_1) via the GS construction are contactomorphic.*

In particular, when $\omega_0 = \omega_1$, the contact structure constructed via GS construction is independent of choices, up to contactomorphism.

Summarizing Theorem 1.2.2 and Proposition 1.2.4, we have

Corollary 1.2.7. *Let (D, ω) be a symplectic divisor with ω exact on the boundary of D. Then D is either a concave divisor or a convex divisor, possibly after a symplectic deformation.*

1.2.3 Symplectic log Calabi-Yau surfaces

In [38], Gross, Hacking and Keel proposed a way to interpret mirror symmetry for Looijenga pair (X, D), where X is a smooth projective surface over $\mathbb{C}$ and D is an effective reduced anti-canonical divisor on X with maximal boundary. Under mirror symmetry, certain symplectic invariants of $X - D$ are conjectured to be related to holomorphic invariants of its mirror. In this regard, Pascaleff showed in [83] that the symplectic cohomology of $X - D$ is, as a vector space, isomorphic to the global sections of the structure sheaf of its mirror. A step towards a deeper understanding of mirror symmetry for Looijenga pair would be a classification of Looijenga pairs, which is described in the work [39]. Since one direction of mirror symmetry concerns about the symplectic invariants of $X - D$ instead of the holomorphic invariants, we would like to establish a classification for 'Looijenga pairs' in the symplectic category. From symplectic point of view, we have the following definition of Looijenga pair.

Definition 1.2.8. A **symplectic log Calabi-Yau surface** (X, D, ω) is a closed symplectic real dimension four manifold (X, ω) together with a symplectic divisor D representing the homology class of the Poincare dual of $c_1(X, \omega)$.

A symplectic Looijenga pair (X, D, ω) is a symplectic log Calabi-Yau surface such that each irreducible component of D is a sphere.

Let (X, D, ω) be a symplectic log Calabi-Yau surface. By Theorem A of [61] or [76] and the adjunction formula, it is easy to show (Lemma 4.2.1) that X is uniruled with base genus 0 or 1, and D is a torus or a cycle of spheres. And if (X, D, ω) is a symplectic Looijenga pair then X is rational.

Similar to studying the moduli space under complex deformation in complex category, we would like to classify symplectic log Calabi-Yau surface up to symplectic deformation equivalence.

Definition 1.2.9. A **symplectic homotopy** (resp. **symplectic isotopy**) of (X, D, ω) is a smooth one-parameter family of symplectic divisors (X, D_t, ω_t) with $(X, D_0, \omega_0) = (X, D, \omega)$ (resp. such that in addition $\omega_t = \omega$ for all t). (X', D', ω') is said to be **symplectic deformation equivalent** to (X, D, ω) if it is symplectomorphic to (X, D_1, ω_1) for some symplectic homotopy (X, D_t, ω_t) of (X, D, ω). The symplectic deformation equivalence is called **strict** if the symplectic homotopy is a symplectic isotopy.

Definition 1.2.10. Two symplectic divisors are said to be **homological equivalent** if there is a diffeomorphsim $\Phi : X^1 \to X^2$ such that $\Phi_*[C_j^1] = [C_j^2]$ for all $j = 1, \dots, k$. The homological equivalence is called **strict** if $\Phi^*[\omega]^2 = [\omega^1]$. We call Φ a (strict) homological equivalence.

We obtain the following classification.

Theorem 1.2.11. *Let (X^i, D^i, ω^i) be symplectic log Calabi-Yau surfaces for $i = 1, 2$. Then (X^1, D^1, ω^1) is (resp. strictly) symplectic deformation equivalent to (X^2, D^2, ω^2) if and only if they are (resp. strictly) homological equivalent.*

Moreover, the symplectomorphism in the (resp. strict) symplectic deformation equivalence has same homological effect as the (resp. strict) homological equivalence.

For related results on symplectic log Calabi-Yau surfaces see [35],[79]. We remark that when D is a smooth divisor, the relative Kodaira dimension $\kappa(X, D, \omega)$ was introduced in [59] and it was noted there that this notion could be extended to nodal divisors. With this extension understood, symplectic Calabi-Yau surfaces have relative Kodaira dimension $\kappa = 0$ (cf. Theorem 3.28 in [59]). Moreover, Theorem 1.2.11 is also valid when $\kappa(X, D, \omega) = -\infty$. This will be also treated in the sequel.

1.3 Lagrangian - Dehn twist long exact sequences

The celebrated Lagrangian cobordism theory introduced by Biran and Cornea in their sequel papers [13][14][15] has achieved great success encapsulating information of the triangulated structures of the derived Fukaya category. A particularly attractive application is that they establish the long-expected relation between Lagrangian surgeries [49][86] and the mapping cones in derived Fukaya categories.

We want to revisit such surgery-cobordism relations with emphasis on applications to Dehn twists. The underlying philosophy of our approach is to understand the functors between Fukaya categories via Lagrangian cobordisms. This functor-level point of view has been exploited in several other contexts by many authors [63][106][2] etc.

We explore this direction through the eyes of Lagrangian corresondences. Intuitively, one may regard Lagrangian correspondences as symplectic mirrors of kernels of Fourier-Mukai transforms. The observation is, almost all exact sequences involving Lagrangian Dehn twists can be interpreted as cone relations between these "kernels". Explicitly, Lagrangian cobordism constructions geometrically realize all these cones on the correspondence level and provides a completely analogous picture on the symplectic side, versus various twist constructions on derived categories. This point of view greatly simplifies the proof of several known exact sequences and leads to new cone relations in Floer theory such as Lagrangian $\mathbb{CP}^n$-twists, verifying a conjecture due to Huybrechts-Thomas.

We designed a new approach to Lagrangian surgeries called the *flow surgery*, which is coordinate-free and easy to compare with other constructions such as Dehn twists. The construction on its own also allows many variants open for future exploration.

1.3.1 Flow surgeries and flow handles

Recall that for two Lagrangians $L_1 \pitchfork L_2 = \{x\}$, their Lagrangian surgery at x is given by adding an explicit Lagrangian handle in the Darboux chart [49][86]. Then a Lagrangian cobordism can be obtained by using "half" of a Lagrangian handle of one dimension higher [13]. This line of thoughts has led to remarkable breakthroughs in both constructions of new examples of Lagrangian submanifolds and cobordism theory.

To implement this construction to Lagrangian "fiber sums" (surgery along clean intersections), the patching of local models requires more delicate consideration on the connection of normal bundles. On top of that, in most of our applications, the main difficulty is to show that the resulting manifold is Hamiltonian isotopic to certain given Lagrangians, usually those obtained by Lagrangian Dehn twists.

Our basic idea to solve both problems at once is to use a reparametrized geodesic flow, mimicking the original construction of Dehn twist by Seidel, to produce a new Lagrangian surgery operation called the **flow surgery** (See Section 5.1.2). This flow surgery recovers the usual Lagrangian surgery when the auxiliary data is chosen appropriately, but has much better flexibility. For example, the resulting Lagrangian handle needs not be diffeomorphic to a puntured ball (or a bundle with punctured-ball fibers in the clean surgery case). Moreover, the Biran-Cornea's cobordism construction via surgeries fits into this framework easily as well.

The main examples we have are the following (see Section 5.2 for relevant definitions).

Theorem 1.3.1. *Let $S^n \subset M$ be a Lagrangian sphere and $S \subset M$ be a Lagrangian submanifold diffeomorphic to either $\mathbb{RP}^n$, $\mathbb{CP}^n$ or $\mathbb{HP}^n$. Let τ_{S^n} and τ_S denote the corresponding Dehn twists. One has the following surgery equalities up to hamiltonian isotopies in $M \times M^-$:*

(1) $(S^n \times (S^n)^-)\#_{\Delta_{S^n},E_2}\Delta_M = Graph(\tau_{S^n}^{-1})$,

(2) $\widetilde{C}\#_{\mathcal{D},E_2}\Delta_M = Graph(\tau_C^{-1})$, *where $C \subset M$ is a spherically coisotropic submanifold.*

(3) $(S \times S^-)\#_{\mathcal{D}^{op},E_2}(S \times S^-)\#_{\Delta_S,E_2}\Delta_M = Graph(\tau_S^{-1})$,

(4) $\widetilde{C}_P\#_{\mathcal{D}^{op},E_2}\widetilde{C}_P\#_{\mathcal{D},E_2}\Delta_M = Graph(\tau_{C_P}^{-1})$, *where $C_P \subset M$ is a projectively coisotropic submanifold.*

The surgery equalities immediately lead to the existence of corresponding Lagrangian cobordisms. Note that in case (1), a similar cobordism construction was established in [2] using Lefschetz fibrations independently.

Remark 1.3.2. Formal proofs will only be given in the case of S^n and $\mathbb{CP}^n$, since $\mathbb{HP}^n$ and $\mathbb{RP}^n$ cases will follow from the proof of $\mathbb{CP}^n$ word-by-word. The common feature for these manifolds we used is the existence of a metric g_S of the following property:

for any point $x \in S$, the injectivity radius x equals π, and $S\backslash B_x(\pi)$ is a smooth closed submanifold.

1.3.2 The Huybrechts-Thomas conjecture and projective twists

There is a natural extension of Dehn twists construction along spheres to arbitrary rank-one symmetric spaces, which is known for a long time. Seidel's long exact sequence associated to a Dehn twist along spheres should be viewed as the mirror of spherical twists in derived categories [95]. Also, such a cone relation on the A-side has become a foundational tool in the study of homological mirror symmetry, especially in the Picard-Lefschetz theory [92].

Since then, it has remained a mystery about how to describe the effect of the Dehn twists along a rank-one symmetric space in Floer theory.

Fortunately, one could again find hints from homological mirror symmetry in this case. On the B-side, Huybrechts-Thomas [43] first defined $\mathbb{P}^n$*-objects* in the derived category. Recall that an object $\mathcal{E} \in D^b(X)$ for a smooth projective variety is called a $\mathbb{P}^n$-object if $\mathcal{E} \otimes \omega_X = \mathcal{E}$ and $Ext^*(\mathcal{E}, \mathcal{E})$ is isomorphic as a graded ring to $H^*(\mathbb{P}^n, \mathbb{C})$. Then they constructed an auto-equivalence called the $\mathbb{P}^n$*-twist* associated to $\mathcal{E}$, which is the Fourier-Mukai transform with kernel

$$Cone(Cone(\mathcal{E}^\vee \boxtimes \mathcal{E}[-2] \xrightarrow{\bar{h}^\vee \times id - id \times \bar{h}} \mathcal{E}^\vee \boxtimes \mathcal{E}) \xrightarrow{ev} \mathcal{O}_\Delta). \tag{1.1}$$

Here $\bar{h} \in hom^2(\mathcal{E}, \mathcal{E})$ is a representative of the generator in cohomology. They then conjectured the $\mathbb{P}^n$-twist is exactly the mirror auto-equivalence of the one induced by a Dehn twist along Lagrangian $\mathbb{CP}^n$ on the derived Fukaya categories. Carried out explicitly in the derived Fukaya category, the conjecture reads:

Conjecture 1.3.3. *Given a monotone/exact Lagrangian $\mathbb{CP}^n$ $S \subset M$ and a compact monotone/exact Lagrangian L, then in $D^\pi Fuk(M)$*

$$\tau_{\mathbb{CP}^n}(L) \cong Cone(Cone(hom(S, L) \otimes S[-2] \xrightarrow{\mu^2(h,-) \times id - id \times h} hom(S, L) \otimes S) \xrightarrow{ev} L) \tag{1.2}$$

Here the right hand side is an iterated mapping cone, and $D^\pi Fuk(M)$ denotes the derived Fukaya category generated by compact Lagrangian branes, and $h \in hom^2(S, S)$ is the Floer cochain in degree 2 representing the dual of hyperplane in cohomology.

Richard Harris studied the problem in A_∞ contexts and formulated the corresponding algebraic twist on A-side [40]. The only missing link to the actual geometry of Lagrangian submanifolds, remains unproved for years.

As an application of the surgery equalities in Theorem 1.3.1, we show the following cone relations:

Theorem 1.3.4. *For any given monotone/exact Lagrangian submanifolds L_0 and L_1, and S diffeomorphic to $\mathbb{CP}^n$ in M, there is an quasi-isomorphism of cochain complexes*

$$\begin{aligned} &CF^*(L_0, \tau_S L_1) = \\ &Cone(CF^*(S, L_1) \otimes CF^{*-2}(L_0, S) \to CF^*(S, L_1) \otimes CF^*(L_0, S) \to CF^*(L_0, L_1)). \end{aligned} \tag{1.3}$$

Note that the right hand side denotes an iterated mapping cones as in [13]. Combining with the cone relation on $D^\pi Fuk(M)$ [14] and M'au-Wehrheim-Woodward functor [63], one verifies

Theorem 1.3.5. *Huybrechts-Thomas conjecture 1.3.3 is true modulo determination of connecting maps.*

The proof of Theorem 1.3.5 follows from the construction of a cobordism representing an iterated cone on the functor level, see Theorem 1.3.1 and Lemma 5.3.20. Our method applies well on $\mathbb{RP}^n$ or $\mathbb{HP}^n$, and should extend to other Lagrangians whose geodesics are closed with rational proportions such as Cayley plane or their finite covers. These are supposed to be the mirror of $\mathbb{P}^n$-like objects except for a change in gradings for non-trivial self-hom's. A family version of projective twist is also given.

Remark 1.3.6. While it is not difficult to find examples of Lagrangian $\mathbb{RP}^n$ in problems in symplectic topology [96][112] etc, the search of interesting examples of Lagrangian $\mathbb{CP}^n$ is more intriguing. In [43] the authors suggested several sources of $\mathbb{P}^n$-objects in derived categories. An interesting instance is given by pull-back sheaves of a Lagrangian fibration on a hyperkähler manifold. From the SYZ point of view, this should correspond to a Lagrangian $\mathbb{CP}^n$ section on the SYZ mirror. While the role of $\mathbb{P}^n$ objects on either side of mirror symmetry remains widely open so far, it is interesting to know whether such objects split generate either side of mirror symmetry.

Remark 1.3.7. In a different direction, the $\mathbb{P}^n$-cone relation should be interested in understanding some basic problems in symplectic topology, such as mapping class groups of a symplectic manifold and the search of exotic Lagrangian submanifolds. For instance, while a Lagrangian $\mathbb{CP}^n$-twist is always smoothly isotopic to identity, it is usually not **Hamiltonian** isotopic to identity. A simplest model result along this line is to generalize Seidel's twisted Lagrangian sphere construction [89]: in the plumbing of three $T^*\mathbb{CP}^n$, the iterated Dehn twists along $\mathbb{CP}^n$ in the middle should generate an infinite subgroup in the symplectic mapping class group.

Remark 1.3.8. With Theorem 1.3.1 the projective twist cone formula easily generalizes to $\mathbb{RP}^n$ and $\mathbb{HP}^n$. The only difference between the formulas is the grading shift of the first term.

The remaining chapters are organized as follows.

- Chapter 2 presents all the symplectic filling obrstructions we get by using symplectic caps, including Theorem 1.1.3 and 1.1.8
- The focus of Chapter 3 is on concrete examples and applications, including exact fillings of unit cotangent bundles (Theorem 1.1.4), the comprehensive study of symplectic divisor neighborhood (Theorem 1.2.2, Proposition 1.2.5 and 1.2.6).
- Chapter 4 describes the classification of log Calabi-Yau surfaces (Theorem 1.2.11).
- Chapter 5 provides a discussion of Lagrangian Dehn twist and proves the induced long exact sequences (Theorem 1.3.1, 1.3.4 and 1.3.5).

Chapter 2

Calabi-Yau and uniruled caps

In this chapter, we provide a comprehensive study of various symplectic caps and prove the structural theorems concerning obstructions to symplectic fillings.

2.1 General discussion

2.1.1 Uniruled manifolds and Calabi-Yau manifolds

Let X be a closed, oriented smooth 4-manifold. Let $\mathcal{E}_X$ be the set of cohomology classes whose Poincaré duals are represented by smoothly embedded spheres of self-intersection -1. X is said to be (smoothly) minimal if $\mathcal{E}_X$ is the empty set. Equivalently, X is minimal if it is not the connected sum of another manifold with $\overline{\mathbb{CP}^2}$.

Suppose ω is a symplectic form compatible with the orientation. (X, ω) is said to be (symplectically) minimal if $\mathcal{E}_\omega$ is empty, where

$$\mathcal{E}_\omega = \{E \in \mathcal{E}_X |\ E \text{ is represented by an embedded } \omega\text{−symplectic sphere}\}.$$

We say that (Z, τ) is a minimal model of (X, ω) if (Z, τ) is minimal and (X, ω) is a symplectic blow up of (Z, τ). A basic fact proved using Taubes' SW theory ([103], [52], [53]) is: $\mathcal{E}_\omega$ is empty if and only if $\mathcal{E}_X$ is empty. In other words, (X, ω) is symplectically minimal if and only if X is smoothly minimal.

For minimal (X, ω), the Kodaira dimension of (X, ω) is defined in the following way in [55] (see also [68] [50]):

$$\kappa^s(X,\omega) = \begin{cases} -\infty & \text{if } K_\omega \cdot [\omega] < 0 \text{ or } \quad K_\omega \cdot K_\omega < 0, \\ 0 & \text{if } K_\omega \cdot [\omega] = 0 \text{ and } \quad K_\omega \cdot K_\omega = 0, \\ 1 & \text{if } K_\omega \cdot [\omega] > 0 \text{ and } \quad K_\omega \cdot K_\omega = 0, \\ 2 & \text{if } K_\omega \cdot [\omega] > 0 \text{ and } \quad K_\omega \cdot K_\omega > 0. \end{cases}$$

Here K_ω is defined as the first Chern class of the cotangent bundle for any almost complex structure compatible with ω.

The invariant κ^s is well defined since there does not exist a minimal (X,ω) with

$$K_\omega \cdot [\omega] = 0, \quad \text{and} \quad K_\omega \cdot K_\omega > 0.$$

This again follows from Taubes' SW theory [55]. Moreover, κ^s is independent of ω, so it is an oriented diffeomorphism invariant of X.

The Kodaira dimension of a non-minimal manifold is defined to be that of any of its minimal models. This definition is well-defined and independent of choice of minimal model so $\kappa^s(X,\omega)$ is well-defined for any (X,ω) (cf. [55]).

Definition 2.1.1. Let (X,ω) be a not necessarily minimal closed symplectic four manifold. We call (X,ω) a **symplectic Calabi-Yau surface** (resp. **symplectic uniruled manifold**) if $\kappa^s(X,\omega) = 0$ (resp. $\kappa^s(X,\omega) = -\infty$).

We sometimes simply call a symplectic Calabi-Yau surface a Calabi-Yau surface and a symplectic uniruled manifold a uniruled manifold. A minimal symplectic Calabi-Yau surface has torsion first Chern class. The first author proved in [54] the following theorem for symplectic Calabi-Yau surfaces (cf. also [8]).

Theorem 2.1.2 (Theorem 1.1 of [54])**.** *If (X,ω) is a minimal symplectic Calabi-Yau surface, then its rational homology is the same as that of the $K3$ surface, the Enriques surface or a torus bundle over torus.*

Remark 2.1.3. If the first integral homology of a rational homology $K3$ symplectic Calabi-Yau surface is not trivial, it admits a finite cover which is also a symplectic Calabi-Yau surface but with Euler characteristic larger than that of the $K3$ surface. Hence, Theorem 2.1.2 implies that a symplectic Calabi-Yau surface with the rational homology of the $K3$ surface is an integral homology $K3$ surface.

For uniruled manifolds, Liu and independently Ohta and Ono proved the following smooth classification. For the symplectic classification, see also [64].

Theorem 2.1.4 ([61] or [76])**.** *If (X,ω) is a minimal uniruled manifold, X is $\mathbb{CP}^2, S^2 \times S^2$ or an S^2-bundle over a Riemann surface of positive genus.*

For a minimal uniruled manifold (X,ω), the base genus is defined to be zero if X is $\mathbb{CP}^2$ or $S^2 \times S^2$, otherwise it is defined to be the genus of the base as an S^2-bundle. For a general uniruled manifold (X,ω), the base genus is defined to be the base genus of any of its minimal models.

In some sense, Calabi-Yau manifolds and uniruled manifolds capture most of the rigidity results for closed symplectic four manifolds. This point of view motivates the definitions of Calabi-Yau caps and uniruled caps to obtain rigidity results for fillings.

We end this subsection with the following lemmas.

Lemma 2.1.5. *Let (X,ω) be a uniruled manifold and $u : \Sigma_g \to X$ be a continuous map from an oriented surface of genus g to X with $(u_*[\Sigma_g])^2 > 0$. Then the base genus of X is less than or equal to g.*

Proof. Without loss of generality, we can assume that X is not of genus 0. We further assume that X is not minimal since blowing-down does not change the base genus. Since X is not minimal, it can be obtained from r-times blow-ups of a product manifold $S^2 \times \Sigma_h$ for some genus h surface. Let f be the class of $[S^2 \times p]$ for some $p \in \Sigma_h$ and s the class of $[p \times \Sigma_h]$ for some $p \in S^2$. Notice that, as a smooth manifold X admits a smooth projection $\pi : X \to \Sigma_h$ with spherical fibers in the class f. We will show that the induced projection $\pi \circ u : \Sigma_g \to \Sigma_h$ has non-zero degree and hence $g \geq h$.

By abuse of notation, $H_2(X,\mathbb{Z})$ is generated by $f, s, e_1, \dots, e_r$, where the e_i are the classes of the exceptional spheres. Since $(a_0 f + a_1 e_1 + \cdots + a_r e_r)^2 \leq 0$ for all $a_i \in \mathbb{Z}$, $u_*[\Sigma_g]$ has non-zero coefficient in s when written as a linear combination over the basis $\{f, s, e_1, \dots, e_r\}$. By observing that this non-zero coefficient is precisely the degree of the map $\pi \circ u$, we conclude that $h \leq g$.

□

Lemma 2.1.6 (Proposition 3.14 of [59])**.** *Let (X,ω) be a non-minimal uniruled manifold and D a symplectic submanifold with positive genus. If $[D] \cdot e > 0$ for all exceptional classes e in X, then $(K_\omega + [D])^2 \geq 0$.*

2.1.2 Relative cohomology pairing

In this subsection, we recall the relative de Rham theory and illustate the well-definedness of several pairings (see eg. [16]).

Given a smooth manifold with boundary X, the relative cochain $\mathcal{C}_k$ consists of pairs (β, α), where β is a k-form on X and α is a $(k-1)$-form on ∂X. The differential d is defined as $d(\beta, \alpha) = (d\beta, \beta|_{\partial X} - d\alpha)$. It is easy to see that $d \circ d = 0$ and it forms a cohomology isomorphic to the usual relative cohomology $H^*(X, \partial X; \mathbb{R})$.

This formulation of relative cohomology can be translated to compactly supported cohomology of $X - \partial X$. For (β, α) a cochain in $\mathcal{C}_k$, we consider a collar $(0,1] \times \partial X$ of ∂X in X. We choose a cutoff function $\chi : (0,1] \times \partial X \to \mathbb{R}$. We want $\chi(r,x) = \chi(r)$ for $(r,x) \in (0,1] \times \partial X$ such that $\chi(r) = 0$ near $r = 0$ and $\chi(r) = 1$ near $r = 1$. Extending by 0, $\chi\alpha$ is a $k-1$-form on X which we also denote as α^c. Then $(\beta, \alpha) - d(\chi\alpha, 0) = (\beta - d(\chi\alpha), 0)$ is another chain level representative of $[(\beta, \alpha)]$ which has compact support in $X - \partial X$. One can show that this translation induce an isomorphism from relative cohomology $H^k(X, \partial X, \mathbb{R})$ to the compactly supported cohomology $H^k_{cpt}(X - \partial X, \mathbb{R})$.

We assume that X has dimension 4 and is connected, oriented. Consider the following pairing:

$$H^2(X;\mathbb{R}) \times H^2(X, \partial X; \mathbb{R}) \to H^4(X, \partial X; \mathbb{R}) = \mathbb{R}$$
$$([A], [(B,b)]) \to \int_X A \wedge B - \int_{\partial X} A \wedge b,$$

where the integral on ∂X is taken with the Stokes boundary orientation. To see it is independent of A, we check that by Stokes Theorem applied to X,

$$\int_X du \wedge B - \int_{\partial X} du \wedge b = \int_X d(du \wedge b) - \int_{\partial X} du \wedge b = 0.$$

To see it is independent of (B, b), we check that by Stokes Theorem applied to both X and ∂X,

$$\int_X A \wedge d\beta - \int_{\partial X} A \wedge (\beta - d\alpha) = \int_{\partial X} A \wedge d\alpha = \int_{\partial X} d(A \wedge \alpha) = 0.$$

Notice that, this pairing is translated from the usual pairing bewteen cohomololgy and compactly supported cohomology (see eg. [16]). By Lefschetz duality, this pairing is non-degenerate.

Consider the following portion of the long exact sequence

$$\cdots H^1(\partial X;\mathbb{R}) \xrightarrow{\partial} H^2(X,\partial X;\mathbb{R}) \xrightarrow{f} H^2(X;\mathbb{R}) \xrightarrow{r} H^2(\partial X;\mathbb{R}) \cdots , \tag{2.1}$$

where at the form level, f sends (A,a) to A, and r sends B to $B|_{\partial X}$. Via the forgetful map f, we also have the pairing

$$H^2(X,\partial X;\mathbb{R}) \times H^2(X,\partial X;\mathbb{R}) \to H^4(X,\partial X;\mathbb{R}) = \mathbb{R}.$$

However, this pairing in general has a kernel.

Lemma 2.1.7. *The pairing $[(A,a)]\cdot[(B,b)]$ is independent of a and b.*

Proof. If b' is another primitive of B, then $b-b'$ is closed. Hence

$$\int_{\partial X} A\wedge(b-b') = \int_{\partial X} da\wedge(b-b') = 0.$$

The same argument applies to the choice of a. □

The kernel is contained in the image of the boundary homomorphism $\partial : H^1(\partial X;\mathbb{R}) \to H^2(X,\partial X;\mathbb{R})$ by the non-degeneracy of the pairing $H^2(X;\mathbb{R})$ and $H^2(X,\partial X;\mathbb{R})$. Actually, the kernel is exactly the image because it is easy to see that it pairs everything to be zero.

From now on, we use P or N instead of X to denote a manifold with boundary, depending on whether it is a cap or a filling. The following simple lemma is the key to relate the caps with the closed manifold.

Lemma 2.1.8. *Let (P,ω_P) be a symplectic cap of (Y,ξ) with a Liouville one form α_P, and (N,ω_N) be a symplectic filling of (Y,ξ) with a Liouville one form α_N. Let $X = N\cup_Y P$ which is a closed manifold. Then for sufficiently large $t>0$, there is a symplectic form ω on X such that $c_1(X)\cdot\omega = c_1(N)\cdot[(\omega_N,\alpha_N)] + c_1(P)\cdot t[(\omega_P,\alpha_P)]$ and $\omega|_N = \omega_N$.*

Proof. We identify $(\partial N, \ker(\alpha_N))$ and $(\partial P, \ker(\alpha_P))$ by a contactomorphism Φ. There is a global positive function $f_{\alpha_P} : \partial N \to \mathbb{R}$ such that $\Phi^*\alpha_P = f_N\alpha_N$. When $t>0$ is large, $\Phi^* t\alpha_P = f_{t\alpha_P}\alpha_N$ is such that $f_{t\alpha_P}(x) > 1$ for all $x\in\partial N$. We fix such a choice of t. Consider the symplectization $(\mathbb{R}\times\partial N, d(e^r\alpha_N), e^r\alpha_N)$ where $r\in\mathbb{R}$. Let $SY = \{(r,x)\in$

$\mathbb{R}\times\partial N|1\leq e^r\leq f_{t\alpha_P}(x)\}$ We equip it with the restricted symplectic form and one form and call it $(SY,\omega_{SY},\alpha_{SY})$. We can glue (N,ω_N,α_N) and $(P,t\omega_P,t\alpha_P)$ by inserting $(SY,\omega_{SY},\alpha_{SY})$, see [28]. Notice that α_{SY} is a globally defined primitive of ω_{SY} on SY and coincides with α_N and $t\alpha_P$ on its two boundary components, respectively. Let the resulting manifold be (X,ω), which is the union of N, SY and P. By multiplying a cutoff function and by abuse of notation, we can extend α_{SY} to be a one form supported in a neighborhood of $SY\subset X$ such that $\alpha_{SY}|_N=\alpha_N^c$ and $\alpha_{SY}|_P=t\alpha_P^c$, where α_N^c and $t\alpha_P^c$ are defined as in the third paragraph of this subsection. Therefore, we have

$$\begin{aligned}
c_1(X)\cdot[\omega] &= c_1(X)\cdot[\omega-d\alpha_{SY}]\\
&= c_1(X)\cdot[\omega_N-d\alpha_N^c]+c_1(X)\cdot t[\omega_P-d\alpha_P^c]\\
&= c_1(N)\cdot[(\omega_N,\alpha_N)]+c_1(P)\cdot t[(\omega_P,\alpha_P)]
\end{aligned}$$

which is simply the sum of the pairings from the cap and the filling. This is clearly true for any t sufficiently large. It completes the proof. $\square$

The following properties are also useful.

Lemma 2.1.9. *Let (P,ω) be a symplectic cap and α a choice of Liouiville one form. Let $\Sigma\subset P$ be a compact embedded surface with boundary $\partial\Sigma\subset\partial P$. Then the followings are true.*

- $[(\omega,\alpha)]^2>0$,
- $[\Sigma]\cdot PD([(\omega,\alpha)])=\int_\Sigma\omega-\int_{\partial\Sigma}\alpha$, *where $\partial\Sigma$ is equipped with the Stokes orientation.*
- *if $c_1(P)\cdot[(\omega,\alpha)]=0$ and $c_1(P)\neq 0\in H^2(P,\mathbb{R})$, then there is a small perturbation (ω',α') of (ω,α) such that (P,ω') is a cap, α' is a Liouiville one form of ω' and $c_1(P)\cdot[(\omega',\alpha')]>0$.*

Proof. For the first bullet, we have $[(\omega,\alpha)]^2=\int_P\omega^2-\int_{\partial P}\alpha\wedge\omega$. The first term is positive because the orientation of P is always chosen to be compatible with ω^2. On the other hand, $\int_{\partial P}\alpha\wedge\omega<0$ because the orientation of ∂P as a contact manifold is determined by $\iota_V(\omega^2)=2\alpha\wedge\omega$ for an inward pointing vector field V, while the Stokes orientation of ∂P is determined by $\iota_{V_{Stoke}}(\omega^2)$ for an outward pointing vector field V_{Stoke}. Therefore, $[(\omega,\alpha)]^2>0$.

The second bullet follows from definition.

For the last bullet, if $c_1(P) \neq 0$ we can find a relative cohomology class $[(A, a)]$ pairs positively with $c_1(P)$ by the non-degeneracy of the pairing between absolute cohomology and relative cohomology. Let (A, a) be a chain level representative of $[(A, a)]$. The result follows by adding $c(A, a)$ to (ω, α) for some small $c > 0$. $\square$

Remark 2.1.10. We could have defined Calabi-Yau cap by the equation $c_1(P)\cdot[(\omega, \alpha)] = 0$ instead of $c_1(P) = 0 \in H^2(P, \mathbb{R})$. In this case, the third bullet of Lemma 2.1.9 implies that if $c_1(P)$ is not torsion, then we can deform the cap to a uniruled cap which gives stronger restrictions to fillings. Therefore, we stick to our defintion of Calabi-Yau caps.

This section provides a discussion of how to get topological restriction to symplectic fillings by having a Donaldson hypersurface in a symplectic cap.

2.1.3 Minimality

Let (N, ω_N) be a convex symplectic manifold. It is called smoothly minimal if there is no smoothly embedded sphere of self-intersection -1. It is called (symplectically) minimal if there is no symplectically embedded sphere of self-intersection -1. For homological reason, any exact/Stein filling is minimal. The following proposition shows that these two are in fact the same notion.

Proposition 2.1.11. *A convex symplectic 4-manifold (N, ω_N) is symplectically minimal if and only if it is smoothly minimal.*

We first recall a result in [56].

Lemma 2.1.12 ([56], Corollary 2.12)**.** *Let (P, ω_P, α_P) be as above and (N, ω_N) a symplectic filling of (Y, ξ). Then any symplectic exceptional class in $(N \cup_Y P, \omega)$ which admits no embedd symplectic representative in $(N, \lambda_P \omega_N)$ pairs positively with $PD[(\omega_P, \alpha_P)]$*

In particular, if (N, ω_N) is (symplectically) minimal, any symplectic exceptional class in $(N \cup_Y P, \omega)$ pairs positively with $PD[(\omega_P, \alpha_P)]$.

Proof of Proposition 2.1.11. Clearly N is symplectically minimal if it is smoothly minimal. To prove the converse, we will assume that N is not smoothly minimal and show that it cannot be symlectically minimal. Let $e \in H_2(N)$ be the class of a smoothly embedded -1 sphere in N.

We glue N along its contact boundary (Y, ξ) with a concave 4-manifold (P, ω_P) to obtain a closed symplectic 4-manifold (X, ω). Further, we can assume that $b^+(P) > 1$ (by [30]). Denote still by e the image of e under the natural map $H_2(N) \to H_2(X)$. Notice that $b^+(X) > 1$ since $b^+(P) > 1$. By a result of Taubes [103], there is a $\omega-$symplectic -1 sphere S in the class e or $-e$.

By Lemma 2.1.12, S pairs positively with $PD[(\omega_P, \alpha_P)] \in H_2(X)$ under the natural map $H_2(P) \to H_2(X)$. This contradicts to the fact that e is represented by a smooth sphere in N. $\square$

By the same argument, we can also obtain the following consequences.

Corollary 2.1.13. *Let (N, ω_N) be a convex symplectic manifold. If there is a smooth -1 sphere in N, there is a symplectic -1 sphere homologous to it up to sign.*

Moreover, the classes of symplectic -1 spheres are pairwise orthogonal.

A natural question is whether the corresponding result of Proposition 2.1.11 is true for concave symplectic 4-manifolds. We remark that removing a ball in a rational 4-manifold with more than two blow-ups gives a counterexample of the corresponding result of Corollary 2.1.13 for concave symplectic 4-manifolds.

2.1.4 Maximality

This subsection discuss the relation between maximal symplectic surface and exceptional curves in a closed symplectic four manifold.

Definition 2.1.14. Let (X, ω) be a closed symplectic four manifold and D be a smooth symplectic surface in X. Then D is called **maximal** if any symplectic exceptional class in (X, ω) pairs positively with $[D]$.

Definition 2.1.15. Let (P, ω_P) be a concave symplectic manifold and D be a smooth symplectic surface in P. Then D is called **maximal** if, for any minimal symplectic filling (N, ω_N) of $Y = \partial P$, D is maximal in $(N \cup_Y P, \omega)$.

Lemma 2.1.16. *A Donaldson hypersurface for (P, ω_P) is a maximal surface.*

Proof. It follows directly from Lemma 2.1.12. $\square$

Lemma 2.1.17. *Let (X,ω) be a non-uniruled closed symplectic four manifold. If D is maximal, then $c_1(X,\omega)^2 \geq c_1(X,\omega)\cdot D$. In particular, the number of exceptional spheres is bounded above by $K \cdot D$.*

Proof. By Taubes, any minimal model $(\tilde{X}, \omega_{\tilde{X}})$ of (X, ω_X) has $c_1 \cdot c_1 \geq 0$ and all the exceptional classes are pairwise orthogonal. Let L be the number of exceptional classes. Then

$$c_1(X,\omega_X)^2 = c_1(\tilde{X},\omega_{\tilde{X}})^2 - L \geq -L.$$

Let $e_1, ..., e_L$ be the exceptional classes. Then by Taubes, $\tilde{K} = K - \sum e_i$ is a GT class and hence represented by a $J-$holomorphic curve for any $\omega_{\tilde{X}}-$compatible J. We pick such a J so that D is $J-$holomorphic. Since D has positive self-intersection, by positivity of intersection, we have $D \cdot \tilde{K} \geq 0$. Therefore we have

$$L \leq (\sum e_i \cdot D) = (K - \tilde{K}) \cdot D \leq K \cdot D = 2g(D) - 2 - D \cdot D,$$

$\square$

Lemma 2.1.18. *Let (X,ω) be a uniruled closed symplectic four manifold. If D is maximal and D is not a sphere, then $c_1(X,\omega)^2 \geq c_1(X) \cdot D + 2 - 2g$, where g is the genus of D.*

Proof. Since D is not a sphere, D is a maximal surface in the sense of [LZ] and hence satisfies $(-c_1(X)+[D])^2 \geq 0$. By adjunction, we have $c_1(X)\cdot D = D\cdot D + 2 - 2g$. Hence the result follows. $\square$

Corollary 2.1.19. *Let D be a maximal symplectic surface with positive genus in a concave symplectic manifold (P,ω). There is a lower bound on $(2\chi + 3\sigma)(N)$ of any minimal strong symplectic filling N of $Y = \partial P$ given by*

$$(2\chi + 3\sigma)(N) \geq c_1(X) \cdot D + 2 - 2g - (2\chi + 3\sigma)(P)$$

Moreover, if $b^+(P) > 1$, we have

$$(2\chi + 3\sigma)(N) \geq c_1(X) \cdot D - (2\chi + 3\sigma)(P)$$

Proof. Notice that, for any minimal strong symplectic filling N of Y, the glued symplectic manifold $X = N \cup_Y P$ satisfies $c_1(X, \omega_X)^2 = 2\chi(X) + 3\sigma(X)$, and

$$\sigma(X) = \sigma(P) + \sigma(N), \quad \chi(X) = \chi(P) + \chi(N).$$

so it suffices to prove that

$$(2\chi + 3\sigma)(X) \geq c_1(X) \cdot D + 2 - 2g$$

in general and

$$(2\chi + 3\sigma)(X) \geq c_1(X) \cdot D$$

when $b^+(P) > 1$, which in turn follows from Lemma 2.1.17 and Lemma 2.1.18. □

2.1.5 Neck-stretching basic

Let (X, ω) be a closed symplectic four manifold and $\mathcal{J}$ the space of ω-compatible almost complex structures. We recall some basic Gromov-Witten theory and neck-stretching techniques (See [23], [17] and [72] for more comprehensive account). For any $J \in \mathcal{J}$ and a (connected) tree T with $|T|$ (finite) vertices, we call $u = (u_i)_{i=1}^{|T|}$ a **closed genus** 0 **nodal** J-holomorphic map modeled on T if for each vertex v_i of T, there exists a J-holomorphic map $u_i : \mathbb{C}P^1 \to (X, \omega, J)$ such that the intersection pattern of $\{u_i(\mathbb{C}P^1)\}$ is given by the tree T (i.e. an edge joining two vertices corresponds to an intersection between the corresponding $u_i(\mathbb{C}P^1)$). A closed genus 0 nodal J-holomorphic map u is one that modeled on some tree T. In this case, we also call u a closed genus 0 nodal J-holomorphic representative for the homology class $[u] = \sum u_{i*}[\mathbb{C}P^1]$. The following proposition is well-known and readers are referred to [64], [72] and the references therein.

Proposition 2.1.20. *Let e be an exceptional class of (X, ω). Then for any $J \in \mathcal{J}$, there is a closed genus 0 nodal J-holomorphic representative u of e.*

Let (Y, ξ) be a separating contact hypersurface in (X, ω). Here, we mean that there is a Liouville vector V defined near Y such that V is transversal to Y. We denote the Liouville one form α and $\xi = \ker(\alpha)$. Let $(P, \omega|_P)$ be the cap and $(N, \omega|_N)$ be the filling of Y obtained by cutting X along Y. Let $Int(P)$ and $Int(N)$ be the corresponding interiors. We call an $\omega|_P$-compatible almost complex structure J^∞ is **cylindrical** if there is a collar $([0,\epsilon)\times Y, \omega = d(e^r\alpha), \alpha)$ of Y in P such that $J^\infty(\xi) = \xi$, $J^\infty(\partial_r) = \partial_{Reeb}$, $J^\infty(\partial_{Reeb}) = -\partial_r$ and $J^\infty|_\xi$ is translational invariant with respect to r, where $r \in [0,\epsilon)$ and ∂_{Reeb} is the Reeb vector field of α. Any cylindrical J^∞ can be extended to a cylindrical compatible almost complex structure on the symplectic completion of P and we denote this extension as J^∞ by abuse of notation. For a cylindrical J^∞, we call $u^\infty = (u_i^\infty)$ a **genus 0 top building** if the u_i^∞ are J^∞-holomorphic maps (finitely many and possibly empty) from genus 0 puntured Riemann surfaces Σ_i to the symplectic completion of P with puntures aysmptotic to Reeb orbits of Y on its negative end. Similarly, we can think of $u_{i*}^\infty([\Sigma_i]) \in H_2(P,\partial P)$ and we denote $\sum u_{i*}^\infty([\Sigma_i])$ as $[u^\infty]$.

Proposition 2.1.21. *(See [23], [17] and cf. [72]) Let $[D]$ be a homology class in $H_2(P,\mathbb{Z})$. For a tubular neighborhood $\mathcal{N}$ of Y in X, there is a sequence $J^k \in \mathcal{J}$ and an $\omega|_P$-compatible cylindrical almost complex structure J^∞ on P such that the following holds.*

- *$J|_{P-\mathcal{N}} = J^\infty|_{P-\mathcal{N}} = J^k|_{P-\mathcal{N}}$ for all k,*
- *if, for all k, $u^k = (u_i^k)$ is a closed genus 0 nodal J^k-holomorphic map representing the same homology class in X, then there is a genus 0 top building $u^\infty = (u_i^\infty)$ to the symplectic completion of P, and*
- *$[D]\cdot[u^k] = [D]\cdot[u^\infty]$, where the first pairing takes place in $H_2(X)$ (ie. $[D]$ represents its image to $H_2(X)$) and the second is the pairing between $H_2(P)$ and $H_2(P,\partial P)$.*

Proof. The first two bullets are part of the compactness result in symplectic field theory. The last bullet is true because the other buildings at the SFT limit of u^k do not contribute to the intersection pairing with $[D]$. □

Corollary 2.1.22. *Let (P,ω_P) be a symplectic cap of (Y,ξ). Assume that $[(\omega,\alpha)]$ is a rational class and let $[D] \in H_2(P,\mathbb{Z})$ be the Lefschetz dual of $c[(\omega,\alpha)]$ for some $c > 0$. Then for any minimal strong symplectic filling N of (Y,ξ) and any exceptional class e in X, we have $[D]\cdot e > 0$.*

Proof. By Proposition 2.1.20, there is a closed genus 0 nodal representative u for any exceptional class e and any $J \in \mathcal{J}$. We apply Proposition 2.1.20 to the choice of J^k

in Proposition 2.1.21, then we get a J^∞ genus 0 top building $u^\infty = (u_i^\infty)$. Notice that, the u_i^∞ are J^∞-holomorphic for all i and they have punctures asymptotic to Reeb orbits. Near each puncture of Σ_i, the domain of u_i^∞, we have a cylindrical-like conformal coordinate from which we can find a small circle C sufficiently close to the puncture such that $(\pi_Y \circ u_i^\infty)^*\alpha|_C < 0$ with respect to the Stoke orientation of C induced by Σ_i. Here, π_Y is the projection from the negative end of the completion of P to the factor Y. This is because we can find C such that $\pi_Y \circ u_i^\infty(C)$ is close to a Reeb orbit of Y and the Stoke orientation of C is different from the orientation induced by $(\pi_Y \circ u_i^\infty)^*\alpha$. Therefore, as in the calculation of first bullet of Lemma 2.1.9 and by the second bullet of Lemma 2.1.9, we must have $u_{i*}^\infty[\Sigma_i] \cdot [D] > 0$. Therefore, Proposition 2.1.21 implies $[D] \cdot e > 0$.

□

2.1.6 A general property for Betti finiteness

For any contact 3-manifold of Stein, exact, or strong Betti finite type, we have the following restriction for simply connected fillings.

Proposition 2.1.23. *If (Y, ξ) is of Stein, exact, or strong Betti finite type, then (Y, ξ) has at most finitely many simply connected Stein, exact or minimal strong, respectively, fillings up to homeomorphism.*

To prove this proposition, we introduce necessary definitions and a lemma. For an integral symmetric bilinear form $Q : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$, let M_Q be a matrix presentation of Q, and let G_Q be the group presented by the matrix M_Q (i.e. the cokernel of the homomorphism $\mathbb{Z}^n \to \mathbb{Z}^n$ given by M_Q). Note that G_Q is independent of the choice of M_Q. Let r_Q, d_Q be the rank of G_Q and the number of elements of $\mathrm{Tor}\,(G_Q)$, respectively.

Though the lemma below might be known to experts, we give a proof since we could not find any reference.

Lemma 2.1.24. *For any finitely generated abelian group G and any positive integer n, there exist at most finitely many isomorphism types of integral symmetric bilinear forms such that their matrix presentations present G and have the size n.*

Proof. Let d denote the number of elements of $\mathrm{Tor}\,(G)$, and put $r = \operatorname{rank} G$. We prove the claim by induction on the number $r \geq 0$. The $r = 0$ case follows from the finiteness of isomorphism types of intersection forms with non-zero determinant. For this fact, see Theorem 1.1 in Chapter 9 of [19]. Note that $d_Q = \det(M_Q)$ in this case.

Assuming the $r = k \geq 0$ case, we prove the $r = k+1$ case. The condition $r \geq 1$ implies $\det(M_Q) = 0$ for any intersection form Q with $G_Q \cong G$. Therefore, there exist integral square matrixes A, B with size n and $|\det(A)| = |\det(B)| = 1$ such that AM_QB is a diagonal matrix which has a zero in a diagonal component. Using this fact, we easily see that there exists a primitive element $x \in \mathbb{Z}^n$ satisfying $Q(x, y) = 0$ for any $y \in \mathbb{Z}^n$. As a consequence, Q has the orthogonal sum decomposition $Q = Q|_{\langle x \rangle} \oplus Q|_H$ for some subgroup H of $\mathbb{Z}^n$. Since $G_Q \cong G_{Q|_{\langle x \rangle}} \oplus G_{Q|_H}$, we see $r_{Q|_H} = k$. Therefore, the assumption on the induction shows the $r = k+1$ case. $\square$

Proof of Proposition 2.1.23. Let (Y, ξ) be a contact 3-manifold of Stein Betti finite type (resp. exact or strong Betti finite type). The intersection form Q of any simply connected compact 4-manifold with the boundary Y satisfies $G_Q \cong H_1(Y; \mathbb{Z})$ (cf. [37]). By the assumption, there are only finitely many possible values of b_2 for Stein (resp. exact or minimal strong) fillings of (Y, ξ). Therefore, according to Lemma 2.1.24, there are only finitely many possible intersection forms of such Stein (resp. exact or minimal strong) fillings. According to a theorem of Boyer (Corollary 0.4 in [18]), for a given connected oriented closed 3-manifold and intersection form, there are at most finitely many topological types of simply connected 4-manifolds which realize the given boundary and the intersection form. Therefore the desired claim follows. $\square$

2.2 Rigidities from symplectic caps

2.2.1 Calabi-Yau caps and uniruled caps

We present the proof of Theorem 1.1.3, 1.1.8 and some immediate consequences in this section.

Proof of Theorem 1.1.3. Let (P, ω_P) be a Calabi-Yau cap with a Liouville contact form α_P for the contact manifold (Y, ξ). We must now establish that (Y, ξ) is of exact Betti finite type.

For an exact symplectic filling (N, ω_N) of (Y, ξ), we also have a Liouville contact form α_N on ∂N making $c_1(N) \cdot [(\omega_N, \alpha_N)] = 0$. Since $c_1(P) \cdot [(\omega_P, \alpha_P)] = 0$, by Lemma

2.1.8, the glued closed symplectic manifold (X,ω) has $c_1(X)\cdot[\omega]=0$. If X is not minimal, we can blow down the exceptional spheres to obtain a minimal model. Since blow-down increases $c_1(X)\cdot[\omega]$, we must have X being non-minimal uniruled or minimal symplectic Calabi-Yau.

In either case we will establish uniform bounds of Betti numbers of X which depend only on P. It then follows from the Mayer-Vietoris sequence there are uniform bounds of Betti numbers of N depending only on P.

Let us start with the case that X is non-minimal uniruled. We will first bound $b_1(X)$ and $b_3(X)$. Since $[(\omega_P,\alpha_P)]^2>0$ by Lemma 2.1.9 and $b_2^+(X)=1$, we have $b_2^+(P)=1$ and $b_2^+(N)=0$. On the other hand, we can find a closed oriented smoothly embedded surface S in P whose homology class is the Lefschetz dual of $c\,[(\omega_P,\alpha_P)]$ for some constant c, by possibly perturbing $[(\omega_P,\alpha_P)]$ to a rational class (cf. [37], Remark 1.2.4). Then $[S]^2>0$ and one should think that S is chosen before gluing with N. Let $g(S)$ denote the genus of S. By Lemma 2.1.5, the base genus of X is less than or equal to $g(S)$ and this gives an upper bound for $b_1(X)$ and hence $b_3(X)$ depending only on P.

To bound $b_2(X)=b_2^+(X)+b_2^-(X)=1+b_2^-(X)$, it suffices to bound $b_2^-(X)$. And to bound $b_2^-(X)$ it suffices to bound $c_1(X)^2$ from below. This simply follows from the just established bound on $b_1(X)$ and the relation

$$c_1(X)^2=2e(X)+3\sigma(X)=4-4b_1(X)+5b_2^+(X)-b_2^-(X)=9-4b_1(X)-b_2^-(X),$$

where we again used the fact $b_2^+(X)=1$ in the last equality.

To bound $c_1(X)^2$ we need to choose the surface S more carefully. We observe that S can be chosen such that $-c_1(P)\cdot[S]+[S]^2\geq 0$ by choosing a larger c. We further assume that $[S]^2\geq g(S)-1$, by possibly choosing an even larger c. The reason is, once S is chosen as above, we consider ν distinct copies of embedded surfaces representing $[S]$ by local perturbation of S. We assume each pair of these ν copies are intersecting transversally and positively. After resolving all the positive intersection points for these ν copies, we get an embedded surface of genus $\nu g(S)+\frac{(\nu-1)\nu}{2}[S]^2-(\nu-1)$ with self-intersection $\nu^2[S]^2$. When ν is large, we get an embedded surface with homology class being a positive multiple of $cPD[(\omega_P,\alpha_P)]$ such that the self-intersection number is greater than the genus. In summary, the surface S is assumed to satisfy $[S]^2\geq g(S)-1$

and $-c_1(P) \cdot [S] + [S]^2 \geq 0$.

Since ω_N is exact, $\omega_P - d\alpha_P^c$ in Lemma 2.1.8 viewed as a closed two form on X represents the same cohomology class as ω. Therefore, S is the Poincaré dual of $c[\omega]$ and any exceptional class pairs positively with $[S]$ in X. By [51], there is a symplectic surface $\widetilde{S} \subset X$ representing $[S]$ and the genus $\widetilde{g}$ of $\widetilde{S}$ is less than or equal to $g(S)$ according the genus minimizing property of symplectic surfaces ([46], [74]). Notice that $\tilde{g}$ is determined by the adjunction formula:

$$2\tilde{g} - 2 = -c_1(X) \cdot [S] + [S]^2 = -c_1(P) \cdot [S] + [S]^2.$$

Thus we have $\widetilde{g} > 0$. We may assume that X is not minimal, otherwise there is nothing to prove. By Lemma 2.1.6, we have $(-c_1(X) + [\widetilde{S}])^2 \geq 0$. Notice that

$$(-c_1(X) + [\widetilde{S}])^2 = c_1(X)^2 + 2(-c_1(X) \cdot [S] + [S]^2) - [S]^2 = c_1(X)^2 + 2(2\widetilde{g} - 2) - s,$$

where the second equality is by adjunction and $s = [S]^2$. Therefore,

$$c_1(X)^2 \geq s - 2(2\widetilde{g} - 2) \geq s - 2(2g(S) - 2).$$

Since $g(S)$ and s only depend on $S \subset P$, the lower bound of $c_1(X)^2$ is independent of N. As a result, $b_2(X)$ is bounded.

Next, we assume X is a minimal symplectic Calabi-Yau surface, Theorem 2.1.2 give a uniform bound on the Betti numbers of X (which are actually independent of P). Hence, $b_1(N), b_2(N), b_3(N)$ are uniformly bounded by the algebraic topology of P. It finishes the proof of the first statement. The second statement is straightforward because X must be a minimal symplectic Calabi-Yau surface.

□

To motivate the definition of a uniruled cap, we recall a Theorem in [61] and [76].

Theorem 2.2.1. *Let (X, ω) be a closed symplectic manifold with $c_1(X) \cdot [\omega] > 0$. Then, X is uniruled (i.e. rational or ruled).*

Therefore, uniruled caps are the counterpart of uniruled manifolds for compact symplectic manifolds with boundary.

Definition 2.2.2. An **adjunction cap** of a contact 3-manifold (Y, ξ) is a compact symplectic 4-manifold (P, ω) with strong concave boundary contactomorphic to (Y, ξ) such that there exist a smoothly embedded (not necessarily symplectic) genus g surface S in P with $[S]^2 \geq \max\{2g-1, 0\}$. If $[S]^2 = 0$, we further require that $[S] \in H_2(P, \mathbb{Q})$ does not lie in the image of $H_2(Y, \mathbb{Q})$ under the natural map induced by inclusion.

The definition of adjunction caps is motivated by the following proposition (Baykur informed us that he is aware of this statement but we can not find an explicit reference so we present an argument here).

Proposition 2.2.3. *Let (X, ω) be a closed symplectic manifold with a smoothly embedded (not necessarily symplectic) genus g surface S having self-intersection $[S]^2 \geq \max\{2g-1, 0\}$. If $[S]^2 = 0$, we also assume that $[S]$ represents a non-trivial class in $H_2(X, \mathbb{Q})$. Then (X, ω) is uniruled.*

Proof. If S is a sphere with $[S]^2 \geq 0$ and $[S]$ represents a non-trivial class in $H_2(X, \mathbb{Q})$, then X is uniruled, by Corollary 2 of [53].

Now suppose $g > 0$ and $[S]^2 \geq 2g-1 > 0$. Then S violates the adjunction inequality for $b_2^+ \geq 2$ symplectic 4-manifolds in [46] and [74]. Therefore we must have $b_2^+(X) = 1$. By the last paragraph in the proof of Theorem A in [51], there is a positive integer n such that $n[S]$ is represented by an embedded connected ω'-symplectic surface C for some symplectic form ω'. The genus g_C of C is given by

$$2g_C - 2 = (n[S])^2 - c_1(X)([C]) = n^2 s - c_1(X) \cdot [C],$$

where $s = [S]^2$. Notice that $n[S]$ has another smooth representative T given by taking $n-1$ perturbed copies of S with pairwise positive distinct intersections and smoothing out the intersection points. The genus of T is given by

$$g_T = ng + \frac{n(n-1)s}{2} - (n-1).$$

Therefore,

$$2g_T - 2 = 2ng + n(n-1)s - 2n = n^2 s + n(2g-2-s) < n^2 s.$$

Since C is a symplectic surface we must have $g_C \leq g_T$. However, if X is not uniruled, we have $c_1(X) \cdot [C] < 0$ by [61] and thus $g_T < g_C$. This is a contradiction and therefore X must be uniruled.

□

Proposition 2.2.4. *Let (P, ω_P) be a uniruled/adjunction cap of (Y, ξ). If (X, ω) is a closed symplectic manifold obtained by gluing a strong symplectic filling (N, ω_N) of (Y, ξ) with (P, ω_P) along (Y, ξ), then (X, ω) is uniruled.*

Proof. First assume (P, ω_P) is a uniruled cap. There is a Liouville contact form α_P making $c_1(P) \cdot [(\omega_P, \alpha_P)] > 0$. For the strong symplectic filling (N, ω_N) of (Y, ξ), we have a Liouville contact form α_N on ∂N making $(\partial N, \ker(\alpha_N))$ contactomorphic to (Y, ξ). By Lemma 2.1.8, when t is taken to be sufficiently large, $c_1(X) \cdot [\omega_X] > 0$ and hence X is uniruled, by Theorem 2.2.1.

Now, we assume (P, ω_P) is an adjunction cap. Let S be the smoothly embedded surface in P such that $s = [S]^2 \geq \max\{2g - 1, 0\}$. If S is a sphere and $[S]^2 = 0$, we have that $[S]$ represents a non-trivial class in $H_2(X, \mathbb{Q})$ by the Mayer-Vietoris sequence and the assumption that $[S] \in H_2(P, \mathbb{Q})$ does not lie in the image of $H_2(Y, \mathbb{Q})$ under the natural map. Hence, X is uniruled, by Proposition 2.2.3. □

We are ready to prove Theorem 1.1.8 as well as its adjunction cap version.

Theorem 2.2.5. *(cf. Theorem 1.1.8) Any uniruled/adjunction contact manifold (Y, ξ) is of strong Betti finite type.*

Proof. Let the glued symplectic manifold between a uniruled/adjunction cap P of Y and a strong symplectic filling N of Y be (X, ω) as before. By Proposition 2.2.4, (X, ω) is uniruled.

Following the proof of Theorem 1.1.3, we have $b_2^+(P) = 1$ and $b_2^+(N) = 0$. Moreover, there is a bound depending only on P for base genus of X.

The next step is to give a bound depending only on P for the number of exceptional curves in X. Similar to the proof of Theorem 1.1.3, we perturb $[(\omega_P, \alpha_P)]$ a little bit such that there is a positive number c and an embedded surface S representing $cPD([(\omega_P, \alpha_P)])$ and $[S]^2 \geq g(S) - 1$. This time, an exceptional class in X does not have to pair positively with $[S]$ a priori because $[S]$ is not Poincaré dual to multiple of ω_X in general. However, by Corollary 2.1.22, we know that any exceptional class indeed pairs positively with $[S]$. After this point, the rest of the proof is the same as the proof of Theorem 1.1.3.

□

Uniruled/adjunction caps behave well with respect to strong cobordisms.

Lemma 2.2.6. *Let* (W, ω_W) *be a strong cobordism with negative end* $(\partial_- W, \ker(\alpha_W^-))$ *and positive end* $(\partial_+ W, \ker(\alpha_W^+))$. *If* $(\partial_+ W, \ker(\alpha_W^+))$ *is uniruled/adjunction, then so is* $(\partial_- W, \ker(\alpha_W^-))$.

Proof. We first assume $(\partial_+ W, \ker(\alpha_W^+))$ is uniruled. Let (P, ω_P) be a uniruled cap for $(\partial_+ W, \ker(\alpha_W^+))$. By an analogue of Lemma 2.1.8, (W, ω_W) and $(P, t\omega_P)$ can be glued symplectically by inserting part of the symplectization of $(\partial_+ W, \ker(\alpha_W))$ for some large t. This glued symplectic manifold is a uniruled cap when t is sufficiently large.

When $(\partial_+ W, \ker(\alpha_W^+))$ is adjunction, we take an adjunction cap (P_+, ω_{P_+}) together with the surface S inside. The statement follows easily if $[S]^2 > 0$. When $[S]^2 = 0$, we define $P_- = W \cup_{\partial_+ W} P_+$ and we want to show that $[S] \in H_2(P_-, \mathbb{Q})$ does not lie in the image of $H_2(\partial_- W, \mathbb{Q})$. By the Mayer-Vietoris sequence and the definition of being an adjunction cap, the natural map

$$f : H_2(P_+, \mathbb{Q}) \to H_2(P_+, \partial_+ W, \mathbb{Q})$$

satisfies $f([S]) \neq 0$. Notice that, f factors through

$$H_2(P_+, \mathbb{Q}) \xrightarrow{g} H_2(P_-, \mathbb{Q}) \xrightarrow{h} H_2(P_-, \partial_- W, \mathbb{Q}) \xrightarrow{H}_2 (P_-, W, \mathbb{Q}) = H_2(P_+, \partial_+ W, \mathbb{Q})$$

In particular, $f([S]) \neq 0$ implies that the image of $g([S])$ under h is non-trivial. By the Mayer-Vietoris sequence again, $g([S])$ does not lie in the image of the natural map from $H_2(\partial_- W, \mathbb{Q})$ to $H_2(P_-, \mathbb{Q})$ and hence P_- is an adjunction cap. □

The following lemma provides a common source of uniruled and adjunction caps.

Lemma 2.2.7. *Suppose* (P, ω_P) *is a cap for* (Y, ξ) *with a closed embedded symplectic surface* S *not intersecting* ∂P, $[S]^2 \geq 0$ *and* $c_1(P)[S] > 0$. *Then, after a symplectic deformation,* (P, ω'_P) *is a uniruled and an adjunction cap for* (Y, ξ).

Proof. Since S is smoothly embedded symplectic surface with non-negative self-intersection, we can do inflation (See [47], [57]) along S to deform the symplectic form. This gives a family of symplectic form ω_t on P such that $[\omega_t] = [\omega_P] + t\iota_*(PD[S])$, where

$\iota_* : H^2(P, \partial P; \mathbb{R}) \to H^2(P; \mathbb{R})$ is the natural map and PD denotes the Lefschetz dual. Let α_P be a choice of Liouville contact form on ∂P with respect to ω_P. Then, α_P is also a Liouville contact form on ∂P with respect to ω_t since inflation is local. Then, $c_1(P) \cdot [(\omega_t, \alpha_P)] = c_1(P) \cdot ([(\omega_P, \alpha_P)] + tPD[S]) > 0$ for sufficiently large t. Hence, we can find ω'_P such that (P, ω'_P) is a uniruled cap for (Y, ξ).

On the other hand, since S is embedded and symplectic, we have $[S]^2 + 2 - 2g(S) = c_1(P)[S] \geq 1$ and $[(\omega_P, \alpha_P)] \cdot [S] > 0$. Here $g(S)$ is the genus of S. Hence $[S]^2 \geq \max\{2g - 1, 0\}$ and $[S] \in H_2(P, \partial P)$ is non-trivial. Thus $[S] \in H_2(P)$ does not lie in the image of $H_2(\partial P)$ under the natural morphism, which implies that (P, ω_P) is an adjunction cap.

□

Theorem 2.2.5 together with an argument of [29] gives the following byproduct. The reader should compare this with Corollary 1.4 in Albers-Bramham-Wendl [6].

Corollary 2.2.8. *If a contact 3-manifold is uniruled/adjunction, then any strong semi-filling of the contact manifold has connected boundary.*

Proof. Suppose, to the contrary, that (Y, ξ) admits a semifilling W with disconnected boundary. Since every contact 3-manifold has a cap with arbitrarily large b_2^+ (see [30]), by capping off the boundary components of W apart from (Y, ξ), we can construct a strong filling N of (Y, ξ) with $b_2^+(N)$ as large as we want. This contradicts to Theorem 2.2.5. □

2.2.2 $Kod(Y, \xi)$

Definition 2.2.9. For a closed co-oriented contact 3-manifold (Y, ξ), the Kodaira dimension is defined as follows.

$$Kod(Y,\xi) = \begin{cases} -\infty & \text{if it admits a uniruled cap} \\ 0 & \text{if it admits a Calabi-Yau cap but admits no uniruled cap} \\ 1 & \text{if it does not admit Calabi-Yau cap or uniruled cap} \end{cases}$$

In this terminology, Theorem 1.1.3 and 1.1.8 imply that there are Betti number bounds for exact (resp. minimal strong) filling (N, ω_N) of contact 3-manifolds (Y, ξ) with $Kod(Y, \xi) = 0$ (resp. $Kod(Y, \xi) = -\infty$).

The followings are either contained in [56] or immediate from the definition.

Lemma 2.2.10. *The contact Kodaira dimension has the following properties.*

- $Kod(Y, \xi) = -\infty$ *if* (Y, ξ) *is overtwisted.*
- *If the positive end of a strong symplectic cobordism has* $Kod = -\infty$*, then so does the negative end.*
- *If* (Y, ξ) *is co-fillable, ie. it is a connected component of the boundary of a strong semi-filling with disconnected boundary, then* $Kod(Y, \xi) \geq 0$*.*
- *If* $c_1(Y, \xi)$ *is not torsion, then* $Kod(Y, \xi) \neq 0$*.*

Proof. Notice that it is observed in [56] that any overtwisted contact structure admits a uniruled cap and hence $Kod(Y, \xi) = -\infty$ if (Y, ξ) is overtwisted .

The second bullet is Lemma 4.6 in [56].

For the third bullet, capping all other positive ends of the semi-filling we obtain a filling of (Y, ξ). If we choose the cap to have $b_2^+ \geq 2$, which always exist, then after possibly blowing down the exceptional spheres, we get a minimal filling of (Y, ξ) with $b_2^+ \geq 2$.

□

Since the unit cotangent bundle of surface with high genus is a maximal element with respect to symplectic cobordism (because it is co-fillable) and it has $Kod = 0$ (see [56]), we generally do not have the inequality in the second bullet for a strong symplectic cobordism.

In light of the bullets 1 and 2, we ask

Question 2.2.11. Suppose (Y, ξ) is a contact 3-manifold.

- If (Y, ξ) is not fillable, do we have $Kod(Y, \xi) = -\infty$?
- If (Y, ξ) is fillable, do we have $Kod(Y, \xi) \geq \min_{(X,\omega)} \kappa(X, \omega)$, where (X, ω) is a closed symplectic 4-manifold containing (Y, ξ) as a separating hypersurface?

Regarding the first bullet, a related question that was asked by Wendl is whether all non-fillable contact manifold is cobordant to an overtwisted contact manifold. The answer is negative since there are non-fillable 3-manifolds with non-trivial contact invariant [?]. However, Wendl proved that any contact 3 manifold with planar torsion is symplectic cobordant to an overtwisted contact manifold, and hence has negative Kodaira dimension [111]. Therefore, our question is a weaker version of Wendl's question.

Regarding the second bullet, it is true when $Kod(Y, \xi) = -\infty$. When $Kod(Y, \xi) = 0$, there is a subtlety if (Y, ξ) is not exactly fillable. When $Kod(Y, \xi) = 1$, the proposed inequality says that there exists a contact embedding in (X, ω) with $\kappa(X, \omega)$ at most 1. Notice that there are abundant (X, ω) with $\kappa = 1$ due to Gompf. Moreover, since many $\kappa = 1$ symplectic manifold have tori fibration, we speculate that whether any fillable contact manifold can be embedded in a $\kappa = 1$ symplectic manifold is related to the question of supporting genus of the contact manifold.

Remark 2.2.12. Zhang introduced the topological Kodaira dimension for closed 3-manifolds $\kappa(Y)$ in [?] using Thurston's 8 geometries and Perelman's solution to the geometrization conjecture. A major property is that $\kappa(Y) \geq \kappa(Y')$ if there is a nonzero degree map from Y to Y'. However, it is not clear that how $\kappa(Y)$ and $Kod(Y, \xi)$ are related. For example, $Kod(Y, \xi_{OT}) = -\infty$ for any overtwisted contact structure ξ_{OT} (Lemma 2.2.10). There are also Stein fillable (Y, ξ) with $\kappa(Y) = 1$ and $Kod(Y, \xi) = -\infty$ (Lemma 2.2.16).

One possible relation is that $\kappa(Y) \geq \min_{\xi} Kod(Y, \xi)$. This is subtle when $\kappa(Y) = 0$. In the example below we show that there are also some torus bundle Y such that $\kappa(Y) = 0$ and $Kod(Y, \xi) = 1$ for some Stein fillable contact structure ξ.

Example 2.2.13. We discuss the contact Kodaira dimension for various torus bundles over circle. In particular, we provide Stein fillable torus bundles with $Kod(Y, \xi) = 1$.

There are many torus bundles with $Kod(Y, \xi) = -\infty$. A detailed study can be found in [35]. A common feature of these examples is that they can be realized as the contact boundary of a divisor cap with the symplectic divisor being a cycle of symplectic spheres or a single symplectic torus.

To give torus bundles with $Kod(Y, \xi) = 0, 1$, we need to first review some basic facts about cusp singularities [62]. Consider an isolated cusp normal surface singularity. Its minimal resolution is a cycle of rational curves with negative definite self-intersection form Q_Γ. We denote the cycle of rational curves as D and regard it as a symplectic divisor. The boundary of a regular neighborhood $N(D)$ of D is a torus bundle over circle Y [75]. Moreover, Y is equipped with the canonical contact structure ξ as a link of complex isolated normal surface singularity. This setup should be viewed as the dual of the one in [35].

D is called *embeddable* if there is a smooth (complex) rational surface with an effective reduced anticanonical divisor whose dual graph is the same as that of D. D is called *smoothable* if the cusp singularity associated to D is smoothable as a complex surface singularity. When D is smoothable, the smoothings are diffeomorphic to the complements of $\check{D}$ in X, where $\check{D}$ is the resolution divisor of the dual cusp singularity associated to D and X is a smooth rational surface so that $\check{D}$ can be embedded into. In particular, smoothings of cusps have $b_2^+ = 1$. We can now completely determine when a cusp is smoothable/embeddable thanks to Hirzebruch-Zagier algorithm which checks embeddability combinatorically, and proofs of Looijenga conjecture which relates smoothablility of a cusp with embeddability of the dual cusp [38], [**?**].

Lemma 2.2.14. *When a cusp is embeddable and smoothable, then its link* (Y, ξ) *has* $Kod(Y, \xi) = 0$*.*

Proof. When a cusp is embeddable, say D is embedded into X, then the complement P of the regular neighborhood $N(D)$ of D in X is a Calabi-Yau cap of (Y, ξ). In this case, $Kod(Y, \xi) \leq 0$. Moreover, D being smoothable implies that (Y, ξ) admits a Stein filling (its smoothings) with $b_2^+ = 1$ so (Y, ξ) cannot admit uniruled cap and $Kod(Y, \xi) = 0$. □

One can use this explicit Calabi-Yau cap to study exact fillings of (Y, ξ) by Theorem 1.1.3 but we do not pursue it here. We remark that one can check explicitly there are many embeddable and smoothable D. On the other hand, there are also many smoothable but not embeddable cusp singularities.

Lemma 2.2.15. *When a cusp is not embeddable but smoothable, then its link* (Y, ξ) *has* $Kod(Y, \xi) = 1$

Proof. As in the proof of Lemma 2.2.14, D being smoothable implies that $Kod(Y, \xi) \neq -\infty$. Suppose (Y, ξ) admits a CY cap (P, ω_P), then $(X = P \cup N(D), \omega)$ is either a minimal Calabi-Yau manifold or a non-minimal uniruled manifold [56]. Since $c_1(D) \neq 0$, $X = P \cup N(D)$ is a non-minimal uniruled manifold. Moreover, $c_1|_P$ is torsion and the Poincare dual of $c_1|_{N(D)}$ is represented by D (this can be checked by adjunction) implies that D represents the Poincare dual of the first Chern class in (X, ω) and hence a symplectic Looijenga pair in the sense of [?]. By the classification result in [?], the embeddability of D as a symplectic divisor in a symplectic rational manifold is the same as that in the complex (Kähler) case. By the assumption that D is not embeddable in the complex sense, we get a contradiction. As a result, $Kod(Y, \xi) = 1$. $\square$

The following lemma provides another illustrative family of examples of contact manifolds with different Kodaira dimensions.

Lemma 2.2.16. *Let $(Y_{g,n}, \xi_{g,n})$ be the canonical contact 3-manifold as the boundary of a neighborhood of a symplectic surface $D_{g,n}$ with genus g and self-intersection number $n > 0$. Then,*

$$Kod(Y_{g,n}, \xi_{g,n}) = \begin{cases} -\infty & n > 2g-2 \\ 0 & n = 2g-2 \\ 1 & n < 2g-2 \end{cases}$$

Moreover, when $n \leq 2g-2$, the bound obtained for $\kappa^{(P_{g,n}, \omega_{P_{g,n}})}(N, \omega_N) \geq 0$ in Corollary 2.1.19 is sharp, where $(P_{g,n}, \omega_{P_{g,n}})$ is the neighborhood of $D_{g,n}$.

Proof. As a neighborhood of $D_{g,n}$, $(P_{g,n}, \omega_{P_{g,n}})$ is a symplectic cap of $(Y_{g,n}, \xi_{g,n})$. When $n > 2g-2$, $(P_{g,n}, \omega_{P_{g,n}})$ is a uniruled cap and hence $Kod(Y_{g,n}, \xi_{g,n}) = -\infty$. When $n = 2g-2$, $(P_{g,n}, \omega_{P_{g,n}})$ is a Calabi-Yau cap. However, $(Y_{g,n}, \xi_{g,n})$ does not admit a uniruled cap because it is co-fillable. When $n < 2g-2$, it admits no Calabi-Yau cap because $c_1(\xi_{g,n})$ is not torsion. To show that it admits no uniruled cap, we consider the Calabi-Yau cap $(P_{g,2g-2}, \omega_{P_{g,2g-2}})$ with $D_{g,2g-2}$ inside. We can do $2g-2-n$ small symplectic blowups along $D_{g,2g-2}$ and result in a symplectic surface D' of genus g and self intersection number n. The neighborhood of D' is symplectic deformation equivalent to $(P_{g,n}, \omega_{P_{g,n}})$. By deleting the neighborhood of D', we get a symplectic cobordism with negative end being $(Y_{g,2g-2}, \xi_{g,2g-2})$ and positive end being $(Y_{g,n}, \xi_{g,n})$. Since $(Y_{g,2g-2}, \xi_{g,2g-2})$ is co-fillable, so is $(Y_{g,n}, \xi_{g,n})$. Hence $Kod(Y_{g,n}, \xi_{g,n}) = 1$.

By tracing the way we obtain the bound in Corollary 2.1.19, the bound for $\kappa^{(P_{g,n}, \omega_{P_{g,n}})}(N, \omega_N) \geq 0$ is sharp if there is a minimal symplectic filling (N, ω_N) such that $\kappa(N \cup_Y P, \omega) = 0$ and every exceptional class e in $(N \cup_Y P, \omega)$ satisfy $e \cdot D_{g,n} = 1$. Notice that, there are symplectic surfaces of any genus in a $K3$ surface. Similar to above, by performing

$2g-2-n$ blow-ups along a symplectic surface of genus g, we get a symplectic surface $\overline{D}$ of genus g and self intersection number n. The complement of a neighborhood of $\overline{D}$ realizes the lower bound obtained in Corollary 2.1.19. $\square$

Chapter 3

Miscellaneous examples and applications

Having established various results in the previous chapter, the current chapter is devoted to application. The first of which partially settle a conjecture about cotangent bundle. Then we study the local behavouir of symplectic divisor in details, which will provide many handful examples for various illustrations.

3.1 Unit cotangent bundle

This section is devoted to the proof of Theorem 1.1.4. We now study the homology and cohomology of exact fillings of the unit cotangent bundles of surfaces. We start with a lemma.

Lemma 3.1.1. *There is a symplectic $K3$ surface which has g disjoint copies of embedded Lagrangian tori representing the same homology class and each of which intersects an embedded Lagrangian sphere transversally at one point.*

Proof. We first recall a construction of a $K3$ surface [37]. Let (T^4, ω_T) be the four torus equipped with the quotient Kahler form induced by the standard Kahler form on $\mathbb{C}^2$ quotient by $\mathbb{Z}^4$. There is an involution $I : T^4 \to T^4$ defined by $I(z, w) = (-z, -w)$ which has 16 fixed points. The quotient $X' = T^4/((z, w) \sim I(z, w))$ has 16 singular points.

After resolving these 16 singular points, which results in 16 spheres of self-intersection -2, we get our $K3$ surface (X, ω_X).

We can write ω_T as $dx \wedge dy + du \wedge dv$. Then there is a family of Lagrangian tori corresponding to x, u coordinates (ie. tangent space spanned by ∂_x, ∂_u) and another family corresponding to y, v coordinates. We call them xu-tori and yv-tori.

For any $g > 1$, pick g disjoint xu-tori. One can choose these g Lagrangian xu-tori in a way that they avoids the 16 fixed points and such that they descend to disjoint embedded Lagrangian tori in the quotient X'. These 16 tori in X' can then be lifted to X.

We consider a yv-torus that passes through 4 of the 16 fixed points. When descended to X', the image of this Lagrangian torus becomes a orbifold Lagrangian sphere with four orbifold points. We call the orbifold sphere yv-orbifold sphere. We claim that we can resolve the 16 orbifold points of X' in a way that yv-orbifold sphere lifts to an embedded Lagrangian sphere in X. Once we established the claim, it is clear that this Lagrangian sphere together with the g Lagrangian tori are the Lagrangians in X we want.

To prove the claim, it suffices to understand the local model for the symplectic resolution at the orbifold points. It turns out that the resolution is symplectically the same as replacing a neighborhood of an orbifold point and gluing back a neighborhood of zero section of T^*S^2, where the yv-orbifold sphere near the orbifold point is identified with a fiber of T^*S^2 after the gluing. Hence, the yv-orbifold sphere can be lifted to X by extending the fibers across the zero section of T^*S^2.

After explaining the effect of the resolution and why the claim follows from it, we now explain how the surgery goes, which turns out to be a routine calculation. We start with a model for T^*S^2. Let $U_1 = U_2 = \mathbb{C}$ and $\phi_{12} : U_1 \backslash \{0\} \to U_2 \backslash \{0\}$ be $\phi_{12}(z) = \frac{1}{z}$. Then $\Phi_{12} = (\phi_{12}^*)^{-1} : T^*(U_1\backslash\{0\}) \to T^*(U_2\backslash\{0\})$ is given by $\Phi_{12}(z, w) = (\frac{1}{z}, -\bar{z}^2 w)$, where $z \in U_1$ and $w \in \mathbb{C}$. The standard Liouville one form on T^*U_1 is given by $\lambda_1 = p_1 dq_1 + p_2 dq_2 = \frac{1}{2}(w d\bar{z} + \bar{w} dz)$, where the identification is $z = q_1 + iq_2, w = p_1 + ip_2$. The Liouville form on T^*U_2 is similar. These give the description of T^*S^2.

We define a double covering $\rho_j : U_j \times (\mathbb{C}\backslash\{0\}) \to (T^*U_j \backslash U_j)$ by

$$\rho_j(\widetilde{z}, \widetilde{w}) = (\widetilde{z}, i\overline{\widetilde{w}^2})$$

for $j = 1, 2$. This double covering can be globalized using the transition map $\widetilde{\Phi}_{12} : (U_1 \backslash \{0\}) \times (\mathbb{C} \backslash \{0\}) \to (U_2 \backslash \{0\}) \times (\mathbb{C} \backslash \{0\})$ given by

$$\widetilde{\Phi}_{12}(\widetilde{z}, \widetilde{w}) = (\frac{1}{\widetilde{z}}, i\widetilde{z}\widetilde{w})$$

in the sense that $\rho_2 \circ \widetilde{\Phi}_{12} = \Phi_{12} \circ \rho_1$ over $(U_1 \backslash \{0\}) \times (\mathbb{C} \backslash \{0\})$. Clearly, $U_j \times (\mathbb{C} \backslash \{0\})$ together with $\widetilde{\Phi}_{12}$ are charts and transition function of the $O(1)$ bundle over $\mathbb{CP}^1$ away from the zero section. Moreover, ρ_j determines a double covering to $T^*S^2 \backslash S^2$. The pull-back one form is given by

$$\widetilde{\lambda}_1 = \rho_1^* \lambda_1 = \frac{1}{2}(i(\overline{\widetilde{w}})^2 d\overline{\widetilde{z}} - i\widetilde{w}^2 d\widetilde{z})$$

We define diffeomorphisms $\Psi_1 : (\mathbb{C} \backslash \{0\}) \times \mathbb{C} \to U_1 \times (\mathbb{C} \backslash \{0\})$ by

$$\Psi_1(\hat{z}, \hat{w}) = (\frac{\hat{w}}{\hat{z}}, \hat{z})$$

and $\Psi_2 : \mathbb{C} \times (\mathbb{C} \backslash \{0\}) \to U_2 \times (\mathbb{C} \backslash \{0\})$ by

$$\Psi_2(\hat{z}, \hat{w}) = (\frac{\hat{z}}{\hat{w}}, i\hat{w})$$

which satisfy $\widetilde{\Phi}_{12} \circ \Psi_1 = \Psi_2$ over $(\mathbb{C} \backslash \{0\}) \times (\mathbb{C} \backslash \{0\})$. Hence Ψ_1, Ψ_2 together give a diffeomorphism from $\mathbb{C}^2 \backslash \{0\}$ to $O(1)$-bundle of $\mathbb{CP}^1$ away from zero section. The differential of the pull-back one form is given by

$$d(\Psi_1^* \widetilde{\lambda}_1) = -i(d\hat{z} \wedge d\hat{w} - d\overline{\hat{z}} \wedge d\overline{\hat{w}})$$

By letting $\hat{z} = u + ix, \hat{w} = y + iv$, we get $d(\Psi_1^* \widetilde{\lambda}_1) = 2(dx \wedge dy + du \wedge dv)$ which is the standard symplectic form up to a constant multiple. In particular, it coincide with the ω_T on T^4 near a fixed point. The yv-torus near a fixed point corresponds to $\hat{z} = 0$, which can be identified as a fiber of $U_2 \times (\mathbb{C} \backslash \{0\})$ under Ψ_2. Finally, note that the involution on T^4 satisfies $\rho_j \circ \Psi_j \circ I = \rho_j \circ \Psi_j$ for both $j = 1, 2$, which tells us that the yv-orbifold sphere near a orbifold point correspond to a fiber of $T^*U_2 \backslash U_2$ as claimed.

□

Proof of Theorem 1.1.4. By Lemma 3.1.1, we have g Lagrangian tori representing the same class A such that each transversally intersects a Lagrangian sphere in a symplectic

$K3$ surface X. Let the homology class of the sphere be B. We smooth out the intersection points by local Lagrangian surgery [86] and result in an embedded Lagrangian genus g surface L.

Let U be the unit cotangent disk bundle and identify it with a Weinstein neighborhood of L. Then, the complement of the interior of U gives a Calabi-Yau cap P for Y. Notice that $[L]$ is in the span of A and B and the intersection form restricted to the subspace spanned by A and B is given by

$$\begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}$$

which is equivalent to

$$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

As a result, the orthogonal complement of this subspace has intersection matrix $-2E_8 \oplus 2H$. On the other hand, $A - (g-2)B$ is orthogonal to $[L] = A + gB$. In other word, the bilinear form $-2E_8 \oplus 2H \oplus (2-2g)$ embeds into the intersection form of P, where $2 - 2g$ corresponds to the direction spanned by the class $A - (g-2)B$, which has self-intersection $2 - 2g$.

Note that $H_2(Y;\mathbb{Z}) = H^1(Y;\mathbb{Z}) = \mathbb{Z}^{2g}$. As a circle bundle, the generators of $H_2(Y;\mathbb{Z})$ are given by a loop from the base L times the circle fibers. It is the boundary of the same loop of the base times the disk fiber in U, which means that $H_2(Y;\mathbb{Z}) \to H_2(U;\mathbb{Z})$ is a zero map. From the long exact sequence

$$0 \to H_2(Y;\mathbb{Z}) \to H_2(U;\mathbb{Z}) \oplus H_2(P;\mathbb{Z}) \to H_2(X;\mathbb{Z}) = \mathbb{Z}^{22}$$

we see that $H_2(Y;\mathbb{Z}) \to H_2(P;\mathbb{Z})$ is an injection. Since $H_2(X;\mathbb{Z})$ has no torsion and $H_2(Y;\mathbb{Z})$ is free, both $H_2(U;\mathbb{Z})$ and $H_2(P;\mathbb{Z})$ do not have torsion. Hence we know that $H_2(P;\mathbb{Z}) = \mathbb{Z}^{2g+21}$ and the intersection matrix of P is given by $-2E_8 \oplus 2H \oplus (2-2g) \oplus (0)^{2g}$, where $(0)^{2g}$ corresponds to the subspace spanned by the image of $H_2(Y;\mathbb{Z})$.

From the intersection form of P, we see that P cannot embed into any uniruled manifold or minimal symplectic Calabi-Yau surface other than a homology $K3$ surface. Let N be any exact filling of Y, the glued symplectic manifold $P \cup N$ has to be a minimal integral homology $K3$ surface denoted as $\overline{X}$, by the first paragraph in the proof of Theorem 1.1.3 and Remark 2.1.3. It implies that N has Euler characteristic $e(N) = 2 - 2g$ and signature $\sigma(N) = 1$. In particular, $b_2(N) \geq 1$.

By the long exact sequence

$$H_4(\overline{X};\mathbb{Z}) \to H_3(Y;\mathbb{Z}) \to H_3(N;\mathbb{Z}) \oplus H_3(P;\mathbb{Z}) \to 0$$

and the fact that the first map is always an isomorphism, we have $H_3(N;\mathbb{Z}) = H_3(P;\mathbb{Z}) = 0$. Next the long exact sequence

$$0 \to H_2(Y;\mathbb{Z}) \to H_2(N;\mathbb{Z}) \oplus H_2(P;\mathbb{Z}) \to \mathbb{Z}^{22}$$

tell us that $H_2(N;\mathbb{Z}) = \mathbb{Z}$ or $H_2(N;\mathbb{Z}) = 0$ because $H_2(P;\mathbb{Z}) = \mathbb{Z}^{2g+21}$ and $H_2(Y;\mathbb{Z}) = \mathbb{Z}^{2g}$. The latter one is ruled out by the fact that $b_2(N) \geq 1$ so $H_2(N;\mathbb{Z}) = \mathbb{Z}$.

Note that the $-2E_8 \oplus 2H$ lattice from P embed into the intersection form of $\overline{X}$. Also, both the generator $[S_N]$ of $H_2(N;\mathbb{Z})$ and the $A-(g-2)B$ class in P lies in the orthogonal complement of $-2E_8 \oplus 2H$ in $\overline{X}$. By the classification of unimodular bilinear form, the orthogonal complement of $-2E_8 \oplus 2H$ in $\overline{X}$ is H. This together with the fact that $[S_N]$ is orthogonal to $A - (g-2)B$ in H implies that $[S_N]$ has self-intersection $k^2(2g-2)$ for some positive integer k (because the primitive class orthogonal to $A - (g-2)B$ has self-intersection $2g - 2$).

Finally, we want to determine $H_1(N;\mathbb{Z})$ using the fact that $[S_N]^2 = k^2(2g-2)$. From the long exact sequence

$$0 \to H^3(N;\mathbb{Z}) \oplus H^3(P;\mathbb{Z}) \to H^3(Y;\mathbb{Z}) \to H^4(\overline{X};\mathbb{Z})$$

and the fact that the last morphism is an isomorphism, we have $H_1(N,Y;\mathbb{Z}) = H^3(N;\mathbb{Z}) = 0$. Since we already know $H_2(N;\mathbb{Z})$ and $H_3(N;\mathbb{Z})$, the Euler characteristic of N implies the rank of $H_1(N;\mathbb{Z})$ is $2g$. On the other hand, since $H_2(N;\mathbb{Z}) = \mathbb{Z}$ is of rank one, we have that $H_2(N,Y;\mathbb{Z})$ is of rank one. In the long exact sequence

$$H_2(N;\mathbb{Z}) \to H_2(N,Y;\mathbb{Z}) \to H_1(Y;\mathbb{Z}) \to H_1(N;\mathbb{Z}) \to 0$$

the map $f : H_2(N;\mathbb{Z}) \to H_2(N,Y;\mathbb{Z})$ is given by multiplication by $[S_N]^2 = k^2(2g-2)$ on the free generators of $H_2(N;\mathbb{Z})$ and $H_2(N,Y;\mathbb{Z})$. The cokernal of f contributes a $\mathbb{Z}/(k^2(2g-2))\mathbb{Z}$ to $H_1(Y;\mathbb{Z})$. Since $H_1(Y;\mathbb{Z}) = \mathbb{Z}^{2g} \oplus \mathbb{Z}/(2g-2)\mathbb{Z}$, the torsion $\mathbb{Z}/(2g-2)\mathbb{Z}$ comes purely from the cokernal of f and $k = 1$. It also implies that $H_2(N,Y;\mathbb{Z}) = \mathbb{Z}$

and $H_1(N;\mathbb{Z}) = \mathbb{Z}^{2g}$. We have now established the integral homology type as well as the intersection form for any exact filling N of Y and hence finished the proof. □

Proof of Corollary 1.1.6. Let Y_g be the standard unit cotangent bundle of an orinetable surface of genus g. We note that Y_g admits a Weinstein filling (See Example 11.12 (2) of [20]) and hence a Stein (See Theorem 13.5 of [20]) filling.

However, Y_g has a semi-filling for $g > 1$, call it W_g, with disconnected contact boundaries (Theorem 1.1 in [65]). We can cap off the other boundary of W_g by caps with arbitrarily large b_2^+ (See e.g. [30]). Hence, after blowing down the exceptional spheres, W_g and these various caps can be glued together to give minimal strong fillings of Y_g with arbitrarily large b_2^+. Thus, Y_g is not of strong Betti finite type. □

3.2 Symplectic divisor neighborhood

Essential topological information of a symplectic divisor can be encoded by its *graph*. The graph is a weighted finite graph with vertices representing the surfaces and each edge joining two vertices representing an intersection between the two surfaces corresponding to the two vertices. Moreover, each vertex is weighted by its genus (a non-negative integer) and its self-intersection number (an integer).

If each vertex is also weighted by its symplectic area (a positive real number), then we call it an *augmented graph*. Sometimes, the genera (and the symplectic area) are not explicitly stated. For simplicity, we would like to assume the symplectic divisors are connected.

In what follow, we call a finite graph weighted by its self-intersection number and its genus (resp. and its area) with no edge coming from a vertex back to itself a graph (resp. an augmented graph). For a graph (resp. an augmented graph) Γ (resp. (Γ, a)), we use Q_Γ to denote the intersection matrix for Γ (resp. and a to denote the area weights for Γ). We denote the determinant of Q_Γ as δ_Γ. Moreover, $v_1, \dots, v_k$ are used to denote the vertices of Γ and s_i, g_i and a_i are self-intersection, genus and area of v_i, respectively.

Notice that, ω being exact on the boundary of a plumbing is equivalent to $[\omega]$ being able to be lifted to a relative cohomological class. Using Lefschetz duality, this is in turn equivalent to $[\omega]$ being able to be expressed as a linear combination $\sum\limits_{i=1}^{k} z_i[C_i]$, where $z_i \in \mathbb{R}$ and $D = C_1 \cup \cdots \cup C_k$. As a result, ω is exact on the boundary of a plumbing if and only if there exist a solution z for the equation $Q_\Gamma z = a$ (See Subsection 3.2.1 for a more detailed discussion).

We also remark that the germ of a symplectic divisor (D, ω) with ω-orthogonal intersections is uniquely determined by its augmented graph (Γ, a) (See [73] and Theorem 3.1 of [34]) and a symplectic divisor can always be made ω-orthogonal after a perturbation (See [36]).

3.2.1 Existence

In this subsection, Proposition 1.2.4 is given via two different approaches, namely, GS construction and McLean's construction.

Existence via the GS construction

Definition 3.2.1. [34] (X, ω, D, f, V) is said to be an **orthogonal neighborhood 5-tuple** if (X, ω) is a symplectic 4-manifold with D being a collection of closed symplectic surfaces in X intersecting ω-orthogonally such that $f : X \to [0, \infty)$ is a smooth function with no critical value in $(0, \infty)$ and with $f^{-1}(0) = D$, and V is a Liouville vector field on $X - D$.

Moreover, if $df(V) > 0$ (resp < 0), then (X, ω, D, f, V) is called a convex (resp concave) neighborhood 5-tuple.

In [34], Gay and Stipsicz constructed a convex orthogonal neighborhood 5-tuple (X, ω, D, f, V) when the augmented graph (Γ, a) of D satisfies the negative GS criterion. We first review their construction and an immediate consequence will be Proposition 1.2.4.

Let z be a vector solving $Q_\Gamma z = a$ with $z \in (-\infty, 0]^k$. Then $z' = (z'_1, \dots, z'_n)^T = \frac{-1}{2\pi} z$ has all entries being non-negative. We remark that the z' we use corresponds to the z in [34].

For each vertex v and each edge e meeting the chosen v, we set $s_{v,e}$ to be an integer. These integers $s_{v,e}$ are chosen such that $\sum\limits_{\text{e meeting v}} s_{v,e} = s_v$ for all v, where s_v is the self-intersection number of the vertex v. Also, set $x_{v,e} = s_{v,e}z'_v + z'_{v'}$, where v' is the other vertex of the edge e.

For each edge $e_{\alpha\beta}$ of Γ joining vertices v_α and v_β, we construct a local model $N_{e_{\alpha\beta}}$ as follows. Let $\mu : \mathbb{S}^2 \times \mathbb{S}^2 \to [z'_\alpha, z'_\alpha + 1] \times [z'_\beta, z'_\beta + 1]$ be the moment map of $\mathbb{S}^2 \times \mathbb{S}^2$ onto its image. We use p_1 for coordinate in $[z'_\alpha, z'_\alpha + 1]$, p_2 for coordinate in $[z'_\beta, z'_\beta + 1]$ and $q_i \in \mathbb{R}/2\pi$ be the corresponding fibre coordinates so $\theta = p_1 dq_1 + p_2 dq_2$ gives a primitive of the symplectic form $dp_1 \wedge dq_1 + dp_2 \wedge dq_2$ on the preimage of the interior of the moment image.

Fix a small $\epsilon > 0$ and let $D_1 = \mu^{-1}(\{z'_\alpha\} \times [z'_\beta, z'_\beta + 2\epsilon])$ be a symplectic disc. Let also $D_2 = \mu^{-1}([z'_\alpha, z'_\alpha + 2\epsilon] \times \{z'_\beta\})$ be another symplectic disc meeting D_1 ω-orthogonal at the point $\mu^{-1}(\{z'_\alpha\} \times \{z'_\beta\})$.

Our local model $N_{e_{\alpha\beta}}$ is going to be the preimage under μ of a region containing $\{z'_\alpha\} \times [z'_\beta, z'_\beta + 2\epsilon] \cup [z'_\alpha, z'_\alpha + 2\epsilon] \times \{z'_\beta\}$.

A sufficiently small δ will be chosen. For this δ, let $R_{e_{\alpha\beta},v_\alpha}$ be the closed parallelogram with vertices $(z'_\alpha, z'_\beta + \epsilon), (z'_\alpha, z'_\beta + 2\epsilon), (z'_\alpha + \delta, z'_\beta + 2\epsilon - s_{v_\alpha,e_{\alpha\beta}}\delta), (z'_\alpha + \delta, z'_\beta + \epsilon - s_{v_\alpha,e_{\alpha\beta}}\delta)$. Also, $R_{e_{\alpha\beta},v_\beta}$ is defined similarly as the closed parallelogram with vertices $(z'_\alpha + \epsilon, z'_\beta), (z'_\alpha + 2\epsilon, z'_\beta), (z'_\alpha + 2\epsilon - s_{v_\beta,e_{\alpha\beta}}\delta, z'_\beta + \delta), (z'_\alpha + \epsilon - s_{v_\beta,e_{\alpha\beta}}\delta, z'_\beta + \delta)$. We extend the right vertical edge of $R_{e_{\alpha\beta},v_\alpha}$ downward and extend the top horizontal edge of $R_{e_{\alpha\beta},v_\beta}$ to the left until they meet at the point $(z'_\alpha + \delta, z'_\beta + \delta)$. Then, the top edge of $R_{e_{\alpha\beta},v_\alpha}$, the right edge of $R_{e_{\alpha\beta},v_\beta}$, the extension of right edge of $R_{e_{\alpha\beta},v_\alpha}$, the extension of top edge of $R_{e_{\alpha\beta},v_\beta}$, $\{z'_\alpha\} \times [z'_\beta, z'_\beta + 2\epsilon]$ and $[z'_\alpha, z'_\alpha + 2\epsilon] \times \{z'_\beta\}$ enclose a region. After rounding the corner symmetrically at $(z'_\alpha + \delta, z'_\beta + \delta)$, we call this closed region R. Now, we set $N_{e_{\alpha\beta}}$ to be the preimage of R under μ.

On the other hand, for each vertex v_α, we also need to construct a local model N_{v_α}. Let g_α be the genus of v_α. We can form a genus g_α compact Riemann surface Σ_{v_α} such that the boundary components one to one correspond to the edges meeting v_α. We denote the boundary component corresponding to $e_{\alpha\beta}$ by $\partial_{e_{\alpha\beta}}\Sigma_{v_\alpha}$. There exists a symplectic form $\bar{\omega}_{v_\alpha}$ and a Liouville vector field $\bar{X}_{v_\alpha}$ on Σ_{v_α} such that when we give the local coordinates $(t, \vartheta_1) \in (x_{v_\alpha,e_{\alpha\beta}} - 2\epsilon, x_{v_\alpha,e_{\alpha\beta}} - \epsilon] \times \mathbb{R}/2\pi\mathbb{Z}$ to the neighborhood of the boundary component $\partial_{e_{\alpha\beta}}\Sigma_{v_\alpha}$, we have that $\bar{\omega}_{v_\alpha} = dt \wedge d\vartheta_1$ and $\bar{X}_{v_\alpha} = t\partial_t$. Now, we form the local model $N_{v_\alpha} = \Sigma_{v_\alpha} \times \mathbb{D}^2_{\sqrt{2\delta}}$ with product symplectic form $\omega_{v_\alpha} = \bar{\omega}_{v_\alpha} + r dr \wedge d\vartheta_2$ and Liouville vector field $X_{v_\alpha} = \bar{X}_{v_\alpha} + (\frac{r}{2} + \frac{z'_{v_\alpha}}{r})\partial_r$, where (r, ϑ_2) is the standard polar coordinates on $\mathbb{D}^2_{\sqrt{2\delta}}$.

Finally, the GS construction is done by gluing these local models appropriately. To be more precise, the preimage of $R_{e_{\alpha\beta},v_\alpha}$ of $N_{e_{\alpha\beta}}$ is glued via a symplectomorphism preserving the Liouville vector field to $[x_{v_\alpha,e_{\alpha\beta}} - 2\epsilon, x_{v_\alpha,e_{\alpha\beta}} - \epsilon] \times \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{D}^2_{\sqrt{2\delta}}$ of N_{v_α} and other matching pieces are glued similarly. When $\delta > 0$ is chosen sufficiently small, this glued manifold give our desired convex orthogonal neighborhood 5-tuple with the symplectic divisor having graph Γ.

We remark the whole construction works exactly the same if all entries of z' are negative. In this case, all entries of z are positive and we get the desired concave orthogonal neighborhood 5-tuple if (Γ, a) satisfies the positive GS criterion. Now, if we have an ω'-orthogonal divisor (D', ω') with augmented graph (Γ, a), which is the same as that of the concave orthogonal neighborhood 5-tuple (X, ω, D, f, V), then there exist neighborhood N' of D' symplectomorphic to a neighborhood of D and sending D' to D (See [73] and [34]). Therefore, a concave neighborhood of D in N give rise to a concave neighborhood of D' in N'. This finishes the proof of Proposition 1.2.4.

Existence in Higher Dimensions via Wrapping Numbers

To understand the geometrical meaning of the GS criteria, we recall wrapping numbers from [72] and [71]. Then, another construction for Proposition 1.2.4 is given.

Let $(D, P(D), \omega)$ be a plumbing of a symplectic divisor. If ω is not exact on the boundary of D, then there is no Liouville flow X near $\partial P(D)$ such that $\alpha = i_X\omega$ and $d\alpha = \omega$. Therefore, D does not have concave nor convex neighborhood.

When ω is exact on the boundary, let α be a 1-form on $P(D) - D$ such that $d\alpha = \omega$. Let α_c be a 1-form on $P(D)$ such that it is 0 near D and it equals α near $\partial P(D)$. Note that $[\omega - d\alpha_c] \in H^2(P(D), \partial P(D); \mathbb{R})$. Let its Lefschetz dual be $-\sum_{i=1}^{k} \lambda_i [C_i] \in H_2(P(D); \mathbb{R})$. We call λ_i the wrapping number of α around C_i.

Also, there is another equivalent interpretation of wrapping numbers. If we symplectically embed a small disc to $P(D)$ meeting C_i positively transversally at the origin of the disc, then the pull-back of α equals $\frac{r^2}{2} d\vartheta + \frac{\lambda_i}{2\pi} d\vartheta + df$, where (r, ϑ) is the polar coordinates of the disc and f is some function defined on the punctured disc. (See the paragraph before Lemma 5.17 of [71]).

From this point of view, we can see that the z_i's in the GS criteria are minus of the wrapping numbers $-\lambda_i$'s for a lift of the symplectic class $[\omega] \in H^2(P(D);\mathbb{R})$ to $H^2(P(D), \partial P(D);\mathbb{R})$. In particular, Q being non-degenerate is equivalent to lifting of symplectic class being unique, which is in turn equivalent to the connecting homomorphism $H^1(\partial P(D);\mathbb{R}) \to H^2(P(D), \partial P(D);\mathbb{R})$ is zero. When Q is degenerate and for a fixed ω, the equation $Qz = a$ having no solution for z is equivalent to $\omega|_{\partial P(D)}$ being not exact. Similarly, when $Qz = a$ has a solution for z, then the solution is unique up to the kernel of Q, which corresponds to the unique lift of ω up to the image of the connecting homomorphism $H^1(\partial P(D);\mathbb{R}) \to H^2(P(D), \partial P(D);\mathbb{R})$.

To summarize, we have

Lemma 3.2.2. *Let (D, ω) be a symplectic divisor. Then, lifts $[\omega - d\alpha_c] \in H^2(P(D), \partial P(D);\mathbb{R})$ of the symplectic class $[\omega]$ are in one-to-one correspondence to the solution z of $Q_D z = a$ via the minus of Lefschetz dual $PD([\omega - d\alpha_c]) = -\sum_{i=1}^{k} \lambda_i [C_i]$ and $z_i = -\lambda_i$.*

Proposition 1.2.4 can be generalized to arbitrary dimension if we apply the constructions in the recent paper of McLean [72]. We first recall an appropriate definition of a symplectic divisor in higher dimension (See [72] or [71]).

Let (W^{2n}, ω) be a symplectic manifold with or without boundary. Let $C_1, \dots, C_k$ be real codimension 2 symplectic submanifolds of W that intersect ∂W trivially (if any). Assume all intersections among C_i are transversal and positive, where positive is defined in the following sense.

(i) For each $I \subset \{1, \dots, k\}$, $C_I = \cap_{i \in I} C_i$ is a symplectic submanifold.

(ii) For each $I, J \subset \{1, \dots, k\}$ with $C_{I \cup J} \neq \emptyset$, we let N_1 be the symplectic normal bundle of $C_{I\cup J}$ in C_I and N_2 be the symplectic normal bundle of $C_{I \cup J}$ in C_J. Then, it is required that the orientation of $N_1 \oplus N_2 \oplus TC_{I\cup J}$ is compatible with the orientation of $TW|_{C_{I\cup J}}$.

We remark that the condition (ii) above guarantees that no three distinct C_i intersect at a common point when W is four dimensions. Therefore, this higher dimension definition coincides with the one we use in four dimension. To make our thesis more consistent, in higher dimension, we call $D = C_1 \cup \dots \cup C_k$ a symplectic divisor if D is moreover connected and the orientation of each C_i is induced from $\omega^{n-1}|_{C_i}$.

Now, for each i, let N_i be a neighborhood of C_i such that we have a smooth projection

$p_i : N_i \to C_i$ with a connection rotating the disc fibers. Hence, for each i, we have a well-defined radial coordinate r_i with respect to the fibration p_i such that C_i corresponds to $r_i = 0$.

Let $\bar{\rho} : [0, \delta) \to [0, 1]$ be a smooth function such that $\bar{\rho}(x) = x^2$ near $x = 0$ and $\bar{\rho}(x) = 1$ when x is close to δ. Moreover, we require $\bar{\rho}'(x) \geq 0$.

A smooth function $f : W - D \to \mathbb{R}$ is called compatible with D if $f = \sum_{i=1}^{k} \log(\bar{\rho}(r_i)) + \bar{\tau}$ for some smooth $\bar{\tau} : W \to \mathbb{R}$ and choice of $\bar{\rho}(r_i)$ as above.

Here is the analogue of Proposition 1.2.4 in arbitrary even dimension.

Proposition 3.2.3 (cf. Propositon 4.1 of [72])**.** *Suppose $f : W^{2n} - D \to \mathbb{R}$ is compatible with D and D is a symplectic divisor with respect to ω. Suppose $\theta \in \Omega^1(W^{2n} - D)$ is a primitive of ω on $W^{2n} - D$ such that it has positive (resp. negative) wrapping numbers for all $i = 1, \dots, k$. Then, there exist $g : W^{2n} - D \to \mathbb{R}$ such that $df(X_{\theta+dg}) > 0$ (resp. $df(-X_{\theta+dg}) > 0$) near D, where $X_{\theta+dg}$ is the dual of $\theta + dg$ with respect to ω.*

In particular, D is a convex (resp. concave) divisor.

This is essentially contained in Propositon 4.1 of [72]–the only new statement is the last sentence. And Proposition 4.1 in [72] is stated only for the case in which wrapping numbers are all positive, however, the proof there goes through without additional difficulty for the other case. We give here the most technical lemma adapted to the case of negative wrapping numbers and ambient manifold being dimension four for the sake of completeness.

We remark that the ω-orthogonal intersection condition is not required in his construction.

Lemma 3.2.4 (cf. Lemma 4.5 of [72])**.** *Given $D = D_1 \cup D_2 \subset (U, \omega)$, where D_1 and D_2 are symplectic 2-discs intersecting each other positively and transversally at a point p. Suppose $\theta \in \Omega^1(U - D)$ is a primitive of ω on $U - D$ such that it has negative wrapping numbers with respect to both D_1 and D_2. Then there exists g such that for all smooth functions $f : U - D \to \mathbb{R}$ compatible wtih D, we have that $df(-X_{\theta+dg}) > c_f \|\theta + dg\| \|df\|$ near D, where $c_f > 0$ is a constant depending on f.*

Also $c_1 \|db\| < \|\theta + dg\| < c_2 \|db\|$ near D for some smooth function b compatible with D, where c_1 and c_2 are some constants.

Proof. By possibly shrinking U, we give a symplectic coordinate system at the intersection point p such that $D_1 = \{x_1 = y_1 = 0\}$ and 0 corresponds to p. Let π_1 be the projection to the x_2, y_2 coordinates. Write $x_1 = r\cos\vartheta$ and $y_1 = r\sin\vartheta$ and let $\tau = \frac{r^2}{2}$. Let $U_1 = U - D_1$ and $\tilde{U_1}'$ be the universal cover of U_1 with covering map α. Give $\tilde{U_1}'$ the coordinates $(\tilde{x_1}, \tilde{y_1}, \tilde{x_2}, \tilde{y_2})$ coming from pulling back the coordinates of $(\tau, \vartheta, x_2, y_2)$ by the covering map. Then, the pulled back symplectic form on $\tilde{U_1}'$ is given by $d\tilde{x_1} \wedge d\tilde{y_1} + d\tilde{x_2} \wedge d\tilde{y_2}$. Hence, we can enlarge $\tilde{U_1}'$ across $\{\alpha^*\tau = \tilde{x_1} = 0\}$ to $\tilde{U_1}$ by identifying $\tilde{U_1}'$ as an open subset of $\mathbb{R}^4$ with standard symplectic form.

Let $L_{\vartheta_0} = \{(\tau, \vartheta_0, x_2, y_2) \in \tilde{U_1} | \tau, x_2, y_2 \in \mathbb{R}\}$, which is a 3-manifold depending on the choice of ϑ_0. Let T be the tangent space of D_2 at 0 and identify it as a 2 dimensional linear subspace in (x_1, y_1, x_2, y_2) coordinates. Then, $l_{\vartheta_0} = \alpha(L_{\vartheta_0} \cap \tilde{U_1}') \cap T$ is an open ray starting from 0 in U because D_1 and D_2 are assumed to be transversal. If we pull back the tangent space of l_{ϑ_0} to the $(\tilde{x_1}, \tilde{y_1}, \tilde{x_2}, \tilde{y_2})$ coordinates in $\tilde{U_1}'$, it is spanned by a vector of the form $(1, 0, a_{\vartheta_0}, b_{\vartheta_0})$ for some $a_{\vartheta_0}, b_{\vartheta_0}$. We identify this vector as a vector at $(0, \vartheta_0, 0, 0)$ and call it v_{ϑ_0}. Notice that the ω-dual of v_{ϑ_0} is $d\tilde{y_1} - b_{\vartheta_0} d\tilde{x_2} + a_{\vartheta_0} d\tilde{y_2}$, for all $\vartheta_0 \in [0, 2\pi]$.

Let X_1 be a vector field on $\tilde{U_1}$ such that $X_1 = \frac{\lambda_1}{2\pi} v_{\vartheta_0}$ at $(\tilde{x_1}, \tilde{y_1}, \tilde{x_2}, \tilde{y_2}) = (0, \vartheta_0, 0, 0)$ for all $\vartheta_0 \in [0, 2\pi]$, where λ_1 is the wrapping number of θ with respect to D_1. We also require the ω-dual of X_1 to be a closed form on $\tilde{U_1}$. This can be done because the ω-dual of X_1 restricted to $\{\tilde{x_1} = \tilde{x_2} = \tilde{y_2} = 0\}$ is closed. Furthermore, we can also assume X_1 is invariant under the $2\pi\mathbb{Z}$ action on $\tilde{y_1}$ coordinate. Note that $d\tilde{x_1}(X_1) = \frac{\lambda_1}{2\pi} < 0$ at $(0, \vartheta_0, 0, 0)$ for all ϑ_0 so we have $d\tilde{x_1}(X_1) < 0$ near $\{\tilde{x_1} = \tilde{x_2} = \tilde{y_2} = 0\}$.

Let the ω dual of X_1 be $\tilde{q_1}$, which is exact as it is closed in $\tilde{U_1}$. Now, $\tilde{q_1}$ can be descended to a closed 1-form q_1 in U_1 under α with wrapping numbers λ_1 and 0 with respect to D_1 and D_2, respectively. We can construct another closed 1-form q_2 in U_2 in the same way as q_1 with D_1 and D_2 swapped around. Notice that $q_1 + q_2$ is a well-defined closed 1-form in $U - D$ with same wrapping numbers as that of θ. Let $\theta' = \theta_1 + q_1 + q_2$ be such that $d(\theta') = \omega$ and θ_1 has bounded norm. Since θ' has the same wrapping numbers as that of θ, we can find a function $g : U - D \to \mathbb{R}$ such that $\theta' = \theta + dg = \theta_1 + q_1 + q_2$.

We want to show that $df(-X_{\theta+dg}) > c_f \|\theta + dg\| \|df\|$ near D. It suffices to show that $df(-X_{q_1+q_2}) > c_f \|q_1 + q_2\| \|df\|$ near D as $\|\theta_1\|$ is bounded. Since $f = \sum_{i=1}^{n} \log(\rho(r_i)) + \bar{\tau}$

for some smooth $\bar{\tau} : M \to \mathbb{R}$, it suffices to show that $\sum_{i=1}^{2}(d\log(x_i'^2 + y_i'^2))(-X_{q_1+q_2}) > c_f\|q_1+q_2\|\|\sum_{i=1}^{2}(d\log(x_i'^2+y_i'^2))\|$, where (x_1', y_1', x_2', y_2') are smooth coordinates adapted to the fibrations used to define compatibility.

To do this, we pick a sequence of points $p_k \in U - D$ converging to 0. Then $\frac{X_{q_1}}{\|q_1\|}$ at p_k converges (after passing to a subsequence) to a vector transversal to D_1 but tangential to D_2. The analogous statement is true for $\frac{X_{q_2}}{\|q_2\|}$. Hence we have $\sum_{i=1}^{2}(d\log(x_i'^2 + y_i'^2))(-X_{q_1+q_2}) > c_f \sum_{i=1}^{2} \|q_i\|\|(d\log(x_i'^2 + y_i'^2))\|$ and thus get the desired estimate (See [72] for details).

On the other hand, $c_1\|db\| < \|\theta + dg\| < c_2\|db\|$ near D for some smooth function b compatible with D is easy to achieve by taking $b = C\sum_{i=1}^{2}(d\log(x_i'^2 + y_i'^2))$ near D. $\square$

Careful readers will find that when constructing a convex neighborhood, the GS construction works when wrapping numbers are all non-negative while McLean's constructions work only when wrapping numbers are all positive. We end this subsection with a lemma saying that the GS construction is not really more powerful than McLean's construction in dimension four.

Lemma 3.2.5. *Let (D^{2n-2}, ω) be a symplectic divisor with $n > 1$. Suppose ω is exact on the boundary with α being a primitive on $P(D) - D$. If the wrapping numbers of α are all non-negative, then all are positive.*

Proof. Suppose the wrapping numbers λ_i of α are all non-negative and $\lambda_1 = 0$. Then, α can be extend over $C_1 - \cup_{1\in I,|I|\geq 2}C_I$, where we recall C_I with $1 \in I$ are the symplectic submanifold of C_1 induced from intersection with other C_i. Therefore,

$$\int_{C_1} \omega^{n-1} = \int_{P(\cup_{1\in I,|I|\geq 2}C_I)} \omega^{n-1} - \int_{\partial P(\cup_{1\in I,|I|\geq 2}C_I)} \alpha \wedge \omega^{n-2},$$

where $P(\cup_{1\in I,|I|\geq 2}C_I)$ is a regular neighborhood of $\cup_{1\in I,|I|\geq 2}C_I$ in C_1. We claim that $\int_{P(\cup_{1\in I,|I|\geq 2}C_I)} \omega^{n-1} - \int_{\partial P(\cup_{1\in I,|I|\geq 2}C_I)} \alpha \wedge \omega^{n-2} \leq 0$ so we will arrive at a contradiction.

We first assume that if $1 \in I$, then $C_I = \emptyset$ except C_1 and $C_{\{1,2\}}$. As a submanifold of C_1, $P(\cup_{1\in I,|I|\geq 2}C_I) = P(C_{\{1,2\}})$ can be symplectically identified with a closed 2-disc bundle over $C_{\{1,2\}}$. For each fibre, $\alpha|_{\text{fibre}} = \frac{r^2}{2}d\vartheta + \frac{\lambda_2}{2\pi}d\vartheta + df$, where (r, ϑ) is the polar coordinates of the disc and f is a smooth function defined on the punctured disc. Without loss of generality, we can assume $P(C_{\{1,2\}})$ is taken such that symplectic connection rotates the fibre and we have a well defined one form $\frac{\lambda_2}{2\pi}d\vartheta$ on $P(C_{\{1,2\}}) - C_{\{1,2\}}$. Then, $\alpha - \frac{\lambda_2}{2\pi}d\vartheta - df$ can be defined over $P(C_{\{1,2\}})$ for some f defined on $P(C_{\{1,2\}}) - C_{\{1,2\}}$ and

$$\begin{aligned}\int_{\partial P(C_{\{1,2\}})}(\alpha - \frac{\lambda_2}{2\pi}d\vartheta)\wedge\omega^{n-2} &= \int_{\partial P(C_{\{1,2\}})}(\alpha - \frac{\lambda_2}{2\pi}d\vartheta - df)\wedge\omega^{n-2}\\ &= \int_{P(C_{\{1,2\}})} rdr\wedge d\vartheta\wedge\omega^{n-2}\\ &= \int_{P(C_{\{1,2\}})}\omega^{n-1}.\end{aligned}$$

Therefore,

$$\begin{aligned}\int_{P(C_{\{1,2\}})}\omega^{n-1} - \int_{\partial P(C_{\{1,2\}})}\alpha\wedge\omega^{n-2} &= -\int_{\partial P(C_{\{1,2\}})}\frac{\lambda_2}{2\pi}d\vartheta\wedge\omega^{n-2}\\ &= -\lambda_2\int_{C_{\{1,2\}}}\omega^{n-2}\leq 0\end{aligned}$$

It is not hard to see that this argument can be generalized to more than two C_I being non-empty, where $1 \in I$. This completes the proof. □

3.2.2 Obstruction

In this subsection we prove Theorem 1.2.5 and Theorem 3.2.11. We first prove Theorem 1.2.5, in which (D, ω) is assumed to be ω-orthogonal. Then the proof for Theorem 3.2.11, which is similar, is sketched.

Energy Lower Bound

Given an ω-orthogonal symplectic divisor $D = C_1 \cup \cdots \cup C_k$ in a 4-manifold (W, ω), for each i, let N_i be a neighborhood of C_i together with a symplectic open disk fibration $p_i : N_i \to C_i$ such that the symplectic connection induced by ω-orthogonal subspace of the fibers rotates the symplectic disc fibres. Hence, for each i, we have a well-defined radial coordinate r_i with respect to the fibration p_i such that C_i corresponds to $r_i = 0$. Also, N_i are chosen such that the disk fibers are symplectomorphic to the standard open symplectic disk with radius ϵ_i. We also assume $\min_{i=1}^k r_i = r_1$ (or simply $r_1 = r_2 = \cdots = r_k$). Moreover, we require $p_{ij} : N_i \cap N_j \to C_{ij}$ to be a symplectic $\mathbb{D}^2 \times \mathbb{D}^2$ fibration such that $p_i|_{N_i \cap N_j}$ is the projection to the first factor and $p_j|_{N_i\cap N_j}$ is the projection to the second factor. Such choice of p_i and N_i exist (See Lemma 5.14 of [71]).

Lemma 3.2.6. *Let $(D = C_1 \cup \cdots \cup C_k, \omega)$ be a symplectic divisor with p_i and N_i as above. There exist an ω-compatible almost complex structure J_N on $N = \cup_{i=1}^k N_i$ such that C_i are J_N-holomorphic, the projections p_i are J_N-holomorphic and the fibers are J_N-holomorphic.*

Proof. Using p_{ij}, we can define a product complex structure on $\mathbb{D}^2 \times \mathbb{D}^2 = N_i \cap N_j$. Since p_{ij} are compatible with p_i and p_j, we can extend this almost complex structure such that $J|_{C_i}$ and $J|_{C_j}$ are complex structures, $(p_l)_*J = J|_{C_l}$ and $J(r_l\partial_{r_l}) = \partial_{\vartheta_l}$ for $l = i, j$, where (r_i, ϑ_i) and (r_j, ϑ_j) are polar coordinates of the disk fiber for p_i and p_j, respectively.

Although ϑ_i and ϑ_j are not well-defined if the disk bundle has non-trivial Euler class, ∂_{ϑ_l} are well-defined for $k = i, j$. Since the almost complex structure J is 'product-like', J is compatible with the symplectic form ω. We call this desired almost complex structure J_N. $\square$

Now, we consider a partial compactification of $N = \cup_{i=1}^k N_i$ in the following sense. Consider a local symplectic trivialization of the symplectic disk bundle induced by p_1, $B_1 \times \mathbb{D}^2$, where $B_1 \subset C_1$ is symplectomorphic to the standard symplectic closed disk with radius τ. We assume that ϵ_1 is sufficiently small with respect to τ (i.e. $\epsilon_1 << \tau$). We recall that $\mathbb{D}^2$ is equipped with a standard symplectic form with radius ϵ_1. Choose a symplectic embedding of $\mathbb{D}^2_{\epsilon_1}$ to S^2_ϵ with ϵ slightly large than ϵ_1, where S^2_ϵ is a symplectic sphere of area $\pi\epsilon^2$. We glue $\cup_{i=1}^k N_i$ with $B_1 \times S^2_\epsilon$ along $B_1 \times \mathbb{D}^2_{\epsilon_1}$ with the identification above. This glued manifold is called $\bar{N}$ and the compatible almost complex structure constructed above can be extended to $\bar{N}$, which we denote as $J_{\bar{N}}$. We further require that $\{q\} \times S^2_\epsilon$ is $J_{\bar{N}}$-holomorphic for every $q \in B_1$.

We want to get an energy uniform lower bound for J-holomorphic curves representing certain fixed homology class, for those J that are equal to $J_{\bar{N}}$ away from a neighborhood of the divisor D. Let $N^\delta = \cup_{i=1}^k \{r_i \leq \delta\} \subset \bar{N}$, where r_i are the radial coordinates for the disk fibration p_i.

Lemma 3.2.7. *Let $\delta_{\min} > 0$ be small and $\delta_{\max} > 0$ be slightly less than ϵ_1 and τ sufficiently large compare to ϵ_1. Let $q_\infty \in B_1 \times S^2_\epsilon$ be a point in $\bar{N} - N$ and the first coordinate of which is the center of B_1. Let J be an ω-compatible almost complex structure such that $J = J_{\bar{N}}$ on $\bar{N} - N^{\frac{\delta_{\min}}{2}}$. If $u : \mathbb{C}P^1 \to \bar{N}$ is a non-constant J holomorphic curve passing through q_∞, then either $u^*\omega([\mathbb{C}P^1]) \geq \pi((\epsilon - \delta_{\min})^2 + \frac{(\delta_{\max}-\delta_{\min})^2}{4})$ or the image of u stays inside $N^{\delta_{\max}} \cup B_1^{\frac{\tau}{2}} \times S^2_\epsilon$, where $B_1^{\frac{\tau}{2}}$ is a closed sub-disk of B_1 with the same center but radius $\frac{\tau}{2}$.*

Proof. Let us assume $u^*\omega([\mathbb{C}P^1]) < \pi((\epsilon - \delta_{\min})^2 + \frac{(\delta_{\max}-\delta_{\min})^2}{4})$. Otherwise, we have nothing to prove. Also, we can assume u intersect $\partial N^{\delta_{\min}}$ and $\partial N^{\delta_{\max}}$ transversally, by slightly adjusting $\delta_{\min}$ and $\delta_{\max}$. Passing to the underlying curve if necessary, we can also assume u is somewhere injective.

Consider the portion of u inside $Int(B_1^{\frac{\tau}{4}}) \times S^2_\epsilon - N^{\delta_{\min}}$. Let Σ_{q_∞} be the connected component of $u^{-1}(Int(B_1^\tau) \times S^2_\epsilon - N^{\delta_{\min}})$ containing q_∞. There is a symplectic 4-ball centered at q_∞ with radius $\epsilon - \delta_{\min}$, $B^4_{\epsilon-\delta_{\min}}(q_\infty) \subset Int(B_1^{\frac{\tau}{2}}) \times S^2_\epsilon - N^{\delta_{\min}}$. By monotonicty lemma, $u|_{\Sigma_{q_\infty}}$ contributes to area of at least $\pi(\epsilon - \delta_{\min})^2$ because u is holomorphic in this region and u is not constant.

Let $\bar{p_1}$ be the projection to the first factor for $Int(B_1^\tau) \times S^2_\epsilon - N^{\delta_{\min}}$. We claim that the image of $\bar{p_1} \circ u|_{\Sigma_{q_\infty}}$ stays inside $B_1^{\frac{\tau}{2}}$. Suppose not, we let $\Sigma_{q_\infty^+}$ to be the component of $u^{-1}(Int(B_1^\tau) \times S^2_\epsilon - N^{\frac{\delta_{\min}}{2}})$ containing q_∞ (i.e. $\Sigma_{q_\infty^+}$ is an extension of Σ_{q_∞}). We can apply monotonicity lemma many h times to balls of radius $\frac{\delta_{\min}}{2}$ to $u|_{\Sigma_{q_\infty^+}}$, where $h\pi\frac{\delta_{\min}^2}{4} \geq \pi((\epsilon - \delta_{\min})^2 + \frac{(\delta_{\max}-\delta_{\min})^2}{4})$, to show that $u|_{\Sigma_{q_\infty^+}}$ has energy larger than $\pi((\epsilon - \delta_{\min})^2 + \frac{(\delta_{\max}-\delta_{\min})^2}{4})$ and hence a contradiction (We require that τ is chosen after $\delta_{\min}$ so we can choose τ such that the corresponding h satisfies the required inequaility).

If the image of u does not stay inside $N^{\delta_{\max}} \cup B_1^{\frac{\tau}{2}} \times S^2_\epsilon$, then there is a point q_* outside this region, lying inside the image of u and $N^{\epsilon_1} - N^{\delta_{\max}}$. We can assume q_* is an injectivity point of u. In particular, it also means that $u^{-1}(\bar{N} - N^{\delta_{\min}})$ is disconnected. Consider the connected component Σ of $u^{-1}(\bar{N} - N^{\delta_{\min}})$ which contains the preimage of q_* under u. Note that $\Sigma \cap \Sigma_{q_\infty} = \emptyset$.

Using one of the projections p_i, depending on the position of q_*, we can identify a neighborhood of q_* as $Int(\mathbb{D}^2_{\delta_{\max}-\delta_{\min}}) \times (Int(\mathbb{D}^2_{\epsilon_1}) - \mathbb{D}^2_{\delta_{\min}})$, where $\mathbb{D}^2_{\delta_{\min}}$ has the same center as $\mathbb{D}^2_{\epsilon_1}$ and they are closed disks with radii $\delta_{\min}$ and ϵ_1, respectively. We call this neighborhood N_{q_*}. Also, we still have $J = J_{\bar{N}}$ and $J_{\bar{N}}$ is the standard split complex structure in N_{q_*}. Similar as before, we can find a symplectic 4-ball inside N_{q_*} with radius $\frac{(\delta_{\max}-\delta_{\min})}{2}$ such that $u|_\Sigma$ passes through the center. Therefore, we can apply monotonicity lemma again to show that there is another contribution of $\pi\frac{(\delta_{\max}-\delta_{\min})^2}{4}$ to the energy of u. Hence, we get a contradiction. $\square$

Theorem 1.2.5 and Theorem 3.2.11

We recall the terminology *GW triple* used in [72]. For a symplectic manifold (W, ω) (possibly non-compact), a homology class $[A] \in H_2(W; \mathbb{Z})$ and a family of compatible almost complex structures $\mathcal{J}$ such that

(1) $\mathcal{J}$ is non-empty and path connected.

(2) there is a relative compact open subset U of W such that for any $J \in \mathcal{J}$, any compact genus 0 nodal J-holomorphic curve representing the class $[A]$ lies inside U.

(3) $c_1(TW)([A]) + n - 3 = 0$.

$GW_0(W, [A], \mathcal{J})$ is called a GW triple. The key property of a GW triple is the following.

Proposition 3.2.8 (cf [72] and the references there-in)**.** *Suppose $GW_0(W, [A], \mathcal{J})$ is a GW triple. Then, the GW invariants $GW_0(W, [A], J_0)$ and $GW_0(W, [A], J_1)$ are the same for any $J_0, J_1 \in \mathcal{J}$. In particular, if $GW_0(W, [A], J_0) \neq 0$, then for any $J \in \mathcal{J}$, there is a nodal closed genus 0 J-holomorphic curve representing the class $[A]$.*

One more technique that we need to use is usually called neck-stretching (See [17] and the references there-in). Given a contact hypersurface $Y \subset W$ separating W with Liouville flow X defined near Y. We call the two components of $W - Y$ as W^- and W^+, where W^- is the one containing D. Then, Y has a tubular neighborhood of the form $(-\delta, \delta) \times Y$ induced by X, which can be identified as part of the symplectization of Y. By this identification, we can talk about what it means for an almost complex structure to be translation invariant and cylindrical in this neighborhood. If one choose a sequence of almost complex structures J_i that "stretch the neck" along Y and a sequence of closed J_i-holomorphic curve u_i with the same domain such that there is a uniform energy bound, then u_i will have a subseguence 'converge' to a J_∞-holomorphic building. The fact that we need to use is the following.

Proposition 3.2.9 (cf [17], [72] and the references there-in)**.** *Suppose we have a sequence of ω-compatible almost complex structure J_i and a sequence of nodal closed genus 0 J_i-holomorphic maps u_i to W representing the same homology class in W such that the image of u_i stays inside a fixed relative compact open subset of W. Assume J_i stretch the neck along a separating contact hypersurface $Y \subset W$ with respect to a Liouville flow X defined near Y. Assume that the image of u_i has non-empty intersection with W^- and W^+, respectively, for all i. Then, there are proper genus 0 J_∞-holomorphic maps (domains are not compact) $u_\infty^- : \Sigma^- \to W^-$ and $u_\infty^+ : \Sigma^+ \to W^+$ such that u_∞^- and u_∞^+ are asymptotic to Reeb orbits on Y with respect to the contact form $\iota_X \omega$.*

In our notations, u_∞^- and u_∞^+ are certain irreducible components of the top/bottom buildings but not necessary the whole top/bottom buildings. Also, u_∞^- does not necessarily refer to the bottom building because we do not declare the direction of the Liouville flow near Y. We are finally ready to prove Theorem 1.2.5. The following which we are going to prove implies Theorem 1.2.5.

Theorem 3.2.10. *Let $D \subset (W, \omega)$ be an ω-orthogonal symplectic divisor with area vector $a = (\omega[C_1], \dots, \omega[C_k])$. Let $z = (z_1, \dots, z_k)$ be a solution of $Q_D z = a$. If one of the z_i is non-positive (resp. positive), there is a small neighborhood $N^{\frac{\delta_{\min}}{2}} \subset W$ of D such that there is no plumbing $(P(D), \omega|_{P(D)}) \subset (N^{\frac{\delta_{\min}}{2}}, \omega|_{N^{\frac{\delta_{\min}}{2}}})$ of D being a capping (resp. filling) of its boundary $(\partial P(D), \alpha)$ with α being the contact form, where α is any primitive of ω defined near $\partial P(D)$ with wrapping numbers $-z$.*

Proof. We first prove the case that one of the z_i is non-positive. Without loss of generality, assume $z_1 \leq 0$. We use the notation in Lemma 3.2.7. In particular, we have symplectic disk fibration $p_i : N_i \to C_i$, the partial compactification $\bar{N}$ and its ω-compatible almost complex structure $J_{\bar{N}}$. We also have $N^\delta = \cup_{i=1}^k \{r_i \leq \delta\} \subset \bar{N}$ and so on. We want to prove the statement with $N^{\delta_{\min}}$ being a small neighborhood of D, where $\delta_{\min}$ is so small such that $\pi((\epsilon - \delta_{\min})^2 + \frac{(\delta_{\max} - \delta_{\min})^2}{4}) > \pi\epsilon^2$ (The constants are fixed in the following order: $\epsilon_1, \epsilon, \delta_{\max}, \delta_{\min}, \tau$. Here, τ is chosen large enough such that Lemma 3.2.7 holds. If τ is too large to find a symplectic disk of radius τ on C_1, then we choose a new set of constants $c\epsilon_1, c\epsilon, c\delta_{\max}, c\delta_{\min}, c\tau$ by scaling them with a small multiple c). We recall that we have ϵ slightly larger than ϵ_1 and ϵ_1 is slightly larger than $\delta_{\max}$. We also recall that we have $B_1 \times S^2_\epsilon \subset \bar{N}$ and $\{q\} \times S^2_\epsilon$ has symplecitc area $\pi\epsilon^2$ for any $q \in B_1$.

Suppose the contrary, assume $(P(D), \omega|_{P(D)}) \subset (N^{\frac{\delta_{\min}}{2}}, \omega|_{N^{\frac{\delta_{\min}}{2}}})$ caps its boundary $(\partial P(D), \alpha)$. Let X be the corresponding Liouville flow near $\partial P(D)$. We do a small symplectic blow-up centered at q_∞ and this blow-up is so small that it is done in $\bar{N} - N$. We call this blown-up manifold $(\bar{N}', \omega_{\bar{N}'})$ and pick a $\omega_{\bar{N}'}$-compatible almost complex structure $J_{\bar{N}'}$ such that the blow-down map is $(J_{\bar{N}'}, J_{\bar{N}})$-holomorphic, the exceptional divisor is $J_{\bar{N}'}$-holomorphic and $J_{\bar{N}'} = J_{\bar{N}}$ in N. Let the exceptional divisor be E and the proper transform of the sphere fiber in $B_1 \times S^2_\epsilon$ containing q_∞ be A. We have $GW_0(\bar{N}', [A], J_{\bar{N}'}) = 1$ by automatic transversality or argue as in the end of Step 4 of the proof of Theorem 6.1 in [72] for higher dimensions. Since blow-up decreases the area, $\omega_{\bar{N}'}([A]) < \omega_{\bar{N}}(Bl_*[A]) = \pi\epsilon^2$, where Bl is the blow-down map. This gives us the energy upper bound. By the same argument as in Lemma 3.2.7, we have that for any $\omega_{\bar{N}'}$-compatible almost complex structure J such that $J = J_{\bar{N}'}$ on $\bar{N}' - N^{\frac{\delta_{\min}}{2}}$, any (nodal) J-holomorphic curve representing the class $[A]$ stays inside a fixed relative compact open subset of $\bar{N}'$. As a result, we have a GW triple $GW_0(\bar{N}', [A], \mathcal{J})$, where $\mathcal{J}$ is the family of compatible almost complex structure that equals $J_{\bar{N}'}$ on $\bar{N}' - N^{\frac{\delta_{\min}}{2}}$. By Proposition 3.2.8, we have a nodal closed genus 0 J-holomorphic curve for any $J \in \mathcal{J}$.

Now, since $P(D) \subset N^{\frac{\delta_{\min}}{2}}$, we can choose a sequence J_i in $\mathcal{J}$ such that it stretches the neck along $(\partial P(D), \alpha)$ and $J_i = J_{\bar{N}'}$ very close to D. We have a corresponding sequence of nodal closed genus 0 J_i-holomorphic curve u_i to $\bar{N}'$. By Proposition 3.2.9, we have a proper genus 0 J_∞-holomorphic maps $u^-_\infty : \Sigma^- \to Int(P(D))$ such that u^-_∞ is asymptotic to Reeb orbits on $\partial P(D)$ with respect to the contact form α. By the direction of the flow, u^-_∞ corresponds to the top building. In general, the top building can be reducible. In our case, since $[A] \cdot [C_1] = 1$ and $[A] \cdot [C_i] = 0$ for $i = 2, \dots, k$, if the top building is reducible, there is some irreducible component lying inside $Int(P(D)) - D$, by positivity of intersection and D being J_∞-holomorphic. Since $\omega_{\bar{N}'}$ is exact on $Int(P(D)) - D$, any irreducible component lying inside $Int(P(D)) - D$ most have non-compact domain and converge asymptotically to Reeb orbits on Y. By the direction of the Reeb flow, we get a contradiction by Stoke's theorem. (cf. Proposition 8.1 of [72] or Step 3 of proof of Theorem 6.1 in [72] or Lemma 7.2 of [1]) Therefore, we conclude that there is only one irreducible component which is exactly u^-_∞ and the image of u^-_∞ intersect C_1 transversally once. Let $q_0 \in \Sigma^-$ be the point that maps to the intersection. Let also $\mathbb{D}^2_{q_0}$ be a Darboux disk around q_0 and $i = u^-_\infty|_{\mathbb{D}^2_{q_0}}$. Now, we want to draw contradiction using the existence of u^-_∞.

Since ω is exact on $\partial P(D)$, it is exact in $N^{\delta_{\min}} - D$. Extend α to be a primitive of ω in $N^{\frac{\delta_{\min}}{2}} - D$ and we still denote it as α. By assumption, $[\omega - d\alpha_c]$ is Lefschetz dual to $\sum_{i=1}^{k} z_i[C_i]$. In particular, $i^*\alpha$ has wrapping number $-z_1$ around q_0 on $\mathbb{D}^2_{q_0} - q_0$. In other words, $[i^*\alpha - \frac{r^2}{2}d\theta]$ is cohomologous to $\frac{-z_1}{2\pi}d\theta$ in $H^1(\mathbb{D}^2_{q_0} - q_0, \mathbb{R})$, where (r, θ) are the polar coordinates. We have

$$i^*\alpha = \frac{r^2}{2}d\theta + \frac{-z_1}{2\pi}d\theta + df$$

for some function f on $\mathbb{D}^2_{q_0} - q_0$. When we choose the extension of α to $N^{\frac{\delta_{\min}}{2}} - D$, we can choose in a way that $i^*\alpha = \frac{r^2}{2}d\theta + \frac{-z_1}{2\pi}d\theta$ because the image of i is away from $\partial P(D)$.

Notice that the $(u^-_\infty)^*\omega$ dual of $(u^-_\infty)^*\alpha$ defines a Liouville vector field X_{Σ^-} on $\Sigma^- - q_0$ away from critical points of u^-_∞ (if any). This Liouville flow equals to the component of X in $T(u^-_\infty(\Sigma^-))$ near $\partial P(D)$, when we write down the decomposition of X into $T(u^-_\infty(\Sigma^-))$-component and its $\omega_{\bar{N}'}$-orthogonal complement component. Here, $T(u^-_\infty(\Sigma^-))$ denotes the tangent bundle of the image of u^-_∞, which is well-defined near $\partial P(D)$. Since u^-_∞ is J_∞-holomorphic and it asymptotic converges to Reeb orbits of $(\partial P(D), \alpha)$, $X = -J_\infty R_\alpha$ has non-zero $T(u^-_\infty(\Sigma^-))$-component near $\partial P(D)$. Moreover, since X points inward with respect to $P(D)$, so is X_{Σ^-} on Σ^- near infinity. In particular, if we take $\partial_\eta P(D)$ to be the flow of $\partial P(D)$ with respect to X for a sufficiently small time, then X_{Σ^-} is pointing inward along $(u^-_\infty)^{-1}(\partial_\eta P(D))$.

On the other hand, $i^*\alpha = \frac{r^2}{2}d\theta + \frac{-z_1}{2\pi}d\theta$. Therefore, the Liouville vector field X_{Σ^-} near q_0 equals $(\frac{r}{2} + \frac{-z_1}{2\pi r})\partial_r$ and hence points outward with respect to $\mathbb{D}^2_{q_0}$ (This is where we use $z_1 \leq 0$). As a result, we get a compact codimension 0 submanifold Σ^-_0 with boundary in $\Sigma^- - q_0$ that has Liouville flow pointing inward along the boundaries and $(u^-_\infty|_{\Sigma^-_0})^*\omega$ has a globally defined primitive $(u^-_\infty|_{\Sigma^-_0})^*\alpha$. It gives a contradiction by Stoks's theorem and $\int_{\Sigma^-_0}(u^-_\infty)^*\omega \geq 0$.

For the other case, we assume z_1 is positive. In this case, the argument is basically the same but we need to use u^+_∞ instead of u^-_∞. We have u^+_∞ intersecting the exceptional divisor E transversally exactly once. By blowing down, we get a corresponding pseudo-holomorphic map $Bl \circ u^+_\infty$. Let the point on Σ^+ that maps to the exceptional divisor be q_0 as before. Then, the $(u^+_\infty)^*\omega$ dual of $(u^+_\infty)^*\alpha$ defines a Liouville vector field on $\Sigma^+ - q_0$ away from critical points of u^+_∞ (if any). Similar as before, we get a contradiction by Stoke's theorem and the fact that u^+_∞ restricted to any subdomian has non-negative energy. This completes the proof.

□

We remark that the same proof can be generalized to higher dimensional ω-orthogonal divisors. We leave it to interested readers.

On the other hand, this is not obvious to the authors that how one can remove the ω-orthogonal assumption in Theorem 3.2.10. Therefore, Theorem 3.2.10 is not enough for our application. To deal with this issue, we have the following Theorem 3.2.11.

Theorem 3.2.11. *Let (D, ω) be a symplectic divisor in a closed symplectic manifold (W, ω). There exists a neighborhood N of D such that there is no concave neighborhood $P(D)$ inside N with the Liouville form α on $\partial P(D)$ having a non-negative wrapping number among its wrapping numbers.*

The proof is in the same vein of that of Theorem 3.2.10. However, we need W to be closed to help us to run the argument this time.

Proof. Same as before, we start with an energy estimate. Suppose α is a primitive of ω defined near D but not defined on D. Let the wrapping numbers of α be $-z_i$. Suppose z_1 is non-positive.

Identify a neighborhood M_1 of C_1 with a symplectic disk bundle over C_1 with symplectic connection rotating the fibers. Assume the fibers are symplecticomorphic to standard symplectic disk of radius ϵ. Pick a point p on C_1 such that there is a Darboux disk of radius 4τ and $\tau > c\epsilon$ for some $c > 0$ to be determined (we can achieve this by choosing a small ϵ in advance). Identify the neighborhood of p as a product of closed disks $\mathbb{D}^2_{4\tau} \times \mathbb{D}^2_{\epsilon}$. Choose ϵ_i for $i = 0, 1, \dots, 5$ to be determined such that $\epsilon > \epsilon_0 > \epsilon_1 > \cdots > \epsilon_5 > 0$. Now, cut out the closed region $\mathbb{D}^2_{2\tau} \times (\mathbb{D}^2_{\epsilon_1} - Int(\mathbb{D}^2_{\epsilon_3}))$ from W and call it W_0. We partially compactify W_0 to be $\overline{W}$ by gluing W_0 and $Int(\mathbb{D}^2_{\tau}) \times S^2_{\epsilon_2}$ along $Int(\mathbb{D}^2_{\tau}) \times Int(\mathbb{D}^2_{\epsilon_3})$

by identifying $Int(\mathbb{D}^2_{\epsilon_3})$ with a choose of symplectic embedding to $S^2_{\epsilon_2}$, where $S^2_{\epsilon_2}$ is a symplectic sphere of symplectic area $\pi\epsilon_2^2$. We define $N_3 \subset N_2 \subset N_1 \subset N \subset \overline{W}$ to be the following subset of $\overline{W}$. Notice that N, N_1 and N_2 are closed (but not compact) and N_3 is open.

$$N = (\mathbb{D}^2_{4\tau} \times \mathbb{D}^2_{\epsilon} - \mathbb{D}^2_{2\tau} \times (\mathbb{D}^2_{\epsilon_1} - Int(\mathbb{D}^2_{\epsilon_3}))) \cup Int(\mathbb{D}^2_{\tau}) \times S^2_{\epsilon_2}$$

$$N_1 = (\mathbb{D}^2_{4\tau} \times (\mathbb{D}^2_{\epsilon} - Int(\mathbb{D}^2_{\epsilon_5})) - \mathbb{D}^2_{2\tau} \times (\mathbb{D}^2_{\epsilon_1} - Int(\mathbb{D}^2_{\epsilon_3}))) \cup Int(\mathbb{D}^2_{\tau}) \times S^2_{\epsilon_2}$$

$$\begin{aligned} N_2 \;\; &= \;\; ((\mathbb{D}^2_{3\tau} \times (\mathbb{D}^2_{\epsilon_0} - Int(\mathbb{D}^2_{\epsilon_4})) - \mathbb{D}^2_{2\tau} \times (\mathbb{D}^2_{\epsilon_1} - Int(\mathbb{D}^2_{\epsilon_3})) - Int(\mathbb{D}^2_{\frac{\tau}{2}}) \times Int(\mathbb{D}^2_{\epsilon_3})) \\ &\quad\; \cup (Int(\mathbb{D}^2_{\tau}) - Int(\mathbb{D}^2_{\frac{\tau}{2}})) \times S^2_{\epsilon_2} \end{aligned}$$

$$N_3 = Int(\mathbb{D}^2_{\tau}) \times S^2_{\epsilon_2} - Int(\mathbb{D}^2_{\tau}) \times \mathbb{D}^2_{\epsilon_3}$$

Then, $\overline{W}$ is our desired manifold to run the argument above.

Let $J_{\overline{W}}$ be an ω-compatible almost complex structure on $\overline{W} - N_3$ such that $J_{\overline{W}}$ is split as a product in $N - N_3$. Then, extend $J_{\overline{W}}$ naturally over N_3 such that it is 'product-like' making the $S^2_{\epsilon_2}$ sphere fibers $J_{\overline{W}}$-holomorphic. We still call this $J_{\overline{W}}$. Let $q_\infty \in N_3$ be a point in N such that it lies in the $S^2_{\epsilon_2}$ sphere fiber at p. Similar as before, for any ω-compatible almost complex structure J such that $J = J_{\overline{W}}$ on N_1 and any closed genus 0 J-holomorphic curve u to $\overline{W}$ passing through q_∞, we must have the image of u stays inside $\overline{W} - N_2$ or the energy of u, $\int_{\mathbb{C}P^1} u^*\omega$, greater than a lower bound depending on ϵ_5 (once c and ϵ_i for $i = 0, \ldots, 4$ are determined). It should be convincing that one can choose a choice of c and ϵ_i such that any J-holomorphic curve representing the class $[S^2_{\epsilon_2}]$, the spherical fiber class at p, and passing through q_∞ has to stay inside $\overline{W} - N_2$. Since W is closed, $\overline{W} - N_2$ is a relative compact open subset which we can use to define the GW triple below.

We claim that D does not have a concave neighborhood $P(D)$ lying inside $W - N_1$ with the Liouville contact form α' defined near $\partial P(D)$ having the same wrapping numbers as that of α. Suppose on the contrary, there were such a $P(D)$. Then, by a C^0 perturbation near the intersection points of C_i in D, we can assume that D is ω-orthogonal and it still lies inside $P(D)$. We do a sufficiently small blow-up at q_∞ as before and let the proper transform of the $S^2_{\epsilon_2}$ fiber containing q_∞ be A. We have a GW triple $GW_0(\overline{W}, [A], \mathcal{J})$, where $\mathcal{J}$ is the family of ω-compatible almost complex structure J such that $J = J_{\overline{W}}$ on N_1. Since D is now ω-orthogonal, we can find $J \in \mathcal{J}$ such that D is J-holomorphic (notice that, there exists symplectic divisor with no almost complex structure making all irreducible components pseudo-holomorphic simultaneously). Then, we find a sequence $J_i \in \mathcal{J}$ making D J_i-holomorphic for all i and stretch the neck along $\partial P(D)$ as before to draw contradiction.

□

3.2.3 Uniqueness

In this subsection we show that any contact structure obtained from the GS construction is contactomorphic to one from the McLean's construction. Then Theorem 1.2.6 follows from the uniqueness of McLean's construction.

In fact, we are going to prove the following more precise version of Theorem 1.2.6.

Proposition 3.2.12. *Suppose $D = \cup_{i=1}^k C_i$ is a symplectic divisor with each intersection point being ω-orthogonal such that the augmented graph (Γ, a) satisfies the positive (resp. negative) GS criterion. Then, the contact structures induced by the positive (resp. negative) GS criterion are contactomorphic, independent of choices made in the construction and independent of a as long as (Γ, a) satisfies positive GS criterion.*

Moreover, if D arises from resolving an isolated normal surface singularity, then the contact structure induced by the negative GS criterion is contactomorphic to the contact structure induced by the complex structure.

On the other hand, if D is the support of an effective ample line bundle, then the contact structure induced by the positive GS criterion is contactomorphic to that induced by a positive hermitian metric on the ample line bundle.

Uniqueness of McLean's construction

We recall the uniqueness part of McLean's construction, which can be regarded as a more complete version of Proposition 3.2.3.

Proposition 3.2.13. *[cf. Corollary 4.3 and Lemma 4.12 of [72]] Suppose $f_0, f_1 : W - D \to \mathbb{R}$ are compatible with D and D is a symplectic divisor with respect to both ω_0 and ω_1 having positive transversal intersections. Suppose $\theta_j \in \Omega^1(W-D)$ is a primitive of ω_j on $W - D$ such that it has positive (resp. negative) wrapping numbers for all $i = 1, \dots, k$ and for both $j = 0, 1$. Suppose, for both $j = 0, 1$, there exist $g_j : W - D \to \mathbb{R}$ such that $df(X^j_{\theta_j + dg_j}) > 0$ (resp. $df(-X^j_{\theta_j+dg_j}) > 0$) near D, where $X^j_{\theta_j+dg_j}$ is the dual of $\theta_j + dg_j$ with respect to ω_j. Then, for sufficiently negative l, we have that $(f_0^{-1}(l), \theta_0 + dg_0|_{f_0^{-1}(l)})$ is contactomorphic to $(f_1^{-1}(l), \theta_1 + dg_1|_{f_1^{-1}(l)})$.*

Moreover, when (W, D, ω) arises from resolving a normal isolated surface singularity, then the link with contact structure induced from complex line of tangency is contactomorphic to this canonical contact structure.

The first thing to note is that the choice of g_j for $j = 0, 1$ always exist (cf. Proposition 3.2.3 above, Proposition 4.1 and Proposition 4.2 in [72]). Moreover, by the definition of compatible function, it also always exist. In other words, Proposition 3.2.13 implies that in dimension four, for any symplectic form ω_0 and ω_1 making D a divisor such that they have primitives θ_0 and θ_1 on $W - D$ with positive (resp. negative) wrapping numbers, the contact structures constructed by McLean's construction with respect to θ_0 and θ_1 are contactomorphic.

Proposition 3.2.13 is literally not exactly the same as Corollary 4.3 and Lemma 4.12 in [72] so we want to make clear why it is still valid after we have made the changes. We remark that if θ_0 and θ_1 have positive wrapping numbers, then $\theta_t = (1-t)\theta_0 + t\theta_1$ has positive wrapping numbers for all t and $f_t = (1-t)f_0 + tf_1$ is compatible with D for all t. As a symplectic divisor, we always assume C_i have positive orientations with respect to the symplectic form for all i. In other words, both $\omega_0|_{C_i}$ and $\omega_1|_{C_i}$ are positive and hence D is a symplectic divisor with respect to $d\theta_t$ for all t. Therefore, we get a deformation of ω_t and the first half of Proposition 3.2.13 with θ_0 and θ_1 having positive wrapping numbers follows from Corollary 4.3 of [72].

The analogous statement for the first half of Proposition 3.2.13 with θ_0 and θ_1 having negative wrapping numbers follows similarly as in the case where θ_0 and θ_1 have positive wrapping numbers.

On the other hand, Lemma 4.12 of [72] requires that the resolution is obtained from blowing up. Although there exist a resolution such that it is not obtained from blowing up in complex dimension three or higher, every resolution for an isolated normal surface singularity can be obtained by blowing up the unique minimal model, where the minimal model is obtained from blowing up the singularity. Therefore, the second half of Proposition 3.2.13 follows.

Proof of Theorem 1.2.6

To prove Proposition 3.2.12 using Proposition 3.2.13, the remaining task is to construct an appropriate disc fibration having a connection rotating fibers for the local models in the GS-construction. Then, the constructions of θ, f and g will be automatic. We give the fibration in the following Lemma.

Lemma 3.2.14. *Let z_1' and z_2' be two positive numbers. Let $\mu : \mathbb{S}^2 \times \mathbb{S}^2 \to [z_1', z_1'+1] \times [z_2', z_2'+1]$ be the moment map of $\mathbb{S}^2 \times \mathbb{S}^2$ onto its image.*

Fix a small $\epsilon > 0$ and let $D_1 = \mu^{-1}(\{z_1'\} \times [z_2', z_2'+2\epsilon])$ be a symplectic disc. Fix a number $s \in \mathbb{R}$ first and then let $\delta > 0$ be sufficiently small. Let Q be the closed polygon with vertices $(z_1', z_2'), (z_1'+\delta, z_2'), (z_1'+\delta, z_2'+2\epsilon-s\delta), (z_1', z_2'+2\epsilon)$. Using the (p_1, q_1, p_2, q_2) coordinates described in the GS-construction above, we define a map $\pi : \mu^{-1}(Q) \to D_1$ by sending (p_1, q_1, p_2, q_2) to $(z_1', *, p_2 + \frac{(2\epsilon - t(p_1,p_2) - \rho(t(p_1,p_2)))p_1}{\delta}, q_2)$, where $\rho : [0, 2\epsilon] \to [0, 2\epsilon - s\delta]$ is a smooth strictly monotonic decreasing function with $\rho(0) = 2\epsilon - s\delta$ and $\rho(2\epsilon) = 0$ such that $\rho'(t) = -1$ for $t \in [0, \epsilon]$ and near $t = 2\epsilon$. This can be done as δ is sufficiently small. Moreover, $t(p_1, p_2)$ is the unique t solving $p_2 - (z_2' + 2\epsilon - t) = \frac{(\rho(t)-(2\epsilon-t))(p_1-z_1')}{\delta}$ and $*$ means that there is no q_1 coordinate above (z_1', x) for any x so q_1 coordinate is not relevant.

Then, we have that π gives a symplectic fibration with each fibre symplectomorphic to $(\mathbb{D}^2_{\sqrt{2\delta}}, \omega_{std})$ and the symplectic connection of π has structural group lies inside $U(1)$. Moroever, fibres are symplectic orthogonal to the base.

Proof. First, we want to explain what $t(p_1, p_2)$ means geometrically. $\rho(2\epsilon - t)$ is an oriented diffeomorphism from $[0, 2\epsilon] \to [0, 2\epsilon - s\delta]$ so it can be viewed as a diffeomorphism from the left edge of Q to the right edge of Q. $p_2 - (z_2' + 2\epsilon - t) = \frac{(\rho(t)-(2\epsilon-t))(p_1-z_1')}{\delta}$, which we call L_t, is the equation of line joining the point $(z_1', z_2' + 2\epsilon - t)$ and $(z_1' + \delta, z_2' + \rho(t))$. Therefore, for a point (p_1, p_2), $t(p_1, p_2)$ is such that $L_{t(p_1,p_2)}$ contains the point (p_1, p_2). Moreover, $p_2 + \frac{(2\epsilon - t(p_1,p_2) - \rho(t(p_1,p_2)))p_1}{\delta}$ is the p_2-coordinate of the intersection between line $L_{t(p_1,p_2)}$ and the left edge of Q, $\{p_1 = z_1'\}$.

To prove the Lemma, we pick κ close to 2ϵ from below such that $\rho'(t) = -1$ for all $t \in [\kappa, 2\epsilon]$. Let Δ be $\pi^{-1}(\mu^{-1}(\{z_1'\} \times [z_2' + 2\epsilon - \kappa, z_2' + 2\epsilon]))$ We give a smooth trivialization of $\pi|_\Delta$ as follows.

Let $\Phi : [0,\kappa] \times \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{D}^2_{\sqrt{2\delta}} \to \Delta$ be given by sending $(t, \vartheta_1, \tau, \vartheta_2)$ to $(p_1, q_1, p_2, q_2) = (z_1' + \tau, -s\vartheta_1 + \vartheta_2, (z_2' + 2\epsilon - t) + \frac{(\rho(t)-(2\epsilon-t))\tau}{\delta}, -\vartheta_1)$, where t, ϑ_1 are the coordinates of $[0, \kappa]$ and $\mathbb{R}/2\pi$, respectively, and $(\tau = \frac{r^2}{2}, \vartheta_2)$ is such that (r, ϑ_2) is the standard polar coordinates of $\mathbb{D}^2_{\sqrt{2\delta}}$. In particular, $\tau \in [0, \delta]$. Note that, Φ is well-defined and it is a diffeomorphism.

Let $\pi_\Phi : [0,\kappa] \times \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{D}^2_{\sqrt{2\delta}} \to [0,\kappa] \times \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{D}^2_{\sqrt{2\delta}}$ be the projection to the first two factors. Then, we have $\pi \circ \Phi = \Phi \circ \pi_\Phi$. Notice that, when $\tau = 0$, the ϑ_2-coordinate degenerates and it corresponds to $p_1 = z_1'$ and the q_1-coordinate degenerates.

To investigate this fibration under the trivialization, we have

$$\begin{aligned}
\Phi^*\omega &= \Phi^*(dp_1 \wedge dq_1 + dp_2 \wedge dq_2) \\
&= d\tau \wedge (-sd\vartheta_1 + d\vartheta_2) \\
&\quad + (-dt + \frac{\tau}{\delta}dt + \frac{\rho'(t)\tau}{\delta}dt + \frac{\rho(t) - (2\epsilon - t)}{\delta}d\tau) \wedge (-d\vartheta_1) \\
&= (1 - \frac{\tau}{\delta} - \frac{\rho'(t)\tau}{\delta})dt \wedge d\vartheta_1 + d\tau \wedge d\vartheta_2 + (\frac{2\epsilon - t - \rho(t)}{\delta} - s)d\tau \wedge d\vartheta_1
\end{aligned}$$

For a fibre, we have t and ϑ_1 being contant so the the symplectic form restricted on the fibre is $d\tau \wedge d\vartheta_2$, which is the standard one. Hence, each fibre is symplectomorphic to $(\mathbb{D}^2_{\sqrt{2\delta}}, \omega_{std})$. When $\tau = 0$, the symplectic form equals $dt \wedge d\vartheta_1$ so the base is symplectic and fibres are symplectic orthogonal to the base. Moreover, the vector space that is symplectic orthogonal to the fibre at a point is spanned by ∂_t and $\partial_{\vartheta_1} - (\frac{2\epsilon - t - \rho(t)}{\delta} - s)\partial_{\vartheta_2}$ so the symplectic connection has structural group inside $U(1)$.

Finally, we remark that $\rho(0) = 2\epsilon - s\delta$ and $\rho'(t) = -1$ when t is close to 0, hence $\Phi^*\omega = dt \wedge d\vartheta_1 + d\tau \wedge d\vartheta_2$. Therefore, when t is close to 0, the trivialization Φ actually coincide with the gluing symplectomorphism in the GS construction from preimage of $R_{e_{\alpha\beta},v_\alpha}$ to $[x_{v_\alpha,e_{\alpha\beta}} - 2\epsilon, x_{v_\alpha,e_{\alpha\beta}} - \epsilon] \times \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{D}^2_{\sqrt{2\delta}}$, up to a translation in t-coordinate. On the other hand, $\pi|_{\mu^{-1}(Q)-\Delta}$ is clearly a symplectic fibration with all the desired properties described in the Lemma as it corresponds to the trivial projection by sending (p_1, q_1, p_2, q_2) to $(z_1', *, p_2, q_2)$. This finishes the proof of this Lemma.

□

We remark that the disc fibration above gives a fibration on the local models $N_{e_{\alpha\beta}}$ and it is compatible with the trivial fibration on the local models N_{v_α} so they give a well-defined fibration after gluing all the local models N_{v_α} and $N_{e_{\alpha\beta}}$. Now, we are ready to prove Proposition 3.2.12.

Proof of Proposition 3.2.12. Let $(D = \cup_{i=1}^k C_i, W, \omega)$ be a symplectic plumbing. First, we assume the intersection form of D is negative definite (or equivalently, the augmented graph satisfies the negative GS criterion). By [34], D satisfies the negative GS criterion. Therfore, by possibly shrinking W, we can assume W is a symplectic plumbing constructed from the negative GS criterion. A byproduct of the construction is the existence of a primitive of ω on $W - D$, θ, given by contracting ω by the Liouville vector field. From the construction, in the N_{v_α} local model, we have $\theta = \iota_{\bar{X}_{v_\alpha} + (\frac{r}{2} + \frac{z'_{v_\alpha}}{r})\partial_r}(\bar{\omega}_{v_\alpha} + rdr \wedge d\vartheta_2) = \iota_{\bar{X}_{v_\alpha}}\bar{\omega}_{v_\alpha} + (\frac{r^2}{2} + z'_{v_\alpha})d\vartheta_2$. When we restrict it to a fibre, we can see that the wrapping numbers of θ with respect to C_{v_α} is $2\pi z'_{v_\alpha}$, which is positive. Here, C_{v_α} is the smooth symplectic submanifold corresponding to the vertex v_α.

Note that we have $\lambda = -z = 2\pi z'$ in our convention above. By tracing back the negative GS construction, we see that Lemma 3.2.14 provides a desired symplectic fibrations we needed to apply Proposition 3.2.13. In particular, this symplectic fibrations give us well-defined r_i-coordinates near the divisor. As a result, one can set $f = \sum_{i=1}^{k} \log(\rho(r_i))$ and $g = 0$ and get that $df(X_\theta) > 0$ near D and $(f^{-1}(l), \theta|_{f^{-1}(l)})$ is precisely the contact manifold obtained from the negative GS criterion. In particular, $(f^{-1}(l), \theta|_{f^{-1}(l)})$ is the canonical one with respect to (W, D).

If we have made another set of choices in the construction, we get that $(\bar{f}^{-1}(l), \theta|_{\bar{f}^{-1}(l)})$ is the canonical one with respect to $(\bar{W}, \bar{D})$. Then, since (W, D) is diffeomorphic to $(\bar{W}, \bar{D})$, we can pull back the compatible function and the 1-form on $(\bar{W}, \bar{D})$ to (W, D). By Proposition 3.2.13, the two contact manifolds are contactomorphic. Same argument works to show that this contact structure is independent of symplectic area a as long as (Γ, a) still satisfies GS criteria. Also, when D is arising from isolated normal surface singularity, contact structure of its link is contactomorphic that induced by GS-criterion, by Proposition 3.2.13, again. This finishes the case when D is negative definite.

Now, we assume that (D, ω) satisfies the positive GS criterion and D is ω-orthogonal. By the same reasoning as bove, the contact structure induced by the positive GS criterion is independent of choices.

Suppose D is also the support of an effective ample line bundle. Pick a hermitian metric $\|.\|$ and a section s with zero being $\sum_{i=1}^{k} z_i C_i$, where $z_i > 0$. Let $f = -\log \|s\|$, $\theta = -d^c f$ and $\omega = d\theta$, where $d^c f = df \circ J_{std}$. Then, θ induces a contact structure on the boundary of plumbing of (D, ω) with negative wrapping numbers (See Lemma 5.19 of [71]). Moreover, f is compatible with D and $df(-X_\theta) > 0$ near D (See Lemma 2.1 of [71] or Lemma 4.12 of [72]). Hence the contact structure induced by θ is contactomorphic to the canonical one by Proposition 3.2.13, which is contactomorphic to the one induced by the positive GS criterion.

□

3.2.4 Examples of Concave Divisors

In this subsection, we are going to see five illuminating examples. The first one is the simplest kind of symplectic divisor. The second one illustrates Theorem 1.2.5 is no longer valid if the plumbing chosen is not close to the divisor. In particular, there is a concave divisor which admits a convex neighborhood but it is not a convex divisor. The third one is a frequently used example when studying Lefschetz fibration. The forth one is a concave divisor with non-fillable contact structure on the boundary. The last one shows that the constructed contact structure on the boundary is not necessarily contactomorphic to the standard one that one might expect if the divisor is concave.

Example 3.2.15. A symplectic surface with self-intersection n admits a concave (resp convex) boundary when $n > 0$ (resp $n < 0$). When $n = 0$, a symplectic form cannot make both the surface symplectic and the restriction to boundary be exact so it has no convex or concave neighborhood. In fact, more is true, by a result of Eliasberg [24], $\mathbb{S}^1 \times \mathbb{S}^2$ cannot be a convex boundary of any symplecyic form on $\mathbb{D}^2 \times \mathbb{S}^2$. In contrast, althought a symplectic torus with self-intersection zero has no concave nor convex neighborhood, a Lagrangian torus has self-intersection zero and has a convex neigborhood.

Example 3.2.16. ([65]) In [65], McDuff constructed a symplectic form on $(S\Sigma_g \times [0,1], \omega)$ such that it has disconnected convex boundary, where $S\Sigma_g$ is a circle boundle of a genus g surface and $g > 1$. The contact structure near $S\Sigma_g \times \{0\}$ is contactomorphic to the concave boudary near a self-intersection $2g-2$ symplectic genus g surface. The contact structure near $S\Sigma_g \times \{1\}$ is contactomorphic to the convex boundary near a Lagrangian genus g surface. If one glues a symplectic closed disc bundle $P(D)$ over a symplectic genus g surface D with $(S\Sigma_g \times [0,1], \omega)$ along $S\Sigma_g \times \{0\}$. One gets a plumbing of the surface with convex boundary. This suggests that a symplectic genus g ($g > 1$) surface can have both concave and convex neighborhood, depending on the symplectic form and the neighborhood. Notice that D is trivially ω-orthogonal. It illustrates that the assumption on $P(D)$ being sufficiently close to D in Theorem 1.2.5 cannot be dropped. Moreover, by Theorem 1.2.5, D is a concave divisor but not a convex divisor although it admits convex neighborhood.

Example 3.2.17. Suppose there is a symplectic Lefschetz fibration (X, ω) over $\mathbb{CP}^1$ with generic fibre F and a symplectic section S of self-intersection $-n$ ($n \geq 0$). Let $D = F \cup S$, then the augmented graph of D always satisfies the positive GS criterion regardless the area weights of the surfaces. Suppose also that S is perturbed to be ω-orthogonal to F. Then, Proposition 1.2.4 (or, Proposition 3.2.3 if one does not want to perturb) shows that D is a concave divisor. In other words, the complement of a concave neighborhood of D is a convex filling of its boundary.

This fits well to the well-known fact that the complement of a regular neighborhood of D is a Stein domain. Moreover, this construction has been successfully used to find exotic Stein fillings [5].

Lemma 3.2.18. *Let (Γ, a) be an augmented graph satisfying the positive GS criterion and D be a realization. Suppose there are two genera zero vertices with self-intersection s_1, s_2 such that either*

(i) they are adjacent to each other and $s_1 > s_2 \geq 1$, or

(ii) they are not adjacent to each other with $s_1 \geq 1$ and $s_2 \geq 0$.

Then, D is a concave divisor but not a capping divisor.

Proof. Suppose on the contrary, the boundary has a convex fillings Y. Then, we can glue D with Y to obtain a closed symplectic 4 manifold W. By McDuff's theorem

[64], W is rational or ruled and hence have $b_2^+ = 1$. For (i), the two spheres generates a positive two dimensional subspace of $H_2(W)$ with respect to the intersection form. Thus, we get a contradiction. For (ii), it suffices to consider tha case $s_1 = 1$ and $s_2 = 0$. By the Theorem in [64], one can assume the sphere with self-intersection 1 represent the hyperplane class H, with respect to an orthonormal basis $\{H, E_1, \dots, E_n\}$ for $H_2(W)$. The two spheres being disjoint implies the one with self-intersection 0 has homology class being a linear combination of exceptional classes. Since the sphere is symplectic, the linear combination is non-trivial. Thus, we get a contradiction. □

Example 3.2.19. Let Γ be a chain of two vertices of genus zero and the self-intersection numbers are 2 and 1, respectively.

$$Q_\Gamma = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix},$$

Then the boundary fundamental group of Γ is the free group generated by e_1 and e_2 modulo the relations $e_1 e_2^1 = e_2^1 e_1$, $1 = e_1^2 e_2$ and $1 = e_1 e_2$. Therefore, the boundary of the plumbing according to Γ has trivial fundamental group and hence diffeomorphic to a sphere. It is easily see that the corresponding augmented graph (Γ, a) satisfies the positive GS criterion if and only if the area weights satisfy $a_1 < a_2 < 2a_1$, where a_i is the area weight of v_i. In other words, if $a_1 < a_2 < 2a_1$, by Proposition 1.2.4 and Lemma 3.2.18, we get an overtwisted contact structure on S^3 (S^3 has only one tight contact structure which is fillable).

3.2.5 Theorem 1.2.2

The proof of Theorem 1.2.2 involes two inputs. The first one is a linear algebraic lemma, which is simple but important. The second one which is called inflation lemma allows us to deform the symplectic form to our desired one so as to apply Propositon 1.2.4.

A key lemma

The following linear algebraic Lemma related to the positive GS criterion will be crucial.

Lemma 3.2.20. *Let Q be a k by k symmetric matrix with off-diagonal entries being all non-negative. Assume that there exist $a \in (0, \infty)^k$ such that there exist $z \in \mathbb{R}^k$ with*

$Qz = a$. Suppose also that Q is not negative definite. Then, there exists $z \in (0,\infty)^k$ such that $Qz \in (0,\infty)^k$.

Proof. When $k = 1$, it is trivial. Suppose the statement is true for (k-1) by (k-1) matrix and now we consider a k by k matrix Q. Let $q_{i,j}$ be the $(i,j)^{th}$-entry of Q. First observe that if $q_{i,i} \geq 0$, for all $i = 1,\ldots,k$, then the statement is true. The reason is that if each row has a positive entry, then $z = (1,\ldots,1)$ works. If there exist a row with all 0, then there is no $a \in (0,\infty)^k$ such that there exist $z \in \mathbb{R}^k$ with $Qz = a$.

Therefore, we might assume $q_{k,k} < 0$. Let $l_j = -\frac{q_{k,j}}{q_{k,k}} \geq 0$, for $j < k$, and let B be the lower triangular matrix given by

$$b_{i,j} = \begin{cases} \delta_{i,j} & \text{if } i \neq k \text{ or } (i,j) = (k,k) \\ l_j & \text{if } i = k \text{ and } (i,j) \neq (k,k) \end{cases}$$

Let $M = B^T Q B$. Then,

$$m_{i,j} = \begin{cases} q_{i,j} - \frac{q_{i,k}q_{k,j}}{q_{k,k}} & \text{if } (i,j) \neq (k,k) \\ q_{k,k} & \text{if } (i,j) = (k,k) \end{cases}$$

In particular, $m_{i,k} = m_{k,j} = 0$, for all i and j less than k. We can write M as a direct sum of a k-1 by k-1 matrix M' with the 1 by 1 matrix $q_{k,k}$ in the obvious way. Notice that the off diagonal entries of M' are all non-negative.

Let $a = (a_1,\ldots,a_k)^T$ and $z = (z_1,\ldots,z_k)^T$ such that $Qz = a$. Let also $\overline{z} = (\overline{z}_1,\ldots,\overline{z}_k)^T = B^{-1}z$ and $\overline{a} = (\overline{a}_1,\ldots,\overline{a}_k)^T = B^T a$. Then, $Qz = a$ is equivalent to $M\overline{z} = \overline{a}$. Here, $\overline{z}_i = z_i$, for $i < k$, and $\overline{z}_k = z_k - \sum_{i=1}^{k-1} l_i z_i$. On the other hand, $\overline{a}_i = a_i + l_i a_k$, for all $i < k$, and $\overline{a}_k = a_k$.

By assumption, there exist $a \in (0,\infty)^k$ such that there exist $z \in \mathbb{R}^k$ with $Qz = a$. So we have $(\overline{a}_1,\ldots,\overline{a}_{k-1})^T \in (0,\infty)^k$ and $M'(z_1,\ldots,z_{k-1})^T = (\overline{a}_1,\ldots,\overline{a}_{k-1})^T$. Apply induction hypothesis, we can find $y \in (0,\infty)^{k-1}$ such that $M'y \in (0,\infty)^{k-1}$. Pick $y_k > 0$ such that $q_{k,k}(y_k - \sum_{i=1}^{k-1} l_i y_i) > 0$ but sufficient close to zero. Then, let $\overline{z} = (y_1,\ldots,y_{k-1},y_k - \sum_{i=1}^{k-1} l_i y_i)^T$ and tracing it back. We have $Q(y_1,\ldots,y_k)^T \in (0,\infty)^k$. □

Regarding the negative GS criterion, we remark that one can show the following. (It is mentioned in [31] with additional assumption but the additional assumption can be removed.) Suppose Q is a symmetric matrix with off-diagonal entries being non-negative. Then, the following statements are equivalent.

(a) For any $a \in (0,\infty)^n$, there exist $z \in (-\infty,0)^n$ satisfying $Qz = a$.

(a2) For any $a \in (0,\infty)^n$, there exist $z \in (-\infty,0]^n$ satisfying $Qz = a$.

(b) There exist $a \in (0,\infty)^n$ such that there exist $z \in (-\infty,0)^n$ satisfying $Qz = a$.

(b2) There exist $a \in (0,\infty)^n$ such that there exist $z \in (-\infty,0]^n$ satisfying $Qz = a$.

(c) Q is negative definite.

The implication from (a) to (b), (a2) to (b2), (a) to (a2), (b) to (b2) are trivial. (c) implying (a2) is Lemma 3.3 of [34] and a moment thought will justify (c)+(a2) implying (a), which is hiddenly used in [34]. (b) implying (c) is similar to the proof of Lemma 3.2.20. To be more precise, one again use induction on the size of Q and change the basis using B. Therefore, an augmented graph (Γ, a) satisfies the negative GS criterion if and only if Q_Γ is negative definite. In particular, when a graph Γ is negative definite, the negative GS criterion is always satisfied, independent of the area weights.

Inflation

Now, it comes the second input.

Lemma 3.2.21. *(Inflation, See [47] and [57]) Let C be a smooth symplectic surface inside (W,ω). If $[C]^2 \geq 0$, then there exists a family of symplectic form ω_t on W such that $[\omega_t] = [\omega] + tPD(C)$ for all $t \geq 0$. If $[C]^2 < 0$, then there exists a family of symplectic form ω_t on W such that $[\omega_t] = [\omega] + tPD(C)$ for all $0 \leq t < -\frac{\omega[C]}{[C]^2}$. Also, C is symplectic with respect to ω_t for all t in the range above. Moreover, if there is another smooth symplectic surface C' intersect C positively and ω-orthogonally, then C' is also symplectic with respect to ω_t for all t in the range above. Here, $PD(C)$ denotes the Poincare dual of C.*

When $[C]^2 < 0$, one can see that $([\omega] + tPD(C))[C] > 0$ if and only if $t < -\frac{\omega[C]}{[C]^2}$. Therefore, the upper bound of t in this case comes directly from $\omega_t[C] > 0$. We remarked that one can actually do inflation for a larger t but one cannot hope for C being symplectic anymore when t goes beyond $-\frac{\omega[C]}{[C]^2}$.

Proof

Proof of Theorem 1.2.2. First of all, we can isotope D to D' such that every intersection of D' is ω_0-orthogonal, using Theorem 2.3 of [36]. Since every intersection of D is transversal and no three of C_i intersect at a common point, such an isotopy can be extended to an ambient isotopy. Now, instead of isotoping D, we can deform ω_0 through the pull back of ω_0 along the isotopy. As a result, we can assume D is ω_0-orthogonal.

Now, we want to construct a family of realizations D_t of Γ, by deforming the symplectic form, such that the augmented graph of D_1 satisfies the positive GS criterion.

Let $D = D_0 = C_1 \cup \cdots \cup C_k$ and let also the area weights of D_0 with respect to ω_0 be a. Since ω is exact on $\partial P(D)$, there exists z such that $Q_\Gamma z = a$. Also, by assumption and Lemma 3.2.20, there exists $\overline{z} \in (0,\infty)^k$ such that $Q_\Gamma \overline{z} = \overline{a} \in (0,\infty)^k$. Let $z^t = z + t(\overline{z} - z)$ and $a^t = a + t(\overline{a} - a) = Q_\Gamma z^t \in (0,\infty)^k$. We want to construct a realization D_1 of Γ with area weights a^1. If this can be done, then the augmented graph of D_1 will satisfy the positive GS criterion.

Observe that, it suffices to find a family of symplectic forms ω_t such that $[\omega_t] = [\omega_0] + t\sum_i(\overline{z_i} - z_i)PD([C_i])$ and a corresponding family of ω_t-symplectic divisor $D_t = C_1 \cup \cdots \cup C_k$. The reason is that C_i has symplectic area equal the i^{th} entry of a^t under the symplectic form $[\omega_t] = [\omega_0] + t\sum_i(\overline{z_i} - z_i)PD([C_i])$. However, we need to modify this natural choice of family a little bit. Without loss of generality, we can assume $\overline{z_i} > z_i$ for all $1 \leq i \leq k$. We can choose a piecewise linear path p^t arbitrarily close to z^t such that each piece is parallel to a coordinate axis and moving in the positive axis direction. Since satisfying the positive GS criterion is an open condition, we can choose p^t such that $Q_\Gamma p^t \in (0,\infty)^k$. The fact that p^t is chosen such that $Q_\Gamma p^t$ is entrywise greater then zero allows us to do inflation along p^t to get out desired family of ω_t and D_t, by Lemma 3.2.21. Therefore, we arrive at a symplectic form ω_1 such that the augmented graph of (D, ω_1), denoted by (Γ, a), satisfies the positive GS criterion. We finish the proof by applying Propositon 1.2.4.

□

Remark 3.2.22. The proof of Theorem 1.2.2 implies that for any $a \in (0,\infty)^k \cap Q_D(0,\infty)^k$, there is a symplectic deformation making the augmented graph of (D, ω_1) to be (Γ, a).

Proof of Corollary 1.2.7. First suppose D is not negative definite. By Theorem 1.2.2, ω being exact on the boundary implies D is a concave divisor after a symplectic deformation. If D is negative definite, then ω is necessarily exact on the boundary with unique lift of $[\omega]$ to a relative second cohomology class. Moroever, the discussion after the proof of Lemma 3.2.20 implies that D satisifes negative GS criterion and hence D is a convex divisor. □

3.2.6 A short summary

We offer a detailed explanation of the flowchart.

Given a divisor (D, ω) (not necessarily ω-orthogonal, see Proposition 3.2.3), the first obstruction of whether D admits a concave or convex neighborhood comes from ω being not exact on the boundary of D. In this case, $[\omega]$ cannot be lifted to a relative second cohomology class and $Q_D z = a$ has no solution for z.

If ω is exact on the boundary, we look at the solutions z for the equation $Q_D z = a$. When Q_D is negative definite (in this case ω is necessarily exact on the boundary), there is a unique solution for z and all the entries for this solution is negative. Therefore, (D, ω) satisfies the negative GS criterion and D is convex (Proposition 1.2.4 or Proposition 3.2.3).

If ω is exact on the boundary but Q_D is not negative definite, the situation becomes a bit more complicated. There might be more than one solution for z (when Q_D is degenerate). If we are lucky that there is one solution z with all entries being positive, then D is concave (Proposition 1.2.4 or Proposition 3.2.3).

However, it is possible that all the solutions z have at least one entry being non-positive. In this case, if D is ω-orthogonal or D lies inside a closed symplectic manifold, there is a small neigborhood N of D such that D has no convex nor conave neighborhood inside N (Theorem 1.2.5 and Theorem 3.2.11). However, we can choose an area vector $\bar{a}$ such that there is a solution $\bar{z}$ for $Q_D \bar{z} = \bar{a}$ with all entries of $\bar{z}$ being positive (Lemma 3.2.20). Geometrically, we can do inflation (Lemma 3.2.21) to deform the symplectic form such that $(D, \bar{\omega})$ has area vector $\bar{a}$. Then, $(D, \bar{\omega})$ is concave (Proposition 1.2.4 or Proposition 3.2.3). This is exactly the proof of Theorem 1.2.2.

Chapter 4

Symplectic log Calabi-Yau surfaces

This chapter is devoted to the classification of symplectic log Calabi-Yau surfaces.

4.1 Symplectic deformation equivalence of marked divisors

We study the symplectic deformation equivalence property in a general setting, which was initiated by Ohta and Ono in [79]. Here we provide details using the notion of marked divisor, which encodes the blow-down information. We will show that the deformation class of marked symplectic divisors is stable under various operations.

4.1.1 Homotopy and blow-up/down of symplectic divisors

Homotopy

Parallel to the two types of homotopy of a symplectic divisor (X, D, ω) mentioned in the introduction,

- Symplectic isotopy (X, D_t, ω), and
- Symplectic homotopy (X, D_t, ω_t).

We also consider the more restrictive homotopies keeping D fixed:

- $D-$symplectic isotopy (X, D, ω_t) with constant $[\omega_t]$, and

- $D-$symplectic homotopy (X, D, ω_t)

To compare these notions we introduce the following terminology.

Definition 4.1.1. Two *symplectic homotopies* are said to be symplectomorphic if they are related by a one parameter family of symplectomorphisms.

Lemma 4.1.2. *A symplectic homotopy (resp. isotopy) of a symplectic divisor is symplectomorphic to a $D-$symplectic homotopy (resp. isotopy) and vice versa.*

Proof. A $D-$symplectic homotopy is a symplectic homotopy by definition, and by Moser lemma a $D-$symplectic isotopy is symplectomorphic to a symplectic isotopy.

On the other hand, a symplectic homotopy (X, D_t, ω_t) gives rise to a smooth isotopy $\Phi : D \times [0,1] \to X$. Since the intersections of D are transversal and no three of the components intersect at a common point, we can apply the smooth isotopy extension theorem to extend Φ to a smooth ambient isotopy $\Phi = \{\Phi_t\} : X \times [0,1] \to X$. Then we get a $D-$symplectic homotopy $(X, D, \Phi_t^* \omega_t)$ which is symplectomorphic to (X, D_t, ω_t) via the one parameter family of symplectomorphisms $\{\Phi_t\}$. Similarly, a symplectic isotopy is symplectomorphic to a $D-$symplectic isotopy.

□

Lemma 4.1.2 converts the effect of a symplectic isotopy (resp. homotopy) to a $D-$symplectic isotopy (resp. homotopy). This simple observation will be repeatedly used.

Toric and non-toric blow-up/down

Throughout this section, we use the following terminology for symplectic blow-up/down of $D \subset (X, \omega)$.

A **toric blow-up** (resp. **non-toric blow-up**) of D is the total (resp. proper) transform of a symplectic blow-up centered at an intersection point (resp. at a smooth point) of D.

Here, for blow-up at a smooth point p on the divisor D, it means that we first do a C^0 small perturbation of D to D' fixing p and then we do a symplectic blow-up of a ball centered at p such that D' coincide, in the local coordinates given by the ball, with a complex subspace. Similarly, for blow-up at an intersection point, a C^0 small perturbation is performed so that D' is ω-orthogonal at p and D' coincide, in the local coordinates given by the ball, with two complex subspaces.

To describe the corresponding blow-down operations, recall that a symplectic sphere with self-intersection -1 is called an exceptional sphere. The homology class of an exceptional sphere is called an exceptional class.

A **toric blow-down** refers to blowing down an exceptional sphere contained in D that intersects exactly two other irreducible components and exactly once for each of them. Moreover, we require that the intersections are positive and transversal. Such an exceptional sphere is called a toric exceptional sphere.

A **non-toric blow-down** refers to blowing down an exceptional sphere not contained in D that intersects exactly one irreducible component of D and exactly once with the intersection being positive and transversal. Such an exceptional spheres is called a non-toric exceptional sphere.

More precisely, for blow-down of a toric or non-toric exceptional sphere E, we first perturb our symplectic divisor D to another symplectic divisor D' (or perturbing E) such that the intersections of D' and E are ω-orthogonal (In the case that E is an irreducible component of D, we require E has ω-orthogonal intersections with all other irreducible components). Then, we will do the symplectic blow-down of E and D' will descend to a symplectic divisor.

Definition 4.1.3. An exceptional class e is called **non-toric** if e has trivial intersection pairing with the classes of all but one irreducible component of D and the only non-trivial pairing is 1.

An exceptional class e is called **toric** if e is homologous to an irreducible component of D such that e pairs non-trivially with the classes of exactly two other irreducible components of D and these two pairings are 1.

Clearly, the homology class of a toric (non-toric) exceptional sphere is a toric (non-toric) exceptional class. Conversely, we have the following observations.

For a toric exceptional class e, the component of D with class e is obviously a toric exceptional sphere in the class e. For a non-toric exceptional class e, we also have exceptional spheres in the class e, at least when D is $\omega-$orthogonal.

Lemma 4.1.4. *(cf. Theorem 1.2.7 of [67]) Let D be an ω-orthogonal symplectic divisor. There is a non-empty subspace $\mathcal{J}(D)$ of the space of ω-tamed almost complex structure making D pseudo-holomorphic such that for any non-toric exceptional class e, there is a residue subset $\mathcal{J}(D,e) \subset \mathcal{J}(D)$ so that e has an embedded J-holomorphic representative for all $J \in \mathcal{J}(D,e)$.*

Proof. It is immediate to prove that e is D-good in the sense of Definition 1.2.4 in [67] if e is non-toric. Theorem 1.2.7 of [67] then implies the result. □

4.1.2 Deformation of marked divisors

When we blow down an exceptional sphere, we encode the process by marking the descended symplectic divisor.

Definition 4.1.5. A marked symplectic divisor consists of a five-tuple

$$\Theta = (X, D, \{p_j\}_{j=1}^{l}, \omega, \{I_j\}_{j=1}^{l})$$

such that

- $D \subset (X, \omega)$ is a symplectic divisor,
- p_j, called centers of marking, are points on D (intersection points of D allowed),
- $I_j : (B(\delta_j), \omega_{std}) \to (X, \omega)$, called coordinates of marking, are symplectic embeddings sending the origin to p_j and with $I_j^{-1}(D) = \{x_1 = y_1 = 0\} \cap B(\delta_j)$ (resp. $I_j^{-1}(D) = (\{x_1 = y_1 = 0\} \cup \{x_2 = y_2 = 0\}) \cap B(\delta_j))$ if p_j is a smooth (resp. an intersection) point of D. Moreover, we require that the image of I_j are disjoint.

If p_j is an intersection point of D, then we define the symplectic embedding $I_j^{re} = I_j \circ re$, where $re(x_1, y_1, x_2, y_2) = (-x_2, -y_2, x_1, y_1)$ interchanges the two subspaces $\{x_1 = y_1 = 0\}$ and $\{x_2 = y_2 = 0\}$. If p_j is a smooth point of D, then we define $I_j^{re} = I_j$. For simplicity, we denote a marked symplectic divisor as (X, D, p_j, ω, I_j) or Θ and also call it a marked divisor if no confusion would arise.

Definition 4.1.6. Let $\Theta = (X, D, p_j, \omega, I_j)$ be a marked divisor. A $D-$**symplectic homotopy** (resp. $D-$**symplectic isotopy**) of Θ is a 4-tuple (X, D, p_j, ω_t) such that ω_t is a smooth family of symplectic forms (resp. cohomologous symplectic forms) on X with $\omega_0 = \omega$ and D being ω_t-symplectic for all t.

If $\Theta^2 = (X^2, D^2, p_j^2, \omega^2, I_j^2)$ is another marked symplectic divisor and there is a symplectomorphism sending the 4-tuple $(X^2, D^2, p_j^2, \omega^2)$ to a 4-tuple (X, D, p_j, ω_1) which is symplectic homotopic (isotopic) to Θ, then we say that Θ and Θ^2 are $D-$**symplectic deformation equivalent** (resp. **strict** $D-$**symplectic deformation equivalent**).

A symplectic divisor can be viewed as a marked divisor with empty markings.

Lemma 4.1.7. *Two symplectic divisors are (strict) deformation equivalent if and only if they are (strict) D-deformation equivalent as marked symplectic divisor.*

Proof. It follows directly from Lemma 4.1.2. To obtain a (strict) D-symplectic deformation equivalence from a (strict) symplectic deformation equivalence, we just have to pre-compose the symplectomorphism from $(X, D, \Phi_1^*\omega_1)$ to (X, D_1, ω_1). The other direction is similar. □

For marked divisors, both $D-$symplectic deformation equivalence and its strict version do not involve the symplectic embeddings I_j. We have the following seemingly stronger definition of deformation.

Definition 4.1.8. Let $\Theta = (X, D, p_j, \omega, I_j)$ be a marked divisor. A **strong** $D-$**symplectic homotopy** (resp. **strong** $D-$**symplectic isotopy**) of Θ is a 5-tuple $(X, D, p_j, \omega_t, I_{j,t})$ such that

- the 4-tuple (X, D, p_j, ω_t) is a $D-$symplectic homotopy (resp. isotopy) of Θ,
- D is ω_t-orthogonal, and
- $I_{j,t} : B(\epsilon_j) \to (X, \omega_t)$ are symplectic embedding sending the origin to p_j, $I_{j,0} = I_j|_{B(\epsilon_j)}$ and $(I_{j,t})^{-1}(D) = \{x_1 = y_1 = 0\} \cap B(\epsilon_j)$ (resp. $(I_{j,t})^{-1}(D) = (\{x_1 = y_1 = 0\} \cup \{x_2 = y_2 = 0\}) \cap B(\epsilon_j))$ if p_j is a smooth point (resp. p_j is an intersection point), for some $\epsilon_j < \delta_j$.

If $\Theta^2 = (X^2, D^2, p_j^2, \omega^2, I_j^2)$ is another marked sympelctic divisor and there is a symplectomorphism sending $(X^2, D^2, p_j^2, \omega^2, (I_j^2)^\#)$ to $(X, D, p_j, \omega_1, I_{j,1})$, where $(I_j^2)^\#$ is the unique choice between I_j^2 and $(I_j^2)^{re}$ such that the symplectomorphism is possible, then we say that Θ and Θ^2 are **strong** $D-$**symplectic deformation equivalent** (resp. **strong strict** $D-$**symplectic deformation equivalent**).

Lemma 4.1.9. *If* $\Theta = (X, D, \{p_j\}_{j=1}^l, \omega, \{I_j\}_{j=1}^l)$ *and* $\Theta^2 = (X^2, D^2, \{p_j^2\}_{j=1}^l, \omega^2, \{I_j^2\}_{j=1}^l)$ *are (strict)* $D-$*symplectic deformation equivalent, then they are strong (strict)* $D-$*symplectic deformation equivalent.*

Proof. Without loss of generality, we assume $\Theta^2 = (X, D, p_j, \omega_1, I_j^2)$. And for ease of the notations, we will only do the case when $l = 1$. For general l it can be done similarly.

Thus we assume Θ and Θ^2 are related by a $D-$symplectic homotopy (X, D, p, ω_t). And we denote I_1 by I and I_1^2 by I^2. Also, the proof is easier when p is a smooth point of D so we only prove the case when p_1 is an intersection point of D. Moreover, by possibly replacing I^2 with $(I^2)^{re}$, we assume the irreducible component of D corresponding to $\{x_1 = y_1 = 0\}$ in chart I is the same as that of I^2.

The idea of the proof goes as follows. First, we find a smooth family of symplectic embeddings of small ball $\Phi_t : (B(\delta), \omega_{std}) \to (X, \omega_t)$ sending the origin to p_1 such that $\Phi_0 = I|_{B(\delta)}$ and $\Phi_1 = I^2|_{B(\delta)}$. Then, we find another family of symplectic forms ω_t' such that the 4-tuple (X, D, p, ω_t') is still a $D-$symplectic homotopy of Θ with $\omega_1' = \omega_1$ and D is ω_t'-orthogonal for all t. The corresponding symplectic embeddings I_t' for (X, D, p, ω_t') will be constructed based on Φ_t such that the 5-tuple $(X, D, p, \omega_t', I_t')$ is a strong $D-$sympelctic homotopy between Θ and Θ^2.

By the one-parameter family version of Moser lemma, there exist a sufficiently small $\epsilon > 0$ and a smooth family of symplectic embeddings $\Phi = \{\Phi_t\} : (B(\epsilon), \omega_{std}) \to (X, \omega_t)$ sending the origin to p for all $t \in [0, 1]$. Moreover, Φ_0 can be chosen to coincide with $I|_{B(\epsilon)}$. This is not yet the Φ_t we want.

Notice that Φ_1 is a symplectic embedding of $(B(\epsilon), \omega_{std})$ to (X, ω_1) sending the origin to p and so is $I^2|_{B(\epsilon)}$. By possibly choosing a smaller ϵ, there is a symplectic isotopy of embeddings from Φ_1 to $I^2|_{B(\epsilon)}$ sending the origin to p for all time, by the trick in Exercise 7.22 of [69] (This is the trick to prove the space of symplectic embeddings of small balls is connected). By smoothing the concatenation of Φ_t with this symplectic isotopy, we can assume that $\Phi_1 = I^2|_{B(\epsilon)}$ as we want.

We actually need a further Φ_t by possibly another concatenation. For this purpose, consider the family of local divisors Let $F_t = \Phi_t^{-1}(D)$ in the standard coordinates in $(B(\epsilon), \omega_{std})$. Let M_t be the ordered 2-tuple of the symplectic tangent spaces to the two branches of F_t at the origin. Since $\Phi_0 = I|_{B(\delta)}$ and $\Phi_1 = I^2|_{B(\delta)}$, M_t is a loop. Let $-M_t$ be the inverse loop of M_t in the space of ordered 2-tuples of positively transversal intersecting two dimensional symplectic vector subspaces. We can find an isotopy of symplectic embeddings Ψ_t from Φ_1 to Φ_1 in (X, ω_1) such that the corresponding ordered 2-tuple of the symplectic tangent spaces of $\Psi_t^{-1}(D)$ at the origin is $-M_t$. By concatenating Φ_t with Ψ_t, we can assume in the beginning that the Φ_t we chose is such that M_t is null-homotopic. This is the Φ_t we want which gives a nice family of Darboux balls in (X, ω_t).

To construct ω_t', we will isotope the one parameter family of local divisors F_t (fixing both ends) to another one parameter family of symplectic divisors $F_{1,t}$ such that it coincides with $F_0 = F_1$ near the origin for all t. First, we perform a one-parameter family of C^1 small perturbations to make F_t coincide with a symplectic vector subspace in a sufficiently small ball $(B(\epsilon_2), \omega_{std})$, where $\epsilon_2 < \epsilon$. In other words, F_t coincides with M_t in $B(\epsilon_2)$. Since M_t is null-homotopic, there is a homotopy $W_{r,t}$ between M_t $(r = 0)$ and the constant path $M_0 = M_1$ $(r = 1)$ such that $W_{r,0} = W_{r,1} = M_0$ for all r. Hence, we can perform a one-parameter family of Lemma 5.10 of [71] (See its proof) to obtain a 3-parameter family of submanifolds $U_{r,s,t}$ in $B(\epsilon_2)$ such that $U_{r,s,t} = W_{s,t}$ outside a fixed small compact set containing the origin, $U_{r,s,t} = W_{r,t}$ close to the origin and $U_{r,r,t} = W_{r,t}$. As in the proof of Lemma 5.10 of [71], from $U_{r,s,t}$ one can construct an $s-$parameter of symplectic isotopy $F_{s,t} \subset B(\epsilon_2)$ such that

- $F_{0,t} = F_t$,
- $F_{s,t}$ is a pair of symplectic submanifolds positively intersecting at the origin for all $s, t \in [0, 1]$,
- $F_{1,t} = F_0 = F_1 = M_0 = M_1$ inside $B(\epsilon_4)$ for all t,
- $F_{s,0} = F_{s,1} = F_0 = F_1$, and
- the isotopy is supported inside $B(\epsilon_3)$,

where $0 < \epsilon_4 < \epsilon_3 < \epsilon_2$.

Due to the last bullet, we obtain a $2-$parameter family of marked divisors $D_{s,t}$ with $D_{0,t} = D_t, D_{s,0} = D_{s,1} = D$, and satisfying the bullets 2 and 3 above near the marked point (recall we assume there is only one marking for simplicity).

The effect of the symplectic isotopy from D_t $(s = 0)$ to $D_{1,t}$ $(s = 1)$ can be converted through symplectomorphism, as in Lemma 4.1.2, to replace (X, D, p, ω_t) $(s = 0)$ by an another $D-$symplectic homotopy (X, D, p, ω_t') $(s = 1)$. More precisely, for the 1-parameter family of isotopy $D_{s,t}$ parameterized by t, we can find a 1-parameter family of ambient isotopy $\Delta = \{\Delta_s\}_{t\in[0,1]} = \{\Delta_{s,t}\}$, $\Delta_{s,t} : X \to X$ extending the 1-parameter family of isotopy $D_{s,t}$ (in particular, for fixed t_0, Δ_{s,t_0} is an ambient isotopy extension of D_{s,t_0}) such that $\Delta_{0,t} = \Delta_{s,0} = \Delta_{s,1} = Id_X$. Then we define $\omega'_t = \Delta^*_{1,t}\omega_t$.

By construction, we have

• $\omega'_i = \omega_i$ for $i = 0, 1$,

• D is positively ω'_t-orthogonal for all t

• there is a family of symplectic embedding $\Phi'_t : B(\epsilon_4) \to (X, \omega_t)$ such that $\Phi_t'^{-1}(D) = F_0$ for all t, and

• $\Phi'_0 = I|_{B(\epsilon_4)}$ and $\Phi'_1 = I^2|_{B(\epsilon_4)}$

In particular, if we let $I'_t = \Phi'_t$, then $(X, D, p, \omega'_t, I'_t)$ is a strong $D-$symplectic homotopy between Θ and Θ^2. The strict version follows similarly. □

The ultimate goal for this section is the following proposition, which will be proved after discussing various operations for marked divisors in the next subsection.

Proposition 4.1.10. *Let $\Theta = (X, D, p_j, \omega, I_j)$ and $\Theta^2 = (X^2, D^2, p^2_j, \omega^2, I^2_j)$ be two marked divisors both with l marked points.*

(i) Up to moving in the $D-$symplectic deformation class, we can blow down a toric or non-toric exceptional class in Θ (and Θ^2) to obtain a marked divisor $\hat{\Theta}$ (resp. $\hat{\Theta}^2$) with an extra marked point (For toric exceptional class, original marked points on the exceptional sphere will be removed after blow-down).

(ii) Moreover, if the blow down divisors $\hat{\Theta}$ and $\hat{\Theta}^2$ are $D-$symplectic deformation equivalent such that the extra marked points correspond to each other in the equivalence, then Θ and Θ^2 are $D-$symplectic deformation equivalent.

4.1.3 Operations on marked divisors

This subsection studies various operations on marked divisors as well as their stability property with respect to $D-$symplectic deformation.

• **Perturbations**

The following fact will be frequently used.

Lemma 4.1.11. *Perturbations of a marked divisor preserve the strict $D-$symplectic deformation class.*

Proof. A perturbation of a marked divisor is simply a symplectic isotopy of the corresponding (unmarked) divisor. By Lemma 4.1.2, the perturbed divisor is symplectomorphic to the original divisor, up to a $D-$symplectic isotopy. $\square$

- **Marking addition**

A marking addition of a marked divisor $(X, D, \{p_j\}_{j=1}^l, \omega, \{I_j\}_{j=1}^l)$ is another marked divisor $(X, D, \{p_j\}_{j=1}^{l+1}, \omega, \{I_j\}_{j=1}^{l+1})$ with the additional marking (p_{l+1}, I_{l+1}).

Lemma 4.1.12. *Let $(X, D, \{p_j\}_{j=1}^l, \omega, \{I_j\}_{j=1}^l)$ be a marked divisor. If the two marked divisors $(X, D, \{p_j\}_{j=1}^l \cup \{q_1\}, \omega, \{I_j\}_{j=1}^l \cup \{I_{q_1}\})$ together with $(X, D, \{p_j\}_{j=1}^l \cup \{q_2\}, \omega, \{I_j\}_{j=0}^l \cup \{I_{q_2}\})$ are obtained by adding markings (q_1, I_{q_1}) and (q_2, I_{q_2}) respectively, then they are strict $D-$symplectic deformation equivalent if*

- *the centers q_1 and q_2 coincide (intersection points of D allowed), or*
- *q_1 and q_2 are distinct smooth points of the same irreducible component.*

Proof. If q_1 and q_2 are the same point of D, then the claim is trivial since Definition 4.1.6 only involves the centers of marking, not the coordinates.

If q_1 and q_2 are smooth points of the same irreducible component, say C_1, then we need to show that the 4-tuple $(X, D, \{p_j\}_{j=1}^l \cup \{q_2\}, \omega)$ is symplectomorphic to a $D-$symplectic isotopy of $(X, D, \{p_j\}_{j=1}^l \cup \{q_1\}, \omega)$. For this purpose, we find a symplectic isotopy of $(D, \omega|_D)$ fixing C_1 setwise, fixing intersection points and $\{p_j\}$ pointwise and moving q_1 to q_2. Using the smooth isotopy extension theorem as in Lemma 4.1.2, this isotopy of symplectic divisor gives rise to a smooth isotopy Φ_t of X. The desired $D-$symplectic isotopy is obtained by taking the $D-$symplectic isotopy to be $(X, D, \{p_j\}_{j=1}^l \cup \{q_1\}, \Phi_t^* \omega)$ and the symplectomorphism to be $\Phi_1 : (X, D, \{p_j\}_{j=1}^l \cup \{q_1\}, \Phi_1^* \omega) \to (X, D, \{p_j\}_{j=1}^l \cup \{q_2\}, \omega)$.

$\square$

We note that marking addition at an intersection point of a marked divisor is not always possible because the intersection might not be ω-orthogonal. However, by Lemma 4.1.11, marking addition at a non-marked intersection point is always possible at the cost of choosing another representative in the strict $D-$symplectic deformation class because a C^0 small perturbation among symplectic divisor suffices to make the intersection point ω-orthogonal ([36]).

- **Marking moving**

Sometimes, it is useful to be able to move an intersection point.

Lemma 4.1.13. *Let* $(X, D = C_1 \cup C_2 \cup \cdots \cup C_k, \{p_j\}_{j=1}^l, \omega, \{I_j\}_{j=1}^l)$ *be a marked divisor. Let* $[C_2]^2 = -1$ *and* $p_1 = C_1 \cap C_2$*. For any smooth point* $\overline{p_1}$ *on* C_2*, there is a marked divisor* $(X, \overline{D} = \overline{C_1} \cup C_2 \cup \cdots \cup C_k, \{\overline{p_1}\} \cup \{p_j\}_{j=2}^l, \omega', \{\overline{I_j}\}_{j=1}^l)$ *such that* $\overline{p_1} = \overline{C_1} \cap C_2$*, where* $\omega' = \omega$ *and* $C_1 = \overline{C_1}$ *away from a small open neighborhood of* C_2*. Moreover, these two marked divisors are in the same* $D-$*symplectic deformation equivalence class.*

Proof. By Lemma 4.1.11 we may assume that the intersection points of D are ω-orthogonal. In particular, if C_j intersects C_2, then C_j coincides with a fiber of the symplectic normal bundle of C_2 when identifying the symplectic normal bundle with a tubular neighborhood of C_2.

Choose an ω-compatible almost complex structure J integrable near C_2 which coincides with $(I_j)_*(J_{std})$ for all j and making the symplectic normal bundle a holomorphic vector bundle. We blows down C_2 and identify the ball obtained by blowing down C_2 as a chart $(B(\epsilon), \omega_{std}, J_{std})$. In this chart, C_j descends to the union of complex vector subspaces V_j each of which corresponds to an intersection point of $C_2 \cap C_j$. On the other hand, $\overline{p_1}$ being a point on C_2 represents a complex vector subspace $V_{\overline{p_1}}$ in this chart. We take a smooth family of complex vector subspaces W_t from V_1 to $V_{\overline{p_1}}$ avoiding V_j for all $j \neq 1$. Applying the trick in Lemma 5.10 of [71] with $N = N' = \emptyset$, $i = 1$, S being the center of $B(\epsilon)$, S_1 being the descended C_1, $W_t = W_1^t$, we obtain an isotopy of symplectic manifolds C^t supported in $B(\epsilon)$ from the descended C_1 (i.e. $C^{t=0}$) to some $C^{t=1} = \tilde{C}_1$ such that C^t coincides with W_t near the origin of $B(\epsilon)$ for all t. By blowing up $B(\epsilon_2) \subset B(\epsilon)$ for some sufficiently small ϵ_2, we can lift this symplectic isotopy to a $D-$symplectic deformation from $(X, D = C_1 \cup C_2 \cup \cdots \cup C_k, \{p_j\}_{j=1}^l, \omega, \{I_j\}_{j=1}^l)$ to $(X, \overline{D} = \overline{C_1} \cup C_2 \cup \cdots \cup C_k, \{\overline{p_1}\} \cup \{p_j\}_{j=2}^l, \omega', \{\overline{I_j}\}_{j=1}^l)$ such that $\overline{p_1} = \overline{C_1} \cap C_2$, where $\overline{C_1}$ is the proper transform of $\tilde{C}_1$. □

- **Canonical blow-up**

Given a marked divisor with l markings, there are l canonical blow-ups we can do, namely, blow-ups using the symplectic embeddings I_j and hence the blow-up size is $B(\delta_j)$. A canonical blow-up of a marked divisor is still a marked divisor with one less the number of p_j's.

Lemma 4.1.14. *If* $\Theta = (X, D, \{p_j\}_{j=1}^l, \omega, \{I_j\}_{j=1}^l)$ *and* $\Theta^2 = (X^2, D^2, \{p_j^2\}_{j=1}^l, \omega^2, \{I_j^2\}_{j=1}^l)$ *are* $D-$*symplectic deformation equivalent, then so are the marked divisors obtained by canonical blow-ups using* I_1 *and* I_1^2.

Proof. By Lemma 4.1.9, Θ and Θ^2 are strong $D-$symplectic deformation equivalent. By blowing up using $I_{1,t}$, we obtain a $D-$symplectic deformation equivalence between the blown-up marked divisors. □

4.1.4 Proof of Proposition 4.1.10

Proof of Proposition 4.1.10. For a non-toric class e, we can find by Lemma 4.1.4, a pseudo-holomorphic representative E such that D is at the same time pseudo-holomorphic, after possibly applying Lemma 4.1.11 to move Θ in the strict $D-$symplectic deformation class. By positivity of intersection, E intersects exactly one irreducible component of D and the intersections is positively transversally once and hence a non-toric exceptional curve. By perturbing E, we can assume E has ω-orthogonal intersection with D. We can get a marked divisor after blowing down E with a marked point corresponds to the contracted E.

For a toric class e, we again apply Lemma 4.1.11 to move Θ in its strict $D-$symplectic deformation class such that every intersection is ω-orthogonal. The irreducible component E of D in the class e is a toric exceptional sphere. Hence, E intersects two other irreducible components of D once. We apply Lemma 4.1.13 to find another representative of Θ in the $D-$symplectic deformation class such that after we blow down the exceptional curve, the intersection point corresponding to the exceptional curve is an ω-orthogonal intersection point so this descended divisor is still a marked divisor (recall, a marking for a marked divisor at an intersection point requires the intersection point is an ω-orthogonal intersection).

Finally, suppose the blow down divisors are $D-$symplectic deformation equivalent. We want to do canonical blow-ups and marking additions to recover our original divisor

D and D^2. Notice that, marking additions are needed because when one blow down a divisor which originally has markings on it, the marking will not persist after the blow-down. Therefore, when we blow up the symplectic ball back, we need marking additions to get back the original marked divisor. We remark that we may not get back exactly the pair of D and D^2 by just canonical blow-ups and marking additions but we can get some pair in the same $D-$symplectic deformation equivalence classes by Lemma 4.1.11.

Since $D-$symplectic deformation equivalence is stable under canonical blow-ups (Lemma 4.1.14) and marking additions (Lemma 4.1.12), we conclude that Θ is $D-$symplectic deformation equivalent to Θ^2.

$\square$

4.2 Minimal models

We first collect some facts, which should be well known to experts.

Lemma 4.2.1. *Let (X, D, ω) be a symplectic Calabi-Yau surface. Then X is rational or an elliptic ruled surface, and D is either a torus or a cycle of spheres. If (X, D, ω) is a Looijenga pair, then (X, ω) is rational.*

Proof. Since D is symplectic and $[D] = PD(c_1(X, \omega))$, we have $c_1(X, \omega)\cdot[\omega] = [D]\cdot[\omega] > 0$. By Theorem A of [61] or [76], X is rational or ruled.

Write $D = C_1 \cup C_2 \cdots \cup C_k$, where each C_i is a smoothly embedded closed symplectic genus g_i surface. By adjunction, we have $[C_i]\cdot[D] = [C_i]^2 + 2 - 2g_i$. Therefore, we have

$$[C_i] \cdot (\sum_{j \neq i} [C_j]) = 2 - 2g_i \geq 0.$$

In particular, we have $g_i \leq 1$ for all i. Since D is connected, D is either a torus or a cycle of spheres. Here a cycle of spheres means that the dual graph is a circle and each vertex has genus 0.

If X is not rational, then X admits an S^2-fibration structure over a Riemann surface of positive genus. After possibly smoothing, we get a torus T representing the class $c_1(X)$. Moreover, $c_1(X)(f) = 2$ where f is the fiber class. The projection from T to the base is of non-zero degree. Therefore, the base genus of X is at most 1. $\square$

For a cycle with k spheres we will also call it a $k-$gon, and a torus a $1-$gon. If we allow some C_i to be positively immersed, then by adjunction we see that the only possibility is a single sphere with one positive double point, which we call a degenerated 1-gon.

The following observations are straightforward.

Lemma 4.2.2. *The operations of toric blow-up, non-toric blow-up, toric blow-down and non-toric blow-down all preserve being symplectic log Calabi-Yau.*

In the next subsection it is convenient to apply a slightly more general version of toric blow-down: Suppose a component C of bi-gon D is an exceptional sphere. The generalized toric blow down of D along C is a nodal symplectic sphere, called a degenerated 1-gon. Notice that the degenerated 1-gon is still an anti canonical divisor.

4.2.1 Minimal reductions

Definition 4.2.3. A symplectic log Calabi-Yau surface (X, D, ω) is called a **minimal model** if either (X, ω) is minimal, or it is a symplectic Looijenga pair with $X = \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$.

Lemma 4.2.4. *Every symplectic log Calabi-Yau surface can be transformed to a minimal model via toric and non-toric blow down.*

Proof. **Non-toric blow-down** Suppose e is an exceptional class intersecting each component of D non-negatively. Then e is a non-toric exceptional class by adjunction.

By Lemma 4.1.4, there is an ω-compatible almost complex structure such that D J-holomorphic (possibly after perturbation of D) and e has an embedded J-holomorphic sphere representative E. Thus we can perform non-toric blow-down along E.

By iterative non-toric blow-downs, we end up with a symplectic log Calabi-Yau surface (X_0, D_0, ω_0) such that any exceptional class pairs negatively with some component of D.

Toric blow-down

If X_0 is not minimal and not diffeomorphic to $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$, then the exceptional class with minimal ω_0-area has an embedded representative, by Lemma 1.2 of [84]. Therefore, this embedded representative must coincide with an irreducible component C of D_0.

Therefore if D_0 is a torus then X_0 must be minimal. So from now on we assume that D_0 is a cycle of spheres, ie. (X_0, D_0, ω_0) is a Looijenga pair.

Suppose that C intersects two other components of D_0 and hence a toric exceptional sphere. In this case we perform toric blow down along C to get another symplectic Looijenga pair (X_0', D_0', ω_0'). We claim that there is no exceptional class in X_0' that pairs all irreducible components of D_0' non-negatively. If there were one, by Lemma 4.1.4, after possibly perturbing D_0' to be $\omega_0'-$orthogonal, then there would be a embedded pseudo-holomorphic representative E_0' intersecting exactly one irreducible component of D_0' transversally at a smooth point. This E_0' can be lifted to the symplectic log Calabi-Yau surface (X_0, D_0, ω_0) because the contraction of E_0 becomes an intersection point of D_0', which is away from E_0'. Contradiction.

If C only intersects with one component of D_0, then D_0 must be a bigon. We claim that $X_0 = \mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ in this case, and we are done with the minimal reduction process according to Definition 4.2.3. To see that we apply a generalized toric low-down along C to obtain (X_0', D_0', ω_0') where D_0' is a degenerated 1-gon. We next show that (X_0', ω_0') is minimal. After possibly perturbing the nodal point of D_0' to be $\omega_0'-$orthogonal so D_0' can be made a pseudo-holomorphic nodal sphere, the analysis above also shows that there is no exceptional class in X_0' that intersects $[D_0']$ non-negatively. Since D_0' represents the Poincáre dual of $c_1(X_0', \omega_0')$, there are also no exceptional class intersecting $[D_0']$ negatively. Thus we have shown that $X_0' = \mathbb{C}P^2$ or $S^2 \times S^2$. If X_0' is $\mathbb{S}^2 \times \mathbb{S}^2$, then D_0' is obtained by blowing down a component of a bi-gon D_0 in $X_0 = \mathbb{C}P^2\#2\overline{\mathbb{C}P^2}$. In this case there are three exceptional class in (X_0, ω_0) with pairwise intersecting number 1. It is simple to check by adjunction that any exceptional class not represented by any of the two components of D_0 is non-toric. But this situation would not appear due to our procedure performing non-toric blow down first. That only leaves the possibility that $X_0' == \mathbb{C}P^2$, from which it follows that $X_0 = \mathbb{C}P^2\#\overline{\mathbb{C}P^2}$.

In summary, we can do iterative toric blow-downs from (X_0, D_0, ω_0) to obtain a symplectic Looijenga pair (X_b, D_b, ω_b) such that either (X_b, ω_b) is minimal or X_b is diffeomorphic to $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$.

□

From Lemma 4.2.1, Lemma 4.2.2, Lemma 4.2.4 and adjunction formula, we can enumerate the minimal symplectic log Calabi-Yau surfaces up to the homology of the irreducible components.

• Case (A): The base genus of X is 1. D is a torus.

• Case (B): $X = \mathbb{C}P^2$. $c_1 = 3H$. Then the symplectic log Calabi-Yau are

$(B1)$ D is a torus,

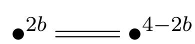
$(B2)$ D consists of a $H-$sphere and a $2H-$sphere, or

$(B3)$ D consists of three $H-$sphere.

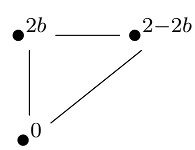
• Case (C): $X = \mathbb{S}^2 \times \mathbb{S}^2$, $c_1 = 2f + 2s$, where f and s are homology class of the two factors. By adjunction, the homology $af + bs$ of any embedded symplectic sphere satisfies $a = 1$ or $b = 1$. Symplectic log Calabi-Yau are

$(C1)$ D is a torus.

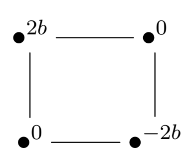
$(C2)$ If D has two irreducible components C_1 and C_2, then the only possible case (modulo obvious symmetry) is $[C_1] = f + bs$ and $[C_2] = f + (2 - b)s$. Its graph is given by



$(C3)$ If D has three irreducible components C_1, C_2 and C_3, then the only possible case (modulo obvious symmetry) is $[C_1] = f + bs$, $[C_2] = f + (1 - b)s$ and $[C_3] = s$. Its graph is given by



$(C4)$ If D has four irreducible components, then the only possible case (modulo obvious symmetry) is $[C_1] = f - bs$, $[C_2] = f + bs$, $[C_3] = s$ and $[C_3] = s$. Its graph is given by

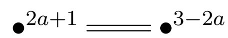


It is not hard to draw contradiction if D has 5 or more irreducible components.

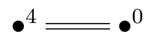
• Case (D): $X = \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$. $c_1 = f + 2s$, where f and s are fiber class and section class, respectively, such that $f^2 = 0$, $f \cdot s = 1$ and $s^2 = 1$. By adjunction, the homology $af + bs$ of an embedded symplectic sphere satisfies $b = 1$ or $b = 2 - 2a$.

$(D1)$ D cannot be a torus because it would not be minimal.

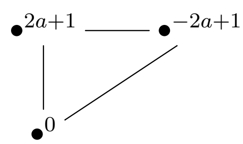
$(D2)$ If D has two irreducible components C_1 and C_2, then the only two possible cases (modulo obvious symmetry) are $([C_1],[C_2]) = (af+s,(1-a)f+s)$ and $([C_1],[C_2]) = (f,2s)$. The graphs are given by



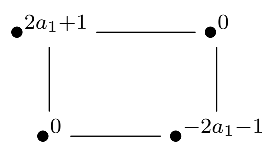
and



$(D3)$ If D has three irreducible components, then the only possible case (modulo obvious symmetry) is $[C_1] = af+s$, $[C_2] = -af+s$ and $[C_3] = f$.



$(D4)$ If D has four irreducible components, then the only possible case (modulo obvious symmetry) is $[C_1] = af+s$, $[C_2] = -(a+1)f+s$, $[C_3] = f$ and $[C_4] = f$.



It is not hard to draw contradiction if D has 5 or more irreducible components.

4.2.2 Deformation class of minimal models

In this section, we study the symplectic deformation class of minimal symplectic log Calabi-Yau surfaces.

Proposition 4.2.5. *Let $(X, D = C_1 \cup \cdots \cup C_k, \omega)$ be a minimal symplectic log Calabi-Yau surface. If $\overline{D} = \overline{C_1} \cup \cdots \cup \overline{C_k} \subset (X,\omega)$ is another symplectic divisor representing the first Chern class such that $[C_i] = [\overline{C_i}]$ for all i. Then (X, D, ω) is symplectic deformation equivalent to $(X, \overline{D}, \omega)$.*

The proof of Proposition 4.2.5 is separated into two cases, Proposition 4.2.6 and Proposition 4.2.9.

Isotopy in rational surfaces

Proposition 4.2.6. *Suppose (X, D, ω) and $(X, \overline{D}, \omega)$ are two homologous minimal symplectic log CY surfaces such that X is rational, then they are symplectic isotopic.*

The proof of Proposition 4.2.6 when D is a torus is given by [98] and Theorem B and Theorem C of [97]. We only need to deal with symplectic Looijenga pairs.

Recall that cohomologous symplectic forms on a rational or ruled 4-manifold are symplectomorphic (cf. [104], [48] and the survey [88]). Therefore it suffices to consider the following 'standard symplectic models' for $\mathbb{S}^2 \times \mathbb{S}^2$, $\mathbb{C}P^2$ and $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$.

• $\mathbb{S}^2 \times \mathbb{S}^2$ model:

When X is diffeomorphic to $\mathbb{S}^2 \times \mathbb{S}^2$, we define the product symplectic form $\omega_\lambda = (1+\lambda)\sigma \times \sigma$ with σ a symplectic form on the second factor with area 1 and $\lambda \geq 0$. Let E_0 be the class of the first factor, F be the class of the second factor and $E_{2k} = E_0 - kF$ for $0 \leq k \leq l$, where l is the integer with $l - 1 < \lambda \leq l$. For $0 \leq k \leq l$, let U_k be the set of ω_λ-compatible almost complex structure such that E_{2k} is represented by an embedded pseudo-holomorphic sphere.

• $\mathbb{C}P^2$ model:

When X is diffeomorphic to $\mathbb{C}P^2$, we use a multiple of the Fubini-Study form, $c\omega_{FS}$.

• $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ model:

When X is diffeomorphic to $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$, we use ω_λ to denote a form obtained by blowing up $(\mathbb{C}P^2, (2+\lambda)\omega_{FS})$ with size $1+\lambda$. So the line class H has area $2+\lambda$ and the exceptional class E_1 has area $1+\lambda$, where $\lambda > -1$. Let $F = H - E_1$ be the fiber class and let also $E_{2k+1} = E_1 - kF$ for $0 \leq k \leq l$, where l is again the integer with $l - 1 < \lambda \leq l$. Similarly, let U_k be the space of ω_λ-compatible almost complex structure such that E_{2k+1} is represented by an embedded pseudo-holomorphic sphere.

Proposition 4.2.7. *(Proposition 2.3 and Corollary 2.8 of [3], see also Proposition 6.4 of [58]) Let (X, ω_λ) be one of the above two cases. For each $0 \leq k \leq l$, U_k is non-empty and path connected. As a result, any two embedded symplectic spheres C_1 and C_2 representing the same class E_j for some $0 \leq j \leq 2l+1$ are symplectic isotopic to each other.*

Lemma 4.2.8. *Let (X, ω_λ) be as in Proposition 4.2.7. Assume $C_1, C_2 \subset X$ are two* *embedded symplectic spheres representing the same class E_j for some $0 \leq j \leq 2l+1$. Then there is a Hamiltonian diffeomorphism of (X, ω_λ) sending C_1 to C_2.*

Proof. By Proposition 4.2.7, we can find a symplectic isotopy $C_t \subset X$ from C_1 to C_2. We can extend this symplectic isotopy from a neighborhood of C_1 to a neighborhood of C_2 by a Moser type argument(See e.g. Chapter 3 of [69]). Our aim is to extend this symplectic isotopy to an ambient symplectic isotopy in order to obtain the result.

We first extend this symplectic isotopy to an ambient diffeomorphic isotopy $\Phi : X \times I \to X$. By considering the pull-back form $\Phi^*\omega_\lambda$, we can identify $C_1 = C_t = C_2$ for all t in the family of symplectic manifold $(X \times \{t\}, \Phi^*\omega_\lambda|_{X\times\{t\}})$, as in Lemma 4.1.2. We denote $\Phi^*\omega_\lambda|_{X\times\{t\}}$ as ω_λ^t. By definition, ω_λ^t is fixed near C_1 for all t. Identify a tubular neighborhood of C_1 with a symplectic normal bundle. Then, choose a smooth family of ω_λ^t-compatible almost complex structure J_t on X such that J_t is fixed near C_1 and the fibers of the normal bundle of C_1 are J_t-holomorphic. Pick a point p_1 on C_1. Let the J_t holomorphic sphere representing the fiber class F and passing through p_1 be C_t^F. Since the fiber class with a single point constraint has Gromov-Witten invariant one or minus one, C_t^F forms a symplectic isotopy by Gromov compactness. By Lemma 3.2.1 of [67] (let C_1 be C^{S_1} and $[C_t^F]$ be B_1), we can assume that the intersection between C_1 and C_t^F is ω_λ^t-orthogonal, after possibly perturbing J_t.

Now, $\Phi(C_1,t) \cup \Phi(C_t^F,t) = C_t \cup \Phi(C_t^F,t)$ is an ω_λ orthogonal symplectic isotopy in (X,ω_λ) (Strictly speaking, C_t^F is the image of another diffeomorphic isotopy Ψ such that $C_t^F = \Psi(C_1^F,t)$ and $C_1 = \Psi(C_1,t)$, then the isotopy we want is $\Phi(\Psi(\cdot,t),t)$). We can extend this symplectic isotopy to a neighborhood of it by another Moser type argument since $\Phi(C_1,t)$ intersects $\Phi(C_t^F,t)$ ω_λ-orthogonally. We have the exact sequence

$$H^1(C_1 \cup C_1^F, \mathbb{R}) = 0 \to H^2(X, C_1 \cup C_1^F, \mathbb{R}) \to H^2(X,\mathbb{R}) \to H^2(C_1 \cup C_1^F, \mathbb{R})$$

where the last arrow is an isomorphism and hence $H^2(X, C_1 \cup C_1^F, \mathbb{R}) = 0$. By Banyaga extension theorem (See e.g. [69]), there is an ambient symplectic isotopy agree with the symplectic isotopy $C_t \cup \Phi(C_t^F,t)$. Finally, this ambient symplectic isotopy is a Hamiltonian isotopy because $H^1(X) = 0$. □

Proof of Proposition 4.2.6. As seen in the previous section, D and $\overline{D}$ have at most four irreducible components. We are going to prove Proposition 4.2.6 by dividing it into the cases of two, three or four irreducible components. The proof for bigons is written with details, while the proof for triangles or rectangles being similar to that of bigons will be sketched.

• Bigons

First, let $(X,\omega) = (\mathbb{S}^2 \times \mathbb{S}^2, c\omega_\lambda)$ for some constant c, $D = C_1 \cup C_2$, $\overline{D} = \overline{C_1} \cup \overline{C_2}$ and $[C_i] = [\overline{C_i}]$ for $i = 1, 2$. Without loss of generality, we may assume $[C_1]^2 \leq [C_2]^2$. From the enumeration, we have $[C_1] = F + (2 - b_1)E_0$ and $[C_2] = F + b_1 E_0$ for some $b_1 \geq 1$, or $[C_1] = (2 - a_1)F + E_0$ and $[C_2] = a_1 F + E_0$ for some $a_1 \geq 1$. We consider the latter case and the first case can be treated similarly.

We first consider $a_1 \geq 2$. By Lemma 4.2.8, after composing a Hamiltonian diffeomorphism, we can assume C_1 and $\overline{C_1}$ completely coincide. Fix an ω-tamed almost complex structure J_0 making $C_1 = \overline{C_1}$ pseudo-holomorphic and integrable near C_1. Consider the set of ω-tamed almost complex structure $\mathcal{J}$ agree with J_0 near C_1. Fix $J \in \mathcal{J}$, we want to inspect all possible degenerations of J-holomorphic nodal curve representing $[C_2]$. By positivity of intersection and positivity of area, the homology class $aF + bE_0$ of any J-holomorphic curve has non-negative coefficient for the E_0 factor (i.e. $b \geq 0$). Therefore, the irreducible components (possibly not simple) of any J-holomorphic curve representing $[C_2]$ give rise to a decomposition $[C_2] = (s_1 F + E_0) + s_2 F + \cdots + s_m F$, where $s_j \geq 0$ for $2 \leq j \leq m$ (by positivity of intersection with $[C_1]$). If $s_1 \leq 0$, then $s_1 F + E_0 = [C_1]$ by positivity of intersection with $[C_1]$. The sum of non-negative Fredholm index of each individual component is given by $Ind_{nodal} = (4s_1 + 2) + 2(\sum_{i=2}^m s_i)$ when $s_1 \geq 0$, and $Ind_{nodal} = 2(\sum_{i=2}^m s_i)$ when $s_1 < 0$ because each component must be a sphere (the index formula for a pseudo-holomorphic curve with class A is $A^2 + c_1([A])$). On the other hand, the index of the class $[C_2]$ is given by $Ind_{C_2} = 2(2a_1) + 2 = 4(\sum_{i=1}^m s_i) + 2 = (4s_1 + 2) + 4(\sum_{i=2}^m s_i)$. If $s_1 \geq 0$ and $m \geq 2$, we have

$$Ind_{nodal} + 2 \leq (4s_1 + 2) + 4(\sum_{i=2}^{m} s_i) = Ind_{C_2}$$

If $s_1 < 0$, we have $s_1 = 2 - a_1$ and hence

$$Ind_{nodal} + 2 = 2(\sum_{i=2}^{m} s_i) + 2 = 2(a_1 - (2 - a_1)) + 2 = 4a_1 - 2 < Ind_{C_2}$$

Therefore, any degeneration happens in codimension two or higher.

Then we can apply the standard pseudo-holomorphic curve argument to obtain a symplectic isotopy from C_2 to $\overline{C_2}$ transversal to C_1 for all time along the isotopy and finish the proof. Since what we could not find references that fit exactly to out situation (Proposition 1.2.9(ii) of [67] is a very closely related one), we provide some details here. We will basically follow [70] together with Lemma 3.2.2 and Proposition 3.2.3 of [67].

We perturb C_2 and $\overline{C_2}$ so that they have $2a_1+1$ distinct intersection points and call these intersection points $\{p_j\}_{j=1}^{2a_1+1}$. We form the universal moduli space for genus 0 curve representing the class $[C_2]$ with $2a_1+1$ point constraints $\{p_j\}_{j=1}^{2a_1+1}$ with respect to the space of almost complex structures $\mathcal{J}$. We want to pick $J, \overline{J} \in \mathcal{J}$ that are regular for all underlying (marked) simple curves appearing in a degeneration of $[C_2]$ except $C_1 = \overline{C_1}$ such that C_2 is J-holomorphic and $\overline{C_2}$ is $\overline{J}$-holomorphic.

To find J and $\overline{J}$, we note the following two facts. For any $J \in \mathcal{J}$ (resp. $\overline{J} \in \mathcal{J}$) making C_2 J-holomorphic (resp. making $\overline{C_2}$ $\overline{J}$-holomorphic), the Fredholm operator taking the point constraints $\{p_j\}_{j=1}^{2a_1+1}$ into account is regular by automatic transversality (See Theorem 3.1 and Proposition 3.2 of [58], and also [42], [44]). On the other hand, for a generic choice of J (resp. $\overline{J}$) making C_1 and C_2 J-holomorphic (resp. $C_1 = \overline{C_1}$ and $\overline{C_2}$ $\overline{J}$-holomorphic), each simple curve other than C_1 and C_2 (resp. other than C_1 and $\overline{C_2}$) in any degeneration has a somewhere injective point away from C_1 and C_2 (resp. away from $\overline{C_1}$ and $\overline{C_2}$) and hence is regular (See Chapter 3.4 of [70]). As a result, we can find $J, \overline{J} \in \mathcal{J}$ as desired.

For such $J, \overline{J}$, there is a regular smooth path $J_t \in \mathcal{J}$ (regular in the sense of Definition 6.2.10 of [70]) such that the parametrized moduli space of J_t-holomorphic curves representing $[C_2]$ and passing through $\{p_j\}_{j=1}^{2a_1+1}$ forms a non-empty one dimensional smooth manifold. Since degeneration happens in codimension 2 or higher, if we choose J_t to be also regular with respect to the lower strata, the one dimensional moduli space is also compact.

Thus, there is a family of embedded J_t-holomorphic spheres C^t all of which passing through $\{p_j\}_{j=1}^{2a_1+1}$. By positivity of intersection, C^t is the only J_t-holomorphic family passing through $\{p_j\}_{j=1}^{2a_1+1}$, hence we have a symplectic isotopy from C_2 to $\overline{C_2}$. Finally, by applying Lemma 3.2.1 of [67] to $\{C^t\}$ to get another symplectic isotopy $\{C^{t'}\}$ transversal to C_1, we get that the intersection pattern of $\{C^{t'}\} \cup C_1$ is unchanged along the symplectic isotopy. This finishes the proof when $a_1 \geq 2$.

The case that $a_1 = 1$ can be treated similarly, which is easier and only requires an analogue of Proposition 4.2.7 and Lemma 4.2.8 for non-negative self-intersection sphere (See e.g Proposition 3.2 of [58]).

Now, we consider $(X, \omega) = (\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}, c\omega_\lambda)$ for some constant c, $D = C_1 \cup C_2$, $\overline{D} = \overline{C_1} \cup \overline{C_2}$ and $[C_i] = [\overline{C_i}]$ for $i = 1, 2$. By the enumeration, there are two possible cases.

The first one is when $[C_1] = [\overline{C_1}] = (1 - a_1)f + s = (2 - a_1)F + E_1$ and $[C_2] = [\overline{C_2}] = a_1 f + s = (a_1 + 1)F + E_1$. By symmetry, it suffices to consider $a_1 \geq 1$. If $a_1 \geq 2$, we apply Lemma 4.2.8 and assume C_1 completely coincides with $\overline{C_1}$. Again, we inspect all possible J-holomorphic degenerations of C_2 for J making C_1 J-holomorphic. A direct index count as before shows that any degeneration of C_2 has at least codimension two. Therefore, the same method applies. The case that $a_1 = 1$ is dealt similarly.

The other case is $[C_1] = [\overline{C_1}] = f = F$ and $[C_2] = [\overline{C_2}] = 2s = 2F + 2E_1$. This cannot cause additional trouble as they have non-negative self-intersection numbers. One can deal with this similar to the previous cases.

The case that $X = \mathbb{C}P^2$ is analogous and easier.

• Triangles and Rectangles

Now, we consider $X = \mathbb{S}^2 \times \mathbb{S}^2$ or $X = \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ and assume D_b has three or four irreducible components. We observe that, there is at most one component with negative self-intersection number and one with positive self-intersection numbers in all cases. Moreover, the homology class of the component with negative self-intersection number is of the form $E_i + jF$ for some j and $i = 0, -1$. If there is a negative self-intersection component, we can apply Lemma 4.2.8 and assume the negative self-intersection components for D and $\overline{D}$ completely coincide. Then we study all the possible J-holomorphic degeneration of the positive curve for J making the negative component J-holomorphic. One can show that the degeneration happens in at least codimension two by index count. Therefore, we can find a relative pseudo-holomorphic isotopy Φ_t from the positive self-intersection component of D to the positive self-intersection component of $\overline{D}$. At the same time, since the remaining components of D and $\overline{D}$ are sphere fibers, which cannot have any pseudo-holomorphic degeneration, the pseudo-holomorphic isotopy Φ_t can be extended to a pseudo-holomorphic isotopy from D to $\overline{D}$. Hence, the result follows when there is a negative self-intersection component. The remaining cases are all similar and simpler, including the case when $X = \mathbb{C}P^2$. $\square$

Elliptic ruled surfaces

In this subsection, we want to finish the proof of Proposition 4.2.5 for the torus type.

Proposition 4.2.9. *Suppose (X, D, ω) and $(X, \overline{D}, \omega)$ are minimal symplectic log Calabi-Yau surfaces such that X is elliptic ruled. Then they are symplectic deformation equivalent to each other.*

We first describe the complement of D following [105]. Any ω-compatible almost complex structure J provides us a J-holomorphic ruling, meaning that there is a sphere bundle map $\pi : X \to \mathbb{T}^2$ such that fibers are J-holomorphic. Usher proves in [105] (Lemma 3.5) that $\pi|_D$ is a two to one covering and in particular D is transversal to the J-holomorphic sphere foliation. If a tubular neighborhood of D is taken out, we have a J-holomorphic annulus foliation, which defines an annulus bundle $X - P(D)$ over the torus $\mathbb{T}^2$. We want to identify this annulus bundle.

Equip the orientation of $\mathbb{T}^2$ such that $\pi|_D$ is orientation preserving, where the orientation of D is given by J. Choose a positively oriented basis $\{t, u\} \in H_1(D, \mathbb{Z})$ and $\{v, w\} \in H_1(\mathbb{T}^2, \mathbb{Z})$ such that $\pi_* t = v$ and $\pi_* u = 2w$. Let $\mathbb{A} = \{z \in \mathbb{C} | \frac{1}{2} \leq |z| \leq 2\}$. The monodromy of this annulus bundle around the loop corresponding to v is orientation preserving and does not flip the boundary. Therefore, the monodromy is isotopic to the identity. Similarly, the monodromy of this annulus bundle around the loop corresponding to w is orientation preserving but flip the boundary components due to $\pi_* u = 2w$. Therefore, the monodromy is isotopic to the map sending z to z^{-1}. This annulus bundle is isomorphic as an annulus bundle to (See the paragraph before Lemma 3.6 of [105])

$$\mathbb{S}^1 \times \frac{\mathbb{R} \times \mathbb{A}}{(x+1, z) \sim (x, z^{-1})}$$

if X is the smoothly trivial sphere bundle, and isomorphic to

$$\frac{\mathbb{R} \times \mathbb{S}^1 \times \mathbb{A}}{(x+1, e^{i\theta}, z) \sim (x, e^{i\theta}, e^{i\theta} z^{-1})}$$

if X is the smoothly non-trivial sphere bundle.

Let $\overline{D}$ be another connected symplectic torus representing $c_1(X)$. For $\overline{D}$, we can also define $\overline{J}$, $\overline{\pi}$, $\overline{\mathbb{T}^2}$, $\overline{t}, \overline{u}, \overline{v}, \overline{w}$ as above. Let $\tau : \mathbb{T}^2 \to \overline{\mathbb{T}^2}$ be a diffeomorphism sending v

and w to $\overline{v}$ and $\overline{w}$, respectively. By construction, the pull-back annulus bundle $\tau^*(\overline{X} - P(\overline{D})) \to \mathbb{T}^2$ has the same monodromy (up to isotopy) as $X - P(D) \to \mathbb{T}^2$ over the one-skeleton. The existence of an annulus bundle isomorphism from $X - P(D)$ to $\tau^*(X - \overline{D})$ covering the identity of $\mathbb{T}^2$ reduces to whether $X - P(D)$ and $\tau^*(\overline{X} - \overline{D})$ are isomorphic annulus bundle (covering some diffeomorphism of the base), which is true because there is only one class of isomorphic annulus bundle for a choice of monodromies over one skeleton (and it is explicitly described above in our case). Therefore, we have a bundle isomorphism $F : X - P(D) \to X - P(\overline{D})$ covering τ. On the other hand, since the image of $\tau_* \circ \pi_*|_{H_1(D,\mathbb{Z})}$ equals the image of $\overline{\pi}_*|_{H_1(\overline{D},\mathbb{Z})}$, there are two lifts of τ to $\tilde{\tau}_i : D \to \overline{D}$ such that $\overline{\pi} \circ \tilde{\tau}_i = \tau \circ \pi$, for $i = 1, 2$. Then, there is a unique way, up to isotopy, to get a sphere bundle isomorphism $\tilde{F} : X \to X$ extending F and $\tilde{\tau}_1$ (or, F and $\tilde{\tau}_2$) by following the pseudo-holomorphic foliation. In particular, we have $\tilde{F}(D) = \overline{D}$.

Using $\tilde{F}$, we can identify $\overline{D} \subset (X, \omega)$ with $D \subset (X, \tilde{F}^*\omega)$. Proposition 4.2.9 will follow if we can find a symplectic deformation equivalence from (X, D, ω) to $(X, D, \tilde{F}^*\omega)$, which can be obtained by the following lemma.

Lemma 4.2.10. *Let $\pi : (X, \omega_i, J_i) \to B$ be a symplectic surface bundle over surface such that J_i is ω_i-compatible and fibers are J_i holomorphic for both $i = 0, 1$. Moreover, we assume the orientation of fibers induced by J_0 and J_1 are the same and the orientation of the total space induced by ω_0^2 and ω_1^2 are the same. Assume $D \subset (X, \omega_i)$ is a J_i holomorphic surface for $i = 0, 1$. and $\pi|_D$ is submersive. Then there is a smooth family of (possibly non-homologous) symplectic forms ω_t on X making D symplectic for all $t \in [0, 1]$ joining ω_0 and ω_1.*

Proof. Fix a point $p \in X$ and consider a non-zero tangent vector $v \in T_pX$ which does not lie in the vertical tangent sub-bundle T_pX^{vert}. Since fibers are J_i holomorphic, we have $Span\{v, J_iv\} \cap T_pX^{vert} = \{0\}$. Choose a volume form (symplectic form) ω_B on B. Since π is a submersion, $\pi_* Span\{v, J_iv\} = T_{\pi(p)}\mathbb{B}$. Therefore, we have $\omega_B(\pi_*(v), \pi_*(J_iv)) \neq 0$. By possibly changing the sign of ω_B, we can assume $\omega_B(\pi_*(v), \pi_*(J_iv)) > 0$. Moreover, this inequality is true for any $v \in T_pX$ not lying in T_pX^{vert}. By continuity, $\omega_B(\pi_*(v), \pi_*(J_iv)) > 0$ for any $p \in X$ and any $v \in T_pX - T_pX^{vert}$ for both $i = 0, 1$.

Now, we apply the Thurston trick. For any $K \geq 0$, we let $\omega_i^K = \omega_i + K\pi^*\omega_B$, which is clearly closed. It is also non-degenerate because it is non-degenerate for the vertical tangent sub-bundle and for any $p \in X$, and any $v \in T_pX - T_pX^{vert}$, we have $\omega_i^K(v, J_iv) = \omega_i(v, J_iv) + K\omega_B(\pi_*(v), \pi_*(J_iv)) > 0$. The first term being greater than zero is by compatibility and the second term being non-negative is due to $K \geq 0$ and the first paragraph. Notice that D is symplectic with respect to ω_i^K for both $i = 0, 1$ because $\pi|_D$ is submersive and D is J_i-holomorphic.

Now, we consider $\omega_t^K = (1-t)\omega_0^K + t\omega_1^K$, which is clearly closed and non-degenerate for TX^{vert}. For $v \in T_pX - T_pX^{vert}$, we have $\omega_t^K(v, J_0v) = (1-t)\omega_0(v, J_0v) + t\omega_1(v, J_0v) + K\omega_B(\pi_*v, \pi_*J_0v)$. We know that the first and the third terms on the right hand side are non-negative but we have no control on the second term. However, there is a sufficiently large K such that $\omega_t^K(v, J_0v) > 0$ for all $p \in X$ and $v \in T_pX - T_pX^{vert}$ and for all t because the sphere subbundle of TX is compact. By smoothening out the piecewise smooth family from ω_0 to ω_0^K, ω_t^K and from ω_1^K to ω_1, we finish the proof.

□

We remark that Lemma 4.2.10 can be viewed as a relative version of Proposition 4.4 in [66] in dimension four.

4.2.3 Proof of Theorem 1.2.11

We are ready to prove Theorem 1.2.11.

Proof of Theorem 1.2.11. Let (X^i, D^i, ω^i) be symplectic log Calabi-Yau surfaces for $i = 1, 2$, which are homology equivalent via a diffeomorphism Φ.

Let $\{e_1, \dots, e_\beta\}$ be a maximal set of pairwisely orthogonal non-toric exceptional classes in X. We can choose an almost complex structure J^1 (possibly after deforming D^1) such that D^1 is J^1-holomorphic and all e_j has embedded J^1-holomorphic representative, by Lemma 4.1.4. Since (X^1, D^1, ω^1) and (X^2, D^2, ω^2) are homological equivalent via Φ, $\{\Phi_*(e_j)\}$ is a maximal set of pairwisely orthogonal non-toric exceptional classes. We can find an ω^2-tamed almost complex structure (possibly after deforming D^2) J^2 such that D^2 is J^2-holomorphic and the $\Phi_*(e_j)$ has embedded J^2-holomorphic representative. After blowing down the J^i-holomorphic representatives of e_j, and $\Phi_*(e_j)$ for all $1 \leq j \leq \beta$, we obtain two symplectic Looijenga pairs $(\overline{X^1}, \overline{D^1}, \overline{\omega^1})$ and $(\overline{X^2}, \overline{D^2}, \overline{\omega^2})$.

Clearly, $(\overline{X^1}, \overline{D^1}, \overline{\omega^1})$ and $(\overline{X^2}, \overline{D^2}, \overline{\omega^2})$ are homological equivalent for some natural choice of diffeomorphism $\overline{\Phi}$. Now, a component in $\overline{D^1}$ is exceptional if and only if the corresponding component in $\overline{D^2}$ is exceptional. By Lemma 4.2.4, we pass to minimal models $(\overline{X^i_b}, \overline{D^i_b}, \overline{\omega^i_b})$ by toric blow-downs. By identifying $\overline{X^1_b}$ and $\overline{X^2_b}$ with a natural choice of diffeomorphism $\overline{\Phi_b}$, the homology classes of the components of $\overline{D^1_b}$ and $\overline{D^2_b}$ are the same.

By Proposition 1.2.15 of [67] or Theorem 2.9 of [22], up to a D-symplectic homotopy (ie. a deformation of $\overline{\omega^2_b}$ keeping $\overline{D^2_b}$ symplectic), we can assume $[\overline{\omega^1_b}] = \overline{\Phi_b}^*[\overline{\omega^2_b}]$. Therefore, $\overline{X^1_b}$ and $\overline{X^2_b}$ are actually symplectomorphic ([104], [48]) and we thus can choose $\overline{\Phi_b}$ to be a symplectomorphism from $(\overline{X^1_b}, \overline{\Phi_b}^{-1}(\overline{D^2_b}), \overline{\omega^1_b})$ to $(\overline{X^2_b}, \overline{D^2_b}, \overline{\omega^2_b})$. Therefore, we conclude that $(\overline{X^1_b}, \overline{D^1_b}, \overline{\omega^1_b})$ and $(\overline{X^2_b}, \overline{D^2_b}, \overline{\omega^2_b})$ are symplectic deformation equivalent, by applying Proposition 4.2.5 to $(\overline{X^1_b}, \overline{D^1_b}, \overline{\omega^1_b})$ and $(\overline{X^1_b}, \overline{\Phi_b}^{-1}(\overline{D^2_b}), \overline{\omega^1_b})$. Further, by Lemma 4.1.7, they are $D-$symplectic deformation equivalent.

Now we record the sequence of non-toric and toric blow-downs by markings $\overline{D^1_b}$ and $\overline{D^2_b}$. As marked divisors, they are $D-$symplectic deformation equivalent by Lemma 4.1.12. Finally, by Proposition 4.1.10 (and viewing unmarked divisors as marked divisors without markings), (X^1, D^1, ω^1) is $D-$symplectic deformation equivalent to (X^2, D^2, ω^2), and hence symplectic deformation equivalent to (X^2, D^2, ω^2) by Lemma 4.1.7. Tracing the steps, we see that the symplectomorphism in the symplectic deformation equivalence between (X^1, D^1, ω^1) and (X^2, D^2, ω^2) has the same homological effect as Φ.

Now, assume (X^1, D^1, ω^1) is *strictly* homological equivalent to (X^2, D^2, ω^2) via a diffeomorphism Φ. It means that Φ is a homological equivalence and $\Phi^*[\omega^2] = [\omega^1]$. We first note that, up to symplectic isotopy of D^1 and D^2, which preserves the strict D-symplectic deformation class (Lemma 4.1.11), we can assume D^i are ω^i-orthogonal. We have shown that there is a $D-$symplectic homotopy (X^1, D^1, ω^1_t) of (X^1, D^1, ω^1) and a symplectomophism $\Psi : (X^1, D^1, \omega^1_1) \to (X^2, D^2, \omega^2)$ with the same homological effect as Φ. Therefore, we have $[\omega^1] = \Phi^*[\omega^2] = \Psi^*[\omega^2] = [\omega^1_1]$. By Theorem 1.2.12 of [67], ω^1_t can be chosen such that $[\omega^1_t]$ is constant for all t. By Corollary 1.2.13 of [67], there is a symplectic isotopy (X^1, D^1_t, ω^1) such that $D^1_0 = D^1$ and (X^1, D^1_1, ω^1) is symplectomorphic to (X^1, D^1, ω^1_1) and hence to (X^2, D^2, ω^2). Therefore, the result follows.

□

In the case $X^1 = X^2 = X$, Theorem 1.2.11 implies the symplectic deformation class of (X, D, ω) is uniquely determined by the homology classes $\{[C_j]\}_{j=1}^k$ modulo the action of diffeomorphism on $H_2(X, \mathbb{Z})$. The fact the the homology classes of D completely determine the symplectic deformation equivalent class can be regarded as in the same spirit of Torelli type theorems in a weak sense.

If $(X^1, \omega^1) = (X^2, \omega^2) = (X, \omega)$, we can take the strict homological equivalence to be identity and hence the symplectomorphism from (X, D^1, ω) to the time-one end of the symplectic isotopy of (X, D^2, ω) in Theorem 1.2.11 has trivial homological action. Therefore, the number of symplectic isotopy classes of homological equivalent log Calabi-Yau surfaces in (X, ω) is bounded above by the number of connected components of Torelli part of the symplectomorphism group of (X, ω).

Chapter 5

Dehn twist long exact sequences

In this chapter, we discuss Lagrangian Dehn twist and prove the induced long exact sequences (Theorem 1.3.1, 1.3.4 and 1.3.5).

5.1 Dehn twist and Lagrangian surgeries

5.1.1 Dehn twist

Let S be a connected closed manifold equipped with a Riemannian metric $g(\cdot, \cdot)$ such that every geodesic is closed of length 2π. We identify T^*S with TS by g and switch freely between the two. The following lemma is well-known.

Lemma 5.1.1. *The Hamiltonian $\varpi : T^*S \to \mathbb{R}$ defined by*

$$\varpi(\xi) = \|\xi\|$$

for all $q \in S$ and $\xi \in T_q^*S$ has its Hamiltonian flow X_ϖ coincides with the normalized geodesic flow on $T^*S \backslash \{0_{section}\}$.

To define Dehn twist, we need to introduce an auxiliary function. We first consider the case when S is not diffeomorphic to a sphere. For $\epsilon > 0$ small, we define a **Dehn twist profile** to be a smooth function $\nu_\epsilon^{Dehn} : \mathbb{R}^+ \to \mathbb{R}$ such that

(1) $\nu_\epsilon^{Dehn}(r) = 2\pi - r$ for $r \ll \epsilon$,

(2) $0 < \nu_\epsilon^{Dehn}(r) < 2\pi$ for all $r < \epsilon$, and

(3) $\nu_\epsilon^{Dehn}(r) = 0$ for $r \geq \epsilon$

Definition 5.1.2. If S is not diffeomorphic to a sphere, the model Dehn twist $(\tau_S, \nu_\epsilon^{Dehn})$ on T^*S is given by

$$\tau_S(\xi) = \phi^{\varpi}_{\nu_\epsilon^{Dehn}(\|\xi\|)}(\xi)$$

on $T^*S - \{0_{section}\}$ and identity on the zero section.

We will simply write τ_S instead of $(\tau_S, \nu_\epsilon^{Dehn})$.

When S is diffeomorphic to a sphere, the **spherical Dehn twist profile** ν_ϵ^{Dehn} is picked with (1)(2) above replaced by

(1') $\nu_\epsilon^{Dehn}(r) = \pi - r$ for $r \ll \epsilon$, and

(2') $0 < \nu_\epsilon^{Dehn}(r) < \pi$ for all $r < \epsilon$

In this case, Dehn twist $(\tau_S, \nu_\epsilon^{Dehn})$ is defined analogously but antipodal map is used to extend smoothly along the zero section instead of the identity map.

Example 5.1.3. Let $(q,p) \in \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R} = T^*S^1$ be equipped with the standard symplectic form $\omega_{S^1} = dp \wedge dq$. For a spherical profile ν_ϵ^{Dehn}, $(\tau_{S^1}, \nu_\epsilon^{Dehn})$ is defined by

$$\tau_{S^1}(q,p) = \begin{cases} (q + \nu_\epsilon^{Dehn}(||p||)\frac{p}{||p||}, p) & \text{for } p \neq 0 \\ (q+\pi, 0) & \text{for } p = 0 \end{cases}$$

Consider the double cover $\iota_{double} : T^*S^1 \to T^*\mathbb{RP}^1 = \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}$ given by

$$\iota_{double}(q,p) = (2q, \frac{1}{2}p) = (\widetilde{q}, \widetilde{p})$$

For $(\widetilde{q}, \widetilde{p}) = \iota_{double}(q,p) \in T^*\mathbb{RP}^1$, we define

$$T(\widetilde{q}, \widetilde{p}) = \iota_{double} \circ \tau_{S^1}(q,p)$$

which is independent of the choice of (q,p) as lift of $(\widetilde{q}, \widetilde{p})$. It is an easy exercise to show that T is Hamiltonian isotopic to $\tau_{\mathbb{RP}^1}$ for the push-forward Dehn twist profile. Also, if we identify $T^*\mathbb{RP}^1$ with T^*S^1 so that τ_{S^1} is well-defined on $T^*\mathbb{RP}^1$, then T is also Hamiltonian isotopic to $\tau_{S^1}^2$ for an appropriate choice of spherical profile.

This example has the following well-known immediate generalizations.

Lemma 5.1.4. *Let $\iota_{double} : T^*S^n \to T^*\mathbb{RP}^n$ be the symplectic double cover obtained by double cover of the zero section. For $(\widetilde{q}, \widetilde{p}) = \iota_{double}(q, p) \in T^*\mathbb{RP}^n$,*

$$T(\widetilde{q}, \widetilde{p}) = \iota_{double} \circ \tau_{S^n}(q, p)$$

is well-defined and T is Hamiltonian isotopic to $\tau_{\mathbb{RP}^n}$ for an appropriate choice of auxiliary function defining $\tau_{\mathbb{RP}^n}$.

If $n > 1$, the choice of auxiliary function defining $\tau_{\mathbb{RP}^n}$ is irrelevant up to Hamiltonian isotopy.

Lemma 5.1.5. *For $T^*S^2 = T^*\mathbb{CP}^1$, $\tau_{S^2}^2$ is Hamiltonian isotopic to $\tau_{\mathbb{CP}^1}$.*

As usual, one may globalize the model Dehn twist.

Definition 5.1.6. A Dehn twist along S in M is a compactly supported symplectomorphism defined by the model Dehn twist as above in a Weinstein neighborhood of S and extended by identity outside.

For more details and the dependence of choices used to define τ_S, see [89] and [91].

5.1.2 Lagrangian surgery through flow handles

Surgery at a point

We first recall the definition of a Lagrangian surgery at a transversal intersection from [49][86] and [13].

Definition 5.1.7. Let $a(s), b(s) \in \mathbb{R}$. A smooth curve $\gamma(s) = a(s) + ib(s) \in \mathbb{C}$ is called **λ-admissible** if

- $(a(s), b(s)) = (-s + \lambda, 0)$ for $s \leq 0$
- $a'(s), b'(s) < 0$ for $s \in (0, \epsilon)$, and
- $(a(s), b(s)) = (0, -s)$ for $s \geq \epsilon$.

The part of a λ-admissible curve with $s \in [0, \epsilon]$ can be captured by $\nu(r) = a(b^{-1}(-r))$. The main property of an admissible curve can be rephrased as follows.

(1) $\nu_\lambda(0) = \lambda > 0$, and $\nu'_\lambda(r) < 0$ for $r \in (0, \epsilon)$.

(2) $\nu_\lambda^{-1}(r)$ and $\nu_\lambda(r)$ has vanishing derivatives of all orders at $r = \lambda$ and $r = \epsilon$, respectively.

Such a function will also be called **λ-admissible**. We will frequently use the two equivalent descriptions of admissibility interchangeably.

We also define a class of **semi-admissible functions**, by relaxing (2) to

(2′) $\nu'_\lambda(0) = -\alpha \in [-\infty, 0]$. Here $\alpha = \infty$ if ν_λ is admissible.

Note that in all definitions of (semi-)admissibility there is an extra variable ϵ. We will see that the dependence on ϵ is not significant in this section: we fix ϵ for each pair of Lagrangian submanifolds (L_1, L_2) once and for all. In any surgery constructions appearing later, the resulting surgery manifold yields a smooth family of isotopic Lagrangian submanifolds as ϵ vaires. As a result we will suppress the dependence of ϵ unless necessary.

Given a λ-admissible curve γ, define the handle

$$H_\gamma = \{(\gamma(s)x_1, \dots, \gamma(s)x_n) | s, x_i \in \mathbb{R}, \sum x_i^2 = 1\} \subset \mathbb{C}^n$$

Lemma 5.1.8. *For an λ-admissible γ, H_γ is a Lagrangian submanifold of* $(\mathbb{C}^n, \sum dx_i \wedge dy_i)$.

Proof. It suffices to observe that $T_{\gamma(s)x}H_\gamma = Span_{\mathbb{R}}\{\gamma'(s)x\} \oplus \gamma(s)T_xS^{n-1}$, for $x = (x_1, \dots, x_n) \in S^{n-1} \subset \mathbb{R}^n$. □

As a consequence, we have

Corollary 5.1.9. *Let $L_1, L_2 \subset (M, \omega)$ be two Lagrangians transversally intersecting at p. Let $\iota : U \to M$ be a Darboux chart with a standard complex structure so that $\iota^{-1}(L_1) \subset \mathbb{R}^n$ and $\iota^{-1}(L_2) \subset i\mathbb{R}^n$, then one can obtain a Lagrangian $L_1\#_pL_2$ by attaching a Lagrangian handle $\iota(H_\gamma)$ to $(L_1 \cup L_2)\backslash\iota(U)$.*

The Lagrangian $L_1\#_pL_2$ is called a **Lagrangian surgery** from L_1 to L_2 following [49, 86]. Note that, the Lagrangian $L_2\#_pL_1$ obtained by performing Lagrangian surgery from L_2 to L_1 is in general not even smoothly isotopic to $L_1\#_pL_2$.

Now, we present an new approach of performing Lagrangian surgery which also motivates the definition of Lagrangian surgery along clean intersections.

Definition 5.1.10. Given the zero section $L \subset T^*L$, a Riemannian metric g on L (hence inducing one on T^*L) and a point $x \in L$, we define the **flow handle** H_ν with respect to a λ-admissible function ν to be

$$H_\nu = \{\phi^{\varpi}_{\nu(\|p\|)}(p) \in T^*L : p \in (T^*_x L)_\epsilon \backslash \{x\}\},$$

where $(T^*_x L)_\epsilon$ denotes the cotangent vectors at $x \in L$ with length $\leq \epsilon$

Remark 5.1.11. $\phi^{\varpi}_{\nu(||p||)}$ is the time-1 Hamiltonian flow of $\widetilde{\nu}(||p||)$, where $\widetilde{\nu}'(s) = \nu(s)$. For this reason, the reader should keep in mind that H_ν is automatically Lagrangian for any choice of admissible ν. For our purpose, the discussion on ν will be more flexible so we suppress the role of the actual Hamiltonian function $\widetilde{\nu}$ unless otherwise specified.

Lemma 5.1.12. *Let $S_\lambda(T^*_x L)$ be the radius λ-sphere in the tangent plane of x. If $exp : S_\lambda(T^*_x L) \to L$ is an embedding, and $\partial H_\nu = exp(S_\lambda(T^*_x L)) \subset L$ divides L into two components, then H_ν glues with exactly one of the components to form a smooth Lagrangian submanifold coinciding with $T^*_x L$ outside a compact set for a λ-admissible ν.*

Proof. The only thing to prove is the smoothness of gluing on $\partial H_\nu = exp(S_\lambda(T^*_x))$. Note that near ∂H_ν, the handle is a smooth section over the open shell $exp(B_\lambda(T^*_x)\backslash B_{\lambda-\delta}(T^*_x))$ which is a smooth open manifold. Moreover, the section has vanishing derivatives for all orders on the boundary due to the assumption of admissibility on $\nu(r)$ near $r = 0$. The conclusion follows.

□

Example 5.1.13. One may match the Lagrangian handle H_γ and flow handle H_ν for admissible γ and its corresponding admissible ν (See Definition 5.1.7 and the paragraph after it) via the identification between $T^*\mathbb{R}^n$ and $\mathbb{C}^n$.

To see this, the flow handle is given by

$$H_\nu = \{\phi^{\varpi}_{\nu(||p||)}(0, p) = (0 + \frac{p}{\|p\|} \cdot \nu(||p||), p) : p \in (T^*_0 \mathbb{R}^n)_\epsilon\}$$

We now identify $T^*\mathbb{R}^n$ with $\mathbb{C}^n$ by sending $(q, p) \mapsto q - ip$, which matches the symplectic form $dp \wedge dq$ and $\frac{1}{-2i} dz \wedge d\bar{z} = dx \wedge dy$. Then by definition

$$\begin{aligned}(\nu(\|p\|)\frac{p}{\|p\|}, p) &\mapsto \nu(\|p\|)\frac{p}{\|p\|} - ip \\ &= (a(s) + ib(s))x\end{aligned}$$

by a change of variable $s = b^{-1}(-||p||)$ and $x = \frac{p}{||p||}$. By this identification, we will simply use H_ν to denote both handles.

Corollary 5.1.14. *Let $L_1, L_2 \subset (M, \omega)$ be two Lagrangians transversally intersecting at p. Under the assumption in Lemma 5.1.12, one can obtain a Lagrangian $L_1\#_p^\nu L_2$ by gluing (1) $L_2\backslash U$, (2) the Lagrangian flow handle H_ν, and (3) an open set in L_1 given by Lemma 5.1.12. For appropriately chosen ν, $L_1\#_p^\nu L_2$ coincides with $L_1\#_p L_2$ defined in Corollary 5.1.9.*

Example 5.1.15. Let $r(p)$ be the injectivity radius of p. For different choices of $\nu(r)$ with $\nu(0) < r(p)$, these handles will define a family of different Lagrangian surgeries which are all Lagrangian isotopic to each other. In the case when L_1 is simply-connected, they are Hamiltonian isotopic.

The situation becomes more interesting when $\nu(0) > r(p)$. Some simple instances are given by $S = \mathbb{RP}^n$, $\mathbb{CP}^n$ or any finite cover of rank-one symmetric space. Take $\mathbb{CP}^n$ and its standard Fubini-Study metric as an example, for any $p \in S$, the flow surgery can be performed for $kr(p) < \nu(0) < (k+1)r(p)$ for any $k \in \mathbb{Z}$. Later we will see that such surgeries are indeed iterated surgeries in the ordinary sense (although surgeries along clean intersections will be involved).

Example 5.1.16. A less standard example is essentially given by exotic spheres in [94]. Given any $f \in Diff^+(S^{n-1})$ and form an exotic sphere $S_f = B_- \cup_f B_+$. There is a Riemannian metric so that all geodesics starting from $0_\pm$ are closed, through each other, and of the same length ([94, Lemma 2.1]). Take $p = 0_- \in B_-$, when λ is below the injectivity radius, the surgery is the original one considered in [86]. When $\nu(0) > r(p)$, the generalized surgery defined above is identified with an iterated surgery along p and $q = 0_+ \in B_+$ in a successive order, which is exactly the family constructed in [94] by the geodesic flow.

The following lemma can be found in [89], but we feel it instructive to sketch the proof from the point of view of flow handles to make our discussion complete.

Lemma 5.1.17 ([89]). *Let $x \in S^n$ be a point and consider $L = \tau_{S^n}(T^*_x S^n) \subset T^*S^n$. Then $S^n \#_x T^*_x S^n$ is Hamiltonian isotopic to L by a compactly supported Hamiltonian.*

Proof. Let $A : S^n \to S^n$ be the antipodal map. We consider open geodesic balls $B_\pi(x)$ and $B_\pi(A(x))$ of radius π centered at x and $A(x)$, respectively. It gives two symplectomorphism $f_x : T^*B_\pi(x) \to T^*S^n \backslash T^*_{A(x)}S^n$ and $f_{A(x)} : T^*B_\pi(A(x)) \to T^*S^n \backslash T^*_x S^n$ under which we have

$$f_x^{-1}(L) = \{\nu_\epsilon^{Dehn}(|p|)\frac{p}{|p|}, p) \in T^*B_\pi : p \in \mathbb{R}^n \backslash \{0\}\}$$

and

$$f_{A(x)}^{-1}(L) = \{(\pi - \nu_\epsilon^{Dehn}(|p|))\frac{p}{|p|}, p) \in T^*B_\pi : p \in B_\epsilon(0)\}$$

On the other hand, suppose $\nu = \nu_\lambda$ is such that $\nu_\lambda(0) = \lambda < \pi = r(x)$. Then $f_x^{-1}(H_{\nu_\lambda})$ is given by

$$f_x^{-1}(H_{\nu_\lambda}) = \{(\nu_\lambda(|p|)\frac{p}{|p|}, p) \in T^*B_\pi : p \in \mathbb{R}^n \backslash \{0\}\} \cup \{(q, 0) \in T^*B_\pi : q \in B_\pi(0) \backslash B_\lambda(0)\}$$

Let $\delta > 0$ be such that $\nu_\epsilon^{Dehn}(r) = \pi - r$ for $r < \delta$. We can pick ν_λ such that $\nu_\lambda(r) = \nu_\epsilon^{Dehn}(r)$ for $r \geq \delta$. The resulting $S^n \#_x T^*_x S^n$ hence coincides with L outside $T^*B_\delta(A(x))$. Inside $T^*B_\delta(A(x))$, even though ν_ϵ^{Dehn} is not an admissible function, both $S^n \#_x T^*_x S^n$ and L are graphs of the zero section. Therefore, $S^n \#_x T^*_x S^n$ is Lagrangian isotopic to L and hence Hamiltonian isotopic to L by a compactly supported Hamiltonian. □

Remark 5.1.18. For semi-admissible ν^α that is not admissible, the gluing with L_1 cannot be smooth in general. Lemma 5.1.17 is an instance when a surgery using a semi-admissible profile ν_ϵ^{Dehn} yields a smooth Lagrangian submanifold. Intuitively, the lemma regards ν_ϵ^{Dehn} as a degenerate case of an admissible function. The point is that, when $\lambda = r(p)$, we only need to glue $Cl(H_\nu)$ with $L_2 \backslash U$, where $Cl(\cdot)$ denotes the closure.

In the case when a semi-admissible function defines a smooth Lagrangian surgery manifold, we will continue to denote it as $L_1 \#_p^{\nu_\lambda^\alpha} L_2$. This applies to other surgeries along clean intersections and will be used several more times in a parametrized version.

Surgery along clean intersection

Let L_1 and L_2 be two Lagrangians in (M, ω) which intersect cleanly at a submanifold D. In other words, we have $T_pD = T_pL_1 \cap T_pL_2$ for all $p \in D$. The following well-known local proposition due to Pozniak allows us to extend the definition of flow handles to this case.

Proposition 5.1.19 ([87]). *Let $L_1, L_2 \subset (M, \omega)$ be two closed embedded Lagrangians with clean intersection at $L_1 \cap L_2 = D$. Then there is a symplectomorphism φ from a neighborhood U of $0_{section}$ in T^*L_1 to M such that $\varphi(0_{section}) = L_1$ and $\varphi^{-1}(L_2) \subset N_D^*$, where $0_{section}$ is the zero section and N_D^* is the conormal bundle of D in L_1.*

Definition 5.1.20. We define the **flow handle for** $D \subset L$ with respect to an admissible function ν to be

$$H_\nu^D = \{\phi^{\varpi}_{\nu(\|\xi\|)}(\xi) \in T^*L : \xi \in (N_D^*)_\epsilon \backslash D\}$$

where $(N_D^*)_\epsilon$ consists of covectors in the conormal bundle of D in L with length $\leq \epsilon$.

Lemma 5.1.21. *Let $S_\lambda(N_D^*)$ be the radius-λ sphere bundle in conormal bundle of D. If $exp : S_\lambda(N_D^*) \to L$ is an embedding, and $\partial H_{\nu_\lambda}^D \subset L$ divides L into two components, then H_ν^D glues with exactly one of the components to form a smooth Lagrangian submanifold coinciding with N_D^* outside a compact set.*

The proof is exactly the same as Lemma 5.1.12 and we omit it. As in the transversal intersection case, the surgery is always well-defined when we choose $\nu(0) = \lambda < r(D)$, the injectivity radius of D along normal directions. Using Proposition 5.1.19 we globalize the construction as follows.

Corollary 5.1.22. *Let $L_1, L_2 \subset (M, \omega)$ be two Lagrangians intersecting cleanly along D. By choosing a metric on L_1, a symplectic embedding $\iota : T_\epsilon^*L_1 \to M$ such that $\iota(0_{section}) = L_1$ and $\iota^{-1}(L_2) \subset N_D^*$, one can obtain a Lagrangian $L_1 \#_D^\nu L_2$ by attaching a Lagrangian flow handle $\iota(H_\nu^D)$ to $(L_1 \backslash U_1) \cup (L_2 \backslash U_2)$, with $U_i \subset L_i$ appropriate open neighborhoods of D, and ϵ being sufficiently small.*

As in Example 5.1.15, we denote $L_1 \#_D^\nu L_2$ by $L_1 \#_D L_2$ if $\lambda < r(D)$.

5.1.3 E_2-flow surgery and its family version

So far we have only used the geodesic flows on the whole T^*L to construct Lagrangian handles, but more flexibility will prove useful in our applications. Heuristically, our previous constructions have taken advantage of the fact that $||p||$ has a well-defined Hamiltonian flow on the whole cotangent bundle except for the zero section. More crucially, the resulting flow handle should have an embedded boundary into L_1 (or at least fiber over its image). Indeed, any Hamiltonian function with such properties will suffice for defining a meaningful Lagrangian handle.

A variant of the flow handle can therefore be defined as follows. Let $L = K_1^{n-m} \times K_2^m$ be a product manifold equipped with product Riemannian metric. Then there is an orthogonal decomposition $T^*L = E_1 \oplus E_2$ given by the two factors respectively. Let $D \subset L$ be of codimension m and transverse to $\{p\} \times K_2$ for all $p \in K_1$. Suppose $\pi_2 : T^*L \to E_2$ be the projection to E_2, one may then use the function $\varpi_\pi = ||\pi_2(\cdot)||_g$ to define a new flow handle. Note that $\varpi_\pi = ||\pi_2(\cdot)||_g$ is smooth on $T^*L \backslash E_1$.

Definition 5.1.23. In the situation above, we define the **E_2-flow handle for** D (or **flow handle along E_2-direction**) with respect to an λ-admissible ν_λ to be

$$H^{D,E_2}_{\nu_\lambda} = \{\phi^{\varpi_\pi}_{\nu_\lambda(\|\pi_2(\xi)\|)}(\xi) \subset T^*L : \xi \in (N^*_D)_{\epsilon,E_2} \backslash D\}.$$

where $(N^*_D)_{\epsilon,E_2}$ consists of covectors ξ in the conormal bundle of D in L such that $\|\pi_2(\xi)\| \leq \epsilon$.

We note that for any point $\xi = (\xi_1, \xi_2) \in E_1 \oplus E_2$, $\phi^{\varpi_\pi}_t(\xi) = (\xi_1, \phi^{\varpi}_t(\xi_2))$ so E_2-flow is the normalized (co)geodesic flow on the second factor and trivial on the first factor.

Let $S_\lambda(E_2|_D)$ be the radius-λ sphere bundle of E_2 over D. We consider $exp^{E_2}_\lambda$: $S_\lambda(E_2|_D) \to L$, which is the exponential map restricted on $S_\lambda(E_2|_D)$ along the leaves of the foliation given by second factor. We define the **E_2-injectivity radius** $r^{E_2}(D)$ of D as the supremum of λ such that $exp^{E_2}_s$ is an embedding for all $s < \lambda$.

Lemma 5.1.24. *Let $D \subset L = K_1 \times K_2$ be of dimension $n - m$ and transversal to $\{p\} \times K_2$ for all $p \in K_1$. If $exp^{E_2}_\lambda : S_\lambda(E_2|_D) \to L$ is an embedding and $\partial H^{D,E_2}_\nu \subset L$ divides L into two components, then H^{D,E_2}_ν glues with exactly one of the components of L to form a smooth Lagrangian submanifold coinciding with N^*_D outside a compact set.*

Proof. The proof is again similar as Lemma 5.1.21. □

Similar to the cases we considered before, if $L_1 = K_1 \times K_2$ and L_2 are Lagrangians cleanly intersecting at D as above, we can add an E_2-flow handle to $L_1 \cup L_2$ outside a tubular neighborhood of D to get a new Lagrangian submanifold for $\lambda < r^{E_2}(D)$. We will denote the resulting Lagrangian submanifold by $L_1 \#_{D,E_2} L_2$, called the **surgery from** L_1 **to** L_2. We remark that H_ν^{D,E_2} coincide with N_D^* when $\|\pi_2(\xi)\| > \epsilon$, so for E_2 flow surgery, we have to take out a neighborhood from N_D^* slightly larger than the ϵ-neighborhood of D in N_D^*. This fact is not essential when ϵ is small.

We now define a family version for E_2-flow surgery. Assume the situation from Definition 5.1.23 that we have a smooth manifold pair (L, D) and a decomposition $T^*L = E_1 \oplus E_2$. Let $(\mathcal{L}, \mathcal{D})$ be another smooth manifold pair so that:

(i) $(\mathcal{L}, \mathcal{D})$ has a compatible fiber bundle structure over a smooth base B, that is,

$$\begin{array}{ccc} L \longrightarrow \mathcal{L} & \quad\text{and,}\quad & D \longrightarrow \mathcal{D} \\ \quad\quad\;\; \downarrow & & \quad\quad\;\; \downarrow \\ \quad\quad\;\; B & & \quad\quad\;\; B \end{array}$$

where the two bundle structures are compatible with the inclusion $\mathcal{D} \hookrightarrow \mathcal{L}$.

(ii) The structure group $G \subset Isom(L)$, the isometry group of L, and it preserves E_1 and E_2.

Assumptions above allow us to glue T^*L via the given bundle data, yielding a symplectic fiber bundle $\mathcal{E} \to B$ with fiber T^*L. All previous symplectic constructions on T^*L are now functorial hence can be glued over B. For example, N_D^*L glues into $N_{\mathcal{D}}^*\mathcal{L}$ hence fits into Pozniak's setting of clean intersection. When $\mathcal{E}$ is regarded as a vector bundle over $\mathcal{L}$, it comes with a natural splitting $\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2$ from local charts. Hence, the E_2-handle H_ν^{D,E_2} can be constructed fiberwisely on $N^*\mathcal{D}$, which gives a smooth handle $\mathcal{H}_\nu \subset \mathcal{E}$. The fact that $\mathcal{H}_\nu \hookrightarrow T^*\mathcal{L} \supset \mathcal{E}$ is indeed a Lagrangian embedding can be check on local charts $\mathcal{U} \subset B$.

Lemma 5.1.25. *For two cleanly intersecting Lagrangians $\mathcal{L}_0, \mathcal{L}_1 \subset (M^{2n}, \omega)$, if $(\mathcal{L}_0, \mathcal{D} = \mathcal{L}_0 \cap \mathcal{L}_1)$ satisfies (i)(ii) above, then family E_2-surgery between $\mathcal{L}_0$ and $\mathcal{L}_1$ can be performed and gives a Lagrangian submanifold $\mathcal{L}_0 \#_{\mathcal{D},E_2}^{\nu} \mathcal{L}_1$ of (M, ω).*

Remark 5.1.26. It is easy to see that our construction works word-by-word as long as there is a decomposition of vector bundle $T^*L = E_1 \oplus E_2$. However, one needs to imposed technical conditions to make $exp_\lambda^{E_2} : S_\lambda \to L$ an embedding even for small λ. An easy condition is to assume E_2 is integrable at least near D, but it should also work in some cases when E_2 is completely non-integrable near D but integrable outside a small neighborhood. Considerations along this line might result in delicate constructions of new Lagrangian submanifolds.

5.2 Perturbations: from surgeries to Dehn twists

This section contains the technical part which passes from Lagrangian surgeries to Dehn twists in several applications. The general idea is the same as Lemma 5.1.17, which may also interpreted as passing from admissible profiles to semi-admissible ones. This is realized as local perturbations of the surgery Lagrangians.

We first explain how this works in the $\mathbb{CP}^n$ case, then give a proof of Theorem 1.3.1(1)(2)(3)(5) using family versions of this observation.

5.2.1 Fiber version

In this section, we are interested in L being $\mathbb{RP}^n$, $\mathbb{CP}^{\frac{m}{2}}$ or $\mathbb{HP}^n$ equipped with the Riemannian metric such that every geodesic is closed of length 2π. All actual proofs will be given only in the case of $\mathbb{CP}^n$ but are easily generalized. Let $x \in L$ be a point and F_x its cotangent fiber inside T^*L. We also let $D = \{y \in L | dist(x, y) = \pi\}$ be the **divisor opposite to** x.

Lemma 5.2.1. *Let $x \in L$ be a point and ν_{λ_i} be λ_i-admissible functions such that $(k-1)\pi < \lambda_i < k\pi$ for some positive integer k for both $i = 1, 2$. Then $L\#_x^{\nu_{\lambda_i}} F_x$ are isotopic for $i = 1, 2$ by a compactly supported Hamiltonian.*

Moreover, if we choose a semi-admissible function $\nu_{k\pi}^\alpha : (0, \infty) \to [0, k\pi)$ that is monotonic decreasing and all orders of derivatives vanish at $r = \epsilon$ such that $\nu_{k\pi}^\alpha(r) = k\pi - \alpha r$ near $r = 0$ $(\alpha \geq 0)$, then $L\#_x^{\nu_{k\pi}^\alpha} F_x$ (See Remark 5.1.18) is a smooth Lagrangian that is isotopic to $L\#_x^{\nu_{\lambda_i}} F_x$ by a compactly supported Hamiltonian.

Furthermore, these Hamiltonian isotopies can be chosen to be invariant under isometric action of L fixing x.

Corollary 5.2.2. *For $\pi < \lambda < 2\pi$ and L being $\mathbb{RP}^n$, $\mathbb{CP}^{\frac{m}{2}}$ or $\mathbb{HP}^n$, $L\#_x^{\nu_\lambda} F_x$ is Hamiltonian isotopic to $\tau_L(F_x)$ for an admissible ν_λ.*

Proof. Observe that when $\alpha = 1$ and $k = 2$, $\nu_{k\pi}^\alpha(r)$ is a Dehn twist profile. The Corollary follows from Lemma 5.2.1. □

Proof of Lemma 5.2.1. For the first statement, we observe that the space of λ-admissible function for $(k-1)\pi < \lambda < k\pi$ is connected. A smooth isotopy $\{\nu_t\}$ from ν_{λ_1} to ν_{λ_2} in this space results in a smooth Lagrangian isotopy from $L\#_x^{\nu_{\lambda_1}} F_x$ to $L\#_x^{\nu_{\lambda_2}} F_x$ since ∂H_{ν^t} does not pass any critical locus. This is a Hamiltonian isotopy because $H^1(L\#_x^{\nu_{\lambda_1}} F_x, \partial^\infty(L\#_x^{\nu_{\lambda_1}} F_x); \mathbb{R}) = 0$ (cf. Example 5.1.15).

For the second statement, we only consider the case that $k = 1$ and $L = \mathbb{CP}^{\frac{m}{2}}$, and the remaining cases are similar. In this case, denote $\nu^\alpha = \nu_\pi^\alpha$, then $Cl(H_{\nu^\alpha}) \backslash H_{\nu^\alpha} = D = \mathbb{CP}^{\frac{m}{2}-1}$. We pick a local chart $U \subset L$ with local coordinates $(q_1, \dots, q_m)$ adapted to D in the sense that $U \cap D = \{q_1 = q_2 = 0\}$ and $c(t) = (tq_1, tq_2, q_3, \dots, q_m)$ are normalized geodesics for any $(q_1, \dots, q_m)$. It induces canonically a Darboux chart T^*U in T^*L. We write a point in T^*U as (q_a, q_b, p_a, p_b), where $q_a = (q_1, q_2)$, $q_b = (q_3, \dots, q_m)$ and similarly for p_a and p_b. Since H_{ν^α} is defined by the geodesic flow, one may directly verify

$$T^*U \cap H_{\nu^\alpha} = \{(q_a, q_b, p_a, 0) | q_a = -\alpha p_a \neq 0\}$$

$$T^*U \cap D = \{(0, q_b, 0, 0)\}$$

Therefore, it is clear that H_{ν^α} and D can be glued smoothly to become $Cl(H_{\nu^\alpha})$. The gluing from H_{ν^α} to $F_x - B_\epsilon$ is the same as the admissible case. It results in a smooth Lagrangian $L\#_x^{\nu^\alpha} F_x$.

Finally, we want to show that $L\#_x^{\nu^\alpha} F_x$ is Hamiltonian isotopic to $L\#_x^{\nu_{\lambda_i}} F_x$. We can assume $\alpha \neq 0$, by a Hamiltonian perturbation if necessary. Locally near D, we have

$$T^*U \cap (H_{\nu^\alpha} \cup D) = \{(-\alpha p_a, q_b, p_a, 0)\} = \{(q_a, q_b, -\frac{1}{\alpha} q_a, 0)\} \tag{5.1}$$

$$T^*U \cap L = \{(q_a, q_b, 0, 0)\} \tag{5.2}$$

It is clear that there is a small $\delta > 0$ such that $(H_{\nu^\alpha} \cup D) \cap T^*B_\delta(D)$ is the graph of $d(-\frac{1}{2\alpha} dist^2(\cdot, D))$ over $B_\delta(D)$, where $B_\delta(D)$ is the δ neighborhood of D in L. Take a smooth decreasing function $f(r):[0,\delta]\to\mathbb{R}$ so that $f=0$ near $r=0$ and $f(r)=-\frac{1}{2\alpha}r$ near $r=\delta$. Denote $f_t(r)=tf(r)-(1-t)\frac{1}{2\alpha}r$.

Then the graph of $d(f_t\circ dist^2(\cdot,D))$ can be patched with $H_{\nu^\alpha}\backslash T^*B_\delta(D)$ to give a Hamiltonian isotopy from $L\#_x^{\nu^\alpha}F_x$ to $L\#_x^{\nu_\lambda}F_x$ for some admissible ν_λ with $0<\lambda<\pi$. We remark that the Hamiltonian isotopy is invariant under the $Isom(L)_x$, isometric group of L fixing x. This concludes the proof.

□

We remark that another point of view of the Lagrangian isotopy from $L\#_x^{\nu^\alpha}F_x$ to $L\#_x^{\nu_\lambda}F_x$ is that it is induced from a smooth isotopy relative to end points from the curve $\{(r,\nu^\alpha(r))\in[0,\epsilon]\times[0,\pi]|r\in(0,\epsilon]\}\cup\{(0,\pi)\}$ to the λ-admissible curve defined by ν_λ.

Later we will see that, when the surgery profile ν_λ has λ exceeding the injectivity radius, there is no cobordism directly associated to such a surgery. To fit such a surgery to the cobordism framework, in general we need to decompose the surgery into several steps. The following lemma shows how this works in the case for $\mathbb{CP}^n$ (which easily generalizes to $\mathbb{RP}^n$ and $\mathbb{HP}^n$).

Lemma 5.2.3. *Let $x\in\mathbb{CP}^{\frac{m}{2}}$ be a point and $F_x\subset T^*\mathbb{CP}^{\frac{m}{2}}$ the corresponding cotangent fiber. Let $D=\{y\in\mathbb{CP}^{\frac{m}{2}}|dist(x,y)=\pi\}$ be the divisor opposite to x. Then there is an embedded Lagrangian $Q_x\subset T^*\mathbb{CP}^{\frac{m}{2}}$ such that*

(1) $Q_x=F_x$ away from a neighborhood of zero section,

(2) Q_x is Hamiltonian isotopic to $\mathbb{CP}^{\frac{m}{2}}\#_xF_x$,

(3) Q_x intersects cleanly with $\mathbb{CP}^{\frac{m}{2}}$ at D,

(4) $\mathbb{CP}^{\frac{m}{2}}\#_DQ_x$ is Hamiltonian isotopic to $\tau_{\mathbb{CP}^{\frac{m}{2}}}(F_x)$

As a result, as far as Hamiltonian isotopy class is concerned, we have $\mathbb{CP}^{\frac{m}{2}}\#_D(\mathbb{CP}^{\frac{m}{2}}\#_xF_x)=\tau_{\mathbb{CP}^{\frac{m}{2}}}(F_x)$.

Proof. Choose a semi-admissible profile ν_π^0 such that $\nu_\pi^0=\pi$ near $r=0$ and let $Q_x=L\#_x^{\nu_\pi^0}F_x$. (1)(3) follows from definition, and (2) is a consequence of Lemma 5.2.1.

To see (4), note that near D, Q_x coincide with the ϵ'-disk conormal bundle at D for some $\epsilon'\ll\epsilon$. Therefore, $\mathbb{CP}^{m/2}\#_D^{\nu_\lambda}Q_x$ is identical to $\mathbb{CP}^{m/2}\#_D^{\nu_{\lambda+\pi}}F_x$ for any $0<\lambda<\pi$

and an appropriate choice of $\nu_{\lambda+\pi}$. The latter is then Hamiltonian isotopic to $\tau_{\mathbb{CP}^{m/2}}(F_x)$ by Corollary 5.2.2. □

Due to the symmetry of $\mathbb{CP}^{m/2}$, we have an alternative description to the Dehn twist of F_{x_0}, yielding another proof for Lemma 5.2.3. Essentially, this description only changes the role of the base and the fiber, but leads to a particularly handy criterion for the isotopy type of $\tau_{\mathbb{CP}^{m/2}}F_{x_0}$, which will be used in later sections. Denote the isometry group of $\mathbb{CP}^{m/2}$ as G and the subgroup of it fixing x_0 as G_{x_0}. There is an induced G_{x_0}-action on $T^*\mathbb{CP}^{m/2}$.

Lemma 5.2.4. *Let $\gamma(t)$ be a normalized geodesic on $\mathbb{CP}^{m/2}$ starting and ending at x_0. Let $c(t)$ be a (rescaled) lift of $\gamma(t)$ in $T^*\mathbb{CP}^{m/2}$, that is, $c(t) = (\gamma(t), f(t)\gamma'(t))$ for some smooth $f(t)$ defined on $[0, 2\pi]$ such that $f(\pi) \neq 0$ and $f(0) > 0$ (recall that $\gamma'(t)$ is identified with its dual). Then the orbit $G_{x_0} \cdot c(t)$ is a Lagrangian which is possibly immersed.*

Moreover, assume further

(a)

$$\frac{d^n(f^{-1}(t))}{dt^n}(f(0)) = 0$$

for all $n \geq 1$, and f is strictly decreasing near $t = 0$,

(b) $f(2\pi) = 0$ and $f'(t) < 0$ when $t \in (2\pi - \delta, 2\pi]$ for some small $\delta > 0$.

Then $G_{x_0} \cdot c(t)$ can be extended to a proper Lagrangian immersion L_f, such that

(i) it overlaps with F_{x_0} along $\{p \in F_{x_0} : \|p\| > f(0)\}$,

(ii) it is isotopic to $\tau_{\mathbb{CP}^{\frac{m}{2}}}F_{x_0}$ through a smooth family of Lagrangian immersions with property (i).

Proof. For the first assertion, $G_{x_0}\cdot c(t)$ is the graph of $d(F\circ dist_{x_0}(\cdot))$ when $t \in (0, \pi)$, and $d(F \circ (2\pi - dist_{x_0}(\cdot)))$ when $t \in (\pi, 2\pi)$, for $F'(t) = f(t)$. The condition (a) guarantees the smoothness of gluing with F_{x_0}, and (b) the smoothness at $t = 2\pi$. The smoothness and Lagrangian properties at the critical set D and x_0 can be checked directly and using that $f(\pi) \neq 0$ and Lagrangian property is a closed condition.

For the last isotopy statement, find an isotopy of smooth functions, within the class of those satisfying $(a)(b)$, from $f(t)$ to some $g(t)$ which is strictly monotonic (decreasing) in $[0, 2\pi]$. This induces an isotopy of Lagrangian immersions from L_f to some L_g.

Consider $L\#_{x_0}^{\nu^{Dehn}} F_{x_0}$ as in Corollary 5.2.2. This Lagrangian and L_g are both G_{x_0}-invariant, it is not hard to check that with $\nu^{Dehn} = g^{-1}$, the two Lagrangians coincide. The conclusion hence follows. $\square$

Remark 5.2.5. As the proof showed, one should heuristically regard $f(t)$ as the inverse function of certain admissible function ν.

The possible immersion points appears if and only if there is $t_0 < \pi$, such that $f(t_0) = -f(2\pi - t_0)$. Otherwise, all above assertions can be improved to the class of embedded Lagrangians.

5.2.2 Product version

In this section we prove Theorem 1.3.1 (1)(3). The proofs here are similar to that in the last section, and should be considered as family versions of it. In this subsection, we use S to denote S^n, $\mathbb{RP}^n$, $\mathbb{CP}^{\frac{m}{2}}$ or $\mathbb{HP}^n$ equipped with the Riemannian metric such that every closed geodesic is of length 2π.

For the moment, let $S \subset (M, \omega)$ be a Lagrangian sphere. One may consider the clean surgery of $L_1 = S \times S^-$ and $L_2 = \Delta$ in $M \times M^-$. In this case, they cleanly intersect at $D = \Delta_S \subset S \times S^-$. In Definition 5.1.23, take $E_2 = S \times (T^*S)^- \subset T^*S \times (T^*S)^-$, $E_1 = T^*S \times S^- \subset T^*S \times (T^*S)^-$ and an π-admissible function ν_π.

Now consider a point $(p, p) \in \Delta$ in a Weinstein neighborhood of L_1, where p can be considered as a point on $T^*_\epsilon S$ for a small $\epsilon > 0$. The flow in Definition 5.1.23 defines a symplectomorphism fixing the first coordinate in $(T^*S \times (T^*S)^-) \backslash E_1$; when restricted to Δ_S, the E_2-flow sends $(p, p) \mapsto (p, \phi^{\varpi}_{\nu_\pi(||p||)}(p))$. This is exactly the graph of τ_S^{-1} (the inverse owes to the negation of symplectic form on M^-), except that we have used an admissible profile for the handle which is not a Dehn twist profile. Lemma 5.2.6 below ensures that this could be compensated by a local Hamiltonian perturbation. Hence modulo Lemma 5.2.6, this shows that $(S \times S^-)\#^{\nu_\pi}_{\Delta_S, E_2}\Delta = Graph(\tau_S^{-1})$. The whole construction applies when S is $\mathbb{RP}^n$, $\mathbb{CP}^{\frac{m}{2}}$ or $\mathbb{HP}^n$, except that the admissible profile has $\nu(0) = 2\pi$.

Lemma 5.2.6. *Let S be S^n, $\mathbb{RP}^n$, $\mathbb{CP}^{\frac{m}{2}}$ or $\mathbb{HP}^n$. Let ν_{λ_i} be λ_i-admissible functions such that $(k-1)\pi < \lambda_i < k\pi$ for some positive integer k for both $i = 1, 2$. Then the E_2-flow surged Lagrangian manifolds $(S \times S^-)\#^{\nu_{\lambda_i}}_{\Delta_S, E_2}\Delta$ above by ν_{λ_i} are Hamiltonian isotopic.*

Moreover, if we choose a semi-admissible function $\nu^{\alpha}_{k\pi}$ such that $\nu^{\alpha}_{k\pi}(r) = k\pi - \alpha r$ near $r = 0$ ($\alpha \geq 0$), then $(S \times S^-)\#^{\nu^{\alpha}_{k\pi}}_{\Delta_S, E_2}\Delta$ is a smooth Lagrangian that is Hamiltonian isotopic to $(S \times S^-)\#^{\nu_{\lambda_i}}_{\Delta_S, E_2}\Delta$.

Furthermore, these Hamiltonian isotopy can be chosen to be $Isom(S)$ invariant, where $Isom(S)$ is the diagonal isometry group in $Isom(S) \times Isom(S)$ acting on $T^*S \times (T^*S)^-$.

We have the following Corollary whose proof is similar to Corollary 5.2.2

Corollary 5.2.7 (cf. Theorem 1.3.1(1))**.** *For $\pi < \lambda < 2\pi$ and S being $\mathbb{RP}^n$, $\mathbb{CP}^{\frac{m}{2}}$ or $\mathbb{HP}^n$ (resp. $0 < \lambda < \pi$ and $S = S^n$), $(S \times S^-)\#^{\nu_\lambda}_{\Delta_S, E_2}\Delta$ is Hamiltonian isotopic to $Graph(\tau_S^{-1})$.*

Proof of Lemma 5.2.6. The proof of the first statement is exactly the same as Lemma 5.2.1. For the second statement, we again only consider the case that $k = 1$ and $S = \mathbb{CP}^{\frac{m}{2}}$ and the remaining cases are similar.

Define $D^{op} = \{(x, y) \in S \times S | dist(x, y) = \pi\}$. Projection to x equips D^{op} with a $\mathbb{CP}^{\frac{m}{2}-1}$-bundle structure over $S = \mathbb{CP}^{\frac{m}{2}}$. Therefore, the neighborhood of D^{op} in $S \times S$ is the total space of a fiber bundle $\widetilde{\mathcal{V}} \to S$, whose fiber is a topological $\mathcal{O}(1)$-bundle $\mathcal{V}$ over $\mathbb{CP}^{\frac{m}{2}-1}$. We pick a local trivialization $U^B \times U^F$ of $\widetilde{\mathcal{V}}$ for $U^B \subset S$ and $U^F \subset \mathcal{V}$. Readers should note that, the product decomposition $U^B \times U^F$ is not compatible with that of $L = S \times S^-$, but $\{q\} \times U^F$ is an open set of the second factor of S for any $q \in U^B$.

Consider a choice of local coordinates $(q^B, q^F) = (q^B_1, \dots, q^B_m, q^F_1, \dots, q^F_m)$ adapted to D^{op} in the sense that $(U^B \times U^F) \cap D^{op} = \{q^F_1 = q^F_2 = 0\}$ and $c(t) = (q^B, tq^F_1, tq^F_2, q^F_3, \dots, q^F_m)$ are normalized geodesics for all (q^B, q^F). It induces canonically a Weinstein neighborhood $T^*(U^B \times U^F)$ in $T^*(S \times S^-)$. We write a point in $T^*(U^B \times U^F)$ as $(q^B, p^B, q^F_a, q^F_b, p^F_a, p^F_b)$, where $q^F_a = (q^F_1, q^F_2)$, $q^F_b = (q^F_3, \dots, q^F_m)$ and similarly for p^F_a and p^F_b. Since H_{ν^α} is defined by the parametrized geodesic flow when restricted on the second factor of T^*S, we have a parametrized version of (5.1)

$$\begin{aligned} &T^*(U^B \times U^F) \cap H_{\nu^\alpha} \\ =&\{(q^B, p^B, -\alpha p_a^F, q_b^F, p_a^F, 0) | p^B \neq 0, \phi^{\varpi}_{\nu^\alpha(\|p^B\|)}(q^B, p^B) = (-\alpha p_a^F, q_b^F, p_a^F, 0)\} \end{aligned} \tag{5.3}$$

Here, both $\phi^{\varpi_B}_{\nu_\pi^\alpha(\|p^B\|)}(q^B, p^B)$ and $(-\alpha p_a^F, q_b^F, p_a^F, 0)$ are considered as points in $T^*_\epsilon S$ although they belong to different factors. Therefore, in $T^*(U^B \times U^F) \cap H_{\nu^\alpha}$, fixing q^B and letting p^B go to 0 linearly leads to fixing q_b^F and letting p_a^F go to zero linearly. All above conclusions can be glued across different charts. Therefore, one can see that H_{ν^α} and D^{op} can be glued smoothly. The fact that H_{ν^α} can be glued smoothly with Δ is because all order of derivatives of ν^α vanish at $r = \epsilon$. It results in a smooth Lagrangian, which we denote as $(S \times S^-)\#^{\nu^\alpha}_{\Delta_S, E_2}\Delta$.

Finally, to show that $(S \times S^-)\#^{\nu^\alpha}_{\Delta_S, E_2}\Delta$ is Hamiltonian isotopic to $(S \times S^-)\#^{\nu_{\lambda_1}}_{\Delta_S, E_2}\Delta$. We can choose $\{\nu^t\}_{t \in [\alpha, \infty)}$ interpolating ν^α and ν_{λ_1} as in the proof of Lemma 5.2.1. This is a smooth Lagrangian isotopy which is invariant under the diagonal $Isom(S)$ action.

□

In parallel to Lemma 5.2.3, we have the following.

Lemma 5.2.8 (Theorem 1.3.1(3))**.** *Let S be $\mathbb{RP}^n$, $\mathbb{CP}^{\frac{m}{2}}$ or $\mathbb{HP}^n$ and $D^{op} = \{(x, y) \in S \times S^- | dist(x, y) = \pi\}$. Up to Hamiltonian isotopy in $T^*S \times (T^*S)^-$, we have $(S \times S^-)\#_{D^{op}, E_2}((S \times S^-)\#_{\Delta_S, E_2}\Delta) = Graph(\tau_S^{-1})$.*

Proof. The proof is similar to that of Lemma 5.2.3 and we again assume $S = \mathbb{CP}^{\frac{m}{2}}$. We use the function ν_π^0 in Lemma 5.2.3 to define $\mathcal{S} = (S \times S)\#^{\nu_\pi^0}_{\Delta_L, E_2}\Delta$, which is Hamiltonian isotopic to $(S \times S^-)\#_{\Delta_S, E_2}\Delta$ by Lemma 5.2.6. Now, $\mathcal{S}$ intersects $S \times S^-$ cleanly along D^{op}. We can perform another E_2-flow surgery using semi-admissible profiles from $S \times S^-$ to $\mathcal{S}$ to obtain the result, by Lemma 5.2.6 and Lemma 5.2.3.

□

5.2.3 Family versions

One may also generalize the above example to the case of family Dehn twists [106]. Recall that a **spherically fibered coisotropic manifold** $i : C^{2n-l} \hookrightarrow M^{2n}$ is a coisotropic submanifold so that there is a fibration $\rho : C \to B^{2n-2l}$ over a symplectic base, while the fibers are null-leaves S^l. In other words, $\rho^*\omega_B = i^*\omega_M$. Moreover, we equip the fibers with round metric such that geodesics are closed of length 2π and ask the structure group of ρ lies in $SO(l+1)$.

A neighborhood U of C can be symplectically identified with $T^*_\epsilon S^l \times_{SO(l+1)} P$, where P is the principal $SO(l+1)$-bundle associated to C and $T^*_\epsilon S^l$ consists of the cotangent vectors with norm less than ϵ. The **family Dehn twist** τ_C can then be defined fiberwisely as the fiberwise Hamiltonian function $\widetilde{\nu}_\epsilon^{Dehn}(||p||)$ (see Remark 5.1.11) is preserved by the structure group. With respect to the fiberwise metric g^v, the function $h(\cdot) = \widetilde{\nu}(|| \cdot ||_{g^v})$ defines a flow along fibers whose time-1 map is the desired Dehn twist (with a continuation over C defined by fiber-wise antipodal map on C).

Now consider a Lagrangian embedding $\widetilde{C} := C \times_B C \hookrightarrow M \times M$. Explicitly, the image of this map is

$$\widetilde{C} = \{(x, y) \in C \times C \subset M \times M : \pi(x) = \pi(y)\},$$

where $\pi : C \to B$ is the S^l-bundle projection. Indeed, $\widetilde{C} = C^t \circ C$ is a composition Lagrangian in the sense of (5.6). Here we have abused the notation by identifying C with its Lagrangian image in $B \times M$ defined by

$$\{(x, y) \in B \times C \subset B \times M : \pi(y) = x\}.$$

Note that $\widetilde{C}$ is a fiber bundle with fiber $S^l \times S^l$ and structural group the diagonal $SO(l+1)$.

Consider a point $(x, p) \in U$ where Dehn twist is performed. Here $x \in B$ and $p \in T^*_\epsilon S^l$: this is an abuse of notation because p is only well-defined up to an $SO(l{+}1)$ action. Any point contained in $\Delta \cap (U \times U^-) \subset M \times M^-$ thus takes the form $((x, p), (x, p))$. In this setting, the graph of τ_C^{-1} in $U \times U^-$ consist of points

$$Graph(\tau_C^{-1}) = \{((x, p), (x, \phi^{\varpi}_{\nu^{Dehn}(||p||)}(p)))|x \in B, p \in T^*_\epsilon S^l\}$$

where p is again only well-defined up to an $SO(l+1)$ action, and $Graph(\tau_C^{-1})$ coincides with Δ outside $U \times U^-$.

Similar as before, we want to realize $Graph(\tau_C^{-1})$ as a surgery from $\widetilde{C}$ to Δ. In this case, we want to perform a family E_2-surgery.

In the notation of Section 5.1.3, let $\mathcal{L} = \widetilde{C}, \mathcal{D} = \Delta \cap \widetilde{C}$. The $S^l \times S^l$-bundle structure (over B) on $\widetilde{C}$ is the needed bundle structure on $\mathcal{L}$. The restriction of Δ on a fiber $T^*_\epsilon \mathcal{L}_b$ for $b \in B$ is precisely $\Delta_{T^*_\epsilon S^l}$, and the fiberwise E_2-flow is taken as the E_2-flow along the second factor of S^l, as described in the previous subsection.

Hence the whole situation on a fiber $T^*_\epsilon \mathcal{L}_b = T^*_\epsilon(S^l \times S^l)$ is identical to the one in the previous subsection and defines a global Lagrangian handle $\mathcal{H}_{\nu_\pi}$ by patching the local trivializations.

By the same token, we can define **projectively fibered coisotropic manifold** which is a coisotropic manifold with null-leaves complex (or real, quaternionic) projective spaces. Family Dehn twists for these spaces are defined similarly.

Lemma 5.2.9. *Let $C \subset (M, \omega)$ be a spherically (resp. projectively) coisotropic submanifold with base B. Let ν_{λ_i} be λ_i-admissible functions such that $(k-1)\pi < \lambda_i < k\pi$ for some positive integer k for both $i = 1, 2$. Then the family E_2-flow surged Lagrangian manifolds $\widetilde{C} \#^{\nu_{\lambda_i}}_{\mathcal{D},E_2} \Delta$ above by ν_{λ_i} are Hamiltonian isotopic.*

Moreover, if we choose a semi-admissible function $\nu^\alpha_{k\pi} : (0, \infty) \to [0, k\pi)$ such that $\nu^\alpha_{k\pi}(r) = k\pi - \alpha r$ near $r = 0$ $(\alpha \geq 0)$, then $\widetilde{C} \#^{\nu^\alpha_{k\pi}}_{\mathcal{D},E_2} \Delta$ is a smooth Lagrangian that is Hamiltonian isotopic to $\widetilde{C} \#^{\nu_{\lambda_i}}_{\mathcal{D},E_2} \Delta$.

Corollary 5.2.10 (Theorem 1.3.1(2)(5))**.** *For spherically (resp. projectively) coisotropic submanifold C, the family E_2-flow clean surgery $\widetilde{C} \#_{\mathcal{D},E_2} \Delta$ (resp. $\widetilde{C} \#_{\mathcal{D}^{op},E_2} \widetilde{C} \#_{\mathcal{D},E_2} \Delta$) is Hamiltonian isotopic to Graph(τ_C^{-1}). Here $\mathcal{D}^{op}$ is a D^{op}-bundle over the base B and D^{op} is as in Lemma 5.2.8.*

Proof of Lemma 5.2.9. We give the proof for the spherical case and the other cases are similar. Since the construction in Lemma 5.2.6 is $SO(l+1)$ invariant, we can apply Lemma 5.2.6 to $\widetilde{C}$ and $\Delta \cap \widetilde{U}$ to obtain the desired Lagrangian isotopy from $\widetilde{C} \#^{\nu_{\lambda_1}}_{\mathcal{D},E_2} \Delta$ to $\widetilde{C} \#^{\nu_{\lambda_2}}_{\mathcal{D},E_2} \Delta$ and from $\widetilde{C} \#^{\nu^\alpha}_{\mathcal{D},E_2} \Delta$ to $\widetilde{C} \#^{\nu_{\lambda_i}}_{\mathcal{D},E_2} \Delta$.

What remains to show is that the Lagrangian isotopies are Hamiltonian isotopies. We prove the case where the Lagrangian isotopy is from $\widetilde{C} \#^{\nu^\alpha}_{\mathcal{D},E_2} \Delta$ to $\widetilde{C} \#^{\nu_{\lambda_i}}_{\mathcal{D},E_2} \Delta$. The other case is similar. Denote the Lagrangian isotopy as $\iota_{\mathcal{L},t} : \mathcal{L} \to M \times M^-$. Notice that the Lagrangian isotopy $\iota_{\mathcal{L},t}$ restricting to each fiber $\iota_{L,t} : L \to T^*S^l \times (T^*S^l)^{-1}$ is a Hamiltonian isotopy and hence an exact isotopy (i.e $\alpha_0 = \iota^*_{L,t}(\omega_{can} \oplus -\omega_{can})(\frac{\partial \iota_{L,t}}{\partial t}, \cdot)$ is exact). Since the fiberwise symplectic form and the isotopy are $SO(l+1)$-invariant, so is α_0 and its primitive. These primitives on fibers can be patched together to form the primitive of $\alpha = \iota_{\mathcal{L},t}^*(\omega_M \oplus -\omega_M)(\frac{\partial \iota_{\mathcal{L},t}}{\partial t}, \cdot)$ and hence $\iota_{\mathcal{L},t}$ is an exact, thus a Hamiltonian isotopy.

Alternatively, one may also patch the Hamiltonian isotopy from Lemma 5.2.6. We will leave the details to the reader.

$\square$

Corollary 5.2.10 is now an immediate consequence of Lemma 5.2.9 by setting $k = 1$ for spherical case and $k = 2$ for the projective cases.

5.3 Gradings

In this section we discuss the gradings in Lagrangian surgeries. We follow mostly the exposition in [7] to review the definition of gradings in subsection 5.3.1. The subsequent subsections provide computation for a sufficient and necessary criterion to perform graded surgeries. Starting from the next section, all surgeries between graded Lagrangian will be graded surgeries. Our discussion stay in the $\mathbb{Z}$-graded case but the corresponding results for $\mathbb{Z}/N$-gradings can be obtained by modifying our argument using the setting in [90] and the statements will be a mod-N reduction of what we have here.

5.3.1 Basic notions

Let (M^{2n}, ω) be an exact symplectic manifold with a primitive one form α for ω, equipped with an ω-compatible almost complex structure J making M pseudo-convex at infinity. We also assume $2c_1(M) = 0$ and fix once and for all a nowhere-vanishing section Ω^2 of $(\Lambda_{\mathbb{C}}^{top}(T^*M, J))^{\otimes 2}$.

Let L be a connected manifold without boundary (not necessarily compact) and $\iota_L : L \to M$ a proper exact Lagrangian immersion (i.e. $\iota_L^*\alpha$ is exact). A **grading** on (L, ι_L) (sometimes simply denoted as ι_L) is defined as a continuous function $\theta_L : L \to \mathbb{R}$ such that $e^{2\pi i \theta_L} = Det_\Omega^2 \circ D\iota_L$, where Det_Ω^2 is defined as

$$Det_\Omega^2(\Lambda_p) = Det_\Omega^2(X_1, \dots, X_n) = \frac{\Omega(X_1, \dots, X_n)^2}{\|\Omega(X_1, \dots, X_n)^2\|} \in S^1$$

for any Lagrangian plane $\Lambda_p \subset T_pM$ at a point p and any choice of a basis $\{X_1, \ldots, X_n\}$ for Λ_p.

Given two transversal Lagrangian planes Λ_0, Λ_1 (of dimension n) at the same point with a choice of θ_0, θ_1 such that $e^{2\pi i \theta_j} = Det^2_\Omega(\Lambda_j)$ for both j, we can identify them as graded Lagrangian vector subspaces of $\mathbb{C}^n$. The index of the pair (Λ_0, θ_0) and (Λ_1, θ_1) is defined as

$$Ind((\Lambda_0, \theta_0), (\Lambda_1, \theta_1)) = n + \theta_1 - \theta_0 - 2Angle(\Lambda_0, \Lambda_1) \tag{5.4}$$

where $Angle(\Lambda_0, \Lambda_1) = \sum_{j=1}^{n} \beta_j$ and $\beta_j \in (0, \frac{1}{2})$ are such that there is a unitary basis $u_1, \ldots, u_n$ of Λ_0 satisfying $\Lambda_1 = Span_{\mathbb{R}}\{e^{2\pi i \beta_j} u_j\}_{j=1}^n$.

In general, when $\Lambda_0 \cap \Lambda_1 = \Lambda \neq \{0\}$, the definition of index for the pair (Λ_0, θ_0) and (Λ_1, θ_1) is the same as above with the definition of $Angle(\Lambda_0, \Lambda_1)$ modified as follows. Pick a path of Lagrangian planes Λ_t from Λ_0 to Λ_1 such that

- $\Lambda \subset \Lambda_t \subset \Lambda_0 + \Lambda_1$ for all $t \in [0, 1]$, and
- the image $\overline{\Lambda_t}$ of Λ_t inside the symplectic vector space $(\Lambda_0 + \Lambda_1)/\Lambda$ is the positive definite path from $\overline{\Lambda_0}$ to $\overline{\Lambda_1}$.

Let β_t be a continuous path of real numbers such that $e^{2\pi i \beta_t} = Det^2_\Omega(\Lambda_t)$. Then, the Lagrangian angle is defined as

$$2Angle(\Lambda_0, \Lambda_1) = \beta_1 - \beta_0$$

Definition 5.3.1. For two graded Lagrangian immersions (ι_{L_1}, θ_1), (ι_{L_2}, θ_2) (not necessarily distinct), and points $p_j \in L_j$ for $j = 1, 2$ such that $\iota_1(p_1) = \iota_2(p_2) = p$, the index for the ordered pair (p_1, p_2) is

$$Ind(p_1, p_2) = Ind(((\iota_1)_* T_{p_1} L_1, \theta_1(p_1)), ((\iota_2)_* T_{p_2} L_2, \theta_2(p_2)))$$

We also use the notation $Ind(L_1|_p, L_2|_p)$ to denote $Ind(p_1, p_2)$ if $\iota_1^{-1}(p) = \{p_1\}$ and $\iota_2^{-1}(p) = \{p_2\}$. Note that if L_1 intersects L_2 cleanly along D and if D is connected, than $Ind(L_1|_p, L_2|_p) = Ind(L_1|_q, L_2|_q)$ for all $p, q \in D$. In this case, we denote the index as $Ind(L_1|_D, L_2|_D)$.

Example 5.3.2. For a graded Lagrangian immersion (ι_L, θ) and an integer k, $\iota_L[k]$ is defined as $\iota_L[k] = (\iota_L, \theta - k)$. In particular, we have

$$Ind(\iota_{L_1}[k]|_D, \iota_{L_2}[k']|_D) = Ind(\iota_{L_1}|_D, \iota_{L_2}|_D) + k - k'$$

Example 5.3.3. Let $M = \mathbb{C}^n$ be equipped with the standard symplectic form, complex structure and complex volume form. Let $L_1 = \mathbb{R}^n = \{y_1 = \cdots = y_n = 0\}$ and $L_2 = \{x_1 = \cdots = x_{n-k} = y_{n-k+1} = \cdots = y_n = 0\}$ be two Lagrangian planes for some $0 \leq k \leq n$. We have $Det^2_\Omega(L_1) = 1$ and $Det^2_\Omega(L_2) = (-1)^{n-k}$. Let $\theta_{L_1} = n-k-1$ and $\theta_{L_2} = \frac{n-k}{2}$ be the grading of L_1 and L_2. Then, we have $Ind(L_1|_0, L_2|_0) = (n) + \frac{n-k}{2} - (n-k-1) - 2(n-k)(\frac{1}{4}) = k+1$.

Remark 5.3.4. For a Lagrangian isotopy $\Phi = (\Phi_t)_{t\in[0,1]} : L \times [0,1] \to (M,\omega)$, if Φ_0 is equipped with grading θ_0, then the induced grading on Φ_1 is defined as follows. There is a uniquely way to extend $\theta_0 : L \times \{0\} \to \mathbb{R}$ continuously to $\theta : L \times [0,1] \to \mathbb{R}$ such that $e^{2\pi i\theta(\cdot,t)} = Det^2_\Omega \circ D\Phi_t(\cdot)$ and the induced grading on Φ_1 is defined by $\theta(\cdot,1)$.

Example 5.3.5. Let $L = \mathbb{R} \subset (\mathbb{R}^2, dx \wedge dy)$ and identify the latter with $\mathbb{C}$ equipped with the standard complex volume form. Consider $h : \mathbb{R} \to \mathbb{R}$ given by $h(q) = c\frac{q^2}{2}$ for some constant c. The graph of dh, $Graph(dh)$, is given by $\{(q, cq) \in T^*L | q \in L\}$. By letting $q = x$ and $p = -y$ to identify $\mathbb{C}$ with T^*L, $Graph(dh)$ is given by $\{(x, -cx) \in \mathbb{C}\}$. Under our convention of Hamiltonian flow, $Graph(dh)$ is the time 1 Hamiltonian flow of L under Hamiltonian $-h \circ \pi : T^*L \to \mathbb{R}$, where $\pi : T^*L \to L$ is the projection. If we give a grading to L and induces it to a grading on $Graph(dh)$ by the Hamiltonian isotopy, then

$$Ind(L|_0, Graph(dh)|_0) = \begin{cases} 1 & \text{if } c \leq 0 \\ 0 & \text{if } c > 0 \end{cases}$$

In short, the index equals the Morse index of h if $c \neq 0$. We call the grading defined above an **induced grading on** $Graph(dh)$.

Example 5.3.6. Let $L = \mathbb{R}^n \subset (\mathbb{C}^n, dx_i \wedge dy_i)$. Consider $h : L \to \mathbb{R}$ given by $h(q) = c\sum_{j=1}^{k} \frac{q_j^2}{2}$. If we let $q_i = x_i$ and $p_i = -y_i$ to identify $\mathbb{C}^n$ with T^*L and give the induced grading to $Graph(dh)$ by a grading of L and the Hamiltonian isotopy induced by $-h\circ\pi$, then

$$Ind(L|_{\mathbb{R}^{n-k}}, Graph(dh)|_{\mathbb{R}^{n-k}}) = \begin{cases} n & \text{if } c \leq 0 \\ n-k & \text{if } c > 0 \end{cases}$$

where $\mathbb{R}^{n-k}$ is the last $n-k$ q_i coordinates.

Corollary 5.3.7. *Let $h : L \to \mathbb{R}$ be a Morse-Bott function with Morse-Bott maximum at critical submanifold D_1 of dimension k_1 and minimum at D_2 of dimension k_2. If $L \subset T^*L$ is graded and $Graph(dh)$ is equipped with grading induced from that of L and the Hamiltonian isotopy induced by $-h \circ \pi$, then $Ind(L|_{D_1}, Graph(dh)|_{D_1}) = n$ and $Ind(L|_{D_2}, Graph(dh)|_{D_2}) = n - k_2$*

Definition 5.3.8. For a Lagrangian immersion (ι_L, θ_L), we define

$$R_L = R_{\iota_L} := \{(p,q) \in L \times L | \iota(p) = \iota(q), p \neq q\}$$

and call it the **set of branch jump** .

Example 5.3.9. Let $M = T^*\mathbb{RP}^n$ be equipped with the canonical one form and symplectic form with $n > 1$. Fix a choice of J and Ω^2. Let $\iota_L : L = S^n \to \mathbb{RP}^n := \underline{L}$ be the double cover of the zero section. Note that we can equip $\underline{L}$ with a grading $\theta_{\underline{L}} : \mathbb{RP}^n \to \mathbb{R}$. The lift of $\theta_{\underline{L}}$ to $\theta_L : L \to \mathbb{R}$ gives a $\mathbb{Z}/2\mathbb{Z}$ invariant grading of L and hence $Ind(p,q) = n$ for any $(p,q) \in R_L$. Converesely, if θ_L is any grading on L, then we must have $\theta_L(p) = \theta_L(q)$ for any $(p,q) \in R$ because $\theta_L(q) - \theta_L(p) \in \mathbb{Z}$ for any $(p,q) \in R$ and $\theta_L(q) - \theta_L(p)$ varies continuously with respect to (p,q).

5.3.2 Local computation for surgery at a point

The grading of Lagrangian surgery in the local model was considered by Seidel [90] already, and we include an account for completeness. Let H_γ be a Lagrangian handle. We equip $\mathbb{C}^n$ with the standard complex volume form $\Omega = dz_1 \wedge \cdots \wedge dz_n$.

Lemma 5.3.10 ([90])**.** *Let $\mathbb{R}^n$ and $i\mathbb{R}^n$ be equipped with gradings θ_r and θ_i, respectively. Then, there is a grading θ_H on H_γ and a unique integer m such that θ_H can be patched with $\theta_r + m$ and θ_i to give a grading on $\mathbb{R}^n \#_0 i\mathbb{R}^n$. If $Ind((\mathbb{R}^n|_0, \theta_r), (i\mathbb{R}^n|_0, \theta_i)) = 1$, we have $m = 0$.*

Proof. As shown in Example 5.1.13, $H_\gamma = H_\nu$ for some flow handle H_ν. Since H_ν is obtained by Hamiltonian flow of $i\mathbb{R}^n$, H_ν is canonically graded by θ_i using the Hamiltonian isotopy. We call this grading θ_H and continuously extend it on $Cl(H_\nu)$. Since $\mathbb{R}^n \cap Cl(H_\nu)$ has one grading induced from θ_r and one induced from θ_H, $\theta_H|_{\mathbb{R}^n \cap Cl(H_\nu)} - \theta_r|_{\mathbb{R}^n \cap Cl(H_\nu)}$ is a locally constant integer-valued function. If $\mathbb{R}^n \cap Cl(H_\nu)$ is connected, then there is a unique integer m such that $\theta_H|_{\mathbb{R}^n\cap Cl(H_\nu)} = \theta_r|_{\mathbb{R}^n\cap Cl(H_\nu)} + m$. If $\mathbb{R}^n \cap Cl(H_\nu)$ is not connected, then $n = 1$ and one can check directly that the same conclusion holds. As a result, this m is the unique integer such that θ_H can be patched with $\theta_r + m$ and θ_i to give a grading on $\mathbb{R}^n\#_0 i\mathbb{R}^n$. In what follows, we want to show that $m = 0$ if $Ind((\mathbb{R}^n|_0, \theta_r), (i\mathbb{R}^n|_0, \theta_i)) = 1$.

Pick a point $x = (x_1, \dots, x_n) \in S^{n-1}$. Let $c(s) = \gamma(s)x \in H_\gamma$ and denote the image curve as $Im(c)$, where γ is an admissible curve (See Definition 5.1.7). The Lagrangian plane Λ_s at $c(s)$ is spanned by $\{\gamma'(s)x\} \cup \{\gamma(s)v_j\}_{j=2}^n$, where $v_j \in T_xS^{n-1}$ forms an orthonormal basis. (See also the proof of Lemma 5.1.8). Therefore, we have

$$Det^2_\Omega(\Lambda_s) = e^{i2(\arg(\gamma'(s))+(n-1)\arg(\gamma(s)))}$$

for all s. There is a unique continuous function $\theta_c : Im(c) \to \mathbb{R}$ such that

- $\theta_c(c(s)) = n - 1$ for $s < 0$,
- $\theta_c(c(s)) = \frac{n}{2}$ for $s > \epsilon$, and
- $e^{2\pi i\theta_c(c(s))} = Det^2_\Omega(\Lambda_s)$ for all s

Therefore, we have $\theta_c - \theta_H|_{Im(c)} \in \mathbb{Z}$ and θ_c describes the change of Lagrangian planes from $\mathbb{R}^n$ to $i\mathbb{R}^n$ along the handle. By comparing with the Example 5.3.3 (for $k = 0$), we can see that if the graded Lagrangians $\mathbb{R}^n$ and $i\mathbb{R}^n$ inside $\mathbb{C}^n$ intersect at the origin of index 1, then $m = 0$. This finishes the proof. $\square$

Corollary 5.3.11 ([90])**.** *Let $\iota_i : L_i \to (M, \omega)$ for $i = 1, 2$ be two graded Lagrangian immersions with grading θ_1 and θ_2, respectively, intersecting transversally at a point p. If $Ind((L_1|_p, \theta_1), (L_2|_p, \theta_2)) = 1$, then $\iota : L_1\#_p L_2 \to (M, \omega)$ can be equipped with a grading θ_{12} extending θ_1 and θ_2. In this case, we call $L_1\#_p L_2$ together with its grading as a surgery from graded L_1 to L_2.*

5.3.3 Local computation for surgery along clean intersection

This subsection discuss the grading for Lagrangian surgery along clean intersection. We start with ordinary clean surgery (See subsubsection 5.1.2).

Lemma 5.3.12. *Let $L_1, N^*_D \subset T^*L_1$ be equipped with gradings θ_r and θ_i, respectively. For any λ-admissible function ν such that $\lambda < r(D)$, there is a grading θ_H on H^D_ν and* *a unique integer m such that θ_H can be patched with $\theta_i, \theta_r + m$ to become a grading on $L_1 \#^{\nu}_{D} N^*_D$.*

Moreover, $m = 0$ if and only if $Ind((L_1|_D, \theta_r), (N^*_D|_D, \theta_i)) = dim(D) + 1$.

Immediately from Lemma 5.3.12, we have

Corollary 5.3.13. *Let $L_1, L_2 \subset (M, \omega)$ be graded Lagrangians cleanly intersecting at D. We can perform a graded surgery $L_1 \#_D L_2$ from L_1 to L_2 along D if and only if $Ind(L_1|_D, L_2|_D) = dim(D) + 1$.*

Proof of Lemma 5.3.12. The first statement of the lemma follows as in the first paragraph of the proof of Lemma 5.3.10. Therefore, we just need to prove that $m = 0$ if and only if $Ind((L_1|_D, \theta_r), (N^*_D|_D, \theta_i)) = dim(D) + 1$. Let $dim(D) = k$.

Pick a Darboux chart such that in local coordinates N^*_D is represented by points of the form $(q, p) = (q_b, 0, 0, p_f) = (q_1, \dots, q_k, 0, \dots, 0, p_{k+1}, \dots, p_n)$, where the first 0 in $(q_b, 0, 0, p_f)$ are the last $n - k$ q_i coordinates and the second 0 are the first k p_i coordinates. We also require $(q_1, \dots, q_k, tq_{k+1}, \dots, tq_n)$ are normalized geodesics on L_1 as t varies, for any $q_1, \dots, q_n$ such that $\sum_{j=k+1}^{n} q_j^2 = 1$. As a result, the handle H^D_ν in local coordinates is given by (here, we suppose that the surgery is supported in a sufficiently small region relative to the Darboux chart)

$$\{\phi^{\varpi}_{\nu(\|p_f\|)}(q_b, 0, 0, p_f) = ((q_b, \nu(\|p_f\|)\frac{p_f}{\|p_f\|}, 0, p_f)|q_b \in \mathbb{R}^k, p_f \in \mathbb{R}^{n-k}\}$$

We consider the standard complex volume form $\Omega = dz_1 \wedge \cdots \wedge dz_n$ in the chart. Let $e_{\pi_2} \in S^{n-k-1} \subset \mathbb{R}^{n-k}$ be a vector in the unit sphere of last $n - k$ p_i coordinates. Let

$$c(r) = (0, \nu(\|re_{\pi_2}\|)\frac{re_{\pi_2}}{\|re_{\pi_2}\|}, 0, re_{\pi_2}) = (0, \nu(r)e_{\pi_2}, 0, re_{\pi_2})$$

be a smooth curve on H^D_ν for $r \in (0, \epsilon]$. We define $c(0) = \lim_{r \to 0^+} c(r)$.

We want to understand how the Lagrangian planes change from L_1 to N^*_D along the handle and it suffices to look at how the Lagrangian planes change along $c(r)$. The Lagrangian plane Λ_r at $c(r)$ is spanned by

$$\{(e_j, 0, 0, 0)\}_{j=1}^{k} \cup \{(0, \nu'(r)e_{\pi_2}, 0, e_{\pi_2})\} \cup \{(0, \nu(r)\frac{e_j^{\perp}}{r}, 0, e_j^{\perp})\}_{j=2}^{n-k}$$

where $e_j \in \mathbb{R}^k$ are coordinate vectors and $e_j^{\perp}$ form an orthonormal basis for orthogonal complement of e_{π_2} in $\mathbb{R}^{n-k}$.

Then we have

$$Det^2_\Omega(\Lambda_r) = e^{i2(\arg(\nu'(r)-\sqrt{-1})+(n-k-1)\arg(\frac{\nu(r)}{r}-\sqrt{-1}))}$$

for all r. Here, the convention we use is still, $z_i = q_i - \sqrt{-1}p_i$. Observe that $\nu'(\epsilon) = \frac{\nu(\epsilon)}{\epsilon} = 0$. When r goes to 0, $\nu'(r)$ decreases monotonically to $-\infty$. Similarly, $\frac{\nu(r)}{r}$ increases monotonically to infinity because r goes to zero and ν is bounded and positive.

In particular, $\arg(\nu'(r) - \sqrt{-1})$ increases from π to $\frac{3\pi}{2}$ as r increases and $\arg(\frac{\nu(r)}{r} - \sqrt{-1}))$ decreases from 2π to $\frac{3\pi}{2}$ as r increases. Therefore, there is a unique continuous function $\theta_c : Im(c) \to \mathbb{R}$ such that

- $\theta_c(c(r)) = n - k - 1$ for $r = 0$,
- $\theta_c(c(r)) = \frac{n-k}{2}$ for $r = \epsilon$, and
- $e^{2\pi i\theta_c(c(r))} = Det^2_\Omega(\Lambda_r)$ for all $r \in [0, \epsilon]$.

By Example 5.3.3, we have $Ind((\mathbb{R}^n|_{\mathbb{R}^k}, n-k-1), (N^*(\mathbb{R}^k)|_{\mathbb{R}^k}, \frac{n-k}{2})) = k+1$. Hence, $m = 0$ if and only if $Ind((L_1|_D, \theta_r), (N^*_D|_D, \theta_i)) = k + 1$. □

For the E_2-flow surgery, we use the setting in subsection 5.1.3 and we have

Lemma 5.3.14. *Suppose $D \subset L = K_1 \times K_2$ is a smooth submanifold of dimension k which is transversal to $\{p\} \times K_2$ for all $p \in K_1$. Let $L, N^*_D \subset T^*L$ be equipped with gradings θ_r and θ_i, respectively. For any λ-admissible function ν such that $\lambda < r^{E_2}(D)$, there is a grading θ_H on H^{D,E_2}_ν and a unique integer m such that θ_H can be patched with $\theta_r + m$ and θ_i to become a grading on $L\#^\nu_{D,E_2}N^*_D$.*

Moreover, we have $m = 0$ if and only if $Ind((L|_D, \theta_r), (N^*_D|_D, \theta_i)) = dim(D) + 1$.

Corollary 5.3.15. *Let $L_1 = K_1 \times K_2, L_2 \subset (M, \omega)$ be graded Lagrangians cleanly intersecting at D. Suppose D is transversal to $\{p\} \times K_2$ for all $p \in K_1$. Then we can perform a graded E_2-flow surgery $L_1\#_{D,E_2}L_2$ from L_1 to L_2 along D if and only if $Ind(L_1|_D, L_2|_D) = dim(D) + 1$.*

Proof of Lemma 5.3.14. As explained before (cf. Lemma 5.3.10 and Lemma 5.3.12), we just need to show that $m = 0$ if and only if $Ind((L|_D, \theta_r), (N^*_D|_D, \theta_i)) = dim(D) + 1$. Again denote $k = dim(D)$.

Pick a chart compatible with the product structure on L and define $q_b = (q_1, \dots, q_k) \in L_1$ and $q_f = (q_{k+1}, \dots, q_n) \in L_2$. We also want that (q_b, tq_f) is a geodesic with velocity one as t varies, for any q_b, q_f such that $|q_f| = 1$. We can also assume the origin belongs to D and denote a basis of the tangent space of D at origin T_0D as $\{w^1, \dots, w^k\}$ and $w^j = w_b^j + w_f^j$, where w_b^j and w_f^j are the q_b and q_f components of w^j, respectively. Since D is transversal to the second factor, we can assume w_b^j are the unit coordinate vectors in the q_b-plane for $1 \le j \le k$. Moreover, there is a function $q_f^D(q_b)$ of q_b near origin such that $(q_b, q_f^D(q_b)) \in D$.

This chart gives a corresponding Darboux chart on T^*L and we define p_b^D as a function of q_b, p_f near origin such that $(q_b, q_f^D(q_b), p_b^D(q_b, p_f), p_f) \in N_D^*$. Note that $p_b^D(\cdot,\cdot)$ is linear on the second factor. Near the origin (close enough to origin such that $q_f^D(q_b)$ is well-defined), the handle H_ν^{D,E_2} in local coordinates is given by

$$\{\phi^{\varpi_\pi}_{\nu(\|p_b\|)}(q_b, q_f^D(q_b), p_b^D(q_b, p_f), p_f) = (q_b, q_f^D(q_b) + \nu(\|p_f\|)\frac{p_f}{\|p_f\|}, p_b^D(q_b, p_f), p_f) | q_b \in \mathbb{R}^k, p_f \in \mathbb{R}^{n-k}\}$$

We consider the standard complex volume form $\Omega = dz_1 \wedge \cdots \wedge dz_n$ in the chart. Let $e_{\pi_2} \in S^{n-k-1} \subset \mathbb{R}^{n-k}$ be a vector in the unit sphere in the p_f coordinates. Let

$$\begin{aligned} c(r) &= \phi^{\varpi_\pi}_{\nu(\|re_{\pi_2}\|)}(0, 0, p_b^D(0, re_{\pi_2}), re_{\pi_2}) \\ &= (0, \nu(\|re_{\pi_2}\|)\frac{re_{\pi_2}}{\|re_{\pi_2}\|}, p_b^D(0, re_{\pi_2}), re_{\pi_2}) \\ &= (0, \nu(r)e_{\pi_2}, p_b^D(0, re_{\pi_2}), re_{\pi_2}) \end{aligned}$$

be a smooth curve in H_ν^{D,E_2} for $r \in (0, \epsilon]$. We define $c(0) = \lim_{r\to 0^+} c(r)$.

The Lagrangian plane Λ_r of H_ν^{D,E_2} at $c(r)$ is spanned by

$$\{(w_b^j, w_f^j, \kappa(r, w^j), 0)\}_{j=1}^k \cup \{(0, \nu'(r)e_{\pi_2}, p_b^D(0, e_{\pi_2}), e_{\pi_2})\} \cup \{(0, \frac{\nu(r)}{r}e_j^\perp, p_b^D(0, e_j^\perp), e_j^\perp)\}_{j=2}^{n-k}$$

where $\kappa(r, w^j) = \partial_{q_j} p_b^D(0, re_{\pi_2}) = r(\partial_{q_j} p_b^D(0, e_{\pi_2}))$ is linear in r and $e_j^\perp$ form an orthonormal basis for orthogonal complement of e_{π_2} in $\mathbb{R}^{n-k}$. We note that $(0, \nu'(r)e_{\pi_2}, p_b^D(0, e_{\pi_2}), e_{\pi_2}) = c'(r)$ and the computation uses the fact that $p_b^D(\cdot,\cdot)$ is linear on the second factor.

Let $\kappa_j(r, w^j)$ be the coefficient of w_b^j-component of $\kappa(r, w^j)$ (Here, we identify the q_b-plane and the p_b-plane). Notice that

$$Det^2_\Omega(\Lambda_r) = e^{i2(\sum_{j=1}^k \arg(1-\kappa_j(r,w^j)\sqrt{-1}) + \arg(\nu'(r)-\sqrt{-1}) + (n-k-1)\arg(\frac{\nu(r)}{r}-\sqrt{-1}))}$$

for all r (Here, we use the fact that w_b^j are unit coordinates vectors and we use the convention $z_i = q_i - \sqrt{-1}p_i$). Let $K(r) = \sum_{j=1}^k \arg(1 - \kappa_j(r, w^j)\sqrt{-1})$.

Similar to Lemma 5.3.12, $\arg(\nu'(r)-\sqrt{-1})$ increases from π to $\frac{3\pi}{2}$ as r increases and $\arg(\frac{\nu(r)}{r}-\sqrt{-1}))$ decreases from 2π to $\frac{3\pi}{2}$ as r increases. Therefore, there is a unique continuous function $\theta_c : Im(c) \to \mathbb{R}$ such that

- $\theta_c(c(r)) = n-k-1+\frac{K(0)}{\pi} = n-k-1$ for $r=0$,
- $\theta_c(c(r)) = \frac{n-k}{2}+\frac{K(\epsilon)}{\pi}$ for $r=\epsilon$, and
- $e^{2\pi i\theta_c(c(r))} = Det^2_\Omega(\Lambda_r)$ for all $r \in [0,\epsilon]$.

On the other hand, we can lift a path of Lagrangian plane Λ_r^N of N_D^* over the path $c_2(r) = (0,0,p_b^D(0,re_{\pi_2}),re_{\pi_2})$ connecting the origin and $c(\epsilon)$. The Lagrangian plane Λ_r^N is spanned by

$$\{(w_b^j, w_f^j, \kappa(r,w^j),0)\}_{j=1}^k \cup \{(0,0,p_b^D(0,e_{\pi_2}),e_{\pi_2})\} \cup \{(0,0,p_b^D(0,e_j^\perp),e_j^\perp)\}_{j=2}^{n-k}$$

Therefore, the grading of N_D^* at origin is the grading of N_D^* at $c(\epsilon)$ subtracted by $\frac{K(\epsilon)}{\pi}$. If we extend θ_c continuously over $Im(c_2)$ (note: $Im(c) \cap Im(c_2) = \{c(\epsilon)\}$), then $\theta_c(c_2(0)) = \frac{n-k}{2}$.

By an analogous calculation as in Example 5.3.3, we have

$$Ind((L_1|_D, n-k-1),(N_D^*|_D, \frac{n-k}{2})) = k+1$$

and by comparing it with θ_c, the result follows.

$\square$

The following is a family version whose proof is similar.

Corollary 5.3.16. *Let $\mathcal{L}_0, \mathcal{L}_1 \subset (M^{2n},\omega)$ as in Lemma 5.1.25 and let the dimension of $\mathcal{D}$ be k. Assume $\mathcal{L}_0, \mathcal{L}_1$ are graded with grading θ_r and θ_i. Then $Ind((\mathcal{L}_1|_\mathcal{D},\theta_r),(N^*\mathcal{D}|_\mathcal{D},\theta_i)) = k+1$ if and only if $\mathcal{L}_0\#^\nu_{\mathcal{D},E_2}\mathcal{L}_1$ has a grading such that the grading restricted to $\mathcal{L}_0, \mathcal{L}_1$ coincide with θ_r and θ_i, respectively.*

5.3.4 Diagonal in product

We recall from [108] how to associate the canonical grading to the diagonal in $M \times M^-$.

For a standard symplectic vector space $(\mathbb{R}^{2n},\omega_{std})$ and its N-fold Maslov cover $Lag^N(\mathbb{R}^{2n},\Lambda_0)$ based at a graded Lagrangian plane Λ_0, we can associate a N-fold Maslov cover $Lag^N(\mathbb{R}^{2n,-} \times \mathbb{R}^{2n}, \Lambda_0^- \times \Lambda_0)$. In particular, $\Lambda_0^- \times \Lambda_0$ is canonically graded. For any Lagrangian plane $\Lambda \subset \mathbb{R}^{2n}$ and a path γ from Λ to Λ_0, the induced path $\gamma^- \times \gamma$ from $\Lambda^- \times \Lambda$ to $\Lambda_0^- \times \Lambda_0$ gives an identification between $Lag^N(\mathbb{R}^{2n,-} \times \mathbb{R}^{2n}, \Lambda_0^- \times \Lambda_0)$ and $Lag^N(\mathbb{R}^{2n,-} \times \mathbb{R}^{2n}, \Lambda^- \times \Lambda)$, independent from the choice of γ. This gives a canonical grading on $\Lambda^- \times \Lambda$.

To give a canonical grading to the diagonal $\Delta \subset \mathbb{R}^{2n,-} \times \mathbb{R}^{2n}$, it suffices to give once and for all an identification between $Lag^N(\mathbb{R}^{2n,-} \times \mathbb{R}^{2n}, \Lambda^- \times \Lambda)$ and $Lag^N(\mathbb{R}^{2n,-} \times \mathbb{R}^{2n}, \Delta)$. This is given by concatenation of two paths

$$(e^{Jt}\Lambda^- \times \Lambda)_{t\in[0,\frac{\pi}{2}]}, \quad (\{(tx + Jy, x + tJy) | x, y \in \Lambda\})_{t\in[0,1]}$$

where J is an ω_{std}-compatible complex structure on $\mathbb{R}^{2n}$. This canonical grading induces a canonical grading on $\Delta_M \subset M^- \times M$ for any symplectic manifold M.

In the following lemma, we consider our symplectic manifold being $M = \mathbb{C}^{n,-}$ and compute the index between a product Lagrangian with the diagonal Δ_M.

Lemma 5.3.17 (c.f. Section 3 of [108])**.** *For any graded Lagrangian subspace $\Lambda \subset \mathbb{C}^{n,-}$, we have*

$$Ind(\Lambda^- \times \Lambda|_{\Delta_\Lambda}, \Delta_{\mathbb{C}^{n,-}}|_{\Delta_\Lambda}) = n$$

where $\Lambda^- \times \Lambda$ and $\Delta_{\mathbb{C}^{n,-}}$ are equipped with their canonical gradings in $\mathbb{C}^n \times \mathbb{C}^{n,-}$.

Proof. It suffices to consider $\Lambda = \mathbb{R}^n \subset \mathbb{C}^{n,-}$ and $J = -J_{std} = -\sqrt{-1}$. Let $z_i = x_i + \sqrt{-1}y_i$ be the coordinates of $\mathbb{C}^n$ and $w_i = u_i + \sqrt{-1}v_i$ be the coordinates of $\mathbb{C}^{n,-}$. We consider the standard complex volume form $\Omega = dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{w}_1 \wedge \cdots \wedge d\bar{w}_n$ on $\mathbb{C}^n \times \mathbb{C}^{n,-}$ and equip $\Lambda^- \times \Lambda$ with grading 0. We have $Det^2(e^{Jt}\Lambda^- \times \Lambda) = e^{-i2nt}$, which induces a grading of $-\frac{n}{2}$ on $e^{J\frac{\pi}{2}}\Lambda^- \times \Lambda$. We also have $Det^2(\{(tx + Jy, x + tJy) | x, y \in \Lambda\}) = e^{-in\pi}$ for all t so the canonical grading on Δ is $-\frac{n}{2}$.

To calculate $Angle(\Lambda^- \times \Lambda, \Delta)$, we observe that $(\Lambda^- \times \Lambda) \cap \Delta = Span\{(\partial_{x_i} + \partial_{u_i})\}_{i=1}^n$. We can use $\Lambda_t = (\Lambda^- \times \Lambda) \cap \Delta + Span\{(t(\partial_{y_i} + \partial_{v_i}) + (1-t)(-\partial_{x_i} + \partial_{u_i}))\}$ from $\Lambda^- \times \Lambda$ to Δ for the calculation of $Angle(\Lambda^- \times \Lambda, \Delta)$. As a result, we have $2Angle(\Lambda^- \times \Lambda, \Delta) = \frac{n}{2}$ and hence

$$Ind(\Lambda^- \times \Lambda|_{\Delta_\Lambda}, \Delta_{\mathbb{C}^{n,-}}|_{\Delta_\Lambda}) = 2n + (-\frac{n}{2}) - 0 - \frac{n}{2} = n$$

$\square$

Corollary 5.3.18. *Let L be a Lagrangian in M. With the canonical gradings of $L \times L \subset M \times M^-$ and $\Delta \subset M \times M^-$, one can perform graded clean surgery to obtain $(L \times L)[1]\#_{\Delta_L, E_2}\Delta$.*

Proof. This is a direct consequence of Lemma 5.3.14 and Lemma 5.3.17. $\square$

Corollary 5.3.19 (cf. Theorem 1.3.1(1))**.** *There is a graded clean surgery identity*

$$(S^n \times S^n)[1]\#_{\Delta_{S^n}, E_2}\Delta = Graph(\tau_{S^n}^{-1})$$

Proof. A direct consequence of Corollary 5.2.7 and Corollary 5.3.18. $\square$

Lemma 5.3.20 (cf. Theorem 1.3.1(3))**.** *There is a graded clean surgery identity*

$$\mathbb{CP}^{\frac{m}{2}} \times \mathbb{CP}^{\frac{m}{2}} \#_{D^{op}, E_2}((\mathbb{CP}^{\frac{m}{2}} \times \mathbb{CP}^{\frac{m}{2}})[1]\#_{\Delta_{\mathbb{CP}^{\frac{m}{2}}}, E_2}\Delta) = Graph(\tau_{\mathbb{CP}^{\frac{m}{2}}}^{-1})$$

Proof. By Corollary 5.3.18, we can obtain a graded Lagrangian $L = (\mathbb{CP}^{\frac{m}{2}} \times \mathbb{CP}^{\frac{m}{2}})[1]\#_{\Delta_{\mathbb{CP}^{\frac{m}{2}}}, E_2}\Delta$. As explained in the proof of Lemma 5.2.6 and Lemma 5.2.8, L is Hamiltonian isotopic to a Lagrangian Q cleanly intersecting with $\mathbb{CP}^{\frac{m}{2}} \times \mathbb{CP}^{\frac{m}{2}}$ along D^{op} such that Q coincide with the graph of a Morse-Bott function with maximum at D^{op} near D^{op}. Therefore, we have $Ind(\mathbb{CP}^{\frac{m}{2}} \times \mathbb{CP}^{\frac{m}{2}}|_{D^{op}}, Q|_{D^{op}}) = 2m - 1$. Here the first term $2m$ follows by Corollary 5.3.7 and the second term -1 comes from the grading shift of the first factor of L. Since D^{op} is of dimension $2m-2$, we get the result by applying Lemma 5.3.14. $\square$

The cases for $\mathbb{RP}^n$ and $\mathbb{HP}^n$ can be computed analogously.

Lemma 5.3.21 (cf. Theorem 1.3.1(3))**.** *There are also graded clean surgery identities*

$$\mathbb{RP}^n \times \mathbb{RP}^n[1]\#_{D^{op}, E_2}((\mathbb{RP}^n \times \mathbb{RP}^n)[1]\#_{\Delta_{\mathbb{RP}^n}, E_2}\Delta) = Graph(\tau_{\mathbb{RP}^n}^{-1})$$

and

$$\mathbb{HP}^n \times \mathbb{HP}^n[-2]\#_{D^{op}, E_2}((\mathbb{HP}^n \times \mathbb{HP}^n)[1]\#_{\Delta_{\mathbb{HP}^n}, E_2}\Delta) = Graph(\tau_{\mathbb{HP}^n}^{-1})$$

where D^{op} are defined similar to Lemma 5.3.20.

For family Dehn twist, we have (See Corollary 5.2.10)

Lemma 5.3.22 (cf. Theorem 1.3.1(2)(5)). *There are graded clean surgery identities*

$$\widetilde{C}_S[1]\#_{\mathcal{D},E_2}\Delta = Graph(\tau_{C_S}^{-1})$$

$$\widetilde{C}_R[1]\#_{\mathcal{D}^{op},E_2}\widetilde{C}_R[1]\#_{\mathcal{D},E_2}\Delta = Graph(\tau_{C_R}^{-1})$$

$$\widetilde{C}_C\#_{\mathcal{D}^{op},E_2}\widetilde{C}_C[1]\#_{\mathcal{D},E_2}\Delta = Graph(\tau_{C_C}^{-1})$$

$$\widetilde{C}_H[-2]\#_{\mathcal{D}^{op},E_2}\widetilde{C}_H[1]\#_{\mathcal{D},E_2}\Delta = Graph(\tau_{C_H}^{-1})$$

where C_S (resp. C_R, C_C, C_H) is a spherically (resp. real projectively, complex projectively, quaternionic projectively) coisotropic submanifold.

5.4 Review of Lagrangian Floer theory, Lagrangian cobordisms and quilted Floer theory

We now fix the conventions for Lagrangian Floer theory, which follows that of [92]. Note that this is different from the homology convention of [13].

Let $L_0, L_1 \subset (M, \omega)$ be a pair of transversally intersecting Lagrangians. For a generic one-parameter family of ω-compatible almost complex structure $\mathbf{J} = J_t$, let

$$\begin{aligned}\mathcal{M}(p_-, p_+) = \{u : \mathbb{R} \times [0,1] \to M :& \\ & u_s(s,t) + J_t(u(s,t))u_t(s,t) = 0, \\ & u(s,0) \in L_0 \text{ and } u(s,1) \in L_1 \\ & \lim_{s\to+\infty} u(s,t) = p_+, \\ & \lim_{s\to-\infty} u(s,t) = p_-\}.\end{aligned} \tag{5.5}$$

Then the Floer cochain complex $CF^*(L_0, L_1)$ is equipped with a differential by counting rigid elements from $\mathcal{M}(p, q)$, i.e.

$$dp_+ = \sum_{p_-\in L_0\cap L_1} \#\mathcal{M}(p_+, p_-)p_-$$

The higher operations are defined analogously by counting holomorphic polygons as in [92]. We refer thereof for the definition of the Fukaya category and will not repeat it here.

Definition 5.4.1. Let $L_i, L'_j \subset (M,\omega)$, $1 \le i \le k$, $1 \le j \le k'$ be a collection of Lagrangian submanifolds. A Lagrangian cobordism V from $(L_1,\dots,L_k)$ to $(L'_{k'},\dots,L'_1)$ is an embedded Lagrangian submanifold in $M \times \mathbb{C}$ so that the following condition hold.

• There is a compact set $K \subset \mathbb{C}$ such that $V-(M\times K) = (\sqcup_{i=1}^{k} L_i \times \gamma_i) \sqcup (\sqcup_{j=1}^{k'} L'_j \times \gamma'_j)$, where $\gamma_i = (-\infty, x_i) \times \{a_i\}$ and $\gamma'_j = (x'_j, \infty) \times \{b'_j\}$ for some x_i, a_i, x'_j, b'_j such that $a_1 < \cdots < a_k$ and $b'_1 < \cdots < b'_{k'}$.

The main result we will utilize from Biran-Cornea's Lagrangian cobordism formalism reads:

Theorem 5.4.2 ([14])**.** *If there exists a monotone (or exact) Lagrangian cobordism from monotone (or exact) Lagrangians $(L_1,\dots,L_k)$ to $(L'_{k'},\dots,L'_1)$, then there is an isomorphism between iterated cones in $\mathcal{D}^\pi\mathcal{F}uk(M)$,*

$$Cone(L_1 \to L_2 \to \cdots \to L_k) \cong Cone(L'_{k'} \to L'_{k'-1} \to \cdots \to L'_1)$$

Here $Cone(L_1 \to L_2 \to \cdots \to L_k) = Cone(\cdots Cone(Cone(L_1 \to L_2) \to L_3) \to \cdots \to L_k)$.

We will dedicate the rest of this section to quilted Floer theory developed in [108][107][109][63].

Definition 5.4.3. Given a sequence of symplectic manifolds $M_0,\dots,M_{r+1}$, a generalized Lagrangian correspondence $\underline{L} = (L_{01},\dots,L_{r(r+1)})$ is a sequence such that $L_{i(i+1)} \subset M_i^- \times M_{i+1}$ are embedded Lagrangian submanifolds for all i. A cyclic generalized Lagrangian correspondence is one such that $M_0 = M_{r+1}$.

For a Lagrangian correspondence $L_{01} \subset M_0^- \times M_1$, $L_{01}^t \subset M_1^- \times M_0$ is defined to be $L_{01}^t = \{(x,y)|(y,x) \in L_{01}\}$. Given two Lagrangian correspondences $L_{01} \subset M_0^- \times M_1$ and $L_{12} \subset M_1^- \times M_2$, the geometric composition is defined as

$$L_{01} \circ L_{12} = \{(x,z)|\exists y \text{ such that } (x,y) \in L_{01} \text{ and } (y,z) \in L_{12}\} \tag{5.6}$$

For the composition to work nicely, we require that:

- $L_{01} \circ L_{12} \subset M_0^- \times M_2$ is embedded,
- $L_{01} \times L_{12}$ intersects $M_0^- \times \Delta \times M_2$ transversally in $M_0^- \times M_1 \times M_1^- \times M_2$.

One is referred to Section 5.2.3 for a non-trivial example of Lagrangian correspondence and comopsition coming from coisotropic embeddings.

For a cyclic generalized Lagrangian correspondence $\underline{L}$, the **quilted Floer cohomology** is defined to be

$$HF(\underline{L}) = HF(L_{01} \times L_{23} \ldots L_{(r-1)r}, L_{12} \times L_{34} \ldots L_{r(r+1)})$$

if r is odd, and

$$HF(\underline{L}) = HF(L_{01} \times L_{23} \ldots L_{r(r+1)}, L_{12} \times L_{34} \ldots L_{(r-1)r} \times \Delta_{M_0})$$

if r is even.

It is worth pointing out that for the quilted Floer cohomology to be well-defined, $\underline{L}$ needs to satisfy a stronger monotonicity condition ([108, Definitnion 4.1.2(b)]). A sufficient condition for this monotonicity to hold for $\underline{L} = (L_0, L_{01}, L_1)$ is when $\pi_1(L_{01}) = 1$ ([108, Lemma 4.1.3]). We refer readers to [108] for further details on monotonicity, as well as orientation, grading, exactness, and so forth for a generalized Lagrangian correspondence. The following theorems summarize main properties that will concern us.

Theorem 5.4.4 (Theorem 5.2.6 of [108]). *For a monotone (or exact) cyclic generalized Lagrangian correspondence $\underline{L}$ such that*

- *M_i are monotone with the same monotonicity constant,*
- *the minimal Maslov numbers of $L_{i(i+1)}$ are at least three,*
- *$M_0 = M_{r+1}$ is a point,*
- *$L_{i(i+1)} = L_i \times L_{i+1}$ for Lagrangians $L_i \subset M_i$ and $L_{i+1} \subset M_{i+1}$ for some $1 \leq i < r$*

then there is a canonical isomorphism

$$HF(\underline{L}) = HF(L_{01}, L_{12}, \ldots, L_{(i-1)i}, L_i) \otimes HF(L_{i+1}, L_{(i+1)(i+2)}, \ldots, L_{r(r+1)})$$

with coefficients in a field.

Theorem 5.4.5 (Theorem 5.4.1 of [108]). *For a cyclic generalized Lagrangian correspondence $\underline{L}$ such that*

- *M_i are monotone with the same monotonicity constant,*
- *the minimal Maslov numbers of $L_{i(i+1)}$ are at least three,*
- *$\underline{L}$ is monotone, relatively spin and graded in the sense of Section 4.3 of [108], and*
- *$L_{(i-1)i} \circ L_{i(i+1)}$ is embedded*

then there is a canonical isomorphism

$$HF(\underline{L}) = HF(L_{01}, L_{12}, \ldots, L_{(i-1)i} \circ L_{i(i+1)}, \ldots, L_{r(r+1)})$$

where the orientation and grading on the right are induced by those on $\underline{L}$.

For a symplectomorphism $\phi \in Symp(M)$, the fixed point Floer cohomology can be defined as

$$HF(\phi) = HF(\Delta, graph(\phi)) = HF(graph(\phi^{-1}), \Delta)$$

where the Lagrangian Floer cohomology take place in $M \times M^-$.

Remark 5.4.6. We follow the convention in [108], where $HF(\phi) = HF(graph(\phi), \Delta)$ in $M^- \times M$. Therefore, we have $HF(\phi) = HF(\Delta, graph(\phi))$ in $M \times M^-$.

5.5 Proof of long exact sequences

We construct Lagrangian cobordisms associated to the surgery identities obtained and deduce the long exact sequences in this section.

Lemma 5.5.1. *Let $L = L_1 \#_D L_2, L_1 \#_{D,E_2} L_2$ or $\mathcal{L}_1 \#_{\mathcal{D},E_2} \mathcal{L}_2$. Then there is a Lagrangian cobordism from L_1 and L_2 (or $\mathcal{L}_1$ and $\mathcal{L}_2$) to L.*

Proof. We give the proof for $L = L_1 \#_{D,E_2} L_2$ and the proof for $L_1 \#_D L_2$ and $\mathcal{L}_1 \#_{\mathcal{D},E_2} \mathcal{L}_2$ are similar. It suffices to consider $M = T^*L_1$ and $L_2 = N^*_D$ is the conormal bundle of D in L_1. As usual, we assume a product metric on L_1 is chosen and $D \pitchfork \{p\} \times K_2$ for all $p \in K_1$ so that the E_2-flow clean surgery can be performed.

First note that $L_1 \times \mathbb{R}$ intersects cleanly with $L_2 \times i\mathbb{R}$ at $D \times \{0\}$. Pick the standard metric on $\mathbb{R}$. We can perform the Lagrangian surgery from $L_1 \times \mathbb{R}$ to $L_2 \times i\mathbb{R}$ resolving the clean intersection by a $(E_2 \oplus \mathbb{R})$-flow handle $H_\nu^{D,E_2 \oplus \mathbb{R}}$, where we canonically identify

$T^*(L_1 \times \mathbb{R})$ as a $E_1 \oplus E_2 \oplus \mathbb{R}$ bundle over $L_1 \times \mathbb{R}$. We note that $E_2 \oplus \mathbb{R}$-flow is well-defined to give a smooth Lagrangian manifold because we stayed inside the injectivity radius (Lemma 5.1.24). Hence we have an embedded Lagrangian cobordism with four ends.

Let $\pi : M \times \mathbb{C} \to \mathbb{C}$ be the projection on second factor and $\pi_H = \pi|_{H_\nu^{D,E_2 \oplus \mathbb{R}}}$. We define $S_+ = \{(x,y) \in \mathbb{R}^2 | y \geq x\}$ and $W = \pi_H^{-1}(S_+)$. A direct check shows that W is a smooth manifold with boundary $\pi_H^{-1}(0) = L$. Let $W_0 = W \cap \pi^{-1}([-3\epsilon, 0] \times [0, 3\epsilon])$. It has three boundary components, namely $L_1 \times \{(-3\epsilon, 0)\}$, $L_2 \times (0, 3\epsilon)$ and $L \times \{(0,0)\}$, while $L \times \{(0,0)\}$ is the only boundary component that is not cylindrical. One then applies a trick due to Biran-Cornea (see Section 6 of [13]). This yields a Hamiltonian perturbation φ supported on $\pi^{-1}([-\epsilon, \epsilon] \times [-\epsilon, \epsilon])$, so that $\varphi(W)$ has all three cylindrical ends. By extending $\phi(\pi_H^{-1}(0))$ to infinity, we get the desired Lagrangian cobordism. $\square$

We call a cobordism obtained by Lemma 5.5.1 a **simple cobordism** . When D is a single point, it reduces to the usual Lagrangian surgery and Lemma 5.5.1 was discussed in Section 6 of [13] in detailed.

Lemma 5.5.2. *Let V be a simple cobordism from L_1, L_2 to L and D is connected. If L_1 and L_2 are exact Lagrangians, then L is exact and V is also exact.*

Proof. We give the proof for $L = L_1 \#_{D,E_2} L_2$. Without loss of generality, we can assume $M = T^*L_1$, $L_2 = N^*D$ is the conormal bundle. We first assume $codim_{L_i}(D) \geq 2$.

Since the E_2-flow handle H_ν^{D,E_2} is obtained by a Hamiltonian flow of $T^*L_2 \backslash E_1$, it is immediate that H_ν^{D,E_2} is an exact Lagrangian because $\mathcal{L}_{X^\nu}\alpha = dK$ for a function K vanishing at infinity boundary of T^*L_2. Let f_1, f_2 and f_H be a primitive of α restricted on L_1, L_2 and H_ν^{D,E_2}, respectively. If D is of codimension two or higher, $(f_i - f_H)|_{L_i \cap \overline{H_\nu^{D,E_2}}}$ are locally constants and hence constants for $i = 1, 2$, where $\overline{H_\nu^{D,E_2}}$ denotes the closure of the handle. Hence, f_1, f_2 and f_H can be chosen such that they match to give a primitive on L.

Now we drop the codimension assumption and only assume $codim_{L_i}(D) \geq 1$. We recall that in the proof of Lemma 5.5.1, the first step for constructing V is to resolve $L_1 \times \mathbb{R}$ and $L_2 \times i\mathbb{R}$ along $D \times \{(0,0)\}$, which has now $codim_{L_i \times \mathbb{R}}(D) \geq 2$. This process preserves exactness by what we just proved. Then we cut the cobordism into a half, do Hamiltonian perturbation near $L \times \{(0,0)\}$ and extend the cylindrical end. All of these steps preserve the exactness of the Lagrangian and hence V is exact. The restriction of V to the fiber over $\{(0,0)\}$ is precisely L, proving the exactness of the surgery. □

Lemma 5.5.3. *Let V be a simple cobordism from L_1, L_2 to L. If L_1 and L_2 are monotone Lagrangians such that either*

(1) $\pi_1(L_1, D) = 1$ or $\pi_1(L_2, D) = 1$, or

(2) $\pi_1(M) = 1$ and D is connected,

then L is monotone and V is also monotone.

Proof. Again we give the proof for $L = L_1 \#_{D,E_2} L_2$ and we first assume that $codim_{L_i}(D) \geq 2$. For convenience we decompose $L = \mathring{L}_1 \cup \mathring{L}_2$. Here $\mathring{L}_2$ is the closure of the image of $L_2 \backslash D$ under the E_2-flow defining the surgery, and $\mathring{L}_1$ is the closure of the complement of $\mathring{L}_2$.

In case (1) it suffices to prove the lemma when $\pi_1(L_2, D) = 1$, since the slight asymmetry of L_1 and L_2 will be irrelevant. First note that $\pi_1(U(D), U(D)\backslash D) = \pi_1(N^*_D, N^*_D \backslash D) = 1$ by our assumption on D, where $U(D)$ is a tubular neighborhood of D in L_2. Since the flow handle H_ν^{D,E_2} is obtained by E_2 direction flow of $N^*D - D$, any path in $\mathring{L}_2$ with ends at H_ν^{D,E_2} can be homotoped to a path in H_ν^{D,E_2}, while H_ν^{D,E_2} in turn retracts to its boundary component that lies on $\mathring{L}_1$.

The upshot is, we can find for any element in $\pi_2(M, L)$ a representative $u : \mathbb{D}^2 \to M$ with boundary completely lie in $\mathring{L}_1$. Since L_1 is monotone, it finishes the proof for L.

Case (2) is similar. Take again any disk $u : \mathbb{D}^2 \to M$ with boundary on L, and assume ∂u intersects $\partial \mathring{L}_2$ transversally. For any segment $I \subset \partial u$ contained in $\mathring{L}_2$ satisfying $\partial I \subset \partial \mathring{L}_2$, one connects the two endpoints of ∂I by $I' \subset \partial \mathring{L}_2$ (the relevant boundary is connected due to the assumption of connectedness and codimension of D). Take any disk $v : \mathbb{D}^2 \to M$ with $\partial v = I \cup I'$. Then one may decompose u so that $[u] = [u - v] + [v]$, so that $\partial v \subset \mathring{L}_2$. By performing such a cutting iteratively, one may assume $\partial(u - v) \subset \mathring{L}_1$. Since ∂v retracts to $L_2 \cap \mathring{L}_2$, the monotonicity follows from that of L_1 and L_2 with such a decomposition.

Now in either case the monotonicity of V is argued in a similar way as Lemma 5.5.2 because all processes involved preserve monotonicity. The restriction to the fiber over the origin again removes the assumption of $codim_{L_i} D \geq 2$. □

Theorem 5.5.4. *Let (M,ω) be a monotone (or exact) symplectic manifold and S^n $(n>1)$ an embedded Lagrangian sphere. For monotone (or exact) Lagrangians L_1 and L_2, there is a long exact sequence*

$$\dots \to HF(S^n, L_2) \otimes HF(L_1, S^n) \to HF(L_1, L_2) \to HF(L_1, \tau_{S^n}(L_2)) \to \dots$$

Proof. By Lemma 5.2.6 and Lemma 5.5.1, there is a Lagrangian cobordism V from $S^n \times S^n$ and the diagonal Δ to $graph(\tau_{S^n}^{-1})$ in $M \times M^-$, where $M^- = (M, -\omega)$. By Lemma 5.5.2, 5.5.3, the monotonicity (exactness) of (M,ω) implies the same property for $S^n \times S^n$, $\Delta \subset M \times M^-$ and the corresponding cobordism V.

In either case, $graph(\tau_{S^n}^{-1})$ is a cone from $S^n \times S^n$ to Δ in the Fukaya category of $M \times M^-$ by Theorem 5.4.2. In particular, we have a long exact sequence

$$\dots \to HF(L_1 \times L_2, S^n \times S^n) \to HF(L_1 \times L_2, \Delta) \to HF(L_1 \times L_2, graph(\tau_{S^n}^{-1})) \to \dots$$

In the language of Lagrangian correspondence, we have $HF(L_1 \times L_2, S^n \times S^n) = HF(L_1, S^n \times S^n, L_2) = HF(L_1, S^n) \otimes HF(S^n, L_2)$ by Theorem 5.4.4, where $(L_1, S^n \times S^n, L_2)$ is a generalized Lagrangian correspondence in $\{pt\} \times M \times M \times \{pt\}$. Similarly, we have $HF(L_1 \times L_2, \Delta) = HF(L_1, \Delta, L_2) = HF(L_1, \Delta \circ L_2) = HF(L_1, L_2)$ by Theorem 5.4.5. Finally, we also have $HF(L_1 \times L_2, graph(\tau_{S^n}^{-1})) = HF(L_1, graph(\tau_{S^n}^{-1}) \circ L_2) = HF(L_1, \tau_{S^n}(L_2))$, by Theorem 5.4.5 again.

We remark that although the results in [108] require a stronger monotonicity assumption on the generalized Lagrangian correspondence, the isomorphisms we need are classical (e.g. it can be proved by hand-crafted correspondence of relevant moduli spaces) and require only monotonicity assumptions on the Lagrangians. □

Corollary 5.5.5. *In the same situation as Lemma 5.5.4, $f \in Symp(M)$, then*

$$\dots \to HF(\tau \circ f) \to HF(f) \to HF(f(S^n), S^n) \to \dots \tag{5.7}$$

Proof. The exact sequence follows from applying the cohomological functor $HF(-, graph(f))$ to the cone given by the cobordism.

□

The above result is predicted by Seidel [93, Remark 2.11] in a slightly different form from here. This is solely due to the cohomological convention we took. In the following theorem, we assume all involved symplectic and Lagrangians have the same monotonicity constant with minimal Maslov number at least three.

Theorem 5.5.6. *Let C be a spherically fibered coisotropic manifold over the base (B, ω_B) in (M, ω). Given Lagrangians L_1 and L_2 and assume the following monotonicity conditions:*

(i) the generalized Lagrangian correspondence (L_1, C^t, C, L_2) is monotone in the sense of [108] and,

(ii) the simple cobordism corresponding to the surgery in Corollary 5.2.10 is monotone.

Then there is a long exact sequence

$$\cdots \to HF(L_1 \times C, C^t \times L_2) \to HF(L_1, L_2) \to HF(L_1, \tau_C(L_2)) \to \dots$$

In particular if the spherical fiber of C has dimension > 1 or $\pi_1(M) = 1$, (ii) is automatic.

Proof. The proof is analogous to Theorem 5.5.4 with Lemma 5.2.6 replaced by Corollary 5.2.10. Here we give a sketch. First, (L_1, C^t, C, L_2) being monotone implies $\tilde{C} = C^t \circ C$ being monotone (See Remark 5.2.3 of [108]). The Lagrangian cobordism in Corollary 5.2.10 is monotone by Lemma 5.5.3. It is not hard to verify $\pi_1(\widetilde{C}, \widetilde{C} \cap \Delta) = 1$ when $codim_M C \geq 2$. Hence, Theorem 5.4.2 applies either in this case or when $\pi_1(M) = 1$, and we obtain long exact sequence

$$\cdots \to HF(L_1 \times L_2, \tilde{C}) \to HF(L_1 \times L_2, \Delta) \to HF(L_1 \times L_2, graph(\tau_C^{-1})) \to \dots$$

With our assumption on monotonicity of (L_1, C^t, C, L_2), we apply Theorem 5.4.5 to obtain the desired result. $\square$

There is a similar result on the fixed point version of family Dehn twist, and we will not state it explicitly here.

We also have two long exact sequences associated to projective Dehn twists. The proof is similar to the proof above (cf. Lemma 5.3.21), which implies Theorem 1.3.4 and hence the long exact sequences. Theorem 1.3.5 also follows from it by applying MWW functor [63]. The family versions and (family) fixed point versions for projective Dehn twists are also similar (cf. Lemma 5.3.22).

Theorem 5.5.7. *Let (M, ω) be a monotone (or exact) symplectic manifold and S a graded embedded Lagrangian complex (or real, quaternionic) projective space. For graded monotone (or exact) Lagrangians L_1 and L_2, there are long exact sequences*

$$\cdots \to HF^*(S, L_2) \otimes HF^*(L_1, S)[-\dagger] \to HF^*(S, L_2) \otimes HF^*(L_1, S) \to H^*(C) \dots$$

and

$$\cdots \to H^*(C) \to HF^*(L_1, L_2) \to HF^*(L_1, \tau_S(L_2)) \to \dots$$

for some chain complex C, where $\dagger = 2, 1, 4$, respectively, for complex, real and quaternionic projective space.

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