# Essays on Endogenously Incomplete Markets

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# Dedication

To my parents, Sunaina and Ravi

## Abstract

The three chapters in this dissertation constitute a study of macroeconomic models with markets that are *endogenously* incomplete. Chapter 2 provides microfoundations for two types of widely used incomplete market models while chapter 3 studies the role for policy in endogenously incomplete models. Chapter 4 examines the existence of multiplicity in such models and shows how this can provide an endogenous source of aggregate fluctuations in such models.

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# Chapter 1

# Introduction

An extensive literature in macroeconomics and international economics uses models with exogenously incomplete markets and financial frictions for a variety of quantitative and policy exercises. In chapter 2, I relax the assumptions of exogenous incompleteness and instead consider general contracting environments in which no restrictions are placed on the types of contracts agents can sign. I show that with three key frictions: private information, voluntary participation and hidden trading, equilibrium outcomes of the contracting environment coincide with those in models with exogenously incomplete markets. In particular, under appropriate assumptions, equilibrium outcomes are identical to either an environment with trades in a risk-free bond subject to occasionally binding debt constraints, or an environment with defaultable debt. The policy implications, however, are very different. For example, equilibrium outcomes in models with exogenously incomplete markets are typically inefficient while the best equilibrium in my environment is efficient. This implies that imposing borrowing limits may be desirable when markets are exogenously incomplete while such policies cannot improve welfare in my model. However, I show that this environment has multiple equilibria and that governments can play an important role as a lender of last resort in ensuring that the best equilibrium occurs.

In chapter 3, I study the role for policy in uniquely implementing the best competitive equilibrium in financial frictions models with multiple equilibria. I consider an environment in which agents with idiosyncratic productivity shocks trade Arrow securities subject to state contingent debt constraints that are determined endogenously in equilibrium. Agents can choose to default on their debt at any time after which they are barred from financial markets in all future periods but can continue to hold fiat money. Debt constraints are determined in equilibrium to have the property that an agent who has borrowed up to this limit is indifferent between defaulting and having access to financial markets in future periods. I characterize the Ramsey equilibrium when the government can control the money supply and show that the policy departs from the Friedman rule. However, I show that the Ramsey policy is also consistent with another equilibrium in which no Arrow securities are traded and debt constraints are zero in all periods. I consider an expanded policy space where in addition to money supply rules, the government can offer some agents access to an overnight risk-free deposit facility with specified interest rates. I show that the set of competitive equilibria is identical to the set when policy only consists of money supply rules. I prove that there exist *Sophisticated* policies as in Atkeson et al. (2010) that uniquely implement any particular competitive equilibrium. These policies are contingent on histories of private actions and require that all continuation outcomes constitute continuation competitive equilibria. The policy stipulates that the government allow access to this deposit scheme only after particular histories with an interest rate that is greater than the return on money.

In chapter 4, I study a class of decentralized contracting environments with multiple equilibria that can endogenously generate aggregate fluctuations. In these models agents sign long term contracts with risk-neutral intermediaries subject to voluntary participation constraints. Agents run production technologies by accumulating capital and hiring labor and are subject to idiosyncratic productivity shocks. They can at any time choose to default on their obligations to the intermediary and live in financial autarky forever. In autarky, agents can continue to run their technology but can no longer sign contracts with intermediaries. I consider two related environments; the first in which agents' actions are observable and the second in which these actions are hidden. In the second case, intermediaries cannot observe agents' choices of capital and labor. I show that in both cases, the decentralized environment has multiple equilibria. While this can be shown directly in the case with observable actions, in the case with hidden actions, the contracting problem is intractable. However, I prove that the set of equilibria is identical to a different environment in which agents trade Arrow securities subject to debt constraints that are determined endogenously in equilibrium. Equilibria can be characterized in the equivalent environment using standard techniques. In these models, multiple equilibria exist due to strategic complementarities in players' actions. For example, in the debt constrained setup, an autarkic equilibrium exists because if all agents expect that there will be no borrowing and lending in the future, no agent will be willing to lend today since the borrower will always choose to default. Similarly, in the contracting environment, if intermediaries believe that agents will not be able to borrow in the future, they will not be willing to lend today. This allows for the possibility of generating sunspot equilibria and hence endogenously generated aggregate fluctuations. To illustrate this, I compute a simple example in which there are sizable and persistent changes to aggregate output and investment due to changes in expectations about the tightness of debt constraints in the future.

# Chapter 2

# Endogenously Incomplete Markets with Equilibrium Default

## 2.1 Introduction

A large and growing literature in macroeconomics and international economics uses models with incomplete markets and financial frictions for a variety of quantitative and policy exercises. Examples include the study of financial and sovereign debt crises, optimal taxation, and bankruptcy laws. The key assumption in these models is that markets are *exogenously* incomplete. In particular, strong assumptions are imposed on the types of contracts agents within the model can sign. Most of these models make one of two assumptions. The first type of assumption is that agents can trade an uncontingent risk-free bond subject to exogenous debt constraints. These include environments studied by Huggett (1993) and Aiyagari (1994). The second type of assumption is that agents can trade defaultable debt contracts. Such models are standard in the international macro and bankruptcy literature.

An alternate view, which I take in this paper, is to relax the assumptions of exogenous incompleteness and instead consider general contracting environments in which no restrictions are placed on the types of contracts agents can sign. I show that there exist informational and commitment assumptions that endogenously generate the types of contracts assumed by much of the applied literature. Next, I show that the best equilibrium in these environments is efficient. Finally, I show that models with endogenous incompleteness have substantially different implications for policy than those with exogenous incompleteness.

I study a dynamic environment with a large number of risk-averse households that receive stochastic endowments each period and seek to share risk with each other. I model trading among households by allowing them to sign contracts with competitive financial intermediaries. The contracting environment is subject to three key frictions: private information, voluntary participation and hidden trading. The first is that households' endowments are private information and not observable to any other household. The second is that household participation in financial markets is voluntary in that in any period they can always choose autarky namely, to not participate in financial markets from then on and consume their endowments in every period. The third is that trades between households and intermediaries are hidden in that they are not observable by other households and other intermediaries. In particular, I allow households to sign contracts with multiple intermediaries in a hidden fashion.

A well known feature of these environments is that risk-sharing is possible only if households that do not repay their debts suffer a cost. In my environment, I assume that if households do not repay their debts as specified in the contract, they are permanently banished from financial markets and forced into autarky. With this assumption I consider two environments. In the first, I assume that financial intermediaries can only offer contracts that induce households to always repay their debts. In the second, I introduce a technology that allows financial intermediaries to temporarily banish households from financial markets. Contracts can specify banishment and banished households consume their endowment and cannot trade with intermediaries. The contract also specifies a re-entry probability after which households are able to sign contracts again.

I show that equilibrium outcomes in the first environment are equivalent to those in a standard incomplete markets model in which households trade a risk-free bond subject to debt constraints. Moreover, these debt constraints are independent of households' histories and thus look exactly like those assumed in models with exogenous incompleteness.

In the second environment, I show that intermediaries will choose use the banishment technology in equilibrium. As a result, equilibrium contracts will feature temporary periods of financial autarky, much like in the sovereign default and bankruptcy literature. Under some sufficient conditions, equilibrium outcomes here are equivalent to those used widely in the sovereign default and bankruptcy literature, for example Eaton and Gersovitz (1981). In equilibrium, intermediaries use banishment as a way of introducing state and history contingency into contracts. In most contracting environments restricting to no-banishment contracts is without loss of generality. However in this environment, since banishment is publicly observable, it incentivizes truthful revelation of types. This might seem counterintuitive since the intermediary can always provide the value associated with autarky to the household without banishment. For example, the transfer scheme in which the intermediary makes zero transfers in all dates and states provides the autarkic value to the household. However, unlike banishment, the household still has the option of signing contracts with other intermediaries. As a result, with private information and hidden trading, such a transfer scheme is not in general incentive compatible. To understand this difference more starkly suppose that intermediaries have to pay an exogenous cost whenever they banish households. One can interpret this as a cost of monitoring households that are banished. In models with exclusive contracts and no hidden trading, equilibria with banishment are always Pareto-inferior to ones with no banishment. In contrast, with hidden trading, equilibria with banishment can Pareto-dominate any equilibrium without banishment. In this sense, hidden trading is necessary to get default in equilibrium.

The second main result of the paper is that the best equilibrium in these environments is efficient. By this I mean that a planner confronted with the same frictions as intermediaries cannot improve overall welfare. In particular, I show that in the presence of hidden markets, the amount of state contingency a planner can offer in a contract is severely limited. As a result, in the first environment with no banishment, for example, the planner cannot do better than offer short-term uncontingent contracts. However, the first welfare theorem does not hold since in general, the environment has multiple equilibria. This multiplicity is due to the presence of strategic complementarities in the actions of intermediaries.

The third set of results concern the lessons for policy. There are three important implications for policy. The first is that policies which might be considered desirable when markets are exogenously incomplete, may no longer improve welfare when markets are endogenous incomplete. For example, I illustrate how in models with exogenously incomplete markets, setting limits on how much households can borrow may increase overall welfare. However, I show that in models with hidden trading, households will use hidden markets to circumvent these limits. As a result, in the environment I study, imposing such limits will not increase overall welfare. Second, because of the multiplicity of equilibria there is an important role for policy to uniquely implement the best equilibrium. I show how simple lender of last resort policies can help achieve this. The third implication for policy is that with banishment, the efficient probability of re-entry is decreasing in the level of the debt defaulted on. This result has important implications for the bankruptcy policy. To understand these implications note that we can re-interpret this environment as one in which intermediaries decide whether or not to banish households and an outside authority enforces banishment and decides the probability of re-entry. Under this interpretation, a bankruptcy policy which allowed the probability of re-entry to depend on the level of defaulted debt can increase welfare.

Finally, I consider the positive implications of the model. As documented by Cruces and Trebesch (2013) in the case of sovereign defaults, larger haircuts are associated with a longer duration of banishment from capital markets. In my environment, even though default is associated with a 100 percent haircut, it is still true that the probability of re-entry is smaller if the level of defaulted debt is larger.

A final point worth noting is that all three frictions i.e. private information, limited commitment and hidden trading, are essential to the nature of the contract. Obviously, without private information, fully state-contingent contracts would be equilibrium outcomes. Without limited commitment, households will never be borrowing constrained. Without hidden trading, contracts will feature history contingency and equilibrium contracts will resemble those in Thomas and Worrall (1990) and Atkeson and Lucas (1992).

**Literature**: This paper is related to a large literature on dynamic contracts and its applications in macroeconomics. Green (1987), Thomas and Worrall (1990), Phelan and Townsend (1991) and Atkeson and Lucas (1992) are some of the important papers studying dynamic environments with private information. In general, efficient contracts in these environments feature history contingency and no banishment/separation. As a result, these contracts are very different from the defaultable debt or the uncontingent borrowing and lending contracts assumed by the applied literature. In contrast, I show that when dynamic private information interacts with limited commitment and hidden trading, the equilibrium contracts are identical to those assumed in standard macroeconomic models.

Allen (1985), Cole and Kocherlakota (2001), Golosov and Tsyvinski (2007) and Ales and Maziero (2014) study dynamic private information environments with hidden trading.[1] While the first three papers assume a technology that allows agents to engage in hidden transactions, Ales and Maziero (2014) study an environment in which agents can sign non-exclusive contracts. The equilibrium contracts in these environments are also very different than those in the environment I study. While contracts in Golosov and Tsyvinski (2007) feature state contingency, the equilibrium contracts resulting from the other three papers are uncontingent. However, these contracts do not have separation on path and no agent is borrowing constrained. In the environment I study, if intermediaries are allowed to banish households in equilibrium, contracts will feature separation on path. If they are not allowed to banish, the equilibria are equivalent to an incomplete markets environment with endogenous debt constraints. In particular, households will be borrowing constrained in equilibrium.

In a recent important paper, Dovis (2014) studies an environment with both hidden types and limited commitment.[2] He shows how one can decentralize the efficient allocation as an equilibrium of a sovereign debt game in which there is suspension of payments to lenders along the equilibrium path. There are two main differences between the environment in this paper and the one studied by Dovis (2014). First, in terms of the contracting problem my environment features hidden trading while in his, contracts are exclusive. This implies that if there was an exogenous cost of banishment, in his environment, separation would not be efficient. As mentioned earlier, even with a positive banishment cost, separation can be efficient in the environment I study. From an observational perspective, the key difference between our environments concerns the probability of re-entry after default. In Dovis (2014), the probability of re-entry is independent of the level of debt defaulted on. In

[1] Bisin and Guaitoli (2004) and Bisin and Rampini (2006) study two period environments with moral hazard (hidden action) and hidden trading. In particular, Bisin and Rampini (2006) find that the ability to seize payoffs from secondary contracts is valuable and interpret this as bankruptcy. However, this requires that output is observable which is not true in my environment since endowments are private information.

[2] See Atkeson (1991), Atkeson and Lucas (1995) and Yared (2010) for other papers with both private information and limited commitment.

contrast, in my environment the probability of re-entry is independent of the household's type but depends on the level of defaulted debt.

The efficient contracts studied by DeMarzo and Sannikov (2006) and Clementi and Hopenhayn (2006) feature inefficient terminations of the risk-sharing relationship on path. The reason for this is the presence of an exogenous outside option available to the lender. In particular, for certain regions of the contract space, the value of the outside technology is strictly greater than the value of firm. If the principal had access to same technology within the firm, separation would not be efficient. However, in the environment I consider, even though the intermediary can replicate the value of banishment or default on path, it is efficient to banish households in equilibrium.

This paper is also related to Hopenhayn and Werning (2008) who study a contracting environment in which the agents have a stochastic outside option that is unobservable to the principal. They show that the efficient contract features separation on path. In their model, separation also arises due the presence of an outside technology that is not available to the principal.[3] In particular, if in their model the principal had access to this technology within the firm and there was a small cost of separation, then default would not be efficient. However, in the environment I study, even though the intermediary can offer the outside option to the household without banishment, the presence of hidden trading implies that separation is necessary to achieve efficiency.

In seminal papers, Prescott and Townsend (1984) and Kehoe and Levine (1993) studied and defined constrained-efficiency for environments with moral hazard and limited commitment[4] respectively. The decentralized environment I study has both incentive compatibility and voluntary participation constraints as in these papers. However, in contrast to both papers, in my environment households can engage in hidden trading. As a result, their welfare theorems do not apply here.

Golosov and Tsyvinski (2007) study a dynamic Mirrleesian environment in which agents can trade a risk-free bond in a hidden market. They find that competitive equilibria are inefficient. The planning problem I study is related in that I also assume that households

[3] Note that in their model if the outside option is observable the efficient allocation will not feature separation as in Albuquerque and Hopenhayn (2004a).

[4] See Kocherlakota (1996), Albuquerque and Hopenhayn (2004a) and Kehoe and Perri (2002) for other papers studying models with limited commitment.

can trade in a hidden fashion. However, unlike their model, the best equilibrium in the environment I study is efficient even though the planner has control of the price in the hidden markets. This is because hidden trading in my environment implies that it is not incentive feasible to introduce any state contingency into contracts. As a result, the best the planner can do is to offer an uncontingent contract

Since the environment I study has multiple equilibria, I consider the role for policy to uniquely implement the best equilibrium. This paper uses techniques and language developed by Atkeson et al. (2010) and Bassetto (2002) which allows us to think about how policy can uniquely implement a desired competitive equilibrium.

The framework developed by Eaton and Gersovitz (1981) has been widely used in the sovereign debt literature[5] . Similar models have also been used to study the effects of changing bankruptcy laws as in Chatterjee et al. (2007) and Livshits et al. (2007). In particular, Chatterjee et al. (2007) use a model with exogenous incompleteness to understand the effects of changing bankruptcy laws. For example, they find substantial welfare gains to enacting a policy which prevents households with above median incomes from declaring bankruptcy. Since the best equilibrium is efficient in the environment I consider, such policies will not in general be welfare improving. However, I show that allowing the probability of re-entry after default to depend on the level of defaulted debt can improve welfare.

This paper is also related and contributes to the vast literature in macroeconomics that uses models with incomplete markets, two important examples of which are Huggett (1993) and Aiyagari (1994)[6] .

While the environment I consider is observationally equivalent to a large class of exogenously incomplete models, the approach to efficiency I take is substantially different. Usually, the approach taken is similar in spirit to Diamond (1967) who exogenously restricts the set of instruments available to the planner. Geanakoplos and Polemarchakis (1986) and more recently Dávila et al. (2012) study such planning problems and conclude that the equilibria with incomplete markets are constrained inefficient. However, I use an

[5] Quantitative versions of this model include Aguiar and Gopinath (2006) and Arellano (2008). See Aguiar and Amador (2014) for a short survey.

[6] These models have been used to study a variety of issues from optimal quantity of government debt by Aiyagari and McGrattan (1998) and more recently to studying the effects of exogenous shocks to the debt constraints as in Guerrieri and Lorenzoni (2011).

example to show that outcomes which would be considered constrained-inefficient when markets are exogenously incomplete are actually constrained-efficient when markets are endogenously incomplete.

The paper proceeds as follows. In section 2.2 I describe the underling contracting environment and define an equilibrium. In section 2.3 and section 2.4, I study environments without and with banishment respectively and prove the equivalence results. Next, section 2.5 studies the efficiency properties of these environments while section 2.6 presents an application of this framework to bankruptcy policy. Finally section 2.7 discusses the role of various assumptions in generating the main results and section 2.8 concludes. Most of the proofs are contained in section B.1.

## 2.2 Environment

Consider an infinite horizon discrete time environment, $t = 1, 2, ...$ with a continuum of infinitely lived households $i \in I$ and a continuum of overlapping $\hat{T} < \infty$ period lived[7] risk-neutral intermediaries/firms born each period. Households are risk-averse with period utility functions $u(c_t)$ where $u : \mathbb{R}_+ \to \mathbb{R}$ is an increasing and strictly concave continuously differentiable function. I also assume that $u$ satisfies Inada conditions, $\lim_{c\to 0} u'(c) = -\infty$ and $\lim_{c\to\infty} u'(c) = 0$. There is a single non-storable consumption good of which households receive a random endowment $\theta_t \in \Theta$, $\theta_t \in \mathbb{R}_{++}$ each period where $\Theta$ is a finite set. Denote the maximal and minimal element of $\Theta$ by $\bar{\theta}$ and $\underline{\theta}$ respectively. The endowment shock is independently and identically distributed over time and households with density function $\pi(\cdot)$. Intermediaries can borrow and lend with each other at a market determined interest rate $\frac{1}{q_t}$ each period.

Households enter into long term contracts with intermediaries in order to smooth their consumption over time and can sign with multiple such intermediaries as described below. An important feature of the contracting environment is that I will endow intermediaries with a banishment technology. Banishment is publicly observable and a banished household consumes its endowment and cannot sign with other intermediaries. Once banished, the intermediary can also choose a probability of re-entry in subsequent periods. I also allow

[7] Intermediaries are assumed to be finitely lived so that their problem is always well defined. See section 2.7 for further discussion.

households to not repay their debts to intermediaries and subsequently live in financial autarky in all future periods.

1. With some probability previously banished households are allowed to contract with intermediaries[8]

2. Types are realized and are private information to households. Households report types to all intermediaries they are currently signed to.

3. Households that are not banished receive transfers from or make transfers to incumbent intermediaries, namely those intermediaries with whom they have pre-existing contracts. At this time households can voluntarily choose to not participate in financial markets and live in financial autarky forever.

4. Intermediaries post contracts.

5. Households observe the offered contracts and can choose to sign with at most one new intermediary.

6. Consumption takes place

An important assumption I make throughout the paper is that any contract signed between a household and intermediary is not observable to any other intermediary. The only outcomes that are publicly observable are posted contracts, banishment histories and whether the household has chosen to not participate in financial markets.

I begin the formal description of the game between intermediaries and households by first describing the information sets available to both types of players. Let $z^t \in Z^t$ denote the public history in period $t$ after previously banished households are stochastically allowed to sign contracts again. The public history $z^t = \left(\mathcal{B}^{t-1}, \left(\gamma^{i,t}\right)_{i \in I}\right)$ consists of the history of posted contracts $\mathcal{B}^{t-1}$ and banishment histories for all households $\left(\gamma^{i,t}\right)_{i \in I}$. An individual/personal history for each household in period $t$, after endowments have been realized, is denoted by $h^t \in H^t$ where $H^t = \mathcal{B}^{t-1} \times \{0,1\}^t \times \Theta^t$. A typical personal history $h^t = \left(B^{t-1}, \gamma^t, \theta^t\right)$ consists of the vector of contracts the household is signed to at the beginning of period $t$, $B^{t-1} = (B_1, ..., B_{t-1})$, banishment histories $\gamma^t$, where $\gamma_t \in \{0,1\}$

[8] Note that the actual signing of a new contract takes place later in the period.

and $\gamma_t = 1$ means that the household is banished from the contracting environment, and histories of endowment realizations $\theta^t$. Recall that households can sign at most one new contract each period. It will also be useful to define personal histories $\omega^t$ in period $t$ after this new contract is signed where $\omega^t \in \Omega^t = H^t \times \mathcal{B}_t$ and $\omega^t = \left(h^t, B_t\right)$. Note that $B_t$ is new the contract signed in period $t$. If the household is not signed to any contract at the beginning of period $t$, I denote the contract history as $B^{t-1} = \varnothing$. Note that endowments and signed contracts are privately observed by the household while $\gamma^t$ is publicly observed. In each period, households report their endowment type $\theta_t$ to intermediaries who use the public history along with the history of reports and $\sigma^{HH}$ to compute $B^{t-1}$. It is without loss of generality to assume that all households with the same $\left(\gamma^t, \theta^t\right)$ have identical contract histories $B^{t-1}$. Given the public history $z^t$, let $\tilde{\zeta}_t \left(h^t\right)$ denote the intermediaries' beliefs of personal histories in period $t$. We can also define the true probability measure on the space of personal histories $\zeta_t \left(h^t\right)$ and $\wp \left(\omega^t\right)$ for histories $h^t$ and $\omega^t$ respectively, which will be constructed after the formal definition of a contract.

A contract $B_t \left(z^t\right)$ offered in period $t$ is defined as follows:

$$B_t \left(z^t\right) = \left({}_t b_{t+s} \left(z^{t+s}, m^s\right), {}_t \delta_{t+s} \left(z^{t+s}, m^s\right), {}_t \mu_{t+s} \left(z^{t+s}, m^s\right) : 0 \le s \le \hat{T} - 1\right)$$

where $m^s$ denotes the history of type reports $(m_t, ..., m_{t+s})$. Denote the space of all such contracts by $\mathbb{B}_t$. In general, given an element of a contract ${}_t x_s$, the left subscript denotes the period in which the contract is agreed to and the right subscript the current period. Here ${}_t b_{t+s} \left(z^{t+s}, m^s\right) \in \mathbb{R}$ denotes the transfers to the households as a function of the history of reported types in period $t+s$, ${}_t \delta_{t+s} \left(z^{t+s}, m^s\right) \in \{0, 1\}$ denotes the banishment decision in period $t+s$ with ${}_t \delta_{t+s} \left(z^{t+s}, m^s\right) = 1$ meaning that the household is banished, and ${}_t \mu_{t+s} \left(z^{t+s}, m^s\right) \in [0, 1]$ is the subsequent probability of re-entry if the household is banished.

Next, I consider the problem of a household. A strategy for a household is $\sigma_t^{HH}$ which maps the appropriate histories into $\{0, 1\} \times \Sigma_t \times \mathcal{B}_t \times \mathbb{R}_+$ where $\Sigma_t$ is the set of type reporting strategies and $\mathcal{B}_t$ denotes the set of posted contracts in period $t$. In each period, the household chooses whether to participate, what to report, whether to sign a new contract and how much to consume. A typical strategy, $\sigma_t^{HH} = \{\varpi_t, \sigma_t, B_t, c_t\}$ where each element depends on the appropriate histories. Let $\varpi_t \in \{0, 1\}$ denote the participation strategy for the household which depends on $h^t$ with $\varpi_t = 0$ implying that the household chooses to not participate in financial markets and consequently live in autarky forever. Let $\Sigma = (\Sigma_t)_{t\geq 1}$ with typical element $\sigma = \left(\{\sigma_t^s\}_{s\leq t}\right)_{t\geq 1}$ where $\sigma_t^s : Z^t \times H^t \to \Theta$ is the household's type reporting strategy in period $t$, to the intermediary associated with contract $B_s$ where $s \leq t$ which depends on $h^t$. In particular note that the household can potentially report different types to different intermediaries. I define the truth-telling strategy $\sigma^*$, to be one that satisfies $\sigma_t^{*s}\left(z^t, h^t\right) = \theta_t$ for all $s$ and $t$ where $\theta_t$ is the household's endowment. Given the structure of the game, if a household is not banished at the initial stage it has the option to sign at most one new contract with another intermediary from the set of posted contracts which also depends on $h^t$. Note however that the consumption strategy depends on the new contract and hence on $\omega^t$. Given a personal history $h^t$ and an associated vector of signed contracts $B^{t-1}$, it will be useful to define the following objects

$$b_t^{old}\left(h^t \mid z^t\right) \equiv \sum_{s<t} {}_s b_t\left(z^t, \left(\sigma_s^s\left(z^s, h^s\right), ..., \sigma_t^s\left(z^t, h^t\right)\right)\right)$$

$$\delta_t\left(h^t \mid z^t\right) \equiv \min\left(\sum_{s\leq t-1} {}_s\delta_{t-1}\left(z^t, \left(\sigma_s^s\left(z^s, h^s\right), ..., \sigma_t^s\left(z^t, h^t\right)\right)\right), 1\right)$$

$$\mu_t\left(h^t \mid z^t\right) \equiv \prod_{s<t} {}_s\mu_t\left(z^t, \left(\sigma_s^s\left(z^s, h^s\right), ..., \sigma_t^s\left(z^t, h^t\right)\right)\right)$$

Here, $b_t^{old}\left(h^t \mid z^t\right)$ denotes the total transfers in period $t$ from contracts signed prior to period $t$ as a function of reports $\left(\sigma_s^s\left(z^s, h^s\right), ..., \sigma_t^s\left(z^t, h^t\right)\right)$, $\delta_t\left(h^t \mid z^t\right)$ denotes the banishment indices of contracts signed prior to $t$ and similarly $\mu_t\left(h^t \mid z^t\right)$ is re-entry probability prescribed by these contracts. In particular, a household is banished if at least one of the contracts it is signed to prescribes banishment. For ease of notation I will subsequently refer to these objects as $b_t^{old}\left(h^t\right)$, $\delta_t\left(h^t\right)$ and $\mu_t\left(h^t\right)$. It is worth noting the difference between $\gamma_t\left(h^t\right)$ and $\delta_t\left(h^t\right)$. $\gamma_t\left(h^t\right)$ denotes the state of banishment at the beginning of the period, after previously banished households are stochastically allowed to sign contracts again. For example, if $\delta_{t-1}\left(h^{t-1}\right) = 1$, and the household was not allowed to contract with intermediaries at the beginning of period $t$, then $\gamma_t\left(h^t\right) = 1$. If it was allowed to sign contract with intermediaries, then $\gamma_t\left(h^t\right) = 0$. When a household is allowed to sign contracts with intermediaries after being banished, it starts afresh, i.e. $B^{t-1} = \varnothing$ and it can sign a new contract with any intermediary. Given the definition of a contract we can now define how the true probabilities of personal histories $h^t$ are constructed.[9] This is done recursively as follows[10] : $\wp\left(\omega^1\right)=\pi\left(\theta_1\right)$ and for all $t>1$

$$\begin{aligned}\wp\left(\omega^t\right)=&\wp\left(\omega^{t-1}\right)\pi\left(\theta_t\right)\left(\left[1-\delta_{t-1}\left(h^{t-1}\right)\right]\mathbf{1}_{B_t}\right.\\&\left.+\delta_{t-1}\left(h^{t-1}\right)\left[\gamma_t\left(h^t\right)\left[1-\mu_t\left(h^{t-1}\right)\right]+\left(1-\gamma_t\left(h^t\right)\right)\mu_t\left(h^{t-1}\right)\mathbf{1}_{B_t}\right]\right)\end{aligned}\tag{2.1}$$

In the first period, $\wp$ is the same as $\pi$. In subsequent periods, the first term $\wp\left(\omega^{t-1}\right)\pi\left(\theta_t\right)$ on the right hand side of (2.1) corresponds to the probability of $h^{t-1}$ times the probability of the current realization of type $\theta_t$ multiplied by an indicator function which indicates that contract $B_t$ has been signed. Next, if the household was not banished last period and $\delta_{t-1}\left(h^{t-1}\right)=0$, the probability is just these two terms. However, if the household was banished in $t-1$ then with probability $\mu_t\left(h^{t-1}\right)$ it is allowed to sign contracts again and with probability $1-\mu_t\left(h^{t-1}\right)$ it is still banished.

For any $t\geq 1$ and $h^t\in H^t$, the household of type $h^t$ chooses a strategy $\sigma_t^{HH}$ to maximize

$$\sum_{s=0}^{\infty}\beta^s\sum_{\omega^{t+s}\in\Omega^{t+s}}\wp\left(\omega^{t+s}\right)u\left(c_{t+s}\left(\omega^{t+s}\right)\right)\tag{2.2}$$

subject to a budget constraints: $\forall s\geq 0,\ h^{t+s}\in H^{t+s}$ such that $\gamma_{t+s}=0$ and $\delta_{t+s}\left(h^{t+s}\right)=0$ i.e, the household is not banished,

$$c_{t+s}\left(h^{t+s}\right)\leq\varpi_{t+s}\left(h^{t+s}\right)\left[\theta_{t+s}+b_{t+s}^{old}\left(h^{t+s}\right)+{}_{t+s}b_{t+s}\left(h^{t+s}\right)\right]+\left(1-\varpi_{t+s}\left(h^{t+s}\right)\right)\theta_{t+s}\tag{2.3}$$

and $c_t\left(\omega^t\right)=\theta_t$ if $\delta_t\left(h^t\right)=1$ or $\gamma_t=1$. The term $b_t^{old}\left(h^t\right)$ denotes the transfers from contracts signed in periods prior to $t$, while ${}_tb_t\left(h^t\right)\equiv{}_tb_t\left(z^t,\sigma_t^t\left(h^t\right)\right)$ denotes the transfers from the contract $B_t$, signed in period $t$. $B_t$ is chosen from the set of posted contracts $\mathcal{B}_t$. With slight abuse of notation I will sometimes denote the sum $b_t^{old}\left(h^t\right)+{}_tb_t\left(h^t\right)$ as $b_t\left(h^t\right)$. Note that if $B^{t-1}=\varnothing$, $b_t^{old}\left(h^t\right)=0$. The second term on the right hand side of the budget constraint says that if the household voluntarily chooses to not participate, it consumes its endowment that period. Denote the value of the above problem when the household is using reporting strategy $\sigma$ by $V_t\left(h^t\right)(\sigma)$.

[9] The probabilites for histories $h^t$ are constructed similarly.

[10] Recall that households with identical type and exclusion histories have the same contract history $B^{t-1}$.

Note that if $\delta_t\left(h^t\right)=1$, the value of a banished household is given by

$$V_t\left(h^t\right)=u\left(\theta_t\right)+\beta\mathbb{E}_t\left[\mu_{t+1}\left(h^t\right)V_{t+1}\left(h^t,\left(\varnothing,0,\theta_{t+1}\right)\right)+\left[1-\mu_{t+1}\left(h^t\right)\right]V_{t+1}\left(h^t,\left(\varnothing,1,\theta_{t+1}\right)\right)\right]$$

Finally, lets consider the problem of an intermediary. A strategy for an intermediary is $\sigma_t^{INT}:Z^t\rightarrow\mathbb{B}_t$ and a typical strategy $\sigma_t^{INT}\left(z^t\right)=B_t$. In each period, without loss of generality, we can consider intermediaries offering one contract for each type $h^t\in H^t$ and so $B_t=\left\{B_t^{h^t}\left(z^t\right)\in h^t\in H^t\right\}$. Here $B_t^{h^t}\left(z^t\right)$ is the contract *intended* for type $h^t$. Since households can choose any one of these contracts, each contract $B_t$ must satisfy self-selection constraints which require that no type has an incentive to choose a contract intended for a different type. In any period $t$, after new contracts are posted, define $\hat{V}_t\left(h^t,B_t^{\hat{h}^t}\left(z^t\right)\right)$ to be the value for type $h^t$ of choosing a contract intended for type $\hat{h}^t$. Clearly, a type $h^t$ can only choose contracts associated with histories consistent with the publicly observable component of its history, $\gamma^t$. Given a history $h^t$, define $H^c\left(h^t\right)$ to be the set of histories with same banishment histories as $h^t$. Contracts must satisfy the following self-selection constraints: for all $t$, $h^t\in H^t$,

$$\hat{V}_t\left(h^t,B_t^{h^t}\left(z^t\right)\right)\geq\hat{V}_t\left(h^t,B_t^{\hat{h}^t}\left(z^t\right)\right)\ \text{for all}\ \hat{h}^t\in H^c\left(h^t\right)\tag{2.4}$$

Second, each contract must satisfy incentive compatibility constraints at each date and history. A contract $B_t$ is incentive compatible if for all $t$, and histories $h^t\in H^t$,

$$V_{t+s}\left(h^{t+s}\right)\left(\sigma^*\right)\geq V_{t+s}\left(h^{t+s}\right)\left(\tilde{\sigma}\right)\ \text{for all}\ \tilde{\sigma}\in\Sigma\tag{2.5}$$

where $V_t\left(h^t\right)(\sigma)$ denotes the value to type $h^t$ of following reporting strategy $\sigma\in\Sigma$ as defined in (2.2). The incentive compatibility constraints are the restrictions that private information places on the set of feasible contracts. In particular, all contracts must have the feature that no household has an incentive to misreport its type in any period. For ease of notation, I will sometimes denote the equilibrium value for a household following the truth telling strategy by $V_t\left(h^t\right)$.

Third, any contract $B_t$ must satisfy voluntary participation constraints at each date $t$, and for each history $h^t$. At the beginning of each period, a household can choose to not repay their debts and thereafter live in autarky forever where it just consumes its endowment each period and cannot sign with new intermediaries. Formally, the voluntary participation constraint is

$$\left[1 - {}_{t}\delta_{t+s}\left(h^{t+s}\right)\right] V_{t+s}\left(h^{t+s}\right)\left(\sigma^{*}\right) \geq \left[1 - {}_{t}\delta_{t+s}\left(h^{t+s}\right)\right] V_{t+s}^{d}\left(h^{t+s}\right) \tag{2.6}$$

where ${}_{t}\delta_{t+s}\left(h^{t+s}\right) \in \{0,1\}$ is the banishment index prescribed by contract $B_t$ in period $t+s$ and $V_{t+s}^{d}\left(h^{t+s}\right)$ is the value of autarky which by assumption depends only on $\theta_t$.This constraint captures the restrictions limited commitment places on the contract. It says that all households not being banished must want to paritcipate in financial markets. I assume that if a household chooses to not participate, they live in autarky in all future periods,[11] i.e.

$$V_{t}^{d}\left(h^{t}\right) = u\left(\theta_t\right) + \frac{\beta}{1-\beta^{2}}\mathbb{E}u\left(\theta'\right)$$

Intermediaries can borrow and lend at market determined rate $\frac{1}{q_t}$. Given public histories, $\sigma^{HH}$, the strategies of future intermediaries and reservation utilities $\left\{\tilde{V}_t\left(h^t\right)\right\}$, each intermediary chooses $\sigma_t^{INT}$ to maximize

$$-\sum_{s=0}^{\hat{T}-1}\left(\prod_{j=0}^{s} q_{t+j}\right)\sum_{h^{t+s}\in H^{t+s}} \tilde{\zeta}_t\left(h^{t+s}\right)\left(\left[1 - {}_{t}\delta_{t+s}\left(h^{t+s}\right)\right] {}_{t}b_{t+s}\left(h^{t+s}\right)\right) \tag{2.7}$$

subject to (2.4), (2.5), (2.6) and participation constraints

$$\hat{V}_t\left(h^t, B_t^{h^t}\left(z^t\right)\right) \geq \underline{V}_t\left(h^t\right) \tag{2.8}$$

Clearly, to attract households, contracts must satisfy the above participation constraints. Of course in equilibrium, $\underline{V}_t\left(h^t\right)$ is such that intermediaries make zero profits.

I now formally define a Perfect Bayesian Equilibrium of the game.

**Definition 1** *A Perfect Bayesian Equilibrium is a sequence of prices* $\{q_t\}_{t\geq 1}$, *reservation utilities* $\left\{\underline{V}_t\left(h^t\right)\right\}_{t\geq 1}$, *strategies* $\left\{\sigma_t^{HH}, \sigma_t^{INT}\right\}_{t\geq 1}$, *and beliefs* $\left\{\tilde{\zeta}_t\right\}_{t\geq 1}$ *such that*

1. *For all* $t, z^t, h^t$, *the strategy* $\sigma_t^{HH}$ *solves the households problem (2.2)*

2. *For all* $t, z^t$, *given prices, reservation utilities,* $\sigma^{HH}$ *and beliefs* $\tilde{\zeta}_t$, *the strategy* $\sigma_t^{INT}\left(z^t\right)$ *solves the intermediaries' problem (2.7)*

[11] This assumption can be relaxed and we can introduce an exogenous probability of re-entry each period after default.

3. *Beliefs satisfy Bayes' rule wherever it applies*

4. *Markets clear: for all* $t \geq 1$,

$$\sum_{h^t \in H^t} \wp\left(\omega^t\right) c_t\left(\omega^t\right) = \sum_{h^t \in H^t} \wp\left(\omega^t\right) \theta_t$$
$$\delta_t\left(h^t\right) c_t\left(\omega^t\right) = \delta_t\left(h^t\right) \theta_t$$

Note that in any equilibrium $\tilde{\zeta}_t\left(h^t\right) = \zeta\left(h^t\right).$ It is also worth noting that in any equilibrium, for all dates $t \leq s \leq \hat{T} - 1,$ and each history $h^t \in H^t$, contracts must satisfy budget feasibility,

$$c_{t+s}\left(h^{t+s}\right) \leq \theta_{t+s} + b_{t+s}^{old}\left(h^{t+s}\right) + {}_{t+s}b_{t+s}\left(h^{t+s}\right) \tag{2.9}$$

where as before $b_{t+s}^{old}\left(h^{t+s}\right)$ denotes the total transfers received from contracts signed prior to period $t+s$, including those associated with $B_t$. ${}_{t+s}b_{t+s}\left(h^{t+s}\right)$ denotes the transfers associated with a potential new (hidden) contract that households can sign in period $t+s$. Both the consumption strategy $c_{t+s}\left(h^{t+s}\right)$ and ${}_{t+s}b_{t+s}\left(h^{t+s}\right)$ can be computed using the household's strategy $\sigma^{HH}$. Note that when characterizing the equilibrium contract, it is without loss of generality to restrict to equilibria in which households sign with only one intermediary at a time. If the equilibrium strategy doesn't satisfy (2.9), either households are not maximizing or markets cannot clear.

As a final point about the setup, note that the space of contracts is very general. Intermediaries can decide to offer short or long term contracts depending on the actions of other intermediaries. A particularly useful contract which will play a central part in thinking about deviations is an uncontingent savings contract. A contract $S_t^{\varepsilon,\delta}$ is called a $\varepsilon\delta-$savings contract if $S_t = \left({}_tb_t\ ,\ {}_tb_{t+1}\right)$ where

$${}_tb_t = -q_t\varepsilon$$
$${}_tb_{t+1} = \delta\varepsilon$$

for any $\varepsilon \geq 0$ and $\delta \leq 1$. Note that the $S_t^{\varepsilon,\delta}$ contract is not contingent on report types. In particular if offered any household can choose to sign it.

In the future, I will sometimes refer to the above as the environment with PI (private information), LC (Limited Commitment) and HT (Hidden Trading).[12]

[12] By hidden trading I mean that households can sign a new contract each period in a hidden fashion.

## 2.3 Contracts without Banishment

In this section, I consider an environment in which intermediaries do not have access to the banishment technology and that they can only offer contracts that induce households to always repay their debts. One interpretation of this assumption is that failure by households to not pay back their debts and consequently not participate imposes a large exogenous cost on intermediaries. Under this assumption we can work with a more familiar personal history space $H^t = \Theta^t \times \mathbb{B}^{t-1}$ and the corresponding probability measure $\pi$. Since by assumption all households with identical endowment histories $\theta^t$ have the same $B^{t-1}$, for ease of notation I will denote a history simply by $\theta^t$ and probability $\pi\left(\theta^t\right)$. The definition of equilibrium without banishment is identical to the one section 2.2 except that $\delta_t\left(\theta^t\right) = 0$ for all $t,\ \theta^t \in \Theta^t$.

The main result in this section says that under the above restrictions, the set of equilibria of the intermediary game is identical to the set in an incomplete markets model where households trade a risk-free bond subject to appropriately chosen debt constraints. I now describe this equivalent environment. For one direction of the equivalence result, namely that any equilibrium of the intermediary game is an equilibrium of the incomplete markets environment, we need only consider a standard model with exogenous debt constraints. For the other direction, we need a way of endogenizing debt constraints and this will require introducing a notion of default into the incomplete markets framework.

There are a continuum of infinitely lived households, $i \in I$, who each receive an i.i.d endowment shock each period $\theta_t \in \Theta$. All households begin the period with an existing stock of debt and after knowing their endowment shock, they can choose to default and live in autarky forever or not in which case they pay their debts and can continue to trade a risk-free bond subject to debt constraints. If a household chooses not to default, it chooses an allocation $\{s_{t+s}, c_{t+s}\}_{s\geq 0}$ to maximize

$$\mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \sum_{\theta^{t+s} \in \Theta^{t+s}} \pi\left(\theta^{t+s}\right) u\left(c_{t+s}\left(\theta^{t+s}\right)\right)$$

subject to budget constraints in each period

$$c_{t+s} + s_{t+s+1} \leq \theta_{t+s} + R_{t+s} s_{t+s} \tag{2.10}$$

and debt constraints

$$s_{t+s+1} \geq -\phi_{t+s} \tag{2.11}$$

Note all agents face identical debt constraints $\phi_t$ which can only depend on calendar time and not a household's type. Denote the value of this problem by $W_t\left(\theta^t, b_t; \Phi_t\right)$ where $\Phi_t = \{\phi_{t+s}\}_{\geq 0}$.

For ease of notation I denote the entire sequence $\left\{c_t^i\left(\theta^t\right)\right\}_{\theta^t \in \Theta^t}$ by $\left\{c_t^i\right\}$. The household's problem at the beginning of date $t$ if it hasn't defaulted in the past is to choose a default strategy $d_t \in \{0, 1\}$ to maximize

$$d_t W_t\left(\theta^t, s_t; \Phi_t\right) + [1 - d_t] V_t^d(\theta_t)$$

where as before, $V_t^d(\theta_t) = u(\theta_t) + \frac{\beta}{1-\beta^2}\mathbb{E}u(\theta')$.

Next, I define an equilibrium concept that endogenizes the sequence of debt constraints. This is similar to the concept introduced by Alvarez and Jermann (2000).

**Definition 2** *A Not-Too-Tight competitive equilibrium is a sequence of interest rates* $\{R_t\}_{t\geq 0}$, *debt constraints* $\{\phi_t\}_{t\geq 0}$, *allocations for households* $\{d_t, c_t, s_t\}_{t\geq 0}$ *such that*

1. *Given prices, the allocations solve each household's problem*
2. *Markets clear,* $\forall t$
$$\sum_{\theta^t \in \Theta^t} \pi\left(\theta^t\right) s_{t+1}\left(\theta^t\right) = 0$$
3. *The sequence* $\{\phi_t\}_{t\geq 0}$ *is chosen to be Not-Too-Tight, i.e.* $\forall t$,
$$W_{t+1}\left(\theta^{t+1}, -\phi_t; \Phi_{t+1}\right) \geq V_{t+1}^d(\theta_{t+1}) \text{ for all } \theta^{t+1}$$
$$W_{t+1}\left(\hat{\theta}^{t+1}, -\phi_t; \Phi_{t+1}\right) = V_{t+1}^d\left(\hat{\theta}_{t+1}\right) \text{ for some } \hat{\theta}^{t+1}$$

Debt constraints are "Not-Too-Tight" if the following property is true; in equilibrium, at each date and given any history $\theta^t$, if this household has borrowed up to this constraint the previous period, it weakly prefers to not default while there exists some type $\hat{\theta}^t$ who is exactly indifferent. The idea is to allow households to hold the maximum amount of debt consistent with no default. The primary difference between the above definition and the one in Alvarez and Jermann (2000) is that unlike their environment, here debt constraints are not state contingent. In particular, in their model, agents trade Arrow securities subject to state contingent debt constraints, while here since markets are incomplete, we have constraints that are independent of states. This equilibrium concept has also been studied by Zhang (1997).

It is worth noting that the usual incomplete markets environment with exogenous debt constraints can also be defined using the model described above.

**Definition 3** *A $\Phi-$competitive equilibrium is a sequence of interest rates $\{R_t\}_{t\geq 0}$ , debt constraints $\Phi = \{\phi_t\}_{t\geq 0}$ , allocations for households $\{d_t, c_t, s_t\}_{t\geq 0}$ such that*

1. *Given prices, the strategies and allocations solve each household's problem*

2. *Markets clear, $\forall t$*

$$\sum_{\theta^t \in \Theta^t} \pi\left(\theta^t\right) s_t\left(\theta^t\right) = 0 \tag{2.12}$$

The first main equivalence result of the paper proves an equivalence between the banishment-free equilibria of the intermediary game defined in the previous section and the model with a risk-free bond and endogenous debt constraints.

**Theorem 1** (Equivalence: No banishment)

1. *A no-banishment equilibrium outcome of the environment with PI, LC and HT, is an equilibrium outcome of the environment with incomplete markets and not-too-tight debt constraints.*

2. *An equilibrium outcome of the environment with incomplete markets and not-too-tight debt constraints is a no-banishment equilibrium outcome of the environment with PI, LC and HT.*

A brief sketch of the proof is as follows. Consider the incomplete markets environment. A sequence of outcomes $\{q, \phi, c, s\}$ is an equilibrium of the incomplete markets environment iff

1. $u'\left(c_t\left(\theta^t\right)\right) q_t \geq \beta \mathbb{E}_t u'\left(c_{t+1}\left(\theta^{t+1}\right)\right)$ for all $t, \theta^t$.
2. $u'\left(c_t\left(\theta^t\right)\right) q_t > \beta \mathbb{E}_t u'\left(c_{t+1}\left(\theta^{t+1}\right)\right) \Rightarrow s_{t+1}\left(\theta^t\right) = -\phi$

3. Given $\{q, \phi\}$, $\{c, s\}$ satisfy the household's budget and debt constraints (2.10) and (2.11)

4. Market clearing conditions (2.12) hold

5. The debt constraints $\{\phi\}$ are chosen to be not-too-tight, i.e.

$$W_{t+1}\left(\theta^{t+1}, -\phi_t; \Phi_{t+1}\right) \geq V^d_{t+1}\left(\theta_{t+1}\right) \text{ for all } \theta^{t+1}$$
$$W_{t+1}\left(\hat{\theta}^{t+1}, -\phi_t; \Phi_{t+1}\right) = V^d_{t+1}\left(\hat{\theta}_{t+1}\right) \text{ for some } \hat{\theta}^{t+1}$$

The proof requires a series of preliminary results. The three main propositions that required to prove Theorem 1 are Proposition 1, Proposition 2 and Proposition 3. The first of these propositions shows that we can construct outcomes that satisfy conditions 1. and 5. above. The second proposition shows that these outcomes satisfy 3 and the final proposition shows 2.

The first main proposition required to prove Theorem 1 says that in any equilibrium, households can only be borrowing constrained and never savings constrained. Further, if a household is borrowing constrained in a period then the voluntary participation constraint binds for some type in the following period.

**Proposition 1** *In any non-autarkic equilibrium of the intermediary game*

1. *For all types $\theta^t$,*
$$u'\left(c_t\left(\theta^t\right)\right) q_t \geq \beta \mathbb{E}_t u'\left(c_{t+1}\left(\theta^{t+1}\right)\right)$$

2. *In any period if for any $\theta^t$,*
$$u'\left(c_t\left(\theta^t\right)\right) q_t > \beta \mathbb{E}_t u'\left(c_{t+1}\left(\theta^{t+1}\right)\right)$$
*then there exists some $\tilde{\theta}^{t+1}$ such that*
$$V_{t+1}\left(\tilde{\theta}^{t+1}\right) = V^d_{t+1}\left(\tilde{\theta}^{t+1}\right)$$

**Proof.** See Appendix A. ■

If a household was savings constrained, at the second stage of the period a new intermediary can offer an $\varepsilon\delta$ savings contract which would make both it and the household strictly better off. In the case in which intermediaries write default-free contracts, they will be unwilling to lend too much since zero profits requires imposing negative transfers in subsequent periods on the household which would worsen its incentives to default. The second part of the proposition shows that if a household is Euler-constrained, then it must be that the voluntary participation constraints bind for some type the following period. The reason for this is clear, if not then an intermediary can increase transfers to the constrained household in the current period and reduce them in the following period in a way so as to make strictly positive profits since the shadow rate of interest of a constrained agent is higher than the market rate.

The second main proposition required for the equivalence result says that we can represent any $\hat{T}$ period contract as a sequence of 2 period contracts, each one of which makes zero profits.

**Proposition 2** *Given an equilibrium of a truncated $T$-period environment with $\hat{T}$ period lived overlapping intermediaries, there exists an equilibrium with 2 period lived intermediaries with same allocations and prices.*

**Proof.** See Appendix A. ■

This proposition establishes that in equilibrium, intermediaries can only offer short term contracts. Given a $\hat{T}-$period contract we set the first period transfers to be the same and in subsequent periods, we split transfers from the original contract into $\zeta_t = -\frac{{}_{t-1}b_t}{q_t} + {}_t b_t$ where ${}_{t-1}b_t$ is the period $t$ transfer from a contract signed in period $t-1$ and ${}_t b_t$ is the transfer from an intermediary born in period $t$. Since the original $\hat{T}$ period lived intermediaries must make zero profits, to show that these 2 period contracts also make zero profits it is sufficient to show that the expected present discounted value of transfers from $\hat{T}-1$ onwards, is independent of the period $\hat{T}-1$ report of endowment. In particular, given a history $\theta^{\hat{T}-2}$, I show that the present discounted value of transfers in $\hat{T}-1$, is independent of $\theta_{\hat{T}-1}$. To see why, suppose we have two types $\left(\theta^{\hat{T}-2}, \theta\right)$ and $\left(\theta^{\hat{T}-2}, \theta'\right)$ with $\theta > \theta'$, but type $\left(\theta^{\hat{T}-2}, \theta'\right)$ receives the higher present discounted value of transfers. There are two cases to consider. The first is that the difference in transfers is front-loaded and that period $\hat{T}-1$ transfers are higher for type $\left(\theta^{\hat{T}-2}, \theta'\right)$. In this case, type $\left(\theta^{\hat{T}-2}, \theta\right)$ will strictly prefer to lie and pretend to be $\left(\theta^{\hat{T}-2}, \theta'\right)$, and save with another intermediary. As mentioned earlier, intermediaries are always willing to over $\varepsilon\delta$ savings contracts and one can be constructed to make both the lying agent and a new intermediary strictly better off. The second case is a little more complicated in the case in which the difference in transfers is back-loaded and both types are Euler-constrained. However, I show that if a lower type weakly prefers the backloaded transfer scheme (which should be true in equilibrium) type $\left(\theta^{\hat{T}-2}, \theta\right)$ will again strictly prefer to lie and pretend to be $\left(\theta^{\hat{T}-2}, \theta'\right)$. On the other hand, if $\left(\theta^{\hat{T}-2}, \theta\right)$ receives the higher present discounted value of transfers, then a perturbation which redistributes to types below $\theta$ increases ex-ante welfare since it increases the amount of insurance in $\hat{T}-1$. Such a perturbation always satisfies voluntary participation constraints since one can show (see Lemma 14) that these constraints only bind for the lowest types. An important property in a 2-period lived intermediary environment is that for all $t$, and histories $\theta^{t-1}$, the present discounted values of equilibrium transfers is independent of $\theta_t$.

The results so far suggest that the equilibria in the intermediary environment are equivalent to one in which agents trade a risk-free bond subject to debt constraints. In particular, any equilibrium with incomplete markets and borrowing constraints must satisfy the constrained Euler equation and the above conditions on the transfers. The next few results will help us prove some properties about the corresponding debt constraints. The third key proposition required to prove Theorem 1 shows that in any period, all Euler-constrained households have identical debt constraints.

**Proposition 3** *For any $t$, and $\theta^t$ such that*

$$u'\left(\theta_t + {}_{t-1}b_t\left(\theta^t\right) + {}_tb_t\left(\theta^t\right)\right) q_t > \beta \mathbb{E}_t u'\left(\theta + {}_tb_{t+1}\left(\theta^t\right) + {}_{t+1}b_{t+1}\left(\theta^{t+1}\right)\right)$$

*it must be that ${}_tb_t\left(\theta^t\right) = \varphi_t$ where $\varphi_t$ is independent of the agent's history.*

**Proof.** See Appendix A. ■

The proof follows from a preliminary result which states that in equilibrium, the value of not defaulting for any two types $\theta^t$ and $\tilde{\theta}^t$ such that $\theta_t + {}_{t-1}b_t = \tilde{\theta}_t + {}_{t-1}\tilde{b}_t$. Notice that here ${}_{t-1}b_t$ corresponds to the transfer in period $t$ from a contract signed in period $t-1$. I prove this using an induction argument. Given that we are working in a truncated economy, consider the last period $T$ in which intermediaries are operational. Since from period $T$ onwards households trade a risk-free bond, the household's value going forward depends on only its current endowment and transfer. Next, suppose the hypothesis is true from period $t+1$ onwards and so we want to establish that it is true in period $t$. For contradiction, suppose we have two histories such that $\theta_t + {}_{t-1}b_t = \theta_t + {}_{t-1}\tilde{b}_t$ but $V_t\left(\theta^t\right) > V_t\left(\tilde{\theta}^t\right)$. The idea of the proof is to show that a deviating intermediary can give agent $\tilde{\theta}^t$ a contract similar to type $\theta^t$, which makes both the household and it strictly better off while still satisfying incentives. The key condition that needs to be checked is that such a contract does not incentivize default the following period. Notice that household $\tilde{\theta}^t s$ incentives to default in period $t+1$ are exactly the same as household $\theta^t$ if they receive the same transfers since the value of the two households going forward is identical by the induction assumption.

The result states that in the environment with 2 period lived intermediaries, the transfers received from new contracts signed in period $t$ are identical for households that are Euler-constrained in period $t$. At the first glance, the result may seem surprising since in general the present discounted value of transfers is not identical across all histories. Suppose we have two households with different histories who are Euler-constrained in period $t$. Given that each contract must make zero profits, contracts offered in period $t$ are of the form $\left(\varphi, -\frac{\varphi}{q_t}\right)$. Competition among intermediaries will force $\varphi$ to be as high as possible consistent with no default the following period for each Euler-constrained household. Then the previous proposition tells us that all agents receiving $\left(\varphi, -\frac{\varphi}{q_t}\right)$ will have exactly the same incentives to default independent of history. As a result, such a contract will always satisfy voluntary participation constraints.

Using these characterization results, we can proceed to proof of the equivalence theorem (see Appendix A). The proof of the first part of the theorem is a direct consequence of the properties proved in the previous section. The necessary and sufficient conditions for an allocation-price pair to constitute a $\Phi-$competitive equilibrium are, for all $t, \theta^t$

$$u'\left(c_t\left(\theta^t\right)\right) \geq \beta R_{t+1} \mathbb{E}_t u'\left(c_{t+1}\left(\theta^{t+1}\right)\right)$$

and

$$u'\left(c_t\left(\theta^t\right)\right) > \beta R_{t+1} \mathbb{E}_t u'\left(c_{t+1}\left(\theta^{t+1}\right)\right)$$
$$\Rightarrow b_{t+1} = -\phi_t$$

Finally, the budget constraint must hold at each date and state. The first two properties are satisfied in equilibrium of the intermediary game as described earlier. The second

follows from the fact that in any equilibrium of the intermediary game, the equilibrium expected present discounted value of transfers, $A_1(\theta_1) = 0$ and for all $t$, and histories $\theta^{t-1}$, $A_t(\theta^{t-1}, \theta) = A_t(\theta^{t-1}, \theta')$ for all $\theta, \theta' \in \Theta$ where

$$A_t(\theta^{t-1}, \theta) \equiv b_t(\theta^{t-1}, \theta_t) + q_t \sum_{\theta' \in \Theta} \pi(\theta') A_{t+1}(\theta^{t-1}, \theta, \theta')$$

For the converse, we need to show that if all intermediaries are offering $\Phi-$contracts[13] , no existing or new intermediary has an incentive to deviate and offer contracts that make positive profits. First consider the case of a new intermediary. The only type of deviating contract we need to consider is one in which an Euler-constrained household at some date receives an increased transfer. Incentive compatibility requires that the contract make a negative transfer in the following period. However, any Not-too-tight competitive equilibrium has the property that some type's voluntary participation constraint binds the following period and therefore this negative transfer cannot be uncontingent. I show that such a deviating contract is never incentive compatible in any period since households are not constrained to reporting the same type to different intermediaries. In particular, they can always report the type that results in the highest transfer to the new intermediary while reporting their true type to the original intermediary. Finally we need to consider the incentives for an existing intermediary to modify its contract. As in the case with the new intermediary, the relevant deviations involve increasing transfers to Euler-constrained households at some $t$, and reducing transfers the following period. Since some type's voluntary participation constraint binds in $t + 1$, the negative transfer must be state-contigent. Consider imposing the negative transfer on those households that are Euler constrained in $t + 1$. Since the lowest type falls into this category, clearly this is not possible since his voluntary participation constraint is binding. On the other hand, if the negative transfers are imposed on those households that are not Euler-constrained, these agents will strictly prefer to lie and pretend to be a lower type. Therefore such contracts are not incentive compatible.

[13] Simple borrowing and lending contract subject to debt constraints

It is worth noting that all three frictions, i.e., private information, limited commitment and hidden trading are necessary to obtain the above characterization. Environments with only private information, for example Atkeson and Lucas (1992) or private information and limited commitment as in Dovis (2014) cannot be decentralized with only a short term uncontingent bond. In particular, it is not true in such environments that the present discounted value of transfers is independent of current type. Environments with private information and hidden trading imply contracts that resemble trade in a risk-free bond as was shown by Allen (1985). Cole and Kocherlakota (2001) Also prove a similar result in an environment with hidden savings. However in both these environments, no agent is Euler-constrained in equilibrium and as a result the efficient allocation cannot be decentralized as an environment with a risk-free bond and binding (endogenous) debt constraints. In particular, the efficient allocation in models with private information and hidden savings will not in general satisfy voluntary participation constraints introduced in the previous sections.

### 2.3.1 Equilibrium Existence and Multiplicity

Next, I consider whether equilibria of the intermediary game without banishment exist. Given the equivalence result, it suffices to prove the existence of a Not-too-tight competitive equilibrium. To show existence, I focus on stationary recursive competitive equilibria and show that these are well defined and exist. The main theorem in this subsection is that are multiple competitive equilibria.

We can write the problem of a household recursively as follows:

$$W\left(\theta, b, \phi; \Phi\right) = \max_{c,b} u\left(c\right) + \beta \mathbb{E} W\left(b', \phi'; \Phi'\right)$$

$$\begin{aligned}&\text{subject to}\\ &c + b' \leq \theta + Rb\\ &b' \geq -\phi\end{aligned}$$

where $\theta$ is the household's current endowment, $b$ its assets and $\phi$, the current debt constraint which is determined by the rule $\phi' = \Phi\left(\phi\right)$ where $\Phi$ is known to all households.

In this case the value of default is given by

$$V^d(\theta) = u(\theta) + \mathbb{E}V^d(\theta')$$

As earlier we can define the notion of a $\Phi-$Recursive competitive equilibrium and finally a Not-Too-Tight RCE. Let $\mathbb{A}$ be the bounded space of assets and $\mathcal{P}(\mathbb{A})$ the set of probability measures on $\mathbb{A}$.

**Definition 4** *A $\Phi-$Recursive Competitive Equilibrium is price function $R(\phi)$, a law of motion $\phi' = \Phi(\phi)$ , a measurable map $G : R_+ \times \mathcal{P}(\mathbb{A})$ ,value functions $W(\theta, b, \phi; \Phi)$, policy functions $b'(\theta, b, \phi)$ such that*

1. *Given $R$ and $\Phi$, the value functions and policy functions solve the households' problems and*

2. *the sequence of distributions generated by $G$ is such that markets clear*

$$\int_{\mathbb{A}\times\Theta} b'(\theta, b, \phi)\, d\lambda(b, \Theta) = 0$$

*where*

$$\lambda' = G(\phi, \lambda)$$

**Definition 5** *A NTT$-$Recursive Competitive Equilibrium is price function $R(\phi)$, a law of motion $\phi' = \Phi(\phi)$ , a measurable map $G : R_+ \times \mathcal{P}(\mathbb{A})$ ,value functions $W(\theta, b, \phi; \Phi)$, policy functions $b'(\theta, b, \phi)$ such that*

1. *Given $R$ and $\Phi$, the value functions and policy functions solve the agents problems and*

2. *the sequence of distributions generated by $G$ is such that markets clear*

$$\int_{\mathbb{A}\times\Theta} b'(\theta, b, \phi)\, d\lambda(b, \Theta) = 0 = 0$$

*where*

$$\lambda' = G(\phi, \lambda)$$

*3. If* $\phi' \in \Phi(\phi)$ *then*

$$W\left(\theta, -\phi', \phi'; \Phi\right) \geq V^d(\theta) \text{ for all } \theta \in \Theta$$
$$W\left(\theta^*, -\phi', \phi'; \Phi\right) = V^d(\theta^*) \text{ for some } \theta^* \in \Theta$$

Define $\eta = \int_{\theta \in \Theta} u'(\theta)\, dF(\theta)$ and let

$$\kappa = \min_{\theta} \frac{u'(\theta) + \beta\eta}{u'(\theta) + \beta\eta + \beta^2\eta}$$

**Theorem 2** (Existence: No banishment) *Under the following sufficient condition*

$$\frac{u'(\bar{\theta})}{\beta\eta} < \kappa$$

*there exist multiple NTT−Recursive Competitive Equilibria.*

**Proof.** See Appendix A. ■

The first step in the proof is to show that given a measurable map $\Phi$, a $\Phi$−RCE always exists. Next, it always true that a $\Phi$−RCE with $\Phi$ being the zero map is NTT-RCE. The reason for this is clear. If debt constraints are zero each period, then in equilibrium agents consume their endowment which trivially implies that the voluntary participation constraint binds for each period and each type. The final and key proposition that completes the proof of Theorem 2 is to show that there exists a NTT-RCE with $\Phi \neq 0$. The idea is to show that for each $\theta$, there exists $\Phi^\theta$, such that debt constraints are $\phi^\theta$ each period and

$$W\left(\theta, -\phi^\theta, \phi^\theta; \Phi^\theta\right) = V^d(\theta)$$

Then setting $\phi = \min_\theta \phi^\theta$ given us a $\Phi$−RCE with debt constraints that are not too tight.

The above result along 1 shows that the intermediary game with no-banishment contracts has multiple equilibria. There exists an equilibrium of the decentralized contracting environment in which all intermediaries offer null contracts to households.. A simple way of understanding this result is to notice a *strategic complementarity* in the actions of intermediaries. In particular, if an intermediary believes that no future intermediary is willing to lend to households, it will be unwilling to lend since the household will choose to default in subsequent periods.

On the surface this might seem a surprising result since one would expect a intermediary to always be able to construct a deviating contract that offers some insurance and hence make positive profits. To see why this is not possible, consider a $\hat{T}$ lived intermediary born at date $t+1$. In the last period of the contract, $\hat{T}$ it must be that ${}_{t+1}b_{\hat{T}}\left(\theta^{\hat{T}}\right) \geq 0$ since no intermediary in the future is offering any insurance. If ${}_{t+1}b_{\hat{T}}\left(\theta^{\hat{T}}\right) < 0$ for any $\theta^{\hat{T}}$ that household will strictly prefer to default. Now consider $\hat{T}-1$. For any $\theta^{\hat{T}-1}$ it must be that ${}_{t+1}b_{\hat{T}-1}\left(\theta^{\hat{T}-1}\right) \leq 0$ since if is strictly positive then in order to preserve incentive compatibility and make positive profits the intermediary will have to set transfers negative for some type in $\hat{T}$. Therefore the only feasible perturbation in $\hat{T}-1$ must be ${}_{t+1}b_{\hat{T}-1}\left(\theta^{\hat{T}-1}\right) < 0$ and ${}_{t+1}b_{\hat{T}}\left(\theta^{\hat{T}-1}\right) > 0$. Note again that if $b^t_{\hat{T}}\left(\theta^{\hat{T}-1}\right)$ depended on $\theta_{\hat{T}}$ incentive compatibility would be violated. The perturbation resembles a savings contract. However if the interest rates are such that $R_{\hat{T}} \leq \frac{u'(\bar{\theta})}{\beta \mathbb{E} u'(\theta)} b$ such a contract would have to offer a return on savings $> R_{\hat{T}}$ which would mean that the intermediary makes negative profits. For any $R \leq R_{\hat{T}}$ the household prefers the transfer schedule ${}_{t+1}b_{\hat{T}-1}\left(\theta^{\hat{T}-1}\right) = 0$, ${}_{t+1}b_{\hat{T}}\left(\theta^{\hat{T}-1}\right) = 0$ to the one offered by the deviating contract. Therefore in $\hat{T}-1$ it must be that ${}_{t+1}b_{\hat{T}-1}\left(\theta^{\hat{T}-1}\right) \geq 0$. A similar argument works in $\hat{T}-2$ and hence for previous periods.

## 2.4 Contracts with Banishment

In this section, I allow intermediaries to use banishment in equilibrium. The main result in this section shows that under sufficient conditions, intermediaries will choose to banish households in equilibrium. As a result, equilibria will feature periods in which households are in financial autarky.

**Proposition 4** *For $\pi(\underline{\theta})$ small and $u(\underline{\theta}) + \frac{\beta}{1-\beta^2}\mathbb{E}u(\theta')$ large enough, any non-autarkic equilibrium features banishment on path.*

**Proof.** See Appendix A. ■

The idea behind the proof is to show that given any equilibrium with no banishment, a deviating intermediary can offer a contract with temporary banishment in some states and make strictly positive profits while making some household strictly better off. As we saw in the previous section (Theorem 1), any equilibrium contract with no banishment takes the form of a simple uncontingent borrowing and lending subject to history independent debt constraints. Consider any such equilibrium and a period $t$, and a type $\left(\theta^{t-1},\theta\right)$ who is Euler-constrained (borrowing constrained). Given that there is no banishment, we can work with the more familiar type spaces $\Theta^t$. One can show (Lemma 14) that this implies that in period $t+1,$ the voluntary participation constraint for type $\left(\theta^{t-1},\theta,\underline{\theta}\right)$ is binding. A deviating intermediary can modify the original contract as follows

$$
\begin{aligned}
{}_t\tilde{b}_t\left(\theta^t\right) &= {}_tb_t\left(\theta^t\right)+\varepsilon \\
{}_t\tilde{b}_{t+1}\left(\theta^t,\theta\right) &= -\frac{\left[{}_tb_{t+1}\left(\theta^t,\theta\right)+R_t\varepsilon\right]}{1-\pi\left(\underline{\theta}\right)} \text{ for all } \theta\neq\underline{\theta}
\end{aligned}
$$

where $b$ corresponds to the original equilibrium contract and. Under this contract, type $\left(\theta^{t-1},\theta,\underline{\theta}\right)$ is banished and in each subsequent period is allowed back into the contracting environment with probability $\lambda=0$ and hence receives value $V_t^d\left(\underline{\theta}\right)$ equal to the value under the original contract. The change in welfare for the household in $t$ is given by

$$
\begin{aligned}
\Delta\left(\theta^{t-1},\theta\right) = {} & u\left(\theta+{}_{t-1}b_t+{}_t\tilde{b}_t\right)+\beta\sum_{\theta'>\underline{\theta}}u\left(\theta+{}_t\tilde{b}_{t+1}+{}_{t+1}b_{t+1}\right) \\
& -u\left(\theta+{}_{t-1}b_t+{}_tb_t\right)-\beta\sum_{\theta'\in\Theta}u\left(\theta+{}_tb_{t+1}+{}_{t+1}b_{t+1}\right)
\end{aligned}
$$

One can use a Taylor approximation to show that $\operatorname{sgn}\left(\Delta\left(\theta^{t-1},\theta\right)\right)\geq\operatorname{sgn}\left(\tilde{\Delta}\left(\theta^{t-1},\theta\right)\right)$ where

$$
\begin{aligned}
\tilde{\Delta}\left(\theta^{t-1},\theta\right) \approx {} & u'\left(\theta+{}_{t-1}b_t+{}_tb_t\right)-\frac{\beta}{1-\pi\left(\underline{\theta}\right)}\sum_{\theta'\in\Theta}\pi\left(\theta'\right)u'\left(\theta'+{}_tb_{t+1}+{}_{t+1}b_{t+1}\right) \\
& +\beta\sum_{\theta'\in\Theta}\pi\left(\theta'\right)\left[u\left(\theta'+\frac{{}_tb_{t+1}}{1-\pi\left(\underline{\theta}\right)}+{}_{t+1}b_{t+1}\right)-u\left(\theta'+{}_tb_{t+1}+{}_{t+1}b_{t+1}\right)\right]
\end{aligned}
$$

which is strictly positive if $\pi\left(\underline{\theta}\right)$ is small enough since the type is Euler-constrained in period $t$. Moreover, the intermediary is as well off as before. As a result a contract can be constructed that makes both the intermediary and the household strictly better off.

The above result might seem surprising since the intermediary can implement the outcomes associated with banishment without actually having to banish the household. For example, consider the following contract

$$\begin{aligned}
\tilde{b}_t\left(\theta^{t-1},\theta\right) &= b_t\left(\theta^{t-1},\theta\right)+\varepsilon \\
\tilde{b}_{t+1}\left(\theta^{t-1},\theta,\theta'\right) &= \frac{b_{t+1}\left(\theta^{t-1},\theta,\theta'\right)-R_{t+1}\varepsilon}{1-\pi\left(\underline{\theta}\right)} \text{for all } \theta'\neq\underline{\theta} \\
\tilde{b}_{t+1}\left(\theta^{t-1},\theta,\underline{\theta}\right) &= b_{t+1}\left(\theta^{t-1},\theta,\underline{\theta}\right)
\end{aligned}$$

Such a contract also gives type $\left(\theta^{t-1},\theta,\underline{\theta}\right)$, the value associated with banishment. However, Proposition 28 implies that such a contract is not incentive compatible since here the present discounted value of transfers to type $\left(\theta^{t-1},\theta,\underline{\theta}\right)$ is larger than that for types $\left(\theta^{t-1},\theta,\theta'\right)$, $\theta'>\underline{\theta}$. As a result, these types will strictly prefer to lie downwards and save with another intermediary.

Suppose we had *exclusive contracts* in that households can only sign contracts with one intermediary at a time. Then the above perturbation can be implemented without banishment on path using the following transfer scheme

$$\begin{aligned}
{}_t\tilde{b}_t\left(\theta^t\right) &= {}_tb_t\left(\theta^t\right)+\varepsilon \\
{}_t\tilde{b}_{t+1}\left(\theta^t,\theta\right) &= -\frac{\left[{}_tb_{t+1}\left(\theta^t,\theta\right)+R_t\varepsilon\right]}{1-\pi\left(\underline{\theta}\right)} \text{ for all } \theta\neq\underline{\theta} \\
{}_t\tilde{b}_{t+1}\left(\theta^t,\underline{\theta}\right) &= 0 \text{ and } 0 \text{ in all future periods}
\end{aligned}$$

As before such a scheme gives type $\left(\theta^t,\underline{\theta}\right)$ a value equal to autarky. Since we know that under the original contract, the present discounted value of transfers to any type must be 0 and $b_{t+1}\left(\theta^{t-1},\theta,\theta'\right)<0$, it must that the under the above contract, the present discounted value of transfers to these types is less than zero. However, unlike the environment with non-exclusive contracts, these types will not strictly prefer to lie since they cannot save and borrow in a hidden fashion. In particular, given a type $\left(\theta^{t-1},\theta,\theta'\right)$, $\theta'\neq\underline{\theta}$, the value of lying is

$$W_{\theta^t}\left(\theta',\underline{\theta}\right)=V^d\left(\theta'\right)\leq W_{\theta^t}\left(\theta',\theta'\right)$$

and so this perturbation preserves incentives. To summarize, the crucial difference in the case with hidden trading (non-exclusive contracts) is that banished households are unable to sign contracts with other intermediaries, which allows banishment to incentivize truthful revelation of types. With hidden trading, banishing a household is equivalent to a contract with transfers equal to zero in all future periods.

A more stark way to distinguish contracts with exclusivity to those without it is to consider an environment in which intermediaries face an exogenous cost of banishment. One way to interpret this cost is to assume that intermediaries need pay an outside regulatory authority to monitor households and make sure that they don't sign contracts with other intermediares while banished. With exclusive contracts, any equilibrium in which households are being banished is Pareto-inferior to one in which they are not since intermediaries can provide the autarkic value to households on path and save the cost. However, with hidden trading, equilibria in which intermediaries pay this cost and banish households may Pareto-dominate all equilibria without banishment. This is the sense in which the hidden trading assumption is necessary to get banishment/default on path.

Next, I provide a characterization of the equilibrium in some special cases. First suppose that intermediaries live for two periods.[14] Then I show that an equilibrium outcome of this environment is also an equilibrium outcome of an Eaton-Gersovitz like environment with short-term defaultable debt and suitably chosen re-entry probabilities. An equilibrium of the intermediary game was defined in the previous section. Next, I define the equivalent environment.

There are continuum of infinitely lived households $i \in I$. Households begin each period $t$, with asset holdings $s_t$. The timing within a period is as follows:

1. At the beginning of period $t$, $\theta_t$ is realized
2. Households choose whether to default or pay back $s_t$.
   - If it pays back, the household can issue new debt, $s_{t+1}$, at corresponding price schedule $\mathcal{Q}_{t+1}(s_{t+1})$
   - If the household defaults, it consumes its endowment in the current period and in future periods is allowed to trade in financial markets with probability $\lambda(s_t)$
3. Households consume

In each period, given state $(\theta_t, s_t)$ households choose $(c_t, s_{t+1})$ to maximize

$$V_t^R(\theta_t, s_t; \mathcal{Q}_t) = u(c_t) + \beta \mathbb{E}_t V_{t+1}^0(\theta_{t+1}, s_{t+1}; \mathcal{Q}_{t+1})$$

[14] A full characterization of the equilibrium with longer lived intermediaries is in progress.

subject to a budget constraint

$$c_t + \mathcal{Q}_t\left(s_{t+1}\right) \leq \theta_t + s_t \tag{2.13}$$

Here $\mathcal{Q}_t\left(s_{t+1}\right)$ is the debt pricing schedule which is taken as given by households. If a household defaults, it consumes its endowment that period and in subsequent periods it can regain access to financial markets with probability $\lambda\left(s\right)$. Notice that the re-entry probability only depends on the level of debt $s$ that was defaulted on and is independent of the household's endowment. Therefore, the value of default is given by

$$V_t^D\left(\theta_t;\lambda\left(s\right)\right) = u\left(\theta_t\right) + \beta\mathbb{E}_t\left[\lambda\left(s\right)V_{t+1}^R\left(\theta_{t+1},s_{t+1};\mathcal{Q}_{t+1}\right) + \left(1-\lambda\left(s\right)\right)V_{t+1}^D\left(\theta_{t+1};\lambda\right)\right]$$

At the beginning of each period, households choose whether to default or not, $d_t = \{0,1\}$ with $d_t = 0$ implying default,

$$V_t^0\left(\theta_t,s_t;\mathcal{Q}_t,\lambda\right) = \max_{d_t\in\{0,1\}} d_t V_t^R\left(\theta_t,s_t;\mathcal{Q}_t\right) + \left[1-d_t\right]V_t^D\left(\theta_t;\lambda\left(s_t\right)\right)$$

Households borrow and lend with a continuum of risk-neutral lenders who have an outside option that yields return $R_{t+1} = \frac{1}{q_t}$ in period $t+1$. Therefore, in order to break even, the price of debt is determined by

$$\begin{aligned} &-\left[1-\Pr\left[V_{t+1}^D > V_{t+1}^R\right]\right]\frac{s}{\mathcal{Q}_t\left(b\right)} \geq -R_{t+1}s \\ &\Rightarrow \mathcal{Q}_t\left(b\right) = \frac{\left[1-\Pr\left[V_{t+1}^D > V_{t+1}^R\right]\right]s}{R_{t+1}} = q_t\left[1-\Pr\left[V_{t+1}^D > V_{t+1}^R\right]\right]s \end{aligned}$$

where

$$\Pr\left[V_{t+1}^D > V_{t+1}^R\right] = \sum_{\theta\in\Theta}\pi\left(\theta\right)\mathbf{1}_{V_t^D\left(\theta;\lambda\left(s\right)\right)>V_t^R\left(\theta,s;\mathcal{Q}_t\right)}$$

determines the probability that the household will default the next period.

**Definition 6** *Given a sequence $\{q_t\}$ and a function $\lambda\left(s\right)$, a competitive equilibrium consists of value functions $V^0, V^D, V^R$, policy functions, $d, s, c$ and a pricing schedule $\mathcal{Q}\left(s\right)$ such that*

1. *Given the pricing schedule, the value functions and policy functions solve the household's problem*

*2. For all* $t,\ \mathcal{Q}_t(s) = q_t\left[1 - \Pr\left[V_{t+1}^D > V_{t+1}^R\right]\right] s$

The next result states that if intermediaries live for two periods, then an equilibrium outcome of the intermediary game (environment with PI, LC and HT) is also an equilibrium outcome of the Eaton-Gersovitz environment defined above.

**Proposition 5** *Suppose intermediaries live for two periods. Then there exists a function* $\lambda(s)$ *such that an equilibrium outcome of the environment with PI, LC and HT is an equilibrium outcome of the EG economy with re-entry probabilities given by* $\lambda(s)$*.*

A sketch of the proof is as follows. Given a sequence $\{q_t\}_{t\geq 1}$ and a function $\lambda(s)$, we know that a sequence of outcomes $\{\mathcal{Q}(s), d, s, c\}$ is an equilibrium of the Eaton-Gersovitz economy if and only if $\exists$ functions $V^0, V^D, V^R$ s.t.

1. $(c_t, s_{t+1}) \in \arg\max_{c,b} V_t^R(\theta_t, s_t; \mathcal{Q}_t) = u(c) + \beta \mathbb{E}_t V_{t+1}^0(\theta_{t+1}, s_{t+1}; \mathcal{Q}_{t+1})$ subject to (2.13)
2. $d \in \arg\max_{\hat{d}_t \in \{0,1\}} \hat{d}_t V_t^R(\theta_t, s_t; \mathcal{Q}_t) + \left[1 - \hat{d}_t\right] V_t^D(\theta_t; \lambda)$
3. $\mathcal{Q}_t(b) = q_t\left[1 - \Pr\left[V_{t+1}^D > V_{t+1}^R\right]\right] s$

The idea is to show that given an equilibrium outcome of the intermediary game we can construct such functions and outcomes that satisfy the above conditions. The four main results required to prove Proposition 5 are Lemma 1, Proposition 6, Proposition 7 and Proposition 8. The first establishes that each contract must make zero profits and intermediaries cannot cross-subsidize between types. The second result shows that households can never be savings constrained in equilibrium, the third that there exists the corresponding price function depends only on the level of debt and the last establishes that the re-entry probability is a function of the level of debt defaulted on.

Given a contract $B_t(z^t) = \left\{B_t^{h^t}(z^t) : h^t \in H^t\right\}$, let ${}_t\mathcal{P}_t(h^t)$ denote the expected present discounted value of transfers associated with contract $B_t^{h^t}\left(z^t\right)$ from period $t$ onwards. Therefore,

$$\begin{aligned}
&{}_t\mathcal{P}_t\left(h^t\right)\\
&=\left[1-{}_t\delta_t\left(h^t\right)\right]\left[{}_tb_t\left(h^t\right)+q_t\sum_{h_{t+1}\in H_{t+1}}\zeta_{t+1}\left(h^t,h_{t+1}\right){}_t\mathcal{P}_{t+1}\left(h^t,h_{t+1}\right)\right]\\
&=\left[1-{}_t\delta_t\left(h^t\right)\right]\cdot\\
&\cdot\left[{}_tb_t\left(h^t\right)+\sum_{s=1}^{\hat{T}}\left(\prod_{j=0}^{s-1}q_{t+j}\right)\sum_{h^{t+s}\in H^{t+s}}\zeta_{t+s}\left(h^{t+s}\right)\left(\left[1-{}_t\delta_{t+s}\left(h^{t+s}\right)\right]{}_tb_{t+s}\left(h^{t+s}\right)\right)\right]
\end{aligned}$$

where ${}_tb_t\left(h^t\right)=0$ if ${}_t\delta_t\left(h^t\right)=1$. When intermediaries live for two periods

$${}_t\mathcal{P}_t\left(h^t\right)=\left[1-{}_t\delta_t\left(h^t\right)\right]\left[{}_tb_t\left(h^t\right)+q_t\sum_{h_{t+1}\in H_{t+1}}\zeta_{t+1}\left(h^t,h_{t+1}\right)\left[1-{}_t\delta_t\left(h^t\right)\right]{}_tb_{t+1}\left(h^t,h_{t+1}\right)\right]$$

**Lemma 1** *In any equilibrium, for any $t$ and any contract offered by an intermediary born at date $t$, ${}_t\mathcal{P}_t\left(h^t\right)=0$ for all $h^t\in H^t$.*

**Proof of Lemma 1.** Suppose not. Clearly, ${}_t\mathcal{P}_t\left(h^t\right)>0$ for all $h^t$ is not possible since the intermediary would making negative profits. On the other hand if ${}_t\mathcal{P}_t\left(h^t\right)\leq 0$ for all $\theta^t$ with strict inequality for some, then a deviating intermediary can offer a contract which transfers a little more to some types and still continue to make positive profits. As a result these types will strictly prefer to sign with the deviating intermediary. Finally, suppose that there exists $h^t$ and $\hat{h}^t$ such that ${}_t\mathcal{P}_t\left(h^t\right)>0$ and ${}_t\mathcal{P}_t\left(\hat{h}^t\right)<0$. Then at the beginning of period $t$, consider a deviating intermediary offering the following contract,

$$\begin{aligned}
{}_t\tilde{\mathcal{P}}_t\left(\hat{h}^t\right)&={}_t\mathcal{P}_t\left(\hat{h}^t\right)+\varepsilon\\
{}_t\tilde{b}_{t+s}\left(h^t\right)&=0\text{ for all }s\geq 0,\ \text{ for }h^t\neq\hat{h}^t
\end{aligned}$$

where $\varepsilon>0$ and small. Notice that types $h^t$ strictly prefer the original contract while types $\hat{h}^t$ strictly prefer ${}_t\tilde{\mathcal{P}}_t$ to ${}_t\mathcal{P}_t$. As a result, these households will strictly prefer to sign with the deviating intermediary who makes a positive profit. ■

In particular, when intermediaries live for two periods, perfect competition implies that each two period contract must make zero profits in equilibrium.

To compute properties of the equilibria in the intermediary game, we will consider the limit of a sequence of truncated economies. In particular, I assume that there exists a finite date $T$, such that from $0 \leq t \leq T$, intermediaries offer contracts and for all $t > T$, those agents who have not defaulted in the past trade a risk free bond subject to exogenous debt constraints $\{\phi_t^e\}_{t>T}$. The claim that we can take such limits is formalized in the appendix.

In the intermediary game, given that types are being banished, we can define *banishment sets* as follows, $D_t\left(h^{t-1}\right) = \left\{h_t \in H_t : \ \delta_t\left(h^{t-1}, h_t\right) = 1\right\}$, i.e. the set of types being banished in equilibrium. Similarly, let $D_t^c\left(h^{t-1}\right)$ denote the complement of that set. Given a random variable $x\left(h^t\right)$, define $\mathbb{E}_{D_t^c(h^{T-1})} x\left(h^t\right) \equiv \sum_{h^t \in H^t} \wp\left(h^t\right)\left[1 - \delta_t\left(h^t\right)\right] x\left(h^t\right)$. The first main result required to prove Proposition 5 says that in equilibrium, households can never be savings constrained.

**Proposition 6** *In any equilibrium of the intermediary game, for all $t$ and $h^t \in H^t$,*

$$q_t \geq \frac{\beta R_{t+1} \mathbb{E}_{D_t^c(h^{t-1},h)} u'\left(c_{t+1}\left(h^{t-1}, h, h'\right)\right)}{u'\left(c_t\left(h^{t-1}, h\right)\right)}$$

**Proof.** See Appendix A. ■

If a household is savings constrained, a deviating intermediary has an incentive to offer it an uncontingent savings contract. Such a contract is always incentive compatible and trivially satisfies voluntary participation constraints.

The next key result required to prove Proposition 5 states that for all types not being banished, their continuation utility depends only on the sum $\theta + {}_{t-1}b_t\left(h^t\right)$.

**Proposition 7** *In equilibrium with two period lived intermediaries, for any $t$ and $h^t, \hat{h}^t$ such that $\delta_t\left(h^t\right) = \delta_t\left(\hat{h}^t\right) = 0$, if $\theta + {}_{t-1}b_t\left(h^t\right) = \hat{\theta} + {}_{t-1}b_t\left(\hat{h}^t\right)$, then*

$$V_t\left(h^t\right) = V_t\left(\hat{h}^t\right)$$

**Proof.** See Appendix A. ■

The result states that in equilibrium, the continuation value for any two types not being banished $h^t$ and $\tilde{h}^t$ such that $\theta_t + {}_{t-1}b_t = \tilde{\theta}_t + {}_{t-1}\tilde{b}_t$ is identical. Notice that here ${}_{t-1}b_t$ corresponds to the transfer in period $t$ from a contract signed in period $t-1$. I prove this using an induction argument. Given that we are working in a truncated economy, consider the last period $T$ in which intermediaries are operational. Since from period $T$ onwards households trade a risk-free bond, the household's value going forward depends on only its current endowment and transfer. Next, suppose the hypothesis is true from period $t+1$ onwards and so we want to establish that it is true in period $t$. For contradiction, suppose we have two histories such that $\theta_t + {}_{t-1}b_t = \theta_t + {}_{t-1}\tilde{b}_t$ but $V_t\left(h^t\right) > V_t\left(\tilde{h}^t\right)$. The idea of the proof is to show that a deviating intermediary can give agent $\tilde{h}^t$ a contract similar to type $h^t$, which makes both the household and it strictly better off while still satisfying incentives. The key condition that needs to be checked is that such a contract does not incentivize default the following period. Notice that household $\tilde{h}^t s$ incentives to default in period $t+1$ are exactly the same as household $h^t$ if they receive the same transfers since the value of the two households going forward is identical by the induction assumption.

An important consequence of the previous result is that the probability of re-entry after banishment in period $t$ is independent of current period reports and depends at most on ${}_{t-1}b_t$ , the period $t$ transfer from the contract signed in period $t-1$. This result will be important when we study an application of the framework to bankruptcy policy.

**Proposition 8** *In any equilibrium,*

1. *For all $t$ and $h^{t-1}$, if $\exists$ $h_t$ and $\hat{h}_t$ such that $\delta_t\left(h^{t-1}, h_t\right) = \delta_t\left(h^{t-1}, \hat{h}_t\right) = 1$ then $\mu_t\left(h^{t-1}, h_t\right) = \mu_t\left(h^{t-1}, \hat{h}_t\right)$.*

2. *If ${}_{t-1}b_{t-1}\left(h^{t-1}\right) = {}_{t-1}b_{t-1}\left(\hat{h}^{t-1}\right)$ for any two histories $h^{t-1}$ and $\hat{h}^{t-1}$ then $D_t\left(h^{t-1}\right) = D_t\left(\hat{h}^{t-1}\right)$ and $\mu_t\left(h^{t-1}\right) = \mu_t\left(\hat{h}^{t-1}\right)$*

3. *If ${}_{t-1}b_{t-1}\left(h^{t-1}\right) \geq {}_{t-1}b_{t-1}\left(\hat{h}^{t-1}\right)$ for any two histories $h^{t-1}$ and $\hat{h}^{t-1}$ then $D_t\left(h^{t-1}\right) \supseteq D_t\left(\hat{h}^{t-1}\right)$ and $\mu_t\left(h^{t-1}\right) \leq \mu_t\left(\hat{h}^{t-1}\right)$*

**Proof.** See Appendix A. ■

The proposition implies that re-entry probabilities for any history $h^t$ depend only on ${}_{t-1}b_t\left(h^{t-1}\right)$ for types not being banished in period $t$. In particular, the probability of re-entry after being banished in $t+1$ is decreasing in the transfer ${}_t b_t\left(h^t\right)$.

Since the value of not being banished depends only on the current shock $\theta$ and $b$, the banishment sets $D_t\left(h^{t-1}\right) = D_t\left(b_t^{t-1}\left(h^{t-1}\right)\right)$ and so in any equilibrium contract, the incentives to banish are only affected by the current shock $\theta_t$ and the transfer ${}_{t-1}b_t\left(h^t\right)$. Since transfers are bounded, there exists $\bar{\phi}, \phi^D$ such that

1. ${}_{t-1}b_t\left(h^t\right) < \bar{\phi}$
2. If ${}_{t-1}b_t\left(h^t\right) < \phi^D$, $D_t\left({}_{t-1}b_t\left(h^t\right)\right) = \varnothing$ and $\hat{R}_t\left({}_{t-1}b_t\left(h^t\right)\right) = R_t = \frac{1}{q_{t-1}}$
3. If ${}_{t-1}b_t\left(h^t\right) \geq \phi^D$, $D_t\left({}_{t-1}b_t\left(h^t\right)\right) \neq \varnothing$ and $\hat{R}_t\left({}_{t-1}b_t\left(h^t\right)\right) = \frac{R_t\,{}_{t-1}b_t\left(\theta^t\right)}{\sum_{h \notin D_t\left({}_{t-1}b_t\left(h^{t-1}\right)\right)} \wp\left(h^{t-1}, h\right)}$

In particular, the intermediary only banishes households in states in which transfers are low and households have some incentive to voluntarily default. Since the contracts for those not being banished are still simple borrowing and lending contracts of the form $(b, -Rb)$ where $Rb$ is independent of current announced type, the household being banished always has the option of lying and not being banished. This idea is made more concrete in the proof of the equivalence result where we see that in a formal sense, banished is equivalent to default by the household.

Using these results, we can now prove the equivalence result, Proposition 5 (see Appendix A. for the proof). The result is a consequence of the characterization results proved earlier. In summary, we showed that equilibrium contracts of the intermediary game when banished is allowed and intermediaries are two period lived, resembled short term defaultable debt. These turn out to be exactly the types of contracts that households are assumed to be able to sign in an EG model. Since in the contracting environment, I allow intermediaries to stochastically allow households back in after being banished, the actual contract resembles short term defaultable debt with stochastic re-entry. In particular, if intermediaries are not allowed to bring households back, then the equivalence would hold for an environment in which after default, households are in autarky forever.

As a final point about the equivalence result, it is worth noting that the first-order condition of the household's problem in the EG environment is also satisfied in the equilibrium of the intermediary game. In the case with a continuous[15] type space, in EG, the first

[15] With a discrete state-space like the one assumed in this paper, the above condition might not always be well defined.

order condition for the household's problem is[16]

$$u'(c(\theta, s)) Q'(s') = \beta \mathbb{E}_t d_{t+1}(\theta', s') u'(c(\theta', s')) \tag{2.14}$$

While in the contracting environment the object $Q'(s')$ doesn't show up directly, it is captured by the multipliers on the incentive/participation constraints. In particular, the first order condition is

$$u'\left(c_t\left(h^t\right)\right) q_t = \beta \mathbb{E}_t\left[1 - \delta_{t+1}\left(h^{t+1}\right)\right] u'\left(c_{t+1}\left(h^{t+1}\right)\right) + \mathbb{E}_t \nu_{t+1}\left(h^{t+1}\right) u'\left(c_{t+1}\left(h^{t+1}\right)\right)$$

where $\nu_{t+1}\left(h^{t+1}\right)$ denotes the incentive constraints on incentive compatibility constraints in period $t+1$. The above equation can be rewritten in the form of (2.14) with $Q' = q_t \left(1 + \frac{\mathbb{E}_t \nu_{t+1}\left(h^{t+1}\right) u'\left(c_{t+1}\left(h^{t+1}\right)\right)}{\beta \mathbb{E}_t\left[1-\delta_{t+1}\left(h^{t+1}\right)\right] u'\left(c_{t+1}\left(h^{t+1}\right)\right)}\right)^{-1}$.

**Existence:** The presence of re-entry choices on the part of intermediaries make the general existence problem quite hard. In Appendix A, using recent results by Auclert and Rognlie (2014), I prove a more limited existence theorem in the case in which intermediaries cannot allow households to re-renter. However, it easy to show that autarky is always an equilibrium of the intermediary game. This suggests that in general, the environment has multiple equilibria.

**Lemma 2** *There exists an equilibrium in which for all $t$ and $h^t \in H^t$, $b_t\left(h^t\right) = 0$*

**Proof.** Suppose all intermediaries offer null contracts i.e. contracts in which transfers are zero in all dates and for all histories. Suppose further that the price in each period $q_t \geq \max_{\theta \in \Theta} \frac{\mathbb{E}u'(\theta')}{u(\theta)}$. It is easy to see that this is an equilibrium of the intermediary game. No intermediary has an incentive to lend to households since given that there is no borrowing and lending in the future, they will choose to default at some future date. On the other hand since $q_t \geq \max_{\theta \in \Theta} \frac{\mathbb{E}u'(\theta')}{u(\theta)}$, no household wishes to save at interest rates $\frac{1}{q_t}$ and hence there exists no profitable savings contract that an intermediary can offer. ■

**Empirical estimates of re-entry probabilities:** In a recent paper, Cruces and Trebesch (2013) construct a new database of haircut estimates for sovereign debt restructurings from 1970 until 2010. A key finding of their paper is that higher haircuts are

[16] Note that $Q(s)$ denotes the price times the debt. If $\tilde{Q}(s)$ was the price, then the term on the left hand side would be $\tilde{Q}'(s)s + \tilde{Q}(s)$.

associated with larger spreads and longer duration of banishment from capital markets. In particular, their data analysis shows that partial re-access to capital markets takes 2.3 years on average for a haircut size less than 30 percent while for haircuts larger than 30 percent, the average duration of banishment more than doubles to 6.1 years. While most models of sovereign default assume a constant probability of re-entry after default, in the environment I consider, re-entry probabilities that depend on the level of defaulted debt arise as part of the profit maximizing contract. While this model features haircuts of a 100 percent, it is still true that the probability of re-entry is weakly decreasing in the level of defaulted debt.

## 2.5 Efficiency

The first step in asking whether the equilibria characterized in the previous sections are efficient is to define the right notion of constrained-efficiency. In environments with private information and limited commitment this is well understood and has been studied by Prescott and Townsend (1984) and Kehoe and Levine (1993). However, the definition of constrained-efficiency is less clear in environments with non-exclusive contracts.

To begin, I consider a setup with a fictitious social planner and continuum of infinitely lived households who receive an unobservable perishable endowment each period. An important feature of the planning environment is that as in the intermediary game, I will allow the planner to temporarily banish households from the mechanism. As a result, we use the same expanded type space to take into account periods of banishment.

An allocation for the planner consists of a sequence $\left\{\delta_t\left(h^t\right), \mu_t\left(h^t\right), c_t\left(\omega^t\right), b_t\left(h^t\right)\right\}_{t\geq 0, h^t\in H^t}$. The first term $\delta_t\left(h^t\right) \in \{0,1\}$ corresponds to an banishment index which indicates if the household is part of the mechanism or not. If the household is not in some period $t$, it cannot receive any transfers from the planner and is also banished from trading in any hidden markets, which will be defined shortly. The planner still keeps track of banished households and can let them back into the mechanism at some future date. The next term $\mu_t\left(h^t\right) \in [0,1]$ corresponds to the re-entry probability chosen by the planner after the agent has been banished. Note that after banishment there is no reporting of types and so the re-entry probability can only depend on the last type reported before banishment. The next two terms correspond to the consumption and transfer sequences to households in the mechanism.

An allocation is *incentive-feasible* if it satisfies the following conditions. First, it must be resource feasible; for each $t$,

$$\sum_{h^t \in H^t} \wp\left(\omega^t\right) c_t\left(\omega^t\right) = \sum_{h^t \in H^t} \wp\left(\omega^t\right) \theta_t$$
$$\delta_t\left(h^t\right) c_t\left(\omega^t\right) = \delta_t\left(h^t\right) \theta_t \tag{2.15}$$

Here the second equation $\delta_t\left(h^t\right) c_t\left(h^t\right) = \delta_t\left(h^t\right) \theta_t$ corresponds to the restriction that all banished households consume their endowment.

Next, the contract must satisfy voluntary participation constraints: for all $t$ and $h \in H^t$,

$$\left[1 - \delta_t\left(h^t\right)\right] V_t\left(h^t\right) \geq \left[1 - \delta_t\left(h^t\right)\right] V_t^d\left(h^t\right) \tag{2.16}$$

I assume that at the beginning of each date, each household can voluntarily default on the planner and consequently live in autarky forever. In autarky, the household consumes its endowment each period. Note that without loss of generality we can restrict attention to outcomes in which the planner does all the banishment and household never voluntarily defaults. Next, the allocation must be incentive compatible

$$V_t\left(h^t\right)\left(\sigma^*\right) \geq \hat{V}_t\left(h^t, \{b\}, \{q\}\right)(\sigma), \tag{2.17}$$

Here $V_t\left(h^t\right)\left(\sigma^*\right)$ denotes the value of the contract to type $h^t$ of following truth-telling strategy $\sigma^*$. $\hat{V}_t\left(h^t, \{b\}, \{q\}\right)(\sigma)$ denotes the value to the household of using reporting strategy $\sigma$ and trading in a *hidden market*. We need to consider two types of hidden markets depending on whether the planner is allowed to banish households or not.

### 2.5.1 Efficiency without Banishment

First, as in the intermediary environment, I restrict the planner to only offer contracts without banishment. Given this, we can restrict ourselves to the usual type spaces $\Theta^t$. In this case, I consider a hidden market in which households can trade a risk free bond subject to *endogenous* debt constraints. Therefore,

$$\hat{V}_t\left(\theta^t;\{b\},\{q\}\right)(\sigma)=\max\sum_{s=0}^{\infty}\beta^s\sum_{\theta^{t+s}\in\Theta^{t+s}}\pi\left(\theta^t\right)u\left(x_{t+s}\left(\theta^{t+s}\right)\right)$$

$$\begin{aligned}&\text{subject to for all } s\geq 0,\ h^{t+s}\\&x_{t+s}\left(\theta^{t+s}\right)+q_{t+s}s_{t+s+1}\left(\theta^{t+s}\right)\geq\theta_{t+s}+b_{t+s}\left(\sigma_{t+s}\left(\theta^{t+s}\right)\right)+s_{t+s}\left(\theta^{t+s-1}\right)\\&s_{t+s+1}\left(\theta^{t+s}\right)\geq-\phi_{t+s}\end{aligned}$$

Here $b_{t+s}\left(\sigma_{t+s}\left(\theta^{t+s}\right)\right)$ denotes the transfer from the planner when type $\theta^{t+s}$ reports $\sigma_{t+s}\left(\theta^{t+s}\right)$, $s_{t+s+1}\left(\theta^{t+s}\right)$, the amount the household saves in period $t+s$ and $\phi_{t+s}$, the debt constraints. We can rewrite $\hat{V}\left(\theta^t;\{b\},\{q\}\right)$ as

$$J_t\left(\theta^t,s_t;\{b\},\{q\},\Phi_t\right)=\max u\left(x_t\right)+\beta\mathbb{E}_tJ_t\left(\theta^{t+1},s_{t+1};\{b\},\{q\},\Phi_t\right)$$

$$\begin{aligned}&\text{subject to}\\&x_t+q_ts_{t+1}\geq\theta_t+b_t\left(\theta^t\right)+s_t\\&s_{t+1}\geq-\phi_t\end{aligned}$$

where $\Phi_t$ denotes the sequence of current and future debt constraints which each household takes as given.

**Definition 7** *An equilibrium in the hidden market given a transfer sequence $\{b\}$ consists of prices $\{q_t\}$, allocations $\{x_t,s_t\}$ and debt constraints $\{\phi_t\}$ such that*

1. *Households solve their problem defined above,*

2. *Markets clear: for all $t$,*

$$\sum_{h^t\in H^t}\pi\left(\theta^t\right)x_t\left(\theta^t\right)=\sum_{h^t\in H^t}\pi\left(\theta^t\right)\left[\theta_t+b_t\left(\theta^t\right)\right]$$

3. *Debt constraints are chosen to be Not-Too-Tight, i.e.*

$$\begin{aligned}&J_t\left(\theta^t,-\phi_t;\{b\},\{q\},\Phi\right)\geq V_t^d\left(\theta^t\right)\ \textit{for all } \theta^t\\&J_t\left(\hat{\theta}^t,-\phi_t;\{b\},\{q\},\Phi\right)=V_t^d\left(\hat{\theta}^t\right)\ \textit{for some } \hat{\theta}^t\end{aligned}$$

The definition of the hidden market is similar in spirit to Golosov and Tsyvinski (2007). In their model, agents traded a risk free bond with the interest rate determined in equilibrium. Here, households trade these bonds subject to debt constraints which along with the interest rates are also determined in equilibrium. I assume that households can also default on their hidden debt obligations. As in the intermediary game, default in the hidden markets is publicly observable and consequently households live in autarky in all future periods. Debt constraints are chosen in equilibrium so that all households weakly prefer not to default on their debt if they have borrowed up to the debt limit the previous period while some household is indifferent between the two options. It is clear that in any constrained-efficient allocation, there will be no trade in these markets. In particular, the efficient allocation will have the property that for any Euler-constrained household, borrowing more in the hidden market will incentivize default the next period. Moreover the price $q_t$ will be such that no household will wish to save in these markets and as a result we a have a well defined equilibrium with no hidden trades.

The idea behind modelling the hidden market this way is as follows: suppose after receiving transfers from the planner, households could sign contracts in a hidden fashion with a continuum of hidden intermediaries subject to incentive and voluntary participation constraints. This game is identical to the one studied in the previous sections and we know that in the case in which intermediaries are not allowed to banish agents, equilibrium contracts are equivalent to trading an uncontingent bond subject to debt constraints.

The main result in this subsection is that the efficient allocation in the case with no banishment can be decentralized as an equilibrium of the intermediary game with no banishment.

**Theorem 3** (Efficiency: No banishment) *The constrained efficient allocation without banishment can be implemented as an equilibrium of the intermediary game without banishment.*

To prove this is result, I first prove properties that any efficient allocation must satisfy. In particular, I show that the planner cannot do better than simple borrowing and lending contracts. Then, I show that if all intermediaries are offering the efficient contract, no incumbent or new intermediary has any incentive to offer a deviating contract.

As in the intermediary game I consider limits of $T-$period truncated environments in which from period 1 to $T$, the planner provides transfers and after $T$ those households that have not defaulted can trade a risk-free bond subject to exogenous debt constraints.

**Proposition 9** *Any $T-$period truncated incentive feasible allocation must satisfy*

$$q_t \geq \beta \frac{\mathbb{E}_{t+1} u'\left(c_{t+1}\left(\theta^{t+1}\right)\right)}{u'\left(c_t\left(\theta^t\right)\right)} \text{ for all } t, \theta^t \in \Theta^t$$

*and*

$$\sum_{t=1}^{T}\left(\prod_{s=1}^{t} q_s\right) b_t\left(\theta^T\right) = 0 \text{ for all } \theta^T \in \Theta^T \tag{2.18}$$

**Proof.** See Appendix A. ■

Notice that the proposition says that in the case with no banishment, the efficient contract is also a simple borrowing and lending contract subject to debt constraints. In particular, the presence of the hidden markets prevents the planner from introducing state-contingency in contracts. The intuition for this is exactly the same as in the intermediary game. If lower types receive a larger present discounted value of transfers, then higher types will lie and use the hidden markets to save. On the other hand if higher types receive a larger present discounted value of transfers then redistribution is welfare increasing. Given that voluntary participation constraints induce some agents to be Euler-constrained in the efficient allocation, the planner will allow agents to borrow the largest amount consistent with no default in the subsequent period. As a result the voluntary participation constraints will be binding for some type in the following period.

These two conditions imply that as in the intermediary game, the efficient contract are simple uncontingent borrowing and lending contracts. The next result provides necessary and sufficient conditions for an allocation to induce an equilibrium of the hidden market with no trades.

**Lemma 3** *A no-banishment allocation induces no trades in the hidden market if and only if for all $t, \theta^t \in \Theta^t$*

$$q_t \geq \beta \frac{\mathbb{E}_{t+1} u'\left(c_{t+1}\left(\theta^{t+1}\right)\right)}{u'\left(c_t\left(\theta^t\right)\right)} \text{ for all } t, \theta^t \in \Theta^t \tag{2.19}$$

*and*

$$\left[q_t - \frac{\beta \mathbb{E}_t u'\left(c_{t+1}\left(\theta^{t+1}\right)\right)}{u'\left(c_t\left(\theta^t\right)\right)}\right] \min_{\tilde{\theta}^{t+1} \in \tilde{\Theta}^{t+1}} \left[V_{t+1}\left(\tilde{\theta}^{t+1}\right) - V_{t+1}^d\left(\tilde{\theta}^{t+1}\right)\right] = 0 \tag{2.20}$$

**Proof.** See Appendix A. ■

The second condition says that if a household is Euler-constrained in period $t$, then it must be that in the following period, the voluntary participation for some type binds. The reason for this is if not then, debt constraints in the hidden market will satisfy the Not-too-tight property. In other words intermediaries will be willing to lend more to agents without fearing default in the subsequent periods.

Next, as in Golosov and Tsyvinski (2007) we can re-write the planner's problem with no banishment as one in which the planner also chooses the prices in the hidden markets subject to additional conditions. Given that we are first restricting the planner to offer allocations without banishment/default, and $\delta_t\left(h^t\right) = 0$ for all $t, h^t \in H^t,$ an allocation in this case is a sequence of transfers $\left\{b_t\left(\theta^t\right)\right\}_{t\geq 0, h^t \in H^t}$ and prices $\{q_t\}_{t\geq 0}$. In this case, (2.15), (2.16) and (2.17) simplify to

$$\sum_{\theta^t \in \Theta^t} \pi\left(\theta^t\right) c_t\left(\theta^t\right) = \sum_{\theta^t \in \Theta^t} \pi\left(\theta^t\right) \theta_t \text{ for all } t \tag{2.21}$$

$$V_t\left(\theta^t\right)\left(\sigma^*\right) \geq \hat{V}_t\left(\theta^t, \{b\}, \{q\}\right)(\sigma) \text{ for all } t, \theta^t \in \Theta^t \tag{2.22}$$

$$V_t\left(\theta^t\right) \geq V_t^d\left(\theta_t; \lambda\right) \text{ for all } t, \theta^t \in \Theta^t \tag{2.23}$$

**Lemma 4** *The constrained efficient allocation* $\left\{c_t\left(h^t\right), b_t\left(h^t\right)\right\}_{t\geq 0, h^t \in H^t}$ *and prices* $\{q_t\}_{t\geq 0}$ *is a solution to the following programming problem*

$$\max_{\{c,b,q\}} \sum_{t=1}^{T} \beta^{t-1} \sum_{\theta^t \in \Theta^t} \pi\left(\theta^t\right) u\left(c_t\left(\theta^t\right)\right)$$

*subject to (2.21), (2.18), (2.23), (2.19), and (2.20).*

To prove Theorem 3, I show that if all intermediaries are offering the efficient contract, no individual intermediary has an incentive to deviate and offer a different contract. In particular, it will not be able to offer some Euler-constrained individuals the option to borrow more since they will default the following period. This establishes that the efficient allocation can be decentralized as an equilibrium of the intermediary game. Note that even though as in Golosov and Tsyvinski (2007), the planner controls the price in the hidden market, he is unable to achieve outcomes better than the best competitive equilibrium. The planner chooses $q_t$ consistent with best competitive equilibrium from the set of equilibria which we know is not a singleton. The reason for this is that incentive compatibility dictates that in any incentive feasible allocation no state contingency is possible. As a result, the best the planner can do is to choose the allocation that corresponds to loosest borrowing constraints which in turn corresponds to the best competitive equilibrium. Unlike Golosov and Tsyvinski (2007), in this model output is not publicly observable. Therefore, the planner cannot use incentives to work to provide state-contingency in contracts as in their paper.

### 2.5.2 Efficiency with Banishment

Consider a planner who is also allowed to banish households and set re-entry probabilities in all future periods. I restrict the planner to only offer two period contracts as in the intermediary game.[17] I assume that households can sign two period contracts with intermediaries in a hidden market. The equilibrium of the hidden market is identical to that described in section 3, taking into account the transfers from the planner. In particular, I assume that intermediaries can offer contracts that *banish households from the hidden market*. If such a household is not banished by the planner, it can continue to receive transfers from the planner but cannot take part in the hidden market. From Proposition 5, we know that any equilibrium contract in the hidden market will be a short-term defaultable debt contract where default constitutes banishment from the hidden market with a chosen re-entry probability. As a result, we can restrict to deviating contracts of the form $\mathbb{D}_t\left(h^t\right)=\left(z_t\left(h^t\right),z_{t+1}\left(h^t\right),\delta^H_{t+1}\left(h^{t+1}\right),\mu^H\left(h^t\right)\right)$ which consists of transfers in period $t,\ t+1$, banishment indices for the hidden market and re-entry probabilities.

Define $\mathcal{V}_t\left(h^t\right)$ to be continuation value for a household of type $h^t$ when it has access to the planner's transfers and the hidden market, $\mathcal{V}^E_t\left(h^t;\mu,\mu^H\right)$ the value for a household banished from the planning problem, and $\mathcal{V}^N_t\left(h^t;\mu^H\right)$, the continuation value for $h^t$ if it is only banished from the hidden market (and not by the planner). In particular

[17] Without this restriction the planner can do better. However, the point of this exercise is to illustrate that the standard pecuniary externality agrument when markets are exogenously incomplete no longer holds with hidden markets of the form described in this section. A full characterization is in progress.

$$\mathcal{V}_t^E\left(h^t;\mu,\mu^H\right)$$
$$= u\left(\theta_t\right) + \beta\mathbb{E}_t\left[\mu\left[\mu^H\mathcal{V}_{t+1}\left(h_t\right) + \left(1-\mu^H\right)\mathcal{V}_{t+1}^N\left(h_{t+1};\mu^H\right)\right] + \left(1-\mu\right)\mathcal{V}_{t+1}^E\left(h^{t+1};\mu,\mu^H\right)\right] \tag{2.24}$$

and

$$\mathcal{V}_t^N\left(h^t;\mu^H\right) = u\left(\theta_t\right) + \beta\mathbb{E}_t\left[\mu^H\mathcal{V}_{t+1}\left(h^{t+1}\right) + \left(1-\mu^H\right)\mathcal{V}_{t+1}^N\left(h^{t+1};\mu^H\right)\right] \tag{2.25}$$

Notice that $\mathcal{V}_t\left(h^t\right)$ will only differ from $V_t\left(h^t\right)$ when the household is trading in the hidden market.

The main result of this subsection is that the efficient allocation with banishment can be implemented as an equilibrium of the intermediary game with two period lived intermediaries and banishment.

**Proposition 10** (Efficiency: With Banishment) *The constrained efficient two-period allocation with banishment can be implemented as an equilibrium of the intermediary game with two period lived intermediaries and banishment.*

To prove the result, I first prove characterization results about the efficient contract and then show that if intermediaries offer such a contract no profitable deviation exists. As in the intermediary game I consider limits of $T-$period truncated environments in which from period 1 to $T$, the planner provides transfers and after $T$ those households that have not defaulted can trade a risk-free bond subject to exogenous debt constraints. In the appendix I show that we can take such limits.

**Proposition 11** *Any $T-$period truncated incentive feasible allocation must satisfy*

$$q_t \geq \beta\frac{\mathbb{E}_{t+1}\left[1-\delta_{t+1}\left(h^{t+1}\right)\right]u'\left(c_{t+1}\left(\omega^{t+1}\right)\right)}{u'\left(c_t\left(\omega^t\right)\right)} \text{ for all } t, h^t\in H^t$$

*and*

$$\sum_{t=1}^{T}\left(\prod_{s=1}^{t}\hat{q}_s\left(h^s\right)\left[1-\delta_s\left(h^s\right)\right]\right)b_t\left(h^T\right) = 0 \text{ for all } \theta^T\in\Theta^T \tag{2.26}$$

*where*

$$\hat{q}_s\left(h^s\right) = q_s\sum_{h_{s+1}\in H_{s+1}}\zeta_{s+1}\left(h^s,h_{s+1}\right)\delta_{s+1}\left(h^s,h_{s+1}\right)$$

**Proof.** See Appendix A. ■

The proposition says that the efficient contract is also a simple short term defaultable debt contract. In particular, the presence of the hidden markets and the fact that the planner can only offer two period contracts prevents the planner from introducing state-contingency in contracts beyond banishment. The intuition for this is similar to that in the intermediary game.

Since we are modelling the hidden market as one in which households can transact with intermediaries, given earlier results about the nature of the equilibrium contracts we need only consider short-term deviating contracts of the form $\left(z_t\left(h^t\right), z_{t+1}\left(h^t\right), \delta_{t+1}^H\left(h^{t+1}\right), \mu^H\left(h^{t+1}\right)\right)$ where $z_t\left(h^t\right)$ and $z_{t+1}\left(h^t\right)$ denote the transfers specified by the hidden contracts, $\delta_{t+1}^H\left(h^{t+1}\right)$, whether the household is banished from the hidden market and $\mu^H$ denotes the probability of re-entry to the hidden market. As I will show, there are two types of deviating contracts to consider. The first is a simple savings contract. The second is a short term defaultable debt contract.

**Proposition 12** *An allocation induces no trades in the hidden market if and only if for all $t, h^t \in h^t$*

$$q_t \geq \beta \frac{\mathbb{E}_{t+1}\left[1-\delta_{t+1}\left(h^{t+1}\right)\right] u'\left(c_{t+1}\left(h^{t+1}\right)\right)}{u'\left(c_t\left(h^t\right)\right)} \text{ for all } t, h^t \in H^t \tag{2.27}$$

*and*

$$u'\left(\theta_t+b_t\left(h^t\right)\right) \hat{q}_t\left(h^t\right) \leq \beta u'\left(\hat{\theta}_{t+1}+b_{t+1}\left(\hat{h}^{t+1}\right)\right) \text{ for all } t, h^t \in H^t \tag{2.28}$$

*where $\hat{q}_t\left(h^t\right)=q_t \sum_{t+1}\left(1-\delta_{t+1}\left(h^{t+1}\right)\right)$ and $\hat{h}^{t+1}$ is such that*

$$V_{t+1}\left(\hat{h}^{t+1}\right)-\mathcal{V}_t^E\left(\hat{h}^{t+1} ; \mu\right) \leq V_{t+1}\left(h^{t+1}\right)-\mathcal{V}_t^E\left(h^{t+1} ; \mu\right) \ \forall h^{t+1} \text{ such that } \delta_{t+1}\left(h^{t+1}\right)=0$$

**Proof.** See Appendix A. ■

We can use the fact that in equilibrium these deviations must be short-term contracts to greatly simplify the types of deviating contracts. As mentioned earlier, the first a simple savings contract that ensures that the in the efficient allocation, no household can be savings constrained. The second type of deviation involves a debt contract which allows the household to borrow a little more in the current period and in the following period, some types are banished from the hidden markets. To understand the intuition for this result, consider the case in which as part of the efficient allocation, for some type $h^t$, there exists a set of $h^{t+1}$ that is being banished by the planner in $t+1$. Let $\hat{h}^{t+1}$ correspond to the type with the smallest $\theta_{t+1}$ not being banished. Suppose that the planner's allocation satisfies $u'\left(\theta_t + b_t\left(h^t\right)\right)\hat{q}_t\left(h^t\right) > \beta u'\left(\hat{\theta}_{t+1} + b_{t+1}\left(\hat{h}^{t+1}\right)\right)$. Then I prove that there exists such a deviating contract that gives the household strictly higher utility and the deviating intermediary breaks even. In this contract, households receive a positive transfer $\hat{q}_t\varepsilon$ in period $t$, a negative transfer in all non-banished states, and and a probability $\mu^H$ that that the household will be allowed to trade in the hidden market even after it can receive transfers from the planner.

Given these characterization results, as in the no-banishment case we can simplify the constrained-efficient programming problem.

**Proposition 13** *The constrained efficient allocation with banishment is the solution to*

$$\max_{\{\delta,\mu,c,b,q\}} \sum_{t=1}^{T} \beta^{t-1} \sum_{h^t \in H^t} \wp\left(h^t\right) u\left(c_t\left(h^t\right)\right)$$

*subject to (2.15), (2.16), (2.26),(2.27) and (2.28).*

To prove Proposition 10, I ask if there exists a profitable deviation if all intermediaries are offering the efficient contract. Since these deviating contracts cannot be state-contingent, the two types of deviating contracts we need to consider are simple savings contracts and one in which intermediary transfers more in the current period and potentially banishes more types the following period. However, given that the efficient contract satisfies the conditions in Proposition 12, both these deviations can never be profitable.

**Stochastic Re-entry after voluntary default:** We can easily modify the voluntary participation constraints in the planning problem to allow for stochastic re-entry after default. In particular, if households default, they are allowed back into the mechanism with probability $\lambda$ each period. Note that this is feature of the technology and in general different from the re-entry probabilities the planner sets after banishment. Denote above planning problem when the default punishment is parameterized by $\lambda$ by $\mathbb{P}\left(\lambda\right)$ and the set of feasible allocation-price pairs as $Feas\left(\lambda\right)$. Let $x^*\left(\lambda\right) \in Feas\left(\lambda\right)$ denote the constrained-efficient allocation-price pair when the punishment is $\lambda$ and $W\left(x^*\left(\lambda\right)\right)$, the ex-ante welfare of the planning problem. The following result is immediate.

**Lemma 5** *If $\lambda' > \lambda$ then $Feas(\lambda') \subseteq Feas(\lambda)$ and $W(x^*(\lambda)) \geq W(x^*(\lambda'))$*

**Proof.** It is straightforward to notice that any $x \in Feas(\lambda')$, satisfies all the constraints in $Feas(\lambda)$ since $V_t^d(\theta_t; \lambda) < V_t^d(\theta_t; \lambda')$. Therefore $x \in Feas(\lambda)$. It is follows that $W(x^*(\lambda)) \geq W(x^*(\lambda'))$ ■

In particular the solution to the constrained-efficient planning problem must satisfy $W(x^*(0)) \geq W(x^*(\lambda))$ for all $\lambda \in [0, 1]$.

While in general characterizing the re-entry decisions is difficult, under some sufficient conditions, if the planner banishes a household, it is never let back in

**Definition 8** *An allocation-price pair $x(\lambda) \in Feas(\lambda)$ is $\mathcal{E}$-constrained if for all $t$ and histories $h^t$, if $\delta_t(h^{t-1}) = 0$ and $\delta_t(h^t) = 1$, then there exists $\tilde{h}$ such that $\delta_t\left(h^{t-1}, \tilde{h}\right) = 0$ and $V_t\left(h^{t-1}, \tilde{h}\right) = V_t^d\left(h^{t-1}, \tilde{h}; \lambda\right)$*

$\mathcal{E}$-constrained allocations are important to the subsequent results since we will show that it is exactly these allocations which can be decentralized as equilibria of an Eaton and Gersovitz (1981) environment. They key argument will rely on the fact that in any $\mathcal{E}$-constrained allocation, the planner will not bring back any banished agent with probability greater than $\lambda$.

**Proposition 14** *If a solution to $\mathcal{P}(\lambda)$ is $\mathcal{E}$-constrained , then for any $t$ and history $h^t$, if $\delta_{t-1}(h^{t-1}) = 0$ and $\delta_t(h^t) = 1$, $\mu_{t+s}(h^t) \leq \lambda$ for all $s > 1$*

**Proof.** Proof: Suppose we have such an $\mathcal{E}$-constrained solution and consider some $t$ and type $h^t$ such that $\delta_t(h^t) = 1$. By assumption, it must be that for some $\left(h^{t-1}, \tilde{h}\right)$ such that $\delta_t\left(h^{t-1}, \tilde{h}\right) = 0$, $V_t\left(h^{t-1}, \tilde{h}\right) = V_t^d\left(h^{t-1}, \tilde{h}; \lambda\right)$. Suppose now that $\mu_{t+s}(h^t) > \lambda$ for some $s > 1$. Notice that type $\left(h^{t-1}, \tilde{h}\right)$ will strictly prefer to lie and pretend to be type $\left(\tilde{\theta}^{t-2}, \underline{\theta}, \tilde{\theta}_t\right)$ since it will receive a value greater than that of defaulting, $V_t^d\left(h^{t-1}, \tilde{h}; \lambda\right)$. As a result, incentive compatibility constraints are violated. ■

The conditions guaranteeing that an allocation is $\mathcal{E}$-constrained can be seen more intuitively in a simple two state example with $\Theta = \left\{\theta^l, \theta^h\right\}$. Here, an allocation-price pair is $\mathcal{E}$-constrained if for all $t$, $\theta^{t-1}$

$$c_t\left(\theta^{t-1},\theta^l\right) \leq c_{t+1}\left(\theta^{t-1},\theta^l,\theta^h\right)$$

and

$$R_t = \frac{u'\left(c_t\left(\theta^{t-1},\theta^h\right)\right)}{\beta\left[\pi u'\left(c_{t+1}\left(\theta^{t-1},\theta^h,\theta^h\right)\right) + (1-\pi)\,u'\left(c_{t+1}\left(\theta^{t-1},\theta^h,\theta^l\right)\right)\right]} < \frac{1}{\beta}$$

Suppose $\delta_{t+1}\left(\theta^{t-1},\theta^l,\theta^l\right) = 1$ so that the planner banishes type $\left(\theta^{t-1},\theta^l,\theta^l\right)$ in period $t+1$. It must be that

$$u'\left(c_t\left(\theta^{t-1},\theta^l\right)\right) \geq \beta\frac{R_{t+1}}{\pi}\pi u'\left(c_{t+1}\left(\theta^{t-1},\theta^l,\theta^h\right)\right)$$

Notice that if the above equation held with an equality then

$$u'\left(c_t\left(\theta^{t-1},\theta^l\right)\right) = \beta R_{t+1}u'\left(c_{t+1}\left(\theta^{t-1},\theta^l,\theta^h\right)\right) < u'\left(c_{t+1}\left(\theta^{t-1},\theta^l,\theta^h\right)\right)$$

so that $c_t\left(\theta^{t-1},\theta^l\right) > c_{t+1}\left(\theta^{t-1},\theta^l,\theta^h\right)$ which is a contradiction. Therefore an $\mathcal{E}$-constrained allocation if $\delta_{t+1}\left(\theta^{t+1}\right) = 1$, then there must exist some $\tilde{\theta}$ such that $\delta_{t+1}\left(\theta^t,\tilde{\theta}\right) = 0$ and $V_{t+1}\left(\theta^t,\tilde{\theta}\right) = V_{t+1}^d\left(\theta^t,\tilde{\theta}\right)$. Suppose now that $\mu_{t+s}\left(\theta^{t+s}\right) > \lambda$ for some $s > 1$. Then notice that type $\left(\theta^t,\tilde{\theta}\right)$ will strictly prefer to lie and pretend to be type $\left(\theta^t,\theta\right)$ since he will receive a value greater than that of default which violated incentive compatibility.

Recall that $x^*(0)$ is the solution to the constrained-efficient planning problem when $\lambda = 0$. The next main result shows that if the solution to $\mathbb{P}(\lambda)$ is $\mathcal{E}$-constrained, then it can be decentralized as an equilibrium of the EG environment.

**Lemma 6** *If the solution $x^*(0)$ is $\mathcal{E}$-constrained, then it can be implemented as an equilibrium of the intermediary game with default punishment being autarky with re-entry probability $\lambda = 0$.*

**Proof.** Follows directly from Proposition 10 ■

The following corollary is immediate from the previous result and Proposition 5.

**Corollary 4** *If the solution $x^*(0)$ is $\mathcal{E}$-constrained, then it can be implemented as an equilibrium of the EG environment default punishment being autarky with re-entry probability $\lambda = 0$.*

As in the intermediary game, the presence of hidden trading opportunities severely limits the amount of insurance the planner can provide. As a result, the planner will choose to banish some types in order to sustain greater ex-ante risk sharing. The planner always has the option of bring this household back in a later period. However we know that this is never the case in any $\mathcal{E}$-constrained allocation from Proposition 14. Therefore, the planner will only choose to banish some type $\theta^t$ in period $t$ if $V_t\left(\theta^t\right) < V_t^d\left(\theta_t;\lambda\right)$. This allows us to implement the allocation in a decentralized environment in which the household voluntarily chooses to default in period $t$ and consequently live in autarky forever.

### 2.5.3 Efficiency with Exogenous Incompleteness

A general result when markets are exogenously incomplete is that equilibrium outcomes are constrained inefficient. This literature considers a planner who restricted from making state-contingent transfers to agents but internalizes the effect of its allocations on prices. Geanakoplos and Polemarchakis (1986) find that equilibrium outcomes are generically inefficient in an exchange economy with multiple goods. In particular, they find that aggregate welfare can be increased if households are induced to save different amounts. More recently, Dávila et al. (2012) find that the equilibria in the model studied by Aiyagari (1994) are also constrained inefficient. Consumers do not internalize the effects of their choices on factor prices which in a model with uninsurable risk implies that there can be oversaving or undersaving relative to the constrained efficient equilibrium.

While the environment I consider is observationally equivalent to a large class of exogenously incomplete models, the approach to efficiency I take in the case with no banishment is substantially different. Rather than exogenously restrict the set of instruments available to the planner, I derive the incompleteness as a consequence of informational and commitment frictions. In this section, I explore whether for two observationally equivalent models, the two notions have different implications for whether the competitive equilibria are efficient. As I show using a simple example, it is possible that outcomes that are considered inefficient when markets are exogenously incomplete are no longer so when they are endogenously incomplete.

Consider a simple two period environment with $t = 1, 2$ and a continuum of households. In period 1, households can receive endowment shocks $\theta_i \in \Theta = (\theta_h, \theta_l)$ with probability

$\pi_i,\ i \in \{h, l\}$. In period 2, households receive endowment shocks $x_i \in \mathbb{X} = (x_h, x_l)$ with probability $\kappa_j,\ j \in \{h, l\}$. The shocks are i.i.d over time and across households. As in previous sections, there are a large number of intermediaries who sign 2 period contracts with households. The timing of the game is follows:

1. Households can sign a contract with a single intermediary before period 1 types are realized

2. In period 1, after types are realized, households receive transfers from original the intermediary

3. Next, households can sign a contract with another intermediary. This contract is unobservable to the original intermediary and vice-versa.

4. At the beginning period 2, households can default on their obligations to the intermediary and receive utility
$$u(x_j) - \psi$$

Note here that since the horizon is finite I need to assume an exogenous cost of default. If $\psi = 0$, no household would ever have an incentive to pay back in period 2. A contract for the date 0 intermediary is $B = \{b_1(i), b_2(i)\}$. While the equilibrium contract is derived in **??**, it suffices to notice from Proposition 5 that the equilibrium is equivalent to one in which households trade a risk free bond subject to debt constraints $\phi$. In particular households choose $s \geq -\phi$ to maximize

$$u(\theta_i - qs_i) + \beta \mathbb{E} u(x_j + s_i)$$

where $q$ and $\phi$ are chosen to clear markets and satisfy not-too-tight restrictions respectively. Moreover from Theorem 3 we know that given $\psi$, the equilibrium outcome is efficient. Under the following parametrization, $\beta = .9;\ \pi_i = 1/2, \kappa_h = .8,\ \theta_l = .3,\ \theta_h = 2,\ x_l = .5,\ x_h = 1.4,$ in Figure 2.1, I plot the change in the ex-ante welfare and debt levels for $\psi \in [0, 2]$.

As one would expect, initially, as $\psi$ increases, welfare increases and for $\psi$ large enough, the change in welfare is zero after the low type ceases to be Euler-constrained. In addition, the endogenous debt levels $\phi$ increase and eventually flatten out. The key portion of

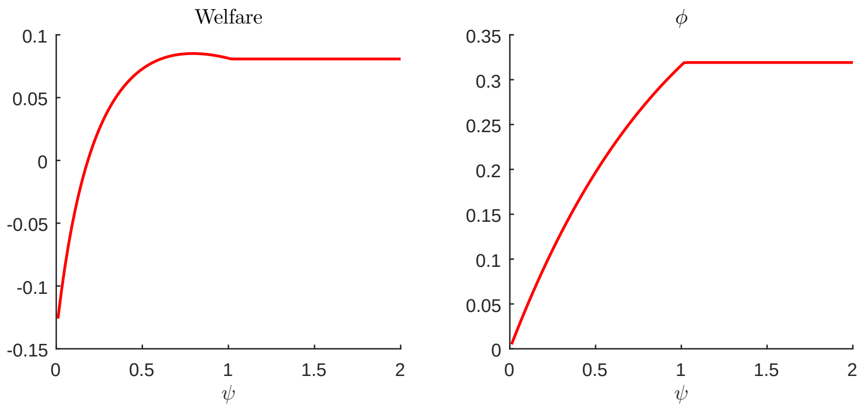


Figure 2.1: Welfare and Debt Levels

Figure 2.1 to notice is the downward sloping part of the welfare plot. In a region around $\psi = 1$, welfare *decreases* as $\psi$ increases. The reason for this is a price effect which redistributes wealth from the period 1 low to the high type. This can be seen easily in the example by computing how the ex-ante welfare

$$W(\psi) = \pi_h \left[u(\theta_h - qs) + \beta \mathbb{E} u(x_j + s)\right] + \pi_l \left[u(\theta_l + qs) + \beta \mathbb{E} u(x_j - s)\right]$$

changes with $\psi$. One can show using simple algebra that

$$W'(\psi) = q'(\psi)\, s \left[-\pi_h u'(\theta_h - qs) + \pi_l u'(\theta_l + qs)\right] + \nu(\psi)$$

where $q'(\psi)$ is the change in price as a function of $\psi$ and $\nu(\psi)$ is the multiplier on the borrowing constraint for the low type. Since risk sharing is imperfect, in general, $u'(\theta_h - qs) \leq u'(\theta_l + qs)$. Further, $q'(\psi) \leq 0$ since interest rates need to rise to clear markets as $\psi$ increases. Given that the multiplier $\nu(\psi) \geq 0$, the change in welfare as $\psi$ is increases is ambiguous. For $\psi$ small enough, $s$ will be small and so the multiplier effect will dominate and hence $W'(\psi) > 0$. However, as we can see from the picture as $\psi$ get larger, $s$ gets larger and $\nu(\psi)$ smaller, which causes the redistribution effect to dominate and $W'(\psi) < 0$.

Suppose we were to take as given the exogenously incomplete market structure and ask if the debt-constrained economy is efficient by considering a planning problem similar to Diamond (1967). For $\phi$ corresponding to the downward sloping portion of the welfare plot, we would conclude that outcomes are inefficient. In this case, imposing additional borrowing limits will implement the desired allocation.

As we have seen, when markets are endogenously incomplete, the outcome is efficient. This is because of hidden trading and in particular the fact that if the planner tried to transfer an amount smaller than $\phi$ to the low type in period 1, its voluntary participation constraints in period 2 would be slack. Therefore, it would use the hidden markets to borrow which would make these additional limits ineffective. In other words, the allocation would no longer satisfy the no-hidden-trades condition in Lemma 3.

The key difference between these two environments is presence of hidden markets. If in the exogenously incomplete world, the assumption is that contracts are observable and exclusivity can be enforced, then the planner should be able do much better than offer uncontingent transfers. However, if we think that the assumption of non-exclusivity is reasonable, then the outcomes are efficient.

### 2.5.4 Unique Implementation

The results in this section so far have two important implications for policy in the context of models with incomplete markets. The first is that interventions which may be desirable when markets are exogenously incomplete, might be ineffective in this environment. In addition, policies like ex-post bailouts will in general reduce welfare by lowering the amount of ex-ante risk-sharing that is possible. The second important message is that there is a role for policy to uniquely implement the best equilibrium. This motivates the use of credible off-equilibrium policies that will ensure that the best outcome will occur on path. To this end, I consider the effect of simply lender of last resort policies.

Consider the intermediary game without banishment. A public history $\omega^t = (q_1, ..q_t)$ consists of a vector of publicly observable variables which in this environment is just the sequence of prices that intermediaries can borrow and lend at. Note that I am assuming that contracts between private agents are still unobservable to any outside authority. A *lender of last resort* policy is vector $G_t = \left\{q_t^G, \phi_t^G\right\}$ which consists of an interest rate $\frac{1}{q_t^G}$ and debt constraint $\phi_t^G$ for all $t \geq 1$.[18] In particular, under such a policy

1. Households can borrow and lend with the government at prices $q_t^G$ subject to debt

[18] The government uses lump-sum taxes to balance its budget.

constraints $\phi_t^G$

2. Intermediaries can borrow and lend with the government at prices $q_t^G$ in an unconstrained fashion.

Given government policy $G_t$, we can define a competitive equilibrium given $\{G_t\}_{t\geq 1}$ in an analogous fashion to section 2.2.

Notice that a lender of last resort policy does not in general depend on the public history $\omega^t$. Using the language of Atkeson et al. (2010) we can define a *sophisticated lender of last resort* policy to be a vector $\mathcal{G}_t\left(\omega^t\right) = \left(q_t^G\left(\omega^t\right), \phi_t^G\left(\omega^t\right)\right)$ that depends on the public history $\omega^t$. Given that we are including a third player into the game, the government, we need to modify the structure of the game. The timing within a period is identical to section 2.2, except that after private transactions have taken place, the government implements a policy $\mathcal{G}_t\left(\omega^t\right)$ and finally private agents transact with the government. I now define the strategies of the players in this game. Given any history we can define a *continuation competitive equilibrium* to one that requires optimality by intermediaries and households. An equilibrium outcome is a collection $a_t = \{B_t, q_t, \mathcal{G}_t\}$ of contracts offered by intermediaries, prices $q_t$ and government policy $\mathcal{G}_t$.

Given public history $\omega^{t-1}$, after transactions between incumbent intermediaries and households take place, intermediaries submit contract schedules to a Walrasian auctioneer who chooses a price $q_t$ to clear the market. Formally, a strategy for an intermediary is $\sigma_I = \left\{B_t\left(\omega^{t-1}\right)(\tilde{q}),\ \forall \tilde{q} \geq 0\right\}$ where $B_t\left(\omega^{t-1}\right)(\tilde{q})$ is the contract the intermediary would offer if the price was $\tilde{q}$. Next, after observing history $\left(\omega^{t-1}, q_t, B_t\right)$ included households choose whether to sign a new contract. Denote their strategy by $\sigma_H$. After observing public history $\left(\omega^{t-1}, q_t\right)$ the government chooses a lending policy $\left(q_t^G\left(\omega^t\right), \phi_t^G\left(\omega^t\right)\right)$. Denote these government's strategy by $\sigma_G$. Finally after observing history $\left(\omega^{t-1}, q_t, B_t, \mathcal{G}_t\right)$ households and intermediaries transact with the government. Denote the strategies by $\sigma_P$. After any history, these strategies induce continuation outcomes in a standard fashion. Given this setup, we can define a *sophisticated equilibrium* as in Atkeson et al. (2010).

**Definition 9** *A sophisticated equilibrium is a collection of strategies $(\sigma_I, \sigma_G, \sigma_P)$ such that after histories the continuation outcomes induced by $(\sigma_I, \sigma_G, \sigma_P)$ constitute a continuation competitive equilibrium.*

We can define a sophisticated outcome to be the equilibrium outcome associated with a sophisticated equilibrium. A policy $\sigma_G^*$ *uniquely implements* a desired competitive equilibrium $a_t^* = \{B_t^*, q_t^*, \mathcal{G}_t^*\}$ if the sophisticated outcome associated with any sophisticated equilibrium of the form $(\sigma_I, \sigma_G^*, \sigma_P)$ coincides with the desired competitive equilibrium. The main result in this section is that there exists a sophisticated lender of last resort policy that uniquely implements the best equilibrium.

**Proposition 15** *Given a desired competitive equilibrium $a^*$, there exists a sophisticated policy that uniquely implements it.*

**Proof.** We know from Theorem 1 that the contract $B_t^*$ is a simple borrowing and lending contract with debt constraints $\phi_t^*$.Consider a history $\left(\omega^{t-1}, \tilde{q}_t\right)$ with $\tilde{q}_t \neq q_t^*$. In this case $\tilde{B}_t \neq B_t^*$. $\tilde{B}_t$ is also an uncontingent contract and is characterized by debt constraints $\tilde{\phi}_t$. As a result to each public history $\tilde{q}_t$ we can associate a private debt constraint $\phi_t^{\tilde{q}_t}$. Consider the following lender of last resort policy: for all $t \geq 0$

$$
\begin{aligned}
&\mathcal{G}_{t+s}^*\left(\omega^t, \omega^s\right) = (0,0) \text{ for all } s \geq 0 \text{ if } \omega^t = \omega^{*t} \\
&\mathcal{G}_{t+s}^*\left(\omega^t\right) = \left(q_{t+s}^*, \max\left(\phi_{t+s}^* - \phi_t^{\tilde{q}_t}, 0\right)\right) \text{ if } \omega^t \neq \omega^{*t} \\
&\mathcal{G}_{t+s}^*\left(\omega^t, \omega^s\right) = \left(q_{t+s}^*, \phi_{t+s}^*\right) \text{ for all } s \geq 0 \text{ if } \omega^t \neq \tilde{\omega}^t
\end{aligned}
$$

where $(q_t^*, \phi_t^*)$ correspond to the price and debt constraint associated with the desired equilibrium. Given strategy $\sigma_G^*$ and associated policy, $\{\mathcal{G}_t^*\}_{t\geq 0}$, it is easy to see that $a^*$ is an equilibrium outcome of the game. We want to show that it is the unique outcome. Given a period $t$, consider whether outcome $\left\{\tilde{B}_t, \tilde{q}_t, \mathcal{G}_t^*\right\}$ with $\tilde{q}_t \neq q_t$ can ever occur on the equilibrium path. It is easy to see that if $\tilde{q}_t \neq q_t$ then arbitrage opportunities exist and so in any equilibrium, it must be that $\tilde{q}_t = q_t^*$. As a result, since the only equilibrium contract consistent with $q^*$ is $B^*$, it must be that $\tilde{B}_t = \tilde{B}_t^*$. Finally, we need to show that the continuation outcomes after any history constitute continuation competitive equilibria. In this case, after an undesirable history $\tilde{\omega}^t$, Euler-constrained households will borrow from the government. In following periods, given $\mathcal{G}_{t+s}^*\left(\omega^t, \omega^s\right)$, market prices will be $q_{t+s}^*$ and private intermediaries will only offer uncontingent savings contracts, and households will only transact with the government. Consider the incentives for any household to default in $t+1$ given this policy. Since the equilibrium outcome $\left(q_{t+s}^*, \phi_{t+s}^*\right)$ is consistent with no default, all households will weakly prefer to pay the government back in all future periods. ■

The policies that uniquely implement the desired equilibrium are simple. After any undesired history $\tilde{\omega}^t$, the government announces a sophisticated lender of last resort policy that allows private agents to borrow and lend with it at prices $\left\{q^*_{t+s}\right\}_{s\geq 0}$. In period $t$, households can borrow up to an amount so that the total debt is at most $\phi^*_t$ while in all future periods, they can borrow the full amount $\phi^*_t$ from the government. After any undesired history, in the continuation equilibrium, households will only transact with the government while intermediaries will offer uncontingent savings contracts. As a result the policy is well defined. It is then easy to see that the only equilibrium consistent with this policy is the desired one since no-arbitrage will ensure that $\tilde{q}_t = q^*_t$.

## 2.6 Application: Optimal Bankruptcy Policies

In this section I present a simple example to illustrate how the framework with endogenously incomplete markets can be useful for thinking about a variety of policy questions. The short term defaultable debt model with stochastic re-entry has been known to match several key aspects of bankruptcy and unsecured credit in the United States. Further, these models have been used to study the effects of changing bankruptcy laws on welfare. The environment with private information, limited commitment and hidden trading has sharp implications for how to design these policies. First Proposition 10 implies that the efficient allocation can be decentralized as in equilibrium in which intermediaries choose the re-entry probabilities after default. More importantly it suggests that the optimal re-entry probabilities should be functions of the *level of debt defaulted on.* In particular, the probability of re-entry after default should be smaller if the level of debt defaulted on is larger. One can re-interpret the environment as follows: intermediaries can only choose whether or not to banish a household while an independent regulatory authority, the government chooses the probability of re-entry. It is straightforward to see that the efficient allocation in both environments are identical and in particular, the efficient allocation can be decentralized as an equilibrium with short-term defaultable debt in which the government optimally chooses the re-entry probability to be a function of the level of debt defaulted upon. While in general, making default punishments a function of the level of debt is always weakly better, in the simple example below, I demonstrate how moving from a system with constant punishments to one in which these are functions of the level of defaulted debt can strictly increase overall welfare in the economy.

Consider a simple two period environment identical to that in subsection 2.5.3 except that in period 1, households can receive endowment shocks $\theta_i \in \Theta = (\theta_h, \theta_m, \theta_l)$ with probability $\pi_i$, $i \in \{h, m, l\}$. In particular, there are three initial types rather than two. The timing is identical, except that analogously to the environment with banishment, I will allow intermediaries to control the severity of punishment after banishment. Therefore, if a household is banished in period 2, it receives

$$u(x_j) - \psi(i, j)$$

A contract for the date 0 intermediary is $B = \{b_1(i), b_2(i), \delta_2(i, j), \psi(i, j)\}$. Since this is a two-period environment, intermediaries cannot choose a re-entry probability. However, I allow them to choose the level of default punishment $\psi(i, j)$.[19] One can show using the results in the previous sections that the best equilibrium is the equilibrium of the following game: Period 0 intermediaries choose $B$ to minimize

$$\sum_i [b_1(i) + q\mathbb{E}_j [1 - \delta_2(i, j)] b_2(i, j)]$$

subject to $\forall i$,

$$\begin{aligned} c_1(i) &= \theta_i + b_1(i) \ , \\ c_2(i, j) &= x_j + b_2(i) \text{ if } \delta_2(i, j) = 0, \\ c_2(i, j) &= x_j \text{ if } \delta_2(i, j) = 1 \end{aligned}$$

$$b_1(i) + q\mathbb{E}_j [1 - \delta_2(i, j)] b_2(i) = 0$$

and for all $(i, j)$

$$[1 - \delta_2(i, j)] u(x_j + b_2(i)) \geq [1 - \delta_2(i, j)] \left(u(x_j) - \psi\left(i, j'\right)\right)$$

$$\sum_i \pi_i [u(c_1(i)) + \beta\mathbb{E}_j [1 - \delta_2(i, j)] c_2(i, j)] \geq \underline{u}$$

[19] The example is constructed to be simple in order to illustrate that default punishments should depend on the level of defaulted debt. This intuition carries over to the infinite hoirzon game with re-entry probabilities.

In equilibrium $q$ is chosen so that markets clear,

$$\sum_i b_1(i) = 0$$

I will first restrict intermediaries to choose $\psi(i,j) = \psi$ and next relax this assumption. Clearly welfare under the latter will always be weakly greater but in a simple numerical illustration I show that ex-ante welfare can be strictly higher. Under the following parametrization, $\beta = .9$, $\theta_h = 2$, $\theta_m = .5$, $\theta_l = .3$, $x_h = 1.4$, $x_l = .5$, $\kappa = .8$, $\pi_i = 1/3$, ex-ante welfare is approximately 5.7% larger when the default costs are allowed to be different. In the best equilibrium, in both cases, $\delta_2(m,l) = \delta_2(l,l)$ and so these types are banished. Using Proposition 5 we can construct an equilibrium of an Eaton-Gersovitz economy with price function, $\mathcal{Q}(b)$ and default cost function $\psi^E(b)$. Figure 2.2 plots these functions for the above example. In this figure, the red lines correspond to variables in the efficient

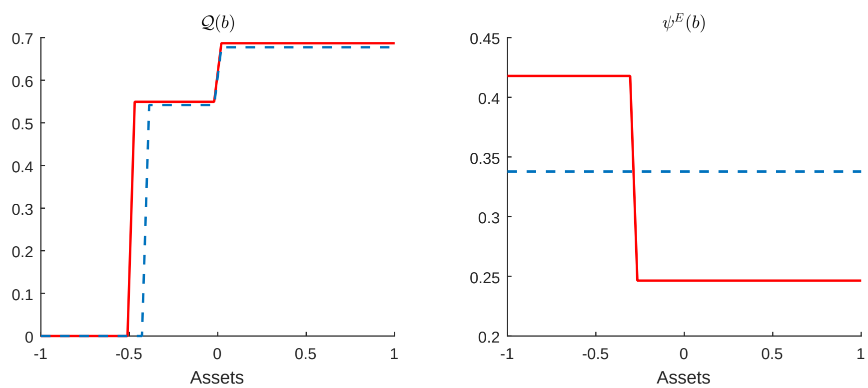


Figure 2.2: Pricing and Default Cost Functions

allocation while the blue dashed lines corresponds to the equilibrium when the punishment is restricted to be independent of the level of debt. As we can see, the optimal default punishment is a function of the level of debt defaulted on. The larger the amount of debt defaulted on, the more stringent the punishment. The intuition for this result is simple. Since types are hidden, the punishment only needs to be large enough to prevent the high type in period 2 from pretending to be the low type. Since the voluntary participation constraints are $u(x_h - b_1(i)) \geq u(x_h) - \psi(i,l)$, a larger amount of debt implies a larger punishment required to keep the high type indifferent. The efficient allocation trades off the benefit of higher consumption in period 1 at the cost of imposing a harsher punishment in period 2. Since the marginal benefit is larger for $\theta_l$ as compared to $\theta_m$ in the efficient allocation, type $\theta_l$ consumes more in period 1 and suffers a larger punishment in low state in period 2 when it defaults.

## 2.7 Discussion of Assumptions

In this section, I discuss the role of some of the assumptions in the model.

1. *Finitely lived intermediaries*: The reason for this is an existence problem. In the model, tighter debt constraints imply lower interest rates or higher $q_t$. In particular, it may be that the value of default is large enough so that the equilibrium debt constraints imply an interest rate that is less than 1. In this case, the present discounted value of transfers to the household is $\infty$ and as a result we cannot have infinitely lived intermediaries in the model. One example of such an environment is Hellwig and Lorenzoni (2009).

2. *i.i.d endowments*: I have assumed that the endowment shocks are independently and identically distributed across time and households. The reason is tractability. Introducing persistent endowment complicates the environment further but would be an interesting extension of the model.

3. *Banishment/Default assumptions*: The assumption that banished or defaulting households cannot sign with a new intermediary is important to the results in the model. Banishment is useful precisely because banished households cannot sign with intermediaries. However note that the problem is well defined even if banished/defaulting households can sign a restricted set of contracts. For example, one can assume that defaulting households are allowed only to save and not borrow, as in Hellwig and Lorenzoni (2009). The equilibrium will still be equivalent to an incomplete markets environment subject to endogenous debt constraints. However, in this case banishment will no longer be a useful tool to introduce state-contingency in contracts.

4. *Restriction to signing with only one new intermediary at a time*: While the environment allows households to sign multiple contracts in a hidden fashion, I only allow them to sign at most one new contract each period. The reason for this is that if all intermediaries posted identical contracts and households could sign multiple hidden contracts, households could in theory borrow an infinite large amount and default the next period.

## 2.8 Conclusion

Models with exogenously incomplete markets have been widely used to study a variety of quantitative questions in macroeconomics and international economics. The purpose of this paper is to complement this literature by providing a framework to think about policy questions in the context of these models. The main advantage of my approach is that unlike the majority of the contracting literature, the resulting contracts are identical to the ones assumed by the applied literature. In particular, I show that both uncontingent contracts with debt constraints and short-term defaultable debt contracts endogenously arise under appropriate assumptions from a contracting environment with private information, limited commitment and hidden trading. If intermediaries are allowed to offer contracts with default on path, the set of equilibrium outcomes in this environment is identical to those in an Eaton-Gersovitz model with short term defaultable debt. If contracts are restricted to not allow banishment, the equilibrium outcomes are identical to a Huggett economy with endogenous debt constraints. I show that the best equilibrium outcome in both cases are efficient but that there are multiple equilibria. This paper has three important implications for policy. The first is that outcomes that might appear inefficient with exogenously incomplete markets may not be so when we explicitly model the underlying frictions. The second is that there is an important role for policy to implement the best equilibrium. The third and final implication is that by explicitly modeling the informational and commitment frictions, we can better understand what kinds of policies will be welfare enhancing. For example, I show that in the context of bankruptcy models that allowing the punishment after default to depend on the level of defaulted debt is welfare improving.

# Chapter 3

# Sophisticated Policies in Models with Endogenous Debt Constraints

## 3.1 Introduction

Is there a role for government intervention in financial markets? If so, what kinds of policies lead to desirable outcomes? To answer these questions, I study a complete markets model with financial frictions in which agents are subject to state contingent debt constraints that are determined endogenously in equilibrium. The key friction underlying this model is that at any time agents can choose to default on their debt and lose access to financial markets in all future periods. Debt constraints are chosen in equilibrium so that an agent who has borrowed up to the limit is indifferent between defaulting and paying back his debt. This allows for the severity of the financial friction to be an equilibrium object which is in sharp contrast to much of the literature that exogenously assumes the form and tightness of debt constraints. A crucial feature of this model is that there are multiple equilibria. For example, there is an equilibrium in which financial markets work well and another in which these frictions are very tight. This generates a potentially important role for policy. The main result of this paper is that one can construct feasible and simple policies that uniquely implement the desired equilibrium.

In the model, agents have stochastic access to a production technology that transforms one unit of labor into one unit of the consumption good. Agents can either be productive or unproductive in which case they do not have access to this technology. All agents are subject to a cash in advance constraint in which the single consumption good must be purchased with money. In addition, they can trade Arrow securities subject to state contingent debt constraints. At the beginning of each period after the state is realized, agents can default on their existing obligations and subsequently be barred from trading Arrow securities in all future periods. They can however continue to hold fiat money after default. As mentioned in the previous paragraph, debt constraints are determined so that the value of paying an amount equal to this limit equals the value of defaulting. These debt constraints are termed "not-too-tight" and were first studied by Alvarez and Jermann (2000). The first main result of the paper concerns the Ramsey outcome when the government can control the money supply and impose uniform lump sum taxes on all agents. I show in this case that the Ramsey policy departs from the Friedman Rule. In general, the government will want to choose the money supply so that the real return on money is lower than the real interest rate in the economy. The reason for this is that agents can hold fiat money even after defaulting. By lowering the real return on money, the government worsens the outcomes for the agent after default. This intuition is similar to much of the work in models with limited commitment where increasing the severity of default punishments allows greater risk-sharing to be sustained in equilibrium. The optimal policy trades off the benefit of harsher punishments after default with worsening the friction due to cash-in-advance constraints on path. However, I show in this environment that the Ramsey policy is also consistent with another competitive equilibrium in which no Arrow securities are traded. Debt constraints are zero for all agents and the prices of the securities adjust so that no agent wishes to save. Agents' only use fiat money to smooth consumption. This equilibrium is similar to the one characterized by Scheinkman and Weiss (1986).

This motivates the study of *Sophisticated Policies* as defined by Atkeson et al. (2010). These policies can differ on and off the equilibrium path and have the property that all continuation outcomes constitute continuation competitive equilibria. The first step in the construction of such policies is to consider a related environment with an equilibrium set that is identical to the one with debt constraints. The reason for this is that it is unclear in the context of this equilibrium concept as to what these not-too-tight debt constraints correspond to. They are neither optimality nor market clearing conditions. In the related setup, competitive intermediaries offer insurance contracts to agents who can default on them at any time. While the state of the world is observable to the intermediaries, agents' actions are not. I show that the set of equilibrium allocations is the same in both environments. This equivalence is useful as it allows us to interpret the "not-too-tight" debt constraints as a consequence of the profit maximizing contracts offered by intermediaries to agents. The policies I consider internalize how these contracts change in anticipation of government intervention after certain histories.

The next step is to consider the set of competitive equilibria when the government has an expanded set of instruments. In addition to money supply rules, I consider a policy instrument which allows the government to offer some agents access to an overnight risk-free deposit facility. Eligible agents can deposit cash with the government overnight at a specified interest rate. Uniform lump sum taxes can be levied to balance the government's budget. The government specifies the contracts eligible for the scheme and allows all agents signed to these contracts who haven't defaulted to use this deposit facility. I prove that the set of competitive equilibria given these policies is identical to the set when the government can only control the money supply. As a result, as long as these policies can only depend on the exogenous states of the world, there are multiple equilibria and the government cannot use the deposit facility to implement equilibria that Pareto-dominate the best competitive equilibrium.

The main result in the paper states that when we allow these policies to also depend on histories of private but publicly observable actions, unique implementation is possible. Given a competitive equilibrium, we can construct Sophisticated policies so that there exists a unique Sophisticated equilibrium corresponding to it. More specifically, under this policy, the government commits to offering a subset of agents access to an overnight risk free savings technology (a risk free bond) after certain histories. As is standard in cash-in-advance environments agents who are productive sell their consumption goods for cash which they hold overnight. Under this savings scheme, agents can deposit this cash with the government and receive a return that is greater than that of money. With the usual restriction on policies, multiple equilibria exist due the strategic complementarities in the actions of intermediaries. In particular, if prices are such that all other intermediaries offer a particular contract $\tilde{C}$ different from the desired one, an individual intermediary finds it optimal to also offer this contract. The idea behind this implementation technique is that policy is chosen so that an individual intermediary no longer has an incentive to go along with contract $\tilde{C}$. As an example suppose the government commits to intervening after the following history; all intermediaries besides a small subset offer $\tilde{C}$ while this small subset offers contract $\hat{C}$ with the property that some agent receives more insurance than that provided by $\tilde{C}$. The government offers all agents receiving $\hat{C}$ who haven't defaulted access to this risk free technology with returns that exceed the return from holding fiat money. Now consider if all intermediaries offering $\tilde{C}$ is an equilibrium. One can show that an individual intermediary has an incentive to offer a contract like $\hat{C}$ which will make both him and the agent strictly better off. Moreover, given this policy agents will not want to default if the return on the risk free savings technology is sufficiently attractive. A similar argument rules out other kinds of equilibria including those with sunspots.

An attractive feature of this policy is its institutional simplicity. The implementation only relies on a risk free instrument . In particular, it does not involve the government having to lend to distressed agents which in general might subject to various moral hazard problems. Moreover, it does not assume that government has a better monitoring technology and so is not subject to the same frictions as private intermediaries. This is contrast to much of the literature that assumes in crisis times that government has the ability to relax financial frictions and enter financial markets and increase the amount of credit in the economy.

**Literature:** There is a large and growing literature on macroeconomic models with financial frictions. Often, these frictions take the form of collateral constraints as first introduced by Kiyotaki and Moore (1997). Recent examples include Gertler and Kiyotaki (2010), Shourideh and Zetlin-Jones (2012) and Buera and Moll (2012). While these models are useful for understanding how exogenous shocks to the tightness of the collateral constraints affect the aggregate economy, they are less useful for understanding the role of policy to mitigate financial crises. This is because the friction is exogenously imposed and as a result the only beneficial policies are those that involve the government having a special ability to relax these frictions in bad times. If we assume that such frictions are derived from a contracting problem with limited commitment or moral hazard, it is unclear why the government should have these abilities when private agents do not.

Endogenous debt constraints of the form studied here were first introduced by Alvarez and Jermann (2000). As in this paper debt constraints are set to be not-too-tight and chosen so that an agent who has borrowed up to the limit the previous period is indifferent between paying back and defaulting. Alvarez and Jermann (2000) show that the efficient allocation from Kehoe and Levine (1993) can be decentralized using complete markets and not-too-tight debt constraints. However, the decentralization is weak as autarky is always an equilibrium in their environment. This is not true in the model I consider. While there is equilibrium multiplicity, the best equilibrium is in general inefficient. This because of the presence of pecuniary externalities in the agent's problem. Since agents can hold money after default, the return on money affects the value of default which is not internalized by agents. This is not true in Alvarez and Jermann (2000) since they assume that after default, agents can only consume their endowments. This is similar to the literature on inefficiencies arising due to prices in the consumption set, a seminal example of which is Golosov and Tsyvinski (2007). However, I show that the set of equilibria is identical to that of a decentralized contracting environment similar to that of Prescott and Townsend (1984) and Golosov and Tsyvinski (2007) in which agents have hidden actions. This provides a similar interpretation for these constraints in the setup I consider. In another related paper, Hellwig and Lorenzoni (2009) study a decentralized environment similar to Alvarez and Jermann (2000) but in which after default, agents can save in Arrow securities. In contrast, I assume that after default agents can save in an uncontingent asset (money). As a result, the punishment after default is more severe than Hellwig and Lorenzoni (2009) but less than Alvarez and Jermann (2000).

This paper uses techniques and language developed by Chari and Kehoe (1990) and expounded upon by Atkeson et al. (2010) (henceforth ACK) which allows us to think about how policy can uniquely implement a desired competitive equilibrium. ACK define the concepts of *Sophisticated policies* and *Sophisticated equilibrium* in which policies depend on the history of private actions and are explicit about how they differ on and off the equilibrium path. This approach requires that all continuation outcomes (including those following a deviation) constitute a continuation competitive equilibrium. It is in sharp contrast to much of the literature that considers policies with the property that no equilibrium exists following a private deviation. This is sometimes referred to as implementation via nonexistence. In a related paper, Bassetto (2002) demonstrates another way in which policy can ensure uniqueness in models with price level indeterminacy. As in ACK, he is explicit about government strategies on and off the equilibrium path and shows that there exist strategies that lead to an equilibrium price level that is pinned down by fiscal variables. However, unlike the standard fiscal theory of the price level, government budget constraints hold both on and off the equilibrium path.

Finally, this paper is related to the literature on multiple equilibria in general equilibrium models. Woodford (1986a) has a simple example of an economy with financing constraints and shows the existence of self-fulfilling fluctuations similar to overlapping generations economies. Moreover he shows that the persistence of these fluctuations is similar to those of business cycle fluctuations. In another important paper, Woodford (1986b) provides conditions in order for a steady state of non-linear model to be indeterminate and proves that indeterminacy is a necessary and sufficiency condition for the existence of sunspot equilibria. In a recent paper Benhabib and Wang (2013) show that one can generate endogenous fluctuations in models with collateral constraints that can match U.S. time series data. Gu et al. (2013) also prove the existence of multiple equilibria in a model with endogenous debt limits and show how one can generate endogenous fluctuations with sunspot dynamics.

The rest of the paper proceeds as follows: Section 2 lays out the model with cash-in-advance and not-too-tight debt constraints and characterizes the multiplicity of equilibria that occurs. In Section 3, I set up the contracting problem and prove an equivalence result, after which I proceed to the construction of Sophisticated Policies. Section 4 contains a discussion of some of the modelling assumptions and Section 5 concludes.

## 3.2 Model

The economy is populated by a two types of agents $I = \{i, j\}$ with each type of equal measure 1. Time is discrete $t = 0, 1, 2, ..$The aggregate state space is $S = \{i, j\}$ with $\Pr[s_0 = i] = \frac{1}{2}$ and subsequently $\Pr\left[s^{t+1} = j \mid s^t = i\right] = \Pr\left[s^{t+1} = i \mid s^t = j\right] = \lambda$. I denote the unconditional probabilities of histories denoted by $\pi\left(s^t\right)$ and the conditional probabilities by $\pi\left(s^{t+1} \mid s^t\right)$. The symbol $\succeq$ is used the denote the partial order on histories. For example, $s^{t'} \succeq s^t$ for $t' \geq t$ denotes a possible continuation of history $s^t$.

Given a random variable $x$, I use the notation $\{x\}_{t'}$ to denote the stochastic process $\left\{x_t\left(s^t\right); \forall\ t' \leq t \leq \infty,\ s^t \in S^t\right\}$.There is a single divisible, nonstorable consumption good. In state $i$ agent $i$ is productive; he can transform 1 unit of labor into 1 unit of the consumption good while $j$ cannot. All agents have preferences over consumptions streams and labor. Agent $i's$ utility in period $t$ and state $s^t$ is denoted by $u\left(c_t^i\left(s^t\right)\right) - l_t^i\left(s^t\right)$ where $u : R_+ \rightarrow R$ is strictly increasing, strictly concave and continuously differentiable.

In each period, agents trade state contingent Arrow securities, and money in certain ways and subject to various constraints to be laid out. To begin with we study a benchmark environment with complete markets.

### 3.2.1 Complete Markets Benchmark

To study the complete markets benchmark I first consider an environment without fiat money in which productive agents can directly consume their produced good. I then show that there exist money supply policies which implement this equilibrium in the environment with fiat money.

Agent $i$ chooses consumption and labor streams $\left\{c_t^i\left(s^t\right), l_t^i\left(s^t\right)\right\}_{s^t \in S^t, t \geq 0}$ to maximize

$$E_0\left[\sum_{t=0}^{\infty}\sum_{s^t \in S^t}\beta^t \pi\left(s^t\right)\left(u\left(c_t^i\left(s^t\right)\right) - l_t^i\left(s^t\right)\right)\right] \tag{3.1}$$

subject to an Arrow-Debreu budget constraint

$$\sum_{t=0}^{\infty}\sum_{s^t} Q_t\left(s^t\right) c_t^i\left(s^t\right) \leq \sum_{t=0}^{\infty}\sum_{s^t} Q_t\left(s^t\right) l_t^i\left(s^t\right)$$

where $Q_t\left(s^t\right)$ is the Arrow-Debreu price and non-negativity constraints on consumption and labor. Given our assumption on productivities we also have that $l^i\left(s^t\right) = 0$ if $s_t \neq i$ for all $i \in I$.

A competitive equilibrium is defined in a standard fashion. Let $g \equiv u'^{-1}$.

**Lemma 7** *If markets are complete,* $c_t^i\left(s^t\right) = g\left(1\right)$ *for all* $i, t, s^t$*. The risk free rate* $R = \frac{1}{\beta}$*.*

**Proof.** See Appendix. ■

Next, I consider a very similar environment to the one above except that agents have to purchase consumption goods with fiat money. In particular, if agent $i$ is productive he sells his labor services for cash which is used in the following period to purchase consumption goods. One interpretation of this assumption is that there is an aggregate technology which transforms 1 unit of labor services into one unit of the consumption good. A productive agent can then sell his labor services to this technology and is compensated in cash. The timing of actions in each period is as follows: First the state $s_t$ is realized following which the agents trade a complete set of Arrow securities and fiat money. Next, each agent splits into an worker and a shopper; the worker if productive can exchange labor services for money and the shopper must use money to purchase consumption goods. Finally, the entrepreneur and shopper reunite and consumption takes place at the end of the period.

Agent $i$ chooses sequences of consumption, labor, money holdings and Arrow securities $\left\{c_t^i\left(s^t\right), l_t^i\left(s^t\right), m_t^i\left(s^t\right), \left\{a_{s_{t+1}}^i\left(s^t\right)\right\}_{s_{t+1}\in S}\right\}_{s^t\in S^t, t\geq 0}$, where $a_{s_{t+1}}^i\left(s^t\right)$ the total amount of a security which pays 1 unit of the consumption good in state $s^{t+1}$ held by agent $i$, to maximize (3.1) subject to period budget constraints (in real terms),

$$\begin{aligned}
& p_t\left(s^t\right) m_t^i\left(s^t\right) + \sum_{s^{t+1}} q_{s_{t+1}}\left(s^t\right) a_{s_{t+1}}^i\left(s^t\right) \\
& \leq \frac{p_t\left(s^t\right)}{p_{t-1}\left(s^{t-1}\right)} l_{t-1}^i\left(s^{t-1}\right) + a_{s_t}^i\left(s^{t-1}\right) + p_t\left(s^t\right)\left[m_{t-1}^i\left(s^{t-1}\right) - \frac{c_{t-1}^i\left(s^{t-1}\right)}{p_{t-1}\left(s^{t-1}\right)}\right] \\
& + T_t\left(s^t\right), \text{ if } s_{t-1} = i,
\end{aligned} \tag{3.2}$$

$$\begin{aligned}
p_t\left(s^t\right) m_t^i\left(s^t\right) + \sum_{s^{t+1}} q_{s_{t+1}}\left(s^t\right) a_{s_{t+1}}^i\left(s^t\right) \leq a_{s_t}^i\left(s^{t-1}\right) + p_t\left(s^t\right)\left[m_{t-1}^i\left(s^{t-1}\right) - \frac{c_{t-1}^i\left(s^{t-1}\right)}{p_{t-1}\left(s^{t-1}\right)}\right] \\
+ T_t\left(s^t\right) \text{ if } s_{t-1} \neq i
\end{aligned} \tag{3.3}$$

a cash-in-advance constraint

$$c_t^i\left(s^t\right) \leq p_t\left(s^t\right) m_t^i\left(s^t\right) \tag{3.4}$$

non-negativity constraints on consumption, labor and money holds, and $l^i\left(s^t\right) = 0$ if $s_t \neq i$. Here $p_t\left(s^t\right)$ is the goods price of money, $q_{s_{t+1}}\left(s^t\right)$ is the price of a security purchased in $s^t$ which pays 1 unit of the consumption good in state $s^{t+1}$ and $T_t\left(s^t\right)$ is a uniform lump-sum tax imposed by government.

A competitive equilibrium given a money supply and transfer policy $\left\{M_t\left(s^t\right), T_t\left(s^t\right)\right\}_{t,s^t}$ is defined in the standard fashion. Next, I show that if the money supply satisfies certain conditions, the complete markets equilibrium outcome characterized above is also an equilibrium outcome in the environment with fiat money for some price sequence.

**Proposition 16** *There exist prices* $\{p_t(s^t)\}$ *such that* $c_t^i(s^t) = g(1)$ *for all* $i, t, s^t$ *and* $R_t = \frac{1}{\beta}$

1. $\liminf_{t\to\infty} M_t = 0$

2. $\inf_t M_t \beta^{-t} = \kappa > 0$

*In such an equilibrium, the nominal interest rate is 1.*

**Proof.** See Appendix. ■

The result serves as a warm-up for the analyses in later sections when agents face state contingent debt constraints. It is similar to one proved by Cole and Kocherlakota (1998) who focused on the kinds of money supply policies that can implement Pareto-optimal allocations in a growth model with cash-in-advance constraints. As I find here, money supply policies that deliver the Friedman Rule achieve this. The basic idea is that the Friedman Rule removes the distortion from having to hold cash from one period to the next. I show that these policies are consistent with the price level $\frac{p_t(s^t)}{p_{t-1}(s^{t-1})} = \frac{1}{\beta}$ which implies that in equilibrium, the multiplier on the cash-advance constraint is 0. As a result it is easy to show that the equilibrium allocation computed in the environment without the CIA constraints satisfy the agent's first order conditions in the above problem. One only needs to check that the transversality conditions hold which are guaranteed by the assumption stated in the proposition.

In subsequent sections I analyze environments in which agents are subject to constraints on how much Arrow securities they can sell. Clearly, ex-ante welfare in these equilibria will be lower than the complete markets benchmark studied here. Moreover, in this setup, the complete markets risk-free rate provides an upper bound on the equilibrium risk free rate in any model with debt constraints. To see this notice that

$$\tilde{R}_t = \frac{1}{\beta}\frac{u'\left(\tilde{c}_t^i\left(s^{t-1}, i\right)\right)}{\lambda u'\left(\tilde{c}_{t+1}^i\left(s^t, j\right)\right) + (1-\lambda)\, u'\left(\tilde{c}_{t+1}^i\left(s^t, i\right)\right)} = \frac{1}{\beta}\frac{1}{\lambda u'\left(\tilde{c}_{t+1}^i\left(s^t, j\right)\right) + (1-\lambda)} \leq \frac{1}{\beta}$$

where $\tilde{R}$ is the equilibrium interest rate in a model with debt constraints. In equilibrium, the interest rate is determined by the unconstrained agents i.e. the savers who in this case are the productive agents. The quasi-linearity assumption implies that for any agent, his marginal utility of consumption in the productive state is always 1. The fact that the agent is constrained implies that in the unproductive state his marginal utility is less than 1. This implies that $\tilde{R}_t \leq \frac{1}{\beta}$ which is the interest rate in the complete markets case.

### 3.2.2 Equilibrium with Debt Constraints and Fiat Money

Next, I consider an environment in which agents must purchase all consumption goods with fiat money and are subject to state contingent debt constraints that are determined in equilibrium. The timing within the period is as follows. After $s_t$ is realized, agents can choose to default on debt. Default is publicly observable and after default, an agent is barred from trading Arrow securities in all future periods. However, he can continue to hold fiat money. Next, the markets for Arrow securities and money open and agents trade with each other after which the agent splits into a shopper and worker. As before, the shopper must purchase consumption goods with money and the worker if productive receives cash in return for labor services. Consumption takes place at the end of the period.

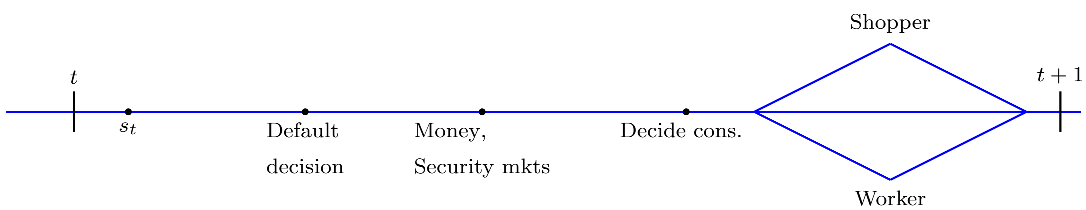


Figure 3.1: Timing.

Before analyzing the competitive equilibrium, a quick note on the default assumption. The literature on limited commitment in financial markets has made various assumptions on the consequences of default for agents. Some of these include permanent autarky (Kehoe and Levine (1993), Alvarez and Jermann (2000)), autarky with a positive probability of being able to regain entry into financial markets (Azariadis and Kaas (2012)), and only being able to save in Arrow securities (Hellwig and Lorenzoni (2009)). I find the introduction of fiat money convenient as allows us to make a natural assumption of allowing agents to hold money after default. As a result, agents can continue to achieve some consumption smoothing even after they default. Furthermore, the assumption of money being "hidden" from any legal authority that can seize financial assets of anyone defaulting is also a reasonable one.

Agent $i$ chooses sequences of consumption, labor, money holdings and Arrow securities $\left\{c_t^i\left(s^t\right), l_t^i\left(s^t\right), m_t^i\left(s^t\right), \left\{a_{s_{t+1}}^i\left(s^t\right)\right\}_{s_{t+1}\in S}\right\}_{s^t\in S^t, t\geq 0}$ to maximize (3.1) subject to budget constraints (4.25), (4.26), state contingent debt constraints

$$a_{s'}^i\left(s^t\right) \geq \phi_{s'}^i\left(s^t\right) \text{ for all } s^t, s' \in S, \tag{3.5}$$

cash-in-advance constraints (3.4) and non-negativity constraints on consumption, labor and money. The debt constraints can be agent and state specific and limit the amount of each Arrow security that an agent can sell. We can define the agent's problem starting at date $t, s^t$ if he hasn't defaulted in the past. Denote the value of this problem as $V_t^{i,c}\left(s^t, a_{s_t}^i\left(s^{t-1}\right); \Phi_t^i(s^t)\right)$ where $\Phi_t^i(s^t) = \left\{\phi_{t'}^i\left(s^{t'}\right)\right\}_{s^{t'}\geq s^t, t'\geq t}$ is the sequence of debt constraints in all future dates and states. As described earlier, the cash-in-advance constraint requires that all purchases of the consumption good must be made with previously accumulated money. As is standard, the CIA constraint will bind in equilibrium if the real risk-free rate is larger than 1.

Given a date $t$ and state $s^t$, an agent who has defaulted chooses a sequence $\left\{c_{t'}^i\left(s^{t'}\right),\ l_{t'}^i\left(s^{t'}\right),\ \tilde{m}_{t+1}^i\left(s^t\right)\right\}_{s^{t'}\succeq s^t, t'\geq t}$ to maximize

$$\sum_{t'\geq t}\sum_{s^{t'}\succeq s^t} \beta^{t'-t}\pi\left(s^{t'} \mid s^t\right)\left[u\left(c_{t'}^i\left(s^{t'}\right)\right) - l_{t'}^i\left(s^{t'}\right)\right]$$

subject to budget constraints,

$$\begin{aligned} & p_t\left(s^t\right)\tilde{m}_t^i\left(s^t\right) \\ & \leq \frac{p_t\left(s^t\right)}{p_{t-1}\left(s^{t-1}\right)} l_{t-1}^i\left(s^{t-1}\right) + p_t\left(s^t\right)\left[\tilde{m}_{t-1}^i\left(s^{t-1}\right) - \frac{c_{t-1}^i\left(s^{t-1}\right)}{p_{t-1}\left(s^{t-1}\right)}\right] + T_t\left(s^t\right),\ \text{if } s_{t-1} = i, \end{aligned} \tag{3.6}$$

cash-in-advance constraints (3.4) and non-negativity constraints on consumption, labor and money. As before $\tilde{m}_t^i\left(s^t\right)$ denotes the agent's money purchases and $p_t\left(s^t\right)$ is the goods price of money. Denote the value of default in $t, s^t$ for agent $i$ by $V_t^{i,d}(s^t; \mathbf{p}_t)$ where $\mathbf{p}_t = \left\{p_{t'}\left(s^{t'}\right)\right\}_{s^{t'}\geq s^t, t'\geq t}$.

**Definition 10** *A Competitive Equilibrium given a money supply policy* $\{M, T\}_0$ *consists of prices* $\left\{p_t\left(s^t\right), \left(q_{s_{t+1}}\left(s^t\right)\right)_{s_{t+1}}\right\}_{t,s^t}$, *debt constraints* $\left\{\phi^i_{s'}\left(s^t\right)\right\}_{i\in I, s^t, s'}$ *and allocations* $\left\{\left(c^i_t\left(s^t\right), l^i_t\left(s^t\right), m^i_t\left(s^t\right), \left\{a^i_{s'}\left(s^t\right)\right\}_{s'\in S}\right)_{i\in I}\right\}_{t,s^t}$ *such that*

- *Given prices and debt constraints, the allocations solve the agents problem*
- *Markets clear*

$$\sum_{i\in I} c^i_t\left(s^t\right) = \sum_{i\in I} l^i_t\left(s^t\right) \text{ for all } t, s^t$$
$$\sum_{i\in I} a^i_{s'}\left(s^t\right) = 0 \text{ for all } t, s^t, s' \in S$$
$$\sum_{i\in I} m^i_t\left(s^t\right) = M_t\left(s^t\right) \text{ for all } t, s^t$$

- *Debt constraints are chosen to be not-too-tight; For all* $i \in I,\ t \geq 0,\ s^t$,

$$V^{i,c}_t\left(s^t, \phi^i_{s_t}\left(s^{t-1}\right); \Phi^i_t(s^t)\right) = V^{i,d}_t(s^t; \mathbf{p}_t)$$

As in Alvarez and Jermann (2000), debt constraints are equilibrium objects and are chosen for each agent $i$ in each date and state to satisfy the above condition. The condition says that debt constraints are chosen so that an an agent who has borrowed up to the limit is indifferent between paying back his debt and defaulting. Loosely speaking, a way to interpret this condition is to consider the effects of weakening the equality to an inequality. If $V^{i,c}_t\left(s^t, \phi^i_{s_t}\left(s^{t-1}\right); \Phi^i_t(s^t)\right) < V^{i,d}_t\left(s^t\right)$, the agent will choose to default in that state as the value of default is strictly greater than repaying his debt. On the other hand, if $V^{i,c}_t\left(s^t, \phi^i_{s_t}\left(s^{t-1}\right); \Phi^i_t(s^t)\right) > V^{i,d}_t\left(s^t\right)$ the agent could borrow a little more and still would prefer not to default. I formalize this intuition in a later section where I consider an environment in which intermediaries offer insurance contracts subject to limited commitment frictions and show that the set of equilibria is identical to environment above.

This paper is interested in whether policy that depends on histories of endogenous events can aid the implementation of desirable outcomes in this model. To motivate the role for policy I first consider the Ramsey outcome when the space of policies are money supply rules and uniform lump sum taxes. In contrast to the preceding section with complete markets I show that the Ramsey policy departs from the Friedman rule. I then show that these policies are also consistent with another Pareto-inferior equilibrium in which debt constraints are zero an all dates and states.

### Ramsey Outcome

The space of policies is given by

$$\mathcal{P}^R = \left\{ \{M\}_0, \{T\}_0 \mid \forall t,\ s^t,\ M_t\left(s^t\right) \in \mathbb{R}_+,\ T_t\left(s^t\right) \in \mathbb{R} \right\}$$

Given a policy $\pi \in \mathcal{P}^R$, let $CE(\pi)$ denote the set of competitive equilibria given this policy. Given an element $\psi \in CE(\pi)$ let $\left\{ \left( c^{i,\psi}, l^{i,\psi}, m^{i,\psi}, \left\{ a_{s'}^{i,\psi} \right\}_{s' \in S} \right)_{i \in I} \right\}_0$ be allocations corresponding to $\psi$. Define

$$U(\pi) = \max_{\psi \in CE(\pi)} \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \sum_{i \in I} \beta^t \pi\left(s^t\right) \left( u\left( c_t^{i,\psi}\left(s^t\right) \right) - l_t^{i,\psi}\left(s^t\right) \right)$$

**Definition 11** *A Ramsey policy is $\pi^* \in \mathcal{P}^R$ such that $\pi^* \in \arg\max_{\pi \in \mathcal{P}^R} U(\pi)$.*

I next characterize some properties of a Ramsey equilibrium. While a full analytical characterization is not possible, I derive what is perhaps the most surprising feature of optimal policy in this environment.

**Proposition 17** *In any Ramsey policy, $\pi^*$ $\exists t,\ s^t,\ s^{t+1}$ such that $\frac{p_{t+1}\left(s^{t+1}\right)}{p_t\left(s^t\right)} \neq \frac{1}{\sum_{s_{t+1}} q_{s_{t+1}}\left(s^t\right)}$.*

In other words, Ramsey policies feature departures from the Friedman rule. The key to understanding this result is to notice that unlike standard cash-in-advance models, the introduction of fiat money is not just a friction. This is because of the assumption that agents can hold money after default and so the return on money affects the value of defaulting. Since the value of default is increasing in the return on money, the Ramsey planner has an incentive to lower this rate since it makes default more costly and could sustain more risk-sharing. However, decreasing the return also worsens the cash-in-advance friction if the rate is different from the market interest rate. I prove that at the allocations and prices associated with the Friedman rule, the first effect dominates and the Ramsey planner does want to introduce a wedge in between market return and the return on money.

To prove this result, I first characterize the best equilibrium consistent with the Friedman Rule. As in Cole and Kocherlakota (1998) I show that this corresponds to the best equilibrium without cash-in-advance constraints. As this equilibrium is stationary, I consider the set of stationary equilibria and prove that moving away from the Friedman rule yields an equilibrium with strictly higher welfare for the planner. As a result the Ramsey policy must feature departures from the Friedman Rule.

The environment without cash-in-advance constraints is identical to the previous one except that now there is no fiat money in the environment. Agents can directly consume their labor output and can continue to trade Arrow securities. I assume that after default, agents can save in a risk-free security whose interest rate is determined by the market.

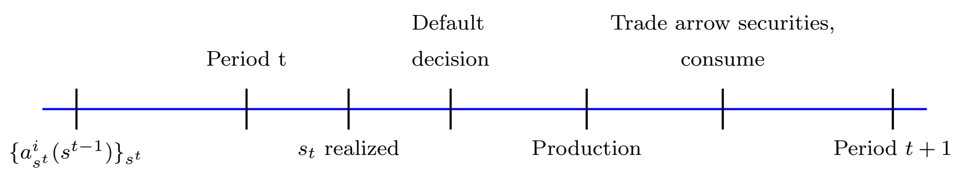


Figure 3.2: Timing

Agent $i$ chooses sequences of consumption, labor, and Arrow securities $\left\{c^i_t\left(s^t\right), l^i_t\left(s^t\right), \left\{a^i_{s_{t+1}}\left(s^t\right)\right\}_{s_{t+1}\in S}\right\}_{s^t\in S^t, t\geq 0}$ to maximize (3.1) subject to budget constraints,

$$c^i_t\left(s^t\right) + \sum_{s^{t+1}} q_{s_{t+1}}\left(s^t\right) a^i_{s_{t+1}}\left(s^t\right) \leq l^i\left(s^t\right) + a^i_i\left(s^{t-1}\right) \text{ if } s_t = i,$$

$$c^i_t\left(s^t\right) + \sum_{s^{t+1}} q_{s_{t+1}}\left(s^t\right) a^i_{s_{t+1}}\left(s^t\right) \leq a^i_j\left(s^{t-1}\right) \text{ if } s_t = j,$$

state contingent debt constraints (3.5) and the usual non-negativity constraints. As mentioned earlier, in each period an agent can default on his existing obligations consequently be barred from trading Arrow securities in all future periods but can continue to save in a risk-free bond. Given a date $t$ and state $s^t$, an agent who has defaulted chooses a sequence $\left\{c^i_{t'}\left(s^{t'}\right),\ l^i_{t'}\left(s^{t'}\right),\ x^i_{t+1}\left(s^t\right)\right\}_{s^{t'}\succeq s^t, t'\geq t}$ to maximize

$$\sum_{t'\geq t}\sum_{s^{t'}\succeq s^t} \beta^{t'-t}\pi\left(s^{t'} \mid s^t\right)\left[u\left(c^i_{t'}\left(s^{t'}\right)\right) - l^i_{t'}\left(s^{t'}\right)\right]$$

subject to budget constraints,

$$c_{t'}^{i}\left(s^{t'}\right)+Q_{t'}(s^{t'})x_{t'+1}^{i}\left(s^{t'}\right)\leq l_{t'}^{i}\left(s^{t'}\right),$$

a constraint that the agent can only save in a risk-free bond,

$$x_{t'+1}^{i}(s^{t'})\geq 0,\text{for all } t'\geq t$$

and other non-negativity constraints. Here $x_{t'}^{i}\left(s^{t'+1}\right)$ denotes the agent's holding of the risk free bond and $Q_{t'}(s^{t'})$ is the market price of a risk free bond. Denote the value of default in $t,s^t$ for agent $i$ by $V_t^{i,d}(s^t;\mathbf{Q}_t)$ where $\mathbf{Q}_t=\left\{Q_{t'}\left(s^{t'}\right)\right\}_{s^{t'}\geq s^t,t'\geq t}$.

**Definition 12** *A Competitive equilibrium consists of prices* $\left\{q_i\left(s^t\right),q_j\left(s^t\right)\right\}_{t,s^t}$, *debt constraints*
$\left\{\phi_{s'}^{i}\left(s^t\right)\right\}_{i,s^t,s'}$ *and allocations* $\left\{\left(c_t^i\left(s^t\right),l_t^i\left(s^t\right),\left\{a_{s'}^{i}\left(s^t\right)\right\}_{s'\in S}\right)_{i\in I}\right\}_{t,s^t}$ *such that*

- *Given prices and debt constraints, the allocations solve the agent's problem*
- *Markets clear*

$$\sum_{i\in I}c_t^i\left(s^t\right)=\sum_{i\in I}l_t^i\left(s^t\right) \text{ for all } t,s^t$$
$$\sum_{i\in I}a_{s'}^{i}\left(s^t\right)=0 \text{ for all } t,s^t,s'\in S$$

- *Debt constraints are chosen to be not-too-tight; For all* $i\in I,\ t\geq 0,\ s^t$,

$$V_t^{i,c}\left(s^t,\phi_{s_t}^{i}\left(s^{t-1}\right);\Phi_t^i(s^t)\right)=V_t^{i,d}(s^t;\mathbf{Q}_t)$$

Next I characterize the best stationary equilibrium in this environment.

**Proposition 18** *A stationary equilibrium exists with*

$$q^c+q^{nc}<1$$

*where*

$$q^c=q_j\left(s^t,i\right)=q_i\left(s^t,j\right)$$
$$q^{nc}=q_i\left(s^t,i\right)=q_j\left(s^t,j\right)$$

**Proof.** See Appendix. ■

A sketch of the proof is as follows. I first conjecture the form of the stationary equilibrium and later confirm that an equilibrium of the form exists. The equilibrium I conjecture has constant prices for each of the two Arrow securities- one that pays off if the state next period is the same and another which pays if the state switches. In addition, consumption and labor is constant for the productive and unproductive types. The first step in the proof is to consider a setup in which after default, agents are allowed to save in Arrow securities. This is the assumption made by Hellwig and Lorenzoni (2009) in an endowment economy. I show that an equilibrium of the conjectured form exists in this environment as well with risk free rate equaling 1. This is similar to the low-interest rate results proved by Hellwig and Lorenzoni (2009). They find that in equilibrium, in order to support borrowing and lending, the present discounted value of endowments must be infinite (as is true when the real interest rate is 1). The reason why this result is useful in proving our result is that one can compute the agents' choices after default in closed form and use this to derive properties of the equilibrium when the agents can only save in a risk free bond. Since the value of default when agents can save in Arrow securities is strictly greater than if agents can only save in a risk free bond, we know that at the debt constraints associated with the former punishment $\tilde{\phi}$

$$V^i\left(i,\tilde{\phi}\right) > V^{i,d}\left(i\right)$$

where $V^i\left(i,\tilde{\phi}\right)$ is the value of not defaulting when agent $i$ owes $\tilde{\phi}$ and $V^{i,d}\left(i\right)$ is the value of default when the agent can save in a risk free bond. One can use continuity arguments to show that when agents can only save in a risk free bond after default, we can sustain a higher level of debt than $\tilde{\phi}$. From an equilibrium relation between the debt constraints and the Arrow security prices I prove that the equilibrium interest rate $q^c + q^{nc}$ is strictly greater than 1.

A natural question that arises is whether the best competitive equilibrium is constrained efficient. In this case, one can just obtain the allocations corresponding to this equilibrium by solving the appropriate planning problem. However, as the next proposition shows, all competitive equilibria are inefficient. As a result we need to work directly with the decentralized environment in order to characterize the set of equilibria.

**Proposition 19** *Any competitive equilibrium is constrained inefficient.*

**Proof.** See Appendix. ■

Next I demonstrate that policies which implement the Friedman rule in the model with cash-in-advance constraints are consistent with an equilibrium where the allocations and Arrow security prices coincide with those in equilibrium characterized above. As mentioned earlier, this is similar to the result in Chari and Kehoe (1990). However in sharp contrast to them, as Proposition 19 suggests, this equilibrium is not constrained efficient. This also suggests that the Friedman rule might not coincide with the Ramsey policy as the Ramsey planner might want use the return on money to introduce distortions in the agents' decisions as a planner solving the constrained efficient planner would. Let $\left\{c_t^i\left(s^t\right), l_t^i\left(s^t\right), a_i^i\left(s^t\right), a_j^i\left(s^t\right), \phi_i^i\left(s^t\right), \phi_j^i\left(s^t\right), q_t^i\left(s^t\right), q_t^j\left(s^t\right)\right\}_{s^t,i}$ correspond to the best equilibrium constructed above

**Proposition 20** *There exist prices* $\left\{p_t\left(s^t\right)\right\}$ *such that* $\left\{c_t^i\left(s^t\right), l_t^i\left(s^t\right), a_i^i\left(s^t\right), a_j^i\left(s^t\right), \phi_i^i\left(s^t\right), \phi_j^i\left(s^t\right)\right\}_{s^t,i}$ *and prices* $\left\{p_t\left(s^t\right), q_t^i\left(s^t\right), q_t^j\left(s^t\right)\right\}$ *constitute a competitive equilibrium if*

1. $\liminf_{t\to\infty} M_t = 0$

2. $\inf_t M_t\beta^{-t} = \kappa > 0$

*In such an equilibrium, the nominal interest rate is 1.*

**Proof.** See Appendix. ■

This equilibrium can be implemented using a deterministic sequence $\{M_t\}_{t=0}^{\infty}$. The proof is similar to that of Proposition 16. I show that these policies are consistent with a price sequence that satisfies $\frac{p_t\left(s^t\right)}{p_{t-1}\left(s^{t-1}\right)} = \frac{1}{Q}$ where $Q = q^c + q^{nc}$ which are defined and computed in Proposition 18. This implies that in equilibrium, the multiplier on the cash-in-advance constraint is zero. The crucial point to note is that since in equilibrium the real return on money equals the return on risk free debt, the value of default if the agent can only hold money is identical to the case in which he can hold only a risk free bond. I can show that the allocation from the economy without money satisfies the agent's first order conditions given these prices and money supply policies. Finally, the Transversality condition for money holds if assumption 1 in the above proposition is satisfied. Assumption

2 guarantees that the money supply is large enough so that the CIA constraints can be satisfied.

I now prove that one can perturb this policy slightly and yield strictly higher welfare for the Ramsey planner. To do this I restrict the planner to choosing from a set of *stationary* equilibria. In a stationary equilibrium $p_t M_t = p_{t+1} M_{t+1}$ and so a money supply rule $\frac{M_{t+1}}{M_t} = \mu$ is equivalent to an interest rate rule $\frac{p_{t+1}}{p_t} = R^m$.

**Lemma 8** *Given a stationary interest rate rule policy $R^m$, an allocation and price pair constitute a stationary competitive equilibrium if*

$$
\begin{aligned}
& c_t^i\left(s^{t-1}, i\right) = c_t^j\left(s^{t-1}, j\right) = g\left(\frac{1}{R^m Q}\right) \\
& c_t^i\left(s^{t-1}, j\right) = c_t^j\left(s^{t-1}, i\right) = g\left(\frac{q^c}{R^m Q \beta \lambda}\right) \\
& l_t^i\left(s^{t-1}, i\right) = l_t^j\left(s^{t-1}, j\right) = g\left(\frac{1}{R^m Q}\right) + g\left(\frac{q^c}{R^m Q \beta \lambda}\right) \\
& q^{nc} = \beta\left(1-\lambda\right) \\
& \frac{\left(1-\beta\left(1-\lambda\right)\right)\left(u\left(g\left(\frac{1}{R^m Q}\right)\right) - \left[g\left(\frac{1}{R^m Q}\right) + g\left(\frac{q^c}{R^m Q \beta \lambda}\right)\right]\right) + \beta \lambda u\left(g\left(\frac{q^c}{R^m Q \beta \lambda}\right)\right)}{\left[1-\beta\left(1-\lambda\right)\right]^2 - \beta^2 \lambda^2} \\
& = V^{i,d}\left(s^t; R^m\right)
\end{aligned} \tag{3.7}
$$

*where*

$$
\begin{aligned}
q^c &= q_i\left(s^{t-1}, j\right) = q_j\left(s^{t-1}, i\right) \\
q^{nc} &= q_j\left(s^{t-1}, j\right) = q_i\left(s^{t-1}, i\right) \\
Q &= q^c + q^{nc}
\end{aligned}
$$

The lemma highlights the distortions introduced by departing from the Friedman Rule. For example, under the Friedman rule ($R^m Q = 1$), a productive agent always consumes $g(1)$ where $g = u'^{-1}$ while in this case it is $g\left(\frac{1}{R^m Q}\right)$. In addition, the money supply policy (which in this case equivalent to an interest rate policy) also affects the right hand side of the not-too-tight constraint (3.7).

Given our restriction to stationary equilibria, we can define the Ramsey policy as solving

$$
\begin{aligned}
&\max_{R^m} W\left(R^m, q^c\left(R^m\right)\right) \\
&= \max_{R^m} u\left(g\left(\frac{1}{R^m\left[q^c\left(R^m\right)+q^{nc}\right]}\right)\right) \\
&- \left[g\left(\frac{1}{R^m\left[q^c\left(R^m\right)+q^{nc}\right]}\right)+g\left(\frac{q^c\left(R^m\right)}{R^m\left[q^c\left(R^m\right)+q^{nc}\right]\beta\lambda}\right)\right] \\
&+ u\left(g\left(\frac{q^c\left(R^m\right)}{R^m\left[q^c\left(R^m\right)+q^{nc}\right]\beta\lambda}\right)\right) \qquad (3.8)
\end{aligned}
$$

where $q^c\left(R^m\right)$ solves (3.7). The next proposition is a restatement of Proposition 17.

**Proposition 21** *In the best equilibrium corresponding to the Friedman rule*

$$\left.\frac{\partial}{\partial R^m} W\left(R^m, q^c\left(R^m\right)\right)\right|_{R^m=\frac{1}{Q}} < 0$$

To understand this result notice that

$$\frac{\partial}{\partial R^m} W\left(R^m, q^c\left(R^m\right)\right) = W_1 + W_2 \cdot q^{c'}\left(R^m\right)$$

Consider the first term on the right hand side of the above equation. This is the value of lowering the return on money holding fixed the market prices. Since this worsens the frictions from the cash-in-advance constraint $W_1 < 0$. Next, the term $W_2 \cdot q^{c'}\left(R^m\right)$ measures the change in welfare given that the prices must adjust via (3.7) in order to constitute a not-too-tight equilibrium. Since lowering $R^m$ reduces the value of default, the price of the arrow security $q^c$ falls in order to satisfy this constraint. Since we are at a constrained equilibrium, lower $q^c$ implies a greater degree of risk sharing and therefore $W_2 \cdot q^{c'}\left(\ R^m\right) > 0$. Finally, one can show that when $R^m = \frac{1}{q^c+q^{nc}}$, $W_1 + W_2 \cdot q^{c'}\left(R^m\right) > 0$ or that the second effect dominates. It is worth noting that if there were no debt constraints, we would not want to depart from the Friedman rule as in that case $q^c = \beta\lambda$ and $q^{c'}\left(\ R^m\right) = 0$. As result setting $R^m < \frac{1}{Q}$ only worsens the friction from the CIA constraints.

From the above analysis it is clear that in any Ramsey equilibrium there must be some trade of Arrow securities. In particular since we can always find policies (for example the Friedman Rule) such that they induce some borrowing and lending among agents, the

Ramsey policy will also have this feature. In general however, the Ramsey policy $\pi^*$ might not be stationary.

The next main result shows that in general, policy $\pi^*$is also consistent with another equilibrium in which there is financial autarky and no Arrow securities are traded. We turn to the characterization of this equilibrium next.

**Proposition 22** *Given a Ramsey policy $\pi^* = \{M^*, T^*\}_0$ there exists another equilibrium in which*

$$\phi^i_{s'}\left(s^t\right) = 0 \textit{ for all } i, s^t, s'$$

**Proof.** See Appendix. ■

To prove this result, I conjecture an equilibrium in which debt constraints are zero and no Arrow securities are traded in any date and state. In such an equilibrium agents use cash balances to achieve limited smoothing. The structure of the conjectured equilibrium is similar to the environment studied by Scheinkman and Weiss (1986) except that it may not be Markov if in general the money supply policies are history dependent. I prove that an equilibrium exists in which the relevant aggregate state variables are $\left(s^t, z^t\right)$ where $z_t = \frac{\left[m^1_{t-1}\left(s^{t-1}\right) - \frac{c^1_{t-1}\left(s^{t-1}\right)}{p_{t-1}\left(s^{t-1}\right)}\right]}{M_t(s^t)}$ is the fraction of the money supply held by agent $i$. While in general proving the existence of equilibria with aggregate uncertainty and incomplete markets is tricky, the assumptions on the state space $S$ along with results from Miao (2006) allow us to prove existence in this setup. The idea is to first show that given continuous price functions for money $p_t\left(s^t, z^t\right)$, the agent's problem can be written as dynamic program and that a unique sequence of value functions and policy functions exist. The next step is to prove that such price functions do exists.

Finally to prove that the conjectured allocation in which securities are not traded constitutes an equilibrium, I show there exist Arrow security prices such that those along with this allocation and the associated money supply sequence constitute an equilibrium with $\phi^i_{s'}\left(s^t\right) = 0$ for all $i, s^t, s'$. The prices of Arrow securities satisfy

$$q_{s'}\left(s^t, z^t\right) = \max_i \left\{ \int_Z \frac{u'\left(c^i\left(s^{t+1}, z^{t+1}\right)\right)}{u'\left(c^i\left(s^t, z^t\right)\right)} Q_{t+1}\left(s', dz_{t+1}, s^t, z^t\right) \right\} \tag{3.9}$$

where $Q_{t+1}$ is the joint distribution of the aggregate states and $q_{s'}\left(s^t, z^t\right)$ is the price of an Arrow security in state $\left(s^t, z^t\right)$ which pays one unit of the consumption good if the state next period $s'$. The consumption policy functions $c^i\left(s^{t+1}, z^{t+1}\right)$ are those computed in the environment without Arrow securities. To see why this constitutes a Competitive Equilibrium, notice that for security prices defined as in (3.9), no agent wishes to save in Arrow securities since for any agent $i$ , $q_{s'}\left(s^t, z^t\right) \geq \int_Z \frac{u'\left(c^i\left(s^{t+1}, z^{t+1}\right)\right)}{u'\left(c^i\left(s^t, z^t\right)\right)} Q_{t+1}\left(s', dz_{t+1}, s^t, z^t\right)$ . Agents would like to borrow, but are constrained from doing so by the zero borrowing limits. As a result, markets for Arrow securities clear and hence we have an equilibrium.

## 3.3 Policy

The key takeaway from the previous section is that policies consistent with the Ramsey outcome which only depend on the exogenous states of the world can only weakly implement the desired equilibrium in models with debt constraints. This motivates the study of Sophisticated Policies as in Atkeson et al. (2010) where policies can also depend on histories of private actions and can differ on and off the equilibrium path. The main result in this section is that there exist such policies that uniquely implement the desired equilibrium. A key requirement of Sophisticated Policies is that all continuation outcomes be continuation competitive equilibria. This ensures that the approach does not achieve implementation via non-existence. To understand the construction of these policies and how they achieve implementation, I first consider a related environment with the property that the set of competitive equilibria is identical to environment above. This environment is formulated in terms of competitive intermediaries offering insurance contracts to agents. The reason I consider implementation in this setup is that the equilibrium conditions determining debt constraints arise naturally from the best responses of intermediaries. Policies will target these best responses and make deviations unprofitable.

### 3.3.1 Contracting framework

Consider a set $J$ of $T-$period lived intermediary and $I$ agents. Time $t = 0, 1, ..$ is discrete and let $S$ be the finite state space and $S^t$ be set of histories till time $t$ with typical element $s^t = (s_0, s_1, ..., s_t)$ (the timing will be described below). The state is known to all agents and the intermediary. The transition probabilities are given by a matrix $\Pi$, with the unconditional probabilities of histories denoted as $\pi\left(s^t\right)$ and the conditional probabilities by

$\pi\left(s^{t+1} \mid s^t\right)$. The symbol $\succeq$ is used the denote the partial order on histories. For example, $s^{t'} \succeq s^t$ for $t' \geq t$ denotes a possible continuation of history $s^t$. Given a random variable $x$, I use the notation $\{x\}_{t'}$ to denote the stochastic process $\left\{x_t\left(s^t\right); \forall\, t' \leq t \leq \infty,\ s^t \in S^t\right\}$. Given a stochastic process for consumption $\left\{c^i\right\}_0$ and labor $\left\{n^i\right\}_0$, the utility for agent $i$ is

$$\sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi\left(s^t\right)\left[u\left(c_t^i\left(s^t\right)\right) - l^i\left(s^t\right)\right]$$

I assume that $u^i$ is strictly increasing, strictly concave and $C^1$. We have $l^i\left(s^t\right) = 0$ if $s_t \neq i$.

Intermediaries are risk-neutral and offer insurance contracts to entrepreneurs. Formally, an intermediary $j$ offers agent $i$ in state $s^t$, a contract

$$C^{i,j}\left(s^t\right) = \left(\left(\left\{c_{t'}^i\left(s^{t'}\right), m_{t'}^i\left(s^{t'}\right), l_{t'}^i\left(s^{t'}\right), \zeta_{t'}^{i,j}\left(s^{t'}\right)\right\}_{s^{t'} \in S^{t'}}\right)_{t' \in \{t, \ldots t+T-1\}}\right)$$

where $\zeta_{t'}^{i,j}\left(s^{t'}\right)$, is the state contingent insurance offered by the intermediary $j$ in state $s^{t'}$.Along with transfers, the contract specifies a set of allocations to the agent. While intermediaries can observe the aggregate state (the history of agent's productivity shocks) and the actions of all other intermediaries, an important friction in the contracting environment is that the entrepreneurs' actions are unobservable. In particular the intermediary cannot observe the agent's consumption and money holding. Define $C^j \equiv \left(C^{i,j}\right)_{i \in I}$. Let $\mathcal{C}$ denote the space of contracts. $\mathcal{C}$ with the sup norm is a Banach space. Let $\mathcal{M}(C)$ denote the space of finite Borel measures over $\mathcal{C}$. Given a contract $C \in \mathcal{C}$, let $x_t^{C,i}\left(s^t\right)$ denote the value at $t, s^t$ of the stochastic process $x$ as specified by contract $C$.

The timing in the last period of a $T-$period contract is as follows[1] : At the beginning of period $t$, after $s_t$ is known, agents decide whether to default on payments owed to the intermediary. Next, intermediaries transfer $\zeta_t^{i,j}\left(s^t\right)$ to agents who have not defaulted in the past following which markets for money open. After this, agents split into a worker and shopper as in the previous section following which (unobservable) consumption takes place. If agents choose to default, they longer receive any insurance from the intermediary they are currently signed to and cannot sign with any other intermediary in the future. Given past beliefs about actions and taking the actions of all other intermediaries' as given,

[1] The other periods are identical, except agents dont sign with new intermediaries the following period

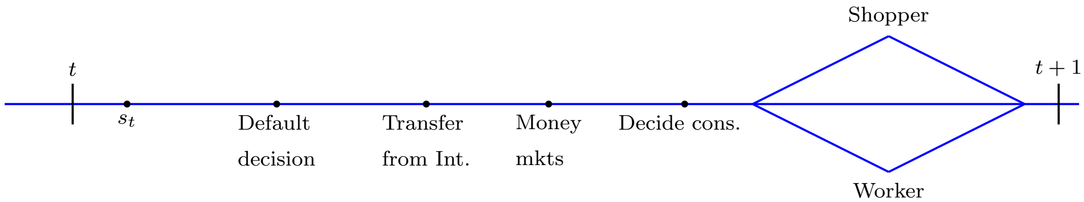


Figure 3.3: Timing.

any feasible contract $C^{i,j}\left(s^t\right)$ that an intermediary $j$ can offer in $s^t$ must satisfy budget constraints for all agents for all $t' \in \{t+1, ..., T-1\}$

$$p_{t'}\left(s^{t'}\right)m^i_{t'}\left(s^{t'}\right) = \frac{p_{t'}\left(s^{t'}\right)}{p_{t'-1}\left(s^{t'-1}\right)}l^i_{t'-1}\left(s^{t'-1}\right)+p_{t'}\left(s^{t'}\right)\left[m^i_{t'-1}\left(s^{t'-1}\right)-\frac{c^i_{t'-1}\left(s^{t'-1}\right)}{p_{t'-1}\left(s^{t'-1}\right)}\right]+\zeta^{j,i}_{t'}\left(s^t\right) \tag{3.10}$$

cash in advance constraints

$$c^i_{t'}\left(s^{t'}\right) \leq p_{t'}\left(s^{t'}\right)m^i_{t'}\left(s^{t'}\right) \text{ for all } i, t', s^{t'}$$

a participation constraint

$$\sum_{t'=t}^{\infty}\sum_{s^{t'}\succeq s^t}\beta^{t'-t}\left[u\left(c^i_{t'}\left(s^{t'}\right)\right)-l^i_{t'}\left(s^{t'}\right)\right] \geq \bar{V}^i_t\left(s^t\right)$$

and incentive compatibility constraints,

$$\sum_{t'=t}^{\infty}\sum_{s^{t'}\succeq s^t}\beta^{t'-t}\left[u\left(c^i_{t'}\left(s^{t'}\right)\right)-l^i_{t'}\left(s^{t'}\right)\right] \geq \hat{V}^i_t\left(s^t, \boldsymbol{\zeta}^{j,i}; \mathbf{p}_t\right) \tag{3.11}$$

where $\boldsymbol{\zeta}^{i,j}_t\left(s^t\right) = \left\{\left\{\zeta^{i,j}_{t'}\left(s^{t'}\right)\right\}_{t\leq t'\leq T-1, s^t\preccurlyeq s^{t'}\preccurlyeq s^{T-1}}, \boldsymbol{\zeta}^{i,j'}_{T-1}\left(s^{T-1}\right)\right\}, \mathbf{p}_t = \left\{p_{t'}\left(s^{t'}\right)\right\}_{s^{t'}\succeq s^t, t'\geq t}$.

$\hat{V}^i_t\left(s^t, \boldsymbol{\zeta}^{i,j}\left(s^t\right); \mathbf{p}_t\right)$ represents the best deviation an agent can undertake given the insurance contract offered by the intermediary and is the solution to

$$\max_{\{\tilde{c}_t, \tilde{m}_t, \Delta\}}\left(1-\Delta^i\left(s^t\right)\right)\left[\begin{array}{c}\sum_{t'=t}^{T-3}\beta^{t'-1}\prod_{\hat{t}=t+1}^{t'-1}\left(1-\Delta^i\left(s^{\hat{t}}\right)\right)\left[\begin{array}{c}\left(1-\Delta^i\left(s^{t'}\right)\right)\left[u\left(\tilde{c}^i_{t'}\left(s^{t'}\right)\right)-\tilde{l}^i_{t'}\left(s^{t'}\right)\right]\\ +\Delta^i\left(s^{t'}\right)V^{i,d}_{t'}\left(s^{t'}; \mathbf{p}_{t'}\right)\end{array}\right]\\ \prod_{\hat{t}=t+1}^{T-2}\left(1-\Delta^i\left(s^{\hat{t}}\right)\right)\left[\begin{array}{c}\left(1-\Delta^i\left(s^{T-1}\right)\right)\hat{V}^i_{T-1}\left(s^{T-1}, \boldsymbol{\zeta}^{i,j'}_{T-1}\left(s^{T-1}\right); \mathbf{p}_{T-1}\right)\\ +\Delta^i\left(s^{T-1}\right)V^{i,d}_{T-1}\left(s^{T-1}; \mathbf{p}_{T-1}\right)\end{array}\right]\end{array}\right]$$

$$+\Delta^i\left(s^t\right)V^{i,d}_t\left(s^t; \mathbf{p}_t\right) \tag{3.12}$$

subject to budget constraints

$$p_{t'}\left(s^{t'}\right)\tilde{m}^i_{t'}\left(s^{t'}\right) = \left[\begin{array}{c} \frac{p_t\left(s^{t'}\right)}{p_{t-1}\left(s^{t'-1}\right)}\tilde{l}^i_{t'-1}\left(s^{t'-1}\right) + p_{t'}\left(s^{t'}\right)\left[\tilde{m}^i_{t'-1}\left(s^{t'-1}\right) - \frac{\tilde{c}^i_{t-1}\left(s^{t'-1}\right)}{p_{t-1}\left(s^{t'-1}\right)}\right] \\ +\prod_{j=0}^{t'}\left[1-\Delta\left(s^j\right)\right]\zeta^{i,j}_{t'}\left(s^{t'}\right) \end{array}\right]$$

$$\text{for } t' = t+1, ..., T-1$$

and CIA constraints

$$\tilde{c}^i_{t'}\left(s^{t'}\right) \le p_{t'}\left(s^{t'}\right)\tilde{m}^i_{t'}\left(s^{t'}\right) \text{ for all } i, t, s^t$$

Here $\Delta^i\left(s^{t'}\right) \in \{0,1\}$ are the agent's default decision in state $s^{t'}$, $\hat{V}^i_{T-1}\left(s^{T-1}, \boldsymbol{\zeta}^{i,j'}_{T-1}\left(s^{T-1}\right); \mathbf{p}_{T-1}\right)$ is his continuation value of the best deviation if the agent chooses to not default in $s^{T-1}$ and sign with some intermediary $j'$, and $V^{i,d}_{t'}\left(s^{t'}; \mathbf{p}_{t'}\right)$ is the value of defaulting in $t'$ and not being able to sign with an intermediary in the future. Given any $t,\ s^t,\ V^{i,d}_{t'}\left(s^{t'}; \mathbf{p}_{t'}\right)$ is defined as the solution to maximizing

$$\sum_{t'=t}^{\infty}\sum_{s^{t'}\succeq s^t}\beta^{t'-t}\left[u\left(c^i_{t'}\left(s^{t'}\right)\right) - l^i_{t'}\left(s^{t'}\right)\right] \tag{3.13}$$

subject to

$$\begin{aligned} & p_t\left(s^t\right)m^i_t\left(s^t\right) \le \frac{p_t\left(s^t\right)}{p_{t-1}\left(s^{t-1}\right)}l^i_{t-1}\left(s^{t-1}\right) + p_t\left(s^t\right)\left[m^i_{t-1}\left(s^{t-1}\right) - \frac{c^i_{t-1}\left(s^{t-1}\right)}{p_{t-1}\left(s^{t-1}\right)}\right] \\ & c^i_t\left(s^t\right) \le p_t\left(s^t\right)m^i_t\left(s^t\right) \\ & l^i\left(s^t\right) = 0 \text{ if } s_t \ne i,\ l^i\left(s^t\right) \ge 0,\ c^i\left(s^t\right) \ge 0, m^i_t\left(s^t\right) \ge 0 \end{aligned} \tag{3.14}$$

As earlier, if the entrepreneur defaults he can continue to hold money after default but is barred from signing with intermediaries in all future periods.

At the end of the current $T$ period contract, if the agent hasn't defaulted in the past, he can sign with a new intermediary who offers him an insurance contract along with recommended allocations. Since actions are unobservable, the agent can choose different levels of consumption and money holdings. The term $\prod_{j=0}^{t}\left[1-\Delta\left(s^{j}\right)\right]\left[\zeta_{t}^{i,j}\left(s^{t}\right)\right]$ captures whether the agent has defaulted in the past or not. Notice that this formulation allows for a rich set of deviations an agent can undertake. For example, the agent can engage in a "double" deviation where he chooses to hold a different amount of money in $s^{t}$, and default the following period. The profit maximizing contract must prevent such deviations.

Intermediary $j$ is risk neutral and maximizes profits

$$-\sum_{i\in I}\sum_{t'=t}^{T}\sum_{s^{t'}\in S^{t'}}\left[\prod_{\hat{t}=t}^{t'}q_{s^{\hat{t}}}\left(s^{\hat{t}-1}\right)\right]\zeta_{t'}^{i,j}\left(s^{t'}\right)$$

Intermediaries can borrow and lend amongst each other at market determined state contingent prices $q_{s_{t+1}}\left(s^{t}\right)$.

**Definition 13** *Given a sequence of money supplies $\left\{M_{t}\left(s^{t}\right)\right\}_{t,s^{t}}$, a competitive equilibrium in the contracting environment consists of prices $\left\{q_{s_{t+1}}\left(s^{t}\right),p_{t}\left(s^{t}\right)\right\}_{t,s^{t}}$, allocations for each intermediary $\left(C^{i,j}\left(s^{t}\right),\bar{V}_{t}^{i}\left(s^{t}\right)\right)_{i,j}$ and a sequence of measures $\left\{\left(\mu_{s^{t}}^{i*}\right)_{i\in I}\right\}_{t,s^{t}}$ such that*

- *Given prices and the actions of other intermediaries, the contract offered by intermediary $j$ solves the problem,*

$$\max_{\{c,k,l,\zeta\}}-\sum_{i\in I}\sum_{t'=t}^{T}\sum_{s^{t'}\in S^{t'}}\left[\prod_{\hat{t}=t}^{t'}q_{s^{\hat{t}}}\left(s^{\hat{t}-1}\right)\right]\zeta_{t'}^{i,j}\left(s^{t'}\right)$$

*subject to*

$$p_{t'}\left(s^{t'}\right)m_{t'}^{i}\left(s^{t'}\right)=\frac{p_{t'}\left(s^{t'}\right)}{p_{t'-1}\left(s^{t'-1}\right)}l_{t'-1}^{i}\left(s^{t'-1}\right)$$

$$+p_{t'}\left(s^{t'}\right)\left[m_{t'-1}^{i}\left(s^{t'-1}\right)-\frac{c_{t'-1}^{i}\left(s^{t'-1}\right)}{p_{t'-1}\left(s^{t'-1}\right)}\right]+\zeta_{t'}^{i,j}\left(s^{t}\right),$$

$$c_{t}^{i}\left(s^{t}\right)\leq p_{t}\left(s^{t}\right)m_{t}^{i}\left(s^{t}\right),$$

$$\sum_{t'=t}^{\infty}\sum_{s^{t'}\succeq s^{t}}\beta^{t'-t}\left[u\left(c_{t'}^{i}\left(s^{t'}\right)-l_{t'}^{i}\left(s^{t'}\right)\right)\right]\geq\bar{V}_{t}^{i}\left(s^{t}\right),$$

$$\sum_{t'=t}^{\infty}\sum_{s^{t'}\succeq s^{t}}\beta^{t'-t}\left[u\left(c_{t'}^{i}\left(s^{t'}\right)-l_{t'}^{i}\left(s^{t'}\right)\right)\right]\geq\hat{V}_{t}^{i}\left(s^{t},\boldsymbol{\zeta}^{i,j}\left(s^{t}\right);\mathbf{p}_{t}\right)$$

- *Intermediaries make zero profits*
- *Markets clear*

$$\sum_{i\in I}\int_{\xi\in\mathcal{C}} c_t^{\xi,i}\left(s^t\right)d\mu_{s^t}^{i*}(\xi)=\sum_{i\in I}\int_{\xi\in\mathcal{C}} l_t^{\xi,i}\left(s^t\right)d\mu_{s^t}^{i*}(\xi)$$
$$\sum_{i\in I}\int_{\xi\in\mathcal{C}} \zeta_t^{\xi,i}\left(s^t\right)d\mu_{s^t}^{i*}(\xi)=0$$
$$\sum_{i\in I}\int_{\xi\in\mathcal{C}} m_t^{\xi,i}\left(s^t\right)d\mu_{s^t}^{i*}(\xi)=M_t^i\left(s^t\right)$$

In the definition, the measure $\mu_{s^t}^{i*}(\xi)$ denotes the measure of agents of type $i$ in equilibrium who are signed to contract $\xi$. One main result of this paper proves an equivalence between the sets of equilibria defined in the previous two sections.

**Theorem 5** *1. Given an equilibrium of the not-too-tight debt constraint economy,* $\left(\{q_{s'},p\}_0\ ,\left\{\left(\phi^i\right)_{i\in I}\right\}_0,\left\{\left(c^i,l^i,m^i,a_{s'}^i\right)_{i\in I}\right\}_0\right)$ *there exist,* $\left(\{\zeta^{i,j}\}_0,\{\bar{V}^i\}_0\right)_{i\in I},\left\{\left(\mu_s^{i*}\right)_{i\in I}\right\}_0$ *such that* $\left(\{q_{s'},p\}_0,C^{i,j}\left(s^t\right)=\left(\left(\left\{c_{t'}^i,m_{t'}^i,l_{t'}^i,\zeta_{t'}^{i,j}\right\}_0\right)\right),\{\bar{V}^i\}_0,\left\{\left(\mu_{s^t}^{i*}\right)_{i\in I}\right\}_0\right)$ *constitute an equilibrium in the contracting environment.*

*2. Given an equilibrium of the contracting environment* $\left(\{q_{s'},p\}_0,C^{i,j}\left(s^t\right)=\left(\left(\left\{c_{t'}^i,m_{t'}^i,l_{t'}^i,\zeta_{t'}^{i,j}\right\}_0\right)\right),\bar{V}_t^i\left(s^t\right),\left\{\left(\mu_{s^t}^{i*}\right)_{i\in I}\right\}_0\right)$, *there exist debt constraints* $\left\{\left(\phi^i\right)_{i\in I}\right\}_0$, *such that* $\left(\{q_{s'},p\}_0\ ,\left\{\left(\phi^i\right)_{i\in I}\right\}_0,\left\{\left(c^i,l^i,m^i,a_{s'}^i\right)_{i\in I}\right\}_0\right)$ *constitute an equilibrium with not-too-tight debt constraints.*

The full proof is in the appendix. Here I give an overview and some intuition for the result. Consider part 1 of the result. Suppose we have an equilibrium allocation and price sequence from the debt constrained problem. To show that these constitute an equilibrium in the contracting environment, we first need to construct a sequence of transfers $\left\{\zeta_{t'}^{i,j}\left(s^{t'}\right)\right\}$. We can do this by setting $\zeta_t^{i,j}\left(s^t\right)=a_{s_t}^i\left(s^{t-1}\right)-\sum_{s_{t+1}}a_{s_{t+1}}^i\left(s^t\right)$, for each $j$, which correspond the agent's net asset holdings at each date and state. Notice that the contract consisting of these transfers along with the allocations from the debt constrained competitive equilibrium is incentive compatible given prices in the contracting problem, since if the agent strictly preferred to default or choose some deviating allocation, he would done so in the contracting problem. Since debt constraints are chosen to prevent default in equilibrium, this choice is incentive compatible. The only concern is if an intermediary could offer a different contract with slightly more insurance that would make him and some agent strictly better off while respecting all other constraints. However, as I show in the proof, if there existed such a contract, then the current debt limits from the debt constrained equilibrium could not have been not-too-tight thus contradicting the assumption that it was.

The key step in proving the converse is to show that in any equilibrium contract satisfies

$$q_{s_{t+1}}\left(s^t\right) \geq \max_{i\in I}\left\{\frac{\beta\pi\left(s^{t+1}\mid s^t\right)u'\left(c^i_{t+1}\left(s^{t+1}\right)\right)}{u'\left(c^i_t\left(s^t\right)\right)}\right\} \text{ for all } t, s^t, s_{t+1}$$

The reason this is important is that we know from the literature on not-too-tight debt constraints that any equilibrium and price schedule must satisfy this condition. What its says is that the price of a Arrow security is determined by the unconstrained buyers of the security, i.e. those with the maximum marginal rates of substitutions. It is exactly for those agents, who are unconstrained, that the first order condition for the security holds with equality. For those that are constrained (sellers), the condition holds with an inequality. To prove that this condition holds in the contracting environment I show that if it did not hold, an intermediary could offer a different contract that would make both the agent and him strictly better off. To see why, consider the case in which $q_{s_{t+1}}\left(s^t\right) > \max_{i\in I}\left\{\frac{\beta\pi\left(s^{t+1}|s^t\right)u'\left(c^i_{t+1}\left(s^{t+1}\right)\right)}{u'\left(c^i_t\left(s^t\right)\right)}\right\}$ and some insurance is being offered. Since insurance is being offered there must exist some agent (receiving a positive transfer) who in $s^{t+1}$ strictly prefers to stay in the contract rather than default. The intermediary can offer this agent a little more transfer $q\left(s^{t+1}\mid s^t\right)\varepsilon$ today at the cost of reducing the transfer to him in $s^{t+1}$ by $\varepsilon$. We can approximate his change in utility using a Taylor expansion

$$\Delta u = \left[q_{s_{t+1}}\left(s^t\right)u'\left(c^i_t\left(s^t\right)\right) - \beta\pi\left(s^{t+1}\mid s^t\right)u'\left(c^i_{t+1}\left(s^{t+1}\right)\right)\right]\varepsilon$$

which is greater than zero since $q_{s_{t+1}}\left(s^t\right)$ is greater than the entrepreneur's marginal rate of substitution. Moreover, given that intermediaries can borrow and lend at prices

$q_{s_{t+1}}\left(s^t\right)$, such a contract is payoff neutral for the intermediary. Similarly, one can construct a deviating contract that makes both the intermediary and the agent strictly better off. This violates the zero profit condition and so is a contradiction. A similar argument applies for the reverse inequality.

The next step in the proof is to show that in any competitive equilibrium of the contracting environment in which full insurance is not being provided, for any consecutive states $\left(s^t, s_{t+1}\right)$, there exists some agent for whom

$$\hat{V}_t^i\left(s^t, \boldsymbol{\zeta}^{i,j}\left(s^t\right); \mathbf{p}_t \mid \delta\left(s^{t+1}\right) = 0\right) = V_t^{i,d}\left(s^t; \mathbf{p}_t\right)$$

The first term in the equality represents the agent's best deviation conditional on not defaulting[2] , and the second, is the value of defaulting. The idea is that if this were not the case, the intermediary could increase the amount of insurance being offered while continuing to respect the incentive compatibility constraints. This will be useful in our construction of not-too-tight debt limits. The construction of equilibrium debt constraints relies on a limiting argument due to Fudenberg and Levine (1983). The idea is to construct truncated allocations of the debt constrained environment, the limit of which converges to an equilibrium with not-too-tight debt constraints. I briefly summarize the construction here. Suppose we have an equilibrium allocation and price sequence from the contracting environment. We can construct a sequence of truncated Arrow security holdings as follows; for each $T$

$$a_{s_T}^{T,i}\left(s^{T-1}\right) = \zeta_T^{i,j}\left(s^T\right) \tag{3.15}$$

$$a_{s_t}^{T,i}\left(s^{t-1}\right) - \sum_{s^{t+1}} q_{s_{t+1}}\left(s^t\right) a_{s_{t+1}}^{T,i}\left(s^t\right) = \zeta_t^{i,j}\left(s^t\right) \text{ for all } t < T \tag{3.16}$$

Next, using the previous result I construct debt constraints equal to the asset holding of the agent $i$ for whom $\hat{V}_t^i\left(s^t, \boldsymbol{\zeta}^{i,j}\left(s^t\right); \mathbf{p}_t \mid \delta\left(s^{t+1}\right) = 0\right) = V_t^{i,d}\left(s^t; \mathbf{p}_t\right)$. For $t > T$ define, the truncated debt constraints and asset holdings to 0 for all agents. To complete the construction of the truncated allocations for $t \leq T$, let the consumption, labor and money holdings correspond to those from the Competitive equilibrium, while after $T$, correspond to the best allocation given 0 debt constraints. Clearly the above allocation does not

[2] Recall, that the agent's actions are hidden and so a deviation might involve choosing allocation different from those recommend and not necessarily defaulting.

constitute a competitive equilibrium with not-too-tight constraints. The rest of the proof involves showing that the limit of these truncated allocations constitute an equilibrium with not-too-tight constraints (given prices and money supply policies). To do this I show that the best deviation by any agent from these truncated allocations is bounded and that the maximal deviation possible converges to 0 as $T \to \infty$. We can then use results from Fudenberg and Levine (1983) to prove this allocation converge to an equilibrium.

We have established that when the space of policies are simply money supply rules, the set of equilibria from the two environments coincide. I now consider a larger space of policies and first define a competitive equilibrium when these policies can only depend on the exogenous state of the world. Next, I prove that set of equilibria is identical to the case in which policies only consist of money supply rules. This result suggests on the surface that expanding the set of policies in this fashion has no bite. However, when I allow these policies to also depend on histories of private actions, I prove that given a competitive equilibrium there exist such policies that uniquely implement it. Formally, let the space of policies be

$$\mathcal{P} = \left\{ \{M\}_0, \{\chi\}_0, \{R^g\}_0, \{\tau\}_0 \mid \forall t,\ s^t,\ \left(M_t\left(s^t\right), R_t^g\left(s^t\right), \tau_t\left(s^t\right)\right) \in \mathbb{R}_+^3,\ \chi_{s^t} : \mathcal{C} \to \{0,1\} \right\}$$

As before $\{M\}_0$ corresponds to the sequence of money supply policies $\left\{M_t\left(s^t\right)\right\}_{t,s^t}$. $\{\chi\}_0$ denotes a sequence of indicator functions that determine whether a particular contract is eligible for the following scheme: all agents signed to this contract who have not defaulted in the past, can deposit the cash received from selling consumption goods (when productive) with the government overnight and receive a return $R_{t+1}^g\left(s^t\right)$ the following period. $\{\tau\}_0$ denotes the sequence of lump sum tax rates that the government can impose on all agents. Given a policy $\pi \in \mathcal{P}$, the intermediaries and agents' problems is almost identical to that described above. A contract which eligible for this savings scheme also stipulates a choice of whether the agent should deposit his labor income with the government. Formally we denote this choice by a sequence indicator function $\{\Sigma\}_0$ where $\Sigma_{s^t} : \mathcal{P} \to \{0,1\}$ and a choice of 1 corresponds to saving with the government. All other agents have no such choice and as a result their decision problems are as above. An eligible agent's budget constraint is

$$p_{t'}\left(s^{t'}\right)m_{t'}^{i}\left(s^{t'}\right)=\mathcal{R}\left(l_{t'-1}^{i}\left(s^{t'-1}\right)\right)+p_{t'}\left(s^{t'}\right)\left[m_{t'-1}^{i}\left(s^{t'-1}\right)-\frac{c_{t'-1}^{i}\left(s^{t'-1}\right)}{p_{t'-1}\left(s^{t'-1}\right)}\right]+\zeta_{t'}^{i,j}\left(s^{t}\right)$$

where $\mathcal{R}\left(l_{t'-1}^{i}\left(s^{t'-1}\right)\right)=R_{t}^{g}\left(s^{t-1}\right)$ if $\Sigma_{s^{t-1}}=1$ and $\mathcal{R}\left(l_{t'-1}^{i}\left(s^{t'-1}\right)\right)=\frac{p_{t}\left(s^{t}\right)}{p_{t-1}\left(s^{t-1}\right)}$ if $\Sigma_{s^{t-1}}=0$. All agents not eligible for the scheme have budget constraints as in (3.10).

**Definition 14** *Given a policy rule $\pi\in\mathcal{P}$,a competitive equilibrium in this contracting environment consists of prices $\left\{q_{s_{t+1}}\left(s^{t}\right),p_{t}\left(s^{t}\right)\right\}_{t,s^{t}}$, allocations for each intermediary $\left(C^{i,j}\left(s^{t}\right),\bar{V}_{t}^{i}\left(s^{t}\right)\right)_{i,j}$ and a sequence of measures $\left\{\left(\mu_{s^{t}}^{i*}\right)_{i\in I}\right\}_{t,s^{t}}$ such that*

- *Given prices and the actions of other intermediaries, the allocation for the $s^{t}$ intermediary solves its problem,*
- *Intermediaries make zero profits*
- *Markets clear*

$$\sum_{i\in I}\int_{\xi\in\mathcal{C}}c_{t}^{\xi,i}\left(s^{t}\right)d\mu_{s^{t}}^{i*}\left(\xi\right)=\sum_{i\in I}\int_{\xi\in\mathcal{C}}l_{t}^{\xi,i}\left(s^{t}\right)d\mu_{s^{t}}^{i*}\left(\xi\right)$$
$$\sum_{i\in I}\int_{\xi\in\mathcal{C}}\zeta_{t}^{\xi,i}\left(s^{t}\right)d\mu_{s^{t}}^{i*}\left(\xi\right)=0$$
$$\sum_{i\in I}\int_{\xi\in\mathcal{C}}m_{t}^{\xi,i}\left(s^{t}\right)d\mu_{s^{t}}^{i*}\left(\xi\right)=M_{t}^{i}\left(s^{t}\right)$$
$$\sum_{i\in I}\int_{\xi\in\mathcal{C}}\chi_{s^{t}}\left(\xi\right)\left[R_{t}^{g}\left(s^{t}\right)-\frac{p_{t}\left(s^{t}\right)}{p_{t-1}\left(s^{t-1}\right)}\right]l_{t-1}^{\xi,i}\left(s^{t-1}\right)d\mu_{s^{t}}^{i*}\left(\xi\right)=2\tau_{t}\left(s^{t}\right)$$

As before, the measure $\mu_{s^{t}}^{i*}\left(\xi\right)$ denotes the measure of agents of type $i$ in equilibrium who are signed to contract $\xi$. The next important result states that in equilibrium if any of the contracts are eligible for the scheme, the rate of return offered by the government must coincide with the market return on money.

**Proposition 23** *In any competitive equilibrium if $\chi\left(C\right)=1$ for any $C$ such that $\mu^{i*}\left(C\right)>0$ for some $i$, $\tau_{t}\left(s^{t}\right)=0$ for all $t,s^{t}$ and $R_{t}^{g}\left(s^{t}\right)=\frac{p_{t}\left(s^{t}\right)}{p_{t-1}\left(s^{t}\right)}$ for all $t,s^{t-1},s^{t}$.*

**Proof.** See Appendix. ■

The reason for this is a simple feasibility argument. Consider the case in which all contracts offered in equilibrium are eligible for the scheme and that $R_t^g\left(s^t\right) > \frac{p_t\left(s^t\right)}{p_{t-1}\left(s^t\right)}$ for some date and state. Then it is easy to see that such a policy can never satisfy the government budget constraint since the total amount owed by the government exceeds the total output in the economy. On the other hand, if only some contracts are eligible then it must be these offer more insurance and as a result will be chosen by all agents and as a result, the same feasibility argument applies.

The result states that the set of equilibria with the expanded set of policy instruments is identical to the set with money supply rules as studied previously. Importantly, it says that these policies do not give the government the ability to implement equilibria better than the best equilibrium with money supply rules. As I will demonstrate in the next section such policies are useful because they help *uniquely* implement particular equilibria rather than expand the equilibrium set.

### 3.3.2 Sophisticated Policies

From the equivalence result proved in the previous section, we know that there are multiple equilibria in the contracting environment given the set of instruments available to the government. In particular, there is an equilibrium in which no intermediary offers any insurance and agents only use money balances to smooth consumption. The goal of this section is to construct policies that uniquely implement the desired equilibrium. I consider Sophisticated Policies as in Atkeson et al. (2010) which allow policy to depend on histories of private actions and to differ on and off the equilibrium path and require that all continuation outcomes be continuation competitive equilibria. There are two key requirements that any Sophisticated policy must satisfy

1. Controllability: A necessary condition for a set of contracts $\left(\tilde{C}^j, \tilde{C}^{-j}\right)$, where $C^{-j} = \left(C^{j'}\right)_{j' \neq j}$ is that $\tilde{C}^j \in BR^j\left(\tilde{C}^{-j}; \Phi^g\right)$ where $BR^j\left(\tilde{C}^{-j}; \Phi^g\right)$ denotes the best response correspondence of intermediary $j$ when other intermediaries are offering $\tilde{C}^{-j}$, conditional on prices, policy $(\Phi^g)$ and past histories. *We say that a Sophisticated policy is controllable if* $\tilde{C}^j \notin BR^j\left(\tilde{C}^{-j}; \Phi^g\right)$ *for* $\left(\tilde{C}^j, \tilde{C}^{-j}\right) \neq \left(C^{j*}, \tilde{C}^{-j*}\right)$ *and*

$C^{j*} \in BR^j\left(\tilde{C}^{-j*}; \Phi^g\right)$.

2. *For all histories (including deviations), the continuation outcomes constitute a continuation competitive equilibrium.*

An overview of the construction is as follows. We now allow the policy instruments defined in the previous section to also depend on endogenous histories. The policy which implements the desired equilibrium has the government intervening after certain histories and allowing certain contracts to be eligible for the risk free savings scheme. To illustrate why history dependence is important suppose we want to uniquely implement the equilibrium in which all intermediaries offer the contract associated with the best equilibrium $C^*$ and none of the agents default. We know that another equilibrium exists in which all intermediaries offer some contract $\tilde{C} \neq C^*$. Consider a history in which all but a positive measure of intermediaries offer $\tilde{C}$ while the small measure offer a contract $\hat{C}$ with the property that $\hat{C}$ offers more insurance than $\tilde{C}$. After such a history, the government allows only $\hat{C}$ to be eligible for the scheme. The role of the scheme is that given current prices and transfers, it raises the value of not defaulting relative to defaulting. As a result, a contract like $\hat{C}$ is incentive compatible and can be constructed to give the intermediaries positive profit. Therefore an allocation in which all intermediaries offer $\tilde{C}$ cannot constitute a competitive equilibrium. While there are other types of deviating allocations, the argument is similar. We formalize this argument below.

The timing of the game is follows (see Figure 4). After $s_t$ realized, agents make default decisions, then intermediaries make transfers to agents following which markets for money open. Next, the government announces which contracts are eligible for the scheme after which the agent splits into a shopper and worker. If eligible, the worker can hold the income from production with the government overnight and finally consumption takes place.

Let $\iota_t = \left(s_t, \left\{\Delta_t^i\right\}_i, \left\{q_{s_{t+1}}\right\}_{s_{t+1}}, \left\{C^{i,j}\right\}_{i,j}, \ p_t, \Phi_t^g, \tau_t\right)$ where $\Delta_t^i$ is agent $i's$ default decision, $C^{i,j}$ is the current contract agent between agent $i$ and intermediary $j$, $\Phi_t^g = \left(M_{t+1}, \chi_t, R_{t+1}^g, \tau_t\right)$ denotes the government policy and $h_t = (h_{t-1}, \iota_t)$ be the public history after period $t$. The public history faced by agents making their default decisions is $h_{dt}\left(h_{t-1}, s_t\right)$ and the default strategy is given by $\sigma_d^i\left(h_{dt}\right) = \Delta_t^i\left(h_{dt}\right)$. Consequently, the history faced by intermediaries when offering contracts is $h_{nt} = \left(h_{t-1}, s_t, \left\{\Delta_t^i\right\}, \left\{q_{s_{t+1}}\right\}_{s_{t+1}}\right)$.

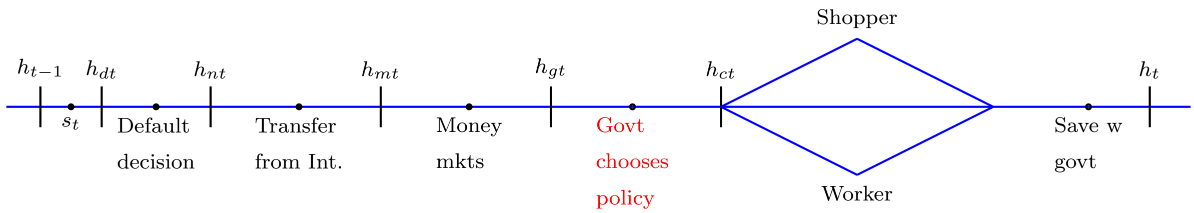


Figure 3.4: Timing

Denote the strategy for intermediary $j$ as $\sigma_n^j = \left\{C^{i,j}\left(h_{nt}\right)\right\}_i$. Next, the history faced by agents when purchasing money in private markets is $h_{mt} = \left(h_{t-1}, s_t, h_{nt}\right)$. We denote the strategy for agent $i$ as $\sigma_a^i = m_t^i\left(h_{mt}\right)(\cdot)$ where $m_t\left(h_{mt}\right)(\cdot)$ is a demand schedule for money

$$m_t^i\left(h_{mt}\right) : R_+ \rightarrow R$$

Agents submit demand schedules to a Walrasian auctioneer who chooses a price $p_t$ to clear the money markets. A price rule as determined by the auctioneer is given by $\sigma_p = \left\{p\left(h_{mt}\right)\right\}$ . The public history faced by the government when setting policy $h_{gt} = \left(h_{t-1}, s_t, \left\{\Delta_t^i\right\}_i, \left\{q_t^{s_{t+1}}\right\}_{s_{t+1}}, \left\{C^i\right\}_i, p_t\right)$. Note that the definition of Sophisticated equilibrium will not require that the government sets its policy optimally. Denote the government strategy by $\sigma_g = \left\{M_{t+1}\left(h_{gt}\right), \chi_t\left(h_{gt}\right), R_{t+1}^g\left(h_{gt}\right), \tau_t\left(h_{gt}\right)\right\}$. Next, before the worker and shopper split up, the agent decides how much consumption goods the shopper should purchase (subject to CIA constraints). Let $h_{ct} = \left(h_{t-1}, h_{gt}\right)$. Denote the consumption strategy by $\sigma_c^i = \left\{c_t^i\left(h_{ct}\right)\right\}$. Finally if eligible, the worker can hold the cash earned from producing consumptions goods with the government overnight. This strategy is given by $\sigma_\Sigma\left(h_{ct}\right) = \Sigma_t^i\left(h_{ct}\right)$.

Strategies induce continuation outcomes $\left\{o_r\left(s^r \mid h_{t-1}; \sigma\right)\right\}$ as follows. Agents' default policy is given by $\Delta^i\left(s^t \mid h_{dt}; \sigma\right) = \Delta_t^i\left(h_{dt}\right)$ which is obtained from $\sigma_d^i$, the intermediary's contract choice is given by $C^{i,j}\left(s^t \mid h_{nt}; \sigma\right) = C^{i,j}\left(h_{nt}\right)$ obtained from $\sigma_n^j$ and the agent's choice of money holdings is given by $m_t^i\left(s^t \mid h_{t-1}; \sigma\right) = m_t^i\left(h_{at}\right)$ obtained from $\sigma_a^i$. The government's policy is determined by $\chi_t\left(s^t \mid h_{gt}; \sigma\right) = \chi_t\left(h_{gt}\right)$, $R_{t+1}^g\left(s^t \mid h_{gt}; \sigma\right) = R_{t+1}^g\left(h_{gt}\right)$ and $\tau_t\left(s^t \mid h_{gt}; \sigma\right) = \tau_t\left(h_{gt}\right)$ obtained from $\sigma_g$. Finally, the agent's decision of how much to consume is given by $c_t^i\left(s^t \mid h_{ct}; \sigma\right) = c_t^i\left(h_{ct}\right)$ obtained from $\sigma_c^i$.

We can define the concept of a continuation competitive equilibrium as follows

**Definition 15** *A continuation competitive equilibrium given a history $h_{gt}$ is a collection of allocations, prices and policies that satisfy*

1. *Agent optimality conditions*
2. *Government budget feasibility*
3. *Market Clearing*
4. *Debt constraints are chosen to be NTT*

Using the previous definition, we can define a Sophisticated Equilibrium.

**Definition 16** *A Sophisticated equilibrium given policies is a collection of strategies for each agent $i \in I$, $\left(\sigma_d^i, \sigma_a^i, \sigma_c^i\right)$, intermediary $j \in J$, $\sigma_n^j$, the government $\sigma^g$ and price rules $\sigma_p$ such that*

1. *Given any history $h_{t-1}$, the continuation outcomes $\{o_r\left(s^r \mid h_{t-1}; \sigma\right)\}$ induced by $\sigma$ constitute a competitive equilibrium and*
2. *Given any history $h_{gt}$ the continuation outcomes $\{o_r\left(s^r \mid h_{gt}; \sigma\right)\}$ constitute a continuation competitive equilibrium*

As mentioned earlier, the key requirement of a Sophisticated equilibrium is that continuation outcomes be continuation competitive equilibria. Competitive equilibria must exist after all histories, and as a result we rule out policies that implement equilibria via non-existence. For example, policies that only feasible on path but are not off path can never be part of a competitive equilibrium.

**Definition 17** *A policy $\sigma_g^*$, uniquely implements a desired competitive equilibrium $\{o_r^*\}$ if the Sophisticated outcome associated with any Sophisticated equilibrium of the form $\left(\sigma_g^*, \sigma_d, \sigma_n, \sigma_a, \sigma_g, \sigma_{a2}, \sigma_p, \sigma_c\right)$ coincides with the desired competitive equilibrium.*

We now turn to the main result in the section which proves that there exists Sophisticated policies that uniquely implement the desired equilibrium.

**Theorem 6** *Given any symmetric competitive equilibrium, there exist Sophisticated policies that uniquely implement it.*

The policies I consider will allow the instruments defined in the previous section to now depend on histories $h_{gt}$. In particular, after such a history some agents will be allowed to hold money overnight with the government at attractive rates. Recall that an agent who is productive at date $t$ and state $s^t$ and who works an amount $l_t\left(s^t\right)$ receives his income $\frac{p_{t+1}\left(s^{t+1}\right)}{p_t\left(s^t\right)}l_t\left(s^t\right)$ the following period. Under this scheme, the agent will be able to hold the income overnight with the government and receive $R^g_{t+1}l_t\left(s^t\right)$ the following period with $R^g_{t+1} \geq \frac{p_{t+1}\left(s^{t+1}\right)}{p_t\left(s^t\right)}$. Let $\mathbf{R}^g_t = \left\{R^g_{t'}\right\}_{t'\geq t}$ denote a sequence of such interest rates. Define $V^{i,g}_t\left(s^t;\mathbf{R}^g_t,\mathbf{p}_t\right)$ to be the value for agent $i$ of the following problem; he maximizes (3.13) subject to budget constraints,

$$p_t\left(s^t\right)m^i_t\left(s^t\right) \leq \mathcal{R}\left(l^i_{t'-1}\left(s^{t'-1}\right)\right) + x^i_t + p_t\left(s^t\right)\left[m^i_{t-1}\left(s^{t-1}\right) - \frac{c^i_{t-1}\left(s^{t-1}\right)}{p_{t-1}\left(s^{t-1}\right)}\right],\ \text{if } s_{t-1} = i,$$

$$p_t\left(s^t\right)m^i_t\left(s^t\right) \leq p_t\left(s^t\right)\left[m^i_{t-1}\left(s^{t-1}\right) - \frac{c^i_{t-1}\left(s^{t-1}\right)}{p_{t-1}\left(s^{t-1}\right)}\right]\ \text{if } s_{t-1} \neq i$$

cash in advance constraints (3.4) and non-negativity constraints $m^i_{t+1}\left(s^t\right) \geq 0$. Note that this is the value to agent $i$ of not being able to trade arrow securities but have access to the overnight scheme. The following lemma will be useful for proving the result.

**Lemma 9** *For all $t, s^t$, there exists a sequence of risk free rates $\mathbf{R}^g_t = \left\{R^g_{t'}\right\}_{t'\geq t}$ with the property that for any agent $i$,*

$$V^{i,g}_{t+1}\left(s^{t+1};\mathbf{R}^g_{t+1},\mathbf{p}_{t+1}\right) > V^{i,d}_{t+1}\left(s^{t+1};\mathbf{p}_{t+1}\right)$$

**Proof.** Recall that $V^{i,d}_{t+1}\left(s^{t+1};\mathbf{p}_{t+1}\right)$ is the value of not participating in financial markets in all future periods for agent $i$ in $s^{t+1}$ and has been defined previously. While in financial autarky, agents can continue to hold money, thereby having access to a savings technology with interest rate $\frac{p_{t+1}}{p_t}$. Notice that under the government scheme the agent's budget constraints are identical except that they can hold (if they choose) their labor income with the government overnight. Clearly if $R^g_{t+1} \geq \frac{p_{t+1}}{p_t}$

$$V^{i,g}_{t+1}\left(s^{t+1};\mathbf{R}^g_{t+1},\mathbf{p}_{t+1}\right) \geq V^{i,d}_{t+1}\left(s^{t+1};\mathbf{p}_{t+1}\right)$$

with this inequality being strict if $R_{t+1} > \frac{p_{t+1}}{p_t}$. ■

**Proof of Theorem 6.** Let the histories associated with the desired equilibrium allocations/prices be denoted by *. We need to consider all possible deviations from the desired equilibrium and construct policy with the property that going along with these deviations is not in the best interest of an individual agent and that the continuation outcomes constitute continuation competitive equilibria.

We first specify government policy after all continuation histories $h_{gt}$ :

1. Consider a history
$\tilde{h}_{gt} = \left( h^*_{t-1}, s_t, \left\{\Delta^i_t\right\}_i, \tilde{p}_t, \left\{\tilde{q}^{s_{t+1}}_t\right\}_{s_{t+1}}, \left\{\tilde{C}^{i,j}\right\}_{j \in J \backslash J^d}, \left\{C^{i,j}\right\}_{j \in J^d}, \left\{\tilde{m}^i\right\}_{i \in I^d}, \left\{m^i\right\}_{i \in I \backslash I^d} \right)$.
In such a history, there is no default, but the equilibrium prices and contracts do not correspond to those of the desired equilibrium. Moreover, all intermediaries besides those in $J^d$ , offer contracts different from the desired ones while all intermediaries $j \in J^d$ offer a contract that with the property that some agent at some date receives strictly more insurance than $\tilde{C}$. These contracts satisfy
$$\frac{\beta\pi\left(s^{t+1} \mid s^t\right) u'\left(c^i_{t+1}\left(s^{t+1}\right)\right)}{u'\left(c^i_t\left(s^t\right)\right)} = \frac{\beta\pi\left(s^{t+1} \mid s^t\right) u'\left(\tilde{c}^i_{t+1}\left(s^{t+1}\right)\right)}{u'\left(\tilde{c}^i_t\left(s^t\right)\right)},$$
$$\text{for all } t \neq t', \left(s^t, s^{t+1}\right) \neq \left(s^{t'}, s^{t'+1}\right), i \neq i'$$
$$\frac{\beta\pi\left(s^{t+1} \mid s^t\right) u'\left(c^{i'}_{t'+1}\left(s^{t'+1}\right)\right)}{u'\left(c^i_{t'}\left(s^{t'}\right)\right)} > \frac{\beta\pi\left(s^{t'+1} \mid s^{t'}\right) u'\left(\tilde{c}^i_{t'+1}\left(s^{t'+1}\right)\right)}{u'\left(\tilde{c}^i_{t'}\left(s^{t'}\right)\right)} \text{ for } i$$
After such a history, the government sets $\chi_t\left(\tilde{h}_{gt}\right)(C) = 1$ and $\chi_t\left(\tilde{h}_{gt}\right)\left(\tilde{C}\right) = 0$ and so offers agents who have signed with intermediaries $j \in J^d$ and who haven't defaulted in the past, the ability hold their income overnight with the government at interest rates $\mathbf{R}^g_t\left(\tilde{h}_{gt}\right)$ in all future periods chosen so that
$$V^{i,g}_{t+1}\left(s^{t+1}; \mathbf{R}^g_{t+1}\left(\tilde{h}_{gt+1}\right), \mathbf{p}_{t+1}\right) > V^{i,d}_{t+1}\left(s^{t+1}; \mathbf{p}_{t+1}\right)$$
The rate of return on the government scheme is greater than the return on money. We know that such a policy exists from the previous lemma. This is financed by a

lumpsum tax on all agents. In order for the scheme to be feasible,

$$\sum_{i\in I}\int_{\xi\in\mathcal{C}}\chi_{s^t}(\xi)\left[R_t^g\left(s^t\right)-\frac{p_t\left(s^t\right)}{p_{t-1}\left(s^{t-1}\right)}\right]l_{t-1}^{\xi,i}\left(s^{t-1}\right)d\mu_{s^t}^{i*}(\xi)$$
$$=\sum_{i\in I}\left[R_t^g\left(s^t\right)-\frac{p_t\left(s^t\right)}{p_{t-1}\left(s^{t-1}\right)}\right]l_{t-1}^{C,i}\left(s^{t-1}\right)\mu_{s^t}^{i*}(C)=2\tau_t\left(s^t\right)$$

Therefore for $\mu_{s^t}^{i*}(C)$ small enough, the policy is always feasible. Since we need only consider infinitesimal deviations, the set $J^d$ can always be made small enough so that feasibility is satisfied.

2. Consider a history $\tilde{h}_{gt}=\left(h_{t-1}^*,s_t,\left\{\tilde{\Delta}_t^i\right\}_i,...\right)$, i.e. one in which all agents but a positive measure $\hat{\mu}^d$ of agents default on their obligations to the intermediaries. After such a history the government sets $\chi\left(\tilde{h}_{gt}\right)(C)=1$ for all contracts offered in equilibrium thereby offering all agents who haven't defaulted access to the savings scheme with high interest rates as described above. As $\hat{\mu}^d\to 0$, the amount owed by the government $\sum_{i\in I}\int_{\xi\in\mathcal{C}}\chi_{s^t}(\xi)\left[R_t^g\left(s^t\right)-\frac{p_t\left(s^t\right)}{p_{t-1}\left(s^{t-1}\right)}\right]l_{t-1}^{\xi,i}\left(s^{t-1}\right)d\mu_{s^t}^{i*}(\xi)\to 0$, since even though all contracts are eligible, most agents are choosing to default and hence cannot avail the scheme.

3. After all other histories, the government chooses $\chi\left(h_{gt}\right)$ so that none of the contracts are eligible for the scheme.

4. Money supply policies are chosen after all histories to be consistent with the Ramsey policy (which departs from the Friedman Rule).

   Next, I show how such a policy uniquely implements the desired equilibrium. We need to consider the set of competitive equilibria different from the desired one in the environment without government intervention and show that they cease to be equilibria given the above policy. There are two relevant deviations to consider. First consider the case in which in some period $\tau$, the government observes a history $\tilde{h}_{g\tau}$ with no history of default, but with $\tilde{h}_{g\tau}\neq h_{g\tau}^*$. For example, it could observe the prices and allocations corresponding to the worst competitive equilibrium (financial autarky).

If we consider symmetric equilibria, then after such a history, the government does not intervene. Now let us check if these allocations can actually constitute a CE. Even though all other intermediaries are offering contract $\tilde{C}$, consider the incentive of an intermediary $j$ to go along with this equilibrium. He knows by offering some agent slightly more insurance, his agents will have access to the savings scheme. As we will see, this allows him to offer some agent a little more insurance and allow him to make strictly positive profits.

Given that full insurance is not being provided, there is some agent $i$, states $s^{\tau}$, $s^{\tau+1}$ such that

$$\tilde{q}_{s_{\tau+1}}\left(s^{\tau}\right) > \frac{\beta\pi\left(s^{\tau+1} \mid s^{\tau}\right) u'\left(\tilde{c}^{i}_{\tau+1}\left(s^{\tau+1}\right)\right)}{u'\left(\tilde{c}^{i}_{\tau}\left(s^{\tau}\right)\right)}$$

Then an intermediary $j$ at $s^{\tau}$ can offer the following contract to agent $i$

$$\begin{aligned}\hat{\zeta}^{i,j}_{\tau}\left(s^{\tau}\right) &= \tilde{\zeta}^{s^{t},i}_{\tau}\left(s^{\tau}\right) + \tilde{q}_{s_{\tau+1}}\left(s^{\tau}\right)\varepsilon_{1}\\ \hat{\zeta}^{i,j}_{\tau+1}\left(s^{\tau+1}\right) &= \tilde{\zeta}^{i,j}_{\tau+1}\left(s^{\tau+1}\right) - \varepsilon_{2}\end{aligned}$$

with the rest of the contract unchanged. For $\varepsilon_1 = \varepsilon_2 = \varepsilon$ small the change in welfare is

$$\left[\tilde{q}_{s_{\tau+1}}\left(s^{\tau}\right) u'\left(\hat{c}^{i}_{\tau}\left(s^{\tau+1}\right)\right) - \beta\pi\left(s^{\tau+1} \mid s^{\tau}\right) u'\left(\hat{c}^{i}_{\tau+1}\left(s^{\tau+1}\right)\right)\right]\varepsilon > 0$$

For $\varepsilon$ small enough this contract gives greater utility to the agent while leaving the intermediary equally well off. In particular we can find $\varepsilon_1, \varepsilon_2$ such that the agent is made strictly better off and

$$-q_{s_{\tau+1}}\left(s^{\tau}\right)\varepsilon_{1} + q_{s_{\tau+1}}\left(s^{\tau}\right)\varepsilon_{2} > 0$$

Recall that since the tilde allocations correspond to a CE for this agent $i$

$$\sum_{t'=\tau+1}^{\infty}\sum_{s^{t'}\succeq s^{\tau}}\beta^{t'-\tau}\left[u\left(\tilde{c}^{i}_{t'}\left(s^{t'}\right) - \tilde{l}^{i}_{t'}\left(s^{t'}\right)\right)\right] = \hat{V}^{i}_{\tau+1}\left(s^{\tau+1}, \tilde{\boldsymbol{\zeta}}^{i,j}\left(s^{\tau+1}\right); \tilde{\mathbf{p}}_{\tau+1}\right)$$

and so if there were no government. intervention after such a history

$$\sum_{t'=\tau+1}^{\infty}\sum_{s^{t'}\succeq s^{\tau}}\beta^{t'-\tau}\left[u\left(\hat{c}^{i}_{t'}\left(s^{t'}\right) - \hat{l}^{i}_{t'}\left(s^{t'}\right)\right)\right] < \hat{V}^{i}_{\tau+1}\left(s^{\tau+1}, \tilde{\boldsymbol{\zeta}}^{i,j}\left(s^{\tau+1}\right); \tilde{\mathbf{p}}_{\tau+1}\right)$$

since the agent will strictly prefer to default and live in autarky forever. However, given government policy after such a history, the intermediary can choose some $\varepsilon_1, \varepsilon_2 \neq 0$ so that incentive compatibility still holds.

To see that all the incentive compatibility constraints are still satisfied notice that as long as $\hat{\zeta}_{t+1}^{s^t,i}\left(s^{t+1}\right)$ is chosen so that

$$u\left(\hat{c}_{\tau+1}^{i}\left(s^{\tau+1}\right)\right)+\beta E_{\tau+1}\left[V_{\tau+2}^{i,g}\left(s^{\tau+2};\mathbf{R}_{\tau+2}^{g}\left(\tilde{h}_{g\tau+2}\right),\mathbf{p}_{\tau+2}\right)\right]\geq V_{t+1}^{i,d}\left(s^{t+1};\mathbf{p}_{t+1}\right)$$

the incentive compatibility constraints are satisfied since

$$\begin{aligned}
&\sum_{t'=\tau+1}^{\infty}\sum_{s^{t'}\succeq s^{\tau}}\beta^{t'-\tau}\left[u\left(\tilde{c}_{t'}^{i}\left(s^{t'}\right)-\tilde{l}_{t'}^{i}\left(s^{t'}\right)\right)\right]\\
&\geq u\left(\hat{c}_{\tau+1}^{i}\left(s^{\tau+1}\right)\right)+\beta E_{\tau+1}\left[\hat{V}_{\tau+1}^{i}\left(s^{\tau+1},\boldsymbol{\zeta}^{i,j}\left(s^{\tau+1}\right);\mathbf{p}_{\tau+1},\mathbf{R}_{\tau+1}^{g},\Delta\left(s^{\tau+1}\right)=0\right)\right]\\
&\geq u\left(\hat{c}_{\tau+1}^{i}\left(s^{\tau+1}\right)\right)+\beta E_{\tau+1}\left[V_{\tau+2}^{i,g}\left(s^{\tau+2};\mathbf{R}_{\tau+2}^{g}\left(\tilde{h}_{g\tau+2}\right),\mathbf{p}_{\tau+2}\right)\right]\\
&\geq V_{\tau+1}^{i,d}\left(s^{\tau+1};\mathbf{p}_{\tau+1}\right)\\
&=\hat{V}_{\tau+1}^{i}\left(s^{\tau+1},\boldsymbol{\zeta}^{i,j}\left(s^{\tau+1}\right);\mathbf{p}_{\tau+1},\mathbf{R}_{\tau+1}^{g},\Delta\left(s^{\tau+1}\right)=1\right)
\end{aligned}$$

As a result, $\tilde{h}_{g\tau}$ can never be part of a competitive equilibrium.

The second kind of equilibria we need to rule out are sunspot equilibria in which agents default due to their beliefs about the types of contracts intermediaries will be willing to offer them in the future. Consider the case in which at date zero we start off with existing contracts in which some agents owe payments to the intermediaries. At date 0, there is a sunspot shock $\iota_0 \in \{0,1\}$ with $\Pr[\iota_0 = 0] = \psi$ which determines if agents stay in the prescribed equilibrium ($\iota_0 = 1$) or if autarky will be played in all future dates in states. If the sunspot is realized, agents who owe payments to the intermediary will strictly prefer to default and live in autarky forever. Moreover, given this no intermediary is willing to lend to agents in the current period As above, we need to consider government. policy after such a history which helps rule it out. Consider the policy after a history in which all agents besides a positive measure have defaulted. Then as described, the government. offers these agents access to the overnight savings scheme with interest rates chosen so that

$$V_{t+1}^{i,g}\left(s^{t+1};\mathbf{R}_{t+1}^{g}\left(\tilde{h}_{gt+1}\right),\mathbf{p}_{t+1}\right)>V_{t}^{i,d}\left(s^{t+1};\mathbf{p}_{t+1}\right)$$

As a result, even if all other agents are defaulting and no intermediary he is still willing to pay the intermediary back in order to be eligible for this government. scheme. Therefore, given that agents are not defaulting, an individual intermediary as an incentive to offer some insurance and so a history consisting of default by all agents and no intermediary offering insurance can never be part of a CE.

Finally we need to check that all continuation histories constitute continuation competitive equilibria. The feasibility of the government policy was shown earlier. To show that all continuation histories constitute continuation competitive equilibria, it suffices to note that since the government policy is feasible, and the prices already constitute a part of a CE, the continuation histories constitute a continuation CE. ■

The proof constructs policies such that given these, there exists a unique symmetric competitive equilibrium corresponding to the desired one. To show this, I consider any other allocation-price pair and show that this cannot be an equilibrium given these policies. There are two types of allocations to consider. The first is one in which there is no default, but all intermediaries offer contracts different from the desired one $C^*$. The second type is one in which there is default, and intermediaries offer some contract. Sophisticated policies must be specified after each history. After a history of no default and intermediaries offering contracts corresponding to the desired ones, the government continues its money supply policy consistent with the Ramsey policy. Next, consider a history in which all intermediaries besides those in some set $J^d$ offer $\tilde{C}$ different from the desired one, while intermediaries in $J^d$ offer contracts $\hat{C}$ which are identical to $\tilde{C}$ except that it offers some agent slightly more insurance. After such a history, the government offers an overnight risk free savings scheme to all agents signed with any of the intermediaries in $J^d$. Next consider a history in which all agents besides those in $I^d$ have defaulted on their obligations. After such a history, the government offers only agents in $I^d$ access to a similar overnight risk free savings technology as described above. After any other history not encompassing the ones specified above, the government continues with the usual Ramsey policy.

To see why how this policy rules out the types of equilibria specified above, first consider the allocation in which there is no default but intermediaries offer contracts $\tilde{C} \neq C^*$. Consider the best response of a particular intermediary $j$, when all other intermediaries are offering contracts $\tilde{C}$ given the Sophisticated policy constructed above.

Intermediary $j$ has an incentive to offer some agent slightly more insurance than $\tilde{C}$ which would make both him and the agent strictly better off. This perturbation is incentive compatible since it is exactly after such histories that the government chooses to intervene and offer its overnight scheme. The agent will not want to default if this savings technology is particularly attractive. As a result, even though all other intermediaries are offering contracts $\tilde{C}$, intermediary $j$ has an incentive to deviate and offer a different contract thereby implying that an equilibrium with contracts $\tilde{C}$ cannot exist. A similar argument works for an allocation with default. This includes sunspot allocations in which each period there is a positive probability that in all future periods, no intermediary will be willing to lend to agents. In such a state all agents who owe payments to intermediaries will strictly prefer to default. However as above, even if all other agents are defaulting and no intermediary in the future is willing to lend, a particular intermediary $j$ will have an incentive to offer some insurance to an agent making both strictly better off. The agent will not want to default if the return on the risk free savings technology offered by the government is sufficiently attractive. As a result, these sunspot outcomes will cease to be equilibria given these policies.

An important point to note is that the government cannot use these Sophisticated policies to achieve outcomes that welfare dominate all competitive equilibria. This is clear from proposition 23 which states that the government can never offer such a savings scheme on the equilibrium path. As a result the set of implementable equilibria correspond to the ones with simple money supply policies.

## 3.4 Discussion

In this section, I discuss some of assumptions and consider some generalizations of the results presented in this paper.

### 3.4.1 Generalizations

The model presented in the earlier sections was deliberately chosen to be simple. The assumptions allowed us to obtain sharp characterizations of the multiplicity and the effect of simple policies. The implementation argument presented in the last section did use any of these simplifying assumptions, for example that there were two types or the quasi-linearity of utility function. If we increase the number of types and assume more general utility functions, very similar policies will work. More generally, such policies will work in environments where the agents can achieve limited consumption smoothing after default.

### 3.4.2 Equivalence Theorem

The results in the previous sections illustrate why the equivalence result is useful. In particular, one way to interpret not-too-tight debt constraints is that they are constraints on debt derived from the profit maximizing incentive compatible contracts offered by intermediaries when the agents have limited commitment and their actions are unobservable. Consider using similar Sophisticated Policies to implement the desired equilibrium in the debt constrained environment. Suppose the government committed to offering a similar risk free savings scheme; after appropriately chosen histories it allowed a subset of agents to save with it at attractive rates. The reason this implemented the desired equilibrium in the contracting environment was that even though no other intermediary was willing to lend to the agent in the future, an individual intermediary was willing to offer some insurance to an agent. The reason the same logic does not quite apply to the debt constrained environment is that in any equilibrium

$$q_{s_{t+1}}\left(s^t\right) = \max_{i \in I}\left\{\frac{\beta\pi\left(s^{t+1} \mid s^t\right) u'\left(c^i_{t+1}\left(s^{t+1}\right)\right)}{u'\left(c^i_t\left(s^t\right)\right)}\right\}$$

In particular, since the Arrow prices equal the unconstrained agent's marginal rate of substitution (i.e. the savers), they have no incentive to lend to the constrained agents at these prices no matter what the interest rate on the savings scheme is. As a result, even though the government policy raises the value of not defaulting relative to default, at the market prices, the unconstrained agent is not willing to lend. Does this imply that these policies fail to implement the desired equilibrium? The answer to this question is no, since the definition of an equilibrium also requires these constraints to be not-too-tight. However since this neither an optimality nor a market clearing constraint in this environment it is unclear how they are affected by policy. This why the contracting environment is particularly suited to understanding implementation. The not-too-tight constraints now arise from the best responses of intermediaries. In that environment, the savings policy by the government achieves unique implementation because *offering no insurance is not a best response of a particular intermediary* even though all other intermediaries are not offering any insurance.

### 3.4.3 Using a Monetary Model

There are a few reasons why considering a monetary economy is attractive. The first is that it allows us to impose a reasonable assumption on what the consequences of default are. As mentioned in the introduction the literature has made various assumptions on what these can be, for example, autarky and being only allowed to save in Arrow securities. Here I assume that while defaulting agents cannot trade Arrow securities, they can continue to hold money. The interpretation is that money is hidden from the eyes of any regulatory authority and thus cannot be seized. As a result, defaulting agents can achieve some level of consumption smoothing. The second attractive feature of introducing money, is that in the bad equilibrium, there is still some consumption smoothing as opposed to pure autarky which is true in the other models with endogenous default.

The third attractive feature is that there is a literature on optimal policy in models with fiat money. This allows us to consider existing policy proposals that were deemed to implement desirable outcomes and see whether such policies continued to do so when we included financial frictions in these models. For example, policies consistent with the Friedman Rule are widely known to be desirable in monetary models. As I showed in previous sections, this is not the case here and moreover, there are also other undesirable equilibria when we include financial frictions. Notice that these equilibria are not the ones we are generally familiar with in monetary models in which there is an indeterminacy of the price level. The multiplicity arises because the endogenous debt constraints induce a strategic complementarity between the actions of agents. This motivates the study alternate policies and particular those that depend also on the histories of private actions.

## 3.5 Conclusion

Any reasonable discussion of government intervention in financial markets cannot be had using models that exogenously impose the form and tightness of financial constraints. In this paper, I provide a framework to think about the role policy can play in mitigating undesirable outcomes that occur in models with financial frictions. To do this I study a financial frictions model in which debt constraints are an equilibrium object. This model has multiple equilibria that differ in how well financial markets work. The main result in the paper shows that in models with endogenous debt constraints, policies that offer attractive savings opportunities after certain histories uniquely implement the best competitive equilibrium. There are two main results that are key to understanding the implementation argument. The first is that the set of equilibria in models of endogenous debt constraints is identical to the set in a decentralized contracting problem in which intermediaries offer insurance contracts to agents whose actions are hidden. The second result is that policies that offer such savings schemes after certain histories raise the value of not defaulting relative to that of defaulting for certain agent. As a result, even though all other intermediaries are offering contracts associated with an undesirable equilibrium, an individual intermediary strictly prefers to offer a different contract which offers more insurance to some agent. As a result, this ceases to be competitive equilibrium.

The environment was deliberately chosen to be simple in order to characterize in closed form the properties of the different equilibria. The implementation argument is very general and can apply to a wider variety of environments. For example, in ongoing work, I study a similar model with capital accumulation and heterogeneous entrepreneurs who has who run production technologies with stochastic productivities. As in this paper, I assume that these entrepreneurs are subject to state contingent debt limits which are determined to be not-too-tight. I show that a similar equivalence result holds for that environment as well and that are multiple equilibria. As a result, similar Sophisticated Policies can implement desirable outcomes in that setup as well.

# Chapter 4

# Multiplicity of Equilibria in Environments with Limited Commitment

## 4.1 Introduction

In this paper, I study decentralized contracting environments with limited commitment in which agents sign long term contracts with risk-neutral intermediaries. The agents are risk-averse entrepreneurs who hire labor and accumulate capital to run a production technology subject to idiosyncratic productivity shocks. Agents sign $T-$ period contracts with intermediaries who can borrow and lend with each other at market determined prices. At the beginning of each period, agents can choose to default on their obligations to the intermediary and consequently live in financial autarky in all future periods. I consider two assumptions on the observability of the agents' actions. The first is that all actions are observable and contracts can be contingent on the level of capital and labor hired by the entrepreneurs. The second is that these actions are hidden. The paper has two main results. The first is that under both assumptions the economy has multiple equilibria and one can generate sunspot equilibria in which there are endogenous fluctuations in aggregate real variables. The second main result is that under the hidden action assumption, the set of equilibria is identical to the set in an environment in which agents trade arrow securities subject to state contingent debt constraints which are "Not-too-Tight" as defined in Alvarez and Jermann (2000).

The decentralized environment is similar to the ones studied in Prescott and Townsend (1984) and Golosov and Tsyvinski (2007). There are a large number of $T-$period lived intermediaries who can borrow and lend in a state contingent fashion with each other. In each period agents who haven't defaulted in the past can sign contracts with an intermediary. In the benchmark case where the agent's actions are observable, a contract consists of a sequence of state contingent transfers and production plans for the agents. Agents run Cobb-Douglass production technologies and are subject to idiosyncratic productivity shocks. They can hire labor and accumulate capital and can themselves work on projects run by other entrepreneurs. At the beginning of each period, any agent can choose to default on her contract and thereafter be disallowed from signing with any intermediary in the future. However, the agent can continue to run the production technology and use her capital stock to achieve some degree of smoothing. Any feasible contract must satisfy resource and voluntary participation constraints. The latter constraints state that at any date and state, the value of not defaulting on the contract must be at least weakly greater than the value of default. The intermediary chooses a feasible contract that maximizes the present discounted value of its profits. I show that this environment has multiple competitive equilibria. The result may seem surprising since in general optimal contracting problems of this nature are thought to be associated with unique equilibria especially in Prescott and Townsend (1984) whether the equilibrium corresponds to unique efficient allocation. However, this is not true in the environment I consider and in general even the equilibrium associated with the highest ex-ante welfare is not in general efficient.

Multiple equilbria exist due to strategic complimentarities in the actions of intermediaries. Intermediaries know that agents' incentives to not default on a contract depend on the types of contracts they will be able to sign with future intermediaries. For example if all intermediaries today believe that no intermediary in the future will be lend to agents, the only feasible contract they can offer is one with no insurance. To see why notice that in the last period of the contract $T$, any agent owing positive transfers to the intermediary will strictly prefer to default on them and therefore any feasible contract cannot have an agent making positive transfers in the last period. Next consider period $T-1$ and a transfer sequence that consists of a negative transfer to the agent in $T-1$ and a positive transfer in $T$. Since in period $T$, the intermediary cannot promise a negative transfer to any agent such a sequence must be financed through the intermediary borrowing and lending. The result in the section states that in equilibrium, the interest rates are such that such transfer sequences are not profitable for the intermediary and therefore the intermediary strictly prefers to offer the zero insurance contract. The next result in this section uses a continuity argument to show that another equilibrium exists in which intermediaries find it optimal to offer some insurance.

Next, I turn to the case in which the agent's actions are hidden. Here, the intermediary cannot observe the agent's actions (capital accumulation, labor etc) and hence a contract cannot be contingent on these as they could in the environment with observable actions. Hidden actions allow for the agents to engage in double deviations in which they deviate from the prescribed allocation in one period and default the following period. The optimal contract must satisfy incentive compatibility constraints that prevent such deviations. The nature of incentive compatibility constraints in this problem makes the characterization of the contract difficult. The first main result in this section proves an equivalence between the set of equilibria in this contracting environment and the set in a different environment where agents trade Arrow securities subject to state contingent debt constraints.

The model with debt constraints turns out to be a a direct extension of the Alvarez and Jermann (2000) state contingent,"not-too-tight" debt constraint setup to the case with idiosyncratic productivity heterogeneity and capital accumulation. Entrepreneurs run constant returns to scale production technologies that are subject to productivity shocks. They hire labor, accumulate capital and can supply their own labor. As is standard in the literature on financial frictions, agents know their next period productivity shock in the current period and so would like to accumulate capital accordingly. Markets are complete and agents can purchase arrow securities subject to state contingent debt constraints. At any date and state, an entrepreneur can choose to default on her obligations and live in financial autarky forever. In autarky, she can continue to run her production technology, accumulate capital and work but can no longer trade arrow securities. Debt constraints are determined endogenously in equilibrium and are chosen to be "not-too-tight" as in Alvarez and Jermann (2000). This means that an entrepreneur who has borrowed up to the limit is indifferent between paying back her (state contingent) debt and defaulting and living autarky forever. In an endowment economy, Alvarez and Jermann (2000) show that "not-too-tight" constraints (weakly) decentralize the efficient Kehoe and Levine (1993) allocation. However, this decentralization fails to hold in the environment I consider. To see why, notice that entrepreneur's default problem depends on equilibirum prices such as the wage rate, which implies that prices show up in the voluntary participation constraints a planner would have to respect. This results in a pecuniary externality due to the fact that agents do not internalize the effect of their choices on their default incentives through equilibrium prices.

The result is significant for two reasons. On one hand, it allows us to study properties of the optimal contract by studying a relatively easier problem. Moreover, the equivalence allows us to interpret a natural extension of not-too-tight debt constraints as being derived from a decentralized contracting problem. This is useful since these constraints can longer be thought of as an implementing the efficient allocation. Importantly, these debt constraints do not arbitrarily restrict the types of contracts borrowers and lenders can sign. The intuition for the equivalence results is as follows- first consider the model with debt constraints. Using the entrepreneurs' choice of asset holdings given constraints and prices it is straightforward to construct an insurance contract that is feasible for an intermediary who is subject to the same prices. I show that if any contract that provides more insurance to the agent and keeps the intermediary equally well off necessarily violates incentive compatibility in that agents will choose to default in some state. Next, consider an equilibrium of the contracting environment. I show that any equilibrium must satisfy a property of any competitive equilibrium with borrowing constraints, namely that the arrow prices must equal the maximum marginal rate of substitution of the agents. This implies that given appropriately constructed borrowing constraints, the consumption allocation from the contracting problem satisfies the inter-temporal optimality constraint in the debt constraint problem. The only involved aspect of the proof is the construction of asset holdings and collateral constraints. To do this I consider a $T$ period truncated allocation in which agents are restricted to autarky after period $T$. I use results from Fudenberg and Levine (1983) and define the notion of an $\varepsilon-$perfect equilibrium where a deviating agent can gain at most $\varepsilon$ by deviating from the truncated allocation. I show that the limit of this allocation as $T \to \infty$ converges to an equilibrium of an economy with not-too-tight debt constraints.

As in the case of observable actions, the hidden action environment also has multiple equilibria. I demonstrate this in a simple two type deterministic model where the entrepreneurs' productivity alternate between high and low. Agents can borrow and lend subject to debt constraints and so markets are complete. The simplicity of the model allows us to highlight the source of multiplicity to be these endogenous debt constraints and not for example market incompleteness where multiplicity is known to exist. It easy to see that autarky is an equilibrium since the value of default trivially equals the value of being able to continue to participate in financial markets. Next, I use continuity properties of entrepreneurs' value functions to show that if autarkic interest rates are low, there exists a positive level of borrowing and lending such that all agents strictly prefer it to autarky. Consequently it is straightforward to prove the existence of a stationary equilibrium with non-zero debt constraints that have the not-too-tight property. The intuition for the existence of multiple equilibria is straightforward- expectations of the future tightness of debt constraints affects the current availability of credit. For example, if all entrepreneurs believe that there will be no borrowing and lending in the future, there will be no borrowing and lending in the present as no agent will be willing to pay back her debt. Similarly, the expectation that constraints will be loose in the future, allows for there to be borrowing and lending in the present period. In addition to the two stationary equilibria, there is an indeterminacy of equilibria (as in Woodford (1986b) etc.). In particular, for a large space of parameters there are a continuum of non-stationary equilibria converging to the autarkic equilibrium. The existence of indeterminacy allows us to construct sunspot equilibria. I show in a numerical example that expectational shocks can affect investment and aggregate output and that the effects are persistent. As a result I show that these models have an endogenous source of aggregate fluctuations.

**Literature:** This paper is related to three strands of literature. First, it draws from and contributes to the theory of optimal contracting, examples of which include Green (1987), Thomas and Worrall (1990), Atkeson and Lucas (1992) and Albuquerque and Hopenhayn (2004a). Albuquerque and Hopenhayn (2004a) study a contracting problem where the agent has a limited liability problem and debt payment cannot be enforced. The show that in the optimal contract, borrowing must be constrained. However in contrast to models with exogenous collateral constraints, they find that the borrowing constraints have a forward looking component in which future profitability affects current access to credit. This paper is also related to a growing subset of the contracting literature that deals with principal-agent problems in which agents can engage in hidden actions. Papers that deal with the hidden action problem include Cole and Kocherlakota (2001), Golosov and Tsyvinski (2007) and Shimer and Werning (2008). Golosov and Tsyvinski (2007) study the problem of providing insurance to agents with hidden types who can borrow and lend in a hidden fashion. Unlike the classic results of Atkeson and Lucas (1992) and Prescott and Townsend (1984), they find that the competitive equilibrium is inefficient. The reason for inefficiency is that prices show up in the consumption sets of agents. Our contracting environment is similar to their decentralized environment but with a few important differences. First, this model only has hidden actions with no hidden types. The equivalence result will not hold with hidden types as agents will not be willing to trade state contingent securities since there is no mechanism enforcing these trades when types are private. In addition, while Golosov and Tsyvinski (2007) consider a single $T$ period intermediary I consider the case in which agents sign consecutive $T$ period contracts with different intermediaries. This introduces a dynamic complementarity between the actions of intermediaries which allows for the equivalence between with the contracting environment. The dynamic complementarity is the primary reason for equilibrium multiplicity in both environments.

The model with endogenous debt constraints is related to the literature on financial frictions originating with Kiyotaki and Moore (1997). More recent examples include Gertler and Kiyotaki (2010), Shourideh and Zetlin-Jones (2012) and Buera and Moll (2012). Shourideh and Zetlin-Jones (2012) Chari (2012) study the flow of funds data and show that if I consider a representative firm, on average, funds flow from firms to households. In particular, the representative firm has more than enough cash in hand to undertake their investment. Consequently they argue that any reasonable financial frictions model must have heterogeneous firms to account for this fact. I incorporate this into our model with collateral constraints by introducing heterogenous entrepreneurs. Buera and Moll (2012) study variants of standard financial frictions models and show how they can be mapped to back to different aggregate wedges. However they assume incomplete markets and model collateral constraints that are linear in the agent's capital stock. As mentioned earlier in the introduction, it is unclear where the restriction on the types of contracts borrowers and lender sign come from. Gertler and Kiyotaki (2010) on the other hand derive their constraint from Kehoe and Levine (1993) like voluntary participation constraints. Unlike the previous papers however they do not have heterogeneous producers.

I choose to model debt constraints differently than the papers mentioned above . I follow Alvarez and Jermann (2000)'s definition of not-too-tight and define credit constraints that are endogenously determined in equilibrium. In particular, the constraints are chosen so that an agent who has borrowed up to the limit the previous period is indifferent between paying back and defaulting and living in financial autarky forever. While Alvarez and Jermann (2000) consider an endowment economy, I consider one with production heterogeneity and capital accumulation in order to relate it to some of the more recent models with financial frictions like Shourideh and Zetlin-Jones (2012) and Buera and Moll (2012). Alvarez and Jermann (2000) show that the efficient allocation from Kehoe and Levine (1993) can be decentralized using complete markets and not-too-tight collateral constraints. The decentralization is weak as they show that autarky is always an equilibrium in their environment. Moreover in a recent paper Bloise et al. (2013) show that the equilibrium is indeterminate and that for any value of aggregate welfare between the efficient allocation and autarky once can construct an equilibrium with not-too-tight constraints that achieve it. I show via an example that the indeterminacy results carry through to a more complicated environment with production heterogeneity and capital accumulation.

Finally, this paper is related to the literature on multiple equilibria in general equilibrium models. Woodford (1986a) has a simple example of an economy with finance constraints and shows the existence of self-fulfilling fluctuations similar to overlapping generations economies. Moreover he shows that the persistence of these fluctuations is similar to those of business cycle fluctuations. In another important paper, Woodford (1986b) provides conditions in order for a steady state of non-linear model to be indeterminate and proves that indeterminacy is a necessary and sufficiency condition for the existence of sunspot equilibria. I use these results to argue the existence of sunspot equilibria in our example. A related to paper to ours is Azariadis and Kaas (2012). They study a simple financial frictions model with two types and linear constraints on debt of the form $d \leq \theta k$. However, unlike some of the other papers, the parameter $\theta$ is an equilibrium object and is determined using a not-too-tight constraint. I take this analysis a step further and do not assume apriori a form on the collateral constraint. Similar to them, I also find multiplicity in this environment and use this fact to generate sunspot equilibria. Another related paper is Gu et al. (2013) who also prove the existence of multiple equilibria in a model with endogenous debt limits and show how one can generate endogenous fluctuations with sunspot dynamics.

The rest of the paper proceeds as follows: In section 2 and 3 I lay out the contracting model and the model with collateral constraints respectively. Then in section 4, I prove an equivalence between the two. Section 5 contains a simple example of the not-too-tight debt constraint economy and shows how one can use multiplicity to generate aggregate fluctuations Section 6 concludes.

## 4.2 Observable Actions

Consider an environment with a representative $T-$period lived intermediary and $I$ entrepreneurs (also referred to as agents). For ease of notation I assume that $T = 2$, but the argument generalizes for any finite $T$. Time $t = 0, 1, ..$ is discrete and let $Z$ be the finite state space and $S^t$ be set of histories till time $t$ with typical element $s^t = (z_0, z_1, ..., z_{t+1})$ (the timing will be described below). The state is known to all agents and the intermediary. The transition probabilities are given by a matrix $\Pi$, with the unconditional probabilities of histories denoted as $\pi\left(s^t\right)$ and the conditional probabilities by $\pi\left(s^{t+1} \mid s^t\right)$. The symbol $\succeq$ is used the denote the partial order on histories. For example, $s^{t'} \succeq s^t$ for $t' \geq t$ denotes a possible continuation of history $s^t$. Given a random variable $x$, I use the notation $\{x\}_{t'}$ to denote the stochastic process $\left\{x_t\left(s^t\right); \forall\ t' \leq t \leq \infty,\ s^t \in S^t\right\}$. Given stochastic process for consumption $\left\{c^i\right\}_0$ and labor $\left\{n^i\right\}_0$, the utility for entrepreneur $i$ is

$$\sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi\left(s^t\right)\left[u\left(c_t^i\left(s^t\right) - v\left(n_t^i\left(s^t\right)\right)\right)\right]$$

I assume that $u^i$ is strictly increasing, strictly concave and $C^1$ and that $v$ is strictly increasing, strictly convex and $C^1$. Entrepreneurs run a production technology where $k_t^i$ is the capital stock at the beginning of the period, $A_t^i\left(s^{t-1}\right)$ is the idiosyncratic productivity shock and $\hat{n}_t^i\left(s^t\right)$ is amount of labor hired.

$$\pi_t^i\left(s^t, k_t\right) = \max_{\hat{n}_t^i(s^t)} \left\{ A_t^i\left(s^{t-1}\right)\left(k_t^i\right)^\alpha \left(\hat{n}_t^i\left(s^t\right)\right)^{1-\alpha} + (1-\delta)\, k_t^i - w_t\left(s^t\right)\hat{n}_t^i\left(s^t\right) \right\}$$

Given constant returns to scale, it is easy to see that

$$\begin{aligned} \hat{n}_t^i\left(s^t\right) &= \rho\left(w_t\left(s^t\right), A_t^i\left(s^{t-1}\right)\right) k_t^i \\ \pi_t^i\left(s^t, k_t\right) &= r\left(w_t\left(s^t\right), A_t^i\left(s^{t-1}\right)\right) k_t^i \end{aligned}$$

where

$$\rho\left(w_t\left(s^t\right), A_t^i\left(s^{t-1}\right)\right) = \left(\frac{(1-\alpha)\, A_t^i\left(s^{t-1}\right)}{w_t\left(s^t\right)}\right)^{\frac{1}{\alpha}} \tag{4.1}$$

$$\begin{aligned} r\left(w_t\left(s^t\right), A_t^i\left(s^{t-1}\right)\right) = {} & A_t^i\left(s^{t-1}\right)\left(\frac{A_t^i\left(s^{t-1}\right)(1-\alpha)}{w_t\left(s^t\right)}\right)^{\frac{1-\alpha}{\alpha}} \\ & - w_t\left(s^t\right)\left(\frac{A_t^i\left(s^{t-1}\right)(1-\alpha)}{w_t\left(s^t\right)}\right)^{\frac{1}{\alpha}} + 1 - \delta \end{aligned} \tag{4.2}$$

This simplifies the problem of the intermediary as $I$ can just replace the intermediaries profit maximization problem with $r\left(w_t\left(s^t\right), A_t^i\left(s^{t-1}\right)\right) k_t^i$.

In this section I consider class of decentralized contracting problems except that competitive intermediaries can now observe and hence control the amount of capital accumulated by the agents/entrepreneurs. I show that there are multiple equilibria in this environment and thus demonstrate that the multiplicity is not special to the hidden trading setup. As mentioned earlier, an advantage of the hidden trading assumption is that I can directly deal with models with debt constraints which are in general easier to solve than contracting problems.

Intermediaries are risk-neutral and offer insurance contracts to entrepreneurs. Formally, given an intermediary born in period $t$ and state $s^t$, a contract is a vector

$$C^{s^t} = \left( \begin{array}{c} c_t^i\left(s^t\right), k_{t+1}^i\left(s^t\right), n_t^i\left(s^t\right), m_t^{i,s^t}\left(s^t\right), \\ \left(\left\{ c_{t'}^i\left(s^{t'}\right), k_{t'+1}^i\left(s^{t'}\right), n_{t'}^i\left(s^{t'}\right), m_{t'}^{i,s^t}\left(s^{t'}\right)\right\}_{s^{t'}\in S^{t'}}\right)_{t'\in\{t+1,\dots t+T-1\}} \\ , \left\{ m_t^{i,s^t}\left(s^{t+T}\right)\right\}_{s^{t+T}\in S^{t+T}} \end{array} \right)$$

where $m_{t'}^{i,s^t}\left(s^{t'}\right)$, is the state contingent insurance offered by the intermediary $s^t$ in state $s^{t'}$. Along with the insurance, the contract also specifies a set of allocations to each intermediary. Here the intermediaries can observe the aggregate state (the history of entrepreneurs productivity shocks), the actions of all other intermediaries and the actions of entrepreneurs.

The timing in the last period of a $T-$period contract is as follows[1] : At the beginning of period $t$, after $z_{t+1}$ is known, entrepreneurs decide whether to default on last period's intermediary. Notice that each entrepreneurs knows her $t+1$ productivity shock in period $t$. This is a commonly used timing assumption in the literature and is assumed by Buera and Moll (2012) and Shourideh and Zetlin-Jones (2012). Next, production takes place following which agents can sign with new intermediaries if they haven't defaulted before while paying back existing ones.

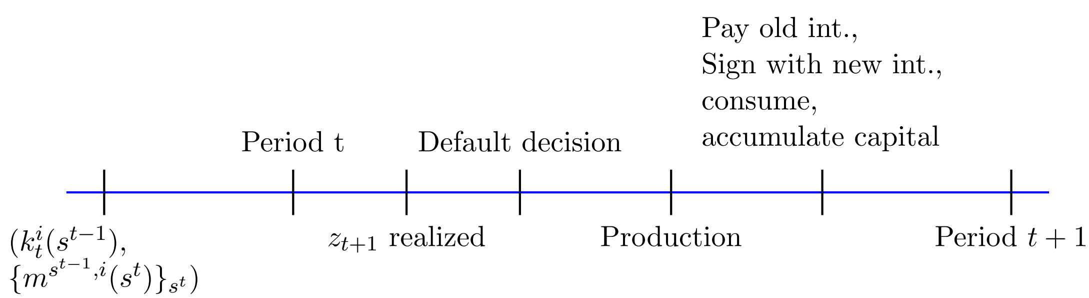


Figure 4.1: Timing

Taking as given the actions of other intermediaries, any feasible contract $C^{s^t}$ that an intermediary born in $s^t$ can offer must satisfy budget constraints in period $t$,

$$c_t^i\left(s^t\right)+k_{t+1}^i\left(s^t\right)=r\left(w_t\left(s^t\right),A_t^i\left(s^{t-1}\right)\right)k_t^i\left(s^{t-1}\right)+w_t\left(s^t\right)n_t^i\left(s^t\right)+m_t^{s^t,i}\left(s^t\right)+m_t^{s^{t-(T-1)},i}\left(s^t\right)$$

for all $t' \in \{t+1,...,T-1\}$. Note that if an agent signs a new contract in period $t$, the previous contract was signed in $t-(T-1)$.

$$c_{t'}^i\left(s^{t'}\right)+k_{t'+1}^i\left(s^{t'}\right)=r\left(w_{t'}\left(s^{t'}\right),A_{t'}^i\left(s^{t'-1}\right)\right)k_{t'}^i\left(s^{t'-1}\right)+w_{t'}\left(s^{t'}\right)n_{t'}^i\left(s^{t'}\right)+m_{t'}^{s^t,i}\left(s^t\right)$$

[1] The other periods are identical, except agents dont sign with new intermediaries the following period

and in period $T-1$,

$$c_{T-1}^{i}\left(s^{T-1}\right)+k_{T}^{i}\left(s^{T-1}\right)=r\left(w_{T-1}\left(s^{T-1}\right),A_{T-1}^{i}\left(s^{T-2}\right)\right)k_{T-1}^{i}\left(s^{T}\right)$$
$$+w_{T-1}\left(s^{T-1}\right)n_{T-1}^{i}\left(s^{T-1}\right)+m_{T-1}^{i,s^{t}}\left(s^{T-1}\right)+m_{T-1}^{i,s^{T-1}}\left(s^{T-1}\right)$$

a participation constraint

$$\sum_{t'=t}^{\infty}\sum_{s^{t'}\succeq s^{t}}\beta^{t'-t}\left[u\left(c_{t'}^{i}\left(s^{t'}\right)-v\left(n_{t'}^{i}\left(s^{t'}\right)\right)\right)\right]\geq\bar{V}_{t}$$

and a voluntary participation constraint,

$$\sum_{t'=t}^{\infty}\sum_{s^{t'}\succeq s^{t}}\beta^{t'-t}\pi\left(s^{t'}\mid s^{t}\right)\left[u\left(c_{t'}^{i}\left(s^{t'}\right)-v\left(n_{t'}^{i}\left(s^{t'}\right)\right)\right)\right]\geq V_{t}^{d}\left(s^{t},k_{t}^{i}\left(s^{t-1}\right);\mathbf{w}_{t}\right)\qquad(4.3)$$

where $\mathbf{w}_{t}=\left\{w_{t'}\left(s^{t'}\right)\right\}_{s^{t'}\succeq s^{t},t'\geq t}$. $V_{t'}^{i,d}\left(s^{t'},k_{t'}\left(s^{t'}\right);\mathbf{w}_{t}\right)$ is the value of defaulting in $t'$ and not being able to sign with an intermediary in the future. Given any $t$, $s^{t}$, $V_{t}^{i,d}\left(s^{t},k_{t}\left(s^{t}\right);\mathbf{w}_{t}\right)$ is defined as the solution to

$$V_{t}^{i,d}\left(s^{t},k^{i};\mathbf{w}_{t}\right)=\max\sum_{t'=t}^{\infty}\sum_{s^{t'}\succeq s^{t}}\beta^{t'-t}\left[u\left(c_{t'}^{i}\left(s^{t'}\right)-v\left(n_{t'}^{i}\left(s^{t'}\right)\right)\right)\right]\qquad(4.4)$$

subject to

$$c_{t}^{i}\left(s^{t}\right)+k_{t+1}^{i}\left(s^{t}\right)\leq r\left(w_{t}\left(s^{t}\right),A_{t}^{i}\left(s^{t-1}\right)\right)k^{i}+w_{t}\left(s^{t}\right)n_{t}^{i}\left(s^{t}\right)$$
$$c_{t'}^{i}\left(s^{t'}\right)+k_{t'+1}^{i}\left(s^{t'}\right)\leq r\left(w_{t'}\left(s^{t\prime}\right),A_{t'}^{i}\left(s^{t'-1}\right)\right)k_{t'}^{i}\left(s^{t'-1}\right)\qquad(4.5)$$
$$+w_{t'}\left(s^{t'}\right)n_{t'}^{i}\left(s^{t'}\right)\text{ for all }t'\geq t$$

In particular, if the entrepreneur defaults she can continue to produce, accumulate capital, hire labor and work but is barred from financial markets in all future periods. This assumption on what happens after default is similar to Alvarez and Jermann (2000).

Intermediaries are risk neutral and maximize profits

$$\max_{\{c,k,n,m\}}-\sum_{i\in I}\sum_{t'=t}^{T}\sum_{s^{t'}\in S^{t'}}\left[\prod_{\hat{t}=t}^{t'}q\left(s^{\hat{t}}\mid s^{\hat{t}-1}\right)\right]m_{t'}^{s^{t},i}\left(s^{t'}\right)$$

Notice that I allow intermediaries to borrow and lend among each other at market determined prices $q\left(s^{t+1}\mid s^{t}\right)$.

**Definition 18** *A competitive equilibrium in this contracting environment is prices* $\left\{q\left(s^{t+1} \mid s^{t}\right), w\left(s^{t}\right)\right\}_{t, s^{t}}$ *, allocations for each intermediary* $\left(C^{s^{t}}, \bar{V}_{t}\right)$ *such that*

- *Given prices and the actions of other intermediaries, the allocation for the* $s^t$ *intermediary solves its problem,*

$$\max_{\{c,k,n,m\}} -\sum_{i\in I}\sum_{t'=t}^{T}\sum_{s^{t'}\in S^{t'}}\left[\prod_{\hat{t}=t}^{t'} q\left(s^{\hat{t}} \mid s^{\hat{t}-1}\right)\right] m_{t'}^{s^{t},i}\left(s^{t'}\right)$$

*subject to*

$$\begin{aligned}
c_t^i\left(s^t\right)+k_{t+1}^i\left(s^t\right) &= r\left(w_t\left(s^t\right), A_t^i\left(s^{t-1}\right)\right) k_t^i\left(s^{t-1}\right)+w_t\left(s^t\right) n_t^i\left(s^t\right) \\
&\quad + m_t^{s^t,i}\left(s^t\right)+m_t^{s^{t-(T-1)},i}\left(s^t\right), \\
c_{t'}^i\left(s^{t'}\right)+k_{t'+1}^i\left(s^{t'}\right) &= r\left(w_{t'}\left(s^{t'}\right), A_{t'}^i\left(s^{t'-1}\right)\right) k_{t'}^i\left(s^{t'-1}\right)+w_{t'}\left(s^{t'}\right) n_{t'}^i\left(s^{t'}\right) \\
&\quad + m_{t'}^{s^t,i}\left(s^t\right) \\
c_{T-1}^i\left(s^{T-1}\right)+k_T^i\left(s^{T-1}\right) &= r\left(w_{T-1}\left(s^{T-1}\right), A_{T-1}^i\left(s^{T-2}\right)\right) k_{T-1}^i\left(s^{T}\right) \\
&\quad + w_{T-1}\left(s^{T-1}\right) n_{T-1}^i\left(s^{T-1}\right)+m_{T-1}^{i,s^t}\left(s^{T-1}\right)+m_{T-1}^{i,s^{T-1}}\left(s^{T-1}\right),
\end{aligned}$$

$$\sum_{t'=t}^{\infty}\sum_{s^{t'}\succeq s^t}\beta^{t'-t}\left[u\left(c_{t'}^i\left(s^{t'}\right)-v\left(n_{t'}^i\left(s^{t'}\right)\right)\right)\right] \geq \bar{V}_t$$

$$\sum_{t'=t}^{\infty}\sum_{s^{t'}\succeq s^t}\beta^{t'-t}\left[u\left(c_{t'}^i\left(s^{t'}\right)-v\left(n_{t'}^i\left(s^{t'}\right)\right)\right)\right] \geq V_t^d\left(s^t, k_t^i\left(s^{t-1}\right); \mathbf{w}_t\right)$$

- *Intermediaries make zero profits*
- *Markets clear*

$$\sum_{i\in I}\left[c_t^i\left(s^t\right)+k_{t+1}^i\left(s^t\right)\right]=\sum_{i\in I}\left[A_t^i\left(s^t\right)\rho\left(w_t\left(s^t\right), A_t^i\left(s^t\right)\right)+1-\delta\right] k_t^i\left(s^{t-1}\right)$$

$$\sum_{i\in I}\rho\left(w_t\left(s^t\right), A_t^i\left(s^{t-1}\right)\right) k_t^i=\sum_{i\in I} n_t^i\left(s^t\right)$$

An important implication of the fact that the intermediary has control of the agents' capital stock is that it can potentially lower it in order to relax future incentive compatibility constraints. It is this control that makes the existence of multiplicity rather surprising.

I first show that there exist prices such that autarky is an equilibrium in the contracting environment and in particular, intermediaries find it optimal to not offer an insurance.

**Proposition 24** *There exist prices* $\left\{q^a\left(s^t \mid s^{t-1}\right)\right\}_{s^t, s^{t-1}}$ *and allocations for the intermediary so that autarky is an equilibrium in the contracting environment. In particular, for all* $i, s^t$, $m_t^{s^t,i}\left(s^t\right) = 0$

**Proof.** Appendix. ■

This proves that there is an equilibrium in which intermediaries offer no insurance. The proof use a construction argument. I construct prices such that given these prices, the profit maximizing contract is one in which no insurance is offered. To do this a consider a setup in which agents accumulate capital and hire labor (and work) but have no access to financial markets. This has a well defined equilibrium which exists and is unique. Using these equilibrium allocations, I construct appropriate Arrow security prices. Then given these prices, I show that the constructed allocations along with zero transfers solve the intermediaries' problem. I use Lagrangian techniques developed by Kehoe and Perri (2002) and Marcet and Marimon (2011) to show that these allocation satisfy the first order conditions of the intermediary's problem.

Next, I show that there is a *conditionally efficient*[2] allocation with borrowing and lending which can be decentralized an equilibrium of the contracting environment.

**Definition 19** *Given wage rates* $\left\{w_t\left(s^t\right)\right\}_{t,s^t}$, *a conditonally efficient allocation is the solution to the problem*

$$\max_{c,n,k} \sum_{i \in I} \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi\left(s^t\right)\left[u\left(c_t^i\left(s^t\right) - v\left(n_t^i\left(s^t\right)\right)\right)\right]$$

*subject to*

$$\begin{aligned}\left(\lambda^c\left(s^t\right)\right): \sum_{i \in I}\left[c_t^i\left(s^t\right) + k_{t+1}^i\left(s^t\right)\right] \leq \\ \sum_i\left[r\left(w_t\left(s^t\right), A_t^i\left(s^{t-1}\right)\right) k_t^i\left(s^{t-1}\right) + w_t\left(s^t\right) n_t^i\left(s^t\right)\right]\end{aligned} \tag{4.6}$$

[2] As in Kehoe and Levine (1993)

*and*

$$\left(\beta^{t}\pi\left(s^{t}\right)\mu^{c,i}\left(s^{t}\right)\right):\sum_{t'=t}^{\infty}\sum_{s^{t'}\succeq s^{t}}\beta^{t'-t}\pi\left(s^{t'}\mid s^{t}\right)\left[u\left(c_{t'}^{i}\left(s^{t'}\right)-v\left(n_{t'}^{i}\left(s^{t'}\right)\right)\right)\right]\geq V_{t}^{d}\left(s^{t},k_{t}^{i}\left(s^{t-1}\right)\right)$$

**Lemma 10** *In any conditionally efficient allocation, for any $t$ and history $s^t$, only the technology with the highest productivity is operated, i.e. for $i^*$ such that $A_t^{i^*}\left(s^{t-1}\right)=\max_{i\in I}A_t^i\left(s^{t-1}\right)$, $k_t^{i^*}\left(s^{t-1}\right),n_t^{i^*}\left(s^t\right)>0$ while $k_t^i\left(s^{t-1}\right),n_t^i\left(s^t\right)=0$ for all $i\neq i^*$.*

**Proof.** The result follows from the fact that the return on investing a unit of capital in any technology is given by $r\left(w_t\left(s^t\right),A_t^i\left(s^{t-1}\right)\right)$ which is increasing in productivity. As a result, only the most efficient technology is used. ■

**Lemma 11** *A conditionally efficient allocation given wage rates $\left\{w_t\left(s^t\right)\right\}_{t,s^t}$ can be decentralized as an equilibrium of the contracting problem if and only if for each $t,s^t$*

$$w_t\left(s^t\right)=\left(\frac{\sum_{i\in I}\left((1-\alpha)A_t^i\left(s^t\right)\right)^{\frac{1}{\alpha}}k_t^i\left(s^{t-1}\right)}{\sum_i n_t^i\left(s^t\right)}\right)^{\frac{1}{\alpha}} \tag{4.7}$$

**Proof.** Appendix. ■

The proposition gives a necessary and sufficient condition for any conditionally efficient allocation to constitute a competitive equilibrium. The condition just follows from labor market clearing $\sum_{i\in I}\rho\left(w_t\left(s^t\right),A_t^i\left(s^t\right)\right)k_t^i\left(s^{t-1}\right)=\sum_i n_t^i\left(s^t\right)$. The first step in the proof is to construct an appropriate sequence of Arrow security prices and show that given these, the contract corresponding to the conditionally efficient allocation solves the intermediary's problem. Finally, the above condition ensures that markets clear.

**Proposition 25** *There exists a non-autarkic competitive equilibrium*

**Proof.** It suffices to prove the existence of a fixed point $w^*$ of the following map: Define the operator $\Gamma:l^\infty\to l^\infty$ such that

$$\Gamma\left(w\right)\left(s^t\right)=\left(\frac{\sum_{i\in I}\left((1-\alpha)A_t^i\left(s^t\right)\right)^{\frac{1}{\alpha}}k_t^i\left(s^{t-1}\right)}{\sum_i n_t^i\left(s^t\right)}\right)^{\frac{1}{\alpha}}$$

where $k_t^i\left(s^{t-1}\right),n_t^i\left(s^t\right)$ are solution to the conditionally efficient planning problem

In general, this map is not a contraction. And so one cannot apply the standard contraction mapping theorem in order to prove the existence of a fixed point. We need to rely on more general arguements.

For some $W$ large but finite, consider the set of infinite sequences where the index set spans $t, s^t$, such that for each $t, s^t$, $w_t\left(s^t\right) \in [0, W]$. Let $\mathcal{W}$ the subset of $l^\infty$ containing such sequences. Then $\mathcal{W}$ is compact in the weak topology by Tychonoff's theorem. Next, I show that the map $\Gamma$ is continuous. Given sequence of wage rates $w$, let correspondence $G(w)$ denote the constraint set of the conditionally efficient planning problem. By the maximum theorem, is straightforward to show that $V_t^d\left(s^t, k_t^i\left(s^{t-1}\right)\right)$ is continuous in $w_t\left(s^t\right)$. As result, $G(w)$ is a continuous correspondence since the utility function is continuous. Then again, by the maximum theorem, the policy functions are continuous in $w_t\left(s^t\right)$ (assuming uniqueness). As a result $\Gamma(w)\left(s^t\right)$ is continuous. Then from Schauder's Fixed Point theorem, I know there exists a fixed point of $\Gamma$ in $\mathcal{W}$. This establishes the existence of an equilibrium different from autarky. ■

## 4.3 Hidden Actions

In this section I present a similar environment to the one in the previous setions except that the agents' actions are hidden. As in the previous section given an intermediary born in period $t$ and state $s^t$, a contract is a vector

$$C^{s^t} = \begin{pmatrix} c_t^i\left(s^t\right), k_{t+1}^i\left(s^t\right), n_t^i\left(s^t\right), m_t^{i,s^t}\left(s^t\right), \\ \left(\left\{c_{t'}^i\left(s^{t'}\right), k_{t'+1}^i\left(s^{t'}\right), n_{t'}^i\left(s^{t'}\right), m_{t'}^{i,s^t}\left(s^{t'}\right)\right\}_{s^{t'} \in S^{t'}}\right)_{t' \in \{t+1, \ldots t+T-1\}} \\ ,\left\{m_t^{i,s^t}\left(s^{t+T}\right)\right\}_{s^{t+T} \in S^{t+T}} \end{pmatrix}$$

where $m_{t'}^{i,s^t}\left(s^{t'}\right)$, is the state contingent insurance offered by the intermediary $s^t$ in state $s^{t'}$. Along with the insurance, the intermediary recommends a set of allocations to each intermediary. However unlike the previous section, while intermediaries can observe the aggregate state (the history of entrepreneurs productivity shocks) and the actions of all other intermediaries, an important friction in the contracting environment is that the entrepreneurs' actions are unobservable. In particular the intermediary cannot see the amount of capital being accumulated by an agent.

As in the previous section a feasible contract must satisfy resource feasiblity and an ex-ante participation constraint. However now, the contract must satisfy an incentive compatibility constraint,

$$\sum_{t'=t}^{\infty}\sum_{s^{t'}\succeq s^{t}}\beta^{t'-t}\left[u\left(c_{t'}^{i}\left(s^{t'}\right)-v\left(n_{t'}^{i}\left(s^{t'}\right)\right)\right)\right]\geq\hat{V}_{t}^{i}\left(s^{t},m^{s^{t},i},k_{t}^{i}\left(s^{t-1}\right),\mathbf{w}_{t}\right) \tag{4.8}$$

where $\mathbf{w}_{t}=\left\{w_{t'}\left(s^{t'}\right)\right\}_{s^{t'}\succeq s^{t},t'\geq t}$. $\hat{V}_{t}^{i}\left(s^{t},m,k_{t}^{i},\mathbf{w}_{t}\right)$ represents the best deviation an agent can undertake and is the solution to

$$\begin{aligned}
&\hat{V}_{t}^{i}\left(s^{t},m,k_{t}^{i},\mathbf{w}_{t}\right)\\
&=\max_{\left\{\tilde{c}_{t},\tilde{k}_{t+1},\Delta\right\}}\left(1-\Delta^{i}\left(s^{t}\right)\right)\left(\sum_{t'=t}^{T-1}\beta^{t'-1}\prod_{\hat{t}=t+1}^{t'-1}\left(1-\Delta^{i}\left(s^{\hat{t}}\right)\right)\left[\begin{array}{c}\left(1-\Delta^{i}\left(s^{t'}\right)\right)u\left(\tilde{c}_{t'}^{i}\left(s^{t'}\right)\right)\\+\Delta^{i}\left(s^{t'}\right)V_{t'}^{i,d}\left(s^{t'},k_{t'}^{i},\mathbf{w}_{t'}\right)\end{array}\right]\right.\\
&\quad\left.+\prod_{\hat{t}=t+1}^{T-1}\left(1-\Delta^{i}\left(s^{\hat{t}}\right)\right)\left[\begin{array}{c}\left(1-\Delta^{i}\left(s^{T}\right)\right)\hat{V}_{T}^{i}\left(s^{T},m,k_{T}^{i}\right)\\+\Delta^{i}\left(s^{T}\right)V_{T}^{i,d}\left(s^{T},k_{T}^{i},\mathbf{w}_{T}\right)\end{array}\right]\right)\\
&\quad+\Delta^{i}\left(s^{t}\right)V_{t}^{i,d}\left(s^{t},k_{t}^{i},\mathbf{w}_{t}\right)
\end{aligned} \tag{4.9}$$

subject to

$$\tilde{c}_{t}^{i}\left(s^{t}\right)+\tilde{k}_{t+1}^{i}\left(s^{t}\right)=r\left(w_{t}\left(s^{t}\right),A_{t}^{i}\left(s^{t-1}\right)\right)k_{t}^{i}+w_{t}\left(s^{t}\right)\tilde{n}_{t}^{i}\left(s^{t}\right)+\prod_{j=0}^{t}\left[1-\Delta\left(s^{j}\right)\right]\left[\begin{array}{c}m_{t}^{s^{t-1},i}\left(s^{t}\right)\\+m_{t}^{s^{t},i}\left(s^{t}\right)\end{array}\right]$$

$$\tilde{c}_{t'}^{i}\left(s^{t}\right)+\tilde{k}_{t'+1}^{i}\left(s^{t'}\right)=\left[\begin{array}{c}r\left(w_{t'}\left(s^{t'}\right),A_{t'}^{i}\left(s^{t'-1}\right)\right)k_{t'}^{i}+w_{t'}\left(s^{t'}\right)\tilde{n}_{t'}^{i}\left(s^{t'}\right)\\+\prod_{j=0}^{t'}\left[1-\Delta\left(s^{j}\right)\right]m_{t'}^{s^{t},i}\left(s^{t'}\right)\end{array}\right],\ t'=t+1,...,T-1$$

$$\tilde{c}_{T}^{i}\left(s^{t}\right)+\tilde{k}_{T+1}^{i}\left(s^{T}\right)=\left[\begin{array}{c}r\left(w_{T}\left(s^{T}\right),A_{T}^{i}\left(s^{T-1}\right)\right)\tilde{k}_{T}^{i}\left(s^{T-1}\right)+w_{T}\left(s^{T}\right)\tilde{n}_{T}^{i}\left(s^{T}\right)\\+\prod_{j=0}^{T}\left[1-\Delta\left(s^{j}\right)\right]\left[m_{T}^{s^{t},i}\left(s^{T}\right)+m_{T}^{s^{T},i}\left(s^{T}\right)\right]\end{array}\right]$$

Here $\Delta^{i}\left(s^{t'}\right)\in\{0,1\}$ are the agent's default decision in $t'$, $r\left(w_{t'}\left(s^{t'}\right),A_{t'}^{i}\left(s^{t'-1}\right)\right)$ is the return on capital accumulated the previous period (profits from production) as defined above, $\tilde{n}^i_{t'}\left(s^{t'}\right)$ is the amount the entrepreneur chooses to work herself in any other production technology, $\hat{V}^i_T\left(s^T, m, k^i_T\right)$ is her continuation value of not defaulting in $T$ given her chosen capital stock (potentially different from intermediaries' recommendations) and $V^{i,d}_{t'}\left(s^{t'}, k_{t'}\left(s^{t'}\right), \mathbf{w}_t\right)$ was defined in (4.4).

In each period, if the agent chooses to not default, she can sign with a new intermediary who offers her an insurance contract along with recommended allocations. Since actions are unobservable, the agent can choose different levels of consumption and capital accumulation. Again, in period $t+1$, she can choose to default on her obligations, take her current capital stock and live in autarky forever. The term $\prod_{j=0}^{t}\left[1-\Delta\left(s^j\right)\right]\begin{bmatrix} m_t^{s^{t-1},i}\left(s^t\right) \\ +m_t^{s^t,i}\left(s^t\right)\end{bmatrix}$ captures whether the agent has defaulted in the past or not. Notice that this formulation allows for a rich set of deviations an agent can undertake. For example, she can engage in a "double" deviation where she chooses to accumulate a different amount of capital in $s^t$, and default the following period. The profit maximizing contract must prevent such deviations.

**Definition 20** *A competitive equilibrium in this contracting environment is prices* $\left\{q\left(s^{t+1}\mid s^t\right), w\left(s^t\right)\right\}_{t,s^t}$, *allocations for each intermediary* $\left(C^{s^t}, \bar{V}_t\right)$ *such that*

- *Given prices and the actions of other intermediaries, the allocation for the $s^t$ intermediary solves its problem,*

$$\max_{\{c,k,n,m\}} -\sum_{i\in I}\sum_{t'=t}^{T}\sum_{s^{t'}\in S^{t'}}\left[\prod_{\hat{t}=t}^{t'} q\left(s^{\hat{t}}\mid s^{\hat{t}-1}\right)\right] m_{t'}^{s^t,i}\left(s^{t'}\right)$$

*subject to*

$$c_t^i\left(s^t\right)+k_{t+1}^i\left(s^t\right)=r\left(w_t\left(s^t\right),A_t^i\left(s^{t-1}\right)\right)k_t^i\left(s^{t-1}\right)+w_t\left(s^t\right)n_t^i\left(s^t\right)+m_t^{s^t,i}\left(s^t\right)$$
$$+m_t^{s^{t-(T-1)},i}\left(s^t\right),$$
$$c_{t'}^i\left(s^{t'}\right)+k_{t'+1}^i\left(s^{t'}\right)=r\left(w_{t'}\left(s^{t'}\right),A_{t'}^i\left(s^{t'-1}\right)\right)k_{t'}^i\left(s^{t'-1}\right)+w_{t'}\left(s^{t'}\right)n_{t'}^i\left(s^{t'}\right)$$
$$+m_{t'}^{s^t,i}\left(s^t\right)$$
$$c_{T-1}^i\left(s^{T-1}\right)+k_T^i\left(s^{T-1}\right)=r\left(w_{T-1}\left(s^{T-1}\right),A_{T-1}^i\left(s^{T-2}\right)\right)k_{T-1}^i\left(s^T\right)$$
$$+w_{T-1}\left(s^{T-1}\right)n_{T-1}^i\left(s^{T-1}\right)+m_{T-1}^{i,s^t}\left(s^{T-1}\right)$$
$$+m_{T-1}^{i,s^{T-1}}\left(s^{T-1}\right),$$
$$\sum_{t'=t}^{\infty}\sum_{s^{t'}\succeq s^t}\beta^{t'-t}\left[u\left(c_{t'}^i\left(s^{t'}\right)-v\left(n_{t'}^i\left(s^{t'}\right)\right)\right)\right]\geq\bar{V}_t$$
$$\sum_{t'=t}^{\infty}\sum_{s^{t'}\succeq s^t}\beta^{t'-t}\left[u\left(c_{t'}^i\left(s^{t'}\right)-v\left(n_{t'}^i\left(s^{t'}\right)\right)\right)\right]\geq\hat{V}_t^i\left(s^t,m^{s^t,i},k_t^i\left(s^{t-1}\right),\mathbf{w}_t\right)$$

- *Intermediaries make zero profits*

- *Markets clear*

$$\sum_{i\in I}\left[c_t^i\left(s^t\right)+k_{t+1}^i\left(s^t\right)\right]=\sum_{i\in I}\left[A_t^i\left(s^t\right)\rho\left(w_t\left(s^t\right),A_t^i\left(s^t\right)\right)+1-\delta\right]k_t^i\left(s^{t-1}\right)$$
$$\sum_{i\in I}\rho\left(w_t\left(s^t\right),A_t^i\left(s^{t-1}\right)\right)k_t^i=\sum_{i\in I}n_t^i\left(s^t\right)$$

## 4.4 Model with Debt Constraints

Consider a model with $i\in I$ entrepreneurs. Time $t=0,1,..$ is discrete and let $Z$ be the finite state space and $S^t$ be set of histories till time $t$ with typical element $s^t=(z_0,z_1,...,z_{t+1})$.

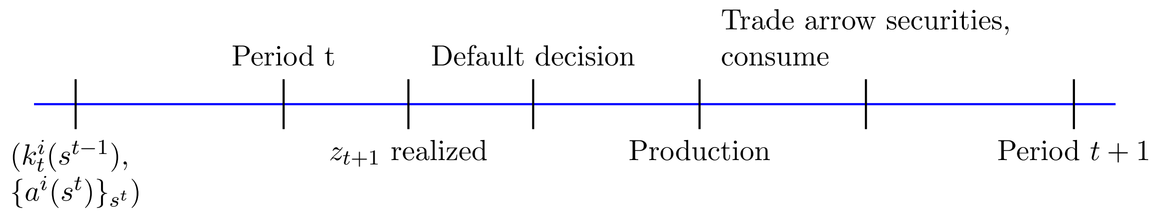


Figure 4.2: Timing

The timing is as follows: each entrepreneur enters the period with capital stock and a vector of arrow securities $\left(k_t^i\left(s^{t-1}\right), \left\{a_t^i\left(s^{t-1}, z_{t+1}\right)\right\}_{z_{t+1}\in Z}\right)$. At the start of period $t$, the state next period $z_{t+1}$ is realized and as a consequence entrepreneurs know their next period productivity shock $A_{t+1}^i\left(s^t\right)$. Next, agents make default decisions on their (state contingent) debt, following which production takes place. Finally, agents make consumption decisions and if they haven't defaulted in the past, purchase arrow securities for the next period.

As described earlier, entrepreneurs are subject to idiosyncratic productivity shocks $A_t^i\left(s^{t-1}\right)$ which are known the previous period. Formally the entrepreneur's ex-ante problem is to maximize her preferences

$$\sum_{t=0}^{\infty}\sum_{s^t\in S^t}\beta^t\pi\left(s^t\right)\left[u\left(c_t^i\left(s^t\right)-v\left(n_t^i\left(s^t\right)\right)\right)\right]$$

subject to the budget constraint at each $t, s^t$

$$\begin{aligned} & c_t^i\left(s^t\right)+\sum_{z_{t+2}\in Z}q\left(s^{t+1}\mid s^t\right)a^i\left(s^t, z_{t+2}\right)+k_{t+1}^i\left(s^t\right) \\ & =\pi_t^i\left(s^t, k_t\right)+w_t\left(s^t\right)n_t^i\left(s^t\right)+a^i\left(s^t\right) \end{aligned} \tag{4.10}$$

where $a^i\left(s^t\right)$ is the current holdings of arrow securities, $\left\{a^i\left(s^t, z_{t+2}\right)\right\}_{z_{t+2}\in Z}$ is the portfolio of securities chosen in period $t$, $k_{t+1}^i\left(s^t\right)$ is the capital stock chosen, $n_t^i\left(s^t\right)$ is the amount the entrepreneur chooses to work (for some other entrepreneur) and $\pi_t^i\left(s^t, k_t\right)$ is the profits from running her production technology,

$$\pi_t^i\left(s^t, k_t\right)=\max_{\hat{n}_t^i(s^t)}\left\{A_t^i\left(s^{t-1}\right)\left(k_t^i\right)^{\alpha}\left(\hat{n}_t^i\left(s^t\right)\right)^{1-\alpha}+(1-\delta)k_t^i-w_t\left(s^t\right)\hat{n}_t^i\left(s^t\right)\right\}$$

Note that $\hat{n}_t^i\left(s^t\right)$ is the amount of labor the entrepreneur chooses to hire for her own production technology. As in Alvarez and Jermann (2000), an entrepreneur's purchases of arrow securities are constrained by state contingent debt constraints

$$a^i\left(s^t, z_{t+2}\right) \geq \phi^i\left(s^t, z_{t+2}\right) \tag{4.11}$$

Notice that the form of these constraints is not prespecified and in general these can be fairly complicated equilibrium objects. They can depend on the entrepreneur's past history of types and choices and her future profitability. This is in sharp contrast to the literature that assumes a linear from on debt/collateral constraints. Using equation (4.2) from the previous section $I$ can simplify the entrepreneurs budget constraint to

$$\begin{aligned} & c_t^i\left(s^t\right) + \sum_{z_{t+2}\in Z} q\left(s^{t+1} \mid s^t\right) a^i\left(s^t, z_{t+2}\right) + k_{t+1}^i\left(s^t\right) \\ & = r\left(w_t\left(s^t\right), A_t^i\left(s^{t-1}\right)\right) k_t^i\left(s^{t-1}\right) + w_t\left(s^t\right) n_t^i\left(s^t\right) + a^i\left(s^t\right) \end{aligned}$$

Let $V_t^{i,c}\left(s^t, k^i, a\left(s^t\right); \Phi^i\left(s^t\right)\right)$ denote the value for an entrepreneur in $s^t$ with capital stock $k^i$ and holdings of arrow securities $a^i\left(s^t\right)$ of being allowed to participate in financial markets at all dates and states in the future given a sequence of borrowing constraints $\Phi^i\left(s^t\right) = \left\{\left(\phi^i\left(s^{t'}\right)\right)_{s^{t'}\in S^{t'}}\right\}_{t'=t}^{\infty}$. At any date and state, the entrepreneur can choose to default on her obligations, take her existing capital stock and live in autarky forever. In autarky, she can continue to produce and work in each period but is barred from financial markets in all future periods. $V^{i,d}\left(s^t, k^i, \mathbf{w}_t\right)$ denote the value of autarky in state $s^t$ which is defined identically to (4.4).

One can now define a competitive equilibrium with debt constraints.

**Definition 21** *A Competitive equilibrium with not-too-tight debt constraints consists of prices*
$\left\{q\left(s^{t+1} \mid s^t\right), w_t\left(s^t\right)\right\}_{t,s^t}$ *,debt constraints* $\left\{\left(\phi^i\right)_{i\in I}\right\}_0$ *and allocations*
$\left\{\left(c_t^i\left(s^t\right), n_t^i\left(s^t\right), k_{t+1}^i\left(s^t\right), \left\{a^i\left(s^t, z_{t+2}\right)\right\}_{s_{t+1}\in S}\right)_{i\in I}\right\}_{t,s^t}$ *such that*

- *Given prices and debt constraints, the allocations for entrepreneur $i$ solve her problem,*

$$V_t^{i,c}\left(s^t, k^i, a\left(s^t\right); \Phi^i\left(s^t\right)\right) = \max_{\{c^i, n^i, k^i, a^i\}_0} \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi\left(s^t\right)\left[u\left(c_t^i\left(s^t\right) - v\left(n_t^i\left(s^t\right)\right)\right)\right] \tag{4.12}$$

*subject to*

$$c_t^i\left(s^t\right) + \sum_{z_{t+2} \in Z} q\left(s^{t+1} \mid s^t\right) a^i\left(s^t, z_{t+2}\right) + k_{t+1}^i\left(s^t\right) \tag{4.13}$$

$$= r\left(w_t\left(s^t\right), A_t^i\left(s^t\right)\right) k_t^i\left(s^{t-1}\right) + w_t\left(s^t\right) n_t^i\left(s^t\right) + a^i\left(s^t\right) \tag{4.14}$$

$$a^i\left(s^t, z_{t+2}\right) \geq \phi^i\left(s^t, s_{t+1}\right)$$

- *Markets clear: for each $t, s^t$*

$$\sum_{i \in I}\left[c_t^i\left(s^t\right) + k_{t+1}^i\left(s^t\right)\right] = \sum_{i \in I}\left[A_t^i\left(s^t\right) \rho\left(w_t\left(s^t\right), A_t^i\left(s^t\right)\right) + 1 - \delta\right] k_t^i\left(s^{t-1}\right)$$

$$\sum_{i \in I} \rho\left(w_t\left(s^t\right), A_t^i\left(s^t\right)\right) k_t^i\left(s^{t-1}\right) = \sum_i n_t^i\left(s^t\right)$$

$$\sum_{i \in I} a^i\left(s^t, z_{t+2}\right) = 0 \textit{ for all } z_{t+2}$$

- *Debt constraints are chosen to be not-too-tight: for each $i, t, s^t$*

$$V_t^{i,c}\left(s^t, k_t^i\left(s^{t-1}\right), \phi^i\left(s^t\right); \Phi^i\left(s^t\right)\right) = V_t^{i,d}\left(s^t, k_t^i\left(s^{t-1}\right), \mathbf{w}_t\right) \tag{4.15}$$

Equation (4.15) is the not-too-tight constraint and is defined in a similar fashion to Alvarez and Jermann (2000). The constraints are chosen so that in each $t$ and state $s^t$, an agent who has borrowed up to the limit $\phi^i\left(s^t\right)$ the previous period is indifferent between paying back her debt and defaulting (and living in autarky for all future periods). As mentioned earlier, $\phi^i\left(s^t\right)$ depends on the history of entrepreneur's productivity shocks and choices as well the future profitability of the firm which is captured in $V_t^{i,c}\left(s^t, k_t^i\left(s^{t-1}\right), \phi^i\left(s^t\right); \Phi^i\left(s^t\right)\right)$. As a result the nature of the debt constraints is similar to that of Albuquerque and Hopenhayn (2004b).

Next, I prove that any equilibrium in this environment is inefficient. As stated in the Introduction, this is because the entrepreneur does not internalize the effects of her capital choice on her future incentives to default. In particular, some agents accumulate too much capital relative to the efficient level.

The efficient allocation is the solution to the following Planning Problem

$$\max_{\{c^i,n^i,k^i,a^i\}} \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \sum_{i \in I} \beta^t \pi\left(s^t\right) \left[u\left(c_t^i\left(s^t\right)\right) - v\left(n_t^i\left(s^t\right)\right)\right]$$

subject to

$$\sum_{i \in I} \left[c_t^i\left(s^t\right) + k_{t+1}^i\left(s^t\right)\right] \leq \sum_{i \in I} \left[A_t^i\left(s^t\right) \rho\left(w_t\left(s^t\right), A_t^i\left(s^t\right)\right) + 1 - \delta\right] k_t^i\left(s^{t-1}\right), \text{ for each } t, s^t$$

$$\sum_{t'=t}^{\infty} \sum_{s^{t'} \succeq s^t} \beta^{t'-t} \pi\left(s^{t'} \mid s^t\right) \left[u\left(c_{t'}^i\left(s^{t'}\right)\right) - v\left(n_{t'}^i\left(s^{t'}\right)\right)\right] \geq V_t^{i,d}\left(s^t, k\left(s^{t-1}\right), \mathbf{w}_t\right),$$

for each $i,\ t, s^t$

$$w_t\left(s^t\right) = \left(\frac{\sum_{i \in I}\left(\left(1-\alpha\right) A_t^i\left(s^{t-1}\right)\right)^{\frac{1}{\alpha}} k_t^i\left(s^{t-1}\right)}{\sum_i n_t^i\left(s^t\right)}\right)^{\alpha}, \text{ for each } t, s^t \tag{4.16}$$

**Proposition 26** *Any competitive equilibrium is constrained-inefficient.*

**Proof.** Let $\lambda\left(s^t\right)$ be the multiplier on the resource constraint for the planner and $\mu^i\left(s^{t+1}\right)$, the multiplier for the voluntary participation constraints. The first order condition for capital $k_{t+1}^i\left(s^t\right)$ is given by

$$\begin{aligned} 0 = {} & \lambda\left(s^t\right) + \sum_i \mu^i\left(s^{t+1}\right) \frac{\partial}{\partial k_{t+1}^i\left(s^t\right)} V_{t+1}^{i,d}\left(s^{t+1}, k_{t+1}^i\left(s^t\right), \mathbf{w}_{t+1}\right) \\ & - \sum_{s^{t+1}} \lambda\left(s^{t+1}\right) \left[A_t^i\left(s^t\right) \rho\left(w_t\left(s^t\right), A_t^i\left(s^t\right)\right) + 1 - \delta\right] \end{aligned}$$

where

$$\begin{aligned} & \frac{\partial}{\partial k_{t+1}^i\left(s^t\right)} V_{t+1}^{i,d}\left(s^{t+1}, k_{t+1}^i\left(s^t\right), \mathbf{w}_{t+1}\right) \\ & = \frac{\partial}{\partial k_{t+1}^i\left(s^t\right)} V_{t+1}^{i,d}\left(s^{t+1}, k_{t+1}^i\left(s^t\right), \left(\frac{\sum_{i \in I}\left(\left(1-\alpha\right) A_{t+1}^i\left(s^t\right)\right)^{\frac{1}{\alpha}} k_{t+1}^i\left(s^t\right)}{\sum_i n_{t+1}^i\left(s^{t+1}\right)}\right)^{\alpha}, \mathbf{w}_{t+2}\right) \end{aligned}$$

since the planner internalizes the effect of capital accumulation on wages using (4.16). Compare this with the problem faced by an entreprenuer in a competitive equilibrium.

The entreprenuer is this case does not internalize the effect her choice of capital has on future debt constraints. Therefore if for some $i$ and $s^t$, $\mu^i\left(s^t\right) > 0$ (which is the case we are interested in), this agent will over-acummulate capital relative to the efficient amount. This implies that equilibrium wages are higher than the efficient level. ■

In the light of this result, the next theorem offers an interpretation of these constraints in environments where they no longer decentralize the efficient allocation.

## 4.5 Equivalence Result

One main result of this paper proves an equivalence between the two sets of equilibria defined in the previous two sections.

**Theorem 7** *1. Given an equilibrium of the not-too-tight debt constraint economy,* $\left(\{q, w_t\}_0, \left\{\left(\phi^i\right)_{i\in I}\right\}_0, \left\{\left(c^i, n^i, k^i, a^i\right)_{i\in I}\right\}_0\right)$ *there exist,* $\left\{\left(m^{s^t,i}, \bar{V}^i\right)_{i\in I}\right\}_0$ *such that* $\left(\{q, w\}_0, \left\{\left(c^i, k^i, n^i, m^{s^t,i}, \bar{V}^i\right)_{i\in I}\right\}_0\right)$ *constitute an equilibrium in the contracting environment.*

*2. Given an equilibrium of the contracting environment* $\left(\{q, w\}_0, \left\{\left(c^i, k^i, n^i, m^{s^t,i}, \bar{V}^i\right)_{i\in I}\right\}_0\right)$, *there exist debt constraints* $\left\{\left(\phi^i\right)_{i\in I}\right\}_0$, *such that* $\left(\{q, w_t\}_0, \left\{\left(\phi^i\right)_{i\in I}\right\}_0, \left\{\left(c^i, n^i, k^i, a^i\right)_{i\in I}\right\}_0\right)$ *constitute an equilibrium with not-too-tight debt constraints.*

I prove this is several steps starting with part 1.

**Proof of part 1.** Consider a competitive equilibrium of the debt constraint problem. Define

$$
\begin{aligned}
m_t^{s^{t-1},i}\left(s_t\right) &= a^i\left(s_t\right) \\
m_t^{s^t,i}\left(s_t\right) &= -\sum_{z_{t+2}} q\left(s^{t+1} \mid s^t\right) a^i\left(s^t, z_{t+2}\right)
\end{aligned}
$$

Now given our proposed $m^i$ along with the allocation $\left\{\left(c_t^i\left(s^t\right), n_t^i\left(s^t\right), k_{t+1}^i\left(s^t\right)\right)_{i\in I}\right\}_{t,s^t}$ from the debt constrained competitive equilibrium it must be that

$$\hat{V}_t^i\left(s^t, m^{s^t,i}; k_t^i\left(s^{t-1}\right)\right) = V_t^{i,c}\left(s^t, k_t^i\left(s^{t-1}\right), a^i\left(s^t\right); \Phi^i\left(s^t\right)\right)$$

where the term on the left hand side is defined using (4.9) and the term on the right hand side using (4.12) . This is because given the construction of $m^i$ and the fact that the allocation satisfies the entrepreneur's optimality conditions in the debt constraint environment, it must be that the best the agent can do given the insurance contract is what she does in the debt constraint environment. Since the debt limits are chosen to be not-too-tight the allocation $\left\{\left(c_t^i\left(s^t\right), n_t^i\left(s^t\right), k_{t+1}^i\left(s^t\right)\right)_{i\in I}\right\}_{t,s^t}$ along with $m$ constructed above satisfies incentive compatibility and so is feasible for the intermediary given the prices.

Suppose that there exists an allocation that is feasible and gives the intermediary strictly higher profit. Note that perfect competition implies that such an allocation must increases both the ex-ante welfare of the firm and the agent. The only way an intermediary can do is to increase the amount of insurance provided to the agent. Consider any such contract with greater insurance that satisfies incentive compatibility.

Then it must be that for some $i, s^t, s^{t+1}$,

$$q\left(s^{t+1} \mid s^t\right) > \frac{\beta\pi\left(s^{t+1} \mid s^t\right) u'\left(c_{t+1}^i\left(s^{t+1}\right)\right)}{u'\left(c_t^i\left(s^t\right)\right)} \tag{4.17}$$

and

$$V_t^{i,c}\left(s^t, k_t^i\left(s^{t-1}\right), a^i\left(s^t\right); \Phi^i\left(s^t\right)\right) > V_t^{i,d}\left(s^t, k_t^i\left(s^{t-1}\right)\right) \tag{4.18}$$

To see why notice that if (B.3) held with an equality, I would have full insurance among agents and as a result an intermediary cannot offer a contract with greater ex-ante insurance. On the other hand if (B.4) held with an equality for all agents in all dates and states in which insurance is being increased, the constraint will be violated for some agent as a result of increased insurance which in turn violates incentive compatibility. However equations (B.3) and (B.4) contradict the not-too-tightness requirement of collateral constraints. ■

The intuition behind this result is fairly straightforward. If the stochastic process for $m$ is chosen to be the entrepreneur's choices of arrow security holdings in all dates and states, then the agent's best deviation given this contract coincides with her value of not defaulting in the competitive equilibrium with collateral constraints. As a result, given this contract, the agent will not want to default nor deviate from the recommended actions. The only thing that needs to be checked is that there is no alternative contract an intermediary can offer which makes him better off while leaving the agents equally as well off. As shown in the proof, the not-too-tight feature of the competitive equilibrium rules such a contract out.

Next, I prove the converse. For the proof, I assume that each intermediary lives for two periods. I do this for presentational purposes but the argument holds for any finite $T < \infty$. The following lemmas will be useful

**Lemma 12** *Consider an equilibrium of the contracting environment. Then*

$$q\left(s^{t+1} \mid s^t\right) \geq \max_{i \in I} \left\{ \frac{\beta \pi \left(s^{t+1} \mid s^t\right) u' \left(c^i_{t+1}\left(s^{t+1}\right)\right)}{u' \left(c^i_t\left(s^t\right)\right)} \right\} \tag{4.19}$$

*with equality if there is some insurance being offered by the intermediary.*

**Proof.** See Appendix A. ■

This lemma gives us a sense of why the second part of the theorem is true. Any equilibrium of the debt constrained environment must satisfy (4.19) and with equality is some agent is constrained. The proof illustrates the fact that if it did not hold an intermediary could offer a different contract that would the agent strictly better off while leaving him equally well off. As an example consider the case in which $q\left(s^{t+1} \mid s^t\right) > \max_{i \in I} \left\{ \frac{\beta \pi \left(s^{t+1} | s^t\right) u' \left(c^i_{t+1}\left(s^{t+1}\right)\right)}{u' \left(c^i_t\left(s^t\right)\right)} \right\}$ and some insurance is being offered. Since insurance is being offered there must exist some agent (receiving a positive transfer) who in $s^{t+1}$ strictly prefers to stay in the contract rather than default. Then the intermediary can offer this agent a little more insurance $q\left(s^{t+1} \mid s^t\right) \varepsilon$ today and the cost of reducing the transfer to her in $s^{t+1}$ by $\varepsilon$. I can approximate her change in utility using a Taylor expansion

$$\Delta u = \left[q\left(s^{t+1} \mid s^t\right) u' \left(c^i_t\left(s^t\right)\right) - \beta \pi \left(s^{t+1} \mid s^t\right) u' \left(c^i_{t+1}\left(s^{t+1}\right)\right)\right] \varepsilon$$

which is greater than zero since $q\left(s^{t+1} \mid s^t\right)$ is greater than the entrepreneur's marginal rate of substitution. Morever, given that intermediaries can borrow and lend at prices $q\left(s^{t+1} \mid s^t\right)$, such a contract is payoff neutral for the intermediary. This violates the zero profit condition and so is a contradiction. A similar argument applies for the reverse inequality.

**Lemma 13** *In the competitive equilibrium, for any* $\left(s^t, s_{t+1}\right)$ *, if full insurance is not being provided by the intermediary, then for some* $i$

$$\hat{V}_t^i\left(s^t, \tilde{m}^{s^t,j}; k_t^j\left(s^{t-1}\right), \delta\left(s^{t+1}\right)=0\right) = \hat{V}_t^i\left(s^t, \tilde{m}^{s^t,j}; k_t^j\left(s^{t-1}\right), \delta\left(s^{t+1}\right)=1\right)$$

**Proof.** See Appendix A. ■

Lemma 22 states that if the intermediary is not offering full insurance, then there exists an agent who is indifferent between defaulting and staying the contract where $\hat{V}_t^i\left(s^t, \tilde{m}^{s^t,j}; k_t^j\left(s^{t-1}\right), \delta\left(s^{t+1}\right)=0\right)$ is the value conditional on the agent not defaulting. The idea is that if this were not the case, the intermediary could increase the amount of insurance being offered while continuing to respect the incentive compatibility constraints. This will be useful in our construction of not-too-tight debt limits.

The proof of the part 2 of the theorem relies on a limiting argument due to Fudenberg and Levine (1983). The idea is to construct truncated allocations of the debt

constraint environment, the limit of which converges to an equilibrium with not-too-tight debt constraints. I turn to this construction next (recall that I are assuming that each intermediary lives for 2 periods for ease of notation).

Let $\left\{\left(c_t^i\left(s^t\right), k_t^i\left(s^t\right), n_t^i\left(s^t\right), m_t^{s^t,i}\left(s^t\right), \bar{V}_t^i\left(s^t\right)\right)_{i\in I}\right\}_{t,s^t}$ be the allocation associated with the competitive equilibrium in the contracting environment. I first construct a truncated $T-$period allocation for the economy with debt constraints as follows: Define $\left\{a^{T,i}\left(s^t\right)\right\}$ using the equations

$$a^{T,i}\left(s^T\right) = m_T^{s^{T-1},i}\left(s^T\right) \tag{4.20}$$

$$a^{T,i}\left(s^t\right) - \sum_{s^{t+1}} q\left(s^{t+1} \mid s^t\right) a^{T,i}\left(s^t, s_{t+1}\right) = m_t^{s^{t-1},i}\left(s^t\right) + m_t^{s^t,i}\left(s^t\right) \text{ for all } t<T \tag{4.21}$$

For all $t<T$, let

$$\phi^{T.i}\left(s^{t+1}\right) = a^i\left(s^{t+1}\right)$$

for $i, s^{t+1}$ such that

$$\hat{V}_t^i\left(s^t, \tilde{m}^{s^t,j}; k_t^j\left(s^{t-1}\right), \delta\left(s^{t+1}\right)=0\right) = \hat{V}_t^i\left(s^t, \tilde{m}^{s^t,j}; k_t^j\left(s^{t-1}\right), \delta\left(s^{t+1}\right)=1\right) \tag{4.22}$$

where I know that there exists $i$ for an $s^t, s^{t+1}$ such that the above holds true from lemma 22 .

For $t > T$ define

$$a^{T,i}\left(s^t\right) = 0,$$
$$\phi^{T,i}\left(s^{t+1}\right) = 0$$

Next, for all $t \leq T$, $c_t^{T,i}\left(s^t\right) = c_t^i\left(s^t\right)$, $n_t^{T,i}\left(s^t\right) = n_t^i\left(s^t\right)$, $k_t^{T,i}\left(s^{t-1}\right) = k_t^i\left(s^{t-1}\right)$ and for $t > T$, $c_t^{T,i}\left(s^t\right)$, $n_t^{T,i}\left(s^t\right)$, $k_t^{T,i}\left(s^{t-1}\right)$ be the best autarkic allocation that can be chosen by agent $i$ given prices. Clearly the above allocation does not constitute a competitive equilibrium with not-too-tight constraints since in period $T$, given that there is no borrowing and lending the future, no agent will be willing to honor her debt. Let $\Delta^i\left(s^t\right) \in \{0,1\}$ denote the agent's default decision in any period and state. Define the sequence $\left\{\Delta^{T,i}\right\}_0$ where

$$\Delta^{T,i}\left(s^t\right) = 0 \text{ for } t \leq T$$
$$\Delta^{T,i}\left(s^t\right) = 1 \text{ for } t > T$$

For each entrepreneur/agent $i$, define

$$\Gamma^i\left(T\right) = \left\{ \begin{array}{c} \left\{c^i, k^i, n^i, a^i, \Delta\right\}_0 : \text{For all } t, s^t, \left( \begin{array}{c} c_t^i\left(s^t\right), k_{t+1}^i\left(s^t\right), n_t^i\left(s^t\right), \\ \left\{a^i\left(s^t, z_{t+2}\right)\right\}_{z_{t+2}}, \Delta^i\left(s^t\right) \end{array} \right) \\ \text{satisfies equations (4.10), (4.11) and for } t' > T \\ \text{correspond to the best autarkic allocation given prices} \end{array} \right\}$$

and $\Gamma\left(T\right) = \prod_{i \in I} \Gamma^i\left(T\right)$. $\Gamma^i\left(T\right)$ consists of choices that are budget feasible for the agent and involve living in autarky after period $T$. $\Gamma\left(\infty\right)$ is the untruncated choice sets for the agent. Clearly, the truncated allocation constructed above in an element of this set for each $i$ given the debt constraints.

Define $x^{i,T} = \left\{c^{T,i}, k^{T,i}, n^{T,i}, a^{i,T}, \Delta^{i,T}\right\}_0$ and $x^{i,*} = \left\{c^i, k^i, n^i, a^i, \Delta^i\right\}_0$ where $\Delta^i\left(s^t\right) = 0$ for all $t$ and $\left\{c^i, k^i, n^i\right\}_0$ correspond to the competitive equilibrium allocation in the contracting problem and $a^i$ is the limit as $T \to \infty$ of asset holdings constructed using (B.8) and (B.9).

For any $i$ consider the best deviation in $\Gamma^i\left(T\right)$ from the truncated allocation constructed above given the debt constraints $\left\{\phi^{T.i}\right\}$. From lemma 21 I know that the in any equilibrium of the contracting environment $q\left(s^{t+1} \mid s^t\right) \geq \max_{i \in I}\left\{\frac{\beta\pi\left(s^{t+1}|s^t\right)u'\left(c_{t+1}^i\left(s^{t+1}\right)\right)}{u'\left(c_t^i\left(s^t\right)\right)}\right\}$. As a result for all $t < T$, conditional on not defaulting the truncated allocation is optimal for each agent given the debt constraints constructed above. Therefore, the best possible deviation (if one exists) involves default by some agent $i$ in some period $t \leq T$. Define $\varepsilon^{i,T} \geq 0$ to be the value of the best possible deviation for agent $i$ in $\Gamma^i(T)$ for any $t \leq T$,

$$V_t^{i,d}\left(s^t, k_t^i\left(s^{t-1}\right)\right) - V_t^{i,c}\left(s^t, k_t^i\left(s^{t-1}\right), a^i\left(s^t\right); \Phi^i\left(s^t\right)\right) = \varepsilon^{i,T}$$

and let $\varepsilon^T = \max_i \varepsilon^{i,T}$. In particular, $\varepsilon^T$ corresponds to the best possible deviation from the truncated allocation that can be achieved by any player who chooses from choice set $\Gamma^i(T)$. Let

$$X_t = R_+ \times R_+ \times [0,1] \times R_+ \times \{0,1\}$$

$X_{t'} = \prod_{t=0}^{t'} X_t$ and $X = \prod_{t=0}^{\infty} X_t$. I have $x^{i,T} \in X$ and $x^{i,*} \in X$. The metric

$$d(x,z) = \sup_t \left[\frac{1}{t}\min\left\{\left|x_t - z_t\right|, 1\right\}\right]$$

induces the product topology on $X$. I can see that $x^{i,T} \to x^{i,*}$ in this metric.

**Definition 22** *A allocation* $\left(\left\{c^{T,i}, k^{T,i}, n^{T,i}, a^{T,i}, \Delta^{T,i}\right\}_0\right)_{i\in I} \in \Gamma(T)$ *is an* $\varepsilon-$*perfect equilibrium* ($\varepsilon -$ *perfect) if for each* $t, s^t$, *any agent i's best deviation in* $\Gamma^i(T)$ *from the prescribed allocation yields her a welfare gain of at most* $\varepsilon$.

Given this definition and the previous discussion I have that $x^T$ is $\varepsilon^T$ perfect in $\Gamma(T)$ and that $x^T \to x^*$. Notice that since the contracting equilibrium allocations satisfy incentive compatibility (4.8), $\varepsilon^T \to 0$ since the agent does not want to deviate. Next, I adapt an argument from Theorem 3.3. in Fudenberg and Levine (1983) which proves that

**Theorem 8** *A sufficient condition that* $x^*$ *be perfect in* $\Gamma(\infty)$ *is that there be a sequences* $\varepsilon^T, x^T$ *such that* $x^T$ *is* $\varepsilon^T-$*perfect in* $\Gamma(T)$ *and as* $T \to \infty$, $\varepsilon^T \to 0$ *and* $x^T \to x^*$.

**Proof.** See Appendix ■

The theorem says that the truncated allocations converge to an equilibrium of the model with debt constraints. The last thing which needs to be checked is that the not-too-tight property is satisfied, but this follows from the construction of the debt limits. In particular if for any agent $i$

$$q\left(s^{t+1} \mid s^t\right) > \frac{\beta\pi\left(s^{t+1} \mid s^t\right) u'\left(c_{t+1}^i\left(s^{t+1}\right)\right)}{u'\left(c_t^i\left(s^t\right)\right)}$$

this agent sets $a^i\left(s^t, s_{t+1}\right) = \phi^i\left(s^t, s_{t+1}\right) < 0$. But then then from (B.10) I see that

$$\hat{V}_t^i\left(s^t, \tilde{m}^{s^t,j}; k_t^j\left(s^{t-1}\right), \delta\left(s^{t+1}\right) = 0\right) = \hat{V}_t^i\left(s^t, \tilde{m}^{s^t,j}; k_t^j\left(s^{t-1}\right), \delta\left(s^{t+1}\right) = 1\right)$$

which proves that the debt constraints are not-too-tight.

## 4.6 An Example

In this section I present a simple example of an economy with not-too-tight debt constraints to illustrate some equilibrium properties and show how one can generate endogenous fluctuations in these models. In addition, one might also be interested in the contracting problem in order to understand the nature of optimal contracts with hidden actions and while this is a fairly intractable problem, I show that the model with debt constraints is easier to solve. An important feature of these models is that there are multiple equilibria. The existence of multiple equilibria provides a natural source for aggregate fluctuations in these models. The model I present is stripped down to highlight the role of endogenous debt constraints in generating multiple equilibria.

Consider a deterministic two-type version of the model presented in section 3. Entrepreneurs' productivity alternate between high and low, $A_t\left(s^t\right) \in \left\{A^h, A^l\right\}$. In addition, I assume that there is another class of hand-to-mouth workers who supply labor inelastically. This is just to make the computation simpler and the multiplicity result is unaffected having the entrepreneur's work themselves on other projects. The entrepreneur's objective function and budget constraint is

$$\sum_{t=0}^{\infty} \beta^t \log\left(c_t^i\right)$$

$$c_t^i + q_t a_{t+1}^i + k_{t+1}^i = r_t\left(A, w_t\right) k_t^i + a_t^i$$

Entrepreneurs are subject to borrowing constraints

$$a_{t+1}^i \geq \phi_{t+1}^i$$

that are determined endogenously. The equilibrium is defined analogously to section 3. The first observation is that autarky is always an equilibrium with not-too-tight collateral constraints. To see why consider the equilibrium requirement that

$$V_t^{i,c}\left(s^t, k_t^i\left(s^{t-1}\right), \phi^i\left(s^t\right); \Phi^i\left(s^t\right)\right) = V_t^{i,d}\left(s^t, k_t^i\left(s^{t-1}\right)\right)$$

In the case of autarky,

$$V_t^{i,c}\left(s^t, k_t^i\left(s^{t-1}\right), \phi^i\left(s^t\right); \Phi^i\left(s^t\right)\right) = V_t^{i,c}\left(s^t, k_t^i\left(s^{t-1}\right), 0; \mathbf{0}\right)$$

which is just the value of default since there is no borrowing and lending in equilibrium. In this equilibrium each entrepreneur solves

$$\begin{aligned}
&\sum_{t=0}^{\infty} \beta^t \log\left(c_t^i\right) \\
&\text{subject to} \\
&c_t^i + q_t a_{t+1}^i + k_{t+1}^i = r_t\left(A, w_t\right) k_t^i + a_t^i \\
&a_{t+1}^i \geq 0
\end{aligned}$$

with the equilibrium defined in a standard fashion.

Given the difference in productivity, the autarkic allocation is inefficient. The efficient allocation would involve only the high productivity technology to be used and a transfer from the entrepreneur owning this technology to the other type. Let the transfer associated with the efficient allocation be $\xi^*$. As is common in most of the literature with limited enforcement (Kehoe and Levine (1993) etc.), I assume that the efficient allocation violates the participation constraint of the high entrepreneur who would rather live in autarky forever than provide the transfer.

Next, I show that there exists a stationary equilibrium with borrowing and lending

**Proposition 27** *If the autarkic return on the low productivity technology* $r\left(A^l, w^{aut}\right) < 1$*, there exists a stationary equilibrium with borrowing and lending, i.e.* $\phi^i < 0$ *for some* $i$*.*

**Proof.** See Appendix. ■

The proposition establishes that there are multiple equilibria in the model. The condition $r\left(A^l, w^{aut}\right) < 1$ ensures that if prices were fixed at the autarkic level, some insurance between the high and low types would be beneficial to the high type. A similar condition is needed in the analogous environment with endowments as in Alvarez and Jermann (2000). As they show in proposition 4.8, if the implied interest rates at autarky are "high" then autarky is the only equilibrium allocation in the model. By high interest rates they mean that the present discounted value of consumption is finite which is ensures if the interest rates at autarky are larger than 1. In our environment with heterogenous productivity, the relevant interest rate is $r\left(A^l, w^{aut}\right)$ since in a stationary autarkic equilibrium this is the interest rate on capital faced by agents who currently have high productivity (and have low productivity the next period). Moreover, in any equilibrium with borrowing and lending, the equilibrium interest rate $R_{t+1}$ must be greater or equal to $r\left(A^l, w_{t+1}\right)$ in order for the current high type to lend to the low type. The stationary equilibrium with borrowing and lending I consider has $R = r\left(A^l, w\right)$.

The intuition for why multiple equilibria exist in this model is the dynamic complementarity present in entrepreneurs' decisions to borrow and lend. For example, if entrepreneurs believe that there will be no borrowing and lending in the future, there will be no borrowing and lending in the present as they will strictly prefer to default on their debts than pay back. Similarly, the expectation that debt constraints will be loose in the future allows for current borrowing and lending. A similar intuition holds in the contracting environment. Here the dynamic complementarity is present in the actions of the intermediaries. If an intermediary today believes that all future intermediaries will not lend to entrepreneurs, it will not be willing to offer any insurance today.

Next I will show that for a large class of parameter values, there is equilibrium indeterminacy in that there are a continuum of non-stationary equilibria converging to the autarkic one. I will use this to construct a sunspot equilibrium and look at the impulse response of current aggregate variables to an expectational shock of the state of collateral constraints the following period. Consider the equilibrium conditions of the model

$$\frac{c_t^h}{\beta c_{t+1}^l} = r\left(A^l, w_{t+1}\right) \tag{4.23}$$

$$\frac{c_t^l}{\beta c_{t+1}^h} = r\left(A^h, w_{t+1}\right) \tag{4.24}$$

$$c_t^h + k_{t+1}^l = r\left(A^h, w_t\right) k_t^h - R_t\phi_t - \phi_{t+1} \tag{4.25}$$

$$c_t^l + k_{t+1}^h = r\left(A^l, w_t\right) k_t^l + R_t\phi_t + \phi_{t+1} \tag{4.26}$$

$$\rho\left(w_t, A^h\right) k_t^h + \rho\left(w_t, A^l\right) k_t^l = 2 \tag{4.27}$$

$$c_t^h + c_t^l + k_{t+1}^h + k_{t+1}^l = r\left(A^l, w_t\right) k_t^l + r\left(A^h, w_t\right) k_t^h \tag{4.28}$$

$$\log c_t^h + \beta \log c_{t+1}^l = a \log r\left(A^h, w_{t+1}\right) k^h + b_t^h - \beta^2 \left[\begin{array}{c} a \log r\left(A^h, w_{t+2}\right) k_{t+2}^h \\ +b_{t+2}^h \end{array}\right] \tag{4.29}$$

$$b_t^h = \log\gamma + \beta\left[a\log\left(r\left(A^l, w_{t+1}\right)(1-\gamma)\right) + b_{t+1}^l\right] \tag{4.30}$$

$$b_t^l = \log\gamma + \beta\left[a\log\left(r\left(A^h, w_{t+1}\right)(1-\gamma)\right) + b_{t+1}^h\right] \tag{4.31}$$

where

$$a = \frac{1}{1-\beta}$$

$$\gamma = 1-\beta$$

(4.23) and (4.24) are the first order conditions for the two types with respect to capital accumulation, (4.25) and (4.26) are the entrepreneurs' budget constraints assuming that the debt constraint is binding, (4.27) is the labor market clearing conditions where $\rho\left(w_t, A^h\right)$ is defined as in (4.1) and (4.29) is the difference of the not-too-tight conditions of the model in $t$ and $t+2$. Note the timing in the model- $k_{t+1}^h$ is the capital invested by the current low productivity agent for use in $t+1$ when she has high productivity. Similarly $k_{t+1}^l$ is the capital invest by the current high productivity agent. The entrepreneur's problem on default has a very simple structure given the log utility assumption and hence takes the form in the equation. In particular, after default, the entrepreneur saves and consumes a constant amount of her wealth in all future periods. Her value of default when she has high productivity and capital stock $k_t^h$ is of the form

$$a \log r\left(A^h, w_{t+1}\right) k_t^h + b_t^h$$

where $b_t^h$ is defined recursively using (4.30) and (4.31). I can linearize this system of equations around the autarkic steady state. Let

$x = \left(\tilde{c}_t^h, \tilde{k}_t^h, \tilde{k}_t^l, \tilde{k}_{t+1}^h, \tilde{k}_{t+1}^l, \tilde{b}_t^h, \tilde{b}_t^l, \tilde{b}_{t+1}^h, \tilde{b}_{t+1}^l\right)$ where the tilde variables are deviations from the steady state. The linear system is of the form $x_{t+1} = Gx_t$. As in standard (see for example Stokey et al. (1989)) the Jordan decomposition allows us the write $G = P\Lambda P^{-1}$ where $\Lambda$ is a diagonal matrix consisting of the eigenvalues of $G$. I know from Woodford (1986b) that if the number of eigenvalues with absolute value greater than 1 is less than the number of forward looking variables, the equilibrium is indeterminate. Furthermore also from Woodford (1986b) I know that indeterminacy of equilibria is a sufficient condition for the existence of sunspot equilibria. Next, I construct an example of a sunspot equilibrium and examine the impulse response from the autarkic steady state to a one time shock to agents expectation of the tightness of collateral constraints the following period. To compute the model I use a technique developed by Farmer and Khramov (2013) which modifies the tools developed by Sims (2002) to deal with indeterminate equilibria[3] . The parameter values are

| | |
|---|---|
| $\beta$ | .98 |
| $\alpha$ | .33 |
| $A^l$ | 1 |
| $A^h$ | 2 |
| $\delta$ | .05 |

Table 4.1: Parameters

Starting at the autarkic steady I consider a one time expectational shock to future collateral constraints of 0.5 and consider the impulse responses of the collateral constraint $\phi_t$, aggregate consumption $c_t = c_t^h + c_t^l$ and the capital accumulated by the current low type (high tomorrow) $k_t^h$.

[3] See the note on computation at the end

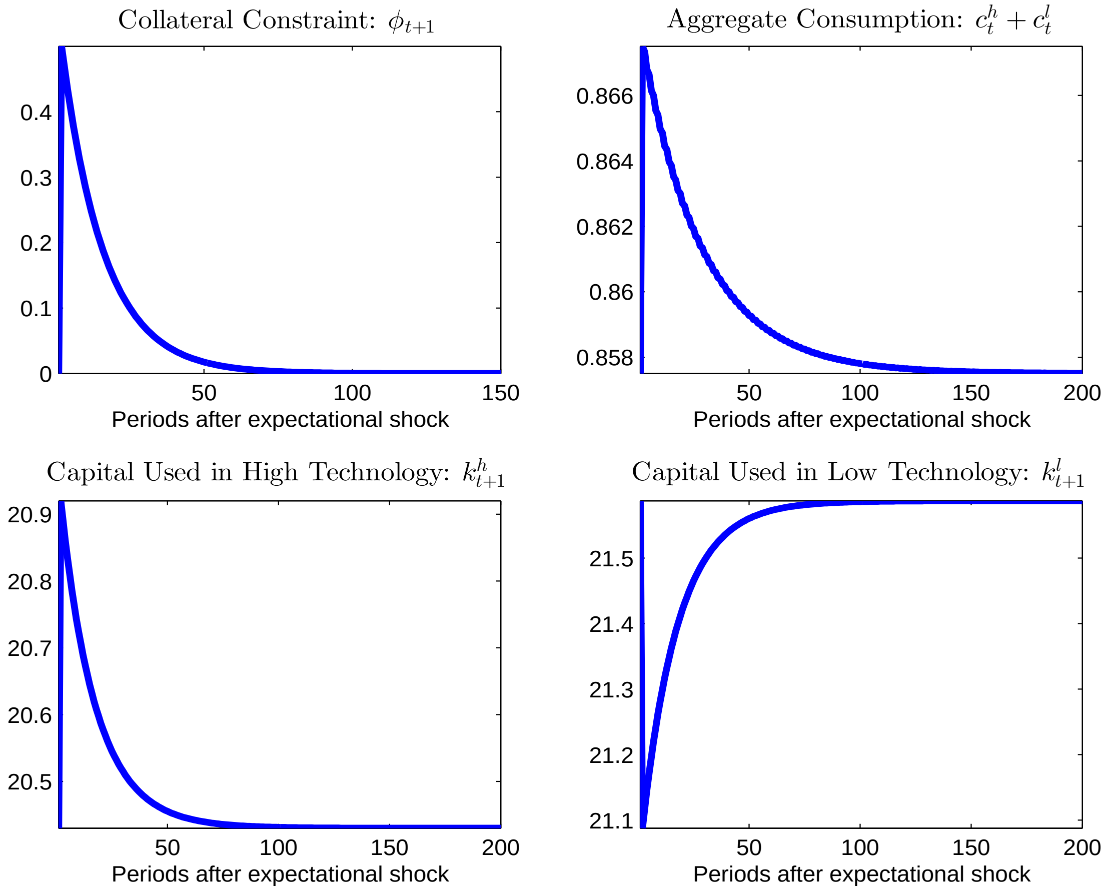


Figure 4.3: Impulse Responses

The impulse responses highlight the persistent effects of a one time expectational shock to future collateral constraints. In the first period of the shock, due to the anticipation of loose constraints in the future, the agents with high technology lend to the agents with low current technology. This reduces the high productivity entrepreneur's investment in capital (recall the timing) and increases the low productivity agent's investment in capital. Additionally aggregate consumption rises since the entrepreneurs with low productivity are consuming more. Note the persistence of the shocks- it takes over a 100 periods for aggregate consumption to return to its original autarkic steady state level.

## 4.7 Conclusion

In this paper, I consider a class of models with debt constraints that can endogenously generate aggregate fluctuations. A crucial difference between the setup I consider and standard models is that I do not arbitrarily restrict the set of contracts borrowers and lenders can sign. In particular, I prove an equivalence between a class of decentralized contracting problems and models with debt constraints that are not-too-tight. I consider an environment in which intermediaries offer insurance contracts to entrepreneurs with heterogeneous productivity where the aggregate state is common knowledge but the actions of the intermediary are unobservable. The debt constraint setup is an extension of that by Alvarez and Jermann (2000) to an environment with heterogenous entrepreneurs and capital accumulation. As in their paper, debt constraints are equilibrium objects.

I view the equivalence result as important for several reasons. First, it offers a new interpretation of not-too-tight collateral constraints in commonly used financial frictions environment. This is in sharp contrast to much of the literature that imposes exogenous constraints on credit. A serious understanding of the origins of collateral constraints is essential for any policy analysis. For example, in our model if agents expect the government to intervene in credit markets in some future date, it will have an effect on the current level of borrowing and lending. Next, it allows us to understand the decentralized contracting environment better which on its own is intractable. As I show in an example, I can compute the equilibria of the debt constrained problem relatively easily. The most interesting feature of both these environments is the equilibrium multiplicity. Entrepreneurs' expectations of the tightness of future credit constraints affect the current state of debt constraints. I construct a simple example to show the effect of such an expectational shock. Our model has no extrinsic uncertainty and highlights the role of this dynamic complementarity present in both models. I see that the shock causes aggregate consumption and the capital used in the high technology to increase while the capital used in the low technology decreases.

As mentioned earlier I view this model as in interesting benchmark in which to contrast the effect of standard policy prescriptions on models with endogeous constraints versus one in which they are exogenous and linear. One such policy recommendation that economists have been proposing is “Macro-prudential” regulation which involves restraining the amount of credit offered by institutions in certain times. While such policy is shown the welfare enhancing in models with exogenous constraints, the mechanism in this paper suggests that agents’ expectations of these policies might result in perversely limiting credit in “good” times as well. If I consider models with exogenous debt constraints, this channel is absent and so is ignored when formulating policy. As a result, models with reduced-form constraints should be used for policy exercises with extreme caution.

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# Appendix A

# Appendix to Chapter 2

## A.1 Appendix: Proofs From the Main Text

This appendix contains proofs from the main text.

### A.1.1 Proofs from Section 3

**Proof of Proposition 1.** Suppose that in some equilibrium, for some $t$ and history $\theta^t$, $u'\left(c_t\left(\theta^t\right)\right) q_t < \beta \mathbb{E}_t u'\left(c_{t+1}\left(\theta^{t+1}\right)\right)$. Recall that $b_t\left(\theta^t\right) = b_t^{old}\left(\theta^t\right) + {}_t b_t\left(\theta^t\right)$ and so in equilibrium $c_t\left(\theta^t\right) = \theta_t + b_t\left(\theta^t\right)$.

Since we are considering symmetric equilibria in which ex-ante identical intermediaries offer the same contract in equilibrium, we consider the incentives for a deviating intermediary to offer a different contract and make strictly positive profits. Consider an intermediary offering a $\varepsilon\delta-$savings contract $S_t^{\varepsilon,\delta}$ for some $\varepsilon > 0$ and $\delta < 1$. Notice that the intermediary makes positive profits whenever this contract is accepted. Since $u'\left(c_t\left(\theta^t\right)\right) q_t < \beta \mathbb{E}_t u'\left(c_{t+1}\left(\theta^{t+1}\right)\right)$, there exists $\varepsilon > 0, \delta < 1$ such that type $\theta^t$ will strictly prefer to sign an $\varepsilon\delta$ savings contract if offered. These contracts are by construction incentive compatible and satisfy voluntary participation constraints. As a result an intermediary offering such a contract will make positive profits which is a contradiction.

The proof of part 2 is also straightforward. Suppose for contradiction we have an agent constrained in period $t$ and in period $t+1$, for all $\tilde{\theta}^{t+1}$,

$$V_{t+1}\left(\tilde{\theta}^{t+1}\right) > V_{t+1}^{d}\left(\tilde{\theta}^{t+1}\right)$$

Consider the following deviating contract

$$\begin{aligned} {}_t\tilde{b}_t &= \delta\varepsilon \\ {}_t\tilde{b}_{t+1} &= -\frac{\varepsilon}{q_t} \end{aligned}$$

where $\varepsilon > 0, \delta < 1$ and the contract is not contingent on reported type. Clearly we can find an $\varepsilon, \delta$ such a constrained household accepting is made strictly better off. Moreover for $\varepsilon, \delta$ small, incentives are preserved for all households since the voluntary participation constraints are assumed to be slack. Since intermediaries make a strictly positive contract by offering such a contract, we have a contradiction. ■

*Proof of Proposition 2:* The proof requires a series of preliminary results.

The first intermediate lemma tells us that we need only consider a relaxed problem and drop all voluntary participation constraints besides those for the lowest type.

**Lemma 14** *In any equilibrium, for at any date t and history $\theta^{t-1}$, if the voluntary participation constraint for type $\left(\theta^{t-1}, \underline{\theta}\right)$ is satisfied, then it is satisfied for all types $\left(\theta^{t-1}, \theta\right)$, $\theta \in \Theta$.*

**Proof of Lemma 14.** Let $W_{\theta^{t-1}}\left(\theta, \hat{\theta}\right) = u\left(\theta + b_t\left(\theta^{t-1}, \hat{\theta}\right)\right) + \beta\mathbb{E}_t V_{t+1}\left(\theta^{t-1}, \hat{\theta}\right)$ be the equilibrium value for type $\left(\theta^{t-1}, \theta\right)$ pretending to be $\left(\theta^{t-1}, \hat{\theta}\right)$. Suppose first that the VP constraint for type $\left(\theta^{t-1}, \underline{\theta}\right)$ is satisfied and that $b_t\left(\theta^{t-1}, \underline{\theta}\right) \leq 0$. Then

$$\begin{aligned} W_{\theta^{t-1}}\left(\theta, \theta\right) &= u\left(c_t\left(\theta^{t-1}, \theta\right)\right) + \beta\mathbb{E}_t V_{t+1}\left(\theta^{t-1}, \theta\right) \\ &\geq W_{\theta^{t-1}}\left(\theta, \underline{\theta}\right) \\ &= W_{\theta^{t-1}}\left(\underline{\theta}, \underline{\theta}\right) + u\left(\theta + b_t\left(\theta^{t-1}, \underline{\theta}\right)\right) - u\left(\underline{\theta} + b_t\left(\theta^{t-1}, \underline{\theta}\right)\right) \\ &= V^{d}\left(\underline{\theta}\right) + u\left(\theta + b_t\left(\theta^{t-1}, \underline{\theta}\right)\right) - u\left(\underline{\theta} + b_t\left(\theta^{t-1}, \underline{\theta}\right)\right) \\ &= V^{d}\left(\theta\right) + u\left(\underline{\theta}\right) - u\left(\theta\right) + u\left(\theta + b_t\left(\theta^{t-1}, \underline{\theta}\right)\right) - u\left(\underline{\theta} + b_t\left(\theta^{t-1}, \underline{\theta}\right)\right) \end{aligned}$$

Since $b_t\left(\theta^{t-1},\underline{\theta}\right)\leq 0$,

$$\begin{aligned}
&u\left(\underline{\theta}\right)-u\left(\theta\right)+u\left(\theta+b_t\left(\theta^{t-1},\underline{\theta}\right)\right)-u\left(\underline{\theta}+b_t\left(\theta^{t-1},\underline{\theta}\right)\right)\\
&=-u'\left(x\right)\left[\theta-\underline{\theta}\right]+u'\left(y\right)\left[\theta-\underline{\theta}\right]\\
&=\left[\theta-\underline{\theta}\right]\left(u'\left(y\right)-u'\left(x\right)\right)\\
&\geq 0
\end{aligned}$$

where $x\in\left[\underline{\theta},\theta\right]$ and $y\in\left[\underline{\theta}+b_t\left(\theta^{t-1},\underline{\theta}\right),\theta+b_t\left(\theta^{t-1},\underline{\theta}\right)\right]$. Next, suppose that $b_t\left(\theta^{t-1},\underline{\theta}\right)>0$. Then the VP constraint for type $\left(\theta^{t-1},\underline{\theta}\right)$ is slack. Suppose that the the VP constraint binds for some other type $\left(\theta^{t-1},\theta\right)$. Then

$$\begin{aligned}
W_{\theta^{t-1}}\left(\underline{\theta},\underline{\theta}\right)&\leq W_{\theta^{t-1}}\left(\theta,\theta\right)+u\left(\underline{\theta}+b_t\left(\theta^{t-1},\underline{\theta}\right)\right)-u\left(\theta+b_t\left(\theta^{t-1},\underline{\theta}\right)\right)\\
&=V^d\left(\theta\right)+u\left(\underline{\theta}+b_t\left(\theta^{t-1},\underline{\theta}\right)\right)-u\left(\theta+b_t\left(\theta^{t-1},\underline{\theta}\right)\right)\\
&=u\left(\underline{\theta}+b_t\left(\theta^{t-1},\underline{\theta}\right)\right)+EV^d\left(\theta'\right)+u\left(\theta\right)-u\left(\theta+b_t\left(\theta^{t-1},\underline{\theta}\right)\right)\\
&<u\left(\underline{\theta}+b_t\left(\theta^{t-1},\underline{\theta}\right)\right)+EV^d\left(\theta'\right)\\
&\leq W_{\theta^{t-1}}\left(\underline{\theta},\underline{\theta}\right)
\end{aligned}$$

which is a contradiction. In particular, if $b_t\left(\theta^{t-1},\underline{\theta}\right)>0$ the the VP constraints for all types $\left(\theta^{t-1},\theta\right)$ are slack. ■

The result states that in general, the voluntary participation constraints will bind for the lowest type $\underline{\theta}$. This is true generally in models with private information and limited commitment, for example in Dovis (2014). Note the binding pattern of these constraints is the opposite of models with only limited commitment such as Kehoe and Levine (1993) and Alvarez and Jermann (2000).

Recall that $b_t\left(\theta^t\right)=b_t^{old}\left(\theta^t\right)+{}_tb_t\left(\theta^t\right)$. For any household define $A_t\left(\theta^{t-1},\theta\right)$ to be the equilibrium expected present discounted value of future transfers for type $\left(\theta^{t-1},\theta\right)$

$$A_t\left(\theta^{t-1},\theta\right)\equiv b_t\left(\theta^{t-1},\theta_t\right)+q_t\sum_{\theta'\in\Theta}\pi\left(\theta'\right)A_{t+1}\left(\theta^{t-1},\theta,\theta'\right)$$

Similarly, given a contract $B_t$, let

$${}_t\mathcal{P}_s\left(\theta^s\right)\equiv{}_tb_s\left(\theta^s\right)+q_t\sum_{\theta'\in\Theta}\pi\left(\theta^s,\theta'\right){}_t\mathcal{P}_{s+1}\left(\theta^s,\theta'\right)$$

denote the expected present discounted value of transfers associated with contract $B_t$ from period $s$ onwards.

The next set of results will be used to prove that in any equilibrium, the expected present discounted value of transfers to households with the same history $\theta^{t-1}$, is independent of their period $t$ reports.

**Lemma 15** *In any equilibrium, for any $t$ and any contract offered by an intermediary born at date $t$, ${}_t\mathcal{P}_t\left(\theta^t\right) = 0$ for all $\theta^t$.*

**Proof of Lemma 15.** Suppose not. Clearly, ${}_t\mathcal{P}_t\left(\theta^t\right) > 0$ for all $\theta^t$ is not possible since the intermediary would making negative profits. On the other hand if ${}_t\mathcal{P}_t\left(\theta^t\right) \leq 0$ for all $\theta^t$ with strict inequality for some, then a deviating intermediary can offer a contract which transfers a little more to some types and still continue to make positive profits. As a result these types will strictly prefer to sign with the deviating intermediary. Finally, suppose that there exists $\theta, \theta'$ such that ${}_t\mathcal{P}_t\left(\theta^{t-1}, \theta\right) > 0$ and ${}_t\mathcal{P}_t\left(\theta^{t-1}, \theta'\right) < 0$. Then at the beginning of period $t$, consider a deviating intermediary offering the following contract,

$$
\begin{aligned}
{}_t\tilde{\mathcal{P}}_t\left(\theta^{t-1}, \theta'\right) &= {}_t\mathcal{P}_t\left(\theta^{t-1}, \theta'\right) + \varepsilon \\
{}_t\tilde{b}_{t+s}\left(\theta^{t-1}, \hat{\theta}\right) &= 0 \text{ for all } s \geq 0, \text{ for } \hat{\theta} \neq \theta'
\end{aligned}
$$

where $\varepsilon > 0$ and small. Notice that types $\theta$ strictly prefer the original contract while types $\theta'$ strictly prefer ${}_t\tilde{\mathcal{P}}_t$ to ${}_t\mathcal{P}_t$. As a result, these households will strictly prefer to sign with the deviating intermediary who makes a positive profit. ■

The lemma shows that all contracts offered by intermediaries must make zero profits and as a result there is no cross subsidization between contracts. The result is a direct consequence of perfect competition among intermediaries. If there is cross subsidization between initial types, a deviating intermediary can offer only the contract that yields positive profits and make strictly positive profits in equilibrium.

In an environment with 2 period lived intermediaries for any $t \geq 0$,

$$
\begin{aligned}
{}_t\mathcal{P}_t\left(\theta^t\right) &= {}_tb_t\left(\theta^t\right) + q_t \sum_{\theta' \in \Theta} \pi\left(\theta^t, \theta'\right) {}_tb_{t+1}\left(\theta^t, \theta'\right) \\
{}_t\mathcal{P}_{t+1}\left(\theta^{t+1}\right) &= {}_tb_{t+1}\left(\theta^{t+1}\right)
\end{aligned}
$$

The final result required for the proof of Proposition 2 shows that higher types will always strictly prefer transfer sequences with a larger present discounted value even if they are Euler-constrained. Given an equilibrium transfer sequence $A$, define

$$A_{\varepsilon_+}\left(\theta^{t-1},\theta_t\right) \equiv b_t\left(\theta^{t-1},\theta_t\right)+\varepsilon+q_t\sum_{\theta'}\pi\left(\theta'\right)\left[A_{t+1}\left(\theta^{t-1},\theta_t,\theta'\right)-a\varepsilon\right]$$
$$A_{\varepsilon_-}\left(\theta^{t-1},\theta_t\right) \equiv b_t\left(\theta^{t-1},\theta_t\right)-\varepsilon+q_t\sum_{\theta'}\pi\left(\theta'\right)\left[A_{t+1}\left(\theta^{t-1},\theta_t,\theta'\right)+a\varepsilon\right]$$

Notice that if $A_{\varepsilon_+}\left(\theta^{t-1},\theta_t\right) > A_t\left(\theta^{t-1},\theta_t\right)$ then it must be that $a < R_{t+1} = \frac{1}{q_t}$ and if $A_{\varepsilon_-}\left(\theta^{t-1},\theta_t\right) > A_t\left(\theta^{t-1},\theta_t\right)$ then $a > R_{t+1}$.

Given a transfer schedule $A$ and the associated transfer sequence, define

$$\begin{aligned} Z_t\left(\theta^t,s_t,b_t;A\right) &= \max_{s_{t+1}} u\left(c_t\right)+\beta\mathbb{E}_t Z_{t+1}\left(\theta^{t+1},s_{t+1},b_{t+1};A\right) \\ & s.t. \\ & c_{t'}+s_{t'+1} \leq \theta_{t'}+b_{t'}\left(\theta^{t'}\right)+R_{t'}s_{t'},\ \forall t' \geq t \\ & s_{t'+1} \geq 0,\ \forall t' \geq t \end{aligned}$$

where $R_{t'} = \frac{1}{q_{t'}}$. Here $Z_t\left(\theta^t,s_t,b_t;A\right)$ denotes the continuation value for a household of type $\theta^t$ who receives transfers according to $A$ and can save at rate $R_{t+s},\ s \geq 0$. The reason this will be useful is that in general, deviating intermediaries are always willing to provide savings contracts since they have no fear of default the following period. Therefore, if it is true that a household can do strictly better by lying and and saving, there exists a deviating contract that makes both the intermediary and household strictly better off. This will be particularly useful in the proof of Proposition 2.

**Lemma 16** *1. If $A_{\varepsilon_+}\left(\theta^{t-1},\theta_t\right) > A\left(\theta^{t-1},\theta_t\right)$ then $Z\left(\left(\theta^{t-1},\theta'\right),0,b_t+\varepsilon;A_{\varepsilon_+}\right) > Z\left(\left(\theta^{t-1},\theta'\right),0,b_t;A\right)$ for all $\theta' > \theta$*

*2. If $A_{\varepsilon_-}\left(\theta^{t-1},\theta_t\right) > A\left(\theta^{t-1},\theta_t\right)$ and $Z\left(\left(\theta^{t-1},\theta'\right),0,b_t-\varepsilon;A_{\varepsilon_-}\right) \geq Z\left(\left(\theta^{t-1},\theta'\right),0,b_t;A\right)$ then $Z\left(\left(\theta^{t-1},\theta'\right),0,b_t-\varepsilon;A_{\varepsilon_-}\right) > Z\left(\left(\theta^{t-1},\theta'\right),0,b_t;A\right)$ for $\theta' > \theta$*

**Proof.** Part 1. To prove this I show that $\frac{\partial}{\partial\varepsilon} Z\left(\left(\theta^{t-1},\theta'\right),0,b_t+\varepsilon;A_{\varepsilon_+}\right)\Big|_{\varepsilon=0} > 0$.

We have that

$$\begin{aligned}
&\frac{\partial}{\partial\varepsilon}Z_t\left(\left(\theta^{t-1},\theta'\right),0,b_t+\varepsilon;A_{\varepsilon_+}\right)=u'\left(\theta'-s_{t+1}+b_t+\varepsilon\right)\left[-\frac{\partial}{\partial\varepsilon}s_{t+1}+1\right]\\
&+\beta\frac{\partial}{\partial\varepsilon}\mathbb{E}_tZ_{t+1}\left(\left(\theta^{t-1},\theta',\theta_{t+1}\right),s_{t+1},b_{t+1}-a\varepsilon\right)\\
&=u'\left(\theta'-s_{t+1}+b_t\left(\theta^t\right)+\varepsilon\right)\left[-\frac{\partial}{\partial\varepsilon}s_{t+1}+1\right]+\beta\left[\mathbb{E}_tZ_{2,t+1}\frac{\partial}{\partial\varepsilon}s_{t+1}-a\mathbb{E}_tZ_{3,t+1}\right]\\
&=u'\left(c_t\left(\theta^{t-1},\theta'\right)\right)\left[-\frac{\partial}{\partial\varepsilon}s_{t+1}+1\right]+\beta\left[\begin{array}{c}R_{t+1}\mathbb{E}_tu'\left(c_{t+1}\left(\theta^{t-1},\theta',\theta_{t+1}\right)\right)\frac{\partial}{\partial\varepsilon}s_{t+1}\\-a\mathbb{E}_tu'\left(c_{t+1}\left(\theta^{t-1},\theta',\theta_{t+1}\right)\right)\end{array}\right]\\
&=\frac{\partial}{\partial\varepsilon}s_{t+1}\left[-u'\left(c_t\left(\theta^{t-1},\theta'\right)\right)+\beta R_{t+1}\mathbb{E}_tu'\left(c_{t+1}\left(\theta^{t-1},\theta',\theta_{t+1}\right)\right)\right]\\
&+u'\left(c_t\left(\theta^{t-1},\theta'\right)\right)-\beta a\mathbb{E}_tu'\left(c_{t+1}\left(\theta^{t-1},\theta',\theta_{t+1}\right)\right)\\
&=-\frac{\partial}{\partial\varepsilon}s_{t+1}\mu_t\left(\theta^{t-1},\theta'\right)+u'\left(c_t\left(\theta^{t-1},\theta'\right)\right)-\beta a\mathbb{E}_tu'\left(c_{t+1}\left(\theta^{t-1},\theta',\theta_{t+1}\right)\right)\\
&>-\frac{\partial}{\partial\varepsilon}s_{t+1}\mu_t\left(\theta^{t-1},\theta'\right)+u'\left(c_t\left(\theta^{t-1},\theta'\right)\right)-\beta R_{t+1}\mathbb{E}_tu'\left(c_{t+1}\left(\theta^{t-1},\theta',\theta_{t+1}\right)\right) \qquad \text{(A.1)}\\
&=-\frac{\partial}{\partial\varepsilon}s_{t+1}\mu_t\left(\theta^{t-1},\theta'\right)+\mu_t\left(\theta^{t-1},\theta'\right)\\
&\geq 0
\end{aligned}$$

where $\mu_t\left(\theta^{t-1},\theta'\right)$ is the multiplier on the non-negative savings constraint. The strict inequality in (A.1) follows since $a<R$ and

$$\begin{aligned}
&c_t+s_{t+1}=\theta_t+b_t+\varepsilon\\
&\Rightarrow\frac{\partial}{\partial\varepsilon}c_t+\frac{\partial}{\partial\varepsilon}s_{t+1}=1\\
&\Rightarrow\frac{\partial}{\partial\varepsilon}s_{t+1}<1
\end{aligned}$$

Part 2. Notice that

$$\frac{\partial}{\partial\varepsilon}Z\left(\left(\theta^{t-1},\theta'\right),0,b_t-\varepsilon;A_{\varepsilon_-}\right)=u'\left(\theta'-s_{t+1}+b_t\left(\theta^t\right)-\varepsilon\right)\left[-\frac{\partial}{\partial\varepsilon}s_{t+1}-1\right]$$
$$+\beta\frac{\partial}{\partial\varepsilon}EZ_{t+1}\left(\left(\theta^{t-1},\theta',\theta_{t+1}\right),s_{t+1},b_{t+1}+a\varepsilon\right)$$
$$=u'\left(c_t\left(\theta^{t-1},\theta'\right)\right)\left[-\frac{\partial}{\partial\varepsilon}s_{t+1}-1\right]+\beta\left[EZ_{2,t+1}\frac{\partial}{\partial\varepsilon}s_{t+1}+aEZ_{3,t+1}\right]$$
$$=u'\left(c_t\left(\theta^{t-1},\theta'\right)\right)\left[-\frac{\partial}{\partial\varepsilon}s_{t+1}-1\right]+\beta\left[\begin{array}{c}REu'\left(c_{t+1}\left(\theta^{t-1},\theta',\theta_{t+1}\right)\right)\frac{\partial}{\partial\varepsilon}s_{t+1}\\+aEu'\left(c_{t+1}\left(\theta^{t-1},\theta',\theta_{t+1}\right)\right)\end{array}\right]$$
$$=\frac{\partial}{\partial\varepsilon}s_{t+1}\left[-u'\left(c_t\left(\theta^{t-1},\theta'\right)\right)+\beta REu'\left(c_{t+1}\left(\theta^{t-1},\theta',\theta_{t+1}\right)\right)\right]$$
$$-u'\left(c_t\left(\theta^t\right)\right)+\beta aEu'\left(c_{t+1}\left(\theta^{t-1},\theta',\theta_{t+1}\right)\right)$$
$$=-\frac{\partial}{\partial\varepsilon}s_{t+1}\mu_t\left(\theta^{t-1},\theta'\right)-u'\left(c_t\left(\theta^{t-1},\theta'\right)\right)+\beta aEu'\left(c_{t+1}\left(\theta^{t-1},\theta',\theta_{t+1}\right)\right)$$

If type $\left(\theta^{t-1},\theta'\right)$ is Euler-unconstrained then clearly

$$-\frac{\partial}{\partial\varepsilon}s_{t+1}\mu_t\left(\theta^{t-1},\theta'\right)-u'\left(c_t\left(\theta^{t-1},\theta'\right)\right)+\beta a\mathbb{E}_tu'\left(c_{t+1}\left(\theta^{t-1},\theta',\theta_{t+1}\right)\right)$$
$$=-u'\left(c_t\left(\theta^{t-1},\theta'\right)\right)+\beta a\mathbb{E}_tu'\left(c_{t+1}\left(\theta^{t-1},\theta',\theta_{t+1}\right)\right)>0$$

since $a>R$. Suppose however that the type $\left(\theta^{t-1},\theta'\right)$ is Euler-constrained at $\varepsilon=0$. Then $s_{t+1}=0$ and so $\beta aEu'\left(c_{t+1}\left(\theta^{t-1},\theta',\theta_{t+1}\right)\right)=\beta aEu'\left(c_{t+1}\left(\theta^{t+1}\right)\right)$ since type $\left(\theta^{t-1},\theta\right)$ will also be Euler-constrained. Moreover since $\theta'>\theta$, we must have that $\mu_t\left(\theta^{t-1},\theta'\right)<\mu_t\left(\theta^{t-1},\theta\right)$. Therefore

$$=-\frac{\partial}{\partial\varepsilon}s_{t+1}\mu_t\left(\theta^{t-1},\theta'\right)-u'\left(c_t\left(\theta^{t-1},\theta'\right)\right)+\beta a\mathbb{E}_tu'\left(c_{t+1}\left(\theta^{t-1},\theta',\theta_{t+1}\right)\right)$$
$$>-\mu_t\left(\theta^{t-1},\theta'\right)-u'\left(c_t\left(\theta^{t-1},\theta'\right)\right)+\beta a\mathbb{E}_tu'\left(c_{t+1}\left(\theta^{t-1},\theta',\theta_{t+1}\right)\right)$$
$$\geq-\mu_t\left(\theta^t\right)-u'\left(c_t\left(\theta^t\right)\right)+\beta a\mathbb{E}_tu'\left(c_{t+1}\left(\theta^{t-1},\theta',\theta_{t+1}\right)\right)$$
$$=-\mu_t\left(\theta^t\right)-u'\left(c_t\left(\theta^t\right)\right)+\beta a\mathbb{E}_tu'\left(c_{t+1}\left(\theta^{t+1}\right)\right)$$
$$\geq 0$$

since by assumption $\frac{\partial}{\partial\varepsilon}Z\left(\left(\theta^{t-1},\theta\right),0,b_t-\varepsilon;A_{\varepsilon-}\right)\big|_{\varepsilon=0}\geq 0$. ■

**Lemma 17** *Given an equilibrium transfer sequence $A$, if for any date and history $\theta^{t-1}$,*

$$Z_t\left(\left(\theta^{t-1},\theta\right),s_t,b_t\left(\theta^{t-1},\tilde{\theta}\right);A\right)>Z_t\left(\left(\theta^{t-1},\theta\right),s_t,b_t\left(\theta^{t-1},\theta\right);A\right)$$

*then there exists a deviating contract that makes both the intermediary and type* $\left(\theta^{t-1},\theta\right)$ *strictly better off.*

**Proof.** It is clear from the definition of $Z$ that such a contract will be savings contract. In particular the deviating intermediary can offer an $\varepsilon\delta$ savings contract that make both it and the household strictly better off. Such a contract will always be incentive compatible and satisfy voluntary participation constraints. ■

**Proof of Proposition 2.** Without loss of generality, we can just consider the truncated $T-$ period economy with a $T$ lived intermediaries. Suppose we have an equilibrium in this environment. Let the equilibrium transfer sequence for the households be denoted by $\left\{\zeta_t\left(\theta^t\right)\right\}_{t,\theta^t}$ where in each period $\zeta_t\left(\theta^t\right)=\zeta_t^{old}\left(\theta^t\right)+{}_t\zeta_t\left(\theta^t\right)$ for all $\theta^t\in\Theta^t$. Let $R_t=\frac{1}{q_t}$ and construct a sequence of contracts for 2 period intermediaries $\left({}_tb_t\left(\theta^t\right),{}_tb_{t+1}\left(\theta^{t+1}\right)\right)$ as follows

$$\begin{aligned}
{}_1b_1\left(\theta_1\right) &= \zeta_1\left(\theta_1\right)\\
&\vdots\\
{}_{t-1}b_t\left(\theta^t\right) &= -R_t\,{}_tb_t\left(\theta^t\right)\\
{}_tb_t\left(\theta^t\right) &= \zeta_t\left(\theta^t\right)-{}_{t-1}b_t\left(\theta^t\right)\\
&\vdots\\
{}_{T-1}b_{T-1}\left(\theta^{T-1}\right) &= \zeta_{T-1}\left(\theta^{T-1}\right)-{}_{T-2}b_{T-1}\left(\theta^{T-1}\right)\\
{}_{T-1}b_T\left(\theta^T\right) &= \zeta_T\left(\theta^T\right)
\end{aligned}$$

We know from Lemma 15 that the expected present discounted value of transfers associated with the sequence $\left\{\zeta_t\left(\theta^t\right)\right\}_{t,\theta^t}$, $A_1\left(\theta^1\right)=0$. By construction[1] ,

$$A_1\left(\theta_1\right)=\zeta_1\left(\theta_1\right)+q_1\sum_{\theta^2}\pi\left(\theta^2\right)A_2\left(\theta^2\right)$$

$$=\zeta_1\left(\theta_1\right)$$

$$+q_1\sum_{\theta_2}\pi\left(\theta_2\right)\left[\zeta_2\left(\theta_0,\theta_1\right)+...\left[...+\sum_{\theta_{T-1}}\pi\left(\theta_{T-1}\right)\left[\zeta_{T-1}\left(\theta^{T-1}\right)+q_{T-1}\sum_{\theta_T}\pi\left(\theta_T\right)\zeta_T\left(\theta^T\right)\right]\right]\right]$$

$$=\ {}_1b_1\left(\theta_1\right)$$

$$+q_1\sum_{\theta_2}\pi\left(\theta_2\right)\left[{}_1b_2+{}_1b_2+...\left[...+\sum_{\theta_{T-1}}\pi\left(\theta_{T-1}\right)\left[..+{}_{T-1}b_{T-1}+q_{T-1}\sum_{\theta_T}\pi\left(\theta_T\right){}_{T-1}b_T\right]\right]\right]$$

$$=q_1\sum_{\theta_2}\pi\left(\theta_2\right)\left[\left[...+...\left[...+\sum_{\theta_{T-1}}\pi\left[{}_{T-2}b_{T-1}+{}_{T-1}b_{T-1}+q_{T-1}\sum_{\theta_T}\pi\left(\theta_T\right){}_{T-1}b_T\right]\right]\right]\right]$$

$$=\prod_{s=1}^{T-2}q_s\sum_{\theta^{T-2}}\pi\left(\theta^{T-2}\right)\sum_{\theta_{T-1}}\pi\left(\theta_{T-1}\right)\left[{}_{T-1}b_{T-1}+q_{T-1}\sum_{\theta_T}\pi\left(\theta_T\right){}_{T-1}b_T\left(\theta^{T-2},\theta_{T-1},\theta_T\right)\right]$$

Since $A_1^1\left(\theta_1\right)=0$,

$$\sum_{\theta^{T-2}}\pi\left(\theta^{T-2}\right)\sum_{\theta_{T-1}}\pi\left(\theta_{T-1}\right)\left[{}_{T-1}b_{T-1}\left(\theta^{T-2},\theta_{T-1}\right)+q_{T-1}\sum_{\theta_T}\pi\left(\theta_T\right){}_{T-1}b_T\left(\theta^T\right)\right]=0$$

We want to show that for all $\theta^{T-2}$,
$\left[{}_{T-1}b_{T-1}\left(\theta^{T-2},\theta_{T-1}\right)+q_{T-1}\sum_{\theta_T}\pi\left(\theta_T\right){}_{T-1}b_T\left(\theta^{T-2},\theta_{T-1},\theta_T\right)\right]$ is independent of $\theta_{T-1}$. By construction, this is equivalent to showing that

$$A_{T-1}\left(\theta^{T-2},\theta_{T-1}\right)=\zeta_{T-1}\left(\theta^{T-2},\theta_{T-1}\right)+q_{T-1}\sum_{\theta_T}\pi\left(\theta_T\right)\zeta_T\left(\theta^{T-2},\theta_{T-1},\theta_T\right)$$

is independent of $\theta_{T-1}$. It is easy to see that $\zeta_T\left(\theta^{T-2},\theta_{T-1},\theta_T\right)$ must be independent of $\theta_T$ else households would always announce the type consistent with the largest transfer. Therefore, ${}_{T-1}b_T\left(\theta^{T-2},\theta_{T-1},\theta_T\right)$ must also be independent of $\theta_T$.

Suppose for some history $\theta^{T-2}$ and $\theta,\theta'\in\Theta$,

[1] Note that I have dropped some of the history dependence, wherever clear, for ease of notation.

$$A_{T-1}\left(\theta^{T-2},\theta\right) > A_{T-1}\left(\theta^{T-2},\theta'\right)$$

First suppose that $\theta < \theta'$. There exists some $\delta > 0$

$$A_{T-1}\left(\theta^{T-2},\theta\right) = A_{T-1}\left(\theta^{T-2},\theta'\right) + \delta$$

Since this excess transfer can either be front or back-loaded, we need to consider two cases. If the transfer is front loaded then

$$A_{T-1}\left(\theta^{T-2},\theta\right) = \zeta_{T-1}\left(\theta^{T-2},\theta'\right) + \varepsilon + q_{T-1}\sum_{\theta'\in\Theta}\pi\left(\theta'\right)\left[\zeta_T\left(\theta^{T-2},\theta'\right) - a\varepsilon\right]$$

where $\varepsilon > 0, a < R_T$ and $\varepsilon - q_{T-1}a\varepsilon = \delta$. Similarly if the transfers are back-loaded then

$$A_{T-1}\left(\theta^{T-2},\theta\right) = \zeta_{T-1}\left(\theta^{T-2},\theta'\right) - \varepsilon + q_1\sum_{\theta'\in\Theta}\pi\left(\theta'\right)\left[\zeta_T\left(\theta^{T-2},\theta'\right) + a\varepsilon\right]$$

where $\varepsilon > 0, a > R_T$ and $\varepsilon - q_{T-1}a\varepsilon = \delta$.

In the first case, the first part of Lemma 16 along with Lemma 17 tells us that type $\left(\theta^{T-2},\theta'\right)$ would strictly prefer to lie and pretend to be type $\left(\theta^{T-2},\theta\right)$ and save with another intermediary and so incentive compatibility constraints are violated. In the second case notice that in any equilibrium type $\left(\theta^{T-2},\theta\right)$ must weakly prefer to tell the truth than announce $\left(\theta^{T-2},\theta'\right)$. As a result this type must weakly prefer transfer scheme $A_{T-1}\left(\theta^{T-2},\theta\right)$ to $A_{T-1}\left(\theta^{T-2},\theta'\right)$. Then part 2 of Lemma 16 along with Lemma 17 implies that the incentive compatibility constraint for $\left(\theta^{t-1},\theta'\right)$ is violated again.

Next, suppose $\theta > \theta'$. We know from Lemma 14 that if the voluntary participation constraint binds, it does so for the lowest type and hence

$$V_{T-1}\left(\theta^{T-2},\theta\right) > V_{T-1}^{d}\left(\theta\right)$$

so that the household with a larger present discounted value of transfers strictly prefers the existing contract to defaulting. In this case consider an intermediary modifying the original contract as follows; for some $\delta > 0$, small

$$\tilde{\zeta}_{T-1}\left(\theta^{T-2},\theta\right) = \zeta_{T-1}\left(\theta^{T-2},\theta\right) - \delta$$
$$\tilde{\zeta}_1\left(\theta^{T-2},\hat{\theta}\right) = \zeta_{T-1}\left(\theta^{T-2},\hat{\theta}\right) + \frac{\delta}{\sum_{\theta'<\theta}\pi\left(\theta^{T-2},\theta'\right)} \text{ for all } \hat{\theta} < \theta$$

Since this provides more insurance in period $T-1$, it increases the expected welfare of agent $\theta^{T-2}$. The perturbation continues to satisfy incentive compatibility and also the participation constraints for $\delta$ small enough.

Clearly, the sequence of constructed transfers is budget feasible and satisfies incentive compatibility and participation constraints. Moreover as demonstrated above, each two-period contract makes 0 profits and hence there is no cross-subsidization. We only need to check that a particular two period intermediary cannot do strictly better. But this is clear since if it could then a $T$ intermediary could just modify its contract and also make positive profits. ■

The next lemma is consequence of the above characterization.

**Lemma 18** *Given a sequence of two-period contracts, for each* $t$, ${}_t\mathcal{P}_t\left(\theta^{t-1},\theta\right) = {}_t\mathcal{P}_t\left(\theta^{t-1},\theta'\right)$ *and* ${}_t\mathcal{P}_{t+1}\left(\theta^{t-1},\theta\right) = {}_t\mathcal{P}_{t+1}\left(\theta^{t-1},\theta'\right)$ *for all* $\theta,\theta' \in \Theta$

**Proof of Lemma 18.** In the last period $T$ of the truncated economy, clearly

$$ {}_{T-1}b_T\left(\theta^{T-1},\theta\right) = {}_{T-1}b_T\left(\theta^{T-1},\theta'\right) $$

for otherwise the household would always announce the type with the highest transfer. In $T-1$, for the $T-1$ intermediary,

$$ {}_{T-1}\mathcal{P}_{T-1}\left(\theta^{T-2},\theta\right) = {}_{T-1}b_{T-1}\left(\theta^{T-2},\theta\right) + q_{T-1}\,{}_{T-1}b_T\left(\theta^{T-1}\right) $$

We know from **??** that competition implies that

$$ \begin{aligned} &{}_{T-1}\mathcal{P}_{T-1}\left(\theta^{T-2},\theta\right) = 0 \\ &\Rightarrow {}_{T-1}b_{T-1}\left(\theta^{T-2},\theta\right) + q_{T-1}\,{}_{T-1}b_T\left(\theta^{T-1}\right) = 0 \end{aligned} $$

And so ${}_{T-1}\mathcal{P}_{T-1}\left(\theta^{T-2},\theta\right) = {}_{T-1}\mathcal{P}_{T-1}\left(\theta^{T-2},\theta'\right)$. Competition similarly implies that for any $t$, ${}_t\mathcal{P}_t\left(\theta^{t-1},\theta\right) = {}_t\mathcal{P}_t\left(\theta^{t-1},\theta'\right)$. Lastly consider $t+1$ and

$$ {}_t\mathcal{P}_{t+1}\left(\theta^{t-1},\theta\right) = {}_tb_{t+1}\left(\theta^{t-1},\theta\right) $$

Since

$$ c_t\left(\theta^t,\theta\right) = \theta + {}_tb_{t+1}\left(\theta^t\right) + {}_{t+1}b_{t+1}\left(\theta^t,\theta\right) $$

and future sequence of transfers ${}_{t+1}\mathcal{P}_{t+1}\left(\theta^t,\theta\right)$ is independent of $\theta$, incentive compatibility implies that ${}_t\mathcal{P}_{t+1}\left(\theta^{t-1},\theta\right) = {}_t\mathcal{P}_{t+1}\left(\theta^{t-1},\theta'\right)$ for all $\theta,\theta' \in \Theta$. ■

The following result follows from the previous two results.

**Proposition 28** *In any equilibrium with $\hat{T}$ lived intermediaries, $2 \leq \hat{T} < \infty$, for all $t$ and $\theta^{t-1}$, $A_t\left(\theta^{t-1},\theta\right) = A_t\left(\theta^{t-1},\theta'\right)$ for all $\theta,\theta' \in \Theta$*

**Proof of Proposition 28.** The result follows from the previous lemmas. We know that the above is true in any model with 2 period lived intermediaries. Moreover since any model with $\hat{T}+1$ lived intermediaries is equivalent to one with the 2 period lived ones, the above must be true. ■

*Proof of Proposition 3:* The proof requires the following result

**Proposition 29** *In any equilibrium with two period lived intermediaries, for any $t$ and $\theta^t,\hat{\theta}^t$ such that $\theta_t + {}_{t-1}b_t\left(\theta^t\right) = \hat{\theta}_t + {}_{t-1}b_t\left(\hat{\theta}^t\right)$,*

$$V_t\left(\theta^t\right) = V_t\left(\hat{\theta}^t\right)$$

**Proof of Proposition 29.** Because of the assumption that after period $T$, households can only trade a risk free bond subject to exogenous debt constraints, it is easy to see that the statement holds in period $T$, since all that matters for the households' choices is the sum $\theta_T + {}_{T-1}b_T\left(\theta^T\right)$. In period $T-1$ suppose $\theta^{T-1}$ and $\hat{\theta}^{T-1}$ such that $\theta_{T-1} + {}_{T-2}b_{T-1}\left(\theta^{T-1}\right) = \hat{\theta}_{T-1} + {}_{T-2}b_{T-1}\left(\hat{\theta}^{T-1}\right)$ and

$$V_{T-1}\left(\theta^{T-1}\right) > V_{T-1}\left(\hat{\theta}^{T-1}\right)$$

For ease of notation denote the corresponding transfers by ${}_{T-2}b_{T-1}$ and ${}_{T-2}\hat{b}_{T-1}$ We need to consider a few cases. Suppose first that for both $\theta^{T-1},\hat{\theta}^{T-1}$

$$u'\left(\theta_{T-1} + {}_{T-2}b_{T-1} + {}_{T-1}b_{T-1}\right) = \beta R_T \mathbb{E}_{T-1} u'\left(\theta_T + {}_{T-1}b_T + \psi_{T+1}\left(\theta_T + {}_{T-1}b_T\right)\right) \tag{A.2}$$

$$u'\left(\hat{\theta}_{T-1} + {}_{T-2}\hat{b}_{T-1} + {}_{T-1}\hat{b}_{T-1}\right) = \beta R_T \mathbb{E}_{T-1} u'\left(\theta_T + {}_{T-1}\hat{b}_T + \psi_{T+1}\left(\theta_T + {}_{T-1}\hat{b}_T\right)\right) \tag{A.3}$$

where $\psi_{T+1}\left(\theta_T + {}_{T-1}b_T\right)$ is the savings choice for the household (given that it is subject to debt constraint $\phi^e_{T+1}$). Since ${}_{T-1}b_{T-1} + \frac{{}_{T-1}b_T}{q_{T-1}} = {}_{T-1}\hat{b}_{T-1} + \frac{{}_{T-1}\hat{b}_T}{q_{T-1}} = 0$ and the savings choice $\psi_{T+1}$ depends only on the sum $\theta_T + {}_{T-1}b_T$ , it must be that ${}_{T-1}b_{T-1} = {}_{T-1}\hat{b}_{T-1}$ and $V\left(\theta^t\right) = V\left(\hat{\theta}^{T-1}\right)$ and so we have a contradiction. Suppose on the other hand that (A.2) holds with equality and (A.3) with strictly inequality. Again, since ${}_{T-1}b_{T-1} + \frac{{}_{T-1}b_T}{q_{T-1}} = {}_{T-1}\hat{b}_{T-1} + \frac{{}_{T-1}\hat{b}_T}{q_{T-1}} = 0$, it must be that ${}_{T-1}b_{T-1} > {}_{T-1}\hat{b}_{T-1}$ . Since the household is Euler-constrained, assume that ${}_{T-1}\hat{b}_{T-1} > 0$. It is easy to see that that giving type $\hat{\theta}^{T-1}$ the contract associated with $\theta^{T-1}$ makes it strictly better off. Consider modifying the original contract so that

$$
\begin{aligned}
{}_{T-1}\tilde{b}_{T-1} &= {}_{T-1}\hat{b}_{T-1} + \varepsilon \\
{}_{T-1}\tilde{b}_{T} &= {}_{T-1}\hat{b}_{T} - \delta\varepsilon
\end{aligned}
$$

where $\varepsilon$ chosen so that ${}_{T-1}\hat{b}_T - \delta\varepsilon \geq {}_{T-1}b_T$ and

$$
\frac{u'\left(\hat{\theta}_{T-1} + {}_{T-2}\hat{b}_{T-1} + {}_{T-1}\hat{b}_{T-1}\right)}{\beta\mathbb{E}_{T-1}u'\left(\theta_T + {}_{T-1}\hat{b}_T + \psi_{T+1}\left(\theta_T + {}_{T-1}\hat{b}_T\right)\right)} > \delta > R_T
$$

This perturbation makes type $\hat{\theta}^{T-1}$ strictly better off. To see that voluntary participation constraints continue to hold for type $\hat{\theta}^{T-1}$ in period $t$, notice that this household's value in period $T$ is exactly the same as $\theta^{T-1}$. Since the original transfer scheme was incentive compatible and satisfied voluntary participation constraints in period $T$, it must be that for all $\theta \in \Theta$

$$
u\left(\theta + {}_{T-1}\tilde{b}_T + \psi_{T+1}\left(\theta + {}_{T-1}\tilde{b}_T\right)\right) + \beta\mathbb{E}_T V_{T+1}\left(\theta, \psi_{T+1}\right) \geq V^d_T\left(\theta\right)
$$

As a result, these constraints continue to hold under this deviation. Finally since $\delta > R_T$, the deviating intermediary makes strictly positive profits. Therefore it must be that ${}_{T-1}b_{T-1} = {}_{T-1}\hat{b}_{T-1}$ . Note that a similar argument holds if both (A.2) and (A.3) hold with inequality and $V_{T-1}\left(\theta^T\right) > V_{T-1}\left(\hat{\theta}^{T-1}\right)$.Given that the property holds for $\hat{T}-1$, assume that this property holds for some $t+1 < \hat{T}-1$. Our goal is to show that the property holds in $t$. Suppose for contradiction we have some $\theta^t, \hat{\theta}^t$ such that $\theta_t + {}_{t-2}b_{t-1}\left(\hat{\theta}^{t-1}\right) = \hat{\theta}_t + {}_{t-2}b_{t-1}\left(\hat{\theta}^{t-1}\right)$ and

$$
V_t\left(\hat{\theta}^t\right) > V_t\left(\hat{\theta}^t\right)
$$

Again, denote the transfers by ${}_{t-2}b_{t-1}$ and ${}_{t-2}\hat{b}_{t-1}$. As before, first consider the case in which both type's Euler equations hold with equality. Suppose ${}_{t}b_{t} < {}_{t}\hat{b}_{t}$. Then it is easy to see that an intermediary can offer an $\varepsilon\delta$ savings contract which will be accepted by this agent making both intermediary strictly better off. To see why notice that since ${}_{t}b_{t} + \frac{{}_{t}b_{t+1}}{q_t} = {}_{t}\hat{b}_{t} + \frac{{}_{t}\hat{b}_{t+1}}{q_t} = 0$ and $V\left(\theta^t\right) > V\left(\hat{\theta}^t\right)$ it must be that there exists some $\varepsilon > 0$ such that the transfer scheme ${}_{t}\hat{b}_{t} - \varepsilon + \frac{{}_{t}\hat{b}_{t+1} + \varepsilon}{q_t}$ makes this type strictly better off. Next suppose that

$$u'\left(\theta_t + {}_{t-1}b_t + {}_{t}b_t\right) = \beta R_{t+1}\mathbb{E}_t u'\left(\theta_{t+1} + {}_{t}b_{t+1} + {}_{t+1}b_{t+1}\right)$$
$$u'\left(\hat{\theta}_t + {}_{t-1}\hat{b}_t + {}_{t}\hat{b}_t\right) > \beta R_{t+1}\mathbb{E}_t u'\left(\theta_{t+1} + {}_{t}\hat{b}_{t+1} + {}_{t+1}\hat{b}_{t+1}\right)$$

As in the period $T-1$ case, consider modifying the original contract

$${}_{t}\tilde{b}_{t} = {}_{t}\hat{b}_{t} + \varepsilon$$
$${}_{t}\tilde{b}_{t+1} = {}_{t}\hat{b}_{t+1} - \delta\varepsilon$$

where

$$\frac{u'\left(\hat{\theta}_t + {}_{t-1}\hat{b}_t + {}_{t}\hat{b}_t\right)}{\beta\mathbb{E}_t u'\left(\theta_{t+1} + {}_{t}\hat{b}_{t+1} + {}_{t+1}\hat{b}_{t+1}\right)} > \delta > R$$

independently of reported type. To see that no agent would choose to default on this intermediary notice that for any type that signs this contract will have the same value in $t+1$ by the induction assumption. Therefore since type $\theta^{t+1}$ preferred not to default under the original contract, type $\hat{\theta}^t$ will not want to default under the deviating contract. If the original contract was incentive compatible, the deviating one will be as well.

Finally, since there exists a type, $\hat{\theta}^t$ who is made strictly better off for some $\delta < 1$, the deviating intermediary makes strictly positive profits. Therefore by induction the claim must hold in period $t$ and by induction for all previous periods as well. ■

**Proof of Proposition 3.** Note that the proposition is written in terms of the equivalent 2 period contracts. We know from Proposition 29 that for all $\theta^t$, $V_t\left(\theta^t\right)$ only depends on $\theta_t + {}_{t-1}b_t\left(\theta^t\right)$. Given the nature of these two period contracts, we consider transfers of the form $\left({}_{t}b_t\left(\theta^t\right), {}_{t}b_{t+1}\left(\theta^t\right)\right) = \left(\varphi\left(\theta^t\right), \frac{-\varphi\left(\theta^t\right)}{q_t}\right)$. Let $\varphi^*$ be largest such $\varphi\left(\theta^t\right)$ given to all households that are Euler-constrained and denote the corresponding history by $\theta^{*t}$. Given some $\varphi\left(\theta^t\right)$ define

$$R^{\varphi\left(\theta^t\right)} = \frac{u'\left(\theta_t + {}_{t-1}b_t\left(\theta^t\right) + \varphi\left(\theta^t\right)\right)}{\beta \mathbb{E}_t u'\left(\theta + \frac{-\varphi\left(\theta^t\right)}{q_t} + {}_{t+1}b_{t+1}\left(\theta^{t+1}\right)\right)}$$

Since this household is Euler-constrained, $R^{\varphi\left(\theta^t\right)} > R_{t+1}$. Suppose there exists an Euler-constrained household $\tilde{\theta}^t$ such that $\varphi\left(\tilde{\theta}^t\right) < \varphi^*$. In this case it must also be that $R^{\varphi\left(\tilde{\theta}^t\right)} > R_{t+1.}$

Consider modifying the original contract as follows

$$\begin{aligned} {}_t\tilde{b}_t\left(\tilde{\theta}^t\right) &= \varphi\left(\tilde{\theta}^t\right) + \varepsilon \\ {}_t\tilde{b}_{t+1}\left(\tilde{\theta}^t\right) &= -\frac{\varphi\left(\tilde{\theta}^t\right)}{q_t} - \frac{\varepsilon}{\hat{q}_t} \end{aligned}$$

where

$$R_{t+1} = \frac{1}{q_t} < \frac{1}{\hat{q}_t} < R^{\varphi\left(\tilde{\theta}^t\right)}$$

Notice that for $\varepsilon$ small, type $\tilde{\theta}^t$ will be made strictly better off by signing such a contract since $R^{\varphi\left(\tilde{\theta}^t\right)} > \frac{1}{\hat{q}_t}$ and the household is Euler-constrained. For $\varepsilon$ small enough, ${}_t\tilde{b}_{t+1}\left(\tilde{\theta}^t\right) \geq \frac{-\varphi^*}{q_t}$. Since we have shown earlier that equilibrium continuation value for any agent going forward only depends on the sum $\theta + {}_tb_{t+1}\left(\theta^t\right)$, if

$$u\left(\theta^* + \frac{-\varphi^*}{q_t} + {}_{t+1}b_{t+1}\left(\theta^{*t+1}\right)\right) + \beta \mathbb{E}_{t+1} V_{t+2}\left(\theta^{*t+2}\right) \geq V^d_{t+1}\left(\theta^{*t+1}\right)$$

then all households accepting the deviating contract will also prefer not to default. To check incentive compatibility, notice that if the original contract was incentive compatible and all other types preferred their transfers to $\left(\varphi^*, \frac{-\varphi^*}{q_t}\right)$, clearly the modified transfer sequence will be incentive compatible as well. Finally, since $\frac{1}{q_t} < \frac{1}{\hat{q}_t}$, the deviating intermediary is also made strictly better off. ■

Using these results, we can proceed to the proof of the equivalence theorem..

**Proof of Theorem 1.** Given an equilibrium of the decentralized contracting problem with equilibrium transfer schedules $s_t\left(\theta^t\right)$, construct the equivalent 2 period contracts (which we proved exists earlier). As a result we have a sequence of transfers

$\left\{ {}_t\zeta_t\left(\theta^t\right), {}_t\zeta_{t+1}\left(\theta^{t+1}\right)\right\}_{\theta^t,t}$. Construct bond holdings after each history for the agent as follows (assume that agents start off with 0 initial wealth)

$$s_2\left(\theta_1\right) = -\ {}_1\zeta_1\left(\theta_1\right)$$
$$\vdots$$
$$s_{t+1}\left(\theta^t\right) = -\ {}_t\zeta_t\left(\theta^t\right)$$
$$\vdots$$

Let the interest rates $\{R_t\}$ be defined such that $R_{t+1} = \frac{1}{q_t}$. Given that the sequence of transfers satisfies the zero profit condition we know that $-R_t s_t\left(\theta^{t-1}\right) = {}_{t-1}\zeta_t\left(\theta^{t-1}\right)$ and therefore the constructed bond holdings satisfy the household's budget constraints. To construct the sequence of debt constraints recall that we showed that in contracting environment, that for any $t$, and $\theta^t$ such that

$$u'\left(\theta_t + {}_{t-1}\zeta_t\left(\theta^{t-1}\right) + {}_t\zeta_t\left(\theta^t\right)\right) > \beta R_{t+1}\mathbb{E}_t u'\left(\theta_{t+1} + {}_t\zeta_{t+1}\left(\theta^t\right) + {}_{t+1}\zeta_{t+1}\left(\theta^{t+1}\right)\right)$$

it must be that ${}_t\zeta_t\left(\theta^t\right) = \varphi_t$ where $\varphi_t$ is independent of the agent's history. Let

$$\phi_{t+1} = \varphi_t$$

for all $t$. The necessary and sufficient conditions for agent optimality in the bond trading economy are

$$u'\left(c_t\left(\theta^t\right)\right) \geq \beta R_{t+1}\mathbb{E}_t u'\left(c_{t+1}\left(\theta^{t+1}\right)\right)$$

with strict inequality if

$$s_{t+1}\left(\theta^t\right) = -\phi_{t+1}$$

along with budget feasibility (which we have already established). We know from earlier results that any allocation from the decentralized contracting environment satisfies exactly these conditions which shows that the constructed allocation is optimal for all agents. It only remains to show that these debt constraints are not-too-tight which follows from Proposition 1 and Proposition 3.

For part 2, consider an equilibrium of the debt constrained environment. Construct transfer schedules for $\hat{T}$ period lived intermediaries as follows

$$\begin{aligned}
{}_1\zeta_1\left(\theta_1\right) &= -s_2\left(\theta_1\right) \\
{}_1\zeta_2\left(\theta^1\right) &= Rs_2\left(\theta_1\right) - s_3\left(\theta^2\right) \\
&\vdots \\
{}_1\zeta_{\hat{T}}\left(\theta^{\hat{T}}\right) &= R_{\hat{T}}s_{\hat{T}}\left(\theta^{\hat{T}-1}\right) \\
{}_{\hat{T}}\zeta_{\hat{T}}\left(\theta^{\hat{T}}\right) &= -s_{\hat{T}+1}\left(\theta^{\hat{T}}\right) \\
&\vdots
\end{aligned}$$

And let $q_t = \frac{1}{R_{t+1}}$. Note that we are constructing an equilibrium in which each intermediary born at date $1, \hat{T}, 2\hat{T}-1, ...$ offers a single contract with transfers as constructed. All intermediaries born at other dates offer simple uncontingent savings contracts. While these will never be signed in equilibrium, a deviating contract that offers some state-contingency will never be profitable since households can always lie and use these savings contracts to smooth any excess transfers. This similar to the "latent contracts" used by Ales and Maziero (2014) to sustain their equilibrium. Suppose these contracts and prices did not constitute an equilibrium. There are two cases to consider:

1. Given prices and the contract offered by this intermediary, no new intermediary has an incentive to offer a contract and make strictly positive profits.

2. The existing intermediary has no incentive to modify its contract and make strictly positive profits.

Consider the first case. Suppose that this was a $\tilde{T}$ period contract that spanned dates $t \to \tilde{T}-1$. Notice that the only way in which households will strictly prefer to sign with such a deviating contract and the intermediary make a positive profit is if it increases increases insurance in some period.

First consider the last period $\tilde{T}-1$. It is easy to see that in this period, the transfers from the intermediary to the household cannot depend on $\theta_{\tilde{T}-1}$ else the household would always announce the type consistent with the highest transfer. Next consider period $\tilde{T}-2$. Suppose the contract made a positive transfer to some type who is Euler constrained in period

$\tilde{T}-1$. Incentive compatibility requires that this type must receive a negative uncontingent transfer in period $\tilde{T}$ otherwise households would lie to get this increased transfer. Since the household is Euler constrained and debt constraints are chosen to be Not-too-tight, we know that some type's voluntary participation constraint holds with equality in $\tilde{T}-1$ and so such a perturbation is not possible. If the household is unconstrained in this state, a perturbation that makes both the intermediary and the agent strictly better off is not possible.

On the other hand, suppose the contract made a negative transfer to some type. Again incentive compatibility dictates that a positive uncontingent transfer be made to this type in period $\tilde{T}-1$. However, this is exactly a pure savings contract and since the households are not savings constrained, this will never be profitable.

Now consider period $\tilde{T}-3$. First, consider a state contingent positive transfer to some type who is constrained. This must be compensated for by a negative transfer in period $\tilde{T}-2$. This transfer cannot be independent of state since some household's voluntary participation constraint binds. It also cannot be state contingent by the previous argument. As before, a negative transfer followed by an uncontingent transfer at date $\tilde{T}-2$ can never make both the intermediary and agent strictly better off. A similar argument holds for all previous periods by induction.

Finally, consider a positive transfer to some type in $\tilde{T}-3$ who is not constrained. Incentive compatibility requires that a negative uncontingent transfer be made in period $\tilde{T}-2$. However such a perturbation can never be welfare enhancing if the present discounted value of transfers is less than zero and so the intermediary can never make a positive profit on this particular deviation.

Next, we need to check that the *existing* intermediary has no incentive to modify its contract given prices. As above, the only such modifications will involve providing some type in some period a little more insurance. Consider a period $t$, and a type $\theta^t$ who is Euler-constrained under the original contract. Given our equilibrium definition, we know that there exists some $\theta^c$ such that the voluntary participation constraint for type $\left(\theta^t, \theta^c\right)$ holds with equality in $t+1$

We consider a deviation in which the intermediary increases the transfer to this household by some $\varepsilon > 0$. It is easy to see that incentive compatibility requires that the intermediary make a negative transfer at some future date, say $t+1$. So there exists some $(\theta^t, \theta^*)$ who receives a negative transfer $\delta$ in $t+1$. Note that the negative transfer cannot be uncontingent since for some type in $t+1$, the voluntary participation constraint holds with equality. Therefore, the transfer $\delta$ must be state contingent. We can group states into two classes; the first $Con_{t+1}(\theta^t)$ are those that are Euler-constrained at $t+1$, i.e.

$$u'\left(c_{t+1}\left(\theta^t,\theta\right)\right) > \beta R_{t+2}\mathbb{E}_{t+1}u'\left(c_{t+2}\left(\theta^t,\theta,\theta'\right)\right)$$

and the second $Uncon_{t+1}(\theta^t)$, those that are not, i.e.

$$u'\left(c_{t+1}\left(\theta^t,\theta\right)\right) = \beta R_{t+2}\mathbb{E}_{t+1}u'\left(c_{t+2}\left(\theta^t,\theta,\theta'\right)\right)$$

We know that $\theta^c \in Con_{t+1}(\theta^t)$. Also, from Proposition 28, it must be that $A_{t+1}(\theta^t,\theta) = A_{t+1}(\theta^t,\theta')$ for any $\theta, \theta' \in Uncon_{t+1}(\theta^t)$. Therefore a negative transfer $\delta'$ will have to be imposed on all such types. However, since under the original contract, $A_{t+1}(\theta^t,\theta)$ is independent of $\theta$, the perturbation implies that $A_{t+1}(\theta^t,\theta) > A_{t+1}\left(\theta^t,\hat{\theta}\right)$ for any $\theta, \hat{\theta}$ in $Con_{t+1}(\theta^t)$ and $Uncon_{t+1}(\theta^t)$ respectively.

Therefore, all types in $Uncon_{t+1}(\theta^t)$ will strictly prefer to lie and announce some type in $Con_{t+1}(\theta^t)$ and save with some other intermediary. ■

### Proofs from Section 3.1

*Proof of Theorem 2*: The first step in the proof is to show that given a measurable map $\Phi$, a $\Phi-$RCE always exists.

**Proposition 30** *For any finite measurable map $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$, a $\Phi-$RCE exists*

**Proof of Proposition 30.** The first step of the proof is to show that given continuous pricing functions $R(\phi)$, there exists a unique list of value functions $W$ and policy functions $b'(\theta, b, \phi)$ that solve the individual household's problems. This part of the proof uses arguments developed in Miao (2006).

Let $\mathbb{A} \subset \mathbb{R}$ be the compact feasible asset space, $\mathbb{D} \subset \mathbb{R}_+$ the compact space of debt constraints and the $\mathbb{V}$ denote the set of uniformly bounded and continuous real valued functions on $\Theta \times \mathbb{A} \times \mathbb{D}$.

Define operator $\mathbb{T}$ as follows: Given some $w \in \mathbb{V}$,

$$(\mathbb{T}w)(\theta, b, \phi) = \max_{b' \in \Gamma(b,\phi)} u\left(\theta - Rb - b'\right) + \beta \mathbb{E}w\left(\theta', b', \phi'; \Phi'\right)$$

where $\Gamma(b, \phi) = [-\phi, \theta + Rb]$. In order to apply the contraction mapping theorem I first show $\mathbb{T}w \in \mathbb{V}$. Boundedness follows. To show continuity, consider a sequence $(\theta, b, \phi)^n \to (\theta, b, \phi)$. Given our restriction to continuous pricing functions, $R(\phi^n) \to R(\phi)$. As a result correspondence $\Gamma$ is continuous. Then first term on the right hand side of the above dynamic program is continuous since $u$ is continuous. Consider second term.

We want to show that

$$|\mathbb{E}w(\theta^n, b^n, \phi^n) - \mathbb{E}w(\theta, b, \phi)| \to 0$$

Since $\mathbb{A} \times \mathbb{D} \times \Theta$ is compact by Tychonoff's theorem, $w$ is uniformly continuous and as a result $w(\theta^n, b^n, \phi^n) \to w(\theta, b, \phi)$ uniformly. As a result we can interchange the limit and integrals. Therefore by Maximum theorem, $\mathbb{T}w$ is also continuous and hence $\mathbb{T}w \in \mathbb{V}$. It is easy to see that the operator satisfies Blackwell's sufficiency conditions. As a result operator $\mathbb{T}$ is a contraction and so by the Contraction Mapping Theorem we have unique sequence of functions $w^*$ and corresponding policy functions $b^{'*}$. Next, we can use the individual policy function to compute the aggregate distribution

$$\lambda(A \times B) = \mu\left(i \in I : \left(b'(i), \theta(i)\right) \in A \times B,\ A \times B = \mathcal{B}(A) \times \mathcal{B}(\Theta)\right)$$

Consequently

$$\lambda'(A \times B) = \int \mu\left(i \in I, \theta'(i) \in A, b'(\theta, b, \phi) \in B\right) d\lambda(\theta, b)$$

which defines the measurable mapping $G$.

Next, it is straightforward to note that the policy functions $b'(\theta, b, \phi)$ are strictly increasing in $R$ for all $b' > -\phi$ and that $b'(\theta, b, \phi) = -\phi$ for $R$ small enough. As a result given $\phi$, for $R(\phi)$ large enough

$$\int_{\mathbb{A} \times \Theta} b'(\theta, b, \phi)\, d\lambda(b, \Theta) > 0$$

and for $R(\phi)$ small enough

$$\int_{\mathbb{A}\times\Theta} b'(\theta, b, \phi)\, d\lambda(b, \Theta) = -\phi < 0$$

As a result continuity implies that there exists $R(\phi)$ such that

$$\int_{\mathbb{A}\times\Theta} b'(\theta, b, \phi)\, d\lambda(b, \Theta) = 0$$

■

Next, it always true that a $\Phi-$RCE with $\Phi$ being the zero map is NTT-RCE

**Lemma 19** *There exists an NTT-RCE in which* $\Phi = 0$

**Proof of Lemma 19.** Consider the $\Phi-$RCE in which $\Phi$ is the zero map i.e. $\phi = 0$ and $\Phi(\phi) = 0$. We know that such an equilibrium exists from the previous lemma.

To show that these also constitutes a NTT-RCE we also need to show that

$$W\left(\theta, 0, 0; \Phi^0\right) = V^d(\theta)$$

which is straightfoward since

$$W\left(\theta, 0, 0; \Phi^0\right) = u(\theta) + \mathbb{E}u\left(\theta'\right) = V^d(\theta)$$

■

The reason for this is clear. If debt constraints are zero each period, then in equilibrium agents consume their endowment which trivially implies that the voluntary participation constraint binds for each period and each type. The final and main proposition that completes the proof of Theorem 2 is to show that there exists a NTT-RCE with $\Phi \neq 0$.

**Proposition 31** *If*

$$\frac{u'\left(\bar{\theta}\right)}{\beta\eta} < \kappa$$

*then there exists a NTT-RCE in which* $\Phi > 0$

**Proof of Proposition 31.** Define $\phi_\varepsilon = \phi + \varepsilon$ and $\Phi_\varepsilon$ such that $\Phi_\varepsilon(\phi + \varepsilon) = \phi' + \varepsilon$.

The first step in the proof is to compute the sign of the following object

$$\lim_{\substack{\phi\to 0\\ \Phi\to \mathbf{0}}} \frac{\partial}{\partial\varepsilon} W\left(\theta, -\phi_\varepsilon, \phi_\varepsilon; \Phi_\varepsilon\right)\bigg|_{\varepsilon=0}$$

In words, this measures the change in equilibrium welfare of the $\Phi-$RCE as we change $\Phi$ from zero the something positive.

In equilibrium we must have from the agent's problem

$$W(\theta,-\phi_{\varepsilon},\phi_{\varepsilon};\Phi_{\varepsilon})=u\left(z-R\phi-R\varepsilon-b'(\theta,-\phi_{\varepsilon},\phi_{\varepsilon})\right)+\beta\mathbb{E}W\left(\theta',b'(\theta,-\phi_{\varepsilon},\phi_{\varepsilon}),\phi';\Phi'_{\varepsilon}\right)$$

where $b'(\theta,-\phi_{\varepsilon},\phi_{\varepsilon},R)$ denote the policy function for bond holdings. We can then compute the following derivative (which is well defined)

$$\begin{aligned}\frac{\partial}{\partial\varepsilon}W(\theta,-\phi_{\varepsilon},\phi_{\varepsilon};\Phi_{\varepsilon})&=u'\left(\theta-R\phi-R\varepsilon-b'\right)\left[-R-R_{\varepsilon}\varepsilon-b'_{\varepsilon}\right]\\&\quad+\beta EW_1\left(\theta',b',\phi'_{\varepsilon};\Phi'_{\varepsilon}\right)b'_{\varepsilon}+\beta\mathbb{E}W_2\left(\theta',b'(\theta,-\phi_{\varepsilon},\phi_{\varepsilon}),\phi'_{\varepsilon};\Phi'_{\varepsilon}\right)\end{aligned}$$

This implies that

$$\begin{aligned}\left.\frac{\partial}{\partial\varepsilon}W(\theta,-\phi_{\varepsilon},\phi_{\varepsilon};\Phi_{\varepsilon})\right|_{\varepsilon=0}&=u'\left(\theta-R\phi-b'(\theta,-\phi,\phi)\right)\left[-R-b'_{\varepsilon}(\theta,-\phi,\phi)\right]\\&\quad+\beta\mathbb{E}W_1\left(\theta',b'(\theta,-\phi,\phi),\phi';\Phi'\right)b'_{\varepsilon}(\theta,-\phi,\phi)\\&\quad+\beta\mathbb{E}W_2\left(b'(\theta,-\phi,\phi),\phi';\Phi'\right)\end{aligned}$$

Given the continuity of the policy and price functions

$$\begin{aligned}\lim_{\substack{\phi\to0\\\Phi\to\mathbf{0}}}\left.\frac{\partial}{\partial\varepsilon}W(\theta,-\phi_{\varepsilon},\phi_{\varepsilon};\Phi_{\varepsilon})\right|&=u'(\theta)\left[-R-b'_{\varepsilon}\right]+\beta\mathbb{E}W_1\left(\theta',0,0;0\right)b'_{\varepsilon}+\beta\mathbb{E}W_2\left(\theta',0,0;0\right)\\&=-Ru'(\theta)-u'(\theta)b'_{\varepsilon}+\beta\mathbb{E}W_1\left(\theta',0,0;0\right)b'_{\varepsilon}+\beta\mathbb{E}W_2\left(\theta',0,0;0\right)\end{aligned}$$

From the first order conditions of the above problem where $\mu(\theta,b,\phi)$ is the multiplier on the debt constraint, we have

$$\beta\mathbb{E}W_1\left(\theta',b'(\theta,b,\phi),\phi';\Phi\right)=u'(c(\theta,b,\phi))-\mu(\theta,b,\phi)$$

we see that

$$\lim_{\substack{\phi\to0\\\Phi\to\mathbf{0}}}\left.\frac{\partial}{\partial\varepsilon}W(\theta,-\phi_{\varepsilon},\phi_{\varepsilon};\Phi_{\varepsilon})\right|_{\varepsilon=0}=-Ru'(\theta)-\mu(\theta,0,0)b'_{\varepsilon}+\beta\mathbb{E}W_2\left(\theta',b'(\theta,0,0),0;0\right)$$

From the complementary slackness condition we have that

$$\begin{aligned}
&\mu\left(z,0,0\right)\left[b'\left(\theta,-\phi,\phi\right)+\phi\right]=0\\
&\Rightarrow \mu_{\varepsilon}\left(\theta,0,0\right)\left[b'\left(\theta,-\phi,\phi\right)+\phi\right]+\mu\left(\theta,0,0\right)\left[b'_{\varepsilon}\left(\theta,-\phi,\phi\right)+\phi_{\varepsilon}\right]=0\\
&\Rightarrow \mu_{\varepsilon}\left(\theta,0,0\right)\left[b'\left(\theta,-\phi,\phi\right)+\phi\right]+\mu\left(\theta,0,0\right)\left[b'_{\varepsilon}\left(\theta,-\phi,\phi\right)+1\right]=0
\end{aligned}$$

As $\phi,\Phi\to 0$ we have

$$\begin{aligned}
&\mu\left(\theta,0,0\right)\left[b'_{\varepsilon}\left(\theta,-\phi,\phi\right)+1\right]=0\\
&\Rightarrow \mu\left(\theta,0,0\right)b'_{\varepsilon}\left(\theta,0,0\right)=-\mu\left(\theta,0,0\right)
\end{aligned}$$

Therefore

$$\lim_{\substack{\phi\to 0\\ \mathbf{\Phi}\to\mathbf{0}}}\frac{\partial}{\partial\varepsilon}W\left(\theta,-\phi_{\varepsilon},\phi_{\varepsilon};\Phi_{\varepsilon}\right)\Big|_{\varepsilon=0}=-Ru'\left(\theta\right)+\mu\left(\theta,0,0\right)+\beta\mathbb{E}W_2\left(\theta',b'\left(\theta,0,0\right),0;0\right)$$

From the first order conditions we have

$$\begin{aligned}
\mu\left(\theta,0,0\right)&=u'\left(\theta\right)-\beta\mathbb{E}W_1\left(\theta',0,0;0\right)\\
&=u'\left(\theta\right)-\beta R\sum_{\theta'\in\Theta}\pi\left(\theta'\right)u'\left(\theta'\right)
\end{aligned}$$

Define $\eta=\sum_{\theta'\in\Theta}\pi\left(\theta'\right)u'\left(\theta'\right)$. Therefore

$$\mu\left(\theta,0,0\right)=u'\left(\theta\right)-\beta R\eta$$

and

$$\begin{aligned}
\mathbb{E}W_2\left(\theta',0,0;0\right)&=\sum_{\theta'\in\Theta}\pi\left(\theta'\right)\mu\left(\theta',0,0\right)\\
&=\sum_{\theta'\in\Theta}\pi\left(\theta'\right)\left[u'\left(\theta'\right)-\beta R\eta\right]\\
&=\eta-\beta R\eta
\end{aligned}$$

As a result

$$\begin{aligned}
\lim_{\substack{\phi\to 0\\ \mathbf{\Phi}\to\mathbf{0}}}\frac{\partial}{\partial\varepsilon}W\left(\theta,-\phi_{\varepsilon},\phi_{\varepsilon};\Phi_{\varepsilon}\right)\Big|_{\varepsilon=0}&=-Ru'\left(\theta\right)+u'\left(\theta\right)-\beta R\eta+\beta\left[\eta-\beta R\eta\right]\\
&=-R\left[u'\left(\theta\right)+\beta\eta+\beta^2\eta\right]+u'\left(\theta\right)+\beta\eta \qquad \text{(A.4)}
\end{aligned}$$

Notice that if

$$R < \frac{u'(\theta) + \beta\eta}{[u'(\theta) + \beta\eta + \beta^2\eta]}$$

then (A.4)$> 0$ since $\eta > 0$.

In any $\Phi-$RCE, when $\phi = 0$ the interest rate must satisfy

$$\begin{aligned} u'\left(\bar{\theta}\right) &\geq \beta R\eta \\ &\Rightarrow R \leq \frac{u'\left(\bar{\theta}\right)}{\beta\eta} \end{aligned}$$

Therefore if

$$\frac{u'\left(\bar{\theta}\right)}{\beta\eta} < \frac{u'(\theta) + \beta\eta}{[u'(\theta) + \beta\eta + \beta^2\eta]}$$

then we know that

$$\lim_{\substack{\phi \to 0 \\ \Phi \to \mathbf{0}}} \frac{\partial}{\partial\varepsilon} W(\theta, -\phi_\varepsilon, \phi_\varepsilon; \Phi_\varepsilon)\Big|_{\varepsilon=0} > 0$$

But since $\frac{u'(\theta)+\beta\eta}{[u'(\theta)+\beta\eta+\beta^2\eta]} \geq \kappa$ by assumption the property is true.

Next notice because of Inada conditions that

$$\lim_{\substack{\phi \to \infty \\ \Phi \to \infty}} W(\theta, -\phi, \phi; \Phi) \to -\infty$$

since eventually, the debt constraints cease to bind for all agents. And so continuity implies that there exists $\phi^\theta$ such that

$$W\left(\theta, -\phi^\theta, \phi^\theta; \Phi^\theta\right) = V^d(\theta)$$

with $\phi > 0$ and $\Phi\left(\phi^\theta\right) = \phi^\theta$. If there are many such we pick the one closest to 0. In this equilibrium all agents are subject to debt constraints $\phi^\theta$ in each period. However it might be that for some $\tilde{\theta}$

$$W\left(\tilde{\theta}, -\phi^\theta, \phi^\theta; \Phi^\theta\right) < V^d\left(\tilde{\theta}\right)$$

and as a result this would cease to be a NTT-RCE. Using a similar procedure, we can construct debt constraints $\phi^{\tilde{\theta}}$ for any $\tilde{\theta}$ such that the above constraint holds with equality. Consider $\phi = \min_\theta \phi^\theta$. By continuity it must be that

$$\frac{\partial}{\partial\varepsilon} W(\theta, -\phi - \varepsilon, \phi + \varepsilon)\Big|_{\varepsilon=0} \leq 0$$

Therefore for all $\theta \in \Theta$,

$$W(\theta, -\phi, \phi; \Phi) \geq W\left(\theta, -\phi^{\theta}, \phi^{\theta}\right) = V^{d}(\theta)$$

which proves the claim. ■

### A.1.2 Proofs from Section 4

**Proof of Proposition 4.** As we saw in Theorem 1, the equilibrium contract when $\delta_t\left(\theta^t\right) = 0$, for all $t, \theta^t$ looks like a simple borrowing contract with exogenous debt limits $\phi$. If $u(\underline{\theta}) + \frac{\beta}{1-\beta^2}\mathbb{E}u(\theta')$ large enough, some household will be borrowing constrained in equilibrium. We now construct a feasible contract with banishment on path that makes both the intermediary and some household strictly better off. Consider the continuation contract at date $t-1$. Suppose there exists some type $\theta^{t-1}$ who is Euler-constrained in $t-1$.In particular,

$$u'\left(c_{t-1}\left(\theta^{t-1}\right)\right) > \beta R_t \mathbb{E}_{t-1} u'\left(c_t\left(\theta^{t-1}, \theta\right)\right)$$

We saw in Lemma 14 that in period $t$,

$$V_t\left(\theta^{t-1}, \underline{\theta}\right) = V^{d}(\underline{\theta}; \lambda)$$

Given that we can transform any $\hat{T}$ period contract into an equivalent 2 period contract $\left\{b_t^t, b_{t+1}^t\right\}$, consider the following deviating contract

$$\begin{aligned}
{}_{t-1}\tilde{b}_{t-1}\left(\theta^{t-1}\right) &= {}_{t-1}b_{t-1}\left(\theta^{t-1}\right) + \varepsilon \\
{}_{t-1}\tilde{b}_t\left(\theta^{t-1}, \theta\right) &= -\frac{\left[{}_{t-1}b_t\left(\theta^{t-1}, \theta\right) + R_t\varepsilon\right]}{1-\pi(\underline{\theta})} \text{ for all } \theta \neq \underline{\theta}
\end{aligned}$$

and the intermediary chooses to banish type $\left(\theta^{t-1}, \underline{\theta}\right)$ .The following is true of the deviating contract:

1. The value to type $\left(\theta^{t-1}, \underline{\theta}\right)$ is identical under both contracts since its participation constraint was binding

2. The present discounted value of transfers for types $\left(\theta^{t-1}, \theta\right)$ is identical for all $\theta \in \Theta$,

   staying in the contract (and not being banished).

3. The equilibrium present discounted value of transfers for type $\theta^{t-1}$ is identical to the one under the original contract since

$$\begin{aligned}
& {}_{t-1}\tilde{b}_{t-1}\left(\theta^{t-1}\right)+\frac{1}{R_t}\left[\sum_{\theta>\underline{\theta}}\pi\left(\theta\right)\left[{}_{t-1}\tilde{b}_t\left(\theta^{t-1},\theta\right)\right]+\pi\left(\underline{\theta}\right)\cdot 0\right] \\
&= {}_{t-1}b_{t-1}\left(\theta^{t-1}\right)+\frac{1}{R_t}\sum_{\theta\in\Theta}\pi\left(\theta\right){}_{t-1}b_t\left(\theta^{t-1},\theta\right)
\end{aligned}$$

Fact 2 implies incentive compatibility in period $t$,continues to hold. The three facts together imply that if the contract is perturbed is a similar fashion for all such constrained types, incentive compatibility also holds in $t-1$.

The change in welfare for this type under this proposed perturbation, $\Delta\left(\theta^{t-1}\right)$ is

$$\begin{aligned}
& u\left(\theta+{}_{t-2}b_{t-1}+{}_{t-1}\tilde{b}_{t-1}\right)+\beta\sum_{\theta'>\underline{\theta}}u\left(\theta+{}_{t-1}\tilde{b}_t+{}_tb_t\right) \\
& -u\left(\theta+{}_{t-2}b_{t-1}+{}_{t-1}b_{t-1}\right)-\beta\sum_{\theta'\in\Theta}u\left(\theta+{}_{t-1}b_t+{}_tb_t\right) \\
& =u\left(\theta+{}_{t-2}b_{t-1}+{}_{t-1}\tilde{b}_{t-1}\right)-u\left(\theta+{}_{t-2}b_{t-1}+{}_{t-1}b_{t-1}\right) \\
& \quad+\beta\sum_{\theta'>\underline{\theta}}u\left(\theta+{}_{t-1}\tilde{b}_t+{}_tb_t\right)-\beta\sum_{\theta'\in\Theta}u\left(\theta+{}_{t-1}b_t+{}_tb_t\right) \\
& \geq u\left(\theta+{}_{t-2}b_{t-1}+{}_{t-1}\tilde{b}_{t-1}\right)-u\left(\theta+{}_{t-2}b_{t-1}+{}_{t-1}b_{t-1}\right) \\
& \quad+\beta\sum_{\theta'\in\Theta}\pi\left(\theta'\right)u\left(\theta+{}_{t-1}\tilde{b}_t+{}_tb_t\right)-\beta\sum_{\theta'\in\Theta}u\left(\theta+{}_{t-1}b_t+{}_tb_t\right)
\end{aligned}$$

where the inequality in the second line follows since $u\left(\theta'+{}_{t-1}\tilde{b}_t+{}_tb_t\right)\geq u\left(\underline{\theta}+{}_{t-1}\tilde{b}_t+{}_tb_t\right)$. Consider the last two terms,

$$\begin{aligned}
& \beta\sum_{\theta'\in\Theta}\pi\left(\theta'\right)u\left(\theta+{}_{t-1}\tilde{b}_t+{}_tb_t\right)-\beta\sum_{\theta'\in\Theta}u\left(\theta+{}_{t-1}b_t+{}_tb_t\right) \\
& =\beta\sum_{\theta'\in\Theta}\pi\left(\theta'\right)u\left(\theta'+\tilde{b}_t^{t-1}+b_t^t\right)-\beta\sum_{\theta'\in\Theta}\pi\left(\theta'\right)u\left(\theta'+\frac{{}_{t-1}b_t}{1-\pi\left(\underline{\theta}\right)}+{}_tb_t\right) \\
& +\beta\sum_{\theta'\in\Theta}\pi\left(\theta'\right)u\left(\theta'+\frac{{}_{t-1}b_t}{1-\pi\left(\underline{\theta}\right)}+{}_tb_t\right)-\beta\sum_{\theta'\in\Theta}\pi\left(\theta'\right)u\left(\theta'+{}_{t-1}b_t+{}_tb_t\right)
\end{aligned}$$

Therefore for $\varepsilon$ small, the sign of the change in welfare is sign of the following expression

$$u'\left(\theta + b_{t-1}^{t-2} + b_{t-1}^{t-1}\right)\varepsilon - \frac{1}{1-\pi\left(\underline{\theta}\right)}\beta\sum_{\theta'\in\Theta}\pi\left(\theta'\right)u'\left(\theta' + \frac{{}_{t-1}b_t\left(\theta^{t-1}\right)}{1-\pi\left(\underline{\theta}\right)} + b_t^t\right)\varepsilon$$
$$+\beta\sum_{\theta'\in\Theta}\pi\left(\theta'\right)u\left(\theta' + \frac{{}_{t-1}b_t\left(\theta^{t-1}\right)}{1-\pi\left(\underline{\theta}\right)} + {}_tb_t\right) - \beta\sum_{\theta'\in\Theta}\pi\left(\theta'\right)u\left(\theta' + {}_{t-1}b_t\left(\theta^{t-1}\right) + {}_tb_t\right)$$

Since this is strictly positive at $\pi\left(\underline{\theta}\right) = 0$, and so for $\pi\left(\underline{\theta}\right) > 0$ small, the change in welfare is strictly positive for this household. Clearly, the intermediary can construct a contract that makes both it and the household strictly better off. ■

**Proof of Proposition 6.** Suppose that in an equilibrium, for some $t$ and history $\theta^t$, $u'\left(c_t\left(h^t\right)\right)q_t < \beta\mathbb{E}_{D_t^c(h^{t-1},h)}u'\left(c_{t+1}\left(h^{t+1}\right)\right)$. Recall that $b_t\left(h^t\right) = b_t^{old}\left(h^t\right) + {}_tb_t\left(h^t\right)$ and so in equilibrium $c_t\left(h^t\right) = \theta_t + b_t\left(h^t\right)$. Since we are considering symmetric equilibria in which ex-ante identical intermediaries offer the same contract in equilibrium, we consider the incentives for a deviating intermediary to offer a deviating contract in period $t$ and make strictly positive profits. Consider an intermediary offering a $\varepsilon\delta-$savings contract $S_t^{\varepsilon,\delta}$ (defined in section 2.2) for some $\varepsilon > 0$ and $\delta < 1$. Notice that the intermediary makes positive profits whenever this contract is accepted. Since $u'\left(c_t\left(h^t\right)\right)q_t < \beta\mathbb{E}_tu'\left(c_{t+1}\left(h^{t+1}\right)\right)$, there exists $\varepsilon > 0, \delta < 1$ such that type $h^t$ will strictly prefer to sign an $\varepsilon\delta$ savings contract if offered. These contracts are incentive compatible and satisfy voluntary participation constraints. As a result an intermediary offering such a contract will make positive profits which is a contradiction. ■

For any history, define $A_t\left(h^{t-1}, h_t\right)$ to be the equilibrium expected present discounted value of future transfers for type $\left(h^{t-1}, h_t\right)$

$$A_t\left(h^t\right) \equiv \left(1-\delta_t\left(h^t\right)\right)\left[b_t\left(h^t\right) + q_t\sum_{h_{t+1}\in H_{t+1}}\zeta\left(h^t, h_{t+1}\right)A_{t+1}\left(h^t, h_{t+1}\right)\right]$$

To compute properties of the equilibria of the intermediary game, we will consider the limit of a sequence of truncated economies. In particular, I assume that there exists a finite date $T$, such that from $0 \leq t \leq T$, intermediaries offer contracts and for all $t > T$, those agents who have not defaulted in the past trade a risk free bond subject to exogenous debt constraints $\{\phi_t^e\}_{t>T}$. The claim that we can take such limits is formalized later in the appendix.

**Proof of Lemma 1.** Suppose not. Clearly, ${}_t\mathcal{P}_t\left(h^t\right) > 0$ for all $h^t$ is not possible since the intermediary would making negative profits. On the other hand if ${}_t\mathcal{P}_t\left(h^t\right) \leq 0$ for all $\theta^t$ with strict inequality for some, then a deviating intermediary can offer a contract which transfers a little more to some types and still continue to make positive profits. As a result these types will strictly prefer to sign with the deviating intermediary. Finally, suppose that there exists $h^t$ and $\hat{h}^t$ such that ${}_t\mathcal{P}_t\left(h^t\right) > 0$ and ${}_t\mathcal{P}_t\left(\hat{h}^t\right) < 0$. Then at the beginning of period $t$, consider a deviating intermediary offering the following contract,

$$
\begin{aligned}
{}_t\tilde{\mathcal{P}}_t\left(\hat{h}^t\right) &= {}_t\mathcal{P}_t\left(\hat{h}^t\right) + \varepsilon \\
{}_t\tilde{b}_{t+s}\left(h^t\right) &= 0 \text{ for all } s \geq 0, \text{ for } h^t \neq \hat{h}^t
\end{aligned}
$$

where $\varepsilon > 0$ and small. Notice that types $h^t$ strictly prefer the original contract while types $\hat{h}^t$ strictly prefer ${}_t\tilde{\mathcal{P}}_t$ to ${}_t\mathcal{P}_t$. As a result, these households will strictly prefer to sign with the deviating intermediary who makes a positive profit. ■

**Proof of Proposition 7.** Because of the assumption that after period $T$, households can only trade a risk free bond subject to exogenous debt constraints, it is easy to see that the statement holds in period $T$, since all that matters for the household's choice is the sum $\theta_T + b_T^{T-1}\left(h^T\right)$. In period $T-1$ suppose $h^{T-1}$ and $\hat{h}^{T-1}$ such that $\theta_{T-1} + b_{T-1}^{T-2}\left(h^{T-1}\right) = \hat{\theta}_{T-1} + b_{T-1}^{T-2}\left(\hat{h}^{T-1}\right)$ and

$$
V_{T-1}\left(h^{T-1}\right) > V_{T-1}\left(\hat{h}^{T-1}\right)
$$

Suppose first that for both $h^{T-1}, \hat{h}^{T-1}$

$$
u'\left(\theta_{T-1} + {}_{T-2}b_{T-1} + {}_{T-1}b_{T-1}\right) = \beta R_T \mathbb{E}_{D_T^c(h^{T-1})} u'\left(\theta_T + {}_{T-1}b_T + \psi_{T+1}\left(\theta_T + {}_{T-1}b_T\right)\right) \tag{A.5}
$$

$$
u'\left(\hat{\theta}_{T-1} + {}_{T-2}\hat{b}_{T-1} + {}_{T-1}\hat{b}_{T-1}\right) = \beta R_T \mathbb{E}_{D_T^c(h^{T-1})} u'\left(\theta_T + {}_{T-1}\hat{b}_T + \psi_{T+1}\left(\theta_T + {}_{T-1}\hat{b}_T\right)\right) \tag{A.6}
$$

where as before, $\psi_{T+1}\left(\theta_T + {}_{T-1}b_T\right)$ is the savings choice for the household (given that it is subject to debt constraint $\phi_{T+1}^e$). Since ${}_{T-1}b_{T-1} + \mathbb{E}_{D_T^c(h^{T-1})} \frac{{}_{T-1}b_T}{q_{T-1}} = {}_{T-1}\hat{b}_{T-1} + \mathbb{E}_{D_T^c(h^{T-1})} \frac{{}_{T-1}\hat{b}_T}{q_{T-1}} = 0$ and the savings choice $\psi_{T+1}$ depends only on the sum $\theta_T + {}_{T-1}b_T$ , it must be that ${}_{T-1}b_{T-1} = {}_{T-1}\hat{b}_{T-1}$ and $V_{T-1}\left(h^{T-1}\right) = V_{T-1}\left(\hat{h}^{T-1}\right)$ and so we have a contradiction. Suppose on the other hand that (A.5) held with an equality and (A.6) with strictly inequality. Since ${}_{T-1}b_{T-1} + \frac{{}_{T-1}b_T}{q_{T-1}} = {}_{T-1}\hat{b}_{T-1} + \frac{{}_{T-1}\hat{b}_T}{q_{T-1}} = 0$, it must be that ${}_{T-1}b_T > {}_{T-1}\hat{b}_T$ and $D_T^c\left(h^{T-1}\right) \subseteq D_T^c\left(\hat{h}^{T-1}\right)$. Since the agent is Euler-constrained, assume that ${}_{T-1}b_T > 0$. Consider modifying the original contract for type $\hat{h}^{T-1}$ so that

$$
\begin{aligned}
{}_{T-1}\tilde{b}_{T-1} &= {}_{T-1}\hat{b}_{T-1} + \varepsilon \\
{}_{T-1}\tilde{b}_T &= -\frac{R_T \; {}_{T-1}\tilde{b}_{T-1}}{\sum_{h_T \in \tilde{D}_T^c} \wp\left(\hat{h}^{T-1}, h_T\right)}
\end{aligned}
$$

where $\varepsilon$ chosen so that ${}_{T-1}\tilde{b}_T \geq {}_{T-1}b_T$. Notice that such a perturbation might incentivize some types to default in period $T$, and so intermediary can choose to banish these types and consequently $D_T^c\left(\hat{h}^{T-1}\right) \supseteq \tilde{D}_T^c$. However, since ${}_{T-1}\tilde{b}_T \geq {}_{T-1}b_T$ the transfer ${}_{T-1}b_T$ is associated with more banishment and hence $D_T^c\left(h^{T-1}\right) \subseteq \tilde{D}_T^c$.

For $\varepsilon$ small enough, this perturbation makes type $\hat{h}^{T-1}$ strictly better off in period $T-1$ and the the intermediary is equally well off. Also by construction the present discounted value of transfers is unchanged for the agent in $T-1$. To check for incentive compatibility we need only consider the households who are constrained and who might lie to get the increased transfer in period $T-1$. However, if the original transfer sequence was incentive compatible, the perturbed one is as well. Moreover, the default incentives in period $T$, are exactly the same as the types considered above and so the intermediary is as well off while the agent is strictly better off. Therefore a contract can be constructed that makes both the intermediary and the household strictly better off. Hence, it must be that ${}_{T-1}b_{T-1} = {}_{T-1}\hat{b}_{T-1}$. Note that a similar argument holds if both (A.5) and (A.6) held with strict inequality. Given that the property holds for $\hat{T}-1$, assume that this property holds for some $t+1 < \hat{T}-1$. Our goal is to show that the property holds in $t$. Suppose for contradiction we have some $h, \hat{h}^t$ such that $\theta_t + {}_{t-1}b_t\left(h^t\right) = \hat{\theta}_t + {}_{t-1}b_t\left(\hat{h}^t\right)$ and

$$
V_t\left(h^t\right) > V_t\left(\hat{h}^t\right)
$$

As before, first consider the case where the Euler equations hold with equality for both types. Suppose ${}_tb_t\left(h^t\right) < {}_tb_t\left(\hat{h}^t\right)$. Then it is easy to see that an intermediary can offer an $\varepsilon\delta$ savings contract to type $\hat{h}^t$ which will be accepted by this household making the intermediary strictly better off. The argument is identical to the one for period $T-1$.

Next suppose that

$$u'\left(\theta_t + {}_{t-1}b_t + {}_tb_t\right) = \beta R_{t+1}\mathbb{E}_{D_t^c(h^{t-1})}u'\left(\theta_{t+1} + {}_tb_{t+1} + {}_{t+1}b_{t+1}\right)$$
$$u'\left(\hat{\theta}_t + {}_{t-1}\hat{b}_t + {}_t\hat{b}_t\right) > \beta R_{t+1}\mathbb{E}_{D_t^c(h^{t-1})}u'\left(\theta_{t+1T} + {}_t\hat{b}_{t+1} + {}_{t+1}\hat{b}_{t+1}\right)$$

As in the period $T-1$ case, the transfer scheme to type $\hat{h}^t$ can be modified so that, it is made strictly better off. We can use a similar argument to show that the intermediary can construct an incentive compatible contract that makes it and the agent strictly better off. Therefore by induction the claim must hold in period $t$ and by induction for all previous periods as well. ■

**Proof of Proposition 8.** Part 1 is a direct consequence of incentive compatibility. If household types are being banished, they will always announce the type consistent with the largest re-entry probability.

Part 2. Consider a period $t$, and $h^t, \hat{h}^t$ such that ${}_{t-1}b_{t-1}\left(h^{t-1}\right) = {}_{t-1}b_{t-1}\left(\hat{h}^{t-1}\right)$. First suppose that $D_t\left(h^{t-1}\right) \subset D_t\left(\hat{h}^{t-1}\right)$. Since ${}_{t-1}b_{t-1}\left(h^{t-1}\right) + q_t\mathbb{E}_{D_t^c(h^{t-1})}\,{}_{t-1}b_t\left(h^{t-1}\right) = {}_{t-1}b_{t-1}\left(\hat{h}^{t-1}\right) + q_t\mathbb{E}_{D_t^c(\hat{h}^{t-1})}\,{}_{t-1}b_t\left(\hat{h}^{t-1}\right) = 0$, it must be that ${}_{t-1}b_t\left(\hat{h}^{t-1}\right) < {}_{t-1}b_t\left(h^{t-1}\right)$. In this case the intermediary can set ${}_{t-1}b_t\left(\hat{h}^{t-1}\right) = {}_{t-1}b_t\left(h^{t-1}\right)$ and $D_t\left(h^{t-1}\right) = D_t\left(\hat{h}^{t-1}\right)$, which leaves the intermediary equally well off but type $\hat{h}^{t-1}$ strictly better off. As a result a deviating contract exists that makes both strictly better off. An identical argument applies if $D_t\left(\hat{h}^{t-1}\right) \subset D_t\left(h^{t-1}\right)$. Given that $D_t\left(\hat{h}^{t-1}\right) = D_t\left(h^{t-1}\right)$, a similar argument implies that $\mu_t\left(h^{t-1}\right) = \hat{\mu}_t\left(h^{t-1}\right)$. If not, the probability of re-entry can be increased for the type making it strictly better off, while still preserving incentives.

Part 3. Suppose that ${}_{t-1}b_{t-1}\left(h^{t-1}\right) \geq {}_{t-1}b_{t-1}\left(\hat{h}^{t-1}\right)$ for two histories $h^{t-1}$ and $\hat{h}^{t-1}$. We prove this by contradiction. There are three cases to consider. First suppose that $D_t\left(h^{t-1}\right) \subset D_t\left(\hat{h}^{t-1}\right)$ and $\mu_t\left(h^{t-1}\right) > \mu_t\left(\hat{h}^{t-1}\right)$. Since ${}_{t-1}b_{t-1}\left(h^{t-1}\right) + \mathbb{E}_{D_t^c(h^{t-1})}\,{}_{t-1}b_t\left(h^{t-1}\right) = {}_{t-1}b_{t-1}\left(\hat{h}^{t-1}\right) + \mathbb{E}_{D_t^c(\hat{h}^{t-1})}\,{}_{t-1}b_t\left(\hat{h}^{t-1}\right) = 0$, $\mathbb{E}_{D_t^c(h^{t-1})}\,{}_{t-1}b_t\left(h^{t-1}\right) \leq \mathbb{E}_{D_t^c(\hat{h}^{t-1})}\,{}_{t-1}b_t\left(\hat{h}^{t-1}\right)$ implies that ${}_{t-1}b_t\left(\hat{h}^{t-1}\right) < {}_{t-1}b_t\left(h^{t-1}\right)$. In this case it is easy to see that the intermediary can increase ${}_{t-1}b_t\left(\hat{h}^{t-1}\right)$, reduce the banishment set and make the household strictly better off while preserving incentives and still making zero profits.

Next suppose that $D_t\left(h^{t-1}\right) \subset D_t\left(\hat{h}^{t-1}\right)$ and $\mu_t\left(h^{t-1}\right) \leq \mu_t\left(\hat{h}^{t-1}\right)$. In this a similar argument to the one above works. Finally suppose that $D_t\left(h^{t-1}\right) \supseteq D_t\left(\hat{h}^{t-1}\right)$ but

$\mu_t\left(h^{t-1}\right) > \mu_t\left(\hat{h}^{t-1}\right)$. Since $D_t\left(h^{t-1}\right) \supseteq D_t\left(\hat{h}^{t-1}\right)$ it must be that ${}_{t-1}b_{t-1}\left(h^{t-1}\right) \leq {}_{t-1}b_{t-1}\left(\hat{h}^{t-1}\right)$. Therefore, the intermediary can increase $\mu_t\left(\hat{h}^{t-1}\right)$ and make the household strictly better off. From Proposition 7 we know that since there is no default after history $h^{t-1}$, this perturbation still preserves default incentives after history $\hat{h}^{t-1}$. ■

**Proof of Proposition 5.** Since we have shown that the types being banished depend only on the current endowment report and transfer, the banishment sets $D_t\left(h^{t-1}\right) = D_t\left(b_t^{t-1}\left(h^{t-1}\right)\right)$ in any equilibrium contract. Since transfers are bounded, there exists $\bar{\phi}, \phi^D$ such that

1. ${}_{t-1}b_t\left(h^t\right) < \bar{\phi}$
2. If ${}_{t-1}b_t\left(h^t\right) < \phi^D$, $D_t\left({}_{t-1}b_t\left(h^t\right)\right) = \varnothing$ and $\hat{R}_t\left({}_{t-1}b_t\left(h^t\right)\right) = R_t\, {}_{t-1}b_t\left(h^{t-1}\right)$
3. If ${}_{t-1}b_t\left(h^t\right) \geq \phi^D$, $D_t\left({}_{t-1}b_t\left(h^t\right)\right) \neq \varnothing$ and $\hat{R}_t\left({}_{t-1}b_t\left(h^t\right)\right) = \frac{R_t\, {}_{t-1}b_t\left(h^{t-1}\right)}{\sum_{h \notin D_t\left({}_{t-1}b_t\left(h^{t-1}\right)\right)} \zeta\left(h^{t-1}, h\right)}$

Given this, construct the equilibrium objects as follows: For any agent with history $h^t = \left(\theta^t, \gamma^t, B^{t-1}\right)$, let $s_t = {}_{t-1}b_t\left(h^{t-1}\right)$ and define

$$
\begin{aligned}
V_t^R\left(\theta_t, s_t\right) &\equiv V_t\left(h^t\right) \\
d_t\left(\theta_t, s_t\right) &\equiv 1 - \delta_t\left(h^t\right) \\
s_{t+1}\left(\theta_t, s_t\right) &\equiv \mathcal{Q}_t^{-1}\left(-\, {}_t b_t\left(h^t\right)\right) \\
\lambda_{t+s}\left(s_t\right) &\equiv \mu_{t+s}\left(h^t\right)
\end{aligned}
$$

where (since $s_t = {}_{t-1}b_t\left(h^{t-1}\right)$ ),

$$
\begin{aligned}
&\mathcal{Q}_t(s) = \frac{s}{R_t} \text{ if } {}_{t-1}b_t\left(h^{t-1}\right) > -\phi^D \\
&\mathcal{Q}_t(s) = -\sum_{h \notin D_t\left({}_{t-1}b_t\left(h^{t-1}\right)\right)} \zeta\left(h^{t-1}, h\right)\ {}_{t-1}b_t\left(h^{t-1}\right) \text{ if } -\bar{\phi} < {}_{t-1}b_t\left(h^{t-1}\right) \leq -\phi^D \\
&\mathcal{Q}_t(s) = 0 \text{ if } {}_{t-1}b_t\left(h^{t-1}\right) < \bar{\phi}
\end{aligned}
$$

and $c\left(\theta_t, b_t\right)$ is determined residually from the budget constraint. Given any other type $\tilde{h}^t$ with $\theta\left(\tilde{h}^t\right) = \theta_t$ and ${}_{t-1}b_t\left(\tilde{h}^t\right) = {}_{t-1}b_t(h)$ we know from Proposition 7 that $V_t\left(h^t\right) = V_t\left(\tilde{h}^t\right)$ and so $V_t^R\left(\theta_t, s_t\right) = V_t^R\left(\tilde{\theta}_t, s_t\right)$ and $d_t\left(\theta_t, s_t\right) = d_t\left(\tilde{\theta}_t, s_t\right)$ since $\mu_{t+s}\left(h^t\right) = \mu_{t+s}\left(\tilde{h}^t\right)$.

Since the value of default is common to both problems it must be that

$$d_t(\theta_t, s_t) = \arg\max_d V_t^0(\theta_{t+1}, s_{t+1}; \mathcal{Q}_{t+1}, \lambda) = \arg\max_d [1-d] V_t^R(\theta_t, s_t; \mathcal{Q}_t) + dV_t^D(\theta_t; \lambda(s))$$

In particular, if $d_t(\theta_t, s_t) = 0$ and $V_t^R(\theta_t, s_t; \mathcal{Q}_t) > V_t^D(\theta_t; \lambda(s))$, the household will strictly prefer to lie and pretend to be a type that is not banished in the intermediary game.

Next, given the constructed interest rate schedule, it must be that the constructed policy functions solve

$$\begin{aligned} V_t^R(\theta_t, s_t; \mathcal{Q}_t) = \max u(c_t) + \beta \mathbb{E}_t V_{t+1}^0(\theta_{t+1}, s_{t+1}; \mathcal{Q}_{t+1}, \lambda) \\ s.t. \\ c_t + \mathcal{Q}_t(s_{t+1}) \leq \theta_t + s_t \end{aligned}$$

since we showed earlier that $b_t^{t-1}\left(h^t\right) = -\hat{R}_t\left({}_{t-1}b_{t-1}\left(h^{t-1}\right)\right)$. If there exists different choice of $b_{t+1}$ that gives the households larger utility, a deviating intermediary can offer such a contract in the original environment and make a strictly positive profit. Finally, by construction, for all $t$

$$\mathcal{Q}_t(s) = \frac{\left[1 - \Pr\left[V_{t+1}^D > V_{t+1}^R\right]\right] s}{R_{t+1}}$$

which proves the result. ■

**Existence:** Define a bounded interval for assets/debt $[\underline{s}, \bar{s}]$. Given functions $V(\theta, s)$ define operator $\mathcal{B}(V)$ that takes in value functions and outputs of a set of default thresholds $D$. For each $\theta$, find $s^*(\theta)$ s.t. $V(\theta, s^*(\theta)) = V^d(\theta)$. If $\sup_b V(\theta, s) < V^d(\theta)$, set $s^*(\theta) = -\infty$ and if $\inf_b V(\theta, b) > V^d(\theta)$, set $s^*(\theta) = \infty$

Given $D = \{s^*(\theta)\}$, define functional $\mathcal{V}(D)$ that takes in a set of thresholds and outputs value functions $V$

$$V(\theta, s; \{s^*(\theta)\})$$

Given a set of thresholds $\{s^*(\theta)\}$ define $V(\theta, s; s^*)$ as follows

$$\begin{aligned} V(\theta, s; s^*) = \left(\max_{c,b'} u(c) + \beta\left[\sum_{\theta'} \pi(\theta') V(\theta', s') \mathbf{1}_{b' \leq b^*(\theta')} + \sum_{\theta'} \pi(\theta') V^d(\theta') \mathbf{1}_{s' > s^*(\theta')}\right]\right) \mathbf{1}_{s \geq s^*(\theta)} \\ + V^d(\theta) \mathbf{1}_{s < s^*(\theta)} \end{aligned}$$

subject to

$$c+\left(\frac{\sum_{\theta'}\pi\left(\theta'\right)\mathbf{1}_{s'\leq s^{*}(\theta')}+\sum_{\theta'}\pi\left(\theta'\right)\chi\mathbf{1}_{s'>s^{*}(\theta')}}{R_{t}}\right)s'=\theta+s$$

Then we can define an operator $T=\mathcal{B}\circ\mathcal{V}$ that maps sets of default thresholds into themselves. Using similar arguments to Auclert and Rognlie (2014), one can show that this operator has a fixed point $B^{*}$. Using this, we can construct an equilibrium debt price schedule as follows

$$Q\left(s'\right)=\frac{1}{R_{t}}\sum_{\theta'}\pi\left(\theta'\right)\mathbf{1}_{b'\leq b^{*}(\theta')}s'$$

This says that given a re-entry probability sequence of interest rates $\{R_{t}\}$ an equilibrium of the EG environment always exists. Since these interest rates are equilibrium objects we need to find a sequence that clears markets at each date. Given the policy functions we can define a measure

$$\nu_{t}\left(A\times B\right)=\mu^{L}\left(i\in\tilde{I}_{t}:\left(b'\left(i\right),\theta\left(i\right)\right)\in A\times B,\ A\times B=\mathcal{B}\left(A\right)\times\mathcal{B}\left(\Theta\right)\right)$$

where $\mu^{L}$ is the lebesgue measure and $\tilde{I}_{t}$ is the set of households who haven't defaulted in time $t$. Then by continuity we know for $R_{t}$ large enough

$$\int_{\mathbb{A}\times\Theta}b'\left(\theta,b\right)d\nu_{t}\left(b,\theta\right)>0$$

while for $R_{t}$ small enough

$$\int_{\mathbb{A}\times\Theta}b'\left(\theta,b\right)d\nu_{t}\left(b,\theta\right)<0$$

Therefore for each $t$, there exists $R_{t}$ that clears the market.

### A.1.3 Proofs from Section 5

This section contains proofs from section 5 of the main text.

#### Proofs from Section 5.1

**Proof of Proposition 9.** The first part is as Proposition 11. Next, given a date

$t$ and history $\theta^{t-1}$, if $\theta > \theta'$ we can use an identical argument as in the intermediary game to show that it must be that $A_t\left(\theta^{t-1},\theta\right) \geq A_t\left(\theta^{t-1},\theta'\right)$ where $A_t\left(\theta^{t-1},\theta\right) = b_t\left(\theta^{t-1},\theta\right) + q_t \sum_{\theta_{t+1}} \pi\left(\theta^{t-1},\theta,\theta_{t+1}\right) A_{t+1}\left(\theta^{t-1},\theta,\theta_{t+1}\right)$ is the expected present discounted value of transfers to type $\left(\theta^{t-1},\theta\right)$. In particular, if this did not hold, type $\theta$ will strictly prefer to lie and pretend to be type $\theta'$ and use the hidden markets to save. Suppose that $A_t\left(\theta^{t-1},\theta\right) > A_t\left(\theta^{t-1},\theta'\right)$. There are two cases to consider. First suppose that $q_t > \frac{\beta \mathbb{E}_t u'\left(c_{t+1}\left(\theta^{t-1},\theta,\theta_{t+1}\right)\right)}{u'\left(c_t\left(\theta^{t-1},\theta\right)\right)}$. Then as in the intermediary game we can find a perturbation which involves a small transfer of wealth between type $\left(\theta^{t-1},\theta\right)$ and the types below that increases ex-ante welfare. The second case to consider is one in which $q_t = \frac{\beta \mathbb{E}_t u'\left(c_{t+1}\left(\theta^{t-1},\theta,\theta_{t+1}\right)\right)}{u'\left(c_t\left(\theta^{t-1},\theta\right)\right)}$. We want to consider a wealth transfer from type $\left(\theta^{t-1},\theta\right)$ that leaves this equation unchanged. Choose $(\varepsilon, a\varepsilon)$ where

$$u'\left(c_t\left(\theta^{t-1},\theta\right)-\varepsilon\right) q - \beta \mathbb{E}_t u'\left(c_{t+1}\left(\theta^{t-1},\theta,\theta_{t+1}\right)-a\varepsilon\right) = 0$$

Modify the transfer sequence as follows: $\tilde{b}_t\left(\theta^{t-1},\theta\right) = b_t\left(\theta^{t-1},\theta\right) - \varepsilon$ and $\tilde{b}_t\left(\theta^{t-1},\theta,\theta_{t+1}\right) = b_t\left(\theta^{t-1},\theta,\theta_{t+1}\right) - a\varepsilon$ for all $\theta_{t+1}$. This constitutes a wealth transfer from type $\left(\theta^{t-1},\theta\right)$ which can be redistributed to lower types. For $\varepsilon$ small, the voluntary participation constraints are still satisfied and the pricing equation is unchanged since given the choice of $a$. As a result any solution to the constrained-efficient problem must satisfy $A_t\left(\theta^{t-1},\theta\right) = A_t\left(\theta^{t-1},\theta'\right)$. Moreover, it must be that $A_1\left(\theta_1\right) = 0$ for all $\theta_1$. These two conditions imply that $\sum_{t=1}^{T}\left(\prod_{s=1}^{t} q_s\right) b_t\left(\theta^T\right) = 0$ for all $\theta^T \in \Theta^T$. ■

**Proof of Lemma 3.** We have already established the first part in an earlier proposition. Next, suppose that

$$q_t > \frac{\beta \mathbb{E}_t u'\left(c_{t+1}\left(\theta^{t+1}\right)\right)}{u'\left(c_t\left(\theta^t\right)\right)}$$

for some type $h^t$ and

$$V_{t+1}\left(\tilde{\theta}^{t+1}\right) - V_{t+1}^d\left(\tilde{\theta}^{t+1}\right) > 0 \text{ for all } \tilde{\theta}^{t+1}$$

In this case, zero debt constraints would no longer be be not-too-tight in the hidden market. More generally, intermediaries can find a deviating contract that makes both it and the household strictly better off. ■

**Proof of Theorem 3.** Given the previous result we know that constrained-efficient allocation with no banishment looks like uncontingent borrowing and lending subject to debt constraints. In particular we can decompose the sequence of efficient transfers $\left\{b_t\left(\theta^t\right)\right\}$ into $b_t\left(\theta^t\right)=\tilde{b}_t\left(\theta^{t-1}\right)+\tilde{b}_t\left(\theta^t\right)$. We can construct contracts for $T$ period lived intermediaries as follows:

$$
\begin{aligned}
&{}_1\zeta_1\left(\theta_1\right)=b_1\left(\theta_1\right)\\
&{}_1\zeta_t\left(\theta^t\right)=b_t\left(\theta^t\right),\ t<T\\
&{}_1\zeta_T\left(\theta^T\right)=\tilde{b}_t\left(\theta^{t-1}\right)\\
&{}_T\zeta_T\left(\theta^T\right)=\tilde{b}_T\left(\theta^T\right)\\
&\vdots
\end{aligned}
$$

while all other intermediaries (for example those born in period $2, T+1, ..$) offer simple uncontingent savings contracts. Given the prices from the planning problem consider the incentives of any particular intermediary to deviate when all other intermediaries are offering the uncontingent contracts constructed above. Given that intermediaries are offering savings contracts, a deviating intermediary cannot offer a contract with state-contingency. Therefore, since the intermediary is restricted to offer no-banishment contracts, the best this intermediary can do is to offer an agent who is Euler constrained the opportunity to borrow more at date $t$. Consider some $t$ and history $\theta^t\in\Theta^t$ such that $u'\left(c_t\left(\theta^t\right)\right)q_t>\beta E_{t+1}u'\left(c_{t+1}\left(\theta^{t+1}\right)\right)$. We know from (2.20) and Lemma 4 that in period $t+1$, $V_{t+1}\left(\theta^t,\theta\right)=V^d_{t+1}\left(\theta^t,\theta\right)$ for some $\theta\in\Theta$. Therefore, the deviating contract will violate voluntary participation constraints for the agent in some state at date $t+1$. Notice that offering a savings contract can never lead to positive profits for any deviating intermediary since it would have to offer a return $\tilde{R}_{t+1}<\frac{1}{q_t}$ no household will ever accept such a contract. ■

**Proofs from Section 5.2**

**Proof of Proposition 11.** The proof of the first part of the proposition is clear. It must be that

$$u'\left(c_t\left(\theta^t\right)\right)q_t\geq\beta\mathbb{E}_{t+1}u'\left(c_{t+1}\left(\theta^{t+1}\right)\right)$$

else, the households will use the hidden markets to save. Next, since we are restricting the planner to only offer two period contracts, it cannot cross-subsidize between types. This along with an initial zero profit condition implies (2.26) . ■

**Proof of Proposition 12.** The proof of the first part is as before. Households can never be savings constrained. Since we know that any equilibrium contract of the hidden market must be short-term it suffices to consider deviating contracts of the following form: intermediaries offer a vector $\mathbb{D}_t\left(h^t\right)=\left(z_t\left(h^t\right),z_{t+1}\left(h^t\right),\delta^H_{t+1}\left(h^{t+1}\right),\mu^H\left(h^t\right)\right)$ which consists of transfers in period $t,\ t+1$, banishment indices for the hidden market and re-entry probabilities. Given a period $t$ and history $h^t$, the best such contract solves the following problem (note that $\delta_{t+1},\delta^H_{t+1}$, and $b_{t+1}$ are functions of $h^{t+1}$):

$$\begin{aligned}
&\max_{\mathbb{D}_t(h^t)} u\left(\theta_t+b_t\left(h^t\right)+z_t\left(h^t\right)\right)\\
&+\beta\sum_{h^{t+1}}\zeta\left(h^{t+1}\right)\left[(1-\delta_{t+1})\left[\left(1-\delta^H_{t+1}\right)u\left(\theta_{t+1}+b_{t+1}+z_{t+1}\left(h^{t+1}\right)\right)+\delta^H_{t+1}\mathcal{V}^N_{t+1}\left(h^{t+1};\mu^H\right)\right]\right.\\
&\left.+\,\delta_{t+1}\mathcal{V}^E_t\left(h^t;\mu_{t+1},\mu^H_{t+1}\right)\right]
\end{aligned}$$

subject to for all $h^{t+1}$

$$\left[1-\delta^H_{t+1}\left(h^{t+1}\right)\right]\mathcal{V}_{t+1}\left(h^{t+1}\right)\geq\left[1-\delta_{t+1}\left(h^{t+1}\right)\right]\mathcal{V}^N_{t+1}\left(h^{t+1};\mu^H_t\left(h^{t-1}\right)\right) \tag{A.7}$$

and

$$z_t\left(h^t\right)+q_t\sum_{h^{t+1}}\left(1-\delta_{t+1}\left(h^{t+1}\right)\right)\left(1-\delta^H_{t+1}\left(h^{t+1}\right)\right)z_{t+1}\left(h^t\right)=0 \tag{A.8}$$

Notice that this contract contains all possible short-term deviating contracts intermediaries can offer. We can substitute (A.8) into the objective function and rewrite the problem as

$$\begin{aligned}
&\max_{\tilde{\mathbb{D}}_t(h^t)} u\left(\theta_t+b_t\left(h^t\right)+\tilde{q}_t\tilde{z}_t\left(h^t\right)\right)\\
&+\beta\sum_{h^{t+1}}\zeta\left(h^{t+1}\right)\left[(1-\delta_{t+1})\left[\left(1-\delta^H_{t+1}\right)u\left(\theta_{t+1}+b_{t+1}-\tilde{z}_t\left(h^t\right)\right)+\delta^H_{t+1}\mathcal{V}^N_{t+1}\left(h^{t+1};\mu^H\right)\right]\right.\\
&\left.+\,\delta_{t+1}\mathcal{V}^E_t\left(h^t;\mu_{t+1},\mu^H_{t+1}\right)\right]
\end{aligned}$$

subject to (A.7). Here $\tilde{q}_t=q_t\sum_{t+1}\left(1-\delta_{t+1}\left(h^{t+1}\right)\right)$ and $\tilde{\mathbb{D}}_t\left(h^t\right)=\left(\tilde{z}_t\left(h^t\right),\delta^H_{t+1}\left(h^{t+1}\right),\mu^H\left(h^{t+1}\right)\right)$. We know from Lemma 14 that we need only consider the constraint for the lowest type $\theta_{t+1}$ such that $\delta_{t+1}\left(h^{t+1}\right)=0$. Denote this type by $\hat{h}^{t+1}$ and let $\eta$ denote the multiplier on this constraint. The first order conditions for $\tilde{z}_t\left(h^t\right)$ and $\mu^H\left(h^t\right)$ are

$$
\begin{aligned}
& u'\left(\theta_t + b_t\left(h^t\right) + \tilde{q}_t \tilde{z}_t\left(h^t\right)\right)\tilde{q}_t \\
& -\beta \sum_{h^{t+1}} \zeta\left(h^{t+1}\right)\left(1-\delta_{t+1}\left(h^{t+1}\right)\right)\left[\left(1-\delta^H_{t+1}\left(h^{t+1}\right)\right) u'\left(\theta_{t+1} + b_{t+1}\left(h^{t+1}\right) - \tilde{z}_t\left(h^t\right)\right)\right] \\
& = -\eta \frac{\partial}{\partial \varepsilon} \mathcal{V}_{t+1}\left(\hat{h}^{t+1}\right)
\end{aligned} \tag{A.9}
$$

and

$$
\begin{aligned}
& \beta \sum_{h^{t+1}} \zeta\left(h^{t+1}\right)\left[\begin{array}{c} \left(1-\delta_{t+1}\left(h^{t+1}\right)\right)\delta^H_{t+1}\left(h^{t+1}\right)\frac{\partial}{\partial \mu^H}\mathcal{V}^N_{t+1}\left(h^{t+1};\mu^H\right) \\ +\delta_{t+1}\left(h^{t+1}\right)\frac{\partial}{\partial \mu^H}\mathcal{V}^E_t\left(h^t;\mu_{t+1},\mu^H_{t+1}\right) \end{array}\right] \\
& = \eta \frac{\partial}{\partial \mu^H}\mathcal{V}^N_{t+1}\left(\hat{h}^{t+1};\mu^H\left(h^t\right)\right)
\end{aligned}
$$

where

$$
\begin{aligned}
& \frac{\partial}{\partial \varepsilon}\mathcal{V}_{t+1}\left(h^{t+1}\right) = u'\left(\theta_{t+1} + b_{t+1}\left(h^{t+1}\right) - \tilde{z}_t\left(h^t\right)\right) \\
& \frac{\partial}{\partial \mu^H}\mathcal{V}^N_{t+1}\left(h^{t+1};\mu^H\right) = \beta \mathbb{E}_t\left[\mathcal{V}_{t+2}\left(h^{t+2}\right) - \mathcal{V}^N_{t+2}\left(h^{t+2};\mu^H\right)\right] \\
& \frac{\partial}{\partial \mu^H}\mathcal{V}^E_t\left(h^t;\mu_{t+1},\mu^H_{t+1}\right) = \beta \mathbb{E}_t \mu\left(h^t\right)\left[\mathcal{V}_{t+2}\left(h^{t+2}\right) - \mathcal{V}^N_{t+2}\left(h^{t+2};\mu^H\right)\right]
\end{aligned}
$$

and the last two equations follow from (2.25) and (2.24) respectively.

Therefore

$$
\eta = \frac{\beta \sum_{h^{t+1}} \zeta\left(h^{t+1}\right)\left[\begin{array}{c} \left(1-\delta_{t+1}\left(h^{t+1}\right)\right)\delta^H_{t+1}\left(h^{t+1}\right)\frac{\partial}{\partial \mu^H}\mathcal{V}^N_{t+1}\left(h^{t+1};\mu^H\right) \\ +\delta_{t+1}\left(h^{t+1}\right)\frac{\partial}{\partial \mu^H}\mathcal{V}^E_t\left(h^t;\mu_{t+1},\mu^H_{t+1}\right) \end{array}\right]}{\frac{\partial}{\partial \mu^H}\mathcal{V}^N_{t+1}\left(\hat{h}^{t+1};\mu^H\left(h^t\right)\right)}
$$

Substituting this into (A.9) we obtain

$$
\begin{aligned}
& u'\left(\theta_t + b_t\left(h^t\right) + \tilde{q}_t \tilde{z}_t\left(h^t\right)\right)\tilde{q}_t = \\
& \beta \sum_{h^{t+1}} \zeta\left(h^{t+1}\right)\left(1-\delta_{t+1}\left(h^{t+1}\right)\right)\left(1-\delta^H_{t+1}\left(h^{t+1}\right)\right) u'\left(\theta_{t+1} + b_{t+1}\left(h^{t+1}\right) - \tilde{z}_t\left(h^t\right)\right) - \\
& \left[\frac{\beta \sum_{h^{t+1}} \zeta\left(h^{t+1}\right)\left[\left(1-\delta_{t+1}\left(h^{t+1}\right)\right)\delta^H_{t+1}\left(h^{t+1}\right)\frac{\partial}{\partial \mu^H}\mathcal{V}^N_{t+1} + \delta_{t+1}\left(h^{t+1}\right)\frac{\partial}{\partial \mu^H}\mathcal{V}^E_{t+1}\right]}{\frac{\partial}{\partial \mu^H}\mathcal{V}^N_{t+1}\left(\hat{h}^{t+1};\mu^H\left(h^t\right)\right)}\right]\cdot \\
& \cdot \frac{\partial}{\partial \varepsilon}\mathcal{V}_{t+1}\left(\hat{h}^{t+1}\right)
\end{aligned}
$$

Suppose that $\delta^H_{t+1}\left(h^{t+1}\right)=0$ for all $h^{t+1}$. This implies that the deviating contract does not banish additional types from the hidden market. Then

$$
\begin{aligned}
&u'\left(\theta_t+b_t\left(h^t\right)+\tilde{q}_t\tilde{z}_t\left(h^t\right)\right)\tilde{q}_t=\\
&\beta\sum_{h^{t+1}}\zeta\left(h^{t+1}\right)\left(1-\delta_{t+1}\left(h^{t+1}\right)\right)u'\left(\theta_{t+1}+b_{t+1}\left(h^{t+1}\right)-\tilde{z}_t\left(h^t\right)\right)-\\
&\left[\frac{\beta\sum_{h^{t+1}}\zeta\left(h^{t+1}\right)\left[\delta_{t+1}\left(h^{t+1}\right)\frac{\partial}{\partial\mu^H}\mathcal{V}^E_{t+1}\left(h^{t+1};\mu_{t+1},\mu^H_{t+1}\right)\right]}{\frac{\partial}{\partial\mu^H}\mathcal{V}^N_{t+1}\left(\hat{h}^{t+1};\mu^H\left(h^t\right)\right)}\right]\frac{\partial}{\partial\varepsilon}\mathcal{V}_{t+1}\left(\hat{h}^{t+1}\right)
\end{aligned}
\tag{A.10}
$$

Consider the last term of the above equation. We have

$$
\begin{aligned}
&-\frac{\beta\sum_{h^{t+1}}\zeta\left(h^{t+1}\right)\left[\delta_{t+1}\left(h^{t+1}\right)\frac{\partial}{\partial\mu^H}\mathcal{V}^E_{t+1}\left(h^{t+1};\mu_{t+1},\mu^H_{t+1}\right)\right]}{\frac{\partial}{\partial\mu^H}\mathcal{V}^N_{t+1}\left(\hat{h}^{t+1};\mu^H\left(h^t\right)\right)}\frac{\partial}{\partial\varepsilon}\mathcal{V}_{t+1}\left(\hat{h}^{t+1}\right)\\
&=-\beta\sum_{h^{t+1}}\zeta\left(h^{t+1}\right)\left[\delta_{t+1}\left(h^{t+1}\right)\frac{\partial}{\partial\varepsilon}\mathcal{V}_{t+1}\left(\hat{h}^{t+1}\right)\right]\frac{\frac{\partial}{\partial\mu^H}\mathcal{V}^E_{t+1}\left(\hat{h}^{t+1};\mu_{t+1},\mu^H_{t+1}\right)}{\frac{\partial}{\partial\mu^H}\mathcal{V}^N_{t+1}\left(\hat{h}^{t+1};\mu^H\left(h^t\right)\right)}\\
&=\beta\sum_{h^{t+1}}\zeta\left(h^{t+1}\right)\left[\delta_{t+1}\left(h^{t+1}\right)u'\left(\hat{\theta}_{t+1}+b_{t+1}\left(\hat{h}^{t+1}\right)-\tilde{z}_t\left(h^t\right)\right)\right]\cdot\\
&\cdot\frac{\frac{\partial}{\partial\mu^H}\mathcal{V}^E_{t+1}\left(\hat{h}^{t+1};\mu_{t+1},\mu^H_{t+1}\right)}{\frac{\partial}{\partial\mu^H}\mathcal{V}^N_{t+1}\left(\hat{h}^{t+1};\mu^H\left(h^t\right)\right)}\\
&=\beta\sum_{h^{t+1}}\zeta\left(h^{t+1}\right)\left[\delta_{t+1}\left(h^{t+1}\right)u'\left(\hat{\theta}_{t+1}+b_{t+1}\left(\hat{h}^{t+1}\right)-\tilde{z}_t\left(h^t\right)\right)\right]\cdot\\
&\cdot\frac{\beta\mathbb{E}_t\mu\left(h^{t+1}\right)\left[\mathcal{V}_{t+2}\left(h^{t+2}\right)-\mathcal{V}^N_{t+2}\left(h^{t+2};\mu^H\right)\right]}{\beta\mathbb{E}_t\left[\mathcal{V}_{t+2}\left(h^{t+2}\right)-\mathcal{V}^N_{t+2}\left(h^{t+2};\mu^H\right)\right]}\\
&\leq\beta\sum_{h^{t+1}}\zeta\left(h^{t+1}\right)\left[\delta_{t+1}\left(h^{t+1}\right)u'\left(\hat{\theta}_{t+1}+b_{t+1}\left(\hat{h}^{t+1}\right)-\tilde{z}_t\left(h^t\right)\right)\right]
\end{aligned}
$$

where the first equality follows since $\mathcal{V}^E_{t+1}\left(h^{t+1};\mu_{t+1},\mu^H_{t+1}\right)=\mathcal{V}^E_{t+1}\left(\hat{h}^{t+1};\mu_{t+1},\mu^H_{t+1}\right)$.

Therefore, from (A.10) we know that

$$\begin{aligned}
&u'\left(\theta_t + b_t\left(h^t\right) + \tilde{q}_t \tilde{z}_t\left(h^t\right)\right)\tilde{q}_t \leq \\
&\beta \sum_{h^{t+1}} \zeta\left(h^{t+1}\right)\left(1-\delta_{t+1}\left(h^{t+1}\right)\right) u'\left(\theta_{t+1} + b_{t+1}\left(h^{t+1}\right) - \tilde{z}_t\left(h^t\right)\right) + \\
&\beta \sum_{h^{t+1}} \zeta\left(h^{t+1}\right)\left[\delta_{t+1}\left(h^{t+1}\right) u'\left(\hat{\theta}_{t+1} + b_{t+1}\left(\hat{h}^{t+1}\right) - \tilde{z}_t\left(h^t\right)\right)\right] \\
&\leq \beta \sum_{h^{t+1}} \zeta\left(h^{t+1}\right)\left(1-\delta_{t+1}\left(h^{t+1}\right)\right) u'\left(\hat{\theta}_{t+1} + b_{t+1}\left(\hat{h}^{t+1}\right) - \tilde{z}_t\left(h^t\right)\right) + \\
&\beta \sum_{h^{t+1}} \zeta\left(h^{t+1}\right)\left[\delta_{t+1}\left(h^{t+1}\right) u'\left(\hat{\theta}_{t+1} + b_{t+1}\left(\hat{h}^{t+1}\right) - \tilde{z}_t\left(h^t\right)\right)\right] \\
&= \beta u'\left(\hat{\theta}_{t+1} + b_{t+1}\left(\hat{h}^{t+1}\right) - \tilde{z}_t\left(h^t\right)\right)
\end{aligned}$$

Therefore, if (2.28) holds, there exists no hidden contract $\mathbb{D}_t\left(h^t\right)$ that makes the household strictly better off. Suppose that $u'\left(\theta_t + b_t\left(h^t\right)\right)\hat{q}_t\left(h^t\right) > \beta u'\left(\hat{\theta}_{t+1} + b_{t+1}\left(\hat{h}^{t+1}\right)\right)$.We have that

$$\begin{aligned}
&u'\left(\theta_t + b_t\left(h^t\right)\right)\tilde{q}_t - \\
&-\beta \sum_{h^{t+1}} \zeta\left(h^{t+1}\right)\left(1-\delta_{t+1}\left(h^{t+1}\right)\right) u'\left(\theta_{t+1} + b_{t+1}\left(h^{t+1}\right)\right) - \\
&\left[\frac{\beta \sum_{h^{t+1}} \zeta\left(h^{t+1}\right)\left[\delta_{t+1}\left(h^{t+1}\right)\frac{\partial}{\partial \mu^H}\mathcal{V}^E_{t+1}\left(h^{t+1}\right)\right]}{\frac{\partial}{\partial \mu^H}\mathcal{V}^N_{t+1}\left(\hat{h}^{t+1};\mu^H\left(h^t\right)\right)}\right] u'\left(\hat{\theta}_{t+1} + b_{t+1}\left(\hat{h}^{t+1}\right)\right) \\
&\geq u'\left(\theta_t + b_t\left(h^t\right)\right)\tilde{q}_t - \\
&-\beta \sum_{h^{t+1}} \zeta\left(h^{t+1}\right)\left(1-\delta_{t+1}\left(h^{t+1}\right)\right) u'\left(\hat{\theta}_{t+1} + b_{t+1}\left(h^{t+1}\right)\right) \\
&-\beta \sum_{h^{t+1}} \zeta\left(h^{t+1}\right)\delta_{t+1}\left(h^{t+1}\right) u'\left(\hat{\theta}_{t+1} + b_{t+1}\left(\hat{h}^{t+1}\right)\right) \\
&= u'\left(\theta_t + b_t\left(h^t\right)\right)\tilde{q}_t - \beta u'\left(\hat{\theta}_{t+1} + b_{t+1}\left(\hat{h}^{t+1}\right) - \tilde{z}_t\left(h^t\right)\right) > 0
\end{aligned}$$

So there exists a deviating contract $\tilde{\mathbb{D}}_t\left(h^t\right) = \left(\varepsilon_t\left(h^t\right), \mu^H\left(h^{t+1}\right)\right)$ that makes the household strictly better off. ■

Given these characterization results, we can prove a version of the Second Welfare Theorem in this environment.

**Proof of Proposition 10.** As we showed in Proposition 11, any solution to the planning problem households can never be savings constrained. Second, using an argument identical to that in decentralized environment with banishment, we can split the sequence of optimal transfers $b_t\left(h^t\right) = -\hat{R}_t\left({}_{t-1}\tilde{b}_t\left(h^{t-1}\right)\right) + {}_t\tilde{b}_t\left(h^t\right)$. As a result we can construct the short term defaultable debt contracts for intermediaries born in period $1, T, 2T-1, ..$ as was done in the intermediary game. All intermediaries born at dates besides these offer simple uncontingent savings contract to deter deviating intermediaries from offering state-contingency contracts. We can show using similar arguments to show that if $\delta_t\left(h^{t-1}, h_t\right) = 1$, then the fact that savers must get a return $R_t = \frac{1}{q_t}$ and incentive-feasiblity imply that $\hat{R}_t\left({}_{t-1}\tilde{b}_t\left(h^{t-1}\right)\right) = \frac{R_t\, {}_{t-1}\tilde{b}_t\left(h^{t-1}\right)}{\sum_{h\notin D_t\left(b_t^{t-1}\left(h^{t-1}\right)\right)}\zeta\left(h^{t-1},h\right)}$ . As before, if $D_t\left(b_t^{t-1}\left(h^{t-1}\right)\right) = \varnothing$, then $\hat{R}_t\left(b\right) = R_t b$. We can then define an allocation for the intermediary game as follows in exactly the same manner in which we constructed two period contracts in the decentralized environment and the banishment policy the same as the one chosen by the planner. By construction these contracts satisfy incentive compatibility, resource feasibility and voluntary participation constraints. Under these contracts, intermediaries make zero profits as well. Suppose that all intermediaries were offering these contracts. Let consider the incentive for a deviating intermediary to offer a contract that makes both it and the household strictly better off. Such a contract can never be a savings contract because of (2.27). As a result we need to consider if a deviating debt contract exists. Since the contract must be short-term i.e. $\mathbb{D}_t\left(h^t\right) = \left({}_t z_t\left(h^t\right), {}_t z_{t+1}\left(h^t\right), {}_t\tilde{\delta}_{t+1\,t+1}\left(h^{t+1}\right), {}_t\tilde{\mu}_{t+1}\left(h^t\right)\right)$ that solves

$$
\begin{aligned}
&\max_{\mathbb{D}_t(h^t)} u\left(\theta_t + b_t\left(h^t\right) + {}_t z_t\left(h^t\right)\right) \\
&+ \beta \sum_{h^{t+1}} \zeta\left(h^{t+1}\right)\left[\left(1-\delta_{t+1}\right)\left[\begin{array}{c}\left(1 - {}_t\tilde{\delta}_{t+1}\right) u\left(\theta_{t+1} + b_{t+1} + {}_t z_{t+1}\right) \\ + {}_t\tilde{\delta}_{t+1}\, \tilde{V}_{t+1}\left(h^{t+1}; {}_t\tilde{\mu}_{t+1}\, \mu_{t+1}\right)\end{array}\right]\right. \\
&\left. + \delta_{t+1}\tilde{V}_{t+1}\left(h^{t+1};\mu_{t+1}\right)\right]
\end{aligned}
$$

where

$$
\begin{aligned}
&\tilde{V}_{t+1}\left(h^{t+1}; {}_t\tilde{\mu}_{t+1}\, \mu_{t+1}\right) = u\left(\theta_{t+1}\right) \\
&+ \beta\mathbb{E}_{t+1}\left[{}_t\tilde{\mu}_{t+1}\, \mu_{t+1} V_{t+2}\left(h_{t+2}\right) + \left(1 - {}_t\tilde{\mu}_{t+1}\, \mu_{t+1}\right)\tilde{V}_{t+2}\left(h^{t+2}; {}_t\tilde{\mu}_{t+1}\, \mu_{t+1}\right)\right]
\end{aligned}
$$

From (2.27), such a contract can never make the household strictly better off. Therefore, these contracts must constitute a competitive equilibrium. ■

### A.1.4 Truncation argument

Let $V_{\infty}^{*}$ denote the ex-ante welfare for the planner under the constrained efficient allocation with no banishment. Now consider two truncated economies in which after period $T$, households trade a risk free bond subject to debt constraints. In the first, the debt constraints after period $T$, are $\phi+\varepsilon$ and the second, the constraints are $\phi-\varepsilon$ for $\phi, \varepsilon > 0$. Let $V_{\infty}^{T,H}\left(\phi+\varepsilon\right)$ denote the welfare to the planner in the first truncated economy and note that $V_{\infty}^{T,H}\left(\phi+\varepsilon\right) = V_{T}^{T,H}\left(\phi+\varepsilon\right) + V_{T+}^{T,H}\left(\phi+\varepsilon\right)$ where $V_{T}^{T,H}\left(\phi+\varepsilon\right)$ denotes aggregate expected welfare from period 1 to $T$ and $V_{T+}^{T,H}\left(\phi+\varepsilon\right)$ the welfare from periods after $T$ . We can similarly define $V_{\infty}^{T,L}\left(\phi-\varepsilon\right) = V_{T}^{T,L}\left(\phi-\varepsilon\right) + V_{T+}^{T,L}\left(\phi-\varepsilon\right)$. We can always find a pair $\left(\phi, \varepsilon\right)$ such that

$$V_{\infty}^{T,L}\left(\phi-\varepsilon\right) \leq V_{\infty}^{*} \leq V_{\infty}^{T,H}\left(\phi+\varepsilon\right)$$

Since we have proved properties of the constrained efficient allocations in the truncated economies, if we can prove that $\lim_{\varepsilon\to 0}\lim_{T\to\infty}\left[V_{\infty}^{T,H}\left(\phi+\varepsilon\right) - V_{\infty}^{T,L}\left(\phi-\varepsilon\right)\right] = 0$, then we know that $V_{\infty}^{*}$ inherits these properties as well. To show this limiting property I show that that for $T$ large enough and $\varepsilon$ small enough, the set of feasible allocations in both truncated economies are identical.

Consider the some $x \in Feas^{T,L}\left(\phi-\varepsilon\right)$. Since $x$ satisfies feasibility and incentive compatibility in $Feas^{T,L}\left(\phi-\varepsilon\right)$, to show that $x \in Feas^{T,H}\left(\phi+\varepsilon\right)$ we just need to show that the voluntary participation constraints are satisfied. But since for all $t$ and types $h^t$

$$V_{t-T}^{T,H}\left(\phi+\varepsilon\right)\left(x\left(h^t\right)\right)+V_{T+}^{T,H}\left(\phi+\varepsilon\right)\left(x\left(h^t\right)\right) \geq V_{t-T}^{T,L}\left(\phi-\varepsilon\right)\left(x\left(h^t\right)\right)+V_{T+}^{T,L}\left(\phi-\varepsilon\right)\left(x\left(h^t\right)\right)$$

where $V_{t-T}^{T,H}\left(\phi+\varepsilon\right)\left(x\left(h^t\right)\right)$ denotes the value for type $h^t$ from period $t$ to $T$, this is clearly satisfied. Hence $Feas^{T,L}\left(\phi-\varepsilon\right) \subseteq Feas^{T,H}\left(\phi+\varepsilon\right)$. Next consider some $x \in Feas^{T,H}\left(\phi+\varepsilon\right)$. As in the previous case we need to show that this satisfies voluntary participation constraints in $Feas^{T,L}\left(\phi-\varepsilon\right)$. Note that in general since $V_{T+}^{T,L}\left(\phi-\varepsilon\right)\left(x\right) < V_{T+}^{T,H}\left(\phi+\varepsilon\right)\left(x\right)$, this will not be satisfied. However as $T\to\infty$ and $\varepsilon\to 0$, $\left[V_{t-T}^{T,H}\left(\phi+\varepsilon\right)\left(x\right) + V_{T+}^{T,H}\left(\phi+\varepsilon\right)\left(x\right)\right] - \left[V_{t-T}^{T,L}\left(\phi-\varepsilon\right)\left(x\right) + V_{T+}^{T,L}\left(\phi-\varepsilon\right)\left(x\right)\right] \to 0$. Therefore for $T$ large enough and $\varepsilon$ small enough, it must be that $x \in Feas^{T,L}\left(\phi-\varepsilon\right)$ and so eventually $Feas^{T,H}(\phi + \varepsilon) \subseteq Feas^{T,L}(\phi - \varepsilon)$.

# Appendix B

# Appendix to Chapter 3

## B.1 Appendix: Proofs From the Main Text

**Proof of Lemma Lemma 7.** The first order conditions from the agent's problem imply

$$\beta^t \pi\left(s^t\right) u'\left(c_t^i\left(s^t\right)\right) = Q_t\left(s^t\right) \mu^i$$

If $s_t = i$,

$$\begin{aligned}
\beta^t \pi\left(s^t\right) &= Q_t\left(s^t\right) \mu^i \\
&\Rightarrow u'\left(c_t^i\left(s^t\right)\right) = 1
\end{aligned}$$

Since markets are complete, marginal rates of substitutions are equalized across agents and states. Therefore

$$\begin{aligned}
\frac{\beta^{t'} \pi\left(s^{t'}\right) u'\left(c_{t'}^i\left(s^{t'}\right)\right)}{\beta^t \pi\left(s^t\right) u'\left(c_t^i\left(s^t\right)\right)} &= \frac{\beta^{t'} \pi\left(s^{t'}\right) u'\left(c_{t'}^j\left(s^{t'}\right)\right)}{\beta^t \pi\left(s^t\right) u'\left(c_t^j\left(s^t\right)\right)} \\
&\Rightarrow \frac{u'\left(c_{t'}^i\left(s^{t'}\right)\right)}{u'\left(c_t^i\left(s^t\right)\right)} = \frac{u'\left(c_{t'}^j\left(s^{t'}\right)\right)}{u'\left(c_t^j\left(s^t\right)\right)} \\
&\Rightarrow u'\left(c_{t'}^i\left(s^{t'}\right)\right) = \frac{1}{u'\left(c_t^j\left(s^t\right)\right)}
\end{aligned}$$

So in a symmetric competitive equilibrium $u'\left(c_{t'}^{i}\left(s^{t'}\right)\right) = u'\left(c_t^{j}\left(s^t\right)\right) = 1$. And so $c^i\left(s^t\right) = g\left(1\right)$ for all $s^t$. Also

$$\frac{Q\left(s^{t+1}\right)}{Q\left(s^t\right)} = \beta\pi\left(s^{t+1} \mid s^t\right)$$

And so the price of a risk free bond is

$$\sum_{s^{t+1}} \frac{Q\left(s^{t+1}\right)}{Q\left(s^t\right)} = \beta$$

and the risk free rate is $\frac{1}{\beta}$. ■

**Proof of Proposition 16.** The first order conditions from the agent $i$'s problem are

$$\begin{aligned}
\beta^t\pi\left(s^t\right)u'\left(c_t^i\left(s^t\right)\right) &= \mu_t^i\left(s^t\right) + \sum_{s^{t+1}} \lambda_{t+1}^i\left(s^t\right)\frac{p_{t+1}\left(s^{t+1}\right)}{p_t\left(s^t\right)} \\
0 &= \lambda_t^i\left(s^t\right)q_{s_{t+1}}\left(s^t\right) - \lambda_{t+1}^i\left(s^{t+1}\right) \\
0 &= \lambda_t^i\left(s^t\right)p_t\left(s^t\right) - \mu_t^i\left(s^t\right)p_t\left(s^t\right) - \sum_{s^{t+1}} \lambda_{t+1}^i\left(s^{t+1}\right)p_{t+1}\left(s^{t+1}\right) \\
\beta^t\pi\left(s^t\right) &= \sum_{s^{t+1}} \lambda_{t+1}^i\left(s^{t+1}\right)\frac{p_{t+1}\left(s^{t+1}\right)}{p_t\left(s^t\right)} \text{if } s_t = i
\end{aligned}$$

Combining, we get

$$\beta^t\pi\left(s^t\right)u'\left(c_t^i\left(s^t\right)\right) = \mu_t^i\left(s^t\right) + \sum_{s^{t+1}} \lambda_{t+1}^i\left(s^t\right)\frac{p_{t+1}\left(s^{t+1}\right)}{p_t\left(s^t\right)} = \lambda_t^i\left(s^t\right)$$

Define

$$\frac{p_{t+1}\left(s^{t+1}\right)}{p_t\left(s^t\right)} = \frac{1}{\beta} \text{ for all } s^t, s^{t+1}$$

and so

$$p_t = p_0\left(\frac{1}{\beta}\right)^t$$

for some constant $p_0$ to be determined.

Conjecture that $c_t^i\left(s^t\right) = g\left(1\right)$ for all $i, t, s^t$. Then

$$\begin{aligned}
\mu_t^i\left(s^t\right) &= \beta^t \pi\left(s^t\right) u'\left(c_t\left(s^t\right)\right) - \sum_{s^{t+1}} \beta^{t+1} \pi\left(s^{t+1}\right) \frac{p_{t+1}\left(s^{t+1}\right)}{p_t\left(s^t\right)} u'\left(c_{t+1}\left(s^{t+1}\right)\right) \\
&= \beta^t \pi\left(s^t\right) \left[u'\left(c_t\left(s^t\right)\right) - \sum_{s^{t+1}} \beta \pi\left(s^{t+1} \mid s^t\right) \frac{p_{t+1}\left(s^{t+1}\right)}{p_t\left(s^t\right)} u'\left(c_{t+1}\left(s^{t+1}\right)\right)\right] \\
&= \beta^t \pi\left(s^t\right) \left[1 - \beta \frac{p_{t+1}}{p_t}\right] \\
&= 0
\end{aligned}$$

and so the multiplier on the cash-in-advance constraint is 0. Given this, notice that conjectured allocation satisfies the agent's first order conditions.

Finally, we need to verify that the transversality conditions holds. The one for state contingent assets follows directly. Notice from our construction of prices

$$\begin{aligned}
&\lim \inf_{t\to\infty} E_{t-1}\left[\beta^t u'\left(c_t^i\left(s^t\right)\right) p_t\left(s^t\right) m_t^i\left(s^t\right)\right] \\
&= \lim \inf_{t\to\infty} E_{t-1}\left[\beta^t p_0 \left(\frac{1}{\beta}\right)^t m_t^i\left(s^t\right)\right] \\
&= p_0 \lim \inf_{t\to\infty} E_{t-1}\left[m_t^i\left(s^t\right)\right] \\
&\leq p_0 \lim \inf_{t\to\infty} M_t \\
&= 0
\end{aligned}$$

Choose $p_0$ so that

$$\frac{2g\left(1\right)}{\kappa} \geq p_0$$

Then

$$\begin{aligned}
&\sum_i c_t^i\left(s^t\right) \leq p_t\left(s^t\right) M_t \\
&\Leftrightarrow M_t \geq \frac{\sum_i c_t^i\left(s^t\right)}{p_t\left(s^t\right)} \Leftrightarrow M_t \geq \frac{2g\left(1\right)}{p_0 \beta^{-t}} \Leftrightarrow \beta^{-t} M_t \geq \frac{2g\left(1\right)}{p_0} \geq \kappa
\end{aligned}$$

Since $\inf M_t \beta^{-t} = \kappa > 0$ the CIA constraints are always satisfied. ■

## Proofs From Section 2.2.2

### Proof of Proposition 18

I conjecture that a stationary equilibrium of the following form exists

$$\begin{aligned}
q^j\left(s^{t-1}, i\right) &= q^i\left(s^{t-1}, j\right) = q^c \\
q^j\left(s^{t-1}, j\right) &= q^i\left(s^{t-1}, i\right) = q^{nc} \\
c^i\left(s^{t-1}, i\right) &= c^h = 1 \\
c^i\left(s^{t-1}, j\right) &= c^l = g\left(\frac{q^c}{\beta\lambda}\right) \\
\phi^i_{s'}\left(s^t\right) &= \phi \\
a^i_j\left(s^{t-1}, i\right) &= \phi \\
a^i_i\left(s^{t-1}, i\right) &= -\phi
\end{aligned}$$

It suffices to check that given these prices, the allocations satisfy the agent's first order conditions, clear markets and the debt limits are not-too-tight. Notice from the budget constraints of the agents

$$\begin{aligned}
c^h + q^c\phi - q^{nc}\phi &= l - \phi \\
c^l - q^c\phi + q^{nc}\phi &= \phi
\end{aligned}$$

we have

$$\begin{aligned}
\phi &= \frac{c^l}{\left[1 + q^c - q^{nc}\right]} \\
&= \frac{g\left(\frac{q^c}{\beta\lambda}\right)}{\left[1 + q^c - q^{nc}\right]}
\end{aligned} \tag{B.1}$$

Therefore

$$\begin{aligned}
& c^h + \left[q^c - q^{nc} + 1\right]\phi = l \\
\Rightarrow\ & c^h + \left[q^c - q^{nc} + 1\right]\frac{g\left(\frac{q^c}{\beta\lambda}\right)}{\left[1 + q^c - q^{nc}\right]} = l \\
\Rightarrow\ & c^h + g\left(\frac{q^c}{\beta\lambda}\right) = l
\end{aligned}$$

and so given this level of $l$, the allocation is feasible for the agents. Given prices, it is easy to see that the allocation satisfies the agent's first order conditions

$$u'\left(c_t^i\left(s^{t-1}, i\right)\right) = 1$$
$$u'\left(c_t^i\left(s^t, j\right)\right) = \frac{q_t^j\left(s^t\right)}{\beta\lambda}$$

The only thing to be checked is that the debt limits satisfy the not-too-tight property. The following lemma will be useful in proving the main result.

**Lemma 20** *If after default, agents can purchase non-negative quantities of Arrow securities, an equilibrium of the conjectured form exists with*

$$q^c + q^{nc} = 1$$

**Proof of Lemma 20.** In our conjectured equilibrium we need only consider the productive agent's incentives to default. In particular, $\phi$ will be chosen so that the productive agent who $\phi$ will be indifferent between defaulting and not. To characterize $\phi$,we want to consider strategies that would replicate the agents allocation had he not defaulted when he is only allowed to save in Arrow securities. The agent's problem after default is

$$V_t^{i,d}(s^t) = \max \sum_{t' \geq t} \sum_{s^{t'} \succeq s^t} \beta^{t'-t} \pi\left(s^{t'} \mid s^t\right)\left[u\left(c_{t'}^i\left(s^{t'}\right)\right) - l_{t'}^i\left(s^{t'}\right)\right]$$

subject to

$$c_t^i\left(s^t\right) + q_i\left(s^t\right) a_i^{i,d}\left(s^t\right) + q_j\left(s^t\right) a_j^{i,d,}\left(s^t\right) \leq l^i\left(s^t\right)$$
$$c_{t'}^i\left(s^{t'}\right) + q_i\left(s^{t'}\right) a_i^i\left(s^{t'}\right) + q_j\left(s^{t'}\right) a_j^i\left(s^{t'}\right) \leq l^i\left(s^{t'}\right) + a_i^i\left(s^{t'-1}\right)$$
$$a_{s'}^{i,d}\left(s^{t'}\right) \geq 0, \text{for all } t' \geq t, s'$$
$$l^i\left(s^{t'}\right) = 0 \text{ if } s_{t'} \neq i$$

Consider the following strategies

$$c^i\left(s^t, i\right) = c^{h,d}$$
$$c^i\left(s^t, j\right) = c^{l,d}$$
$$a_j^{i,d}\left(s^{t-1}, i\right) = a_j^{i,d}\left(s^{t-1}, j\right) = x$$
$$a_i^{i,d}\left(s^{t-1}, i\right) = a_i^{i,d}\left(s^{t-1}, j\right) = 0$$

for some $x > 0$. For the strategy to be feasible and optimal, it must be that

$$c^{h,d} + q^c x = l$$
$$c^{l,d} + q^{nc} x = x$$

and

$$u'\left(c^{h,d}\right) = 1, \; c^{l,d} = g\left(\frac{q^c}{\beta\lambda}\right)$$

Therefore

$$x = \frac{g\left(\frac{q^c}{\beta\lambda}\right)}{(1 - q^{nc})}$$

and so

$$c^{h,d} + q^c \frac{g\left(\frac{q^c}{\beta\lambda}\right)}{(1 - q^{nc})} = l$$
$$\Rightarrow l = c^{h,d} + q^c \frac{g\left(\frac{q^c}{\beta\lambda}\right)}{(1 - q^{nc})}$$

Notice that $c^{h,d} = c^h$, $c^{l,d} = c^l$ and

$$l^d = g(1) + q^c \frac{g\left(\frac{q^c}{\beta\lambda}\right)}{(1 - q^{nc})}$$

Therefore, the value of default us

$$V^{i,d}(i) = \frac{(1 - \beta(1-\lambda))\left[u(g(1)) - g(1) - q^c \frac{g\left(\frac{q^c}{\beta\lambda}\right)}{(1-q^{nc})}\right] + \beta\lambda u\left(g\left(\frac{q^c}{\beta\lambda}\right)\right)}{[1 - \beta(1-\lambda)]^2 - \beta^2\lambda^2}$$

while the value of not defaulting is given by

$$V^{i}(i,\phi) = \frac{(1 - \beta(1-\lambda))\left[u(g(1)) - g(1) - g\left(\frac{q^c}{\beta\lambda}\right)\right] + \beta\lambda u\left(g\left(\frac{q^c}{\beta\lambda}\right)\right)}{[1 - \beta(1-\lambda)]^2 - \beta^2\lambda^2}$$

We see that if $q^c = 1 - q^{nc}$

$$V^{i}(i,\phi) = V^{i,d}(i)$$

Finally, we construct debt constraints that are not-too-tight using (B.1).

Since the first order conditions and budget constraints are all satisfied, we conclude that the an equilibrium of the conjectured form exists with not-too-tight debt constraints. ■

**Proof of Proposition 18.** Consider the agent's problem after default

$$\max E\left[\sum_{t=0}^{\infty}\sum_{s^t}\beta^t\pi\left(s^t\right)\left(u\left(c_t^{i,d}\left(s^t\right)\right)-l_t^{i,d}\left(s^t\right)\right)\right]$$

$$\begin{aligned}
&s.t.\\
&c_t^{i,d}\left(s^t\right)+Qx_{t+1}^{i,d}\left(s^t\right)=l_t^{i,d}\left(s^t\right)+x_t^{i,d}\left(s^{t-1}\right)\ \text{ if } s_t=i\\
&c_t^{i,d}\left(s^t\right)+Qx_{t+1}^{i,d}\left(s^t\right)=x_t^{i,d}\left(s^{t-1}\right)\ \text{ if } s_t\neq i\\
&x_{t+1}^{i}\geq 0
\end{aligned}$$

where $Q = q^c + q^{nc}$. We can write the problem recursively and denote the value by $V^{i,d}(s,x;q^c)$. As earlier we are looking for equilibria that are not-too-tight, i.e.

$$V^i(i,\phi)=V^{i,d}(i;q^c)$$

which implies

$$\begin{aligned}
V^i(i,\phi)&=\frac{(1-\beta(1-\lambda))\left(u\left(c^h\right)-l\right)+\beta\lambda u\left(c^l\right)}{[1-\beta(1-\lambda)]^2-\beta^2\lambda^2}\\
&=\frac{(1-\beta(1-\lambda))\left[u(g(1))-g(1)-g\left(\frac{q^c}{\beta\lambda}\right)\right]+\beta\lambda u\left(g\left(\frac{q^c}{\beta\lambda}\right)\right)}{[1-\beta(1-\lambda)]^2-\beta^2\lambda^2}
\end{aligned}$$

Let $z=\frac{q^c}{\beta\lambda}$. Then we have that

$$\begin{aligned}
\frac{\partial}{\partial z}V^i(i,\phi)&=\frac{\partial}{\partial z}\left[\frac{(1-\beta(1-\lambda))[u(g(1))-g(1)-g(z)]+\beta\lambda u(g(z))}{[1-\beta(1-\lambda)]^2-\beta^2\lambda^2}\right]\\
&=\frac{-(1-\beta(1-\lambda))g'\left(\frac{q^c}{\beta\lambda}\right)+\beta\lambda u'\left(g\left(\frac{q^c}{\beta\lambda}\right)\right)g'\left(\frac{q^c}{\beta\lambda}\right)}{[1-\beta(1-\lambda)]^2-\beta^2\lambda^2}\\
&=\frac{[-(1-\beta(1-\lambda))+q^c]\,g'\left(\frac{q^c}{\beta\lambda}\right)}{[1-\beta(1-\lambda)]^2-\beta^2\lambda^2}
\end{aligned}$$

Since $g'\left(\frac{q^c}{\beta\lambda}\right) < 0$ we have

$$\begin{aligned}
\frac{\partial}{\partial z} V^i(i,\phi)\,|_{q^c>1-\beta(1-\lambda)} &> 0 \\
\frac{\partial}{\partial z} V^i(i,\phi)\,|_{q^c=1-\beta(1-\lambda)} &= 0 \\
\frac{\partial}{\partial z} V^i(i,\phi)\,|_{q^c<1-\beta(1-\lambda)} &< 0
\end{aligned}$$

To see how $V^{i,d}(s,0;q^c)$ changes with $q^c$ notice that after default, we can compute the agent's consumption in states in which he is unproductive from the Euler equations. Suppose he defaults in period $t$,when productive. Then his consumption the following period if the state is $j$ is given by

$$\begin{aligned}
1 &= \beta R\left[\lambda u'\left(c^{i,d}_{t+1}\left(s^t,j\right)\right) + (1-\lambda)\right] \\
&\Rightarrow c^{i,d}_{t+1}\left(s^t,j\right) = g\left(\frac{1-\beta R(1-\lambda)}{\beta R\lambda}\right)
\end{aligned}$$

where $\frac{1}{R} = Q = q^c + q^{nc}$.

Similarly

$$\begin{aligned}
u'\left(c^{i,d}_{t+1}\left(s^t,j\right)\right) &= \beta R\left[\lambda + (1-\lambda)\,u'\left(c^{i,d}_{t+2}\left(s^{t+1},j\right)\right)\right] \\
&\Rightarrow c^{i,d}_{t+2}\left(s^{t+1},j\right) = g\left(\frac{\frac{1-\beta R(1-\lambda)}{\beta R\lambda} - \beta R\lambda}{\beta R(1-\lambda)}\right)
\end{aligned}$$

Clearly $\frac{\partial c^{i,d}_{t+n}\left(s^{t+n-1},j\right)}{\partial q^c} < 0$ for all $n$. Since $c^{i,d}_{t+n}\left(s^{t+n-1},i\right) = g(1)$, we have that $\frac{\partial V^{i,d}(s,0;q^c)}{\partial q^c} < 0$ for all agents $i$.

Let $\tilde{V}^{i,d}(s,0;q^c)$ denote the value of default when agents can save in Arrow securities. Now since

$$V^i(i,\phi) = \tilde{V}^{i,d}(s,0;1-\beta(1-\lambda)) > V^{i,d}(s,0;1-\beta(1-\lambda))$$

and

$$\begin{aligned}
\frac{\partial V^{i,d}(s,0;q^c)}{\partial q^c} &< 0 \\
\frac{\partial V^i(i,\phi)}{\partial q^c}\,|_{q^c=(1-\beta(1-\lambda))} &= 0
\end{aligned}$$

any $q^{c*}$ such that

$$V^i(i,\phi) = V^{i,d}(s;q^{c*})$$

must have the property that

$$q^{c*} < 1-\beta(1-\lambda)$$

Therefore

$$q^{nc} + q^{*c} < 1$$

Since $V^i(i,\phi) - V^{i,d}(s;q^c) > 0$ for $q^c = 1-\beta(1-\lambda)$ and $V^i(i,\phi) - V^{i,d}(s;q^c) < 0$ for $q^c$ small enough, we know that such a $q^{c*}$ exists.

Finally, similar to the previous lemma, we can construct not-too-tight debt limits using $(B.1)$. ■

**Remark 1** *How do we show that $V^i(i,\phi) - V^{i,d}(s,0;q^c) < 0$ for $q^c$ small enough? Consider supporting the complete markets allocation as an equilibrium. Then $\left(c^h,c^l,l\right) = (g(1),g(1),2g(1))$ and $R = \frac{1}{\beta}$. Consider the value of default when productive. Had the agent not defaulted he would have made a payment of $g(1)$ to the unproductive agent. Consider the following strategy when he defaults; he saves $g(1)$ in the risk free bond. As a result his return next period is $\frac{g(1)}{\beta}$. In this period no matter what the state, he consumes $g(1)$ and does not work and saves $\frac{g(1)}{\beta} - g(1) = \frac{g(1)(1-\beta)}{\beta}$. The following period he does the same consuming $g(1)$ and saving $\frac{g(1)(1-\beta)}{\beta^2} - g(1) = \frac{g(1)\left(1-\beta-\beta^2\right)}{\beta^2}$ and so on.. Notice that as long as $\frac{g(1)\left(1-\beta-\beta^2-\beta^3-..\beta^t\right)}{\beta^t} \geq 0$, the strategy is feasible and the agent does not have to work in any future period after the first. Since $1-\beta-\beta^2-... = \frac{1-2\beta}{\beta}$, if $\beta \leq \frac{1}{2}$, the value of default is strictly greater than paying for the productive agent. As a result a sufficient condition for $V^i(i,\phi) - V^{i,d}(s,0;q^c) < 0$ for $q^c$ low enough is $\beta \leq \frac{1}{2}$. Recall that the complete markets benchmark had $\left(c^h,c^l,l\right) = (g(1),g(1),2g(1))$. Here $\left(c^h,c^l,l\right) = \left(1, g\left(\frac{q^c}{\beta\lambda}\right), g(1) + g\left(\frac{q^c}{\beta\lambda}\right)\right)$. Notice that $g\left(\frac{q^c}{\beta\lambda}\right) < g(1)$ since $q^c > \beta\lambda$. As a result, the above equilibrium is the best stationary equilibrium since any equilibrium with higher ex-ante welfare, must have a lower $q^c$ which would violate the not-too-tight constraint. It is also worth noting that in any equilibrium $Q = q^c + q^{nc} = \frac{\beta\lambda}{c^l} + \beta(1-\lambda) \geq \beta$ since $c^l \leq 1$. $c^l = 1$ is the solution to the complete markets problem.*

**Proof of Proposition 19.** As in Golosov and Tsyvinski (2007) and Kehoe and Levine (1993) we can write constrained efficient problem as choosing sequences consumption, labor and price sequences $\left\{ \left( c_t^i \left( s^t \right), l_t^i \left( s^t \right) \right)_{i \in I}, \left\{ q_{s_{t+1}} \left( s^t \right) \right\}_{s_{t+1} \in S} \right\}_{t \geq 0, s^t \in S^t}$ to maximize

$$\sum_{t=0}^{\infty} \sum_{s^t} \sum_{i \in I} \beta^t \pi \left( s^t \right) \left[ u \left( c_t^i \left( s^t \right) \right) - l_t^i \left( s^t \right) \right]$$

subject to resource feasibility,

$$\sum_{i \in I} c_t^i \left( s^t \right) = \sum_{i \in I} l_t^i \left( s^t \right)$$

voluntary participation constraints

$$\sum_{t'=t}^{\infty} \sum_{s^{t'} \succeq s^t} \beta^{t'-t} \left[ u \left( c_{t'}^i \left( s^{t'} \right) \right) - l_{t'}^i \left( s^{t'} \right) \right] \geq V_t^{i,d} \left( s^t, \mathbf{q}_t \right)$$

where $\mathbf{q}_t = \left\{ q_{s_{t'+1}} \left( s^{t'} \right) \right\}_{t' \geq t, s^{t'} \geq s^t}$ is defined by

$$q_{s_{t'+1}} \left( s^{t'} \right) = \max_{i \in I} \left\{ \beta \pi \left( s^{t'+1} \mid s^{t'} \right) \frac{u' \left( c_{t'+1}^i \left( s^{t'+1} \right) \right)}{u' \left( c_{t'}^i \left( s^{t'} \right) \right)} \right\}$$

Let $\mu \left( s^t \right)$ be the multiplier on the resource constraint, $\kappa_{s_{t+1}} \left( s^t \right)$ on the pricing equation and $\eta^i \left( s^t \right)$ on the voluntary participation constraint. The first order condition with respect to $c_t^i \left( s^t \right)$ yields

$$\beta^t \pi \left( s^t \right) u' \left( c_t^i \left( s^t \right) \right) =$$

$$\mu \left( s^t \right) - \sum_{s_{t+1}} \kappa_{s_{t+1}} \left( s^t \right) \mathbf{1}_{q_{s_{t+1}} \left( s^t \right) = \beta \pi \left( s^{t+1} \mid s^t \right) \frac{u' \left( c_{t+1}^i \left( s^{t+1} \right) \right)}{u' \left( c_t^i \left( s^t \right) \right)}} \beta \pi \left( s^{t+1} \mid s^t \right) \frac{u' \left( c_{t+1}^i \left( s^{t+1} \right) \right)}{\left( u' \left( c_t^i \left( s^t \right) \right) \right)^2} u'' \left( c_t^i \left( s^t \right) \right)$$

$$- \sum_{t' \leq t} \eta^i \left( s^{t'} \right) \pi \left( s^t \mid s^{t'} \right) u' \left( c_t^i \left( s^t \right) \right)$$

and with respect to $l_t^i \left( s^t \right)$

$$\beta^t \pi \left( s^t \right) = \mu \left( s^t \right) - \sum_{t' \leq t} \eta^i \left( s^{t'} \right) \pi \left( s^t \mid s^{t'} \right)$$

We can combine these equations to obtain

$$\left[\beta^t \pi\left(s^t\right)+\sum_{t' \leq t} \eta^i\left(s^{t'}\right) \pi\left(s^t \mid s^{t'}\right)\right]=\mu\left(s^t\right)$$

and

$$u'\left(c_t^i\left(s^t\right)\right)=1-$$
$$\frac{1}{\mu\left(s^t\right)} \sum_{s_{t+1}} \kappa_{s_{t+1}}\left(s^t\right) \mathbf{1}_{q_{s_{t+1}}\left(s^t\right)=\beta \pi\left(s^{t+1} \mid s^t\right) \frac{u'\left(c_{t+1}^i\left(s^{t+1}\right)\right)}{u'\left(c_t^i\left(s^t\right)\right)}} \beta \pi\left(s^{t+1} \mid s^t\right) \frac{u'\left(c_{t+1}^i\left(s^{t+1}\right)\right)}{\left(u'\left(c_t^i\left(s^t\right)\right)\right)^2} u''\left(c_t^i\left(s^t\right)\right)$$

Notice that if $s_t=i$, $\kappa_{s_{t+1}}\left(s^t\right) \mathbf{1}_{q_{s_{t+1}}\left(s^t\right)=\beta \pi\left(s^{t+1} \mid s^t\right) \frac{u'\left(c_{t+1}^i\left(s^{t+1}\right)\right)}{u'\left(c_t^i\left(s^t\right)\right)}}>0$ for $s_{t+1}=j$ since

$$\beta \pi\left(s^{t+1} \mid s^t\right) \frac{u'\left(c_{t+1}^i\left(s^t, j\right)\right)}{u'\left(c_t^i\left(s^{t-1}, i\right)\right)} \geq \beta \pi\left(s^{t+1} \mid s^t\right) \frac{u'\left(c_{t+1}^j\left(s^t, j\right)\right)}{u'\left(c_t^j\left(s^{t-1}, i\right)\right)}$$

However, in any competitive equilibrium if $s_t=i$, $u'\left(c_t^i\left(s^t\right)\right)=1$. As a result, any competitive equilibrium is constrained-inefficient. ∎

**Proof of Proposition 20.** The first order conditions from the agent $i'$s problem are

$$\begin{aligned}
l^i\left(s^t\right) &: \beta^t \pi\left(s^t\right)=\sum_{s^{t+1}} \frac{p_{t+1}\left(s^{t+1}\right)}{p_t\left(s^t\right)} \lambda^i\left(s^{t+1}\right) \\
c_t^i\left(s^t\right) &: \beta^t \pi\left(s^t\right) u'\left(c_t^i\left(s^t\right)\right)=\mu^i\left(s^t\right)+\sum_{s^{t+1}} \frac{p_{t+1}\left(s^{t+1}\right)}{p_t\left(s^t\right)} \lambda^i\left(s^{t+1}\right) \\
c_{t+1}^i\left(s^{t+1}\right) &: \beta^{t+1} \pi\left(s^{t+1}\right) u'\left(c_{t+1}^i\left(s^{t+1}\right)\right)=\mu^i\left(s^{t+1}\right)+\sum_{s^{t+2}} \frac{p_{t+1}\left(s^{t+2}\right)}{p_t\left(s^{t+1}\right)} \lambda^i\left(s^{t+2}\right) \\
a_i^i\left(s^t\right) &: 0=\lambda^i\left(s^t\right) q_t^i\left(s^t\right)-\eta_i^i\left(s^t\right)-\lambda^i\left(s^{t+1}\right) \\
m_t^i\left(s^t\right) &: 0=p_t\left(s^t\right) \lambda^i\left(s^t\right)-p_t\left(s^t\right) \mu^i\left(s^t\right)-\sum_{s^{t+1}} p_{t+1}\left(s^{t+1}\right) \lambda^i\left(s^{t+1}\right)
\end{aligned}$$

Therefore

$$\lambda^i\left(s^t\right)=\mu^i\left(s^t\right)+\sum_{s^{t+1}} \frac{p_{t+1}\left(s^{t+1}\right)}{p_t\left(s^t\right)} \lambda^i\left(s^{t+1}\right)$$

which implies

$$\begin{aligned}
\beta^t \pi\left(s^t\right) u'\left(c_t^i\left(s^t\right)\right) &= \lambda^i\left(s^t\right) \\
\beta^{t+1} \pi\left(s^{t+1}\right) u'\left(c_{t+1}^i\left(s^{t+1}\right)\right) &= \lambda^i\left(s^{t+1}\right) \\
&\Rightarrow \pi\left(s^t\right) u'\left(c_t^i\left(s^t\right)\right) q_t^i\left(s^t\right) - \eta_i^i\left(s^t\right) = \beta \pi\left(s^{t+1}\right) u'\left(c_{t+1}^i\left(s^{t+1}\right)\right) \\
&\Rightarrow q_t^i\left(s^t\right) u'\left(c_t^i\left(s^t\right)\right) = \beta \pi\left(s^{t+1} \mid s^t\right) u'\left(c_{t+1}^i\left(s^{t+1}\right)\right) + \frac{\eta_i^i\left(s^t\right)}{\beta^t \pi\left(s^t\right)}
\end{aligned}$$

Consider the allocation $\left\{c_t^i\left(s^t\right), l_t^i\left(s^t\right), a_i^i\left(s^t\right), a_j^i\left(s^t\right), \phi_i^i\left(s^t\right), \phi_j^i\left(s^t\right)\right\}_{s^t, i}$ and prices $\left\{q_t^i\left(s^t\right), q_t^j\left(s^t\right)\right\}_{s^t}$ from the above equilibrium. We want to show that there exist prices $\left\{p_t\left(s^t\right)\right\}$ such that these constitute an equilibrium for the problem above. It suffices to show that the allocations and prices satisfy the above first order conditions along with the TVCs

$$\begin{aligned}
\lim\inf_{t\to\infty} E_{t-1}\left[\beta^t u'\left(c_t^i\left(s^t\right)\right) p_t\left(s^t\right) m_t^i\left(s^t\right)\right] &= 0 \text{ for all } i \\
\lim\inf_{t\to\infty} E_{t-1}\left[\beta^t u'\left(c_t^i\left(s^t\right)\right)\left[a_{s'}^i\left(s^t\right) - \phi_{s'}^i\left(s^t\right)\right]\right] &= 0 \text{ for all } i \text{ and } s'
\end{aligned}$$

Define

$$\frac{p_{t+1}\left(s^{t+1}\right)}{p_t\left(s^t\right)} = \frac{p_{t+1}}{p_t}\frac{1}{Q_t} = \frac{1}{Q} \text{ for all } s^t, s^{t+1}$$

where

$$Q = q^c + q^{nc} = \beta\lambda u'\left(c^l\right) + \beta\left(1-\lambda\right)$$

and so

$$p_t = p_0\left(\frac{1}{Q}\right)^t$$

for some constant $p_0$ to be determined.

Then

$$\begin{aligned}
\mu^i(i) &= \beta^t \pi\left(s^t\right) u'\left(c_t\left(s^t\right)\right) - \sum_{s^{t+1}} \beta^{t+1} \pi\left(s^{t+1}\right) \frac{p_{t+1}\left(s^{t+1}\right)}{p_t\left(s^t\right)} u'\left(c_{t+1}\left(s^{t+1}\right)\right) \\
&= \beta^t \pi\left(s^t\right) u'\left(c_t\left(s^t\right)\right) - \sum_{s^{t+1}} \beta^{t+1} \pi\left(s^{t+1}\right) \frac{p_{t+1}\left(s^{t+1}\right)}{p_t\left(s^t\right)} u'\left(c_{t+1}\left(s^{t+1}\right)\right) \\
&= \beta^t \pi\left(s^t\right) \left[u'\left(c_t\left(s^t\right)\right) - \sum_{s^{t+1}} \beta \pi\left(s^{t+1} \mid s^t\right) \frac{p_{t+1}\left(s^{t+1}\right)}{p_t\left(s^t\right)} u'\left(c_{t+1}\left(s^{t+1}\right)\right)\right] \\
&= \beta^t \pi\left(s^t\right) \left[1 - \beta\lambda \frac{p_{t+1}}{p_t} u'\left(c^l\right) - \beta(1-\lambda) \frac{p_{t+1}}{p_t}\right] \\
&= \beta^t \pi\left(s^t\right) \left[1 - \frac{p_{t+1}}{p_t} \left(\beta\lambda u'\left(c^l\right) + \beta(1-\lambda)\right)\right] \\
&= \beta^t \pi\left(s^t\right) \left[1 - \frac{p_{t+1}}{p_t} Q\right] \\
&= 0
\end{aligned}$$

and so the multiplier on the cash-in-advance constraint is 0.

As a result, the choices for $c$ and $l$ satisfy the first order conditions of the problem. Similarly the first order conditions for Arrow securities are satisfied with $\eta_i^i(i) = \eta_i^i(j) > 0$ and $\eta_i^j(i) = \eta_i^j(j) = 0$.

Finally, we need to verify that the transversality conditions hold. The one for state contingent assets follows directly. Notice from our construction of prices

$$\begin{aligned}
&\lim\inf_{t\to\infty} E_{t-1}\left[\beta^t u'\left(c_t^i\left(s^t\right)\right) p_t\left(s^t\right) m_t^i\left(s^t\right)\right] \\
&= \lim\inf_{t\to\infty} \left[\beta^t \lambda u'\left(c^i(j)\right) p_t(j)\, m_t^i(j) + \beta^t (1-\lambda)\, p_t(i)\, m_t^i(i)\right] \\
&\leq \lim\inf_{t\to\infty} \beta^t p_t \left[\lambda u'\left(c^i(j)\right) + (1-\lambda)\right] M_t \\
&= \lim\inf_{t\to\infty} \beta^{t-1} p_t \left[\beta\lambda u'\left(c^i(j)\right) + \beta(1-\lambda)\right] M_t \\
&= \lim\inf_{t\to\infty} \beta^{t-1} \frac{1}{Q^t} Q M_t \\
&= \lim\inf_{t\to\infty} \left(\frac{\beta}{Q}\right)^t M_t \\
&\leq \lim\inf_{t\to\infty} M_t
\end{aligned}$$

Since $Q \geq \beta$. Choose $p_0$ so that

$$\frac{\left(1+c^l\right)}{\kappa} \geq p_0$$

Then

$$\begin{aligned}
\sum_i c_t^i\left(s^t\right) &\leq p_t\left(s^t\right) M_t \\
&\Leftrightarrow M_t \geq \frac{\sum_i c_t^i\left(s^t\right)}{p_t\left(s^t\right)} \\
&\Leftrightarrow M_t \geq \frac{1+c^l}{p_0\left(\frac{1}{Q}\right)^t} \geq \frac{1+c^l}{p_0\beta^{-t}} \\
&\Leftrightarrow \beta^{-t} M_t \geq \frac{1+c^l}{p_0} \geq \kappa
\end{aligned}$$

As a result, the second assumption ensures that the CIA constraints are always satisfied.

■

**Proof of Lemma 8.** The first order conditions from the agents' problem are

$$\begin{aligned}
l^i\left(s^t\right) &: \beta^t \pi\left(s^t\right) = \sum_{s^{t+1}} \frac{p_{t+1}\left(s^{t+1}\right)}{p_t\left(s^t\right)} \lambda^i\left(s^{t+1}\right) \\
c_t^i\left(s^t\right) &: \beta^t \pi\left(s^t\right) u'\left(c_t^i\left(s^t\right)\right) = \mu^i\left(s^t\right) + \sum_{s^{t+1}} \frac{p_{t+1}\left(s^{t+1}\right)}{p_t\left(s^t\right)} \lambda^i\left(s^{t+1}\right) \\
c_{t+1}^i\left(s^{t+1}\right) &: \beta^{t+1} \pi\left(s^{t+1}\right) u'\left(c_{t+1}^i\left(s^{t+1}\right)\right) = \mu^i\left(s^{t+1}\right) + \sum_{s^{t+2}} \frac{p_{t+1}\left(s^{t+2}\right)}{p_t\left(s^{t+1}\right)} \lambda^i\left(s^{t+2}\right) \\
a_i^i\left(s^t\right) &: 0 = \lambda^i\left(s^t\right) q_t^i\left(s^t\right) - \eta_i^i\left(s^t\right) - \lambda^i\left(s^{t+1}\right) \\
m_t^i\left(s^t\right) &: 0 = p_t\left(s^t\right) \lambda^i\left(s^t\right) - p_t\left(s^t\right) \mu^i\left(s^t\right) - \sum_{s^{t+1}} p_{t+1}\left(s^{t+1}\right) \lambda^i\left(s^{t+1}\right)
\end{aligned}$$

$$\begin{aligned}
\lambda^i\left(s^t\right) &= \mu^i\left(s^t\right) + \sum_{s^{t+1}} \frac{p_{t+1}\left(s^{t+1}\right)}{p_t\left(s^t\right)} \lambda^i\left(s^{t+1}\right) \\
&\Rightarrow \mu^i\left(s^t\right) = \beta^t \pi\left(s^t\right) u'\left(c_t^i\left(s^t\right)\right) - \sum_{s^{t+1}} \beta^{t+1} \pi\left(s^{t+1}\right) \frac{p_{t+1}\left(s^{t+1}\right)}{p_t\left(s^t\right)} u'\left(c_{t+1}^i\left(s^{t+1}\right)\right)
\end{aligned}$$

Combining

$$
\begin{aligned}
&\frac{1}{u'\left(c_t^i\left(s^t\right)\right)} \\
&= \frac{\sum_{s^{t+1}} \frac{p_{t+1}\left(s^{t+1}\right)}{p_t\left(s^t\right)} \lambda^i\left(s^{t+1}\right)}{\mu^i\left(s^t\right) + \sum_{s^{t+1}} \frac{p_{t+1}\left(s^{t+1}\right)}{p_t\left(s^t\right)} \lambda^i\left(s^{t+1}\right)} \\
&= \frac{\sum_{s^{t+1}} \frac{p_{t+1}\left(s^{t+1}\right)}{p_t\left(s^t\right)} \beta^{t+1} \pi\left(s^{t+1}\right) u'\left(c_{t+1}^i\left(s^{t+1}\right)\right)}{\begin{array}{c}\beta^t \pi\left(s^t\right) u'\left(c_t^i\left(s^t\right)\right) - \sum_{s^{t+1}} \beta^{t+1} \pi\left(s^{t+1}\right) \frac{p_{t+1}\left(s^{t+1}\right)}{p_t\left(s^t\right)} u'\left(c_{t+1}^i\left(s^{t+1}\right)\right) \\ + \sum_{s^{t+1}} \frac{p_{t+1}\left(s^{t+1}\right)}{p_t\left(s^t\right)} \beta^{t+1} \pi\left(s^{t+1}\right) u'\left(c_{t+1}^i\left(s^{t+1}\right)\right)\end{array}} \\
&= \frac{p_{t+1}}{p_t} \sum_{s^{t+1}} \beta \pi\left(s^{t+1} \mid s^t\right) \frac{u'\left(c_{t+1}^i\left(s^{t+1}\right)\right)}{u'\left(c_t^i\left(s^t\right)\right)}
\end{aligned}
$$

In this model when $s_t = i$,

$$
\frac{p_{t+1}}{p_t} \sum_{s^{t+1}} \beta^{t+1} \pi\left(s^{t+1} \mid s^t\right) \frac{u'\left(c_{t+1}^i\left(s^{t+1}\right)\right)}{u'\left(c_t^i\left(s^t\right)\right)} = \frac{p_{t+1}}{p_t} Q\left(s^t\right)
$$

Therefore in a stationary equilibrium

$$
\begin{aligned}
u'\left(c_t^i\left(s^t\right)\right) &= \frac{1}{R^m Q} \\
\Rightarrow c^i\left(s^t, i\right) &= c^j\left(s^t, j\right) = g\left(\frac{1}{R^m Q}\right)
\end{aligned}
$$

Then given our conjecture

$$
\begin{aligned}
q^c \frac{1}{R^m Q} &= \beta \lambda u'\left(c_{t+1}^i\left(s^t, j\right)\right) = \beta \lambda u'\left(c_{t+1}^j\left(s^t, i\right)\right) \\
\Rightarrow c_{t+1}^i\left(s^t, j\right) &= c_{t+1}^j\left(s^t, i\right) = g\left(\frac{q^c}{R^m Q \beta \lambda}\right) \\
\Rightarrow l &= g\left(\frac{1}{R^m Q}\right) + g\left(\frac{q^c}{R^m Q \beta \lambda}\right)
\end{aligned}
$$

Clearly $q^{nc} = \beta\left(1-\lambda\right)$ and finally $q^c$ is determined from

$$
\frac{\left(1-\beta\left(1-\lambda\right)\right)\left(u\left(g\left(\frac{1}{R^m Q}\right)\right) - \left[g\left(\frac{1}{R^m Q}\right) + g\left(\frac{q^c}{R^m Q \beta \lambda}\right)\right]\right) + \beta \lambda u\left(g\left(\frac{q^c}{R^m Q \beta \lambda}\right)\right)}{\left[1-\beta\left(1-\lambda\right)\right]^2 - \beta^2 \lambda^2} = V^{i,d}\left(s^t; R^m\right)
$$

■

**Proof of Proposition 21.** Differentiating (3.8) wrt $R^m$

$$\frac{\partial}{\partial R^m} W\left(R^m, q^c\left(R^m\right)\right)$$

$$= \begin{array}{c} q^{c'}\left(R^m\right)\left[g'\left(\frac{1}{R^m Q}\right)\left(-\frac{1}{R^m Q^2}\right)\left[\frac{1}{R^m Q}-1\right]+g'\left(\frac{q^c}{R^m Q\beta\lambda}\right)\left(\frac{1}{R^m\beta\lambda}\frac{Q-q^c}{Q^2}\right)\left[\frac{q^c}{\beta\lambda R^m Q}-1\right]\right] \\ +\left[g'\left(\frac{1}{R^m Q}\right)\left(-\frac{1}{R^{m2} Q}\right)\left[\frac{1}{R^m Q}-1\right]+g'\left(\frac{q^c}{R^m Q\beta\lambda}\right)\left[-\frac{q^c}{R^{m2} Q\beta\lambda}\right]\left[\frac{q^c}{\beta\lambda R^m Q}-1\right]\right] \end{array}$$

$$= \begin{array}{c} q^{c'}\left(R^m\right)\left[g'\left(\frac{1}{R^m Q}\right)\left(\frac{1}{R^m Q^2}\right)\left[1-\frac{1}{R^m Q}\right]+g'\left(\frac{q^c}{R^m Q\beta\lambda}\right)\left(\frac{1}{R^m\beta\lambda}\frac{Q-q^c}{Q^2}\right)\left[\frac{q^c}{\beta\lambda R^m Q}-1\right]\right] \\ +\left[g'\left(\frac{1}{R^m Q}\right)\left(\frac{1}{R^m Q^2}\right)\left[1-\frac{1}{R^m Q}\right]+g'\left(\frac{q^c}{R^m Q\beta\lambda}\right)\left[\frac{q^c}{R^{m2} Q\beta\lambda}\right]\left[1-\frac{q^c}{\beta\lambda R^m Q}\right]\right] \end{array}$$

Therefore, under Friedman Rule i.e. $R^m = \frac{1}{Q}$

$$\left.\frac{\partial}{\partial R^m} W\left(R^m, q^c\left(R^m\right)\right)\right|_{R^m=\frac{1}{Q}}$$
$$= q^{c'}\left(R^m\right)\left[g'\left(\frac{q^c}{\beta\lambda}\right)\left(\frac{1}{\beta\lambda}\frac{Q-q^c}{Q}\right)\left[\frac{q^c}{\beta\lambda}-1\right]\right]+\left[g'\left(\frac{q^c}{\beta\lambda}\right)\left[\frac{q^c}{R^m\beta\lambda}\right]\left[1-\frac{q^c}{\beta\lambda}\right]\right]$$

Define

$$A = \left[g'\left(\frac{q^c}{\beta\lambda}\right)\left(\frac{1}{\beta\lambda}\frac{Q-q^c}{Q}\right)\left[\frac{q^c}{\beta\lambda}-1\right]\right]$$
$$B = \left[g'\left(\frac{q^c}{\beta\lambda}\right)\left[\frac{q^c}{R^m\beta\lambda}\right]\left[1-\frac{q^c}{\beta\lambda}\right]\right]$$

In any constrained equilibrium $q^c > \beta\lambda$ and therefore (note that $g'\left(\frac{q^c}{\beta\lambda}\right) < 0$) $A < 0$ and $B > 0$

Next we need to compute and sign $q^{c'}\left(R^m\right)$. To do this totally differentiate the not-too-tight constraint constraint

$$\frac{\partial}{\partial R^m}\frac{\left(1-\beta\left(1-\lambda\right)\right)\left(u\left(g\left(\frac{1}{R^m Q}\right)\right)-\left[g\left(\frac{1}{R^m Q}\right)+g\left(\frac{q^c}{R^m Q\beta\lambda}\right)\right]\right)+\beta\lambda u\left(g\left(\frac{q^c}{R^m Q\beta\lambda}\right)\right)}{\left[1-\beta\left(1-\lambda\right)\right]^2-\beta^2\lambda^2}$$
$$= \frac{\partial}{\partial R^m} V^{i.d}\left(s^t; R^m\right)$$
$$\Rightarrow \frac{q^{c'}\left(R^m\right) C + D}{\left[1-\beta\left(1-\lambda\right)\right]^2-\beta^2\lambda^2} = \frac{\partial}{\partial R^m} V^{i.d}\left(s^t; R^m\right)$$

where

$$C=\left[\begin{array}{c}\left(1-\beta\left(1-\lambda\right)\right)g'\left(\frac{1}{R^mQ}\right)\left(-\frac{1}{R^mQ^2}\right)\left[\frac{1}{R^mQ}-1\right]\\+g'\left(\frac{q^c}{R^mQ\beta\lambda}\right)\left(\frac{1}{R^m\beta\lambda}\frac{Q-q^c}{Q^2}\right)\left[\frac{q^c}{R^mQ}-\left(1-\beta\left(1-\lambda\right)\right)\right]\end{array}\right]$$

$$D=\left[\begin{array}{c}\left(1-\beta\left(1-\lambda\right)\right)g'\left(\frac{1}{R^mQ}\right)\left(-\frac{1}{R^{m2}Q}\right)\left[\frac{1}{R^mQ}-1\right]\\+g'\left(\frac{q^c}{R^mQ\beta\lambda}\right)\left[-\frac{q^c}{R^{m2}Q\beta\lambda}\right]\left[\frac{q^c}{R^mQ}-\left(1-\beta\left(1-\lambda\right)\right)\right]\end{array}\right]$$

Therefore

$$q^{c'}\left(R^m\right)=\frac{\frac{\partial}{\partial R^m}V^{i.d}\left(s^t;R^m\right)-\frac{D}{\left[1-\beta(1-\lambda)\right]^2-\beta^2\lambda^2}}{\frac{C}{\left[1-\beta(1-\lambda)\right]^2-\beta^2\lambda^2}}$$

At the Friedman Rule

$$q^{c'}\left(R^m\right)\Big|_{R^m=\frac{1}{Q}}=\frac{\frac{\partial}{\partial R^m}V^{i.d}\left(s^t;R^m\right)-\frac{\left[g'\left(\frac{q^c}{\beta\lambda}\right)\left[-\frac{q^c}{Q\beta\lambda}\right]\left[q^c-(1-\beta(1-\lambda))\right]\right]}{1-\beta^2}}{\frac{g'\left(\frac{q^c}{\beta\lambda}\right)\left(\frac{1}{\beta\lambda}\frac{Q-q^c}{Q}\right)\left[q^c-(1-\beta(1-\lambda))\right]}{1-\beta^2}}$$

Signing the terms

$$q^{c'}\left(R^m\right)=\frac{\left(1-\beta^2\right)\underbrace{\frac{\partial}{\partial R^m}V^{i.d}\left(s^t;R^m\right)}_{>0}-\underbrace{\left[g'\left(\frac{q^c}{\beta\lambda}\right)\left[\frac{q^c}{R^m\beta\lambda}\right]\left[\left(1-\beta\left(1-\lambda\right)\right)-q^c\right]\right]}_{<0}}{\underbrace{\left[g'\left(\frac{q^c}{\beta\lambda}\right)\left(\frac{1}{\beta\lambda}\frac{Q-q^c}{Q}\right)\left[q^c-\left(1-\beta\left(1-\lambda\right)\right)\right]\right]}_{>0}}$$

and so $q^{c'}\left(R^m\right)>0$.

Given this we now know that

$$\begin{aligned}&\underbrace{q^{c'}\left(R^m\right)A}_{<0}+\underbrace{B}_{>0}\\&=g'\left(\frac{q^c}{\beta\lambda}\right)\left[\frac{q^c}{\beta\lambda}-1\right]\left[q^{c'}\left(R^m\right)\left(\frac{1}{\beta\lambda}\frac{Q-q^c}{Q}\right)-\frac{q^c}{R^m\beta\lambda}\right]\end{aligned}\tag{B.2}$$

Need to sign $\left[q^{c'}\left(R^m\right)\left(\frac{1}{\beta\lambda}\frac{Q-q^c}{Q}\right)-\frac{q^c}{R^m\beta\lambda}\right]$. Substituting (B.2) we get

$$\left[q^{c'}\left(R^m\right)\left(\frac{1}{\beta\lambda}\frac{Q-q^c}{Q}\right)-\frac{q^c}{R^m\beta\lambda}\right]$$
$$=\frac{\left[1-\beta\left(1-\lambda\right)\right]^2-\beta^2\lambda^2\frac{\partial}{\partial R^m}V^{i.d}\left(s^t;R^m\right)-\left[g'\left(\frac{q^c}{\beta\lambda}\right)\left[\frac{q^c}{R^m\beta\lambda}\right]\left[\left(1-\beta\left(1-\lambda\right)\right)-q^c\right]\right]}{\left[g'\left(\frac{q^c}{\beta\lambda}\right)\left(\frac{1}{\beta\lambda}\frac{Q-q^c}{Q}\right)\left[q^c-\left(1-\beta\left(1-\lambda\right)\right)\right]\right]}\cdot$$
$$\cdot\left(\frac{1}{\beta\lambda}\frac{Q-q^c}{Q}\right)-\frac{q^c}{R^m\beta\lambda}$$
$$>\left[\frac{\left[g'\left(\frac{q^c}{\beta\lambda}\right)\left[\frac{q^c}{R^m\beta\lambda}\right]\left[q^c-\left(1-\beta\left(1-\lambda\right)\right)\right]\right]}{\left[g'\left(\frac{q^c}{\beta\lambda}\right)\left(\frac{1}{\beta\lambda}\frac{Q-q^c}{Q}\right)\left[q^c-\left(1-\beta\left(1-\lambda\right)\right)\right]\right]}\left(\frac{1}{\beta\lambda}\frac{Q-q^c}{Q}\right)-\frac{q^c}{R^m\beta\lambda}\right]$$
$$=\left[\frac{q^c}{R^m\beta\lambda}-\frac{q^c}{R^m\beta\lambda}\right]=0$$

Therefore

$$\underbrace{q^{c'}\left(R^m\right)\left[g'\left(\frac{q^c}{\beta\lambda}\right)\left(\frac{1}{\beta\lambda}\frac{Q-q^c}{Q}\right)\left[\frac{q^c}{\beta\lambda}-1\right]\right]}_{<0}+\underbrace{\left[g'\left(\frac{q^c}{\beta\lambda}\right)\left[\frac{q^c}{R^m\beta\lambda}\right]\left[1-\frac{q^c}{\beta\lambda}\right]\right]}_{>0}>0$$

and

$$\left.\frac{\partial}{\partial R^m}W\left(R^m,q^c\left(R^m\right)\right)\right|_{R^m=\frac{1}{Q}}<0$$

■

**Proof of Proposition 22**

As earlier, I conjecture the form of the equilibrium and then verify that all the equilibrium conditions hold. To construct the guess, I consider an environment which only money can be held and traded. In this setup agent $i$ maximizes (3.1) subject to budget constraints

$$p_t\left(s^t\right)m_t^i\left(s^t\right)\leq\frac{p_t\left(s^t\right)}{p_{t-1}\left(s^{t-1}\right)}l_{t-1}^i\left(s^{t-1}\right)+p_t\left(s^t\right)\left[m_{t-1}^i\left(s^{t-1}\right)-\frac{c_{t-1}^i\left(s^{t-1}\right)}{p_{t-1}\left(s^{t-1}\right)}\right]$$
$$+T_t\left(s^t\right)\text{ if }s_{t-1}=i$$
$$p_t\left(s^t\right)m_t^i\left(s^t\right)\leq p_t\left(s^t\right)\left[m_{t-1}^i\left(s^{t-1}\right)-\frac{c_{t-1}^i\left(s^{t-1}\right)}{p_{t-1}\left(s^{t-1}\right)}\right]+T_t\left(s^t\right)\ \text{ if }s_{t-1}\neq i,$$

cash-in-advance constraints (3.4) and non-negativity constraints. This equilibrium is similar to the one characterized by Scheinkman and Weiss (1986) except that since in general the money supply policy $\left\{M_t\left(s^t\right)\right\}_{t,s^t}$ might be history dependent the equilibrium need not be Markovian. Define $z_t = \frac{\left[m^1_{t-1}\left(s^{t-1}\right)-\frac{c^1_{t-1}\left(s^{t-1}\right)}{p_{t-1}\left(s^{t-1}\right)}\right]}{M_{t-1}\left(s^{t-1}\right)}$ to be the fraction of fiat money held by type $i$ in $s^t$. Notice that this problem is an incomplete markets environment with aggregate uncertainty due the government policy. I prove that there exists an equilibrium in which the aggregate state variables are $\left(s^t, z^t\right) \in S^t \times Z^t$ where $S$ is the exogenous state space as defined before endowed with the discrete topology and $Z = [0,1]$.

Given any agent $i$, let the individual state variables be $\left(s^t, \xi^i_t, z^t\right)$ where $\xi^i_t$ is the amount of fiat money agent $i$ brings over from the previous period. We can analyze the agent's consumption and money holdings using the following dynamic program

$$\begin{aligned} V^i_t\left(s^t, \xi^i_t, z_t\right) = \sup_{l^i_t,\ \xi^i_{t+1}} & u\left(l^i_t + p_t\left(s^t, z_t\right)\xi^i_t - p_t\left(s^t, z_t\right)\xi^j_{t+1} - T_t\right) - l^i_t \\ & + \beta \int_{S\times Z} V_{t+1}\left(s^{t+1}, \xi^i_{t+1}, z^{t+1}\right) Q_{t+1}\left(ds_{t+1}, dz_{t+1}, s^t, z^t\right) \end{aligned}$$

where $Q_{t+1}$ is joint distribution of $s$ and $z$. Notice that

$$z_{t+1}\left(s^t, z^t\right) = \frac{M_t\left(s^t\right) - \xi^j_{t+1}}{M_{t+1}\left(s^{t+1}\right)}$$

and as a result $Q_{t+1}$ computed using the distribution of the money supply policies $\left\{M_t\left(s^t\right)\right\}$. I assume that the Ramsey policy induces a joint distribution $Q$ over $S$ and $Z$ such that it is appropriately measurable and satisfies the Feller property.[1]

Given our assumptions on state space $S$ we can simplify the structure of this problem as follows; If $s_t = i$ then we know from the agents first order conditions $c^i_t\left(s^t\right) = g(1)$. Therefore

$$l^i_t\left(s^t, \xi^i_t, z_t\right) = g(1) + \left[p_t\left(s^t, z_t\right)\xi^j_t - p_t\left(s^t, z_t\right)\xi^j_{t+1}\right] + 2T_t\left(s^t\right)$$

and so

[1] Given any continuous function $f$, $\int f\left(s', z'\right) Q_{t+1}\left(ds', dz', \cdot\right)$ is a continuous function on $S \times Z$.

$$V_t^i\left(s^t,\xi_t^i,z_t\right)=\sup u\left(g\left(1\right)\right)-\left[g\left(1\right)+p_t\left(s^t,z_t\right)\xi_t^j-p_t\left(s^t,z_t\right)\xi_{t+1}^j+2T_t\left(s^t\right)\right]$$
$$+\beta\int_{S\times Z}V_{t+1}\left(s^{t+1},\xi_{t+1}^i,z_{t+1}\left(s^t,z_t\right)\right)Q_{t+1}\left(ds_{t+1},dz_{t+1},s^t,z^t\right)$$

$$V_t^j\left(s^t,\xi_t^j,z_t\right)=\sup_{\xi_{t+1}^j\in\left[0,p_t\left(s^t,z^t\right)\xi_t^j\right]}u\left(p_t\left(s^t,z^t\right)\xi_t^j-p_t\left(s^t,z^t\right)\xi_{t+1}^j-T_t\left(s^t\right)\right)$$
$$+\beta\int_{S\times Z}V_{t+1}\left(s^{t+1},\xi_t^j,z_{t+1}\left(s^t,z_t\right)\right)Q_{t+1}\left(ds_{t+1},dz_{t+1},s^t,z^t\right)$$

The policy correspondence is given by $\gamma^\xi\left(s^t,\xi_t^j,z_t\right)$. The first step of the proof is to show that given continuous pricing functions $p_t\left(s^t,z^t\right)$, there exists a unique sequence of value functions $\left\{V^i\right\}_0$, $\left\{V^j\right\}_0$ and policy functions $\gamma^\xi\left(s^t,\xi_t^j,z_t\right)$ that solve the individual agents' problems. This part of the proof uses arguments developed in Miao (2006).

Let $\mathbb{V}$ denote the set of uniformly bounded and continuous real valued functions on $\mathbb{S}^t\times\mathbb{A}\times\mathbb{Z}^t$ and let $\mathbb{V}^\infty$ denote the set of sequences $v=(v_0,v_1,..)$ of such functions. $\mathbb{V}^\infty$ is a complete metric space with the norm

$$||v||=\sup_{(t.s^t,\xi_t,z^t)}\left|v_t\left(s^t,\xi_t,z^t\right)\right|$$

Define operator $\mathbb{T}$ as follows

$$(\mathbb{T}v)_t\left(s^t,\xi_t,z^t\right)=\max_{l_t,\ \xi_{t+1}}u\left(l_t+p_t\left(s^t,z_t\right)\xi_t-p_t\left(s^t,z_t\right)\xi_{t+1}-T_t\right)-l_t$$
$$+\beta\int_{S\times Z}V_{t+1}\left(s^{t+1},\xi_{t+1}^i,z^{t+1}\right)Q_{t+1}\left(ds_{t+1},dz_{t+1},s^t,z^t\right)$$

In order to apply the contraction mapping theorem I first show $\mathbb{T}v\in\mathbb{V}^\infty$. Boundedness follows. To show continuity, consider a sequence $\left(s^t,\xi_t,z_t,\xi_{t+1}\right)^n\to\left(s^t,\xi_t,z_t,\xi_{t+1}\right)$. Since $S$ is countable with the discrete topology, $\left(s^t\right)^n=s^t$ for $n$ sufficiently large. Given our restriction to continuous pricing functions $p_t\left(s^t,z^t\right)^n\to p_t\left(s^t,z^t\right)$. As a result correspondence $\Gamma$ is continuous. Then first term on the right hand side of the above dynamic program is continuous since $u$ is continuous. Consider second term. For $n$ sufficiently large

$$\begin{aligned}&\int_{S\times Z} v_{t+1}\left(\left(s^{t+1}\right)^n, \left(\xi_{t+1}\right)^n, z_{t+1}^n\right) Q_{t+1}\left(ds_{t+1}, dz_{t+1}, \left(s^t\right)^n, z_t^n\right)\\ &= \int_{S\times Z} v_{t+1}^i\left(\left(s^{t+1}\right)^n, \left(\xi_{t+1}\right)^n, z_{t+1}^n\right) Q_{t+1}\left(ds_{t+1}, dz_{t+1}, s^t, z_t^n\right)\end{aligned}$$

Notice that

$$\begin{aligned}&\left|\begin{array}{c}\int_{S\times Z} v_{t+1}^i\left(\left(s^{t+1}\right)^n, \left(\xi_{t+1}\right)^n, z_{t+1}^n\right) Q_{t+1}\left(ds_{t+1}, dz_{t+1}, s^t, z_t^n\right)\\ -\int_{S\times Z} v_{t+1}^i\left(s^{t+1}, \xi_{t+1}, z_{t+1}\right) Q_{t+1}\left(ds_{t+1}, dz_{t+1}, s^t, z_t\right)\end{array}\right|\\ &\leq \left|\begin{array}{c}\int_{S\times Z} v_{t+1}^i\left(\left(s^{t+1}\right)^n, \left(\xi_{t+1}\right)^n, z_{t+1}^n\right) Q_{t+1}\left(ds_{t+1}, dz_{t+1}, s^t, z_t^n\right)\\ -\int_{S\times Z} v_{t+1}^i\left(s^{t+1}, \xi_{t+1}, z_{t+1}\right) Q_{t+1}\left(ds_{t+1}, dz_{t+1}, s^t, z_t^n\right)\end{array}\right|\\ &+\left|\begin{array}{c}\int_{S\times Z} v_{t+1}^i\left(s^{t+1}, \xi_{t+1}, z_{t+1}\right) Q_{t+1}\left(ds_{t+1}, dz_{t+1}, s^t, z_t^n\right)\\ -\int_{S\times Z} v_{t+1}^i\left(s^{t+1}, \xi_{t+1}, z_{t+1}\right) Q_{t+1}\left(ds_{t+1}, dz_{t+1}, s^t, z_t\right)\end{array}\right|\end{aligned}$$

Since $\mathbb{S}^t \times \mathbb{A} \times \mathbb{Z}^t$ is compact by Tychonoff's theorem, $v_{t+1}$ is uniformly continuous and as a result $v_{t+1}^i\left(\left(s^{t+1}\right)^n, \left(\xi_{t+1}\right)^n, \left(z^{t+1}\right)^n\right) \to v_{t+1}^i\left(s^{t+1}, \xi_{t+1}, z^{t+1}\right)$ uniformly. As a result we can interchange the limit and integrals. Next, the Feller property ensures that second term goes to 0 and $n \to \infty$. Therefore by Maximum theorem, $\mathbb{T}v$ is also continuous and hence $\mathbb{T}v \in \mathbb{V}^\infty$. It is easy to see that the operator satisfies Blackwell's sufficiency conditions. As a result operator $\mathbb{T}$ have a contraction and so by CMT we have unique sequence of functions $v^*$ corresponding policy functions $\gamma^{\xi *}$

The next step is to prove the existence of these pricing functions. Define

$$\Lambda_t = p_t\left(s^t, z_t\right) u'\left(c_t^i\left(\xi_{t+1}\left(s^t, \xi_t^i, z_t\right)\right)\right)$$

Notice that

$$\begin{aligned}&c_t^i\left(\xi_{t+1}\left(s^t, \xi_t^i, z_t\right)\right) = g\left(1\right) \text{ if } s_t = i\\ &c_t^i\left(\xi_{t+1}\left(s^t, \xi_t^i, z_t\right)\right) = \left[p_t\left(s^t, z_t\right)\xi_t^j - p_t\left(s^t, z_t\right)\xi_{t+1}^j\left(s^t, \xi_t^j, z_t\right)\right]\end{aligned}$$

We can write

$$\xi_t^j = M_{t-1}\left(s^{t-1}\right) - M_t\left(s^t\right) z_t\left(s^t\right)$$

Therefore $\Lambda_t$ depends only on $\left(s^t, z_t\right)$. Agent $i'$s Euler condition implies that

$$u'\left(c^i\left(s^t, z\right)\right) p_t\left(s^t, z^t\right) = \beta \int_{S\times Z} u'\left(c^i\left(s^{t+1}, z_{t+1}\right)\right) p\left(s^{t+1}, z^t\right) Q_{t+1}\left(ds_{t+1}, dz_{t+1}, s^t, z^t\right)$$

$$\Rightarrow \Lambda_t\left(s^t, z^t\right) = \beta \int_{S\times Z} \Lambda_{t+1}\left(s^{t+1}, z^{t+1}\right) Q_{t+1}\left(ds_{t+1}, dz_{t+1}, s^t, z^t\right)$$

Let $\mathbb{V}^p$ denote the set of uniformly bounded and continuous real valued functions on $\mathbb{S}^t \times \mathbb{Z}^t$ and let $(\mathbb{V}^p)^\infty$ denote the set of sequences $v = (v_0, v_1, ..)$ of such functions. $(\mathbb{V}^p)^\infty$ is a complete metric space with the norm

$$||v|| = \sup_{(t.s^t, \xi_t, z_t)} \left|v_t\left(s^t, z^t\right)\right|$$

Given $\Lambda \in (\mathbb{V}^p)^\infty$ define operator $\mathbb{K}$

$$(\mathbb{K}\Lambda)_t\left(s^t, z^t\right) = \beta \int_{S\times Z} \Lambda_{t+1}\left(s^{t+1}, z^{t+1}\right) Q_{t+1}\left(ds_{t+1}, dz_{t+1}, s^t, z^t\right)$$

Since $\Lambda$ is bounded $\mathbb{K}\Lambda$ is also bounded. To show continuity consider a sequence $\left(s^{t+1}, z^{t+1}\right)^n \rightarrow \left(s^{t+1}, z^{t+1}\right)$. For $n$ large enough have$\left(s^{t+1}, \left(z^{t+1}\right)^n\right)$. Then

$$\begin{aligned}
&\left| \begin{array}{l} \int_{S\times Z} \Lambda_{t+1}\left(s^{t+1}, z^{t+1}\right)^n Q_{t+1}\left(ds_{t+1}, dz_{t+1}, \left(s^t, z^t\right)^n\right) \\ -\int_{S\times Z} \Lambda_{t+1}\left(s^{t+1}, z^{t+1}\right) Q_{t+1}\left(ds_{t+1}, dz_{t+1}, s^t, z^t\right) \end{array} \right| \\
&\leq \left| \begin{array}{l} \int_{S\times Z} \Lambda_{t+1}\left(s^{t+1}, \left(z^{t+1}\right)^n\right) Q_{t+1}\left(ds_{t+1}, dz_{t+1}, \left(s^t, z^t\right)^n\right) \\ -\int_{S\times Z} \Lambda_{t+1}\left(s^{t+1}, z^{t+1}\right) Q_{t+1}\left(ds_{t+1}, dz_{t+1}, \left(s^t, z^t\right)^n\right) \end{array} \right| \\
&+ \left| \begin{array}{l} \int_{S\times Z} \Lambda_{t+1}\left(s^{t+1}, z^{t+1}\right) Q_{t+1}\left(ds_{t+1}, dz_{t+1}, \left(s^t, z^t\right)^n\right) \\ -\int_{S\times Z} \Lambda_{t+1}\left(s^{t+1}, z^{t+1}\right) Q_{t+1}\left(ds_{t+1}, dz_{t+1}, s^t, z^t\right) \end{array} \right|
\end{aligned}$$

As before, the first term vanishes since $\Lambda$ is uniformly continuous and second term by the Feller property. Thus $\mathbb{K}\Lambda$ is continuous. Since the operator satisfies Blackwell's sufficiency conditions it is a contraction. hence there exists unique fixed point $\Lambda^*$. Then given early result of existence of policy functions can compute pricing functions

$$p\left(s^t, z_t\right) = \frac{\Lambda_t^*\left(s^t, z_t\right)}{u'\left(c\left(s^t, z\right)\right)}$$

The last step in the proof is to show that there exist price functions for Arrow securities such that those along with the above policy and price functions constitute a not-too-tight equilibrium when debt constraints are zero. In this case each agent solves the same problem as described in Section 2.2.2 with

$$a^i_{s'}\left(s^t\right) \geq 0 \text{ for all } s^t, s' \in S$$

Here, an agent can only save in Arrow securities. To have an equilibrium with this property, market clearing dictates that no agent must wish to save in these securities. Therefore, we construct Arrow security prices as follows; for any $\left(s^t, s'\right)$

$$q_{s'}\left(s^t, z^t\right) = \max_i \left\{ \int_{S\times Z} \frac{u'\left(c^i\left(s^{t+1}, z^{t+1}\right)\right)}{u'\left(c^i\left(s^t, z^t\right)\right)} Q_{t+1}\left(ds_{t+1}, dz_{t+1}, s^t, z^t\right) \right\}$$

At these prices, no agent will strictly prefers to save in Arrow securities since for all agents

$$q_{s'}\left(s^t, z^t\right) \geq \int_Z \frac{u'\left(c^i\left(s^{t+1}, z^{t+1}\right)\right)}{u'\left(c^i\left(s^t, z^t\right)\right)} Q_{t+1}\left(s', dz_{t+1}, s^t, z^t\right)$$

Agents would like to borrow but are constrained from doing so. As a result, markets clear which verifies that our guess is indeed an equilibrium. These debt constraints are trivially not-too-tight since in equilibrium there are no Arrow securities being traded.

### B.1.1 Proofs From Section 3.1

#### Proof of Theorem 1

**Proof of Part 1.** Consider a competitive equilibrium of the debt constrained problem. Define

$$\zeta^{i,j}_t\left(s_t\right) = a^i\left(s_t\right) - \sum_{s_{t+1}} q_{s_{t+1}}\left(s^t\right) a^i_{s_{t+1}}\left(s^t\right) \text{ for all } j$$

Now given our proposed $\zeta^{i,j}$ along with the allocation $\left\{\left(c^i_t\left(s^t\right), l^i_t\left(s^t\right), m^i_t\left(s^t\right)\right)_{i\in I}\right\}_{t,s^t}$ from the debt constrained competitive equilibrium it must be that

$$\hat{V}^i_t\left(s^t, \boldsymbol{\zeta}^{i,j}\left(s^t\right); \mathbf{p}_t\right) = V^{i,c}_t\left(s^t, a^i_{s_t}\left(s^{t-1}\right); \Phi^i\left(s^t\right)\right)$$

where the term on the left hand side is defined using (4.9) and the term on the right hand side using (4.12) . This is because given the construction of $\zeta^{i,j}$ and the fact that the allocation satisfies the agent's optimality conditions in the debt constrained environment, it must be that the best the agent can do given the insurance contract is what she does in the debt constrained environment. Since the debt limits are chosen to be not-too-tight the allocation $\left\{\left(c_t^i\left(s^t\right), l_t^i\left(s^t\right), m_t^i\left(s^t\right)\right)_{i\in I}\right\}_{t,s^t}$ along with $\zeta_t^{i,j}$ constructed above satisfies incentive compatibility and so is feasible for the intermediary given the prices $\left\{q_{s_{t+1}}\left(s^t\right)\right\}_{s_{t+1},s^t}$.

Suppose that there exists an allocation that is feasible and gives the intermediary strictly higher profit. Note that perfect competition implies that such an allocation must increases both the ex-ante welfare of the firm and the agent. The only way an intermediary can do is to increase the amount of insurance provided to the agent. Consider any such contract with greater insurance that satisfies incentive compatibility.

Then it must be that for some $i, s^t, s^{t+1}$,

$$q_{s_{t+1}}\left(s^t\right) > \frac{\beta\pi\left(s^{t+1} \mid s^t\right) u'\left(c_{t+1}^i\left(s^{t+1}\right)\right)}{u'\left(c_t^i\left(s^t\right)\right)} \tag{B.3}$$

and

$$V_t^{i,c}\left(s^{t+1}, a_{s_{t+1}}^i\left(s^t\right); \Phi^i\left(s^{t+1}\right)\right) > V_{t+1}^{i,d}\left(s^{t+1}\right) \tag{B.4}$$

To see why notice that if (B.3) held with an equality, marginal rates of substitution would be equated amongst all agents and as a result an intermediary cannot offer a contract with greater ex-ante insurance. On the other hand if (B.3) held and (B.4) held with equality increasing the amount of insurance between those states would result in the agent strictly preferring to default in $s_{t+1}$ and hence would violate incentive compatibility. However equations (B.3) and (B.4) contradict the not-too-tightness requirement of debt constraints.

Given this constructed contract, set $\mu^{*i}(C) = \delta_C$ and let all intermediaries $j$ offer only this contract. ■

Next, I prove the converse. Assume first that in the equilibrium $\mu^{*i}(C) = \delta_C$ for some $C \in \mathcal{C}$. The following lemmas will be useful

**Lemma 21** *Consider an equilibrium of the contracting environment. Then*

$$q_{s_{t+1}}\left(s^t\right) \geq \max_{i\in I}\left\{\frac{\beta\pi\left(s^{t+1} \mid s^t\right) u'\left(c_{t+1}^i\left(s^{t+1}\right)\right)}{u'\left(c_t^i\left(s^t\right)\right)}\right\} \text{ for all } s_{t+1} \tag{B.5}$$

*with equality if there is some insurance being offered by the intermediary.*

**Proof of Lemma 21.** I prove this by contradiction. Suppose that in the competitive equilibrium, for some $i, s^t, s^{t+1}$

$$q_{s_{t+1}}\left(s^t\right) < \frac{\beta\pi\left(s^{t+1} \mid s^t\right) u'\left(c_{t+1}^i\left(s^{t+1}\right)\right)}{u'\left(c_t^i\left(s^t\right)\right)} \tag{B.6}$$

Consider the following contract intermediary $s^t$ can offer to this agent

$$\begin{aligned}\tilde{\zeta}_t^{i,j}\left(s^t\right) &= \zeta_t^{i,j}\left(s^t\right) - q_{s_{t+1}}\left(s^t\right)\varepsilon \\ \tilde{\zeta}_{t+1}^{i,j}\left(s^{t+1}\right) &= m_t^{s^t,i}\left(s^{t+1}\right) + \varepsilon\end{aligned}$$

where $\varepsilon > 0$ and with the rest of the contract being unchanged. For $\varepsilon$ small but positive the change in welfare to agent $i$ is

$$\left[-q_{s_{t+1}}\left(s^t\right) u'\left(c_t^i\left(s^{t+1}\right)\right) + \beta\pi\left(s^{t+1} \mid s^t\right) u'\left(c_{t+1}^i\left(s^{t+1}\right)\right)\right]\varepsilon > 0$$

because of (C.1), while the change in intermediary welfare is

$$q_{s_{t+1}}\left(s^t\right)\varepsilon - q_{s_{t+1}}\left(s^t\right)\varepsilon = 0$$

As a result an intermediary can offer a deviating contract and make strictly positive profit which contradicts the definition of a competitive equilibrium.

Therefore,

$$q_{s_{t+1}}\left(s^t\right) \geq \max_{i \in I}\left\{\frac{\beta\pi\left(s^{t+1} \mid s^t\right) u'\left(c_{t+1}^i\left(s^{t+1}\right)\right)}{u'\left(c_t^i\left(s^t\right)\right)}\right\}$$

Now consider an equilibrium in which

$$q_{s_{t+1}}\left(s^t\right) > \max_{i \in I}\left\{\frac{\beta\pi\left(s^{t+1} \mid s^t\right) u'\left(c_{t+1}^i\left(s^{t+1}\right)\right)}{u'\left(c_t^i\left(s^t\right)\right)}\right\}$$

and some insurance is being offered between periods $t$ and $t+1$ . Since insurance is being offered there must exist some agent $j$ such that

$$\hat{V}_t^i\left(s^t, \boldsymbol{\zeta}^{i,j}\left(s^t\right); \mathbf{p}_t, \delta\left(s^{t+1}\right) = 0\right) > \hat{V}_t^i\left(s^t, \boldsymbol{\zeta}^{i,j}\left(s^t\right); \mathbf{p}_t, \delta\left(s^{t+1}\right) = 1\right) \tag{B.7}$$

where $\hat{V}_t^i\left(s^t, \boldsymbol{\zeta}^{i,j}\left(s^t\right); \mathbf{p}_t, \delta\left(s^{t+1}\right) = 0\right)$ is the agent's best deviation conditional on not defaulting while the term on the right hand side is the best deviation conditional on defaulting. Inequality (C.2) says that the value for some agent $j$ (who receives positive transfers) staying in the contract is strictly greater than defaulting. Consider the following contract an intermediary could offer agent $j$

$$\tilde{\zeta}_t^{i,j}\left(s^t\right) = \zeta_t^{i,j}\left(s^t\right) + q_{s_{t+1}}\left(s^t\right)\varepsilon$$
$$\tilde{\zeta}_{t+1}^{i,j}\left(s^{t+1}\right) = m_t^{s^t,i}\left(s^{t+1}\right) - \varepsilon$$

For $\varepsilon$ small the change in welfare is

$$\left[q_{s_{t+1}}\left(s^t\right)u'\left(c_t^i\left(s^{t+1}\right)\right) - \beta\pi\left(s^{t+1} \mid s^t\right)u'\left(c_{t+1}^i\left(s^{t+1}\right)\right)\right]\varepsilon > 0$$

while

$$\hat{V}_t^i\left(s^t, \boldsymbol{\zeta}^{i,j}\left(s^t\right); \mathbf{p}_t, \delta\left(s^{t+1}\right) = 0\right) \geq \hat{V}_t^i\left(s^t, \boldsymbol{\zeta}^{i,j}\left(s^t\right); \mathbf{p}_t, \delta\left(s^{t+1}\right) = 1\right)$$

And so in period $t$ and state $s^t$, the agent is strictly better off under this contract. By a similar argument to above, since such a perturbation is welfare neutral to the intermediary he can offer a contract and make strictly positive profits contradiction the competitive equilibrium assumption. This proves the claim. ■

**Lemma 22** *In the competitive equilibrium, for any* $\left(s^t, s_{t+1}\right)$, *if full insurance is not being provided by the intermediary, then for some* $i$

$$\hat{V}_t^i\left(s^t, \boldsymbol{\zeta}^{i,j}\left(s^t\right); \mathbf{p}_t, \delta\left(s^{t+1}\right) = 0\right) = \hat{V}_t^i\left(s^t, \boldsymbol{\zeta}^{i,j}\left(s^t\right); \mathbf{p}_t, \delta\left(s^{t+1}\right) = 1\right)$$

**Proof of Lemma 22.** Suppose not and that full insurance is not being provided. Then there is some agent $i$, states $s^t$, $s^{t+1}$ such that

$$q_{s_{t+1}}\left(s^t\right) > \frac{\beta\pi\left(s^{t+1} \mid s^t\right)u'\left(c_{t+1}^i\left(s^{t+1}\right)\right)}{u'\left(c_t^i\left(s^t\right)\right)}$$

and by assumption

$$\hat{V}_t^i\left(s^t, \boldsymbol{\zeta}^{i,j}\left(s^t\right); \mathbf{p}_t, \delta\left(s^{t+1}\right) = 0\right) > \hat{V}_t^i\left(s^t, \boldsymbol{\zeta}^{i,j}\left(s^t\right); \mathbf{p}_t, \delta\left(s^{t+1}\right) = 1\right)$$

Then an intermediary at $s^t$ can offer the following contract to agent $i$

$$\tilde{\zeta}_t^{i,j}\left(s^t\right) = \zeta_t^{i,j}\left(s^t\right) + q_{s_{t+1}}\left(s^t\right)\varepsilon$$
$$\tilde{\zeta}_{t+1}^{i,j}\left(s^{t+1}\right) = m_t^{s^t,i}\left(s^{t+1}\right) - \varepsilon$$

with the rest of the contract unchanged. For $\varepsilon$ small enough this contract gives the agent greater utility to the agent while leaving the intermediary equally well off. To see that all the incentive compatibility constraints are still satisfied notice that since $\frac{\partial \hat{V}_{t+1}^{i}\left(s^{t+1},\boldsymbol{\zeta}^{i,j}\left(s^{t}\right);\mathbf{p}_{t}\right)}{\partial \zeta_{t+1}^{i,j}\left(s^{t+1}\right)} > 0$, and $\hat{V}_{t+1}^{i}\left(s^{t+1},\boldsymbol{\zeta}^{i,j}\left(s^{t}\right);\mathbf{p}_{t}\right)$ is continuous for $\varepsilon$ small enough the voluntary participation constraint in $s^{t+1}$ still holds while in $s^{t}$, the value of not defaulting increases. As a result there exists a contract with gives the intermediary positive profit which contradicts the allocation/price being a competitive equilibrium ■

The proof of the part 2 of the theorem relies on a limiting argument due to Fudenberg and Levine (1983). The idea is to construct truncated allocations of the debt

constrained environment, the limit of which converges to an equilibrium with not-too-tight debt constraints. I turn to this construction next.

Let $\left\{\left(c_{t}^{i}\left(s^{t}\right),m_{t}^{i}\left(s^{t}\right),l_{t}^{i}\left(s^{t}\right),\zeta_{t}^{i,j}\left(s^{t}\right),\bar{V}_{t}^{i}\left(s^{t}\right)\right)_{i\in I}\right\}_{t,s^{t}}$ be the allocation associated with the competitive equilibrium in the contracting environment. I first construct a truncated $T-$period allocation for the economy with debt constraints as follows: Define $\left\{a^{T,i}\left(s^{t}\right)\right\}$ using the equations

$$a_{s_{T}}^{T,i}\left(s^{T-1}\right)=\zeta_{T}^{i,j}\left(s^{T}\right) \tag{B.8}$$

$$a_{s_{t}}^{T,i}\left(s^{t-1}\right)-\sum_{s^{t+1}}q_{s_{t+1}}\left(s^{t}\right)a_{s_{t+1}}^{T,i}\left(s^{t}\right)=\zeta_{t}^{i,j}\left(s^{t}\right)\ \text{ for all } t<T \tag{B.9}$$

For all $t<T$, let

$$\phi_{s_{t+1}}^{T,i}\left(s^{t}\right)=a_{s_{t+1}}^{i,T}\left(s^{t}\right)$$

for $i,s^{t+1}$ such that

$$\hat{V}_{t}^{i}\left(s^{t},\boldsymbol{\zeta}^{i,j}\left(s^{t}\right);\mathbf{p}_{t},\delta\left(s^{t+1}\right)=0\right)=\hat{V}_{t}^{i}\left(s^{t},\boldsymbol{\zeta}^{i,j}\left(s^{t}\right);\mathbf{p}_{t},\delta\left(s^{t+1}\right)=1\right) \tag{B.10}$$

where we know that there exists $i$ for an $s^{t},s^{t+1}$ such that the above holds true from lemma 22 .

For $t>T$ define

$$a_{s^{t}}^{T,i}\left(s^{t-1}\right)=0,$$
$$\phi_{s^{t}}^{T,i}\left(s^{t-1}\right)=0$$

Next, for all $t\leq T$, $c_{t}^{T,i}\left(s^{t}\right)=c_{t}^{i}\left(s^{t}\right)$, $l_{t}^{T,i}\left(s^{t}\right)=l_{t}^{i}\left(s^{t}\right)$, $m_{t}^{T,i}\left(s^{t}\right)=\ m_{t}^{i}\left(s^{t}\right)$ and for $t>T$, $c_{t}^{T,i}\left(s^{t}\right)$, $l_{t}^{T,i}\left(s^{t}\right)$, $m_{t}^{T,i}\left(s^{t}\right)$ be the best allocation that can be chosen by agent $i$

given prices and the constructed debt constraints. Clearly the above allocation does not constitute a competitive equilibrium with not-too-tight constraints since in period $T$, given that there is no borrowing and lending the future, no agent will be willing to honor her debt. Let $\Delta^i\left(s^t\right) \in \{0,1\}$ denote the agent's default decision in any period and state. Define the sequence $\left\{\Delta^{T,i}\right\}_0$ where

$$\Delta^{T,i}\left(s^t\right) = 0 \text{ for } t \leq T$$
$$\Delta^{T,i}\left(s^t\right) = 1 \text{ for } t > T$$

For each entrepreneur/agent $i$, define

$$\Gamma^i(T) = \left\{ \begin{array}{c} \left\{c^i, m^i, l^i, a^i, \Delta\right\}_0 : \\ \text{For all } t, s^t,\ \left(c_t^i\left(s^t\right), m_t^i\left(s^t\right), l_t^i\left(s^t\right), \left\{a_{s_{t+1}}^i\left(s^t\right)\right\}_{s_{t+1}}, \Delta^i\left(s^t\right)\right) \\ \text{satisfies equations (4.25), (4.26) and (3.5) and} \\ \text{for } t' > T \text{ correspond to the best allocation given prices and debt constraints} \end{array} \right\}$$

and $\Gamma(T) = \prod_{i\in I} \Gamma^i(T)$. $\Gamma^i(T)$ consists of choices that are budget feasible for the agent and involve living in financial autarky after period $T$. $\Gamma(\infty)$ is the untruncated choice sets for the agent. Clearly, the truncated allocation constructed above in an element of this set for each $i$ given the debt constraints.

Define $x^{i,T} = \left\{c^{T,i}, m^{T,i}, l^{T,i}, a^{i,T}, \Delta^{i,T}\right\}_0$ and $x^{i,*} = \left\{c^i, m^i, l^i, a^i, \Delta^i\right\}_0$ where $\Delta^i\left(s^t\right) = 0$ for all $t$ and $\left\{c^i, m^i, l^i\right\}_0$ correspond to the competitive equilibrium allocation in the contracting problem and $a^i$ is the limit as $T \to \infty$ of asset holdings constructed using (B.8) and (B.9).

For any $i$ consider the best deviation in $\Gamma^i(T)$ from the truncated allocation constructed above given the debt constraints $\left\{\phi^{T,i}\right\}$. From lemma 21 we know that in any equilibrium of the contracting environment $q_{s_{t+1}}\left(s^t\right) \geq \max_{i\in I}\left\{\frac{\beta\pi\left(s^{t+1}|s^t\right)u'\left(c_{t+1}^i\left(s^{t+1}\right)\right)}{u'\left(c_t^i\left(s^t\right)\right)}\right\}$. As a result for all $t < T$, conditional on not defaulting the truncated allocation is optimal for each agent given the debt constraints constructed above. Therefore, the best possible deviation (if one exists) involves default by some agent $i$ in some period $t \leq T$. Define $\varepsilon^{i,T} \geq 0$ to be the value of the best possible deviation for agent $i$ in $\Gamma^i(T)$ for any $t \leq T$,

$$V_t^{i,d}\left(s^t\right) - V_t^{i,c}\left(s^t, a_{s_t}^i\left(s^{t-1}\right); \Phi^i\left(s^t\right)\right) = \varepsilon^{i,T}$$

and let $\varepsilon^T = \max_i \varepsilon^{i,T}$. In particular, $\varepsilon^T$ corresponds to the best possible deviation from the truncated allocation that can be achieved by any player who chooses from choice set $\Gamma^i(T)$. Let

$$X_t = R_+ \times R_+ \times R_+ \times R_+ \times \{0,1\}$$

$X_{t'} = \prod_{t=0}^{t'} X_t$ and $X = \prod_{t=0}^{\infty} X_t$. I have $x^{i,T} \in X$ and $x^{i,*} \in X$. The metric

$$d(x,z) = \sup_t \left[\frac{1}{t}\min\{|x_t - z_t|, 1\}\right]$$

induces the product topology on $X$. We can see that $x^{i,T} \to x^{i,*}$ in this metric.

**Definition 23** *A allocation $\left(\left\{c^{T,i}, m^{T,i}, l^{T,i}, a^{T,i}, \Delta^{T,i}\right\}_0\right)_{i\in I} \in \Gamma(T)$ is an $\varepsilon-$perfect equilibrium ($\varepsilon -$ perfect) if for each $t, s^t$, any agent i's best deviation in $\Gamma^i(T)$ from the prescribed allocation yields her a welfare gain of at most $\varepsilon$.*

Given this definition and the previous discussion we have that $x^T$ is $\varepsilon^T$ perfect in $\Gamma(T)$ and that $x^T \to x^*$. Notice that since the contracting equilibrium allocations satisfy incentive compatibility (4.8), $\varepsilon^T \to 0$ since the agent does not want to deviate. Next, I adapt an argument from Theorem 3.3. in Fudenberg and Levine (1983) and prove that

**Theorem 9** *A sufficient condition that $x^*$ be perfect in $\Gamma(\infty)$ is that there be a sequences $\varepsilon^T, x^T$ such that $x^T$ is $\varepsilon^T-$perfect in $\Gamma(T)$ and as $T \to \infty$, $\varepsilon^T \to 0$ and $x^T \to x^*$.*

Similar to $\varepsilon^T$, I define $w^T$ to be the greatest variation in any agent's payoff due to events after $T-1:$ for any $t, s^t$ let

$$\tilde{V}_t^i\left(s^t, a_{s_t}^i\left(s^{t-1}\right); \Phi^i\left(s^t\right)\right) = \max\left\{V_t^{i,d}\left(s^t\right) V_t^{i,c}\left(s^t, a_{s_t}^i\left(s^{t-1}\right); \Phi^i\left(s^t\right)\right)\right\}$$

and define

$$w^t = \max_{i\in I}\left(\max_{\substack{x_1,x_2\in X\\ x_1,x_2 \text{ feas}\\ x_1(T-1)=x_2(T-1)}} \tilde{V}_0^i\left(s_0, a_0^i; \Phi^i\left(s_0\right)\right)\right)$$

The restriction $x_1(T-1) = x_2(T-1)$ just means that the allocation is identical for all dates and states prior to period $T$.The following two lemmas whose proofs can be found in Fudenberg and Levine (1983) will be useful,

**Lemma 23** *If $x$ is $\varepsilon-$perfect in $\Gamma(T)$ then $x$ is $\left(\varepsilon + w^T\right)$ perfect in $\Gamma(\infty)$*

**Lemma 24** *Let $x$ be $\varepsilon-$perfect in $\Gamma(\infty)$ and $x \to x^*$. Then $x^*$ is also $\varepsilon$-perfect.*

**Proof of Theorem 9.** From lemma 25 we know that $x^T$ is $\varepsilon^T + w^T$-perfect in $\Gamma(\infty)$. Since $\varepsilon^T + w^T \to 0$ (because $x^T \to x^*$) for each $\delta > 0$ there is some $T^*$such that $\varepsilon^{T'} + w^{T'} < \delta$ for all $T' > T^*$. From lemma 26, $x^*$ is $\delta-$perfect in $\Gamma(\infty)$. Since this is true for all $\delta > 0$, $x^*$ is perfect in $\Gamma(\infty)$. ■

The theorem says that the truncated allocations converge to an equilibrium of the model with debt constraints. The last thing which needs to be checked is that the not-too-tight property is satisfied, but this follows from the construction of the debt limits. In particular if for any agent $i$

$$q_{s_{t+1}}\left(s^t\right) > \frac{\beta\pi\left(s^{t+1} \mid s^t\right) u'\left(c^i_{t+1}\left(s^{t+1}\right)\right)}{u'\left(c^i_t\left(s^t\right)\right)}$$

this agent sets $a^i_{s_{t+1}}\left(s^t\right) = \phi^i_{s_{t+1}}\left(s^t\right) < 0$. But then then from (B.10) I see that

$$\hat{V}^i_t\left(s^t, \boldsymbol{\zeta}^{i,j}\left(s^t\right); \mathbf{p}_t, \delta\left(s^{t+1}\right) = 0\right) = \hat{V}^i_t\left(s^t, \boldsymbol{\zeta}^{i,j}\left(s^t\right); \mathbf{p}_t, \delta\left(s^{t+1}\right) = 1\right)$$

which proves that the debt constraints are not-too-tight.

Recall that the proof assumed that $\mu^{*i}$ was a dirac measure. If in equilibrium, multiple contracts are offered, the proof proceeds in an identical fashion except that the debt constraints associated with each contract will be associated with a corresponding measure of agents in the debt-constrained environment.

**Proof of Proposition 23.** Suppose not. There are two possibilities to consider. The first is one in which in equilibrium all agents are eligible for this savings scheme i.e $\chi(C) = 1$ and $\mu^{i*}(C) = 1$ for all $i \in I$. The only relevant case we need to consider is if for any $t, s^t$ $R^g_t\left(s^t\right) > \frac{p_t\left(s^t\right)}{p_{t-1}\left(s^t\right)}$ and $\tau_t\left(s^t\right) > 0$. It is straightforward to notice that this savings scheme is equivalent to a subsidy on labor. To see this notice since all productive agents strictly prefer to hold their output with government, the optimal contract must satisfy

$$l^i\left(s^t\right) : \beta^t\pi\left(s^t\right) = R^g \sum_{s^{t+1}} \lambda^i\left(s^{t+1}\right)$$

$$\beta^t\pi\left(s^t\right) u'\left(c^i_t\left(s^t\right)\right) = \frac{1}{\beta} \sum_{s^{t+1}} \lambda^i\left(s^{t+1}\right)$$

Here I have assumed that $R_t^g\left(s^t\right)$ can always be chosen so that the CIA constraint is slack.

Therefore

$$\frac{1}{u'\left(c_t^i\left(s^{t-1},i\right)\right)} = \beta R^g$$

and so one can reinterpret this as the agents believing that they now have a production technology that transforms one unit of labor into $\beta R^g > 1$ units of output. If this scheme is offered on the equilibrium path, all productive agents will choose it and as a result while the total amount of consumption goods produced in $t, s^t$ is $l_t^i\left(s^t\right)$, the amount of consumption goods owed by the government owes is

$$\sum_{i\in I}\int_{\xi\in\mathcal{C}}\chi_{s^t}\left(\xi\right)R_t^g\left(s^t\right)l_{t-1}^{\xi,i}\left(s^{t-1}\right)d\mu_{s^t}^{i*}\left(\xi\right) = \sum_{i\in I}R_t^g\left(s^t\right)l_{t-1}^{C,i}\left(s^{t-1}\right)$$

which is strictly greater than the total amount of consumption goods in the economy $\sum_{i\in I}\frac{p_t\left(s^t\right)}{p_{t-1}\left(s^{t-1}\right)}l_{t-1}^{C,i}\left(s^{t-1}\right)$. As a result if all contracts offered in equilibrium are eligible, such a scheme is never feasible.

Now consider a second possibility in which not all contracts offered in equilibrium are eligible for the scheme. As a result policy may now satisfy feasibility. For any contract $\hat{C}$ such that $\chi\left(\hat{C}\right) = 1$, it must be that if $\chi\left(\hat{C}\right) = 0$, the contract is no longer feasible. Else we could just set $\chi\left(\hat{C}\right) = 0$ for all such contracts and so vacuously $\tau_t\left(s^t\right) = 0$. As a result we can restrict to the case in which all contracts eligible for the scheme offer strictly greater insurance than those that are not. Therefore all agents will strictly prefer signing with firms offering eligible contracts. Since intermediaries can only distinguish types but not within types, as in the previous case all agents will be eligible for the scheme which is infeasible. ■

# Appendix C

# Appendix to Chapter 4

## C.1 Proofs From the Main Text

**Proof of Lemma 21.** I prove this by contradiction. Suppose that in the competitive equilibrium, for some $i, s^t, s^{t+1}$

$$q\left(s^{t+1} \mid s^t\right) < \frac{\beta\pi\left(s^{t+1} \mid s^t\right) u'\left(c^i_{t+1}\left(s^{t+1}\right)\right)}{u'\left(c^i_t\left(s^t\right)\right)} \tag{C.1}$$

Consider the following contract intermediary $s^t$ can offer to this agent

$$\begin{aligned}
\tilde{m}^{s^t,i}_t\left(s^t\right) &= m^{s^t,i}_t\left(s^t\right) - q\left(s^{t+1} \mid s^t\right)\varepsilon \\
\tilde{m}^{s^t,i}_{t+1}\left(s^{t+1}\right) &= m^{s^t,i}_t\left(s^{t+1}\right) + \varepsilon
\end{aligned}$$

where $\varepsilon > 0$ and with the rest of the contract being unchanged. For $\varepsilon$ small but positive the change in welfare to agent $i$ is

$$\left[-q\left(s^{t+1} \mid s^t\right) u'\left(c^i_t\left(s^{t+1}\right)\right) + \beta\pi\left(s^{t+1} \mid s^t\right) u'\left(c^i_{t+1}\left(s^{t+1}\right)\right)\right]\varepsilon > 0$$

because of (C.1), while the change in intermediary welfare is

$$q\left(s^{t+1} \mid s^t\right)\varepsilon - q\left(s^{t+1} \mid s^t\right)\varepsilon = 0$$

As a result an intermediary can offer a deviating contract and make strictly positive profit which contradicts the definition of a competitive equilibrium.

Therefore,

$$q\left(s^{t+1} \mid s^t\right) \geq \max_{i \in I} \left\{ \frac{\beta\pi\left(s^{t+1} \mid s^t\right) u'\left(c^i_{t+1}\left(s^{t+1}\right)\right)}{u'\left(c^i_t\left(s^t\right)\right)} \right\}$$

Now consider an equilibrium in which

$$q\left(s^{t+1} \mid s^t\right) > \max_{i \in I} \left\{ \frac{\beta\pi\left(s^{t+1} \mid s^t\right) u'\left(c^i_{t+1}\left(s^{t+1}\right)\right)}{u'\left(c^i_t\left(s^t\right)\right)} \right\}$$

and some insurance is being offered between periods $t$ and $t+1$ . Since insurance is being offered there must exist some agent $j$ such that

$$\hat{V}^j_t\left(s^t, m^{s^t,j}; k^j_t\left(s^{t-1}\right), \delta\left(s^{t+1}\right) = 0\right) > \hat{V}^j_t\left(s^t, m^{s^t,j}; k^j_t\left(s^{t-1}\right), \delta\left(s^{t+1}\right) = 1\right) \qquad \text{(C.2)}$$

where $\hat{V}^j_t\left(s^t, m^{s^t,j}; k^j_t\left(s^{t-1}\right), \delta\left(s^{t+1}\right) = 0\right)$ is the agent's best deviation conditional on not defaulting while the term on the right hand side is the best deviation conditional on defaulting. Inequality (C.2) says that the value for some agent $j$ (who receives positive transfers) staying in the contract is strictly greater than defaulting. Consider the following contract an intermediary could offer agent $j$

$$\begin{aligned} \tilde{m}^{s^t,j}_t\left(s^t\right) &= m^{s^t,j}_t\left(s^t\right) + q\left(s^{t+1} \mid s^t\right)\varepsilon \\ \tilde{m}^{s^t,j}_{t+1}\left(s^{t+1}\right) &= m^{s^t,j}_{t+1}\left(s^{t+1}\right) - \varepsilon \end{aligned}$$

For $\varepsilon$ small the change in welfare is

$$\left[q\left(s^{t+1} \mid s^t\right) u'\left(c^i_t\left(s^{t+1}\right)\right) - \beta\pi\left(s^{t+1} \mid s^t\right) u'\left(c^i_{t+1}\left(s^{t+1}\right)\right)\right]\varepsilon > 0$$

while

$$\hat{V}^j_t\left(s^t, \tilde{m}^{s^t,j}; k^j_t\left(s^{t-1}\right), \delta\left(s^{t+1}\right) = 0\right) \geq \hat{V}^j_t\left(s^t, \tilde{m}^{s^t,j}; k^j_t\left(s^{t-1}\right), \delta\left(s^{t+1}\right) = 1\right)$$

And so in period $t$ and state $s^t$, the agent is strictly better off under this contract. By a similar argument to above, since such a pertubation is welfare neutral to the intermediary, it can offer a contract and make strictly positive profits contradiction the competitive equilibrium assumption. This proves the claim. ■

**Proof of Lemma 22.** Suppose not and that full insurance is not being provided. Then there is some agent $i$, states $s^t$, $s^{t+1}$ such that

$$q\left(s^{t+1} \mid s^t\right) > \frac{\beta\pi\left(s^{t+1} \mid s^t\right) u'\left(c^i_{t+1}\left(s^{t+1}\right)\right)}{u'\left(c^i_t\left(s^t\right)\right)}$$

and by assumption

$$\hat{V}_t^i\left(s^t, \tilde{m}^{s^t,j}; k_t^j\left(s^{t-1}\right), \delta\left(s^{t+1}\right)=0\right) > \hat{V}_t^i\left(s^t, \tilde{m}^{s^t,j}; k_t^j\left(s^{t-1}\right), \delta\left(s^{t+1}\right)=1\right)$$

Then an intermediary at $s^t$ can offer the following contract to agent $i$

$$\begin{aligned}\tilde{m}_t^{s^t,i}\left(s^t\right) &= m_t^{s^t,i}\left(s^t\right) + q\left(s^{t+1} \mid s^t\right)\varepsilon \\ \tilde{m}_{t+1}^{s^t,i}\left(s^{t+1}\right) &= m_{t+1}^{s^t,i}\left(s^{t+1}\right) - \varepsilon\end{aligned}$$

with the rest of the contract unchanged. For $\varepsilon$ small enough this contract gives the agent greater utility to the agent while leaving the intermediary equally well off. To see that all the incentive compatibility constraints are still satisfied notice that since $\frac{\partial \hat{V}_{t+1}^i\left(s^{t+1}, k_{t+1}, m_t^i\right)}{\partial m^i\left(s^{t+1}\right)} > 0$, and $\hat{V}_{t+1}^i\left(s^{t+1}, k_{t+1}, m_t^i\right)$ is continuous for $\varepsilon$ small enough the voluntary participation constraint in $s^{t+1}$ still holds while in $s^t$, the value of not defaulting increases. As a result there exists a contract with gives the intermediary positive profit which contradicts the allocation/price being a competitive equilibrium ■

## Proof of Theorem 9

Similar to $\varepsilon^T$, I define $w^T$ to be the greatest variation in any agent's payoff due to events after $T-1$ : for any $t, s^t$ let

$$\tilde{V}_t^i\left(s^t, k_t^i\left(s^{t-1}\right), a^i\left(s^t\right); \Phi^i\left(s^t\right)\right) = \max\left\{V_t^{i,d}\left(s^t, k_t^i\left(s^{t-1}\right)\right) V_t^{i,c}\left(s^t, k_t^i\left(s^{t-1}\right), a^i\left(s^t\right); \Phi^i\left(s^t\right)\right)\right\}$$

and define

$$w^t = \max_{i \in I}\left(\max_{\substack{x_1, x_2 \in X \\ x_1, x_2 \text{ feas} \\ x_1(T-1)=x_2(T-1)}} \tilde{V}_0^i\left(s_0, k_0^i, a_0^i; \Phi^i\left(s_0\right)\right)\right)$$

The restriction $x_1(T-1) = x_2(T-1)$ just means that the allocation is identical for all dates and states prior to period $T$.The following two lemmas whose proofs can be found in Fudenberg and Levine (1983) will be useful,

**Lemma 25** *If $x$ is $\varepsilon-$perfect in $\Gamma(T)$ then $x$ is $\left(\varepsilon + w^T\right)$ perfect in $\Gamma(\infty)$*

**Lemma 26** *Let $x$ be $\varepsilon-$perfect in $\Gamma(\infty)$ and $x \to x^*$. Then $x^*$ is also $\varepsilon$-perfect.*

**Proof of Theorem 9.** From lemma 25 we know that $x^T$ is $\varepsilon^T + w^T$-perfect in $\Gamma(\infty)$. Since $\varepsilon^T + w^T \to 0$ (because $x^T \to x^*$) for each $\delta > 0$ there is some $T^*$such that $\varepsilon^{T'} + w^{T'} < \delta$ for all $T' > T^*$. From lemma 26, $x^*$ is $\delta-$perfect in $\Gamma(\infty)$. Since this is true for all $\delta > 0$, $x^*$ is perfect in $\Gamma(\infty)$. ■

**Proof of Proposition 27.** Consider the following problem for the entrepreneur in which she is not allowed to borrow and lend, but is subject to a state contingent transfer $\xi > 0$[1].

$$\begin{aligned}
&\max_{\{c,k\}} \sum_{t=0}^{\infty} \beta^t u(c_t) \\
&s.t. \\
&c_t + k_{t+1} = r(A_t, w_t) k_t - \xi 1_{A_t = A^h} + \xi 1_{A_t = A^l}
\end{aligned}$$

An equilibrium is defined in a straightforward manner with the hand to mouth workers. It is easy to show that this economy has a stationary equilibrium $\left(c^h(\xi), c^l(\xi), k^h(\xi), k^l(\xi)\right)$ where$\left(c^l(\xi), k^h(\xi)\right)$ is chosen by the agent with high current productivity and $\left(c^l(\xi), k^h(\xi)\right)$ by the entrepreneur with low current productivity. Let $\left(c^h(0), c^l(0), k^h(0), k^l(0)\right)$ be the autarkic stationary allocation.

Define the value for the high agent in the stationary equilibrium as $V\left(h, k^h(\xi), \xi\right)$.

Next, I consider an alternate world without these exogenous state contingent transfers and consider the relative payoffs for the agent with high productivity in both environments. Let the value of autarky for an agent starting off with capital stock $k^h(\xi)$ with her current productivity is high be $V\left(h, k^h(\xi), 0\right)$. Define the map

$$\Lambda(\xi) = V\left(h, k^h(\xi), \xi\right) - V\left(h, k^h(\xi), 0\right)$$

We are interested in how this difference changes $\xi$ changes. Consider the effect of increasing $\xi$ around the autarkic allocation. Totally differentiating $\Lambda$,

$$\begin{aligned}
&V_k\left(h, k^h(0), 0\right) k^{h\prime}(0) + V_\xi\left(h, k^h(0), 0\right) - V_k\left(h, k^h(0), 0\right) \\
&= V_\xi\left(h, k^h(0), 0\right)
\end{aligned}$$

[1] One can even think of this as an endowment shock.

Notice that$V_\xi\left(h, k^h(0), 0\right)$ measures the change in the continuation value for the agent when $\xi$ is increased slightly if the her capital holdings was fixed at the autarkic level. This can be written as

$$\begin{aligned} V_\xi\left(h, k^h(0), 0\right) &= \frac{d}{d\xi}\left(u\left(c^h(0)-\xi\right)+\beta u\left(c^l(0)+\xi\right)\right)\Big|_{\xi=0} \\ &= -u'\left(c^h(0)\right)+\beta u'\left(c^l(0)\right) \end{aligned}$$

Since $\frac{u'\left(c^h(0)\right)}{\beta u'\left(c^l(0)\right)} = r\left(A^l, w^{aut}\right) < 1,\ V_\xi\left(h, k^h(0), 0\right) > 0$ for $\xi$ small but positive and so for this $\xi,\ \Lambda(\xi) > 0$. Also for $\xi$ large enough $\Lambda(\xi) < 0$. Since $\Lambda$ is continuous in $\xi$, by the intermediate value theorem there exists some $\bar{\xi}$ such that $\Lambda\left(\bar{\xi}\right) = 0$.

One can easily reinterpret this setup as one with not-too-tight collateral constraints. Set

$$\begin{aligned} R &= r\left(A^l, w\right) \\ q &= \frac{1}{R} \\ a = \phi &= \frac{\bar{\xi}}{1-q} \end{aligned}$$

And so the entrepreneur with high productivity lends $\phi$ to the agent with low productivity. Given our construction and the fact that $a - qa = \bar{\xi}$, I know that $\phi$ is not-too-tight in that the high productivity agent who owes $\phi$, is indifferent between defaulting and paying back. ■

**Proof of Proposition 24.** To prove this result, we will construct prices and allocations so that these along with the no insurance contract constitute a competitive equilibrium.

First consider an environment in which agents accumulate capital and hire labor but do not have access to financial markets. Given a sequence of wage rates $\left\{w_t\left(s^t\right)\right\}$, each agent solves

$$\max_{\{c^i, n^i, a^i\}_0} \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi\left(s^t\right)\left[u\left(c_t^i\left(s^t\right) - v\left(n_t^i\left(s^t\right)\right)\right)\right] \tag{C.3}$$

subject to

$$c_t^i\left(s^t\right) + k_{t+1}^i\left(s^t\right) \leq r\left(w_t\left(s^t\right), A_t^i\left(s^t\right)\right) k_t^i\left(s^{t-1}\right) + w_t\left(s^t\right) n_t^i\left(s^t\right) \tag{C.4}$$

An equilibrium for this setup is defined in the standard fashion and exists. Denote the solution to it with superscript $a$. Define a sequence of Arrow security prices as follows

$$q^a\left(s^{t+1} \mid s^t\right) = \max_{i\in I}\left\{\frac{\beta u'\left(c_{t+1}^{a,i}\left(s^{t+1}\right)\right)}{u'\left(c_t^{a,i}\left(s^t\right)\right)}\right\}$$

Now consider the contracting environment in which intermediaries can borrow and lend with each other at prices$\left\{q^a\left(s^{t+1} \mid s^t\right)\right\}_{s^{t+1},s^t}$ and the wage rates are $\left\{w_t^a\left(s^t\right)\right\}_{t,s^t}$. We will show that the allocation constructed above form an equilibrium in this environment and in particular intermediaries offer no insurance to agents. Consider the period 0 intermediary's problem (the multipliers on the constraints are in parentheses)

$$\max_{\{c,k,n,m\}}\sum_i\sum_{t=0}^{T}\sum_{s^t}\prod_{t'=1}^{t} q^a\left(s^{t'} \mid s^{t'-1}\right) m_{t'}^i\left(s^{t'}\right)$$

$s.t.$

$$\left(\lambda^i\left(s^t\right)\right): c_t^i\left(s^t\right) + k_{t+1}^i\left(s^t\right) \le r\left(w_t\left(s^t\right), A_t^i\left(s^{t-1}\right)\right) k_t^i\left(s^{t-1}\right) + w_t\left(s^t\right) n_t^i\left(s^t\right) - m_t^i\left(s^t\right)$$

$$\left(\beta^t\pi\left(s^t\right)\mu^i\left(s^t\right)\right): \sum_{t'=t}^{\infty}\sum_{s^{t'}\succeq s^t}\beta^{t'-t}\pi\left(s^{t'} \mid s^t\right)\left[u\left(c_{t'}^i\left(s^{t'}\right) - v\left(n_{t'}^i\left(s^{t'}\right)\right)\right)\right] \ge V_t^d\left(s^t, k_t^i\left(s^{t-1}\right)\right)$$

$$\left(\nu^i\right): \sum_{t'=0}^{\infty}\sum_{s^{t'}}\beta^{t'-t}\pi\left(s^{t'}\right)\left[u\left(c_{t'}^i\left(s^{t'}\right) - v\left(n_{t'}^i\left(s^{t'}\right)\right)\right)\right] \ge \bar{V}$$

$$(\eta): \sum_i\sum_{t=0}^{T}\sum_{s^t}\prod_{t'=1}^{t} q^a\left(s^{t'} \mid s^{t'-1}\right) m_{t'}^i\left(s^{t'}\right) \ge 0$$

As in Kehoe and Perri (2002), we can define $M^i\left(s^t\right) = M^i\left(s^{t-1}\right) + \mu^i\left(s^t\right)$ and $M^i\left(s_0\right) =$

$\nu^i + \mu^i(s_0)$. Then we can re-write the problem using a "partial" Lagrangian,

$$\max_{\{c,k,n,m\}} \sum_i \sum_{t=0}^{T} \sum_{s^t} \prod_{t'=1}^{t} q^a\left(s^{t'} \mid s^{t'-1}\right) m_{t'}^i\left(s^{t'}\right)$$
$$+\sum_i \sum_{t=0}^{T} \sum_{s^t} \beta^t \pi\left(s^t\right) M^i\left(s^t\right)\left[u\left(c_t^i\left(s^t\right) - v\left(n_t^i\left(s^t\right)\right)\right)\right]$$
$$-\sum_i \sum_{t=0}^{T} \sum_{s^t} \beta^t \pi\left(s^t\right) \mu^i\left(s^t\right) V_t^d\left(s^t, k_t^i\left(s^{t-1}\right)\right) - \sum_i \nu^i \bar{V}$$

subject to

$$\left(\lambda^i\left(s^t\right)\right) : c_t^i\left(s^t\right) + k_{t+1}^i\left(s^t\right) \leq r\left(w_t\left(s^t\right), A_t^i\left(s^{t-1}\right)\right) k_t^i\left(s^{t-1}\right) + w_t\left(s^t\right) n_t^i\left(s^t\right) - m_t^i\left(s^t\right)$$
$$(\eta) : \sum_i \sum_{t=0}^{T} \sum_{s^t} \prod_{t'=1}^{t} q^a\left(s^{t'} \mid s^{t'-1}\right) m_{t'}^i\left(s^{t'}\right) \geq 0$$

The necessary first order conditions of the problem are

$$0 = \lambda^i\left(s^t\right) - \beta^t \pi\left(s^t\right) M^i\left(s^t\right) u'\left(c_t^i\left(s^t\right)\right) \tag{C.5}$$
$$0 = \lambda^i\left(s^{t+1}\right) - \beta^{t+1} \pi\left(s^{t+1}\right) M^i\left(s^{t+1}\right) u'\left(c_{t+1}^i\left(s^{t+1}\right)\right) \tag{C.6}$$
$$0 = -\beta^t \pi\left(s^t\right) M^i\left(s^t\right) v'\left(n_t^i\left(s^t\right)\right) - \lambda^i\left(s^t\right) w_t\left(s^t\right) \tag{C.7}$$
$$0 = \lambda^i\left(s^t\right) - \sum_{s^{t+1}} \left[\begin{array}{c} \lambda^i\left(s^{t+1}\right) r\left(w_{t+1}\left(s^{t+1}\right), A_{t+1}^i\left(s^t\right)\right) \\ -\beta^{t+1}\pi\left(s^{t+1}\right)\mu^i\left(s^{t+1}\right)\frac{\partial V_{t+1}^d\left(s^{t+1}, k_{t+1}^i\left(s^t\right)\right)}{\partial k_{t+1}^i\left(s^t\right)} \end{array}\right] \tag{C.8}$$
$$\prod_{t'=0}^{t} q\left(s^{t'} \mid s^{t'-1}\right) = \lambda^i\left(s^t\right) - \eta \prod_{t'=0}^{t} q\left(s^t \mid s^{t-1}\right) \tag{C.9}$$
$$\prod_{t'=0}^{t+1} q\left(s^{t'} \mid s^{t'-1}\right) = \lambda^i\left(s^{t+1}\right) - \eta \prod_{t'=0}^{t+1} q\left(s^{t'} \mid s^{t'-1}\right) \tag{C.10}$$

Combining (C.5), (C.6), (C.9) and (C.10) we obtain

$$\frac{M^i\left(s^t\right) u'\left(c_t^i\left(s^t\right)\right)}{M^i\left(s^{t+1}\right)\beta\pi\left(s^{t+1} \mid s^t\right) u'\left(c_{t+1}^i\left(s^{t+1}\right)\right)} = \frac{1}{q\left(s^{t+1} \mid s^t\right)} \tag{C.11}$$

Next using (C.8) we get

$$0 = u'\left(c_t^i\left(s^t\right)\right) - \sum_{s^{t+1}} \left[\begin{array}{c} \frac{M^i\left(s^{t+1}\right)}{M^i\left(s^t\right)}\pi\left(s^{t+1} \mid s^t\right)\beta u'\left(c_{t+1}^i\left(s^{t+1}\right)\right) r\left(w_{t+1}\left(s^{t+1}\right), A_{t+1}^i\left(s^t\right)\right) \\ -\frac{\beta^{t+1}\pi\left(s^{t+1}\right)\mu^i\left(s^{t+1}\right)}{\beta^t\pi\left(s^t\right)M^i\left(s^t\right)}\frac{\partial V_t^d\left(s^{t+1}, k_{t+1}^i\left(s^t\right)\right)}{\partial k_{t+1}^i\left(s^t\right)} \end{array}\right] \tag{C.12}$$

Finally, the intra-temporal equation is

$$\frac{u'\left(c_{t+1}^{i}\left(s^{t+1}\right)\right)}{-v'\left(n_{t}^{i}\left(s^{t}\right)\right)}=\frac{1}{w_{t}\left(s^{t}\right)} \tag{C.13}$$

In summary we want to show that autarkic allocation satisfies equations (C.11), (C.12) and (C.13 ). (C.13) follows straight away from the intra-temporal first order condition from the model with debt constraints. Next, consider (C.12) for any set of non-negative multipliers

$$u'\left(c_{t}^{a,i}\left(s^{t}\right)\right)-\sum_{s^{t+1}}\left[\begin{array}{c}\frac{M^{i}\left(s^{t+1}\right)}{M^{i}\left(s^{t}\right)}\pi\left(s^{t+1}\mid s^{t}\right)\beta u'\left(c_{t+1}^{a,i}\left(s^{t+1}\right)\right)r\left(w_{t+1}^{a}\left(s^{t+1}\right),A_{t+1}^{i}\left(s^{t}\right)\right)\\ -\frac{\beta^{t+1}\pi\left(s^{t+1}\right)\mu^{i}\left(s^{t+1}\right)}{\beta^{t}\pi\left(s^{t}\right)M^{i}\left(s^{t}\right)}\frac{\partial V_{t}^{d}\left(s^{t+1},k_{t+1}^{a,i}\left(s^{t}\right)\right)}{\partial k_{t+1}^{a,i}\left(s^{t}\right)}\end{array}\right] \tag{C.14}$$

$$\Rightarrow u'\left(c_{t}^{a,i}\left(s^{t}\right)\right)-\sum_{s^{t+1}}\left[\begin{array}{c}\frac{M^{i}\left(s^{t+1}\right)}{M^{i}\left(s^{t}\right)}\pi\left(s^{t+1}\mid s^{t}\right)\beta u'\left(c_{t+1}^{a,i}\left(s^{t+1}\right)\right)r\left(w_{t+1}^{a}\left(s^{t+1}\right),A_{t+1}^{i}\left(s^{t}\right)\right)\\ -\frac{\beta\pi\left(s^{t+1}\right)\mu^{i}\left(s^{t+1}\right)}{\pi\left(s^{t}\right)M^{i}\left(s^{t}\right)}u'\left(c_{t+1}^{a,i}\left(s^{t+1}\right)\right)r\left(w_{t+1}^{a}\left(s^{t+1}\right),A_{t+1}^{i}\left(s^{t}\right)\right)\end{array}\right] \tag{C.15}$$

where the second line follows from the Envelope Theorem. Then we can rewrite the above expression as

$$u'\left(c_{t}^{a,i}\left(s^{t}\right)\right)-\sum_{s^{t+1}}\left[\pi\left(s^{t+1}\mid s^{t}\right)\beta u'\left(c_{t+1}^{a,i}\left(s^{t+1}\right)\right)r\left(w_{t+1}^{a}\left(s^{t+1}\right),A_{t+1}^{i}\left(s^{t}\right)\right)\right]$$

which equals 0 since the autarkic allocation satisfies the Euler equation of the model with debt constraints. I construct the multipliers as follows: For any $s^{t},s^{t+1}$ for any $i$, such that

$$q^{a}\left(s^{t+1}\mid s^{t}\right)=\frac{\beta\pi\left(s^{t+1}\mid s^{t}\right)u'\left(c_{t+1}^{a,i}\left(s^{t+1}\right)\right)}{u'\left(c_{t}^{a,i}\left(s^{t}\right)\right)}$$

set $\mu^{i}\left(s^{t+1}\right)=0$. Otherwise define $\mu$ recursively so that

$$\frac{M^{i}\left(s^{t}\right)u'\left(c_{t}^{a,i}\left(s^{t}\right)\right)}{M^{i}\left(s^{t+1}\right)\beta\pi\left(s^{t+1}\mid s^{t}\right)u'\left(c_{t+1}^{a,i}\left(s^{t+1}\right)\right)}=\frac{1}{q^{a}\left(s^{t+1}\mid s^{t}\right)}$$

then define $\lambda^{i}\left(s^{t}\right)$ from

$$0=\lambda^{i}\left(s^{t}\right)-\beta^{t}\pi\left(s^{t}\right)M^{i}\left(s^{t}\right)u'\left(c_{t}^{i}\left(s^{t}\right)\right)$$

Then by construction, the autarkic allocation satisfies (C.11). Therefore given our constructed prices, the profit maximizing contract is one that offers no insurance to agents.

■

**Proof of Lemma 11.** For each $i$ define the transfers as follows

$$m_t^i\left(s^t\right) = r\left(w_t\left(s^t\right), A_t^i\left(s^{t-1}\right)\right) k_t^i\left(s^{t-1}\right) + w_t\left(s^t\right) n_t^i\left(s^t\right) - c_t^i\left(s^t\right) - k_{t+1}^i\left(s^t\right)$$

Define the arrow security prices as

$$q\left(s^{t+1} \mid s^t\right) = \max_{i \in I}\left\{\frac{\beta u'\left(c_{t+1}^i\left(s^{t+1}\right)\right)}{c_t^i\left(s^t\right)}\right\}$$

Define the multipliers on the incentive constraint as those corresponding the multipliers on the voluntary participation constraints from the planning problem. Next define the multipliers on the budget constraint to be $\lambda^i\left(s^t\right) = \lambda^c\left(s^t\right) \prod_{t'=0}^{t} q\left(s^{t'} \mid s^{t'-1}\right)$.

To see that the first order conditions of the intermediary's problem is satisfied notice that (C.11) and (C.12) follow from the above construction and the intra-temporal condition from the conditionally efficient planning problem. From Lemma 10, I know that only the most efficient technology is operated. The Euler equation from the planning problem (given that only the most efficient technology is operated) is

$$u'\left(c_t^i\left(s^t\right)\right) - \sum_{s^{t+1}}\left[\begin{array}{c}\frac{M^{c,i}\left(s^{t+1}\right)}{M^{c,i}\left(s^t\right)}\pi\left(s^{t+1} \mid s^t\right)\beta u'\left(c_{t+1}^i\left(s^{t+1}\right)\right) r\left(w_{t+1}\left(s^{t+1}\right), A_{t+1}^i\left(s^t\right)\right) \\ -\frac{\beta^{t+1}\pi\left(s^{t+1}\right)\mu^{c,i}\left(s^{t+1}\right)}{\beta^t\pi\left(s^t\right)M^{c,i}\left(s^t\right)}\frac{\partial V_t^d\left(s^{t+1}, k_{t+1}^i\left(s^t\right)\right)}{\partial k_{t+1}^i\left(s^t\right)}\end{array}\right]$$

But since the multipliers are chosen to be identical, the conditionally efficient allocation satisfies the corresponding first order condition from the intermediary's problem. Finally I need to check that the market clearing conditions are satisfied. The security market clearing follows directly from the resource constraint. To see that the labor market clears, notice that the labor market clearing condition is

$$\sum_{i \in I} \rho\left(w_t\left(s^t\right), A_t^i\left(s^t\right)\right) k_t^i\left(s^{t-1}\right) = \sum_i n_t^i\left(s^t\right)$$

where $\rho\left(w_t\left(s^t\right), A_t^i\left(s^t\right)\right)$ is defined as in (4.1). However this follows from (4.7) since

$$\sum_{i\in I}\rho\left(w_t\left(s^t\right), A_t^i\left(s^t\right)\right)k_t^i\left(s^{t-1}\right)=\sum_i n_t^i\left(s^t\right)$$

$$\Rightarrow\sum_{i\in I}\left(\frac{(1-\alpha)A_t^i\left(s^t\right)}{w_t\left(s^t\right)}\right)^{\frac{1}{\alpha}}k_t^i\left(s^{t-1}\right)=\sum_i n_t^i\left(s^t\right)$$

$$\Rightarrow w_t\left(s^t\right)=\left(\frac{\sum_{i\in I}\left((1-\alpha)A_t^i\left(s^t\right)\right)^{\frac{1}{\alpha}}k_t^i\left(s^{t-1}\right)}{\sum_i n_t^i\left(s^t\right)}\right)^{\frac{1}{\alpha}}$$

■

## C.2 A Note on Computation

One commonly used method to solve DSGE models and look at impulse responses around a deterministic steady state is to linearize the system and use techniques developed by Sims (2002) and other authors. However when the number of non-predetermined (forward looking) variables exceeds the number of explosive eigenvalues of the system, the technique is no longer applicable. Farmer and Khramov (2013) develop a way to append the standard approach to solve the model. The idea is simple- take some of the expectational variables and treat them as exogenous processes. As an example consider a linearized system in which there is one less explosive eigenvalues than forward looking variables (as is true in our model). As shown in Sims (2002), the way to incorporate expectational variables for example $E_t x_{t+1}$ is to define a new variable $y_t = E_t x_{t+1}$ and include a new equation in system $x_{t+1} = y_t + \eta_{t+1}$ where $\eta_t$ is an endogenous shock process. Farmer and Khramov (2013) deal with the indeterminacy issue by redefining $\eta_t$ as an *exogenous* mean zero process and specifying a variance for it. This reduces the dimension of the forward looking variables by one thus allowing us to apply standard techniques. Note that this is only one such solution- for example specifying a different variance will result in different dynamics. In the context of our simple model, the variable I choose to redefine is the expectation of the following periods collateral constraint. In the linearized system associated with (4.23)-(4.31) this would involve defining redefining the $E_t\tilde{\phi}_{t+2}$ term where the expectations is purely in terms of a “sunspot” variable. Our results confirm the intuition laid out in the previous sections that such sunspot equilibria can exist.