Combinatorial constructions motivated by K-theory of Grassmannians

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Rebecca Patrias

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Dedication

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Abstract

Motivated by work of Buch on set-valued tableaux in relation to the K-theory of the Grassmannian, Lam and Pylyavskyy studied six combinatorial Hopf algebras that can be thought of as K-theoretic analogues of the Hopf algebras of symmetric functions, quasisymmetric functions, noncommutative symmetric functions, and the Malvenuto-Reutenauer Hopf algebra of permutations. They described the bialgebra structure in all cases that were not yet known but left open the question of finding explicit formulas for the antipode maps. We give combinatorial formulas for the antipode map in these cases.

Next, using the Hecke insertion of Buch-Kresch-Shimozono-Tamvakis-Yong and the K-Knuth equivalence of Buch-Samuel in place of Robinson-Schensted and Knuth equivalence, we introduce a K-theoretic analogue of the Poirier-Reutenauer Hopf algebra of standard Young tableaux. As an application, we rederive the K-theoretic Littlewood-Richardson rules of Thomas-Yong and of Buch-Samuel.

Lastly, we define a K-theoretic analogue of Fomin's dual graded graphs, which we call dual filtered graphs. The key formula in this definition is $DU - UD = D + I$. Our major examples are K-theoretic analogues of Young's lattice, of shifted Young's lattice, and of the Young-Fibonacci lattice. We suggest notions of tableaux, insertion algorithms, and growth rules whenever such objects are not already present in the literature. We also provide a large number of other examples. Most of our examples arise via two constructions, which we call the Pieri construction and the Möbius construction. The Pieri construction is closely related to the construction of dual graded graphs from a graded Hopf algebra as described by Bergeron-Lam-Li, Lam-Shimozono, and Nzeutchap. The Möbius constructions is more mysterious but also potentially more important as it corresponds to natural insertion algorithms.

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Chapter 1

Introduction

The common theme in this thesis is K-theory combinatorics, so let us first discuss what is meant by this. We begin by describing the connection between classical combinatorics and the cohomology of the Grassmannian.

1.1 Schur functions and cohomology of the Grassmannian

The *Schur functions* $\{s_\lambda\}$ are symmetric functions that can be defined combinatorially as generating functions over semistandard Young tableaux of a fixed shape. (See Section 2.0.1 for details.) They form a basis for the ring of symmetric functions and have connections to representation theory and Schubert calculus; we discuss the latter connection below. We denote the Schur function associated to a Young diagram λ by s_λ.

A natural question to ask is how to expand a product of Schur functions in terms of Schur functions. Since we have this nice combinatorial interpretation of the Schur functions in terms of Young tableaux, we have many combinatorial rules for this multiplication. We call such rules *Littlewood-Richardson rules.* Littlewood-Richardson rules will come up at various points in the following chapters, for example in Corollaries 4.2.4 and 4.2.5. We next discuss the importance of Littlewood-Richardson rules in Schubert calculus.

Recall that the Grassmannian $Gr(k, n)$ is the space of k-planes in $\mathbb{C}^n$. For example, $Gr(1, 3)$ is the space of lines through the origin in $\mathbb{C}^3$ and can be identified with the projective space $\mathbb{P}^2$. The Grassmannian can be decomposed into subsets called *Schubert cells* χ_λ, and the closure of these subsets are called *Schubert varieties*, denoted by $\overline{\chi_\lambda}$. The set of Schubert cells and the set of Schubert varieties are indexed by partitions that fit inside of an $k \times (n-k)$ box and respect the containment order on partitions: $\overline{\chi_\lambda} = \cup_{\mu \subseteq \lambda} \chi_\mu$. For example, for $Gr(3,5)$, we consider Schubert varieties $\overline{\chi}_{\ydiagram{1}}, \overline{\chi}_{\ydiagram{2}}, \overline{\chi}_{\ydiagram{1,1}}, \overline{\chi}_{\ydiagram{1,1,1}}, \overline{\chi}_{\ydiagram{2,1}}, \overline{\chi}_{\ydiagram{2,2}}, \overline{\chi}_{\ydiagram{2,2,1}}$, and $\overline{\chi}_{\ydiagram{2,2,2}}$. An example of a decomposition of a Schubert variety into Schubert cells is $\overline{\chi}_{\ydiagram{2,1}} = \chi_{\ydiagram{2,1}} \cup \chi_{\ydiagram{2}} \cup \chi_{\ydiagram{1,1}} \cup \chi_{\ydiagram{1}} \cup \chi_{\varnothing}$.

We would like to study the Grassmannian, a geometric object, from an algebraic perspective, so we turn to cohomology theory. Both cohomology theory and homology theory assign a group—the *cohomology group* or *homology group*—to our geometric object, which serves as an algebraic invariant. In contrast to homology groups, cohomology groups have a natural product structure. Using this product structure, we now have a *cohomology ring* assigned to our geometric object. We focus on cohomology instead of homology because this product structure makes cohomology a stronger invariant, which allows for a finer differentiaion between objects.

The cohomology ring of the Grassmannian is spanned by *Schubert classes* σ_λ, and there is one such class coming from every Schubert variety $\overline{\chi}_\lambda$. So, for for $Gr(3,5)$, we have Schubert classes $\sigma_{\ydiagram{1}}, \sigma_{\ydiagram{2}}, \sigma_{\ydiagram{1,1}}, \sigma_{\ydiagram{1,1,1}}\sigma_{\ydiagram{2,1}}, \sigma_{\ydiagram{2,2}}, \sigma_{\ydiagram{2,2,1}}$, and $\sigma_{\ydiagram{2,2,2}}$. The interesting question to ask is how to multiply Schubert classes in the cohomology ring and write the result in terms of Schubert classes. It turns out that computing $\sigma_\lambda \cdot \sigma_\mu$ is equivalent to asking how to decompose the intersection of Schubert varieties $\overline{\chi}_\lambda \cap \overline{\chi}_\mu$ into Schubert varieties.

Unsatisfied with having transformed a geometric question into an algebraic question, we now want to rephrase this combinatorially using the Schur functions. Using the fact that the cohomology ring of the Grassmannian $Gr(k,n)$ is isomorphic to a quotient of the ring of symmetric functions, we see that multiplying Schubert classes comes down to multiplying Schur functions, i.e. to Littlewood-Richardson rules. For example, if we want to compute $\sigma_{\ydiagram{2,1}} \cdot \sigma_{\ydiagram{1}}$ in $Gr(3,5)$, we first compute using a Littlewood-Richardson rule that

$$s_{\ydiagram{2,1}} s_{\ydiagram{1}} = s_{\ydiagram{3,1}} + s_{\ydiagram{2,2}} + s_{\ydiagram{2,1,1}}.$$

We then ignore everything that is indexed by partition that does not fit inside of a 3×2 box to get

$$\sigma_{\ydiagram{2,1}} \sigma_{\ydiagram{1}} = \sigma_{\ydiagram{2,2}} + \sigma_{\ydiagram{2,1,1}}.$$

1.2 Stable Grothendieck polynomials and K-theory of the Grassmannian

The *stable Grothendieck polynomials* $\{G_\lambda\}$ are what we call a K-theoretic analogue of the Schur functions because they play the role of Schur functions in K-theory of the Grassmannian. They are symmetric functions in that can be defined combinatorially as generating functions over set-valued semistandard Young tableaux of a fixed shape. A set-valued semistandard tableaux is a Young diagram where boxes are filled with finite, nonempty subsets of positive integers with rows weakly increasing and columns strictly increasing. (See Section 3.9.2 for details.)

As with Schur functions, a useful question is to answer is how to expand a product of stable Grothendieck polynomials in terms of stable Grothendieck polynomials. We will refer to such combinatorial rules as K-theoretic Littlewood-Richardson rules. These rules also have a connection to geometry, which we explain below.

Our goal now is to study K-theory of the Grassmannian, where K-theory is a generalized cohomology theory. Instead of the cohomology ring, we are interested in the K-group of $Gr(n, k)$, which also has a natural product. The K-group is again spanned by K*-classes* indexed by partitions inside of the $k \times (n-k)$ box. We denote the K-class indexed by partition λ by $\mathcal{O}_\lambda$. As before, we are interested in knowing how to multiply K-classes, and we can do this using stable Grothendieck polynomials. The key fact is the ring spanned by $\{\mathcal{O}_\lambda\}$ is isomorphic to a quotient of the ring spanned by $\{G_\lambda\}$.

For example, to compute $\mathcal{O}_{\ydiagram{2,1}} \cdot \mathcal{O}_{\ydiagram{1}}$, we first compute

$$G_{\ydiagram{2,1}} G_{\ydiagram{1}} = G_{\ydiagram{3,1}} + G_{\ydiagram{2,2}} + G_{\ydiagram{2,1,1}} - G_{\ydiagram{3,2}} - G_{\ydiagram{3,1,1}} - G_{\ydiagram{2,2,1}} + G_{\ydiagram{3,2,1}}$$

using the K-theoretic Littlewood-Richardon rules. We then ignore all terms indexed by partitions that do not fit inside of a 3×2 box—reflecting that the K-theory ring isomorphic to a quotient of the ring of stable Grothendieck polynomials—to see that

$$\mathcal{O}_{\ydiagram{2,1}} \cdot \mathcal{O}_{\ydiagram{1}} = \mathcal{O}_{\ydiagram{2,2}} + \mathcal{O}_{\ydiagram{2,1,1}} - \mathcal{O}_{\ydiagram{2,2,1}}.$$

Because of this property, we call the stable Grothendieck polynomials *polynomial representatives* for the K-theory of $Gr(k, n)$. Similarly, we call the Schur functions polynomial representatives for the cohomology of $Gr(k, n)$.

Let $|\lambda|$ denote the number of boxes in λ. Notice that the terms in $G_\lambda G_\mu$ and $\mathcal{O}_\lambda \cdot \mathcal{O}_\mu$ indexed by some ν with $|\nu| = |\lambda| + |\mu|$ coincide with the terms of $s_\lambda s_\mu$ and $\sigma_\lambda \cdot \sigma_\mu$. This shows that the information about the cohomology can be recovered from the information about the K-theory.

1.3 Overview

This relationship between the stable Grothendieck polynomials and the K-theory of the Grassmannian is why the related combinatorics is called K-theory combinatorics. As we will see, many things from classical combinatorics can be naturally tweaked and transferred to this K-theoretic setting. In this thesis, we focus on three of these generalizations: K-theoretic combinatorial Hopf algebras, a K-theoretic Poirier-Reutenauer bialgebra which leads to a proof of a K-theoretic Littlewood-Richardson rule, and K-theoretic version of Fomin's dual graded graphs.

In Chapter 2, we discuss preliminary definitions: tableaux, Schur functions, compositions, and the Robinson-Schensted-Knuth correspondence. Further background will be introduced throughout this document as needed.

Chapter 3 focuses on the K-theoretic analgues of combinatorial Hopf algebras introduced by Lam and Pylyavskyy in [LP07]. Their paper leaves open the problem of finding combinatorial antipode formulas in many of these Hopf algebras, and the main goal of this chapter is to give several such formulas. We begin by defining a Hopf algebra, and we include descriptions of the classical combinatorial Hopf algebras as well as their K-theoretic analogues.

Chapter 4 introduces a K-theoretic analogue of the Poirier-Reutenauer Hopf algebra on standard Young tableaux, which we call KPR. We first describe Poirier-Reutenauer and Knuth equivalence. We then use K-Knuth equivalence and Hecke insertion in place of the usual Knuth equivalence and Robinson-Schensted to define the bialgebra KPR. We use its bialgebra structure to prove a known Littlewood-Richardson rule for stable Grothendieck polynomials.

Chapter 5 discusses a K-theoretic analgoue of Fomin's dual graded graphs [Fom94].

We define our analogue, which we call *dual filtered graphs*; discuss two main constructions; and show many examples where dual filtered graphs appear naturally. In addition, we use dual filtered graphs to create natural K-theoretic analogues of Fomin's Young-Fibonacci insertion and of Sagan-Worley insertion. For each K-theoretic insertion algorithm described, we give the growth rules describing the insertion algorithm locally. We also describe enumerative results using the up and down operators analogous to results using these operators in the dual graded graph setting. The chapter ends with a summary of conjectures relating to dual filtered graphs and k-Schur functions.

Chapters 3 through 5 are largely independent of each other and may be read in any order. In the few occasions where we require definitions from previous chapters, we direct the reader to the proper section. To this end, each chapter contains its own introduction giving background and motivation specialized to the topic of that chapter.

1.4 Summary of the main results

- Theorem 3.11.7 gives combinatorial antipode formulas for the stable Grothendieck polynomial G_λ as well for polynomials in the dual Hopf algebra using the notion of a *hook-restricted plane partition.*

- Section 4.4 describes the bialgebra KPR.

- Theorem 4.7.1 and Theorem 4.7.4 give Littlewood-Richardson rules for stable Grothendieck polynomials. These rules follow from work of Thomas-Young and Buch-Samuel, but our proof using KPR is new.

- Section 5.3.1 defines a dual filtered graph.

- Theorem 5.3.3 algebraically shows that graphs built using our *Pieri construction* are always dual filtered graphs.

- Section 5.3.4 describes our *Möbius construction* of a dual filtered graph starting from a dual graded graph. When we use this construction in later examples, we must prove case-by-case that the result is a dual filtered graph.

- Section 5.4 shows how to build dual Pieri and Möbius dual filtered graphs based on Young's lattice, Schur functions, Hecke insertion, and stable Grothendieck polynomials. We give growth rules for Hecke insertion in Section 5.4.3. We also show an instance of our *Möbius via Pieri phenomenon*.

- Section 5.5 builds dual filtered graphs starting from shifted Young's lattice. We define a K-theoretic analogue of Sagain-Worley insertion using the Möbius construction in Section 5.5.3 and describe its growth rules. This new insertion algorithem coincides with the K-theoretic shifted jeu-de-taquin of Clifford, Thomas, and Yong [CTY14].

- Section 5.6 creates a dual filtered graph based on the Young-Fibonacci lattice. We then use this graph to define a K-theoretic analogue of Fomin's Young-Fibonacci insertion and give the corresponding growth rules.

Chapter 2

Preliminaries

2.0.1 Partitions, tableaux, and Schur functions

A *Young diagram* or *partition* is a finite collection of boxes arranged in left-justified rows such that the lengths of the rows are weakly decreasing from top to bottom. We denote the shape of a Young diagram λ by $(\lambda_1, \lambda_2, \ldots, \lambda_k)$, where λ_i is the length of row i of λ. A *Young tableau* is a filling of the boxes of a Young diagram with positive integers, where the fillings weakly increase in rows and columns. Let $[k]$ denote the set $\{1, \ldots, k\}$. We call a Young tableau a *standard Young tableau* if it is filled with positive integers $[k]$ for some k and if each integer appears exactly once. We call a Young tableau a *semistandard Young tableau* if the entries increase weakly across rows and strictly down columns. The tableau shown below on the left is an example of a standard Young tableau, and both tableaux below are examples of semistandard Young tableaux. The tableau on the left has shape $(3, 3, 2)$ and the tableau on the right has shape $(4, 2, 2)$.

$$\begin{array}{|c|c|c|}\hline 1 & 4 & 6 \\ \hline 2 & 5 & 7 \\ \hline 3 & 8 \\ \cline{1-2}\end{array} \qquad\qquad \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 5 \\ \hline 2 & 2 \\ \cline{1-2} 3 & 8 \\ \cline{1-2}\end{array}$$

Given two partitions, λ and μ, we say that $\mu \subseteq \lambda$ if $\lambda_i \geq \mu_i$ for each μ_i. We then define the *skew diagram* λ/μ to be the set of boxes of λ that do not belong to μ. If the shape of T is λ/μ where μ is the empty shape, we say that T is of *straight shape.* The definitions of Young tableaux, standard Young tableaux, and semistandard Young tableaux extend naturally to skew diagrams. For example, the figure below shows a standard Young tableau of skew shape $(3,3,1)/(2,1)$.

$$\begin{array}{ccc} & & \boxed{3} \\ & \boxed{1} & \boxed{4} \\ \boxed{2} & & \end{array}$$

We define the *Schur function* $s_\lambda = s_\lambda(x_1, x_2, x_3, \ldots)$ by

$$s_\lambda = \sum_T x^T,$$

where we sum over all semistandard Young tableaux of shape λ and

$$x^T = \prod_{i \geq 1} x_i^{\# i \text{ in } T}.$$

For example,

$$s_{(2,1)} = x_1^2 x_2 + 2x_1 x_2 x_3 + x_2^2 x_3 + x_5^2 x_8 + 2x_1 x_4 x_9 + \ldots,$$

where the terms shown correspond to the semistandard tableaux shown below.

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|}\hline 5 & 5 \\ \hline 8 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 9 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|}\hline 1 & 9 \\ \hline 4 \\ \cline{1-1} \end{array}$$

2.0.2 Compositions

A composition of n is an ordered arrangement of positive integers that sum to n. For example, (3), $(1,2)$, $(2,1)$, and $(1,1,1)$ are all of the compositions of 3.

If $S = \{s_1, s_2, \ldots, s_k\}$ is a subset of $[n-1]$, we associate a composition, $\mathcal{C}(S)$, to S by

$$\mathcal{C}(S) = (s_1, s_2 - s_1, s_3 - s_2, \ldots, n - s_k).$$

To composition α of n, we associate $S_\alpha \subset [n-1]$ by letting

$$S_\alpha = \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \alpha_2 + \ldots + \alpha_{k-1}\}.$$

Recall the the *descent set* of a permutation w, $Des(w)$, is define by

$$Des(w) = \{i \mid w_i > w_{i+1}\}.$$

We may extend this correspondence to permutations by letting $\mathcal{C}(w) = \mathcal{C}(Des(w))$, where $w \in \mathfrak{S}_n$.

For example, if $S = \{1,4,5\} \subset [6-1]$, $\mathcal{C}(S) = (1, 4-1, 5-4, 6-5) = (1,3,1,1)$. Conversely, given composition $\alpha = (1,3,1,1)$, $S_\alpha = \{1, 1+3, 1+3+1\} = \{1,4,5\}$. For $w = 132 \in \mathfrak{S}_3$, $Des(w) = \{2\}$ and $\mathcal{C}(w) = (2,1)$.

2.0.3 The Robinson-Schensted-Knuth correspondence

There is a well-known combinatorial correspondence between words in the alphabet of positive integers and pairs consisting of a semistandard Young tableau and a standard Young tableau of the same shape called the Robinson-Schensted-Knuth (RSK) correspondence. We briefly review the correspondence here and refer the reader to [Sta99] for more information.

Given a word w, the RSK correspondence maps w to a pair of tableaux via a row insertion algorithm consisting of inserting a positive integer into a tableau. The algorithm for inserting positive integer k into a row of a semistandard tableau is as follows. If k is greater than or equal to all entries in the row, add a box labeled k to the end of the row. Otherwise, find the first y in the row with $y > k$. Replace y with k in this box, and proceed to insert y into the next row. To insert k into semistandard tableau P, we start by inserting k into the first row of P. For example, if we insert 3 into the tableau below on the left, it first bumps the 5 in the first row. Next, the 5 is inserted into the second row, which bumps the 6. Finally, the 6 is inserted into the third row, and we obtain the tableau on the right.

$$\begin{array}{|c|c|c|c|}\hline 1 & 3 & 5 & 6 \\ \hline 6 \\ \cline{1-1} \end{array} \qquad \qquad \begin{array}{|c|c|c|c|}\hline 1 & 3 & 3 & 6 \\ \hline 5 \\ \cline{1-1} 6 \\ \cline{1-1} \end{array}$$

To create the *insertion tableau* of a word $w = w_1 w_2 \cdots w_r$, we first insert w_1 into the empty tableau, insert w_2 into the result of the previous insertion, insert w_3 into the result of the previous insertion, and so on until we have inserted all letters of w. We denote the resulting insertion tableau by $P(w)$. The insertion tableau will always be a semistandard tableau.

To obtain a standard Young tableau from w, we define the *recording tableau*, $Q(w)$, of w by labeling the box of $P(w_1 \cdots w_s)/P(w_1 \cdots w_{s-1})$ by s. For example, $w = 14252$ has insertion and recording tableau

$$P(w) = \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 4 & 5 \\ \cline{1-2} \end{array} \qquad \qquad Q(w) = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2} \end{array}.$$

The RSK correspondence described above is a bijection between words consisting of positive integers and pairs (P, Q), where P is a semistandard Young tableau and Q is a standard Young tableau of the same shape.

We will often restrict this correspondence to permutations instead of more general words. In this setting, we have a bijection between permutations and pairs (P, Q), where P and Q are standard Young tableaux. We will still call this restricted version RSK.

Chapter 3

Combinatorial Hopf algebras and antipode maps

3.1 Introduction

A Hopf algebra is a structure that is both an associative algebra with unit and a coassociative coalgebra with counit. The algebra and coalgebra structures are compatible, which makes it a bialgebra. To be a Hopf algebra, a bialgebra must have a special anti-endomorphism called the antipode, which must satisfy certain properties.

Hopf algebras arise naturally in combinatorics. Notably, the symmetric functions (Sym), quasisymmetric functions (QSym), noncommutative symmetric functions (NSym), and the Malvenuto-Reutenauer algebra of permutations (MR) are Hopf algebras, which can be arranged as shown in Figure 3.1.

$$\begin{tikzcd}
\mathrm{MR} \arrow[r, dash] & \mathrm{MR} \arrow[d, two heads] \\
\mathrm{NSym} \arrow[u, hook] \arrow[r, dash] \arrow[d, two heads] & \mathrm{QSym} \\
\mathrm{Sym} \arrow[r, dash] & \mathrm{Sym} \arrow[u, hook]
\end{tikzcd}$$

Figure 3.1: Diagram of combinatorial Hopf algebras

Through the work of Lascoux and Schützengerger [LSb], Fomin and Kirillov [FK96], and Buch [Buc02], symmetric functions known as stable Grothendieck polynomials were discovered and given a combinatorial interpretation in terms of set-valued tableaux. They originated from Grothendieck polynomials, which serve as representatives of K-theory classes of structure sheaves of Schubert varieties. As mentioned in the introduction, the stable Grothendieck polynomials play the role of Schur functions in the K-theory of Grassmannians. They also determine a K-theoretic analogue of the symmetric functions, which we call the multi-symmetric functions and denote $\mathfrak{m}\mathrm{Sym}$.

In [LP07], Lam and Pylyavksyy extend the definition of P-partitions to create P-set-valued partitions, which they use to define a new K-theoretic analogue of the Hopf algebra of quasisymmetric functions called the Hopf algebra of multi-quasisymmetric functions. The entire diagram may be extended to give the diagram in Figure 3.2.

$$\begin{tikzcd}
\mathfrak{M}\mathrm{MR} \arrow[r, dash] & \mathfrak{m}\mathrm{MR} \arrow[d, two heads] \\
\mathfrak{M}\mathrm{NSym} \arrow[u, hook] \arrow[r, dash] \arrow[d, two heads] & \mathfrak{m}\mathrm{QSym} \\
\mathfrak{M}\mathrm{Sym} \arrow[r, dash] & \mathfrak{m}\mathrm{Sym} \arrow[u, hook]
\end{tikzcd}$$

Figure 3.2: Diagram of K-theoretic combinatorial Hopf algebras of Lam and Pylyavskyy

Using Takeuchi's formula [Tak71], they give a formula for the antipode for $\mathfrak{M}\mathrm{MR}$ but leave open the question of an antipode for the remaining Hopf algebras. In this chapter, we give new combinatorial formulas for the antipode maps of $\mathfrak{M}\mathrm{NSym}$, $\mathfrak{m}\mathrm{QSym}$, $\mathfrak{m}\mathrm{Sym}$, and $\mathfrak{M}\mathrm{Sym}$.

Remark 3.1.1. As there is no sufficiently nice combinatorial formula for the antipode map in the Malvenuto-Reutenauer Hopf algebra of permutations, we do not attempt a formula for its K-theoretic versions: $\mathfrak{M}\mathrm{MR}$ and $\mathfrak{m}\mathrm{MR}$.

3.1.1 Chapter overview

After a brief introduction to Hopf algebras, we introduce the Hopf algebra $\mathfrak{m}\mathrm{QSym}$ in Section 3.4. Next, we introduce $\mathfrak{M}\mathrm{NSym}$ in Section 3.6. We present results concerning the antipode map in $\mathfrak{MNSym}$ and $\mathfrak{m}$QSym, namely Theorems 3.6.6 and 3.6.8. In Section 3.7, we present an additional basis for $\mathfrak{m}$QSym, give analogues of results in [LP07] for this new basis, and give an antipode formula in $\mathfrak{m}$QSym involving the new basis in Theorem 3.7.6. Lastly, we introduce the Hopf algebras of multi-symmetric functions, $\mathfrak{m}$Sym, and of Multi-symmetric functions, $\mathfrak{M}$Sym in Sections 3.9 and 3.10. We end with Theorems 3.11.2, 3.11.3, and 3.11.7, which describe antipode maps in these spaces.

3.2 Hopf algebra basics

3.2.1 Algebras and coalgebras

First we build a series of definitions leading to the definition of a Hopf algebra. For more information, see [JR79, Mon93, GR14, Swe69].

In this section, k will usually denote a field, although it may also be a commutative ring. In all later sections we take $k = \mathbb{Z}$. All tensor products are taken over k.

Definition 3.2.1. *An associative k-algebra A is a k-vector space with associative operation $m : A \otimes A \to A$ (the product) and unit map $\eta : k \to A$ with $\eta(1_k) = 1_A$ such that the following diagrams commute:*

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "1 \otimes m"] \arrow[d, "m \otimes 1"] & A \otimes A \arrow[d, "m"] \\ A \otimes A \arrow[r, "m"] & A \end{tikzcd} \qquad \begin{tikzcd} k \otimes A \arrow[r, leftrightarrow, "\cong"] \arrow[dr, "\eta \otimes 1"'] & A \arrow[r, leftrightarrow, "\cong"] & A \otimes k \arrow[dl, "1 \otimes \eta"] \\ & A \otimes A \arrow[u, "m"] & \end{tikzcd}$$

where we take the isomorphisms sending $a \otimes k$ to ak and $k \otimes a$ to ka.

The first diagram tells us that m is an associative product and the second that $\eta(1_k) = 1_A$.

Definition 3.2.2. *A co-associative coalgebra C is a k-vector space with k-linear map $\Delta : C \to C \otimes C$ (the coproduct) and a counit $\epsilon : C \to k$ such that the following diagrams commute:*

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"] & C \otimes C \arrow[d, "\Delta \otimes 1"] \\ C \otimes C \arrow[r, "1 \otimes \Delta"] & C \otimes C \otimes C \end{tikzcd} \qquad \begin{tikzcd} k \otimes C \arrow[r, leftrightarrow, "\cong"] & C \arrow[r, leftrightarrow, "\cong"] \arrow[d, "\Delta"] & C \otimes k \\ & C \otimes C \arrow[ul, "\epsilon \otimes 1"] \arrow[ur, "1 \otimes \epsilon"'] & \end{tikzcd}$$

The diagram on the left indicates that Δ is *co-associative*. Note that these are the same diagrams as in the Definition 3.2.1 with all of the arrows reversed.

It is often useful to think of the product as a way to combine two elements of an algebra and to think of the coproduct as a sum over ways to split a coalgebra element into two pieces. When discussing formulas involving Δ, we will use Sweedler notation as shown below:

$$\Delta(c) = \sum_{(c)} c_1 \otimes c_2 = \sum c_1 \otimes c_2.$$

This is a common convention that will greatly simplify our notation.

Example 3.2.3. To illustrate the concepts just defined, we give the example of the shuffle algebra, which is both an algebra and coalgebra.

Let I be an alphabet and $\bar{I}$ be the set of words on I. We declare that words on I form a k-basis for the shuffle algebra.

Given two words $a = a_1 a_2 \cdots a_t$ and $b = b_1 b_2 \cdots b_n$ in $\bar{I}$, define their product, $m(a \otimes b)$, to be the *shuffle product* of a and b. That is, $m(a \otimes b)$ is the sum of all $\binom{t+n}{n}$ ways to interlace the two words while maintaining the relative order of the letters in each word. For example,

$$m(a_1 a_2 \otimes b_1) = a_1 a_2 b_1 + a_1 b_1 a_2 + b_1 a_1 a_2.$$

We may then extend by linearity. It is not hard to see that this multiplication is associative.

The unit map for the shuffle algebra is defined by $\eta(1_k) = \emptyset$, where $\emptyset$ is the empty word. Note that $m(a \otimes \emptyset) = m(\emptyset \otimes a) = a$ for any word a.

For a word $a = a_1 a_2 \cdots a_t$ in $\bar{I}$, we define

$$\Delta(a) = \sum_{i=0}^{t} a_1 a_2 \cdots a_i \otimes a_{i+1} a_{i+2} \cdots a_t$$

and call this the *cut coproduct* of a. For example, given a word $a = a_1a_2$,

$$\Delta(a) = \emptyset \otimes a_1a_2 + a_1 \otimes a_2 + a_1a_2 \otimes \emptyset.$$

The counit map is defined by letting ϵ take the coefficient of the empty word. Hence for any nonempty $a \in \bar{I}$, $\epsilon(a) = 0$.

3.2.2 Morphisms and bialgebras

The next step in defining a Hopf algebra is to define a bialgebra. For this, we need a notion of compatibility of maps of an algebra (m, η) and maps of a coalgebra (Δ, ϵ). With this as our motivation, we introduce the following definitions.

Definition 3.2.4. *If A and B are k-algebras with multiplication m_A and m_B and unit maps η_A and η_B, respectively, then a k-linear map $f : A \to B$ is an algebra morphism if $f \circ m_A = m_B \circ (f \otimes f)$ and $f \circ \eta_A = \eta_B$.*

Definition 3.2.5. *Given k-coalgebras C and D with comultiplication and counit Δ_C, ϵ_C, Δ_D, and ϵ_d, k-linear map $g : C \to D$ is a coalgebra morphism if $\Delta_D \circ g = (g \otimes g) \circ \Delta_C$ and $\epsilon_D \circ g = \epsilon_C$.*

Given two k-algebras A and B, their tensor product $A \otimes B$ is also a k-algebra with $m_{A\otimes B}$ defined to be the composite of

$$A \otimes B \otimes A \otimes B \xrightarrow{1 \otimes T \otimes 1} A \otimes A \otimes B \otimes B \xrightarrow{m_A \otimes m_B} A \otimes B,$$

where $T(b \otimes a) = a \otimes b$. For example, we have

$$m_{A\otimes B}((a \otimes b) \otimes (a' \otimes b')) = m_A(a \otimes a') \otimes m_B(b \otimes b').$$

The unit map in $A \otimes B$, $\eta_{A\otimes B}$, is given by the composite

$$k \longrightarrow k \otimes k \xrightarrow{\eta_A \otimes \eta_B} A \otimes B.$$

Similarly, given two coalgebras C and D, their tensor product $C \otimes D$ is a coalgebra with $\Delta_{C\otimes D}$ the composite of

$$C \otimes D \xrightarrow{\Delta_C \otimes \Delta_D} C \otimes C \otimes D \otimes D \xrightarrow{1 \otimes T \otimes 1} C \otimes D \otimes C \otimes D,$$

and the counit $\epsilon_{A \otimes B}$ is the composite

$$C \otimes D \xrightarrow{\epsilon_C \otimes \epsilon_D} k \otimes k \longrightarrow k.$$

Definition 3.2.6. *Given A that is both a k-algebra and a k-coalgebra, we call A a k-bialgebra if (Δ, ϵ) are morphisms for the algebra structure (m, η) or equivalently, if (m, η) are morphisms for the coalgebra structure (Δ, ϵ).*

Example 3.2.7. The shuffle algebra is a bialgebra. We can see, for example, that

$$\begin{aligned}
\Delta \circ m_A(a_1 \otimes b_1) &= \Delta(a_1 b_1 + b_1 a_1) \\
&= \emptyset \otimes a_1 b_1 + a_1 \otimes b_1 + a_1 b_1 \otimes \emptyset + \emptyset \otimes b_1 a_1 + b_1 \otimes a_1 + b_1 a_1 \otimes \emptyset \\
&= \emptyset \otimes (a_1 b_1 + b_1 a_1) + b_1 \otimes a_1 + a_1 \otimes b_1 + (a_1 b_1 + b_1 a_1) \otimes \emptyset \\
&= m_A(\emptyset \otimes \emptyset) \otimes m_A(a_1 \otimes b_1) + m_A(\emptyset \otimes b_1) \otimes m_A(a_1 \otimes \emptyset) \\
&\quad + m_A(a_1 \otimes \emptyset) \otimes m_A(\emptyset \otimes b_1) + m_A(a_1 \otimes b_1) \otimes m_A(\emptyset \otimes \emptyset) \\
&= m_{A \otimes A}((\emptyset \otimes a_1 + a_1 \otimes \emptyset) \otimes (\emptyset \otimes b_1 + b_1 \otimes \emptyset)) \\
&= m_{A \otimes A} \circ (\Delta(a_1) \otimes \Delta(b_1)).
\end{aligned}$$

This is evidence that the coproduct, Δ, is an algebra morphism.

3.2.3 The antipode map

A Hopf algebra is a bialgebra equipped with an additional map called the antipode map. On our way to defining the antipode map, we must first introduce an algebra structure on k-linear algebra maps that take coalgebras to algebras.

Definition 3.2.8. *Given coalgebra C and algebra A, we form an associative algebra structure on the set of k-linear maps from C to A, $Hom_k(C, A)$, called the convolution algebra as follows: for f and g in $Hom_k(C, A)$, define the product, $f * g$, by*

$$(f * g)(c) = m \circ (f \otimes g) \circ \Delta(c) = \sum f(c_1) g(c_2),$$

where $\Delta(c) = \sum c_1 \otimes c_2$.

Note that $\eta \circ \epsilon$ is the two-sided identity element for $*$ using this product. We can easily see this in the shuffle algebra from Example 3.2.7 if we remember that $(\eta \circ \epsilon)(a) = \eta(0) = 0$ for all words $a \neq \emptyset$. Let c be a word in the shuffle algebra, then

$$(f * (\eta \circ \epsilon))(c) = \sum f(c_1)(\eta \circ \epsilon)(c_2) = f(c) = \sum (\eta \circ \epsilon)(c_1) f(c_2) = ((\eta \circ \epsilon) * f)(c)$$

because $c_1 = c$ when $c_2 = \emptyset$ and $c_2 = c$ when $c_1 = \emptyset$.

If we have a bialgebra A, then we can consider this convolution structure to be on $End_k(A) := Hom_k(A, A)$.

Definition 3.2.9. *Let $(A, m, \eta, \Delta, \epsilon)$ be a bialgebra. Then $S \in End_k(A)$ is called an antipode for bialgebra A if*

$$id_A * S = S * id_A = \eta \circ \epsilon,$$

where $id_A : A \to A$ is the identity map.

In other words, the endomorphism S is the two-sided inverse for the identity map id_A under the convolution product. Equivalently, if $\Delta(a) = \sum a_1 \otimes a_2$,

$$(S * id_A)(a) = \sum S(a_1) a_2 = \eta(\epsilon(a)) = \sum a_1 S(a_2) = (id_A * S).$$

Because we have an associative algebra, this means that if an antipode exists, then it is unique.

Example 3.2.10. In the shuffle algebra, we define the antipode of a word by

$$S(a_1 a_2 \cdots a_t) = (-1)^t a_t a_{t-1} \cdots a_2 a_1$$

and extend by linearity. We can see an example of the defining property by computing

$$\begin{aligned}
(id * S)(a_1 a_2) &= m(id(\emptyset) \otimes S(a_1 a_2)) + m(id(a_1) \otimes S(a_2)) + m(id(a_1 a_2) \otimes S(\emptyset)) \\
&= -a_2 a_1 - m(a_1 \otimes a_2) + a_1 a_2 \\
&= -a_2 a_1 - (a_1 a_2 + a_2 a_1) + a_1 a_2 \\
&= 0 \\
&= \eta(\epsilon(a_1 a_2)).
\end{aligned}$$

We end this section with two useful properties that we use in later sections. The first is a well-known property of the antipode map for any Hopf algebra.

Proposition 3.2.11. *Let S be the antipode map for Hopf algebra A. Then S is an algebra anti-endomorphism: $S(1) = 1$, and $S(ab) = S(b)S(a)$ for all a, b in A.*

The second property allows us to translate antipode formulas between certain Hopf algebras.

Lemma 3.2.12. *Suppose we have two bialgebra bases, $\{A_\lambda\}$ and $\{B_\mu\}$, that are dual under a pairing and such that the structure constants for the product of the first basis are the structure constants for the coproduct of the second basis and vice versa. In other words, $\langle A_\lambda, B_\mu\rangle = \delta_{\lambda,\mu}$, $A_\lambda A_\mu = \sum_\nu f^\nu_{\lambda,\mu} A_\nu$ and $\Delta(B_\lambda) = \sum_{\mu,\nu} f^\lambda_{\mu,\nu} B_\mu \otimes B_\nu$, and $\Delta(A_\lambda) = \sum_{\mu,\nu} h^\lambda_{\mu,\nu} A_\mu \otimes A_\nu$ and $B_\lambda B_\mu = \sum_\nu h^\nu_{\lambda,\mu} B_\nu$. If*

$$S(A_\lambda) = \sum_\mu e_{\lambda,\mu} A_\mu$$

for S satisfying $0 = \sum h^\lambda_{\mu,\nu} S(A_\mu) A_\nu$, then

$$S(B_\mu) = \sum_\lambda e_{\lambda,\mu} B_\lambda$$

satisfies $\sum f^\lambda_{\mu,\nu} S(B_\mu) B_\nu = 0$.

Proof. Indeed,

$$\begin{aligned}
\left\langle \sum_{\mu,\nu} f^{\lambda}_{\mu,\nu} S(B_\mu) B_\nu, A_\tau \right\rangle &= \left\langle \sum_{\mu,\nu,\gamma} f^{\lambda}_{\mu,\nu,\gamma} k_{\gamma,\mu} B_\gamma B_\nu, A_\tau \right\rangle \\
&= \left\langle \sum_{\mu,\nu,\gamma,\rho} f^{\lambda}_{\mu,\nu,\gamma} k_{\gamma,\mu} h^{\rho}_{\gamma,\nu} B_\rho, A_\tau \right\rangle \\
&= \sum_{\mu,\nu,\gamma} f^{\lambda}_{\mu,\nu,\gamma} k_{\gamma,\mu} h^{\tau}_{\gamma,\nu} \\
&= \left\langle B_\lambda, \sum_{\rho,\mu,\nu,\gamma} h^{\tau}_{\gamma,\nu} k_{\gamma,\mu} f^{\rho}_{\mu,\nu} A_\rho \right\rangle \\
&= \left\langle B_\lambda, \sum_{\mu,\nu,\gamma} h^{\tau}_{\gamma,\nu} k_{\gamma,\mu} A_\mu A_\nu \right\rangle \\
&= \left\langle B_\lambda, \sum_{\nu,\gamma} h^{\tau}_{\gamma,\nu} S(A_\gamma) A_\nu \right\rangle \\
&= 0
\end{aligned}$$

by assumption. □

3.3 The Hopf algebra of quasisymmetric functions

3.3.1 Monomial quasisymmetric functions

We give a brief background on quasisymmetric functions, which were introduced by Gessel [Ges84] and stemmed from work of Stanely [Sta72]. An understanding of QSym will be useful for understanding $\mathfrak{m}$QSym. See [GR14] and [Sta11] for further details.

A quasisymmetric function, f, is a formal power series in $\mathbb{Z}[[x_1, x_2, \ldots]]$ of bounded degree such that

$$[x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_k}^{a_k}]f = [x_{j_1}^{a_1} x_{j_2}^{a_2} \cdots x_{j_k}^{a_k}]f$$

whenever $i_1 < i_2 < \ldots < i_k$, $j_1 < j_2 < \ldots < j_k$, and $a_1, a_2, \ldots, a_k \in \mathbb{P}$. For example,

$$\sum_{i_1<i_2<i_3} x_{i_1}^2 x_{i_2} x_{i_3} = x_1^2 x_2 x_3 + x_1^2 x_2 x_4 + x_3^2 x_6 x_8 + \ldots$$

is quasisymmetric while $\sum_{i_1<i_2} x_1^2 x_{i_1} x_{i_2}$ is not since the coefficient of $x_1^2 x_2 x_3$ is 1 and the coefficient of $x_2^2 x_3 x_4$ is 0. Let $QSym$ denote the ring of quasisymmetric fuctions.

Recall the definition of compositions from Section 2.0.2. *QSym* has two distinguished $\mathbb{Z}$-bases, the monomial quasisymmetric functions and the fundamental quasisymmetric functions, both indexed by compositions of n.

Definition 3.3.1. *Let $\alpha = (\alpha_1, \ldots, \alpha_k)$ be a composition of n. The monomial quasisymmetric function M_α is defined as*

$$M_\alpha = \sum_{i_1 < i_2 < \ldots < i_k} x_{i_1}^{\alpha_1} \cdots x_{i_k}^{\alpha_k}.$$

Note that QSym is graded with n^{th} degree spanned by the set of M_α where α is a composition of n. Since there are finitely many such α, QSym is graded of finite type. In other words, QSym $= \bigoplus_{n \geq 0} V_n$ where each V_n is finite dimensional over $\mathbb{Z}$.

It is easy to see that all symmetric functions are quasisymmetric. We have a Hopf morphism $i : \mathrm{Sym} \hookrightarrow \mathrm{QSym}$, where i takes the monomial symmetric function m_λ to $\sum\limits_{\lambda(\alpha)=\lambda} M_\alpha$, where the sum is over compositions α with some permutation of the α_i giving λ.

3.3.2 (P, θ)-partitions

In order to define the second basis of QSym, the fundamental quasisymmetric functions, we discuss definitions regarding (P, θ)-partitions. Defining the fundamental quasisymmetric functions in this way will be the most useful for understanding the K-theoretic analogue.

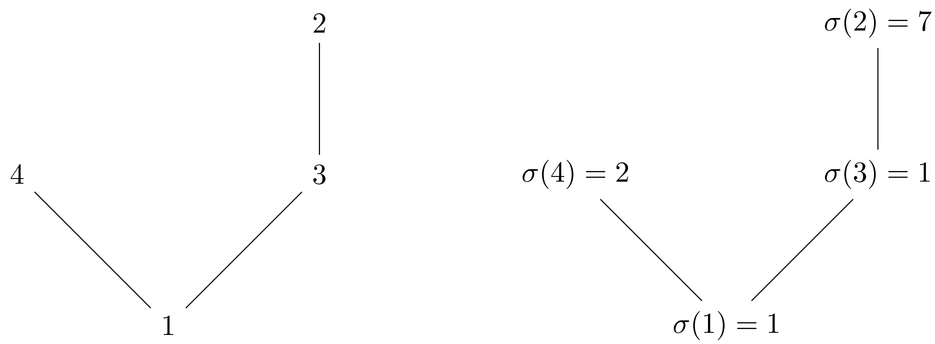
Let P be a finite poset with n elements and $\theta : P \to [n]$ a bijective labeling of its elements.

Definition 3.3.2. *Let (P, θ) be a poset with a bijective labeling. A (P, θ)-partition is a map $\sigma : P \to \mathbb{P}$ such that for each covering relation $s \lessdot t$ in P,*

1. *$\sigma(s) \leq \sigma(t)$ if $\theta(s) \leq \theta(t)$,*

2. *$\sigma(s) < \sigma(t)$ if $\theta(s) > \theta(t)$.*

Example 3.3.3. The diagram on the left shows an example of a poset P with a bijective labeling θ. The diagram on the right shows a valid (P, θ)-partition σ, where we identify an element of P with its labeling. Note that we must have $\sigma(3) < \sigma(2)$ because 2 is greater than 3 in the poset P.



Given poset P with labelling θ, we let $\mathcal{A}(P, \theta)$ denote the set of all (P, θ)-partitions. For each element $i \in \mathbb{P}$, let $\sigma^{-1}(i) = \{x \in P \mid i = \sigma(x)\}$. We now define $K_{P,\theta} \in \mathbb{Z}[[x_1, x_2, \ldots]]$ by

$$K_{P,\theta} = \sum_{\sigma \in \mathcal{A}(P,\theta)} x_1^{\#\sigma^{-1}(1)} x_2^{\#\sigma^{-1}(2)} \cdots.$$

For example, the (P, θ)-partition in Example 3.3.3 contributes the monomial $x_1^2 x_2 x_7$ to $K_{P,\theta}$.

3.3.3 Fundamental quasisymmetric functions

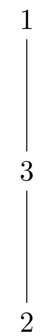
Recall the background on compositions in Section 2.0.2. Given a composition α of n, we write w_α to denote any permutation in $\mathfrak{S}_n$ with $\mathcal{C}(w_\alpha) = \alpha$. We may now define the fundamental quasisymmetric function L_α.

Definition 3.3.4. *Let P be a finite chain $p_1 < p_2 < \ldots < p_k$, $w \in S_k$ a permutation, and α the composition of n associated to the descent set of w. We label P using w with $\theta(p_i) = w_i$. Then*

$$L_\alpha = K_{(P,w)} = \sum_{\sigma \in \mathcal{A}(P,w)} x_1^{\#\sigma^{-1}(1)} x_2^{\#\sigma^{-1}(2)} \cdots.$$

It is not hard to see that $K_{(P,w)}$ depends only on α.

Example 3.3.5. For composition $\alpha = (2, 1)$, we may take $w_\alpha = 231$. This produces the labeled poset below.



Then $L_{(2,1)} = x_1^2x_2 + x_1^2x_3 + x_1x_2x_3 + x_1x_2x_4 + x_2^2x_3 + \ldots$, an infinite sum where all terms are of degree 3.

We can expand each basis of QSym in terms of the other using the following proposition.

Proposition 3.3.6. *Each of the two bases for* QSym *can be written in terms of the other as follows:*

$$L_\alpha = \sum_{\alpha \leq \beta} M_\beta$$

$$M_\alpha = \sum_{\alpha \leq \beta} (-1)^{\ell(\alpha)-\ell(\beta)} L_\beta$$

where $\alpha \leq \beta$ if $Des(w_\beta) \subset Des(w_\alpha)$, and $\ell(\alpha)$ is the number of parts in composition α.

We obtain the first expansion by choosing each weak inequality determined by labeling θ to be weak or strict for a given σ. The second expansion may be derived from the first using Möbius inversion.

Given a poset P, define the set of *linear extensions* of P, $\mathcal{L}(P)$, to be the set of all extensions of P to a linear order. For example, the poset in Example 3.3.3 has $\mathcal{L}(P) = \{1324, 1342, 1432\}$. Note that an element in $\mathcal{L}(P)$ will always be a permutation. The following result due to Stanley will be important in defining the Hopf algebra structure for $QSym$. One proof technique is similar to that of Theorem 3.7.4 and will be omitted here.

Theorem 3.3.7 ([Sta72])**.** *The function $K_{(P,\theta)}$ is quasisymmetric and may be expanded in terms of the basis of fundamental quasisymmetric functions as $K_{(P,\theta)} = \sum_{w \in \mathcal{L}(P)} L_{\mathcal{C}(w)}$.*

3.3.4 Hopf structure

For the remainder of this section, we will focus on the basis of fundamental quasisymmetric functions and will define the Hopf algebra structure using this basis.

Theorem 3.3.8. *Let $\alpha = (a_1, \dots, a_n)$ with associated permutation w_α and $\beta = (b_1, \dots, b_r)$ with permutation w_β. Then*

$$L_\alpha L_\beta = \sum_{u \in \mathrm{Sh}(w_\alpha, w_\beta + n)} L_{\mathcal{C}(u)},$$

where $(w_\beta + n)_i = (w_\beta)_i + n$ and $\mathrm{Sh}(w_\alpha, w_\beta + n)$ denotes the set of shuffles of w_α and $w_\beta + n$.

The idea of this proof is as follows. To find $L_\alpha L_\beta$, we take P to be the disjoint union of two chain posets P_α and P_β labelled by w_α and $w_\beta + n$. Then from Definition 3.3.4 and Theorem 3.3.7,

$$L_\alpha L_\beta = K_{(P_\alpha, w_\alpha)} K_{(P_\beta, w_\beta)} = K_{(P, (w_\alpha, w_\beta))} = \sum_{w \in \mathcal{L}(P)} L_{\mathcal{C}(w)} = \sum_{w \in \mathrm{Sh}(w_\alpha, w_\beta + n)} L_{\mathcal{C}(w)}.$$

The ring QSym has a natural coproduct obtained by first introducing a second variable set $y_1 < y_2 < y_3 \dots$ and ordering $x_1 < x_2 < \dots < y_1 < y_2 < \dots$. With this ordering, any quasisymmetric function f determines $f(x, y) \in \mathbb{Z}[[x_1, x_2, \dots, y_1, y_2 \dots]]$. Next, write $f(x, y)$ as an element of $\mathbb{Z}[[x_1, x_2, \dots]] \otimes \mathbb{Z}[[y_1, y_2, \dots]]$. Through this process, we obtain coproduct $\Delta : \mathrm{QSym} \to \mathrm{QSym} \otimes \mathrm{QSym}$, where $\Delta(f) = f(x, y)$.

Theorem 3.3.9. *The coproduct of L_α in* QSym *is*

$$\Delta(L_\alpha) = L_\alpha(x, y) = \sum_{\substack{\alpha = \beta \lhd \gamma \text{ or} \\ \alpha = \beta \rhd \gamma}} L_\beta \otimes L_\gamma,$$

where $\alpha \lhd \beta = (\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_m)$ and $\alpha \rhd \beta = (\alpha_1, \dots, \alpha_k + \beta_1, \beta_2, \dots, \beta_m)$.

To prove this, we notice that

$$L_\alpha(x, y) = \sum_{k=0}^{n} \sum_{\substack{1 \le i_1 \le \dots \le i_k, \\ 1 \le i_{k+1} \le \dots \le i_n, \\ i_j < i_{j+1} \text{ if } j \in Des(w_\alpha)}} x_{i_1} x_{i_2} \cdots x_{i_k} y_{i_{k+1}} y_{i_{k+2}} \cdots y_{i_n}.$$

If $i_k \in Des(w_\alpha)$, then we obtain a term $L_\beta(x)L_\gamma(y)$ where $\beta \lhd \gamma = \alpha$, and if i_k is not in $Des(w_\alpha)$, we have $L_\beta(x)L_\gamma(y)$ where $\beta \rhd \gamma = \alpha$. For example, if we are looking at $\Delta(L_{(2,1)})$ with $w_\alpha = 132$, there are two types of (P, w)-partitions:

$$\begin{array}{ccc}
\sigma_1(2) = 1_y & \qquad\qquad & \sigma_2(2) = 2_y \\
\Big| & & \Big| \\
\sigma_1(3) = 1_x & & \sigma_2(3) = 1_y \\
\Big| & & \Big| \\
\sigma_1(1) = 1_x & & \sigma_2(1) = 1_x
\end{array}$$

Then σ_1 will contribute to the term $L_{(2)}(x)L_{(1)}(y)$ with $(2) \lhd (1) = (2,1)$, and σ_2 contibutes to $L_{(1)}(x)L_{(1,1)}(y)$ with $(1) \rhd (1,1) = (2,1)$.

Example 3.3.10. We compute $L_{(2,1)}L_{(1)}$ by first choosing $w_{(2,1)} = 231$ and $w_{(1)} = 1$. Then the shuffles of $w_{(2,1)}$ and $w_{(1)} + 3$ are 2314, 2341,2431, and 4231, so

$$L_{(2,1)}L_{(1)} = L_{(2,2)} + 2L_{(2,1,1)} + L_{(1,2,1)}.$$

Also,

$$\Delta(L_{(2,1)}) = 1 \otimes L_{(2,1)} + L_{(1)} \otimes L_{(1,1)} + L_{(2)} \otimes L_{(1)} + L_{(2,1)} \otimes 1.$$

With the addition of the unit map in QSym sending quasisymmetric function f to its evaluation at $(0, 0, 0 \ldots)$, $f(0, 0, 0, \ldots)$, and the counit taking the coefficient of $1 = L_\emptyset$, we have completed our description of QSym as a bialgebra. QSym becomes a Hopf algebra with the antipode given below.

Let $\alpha = (\alpha_1, \ldots, \alpha_k)$ be a composition of n, define $rev(\alpha) = (\alpha_k, \ldots, \alpha_1)$. Now let $\omega(\alpha)$ be the unique composition of n whose partial sums $S_{(\omega(\alpha))}$ form the complementary set within $[n-1]$ to the partial sums $S_{(rev(\alpha))}$. Alternatively, we may think of the ribbon shape α as λ/μ for tableaux λ and μ. Then $\omega(\alpha)$ to be the ribbon shape λ^t/μ^t, where λ^t and μ^t are the transposes of λ and μ, respectively. The number of blocks in each row of $\omega(\alpha)$ reading from bottom to top corresponds to the number of blocks in each column of α reading from right to left. For example, if $\alpha = (2, 1, 1, 3)$, $\omega(\alpha) = (1, 1, 4, 1)$.

Theorem 3.3.11. *We have* $S(L_\alpha) = (-1)^{|\alpha|} L_{\omega(\alpha)}$.

One method used to prove this is to first prove an antipode formula for the monomial quasisymmetric functions by induction on the number of parts in α, and then use Theorem 3.3.6 to obtain the desired formula for the fundamental quasisymmetric functions [Ehr96, Proposition 3.4].

3.4 The Hopf algebra of multi-quasisymmetric functions

The multi-quasisymmetric functions ($\mathfrak{m}$QSym) are a K-theoretic analogue of the Hopf algebra of quasisymmetric functions (QSym).

In what follows, we say that a set $\{A_\lambda\}$ *continuously spans* space A if everything in A can be written as a (possibly infinite) linear combination of A_λ's. Here, we assume that $\{A_\lambda\}$ comes with a natural filtration and that each filtered component is finite. Then we may talk about continuous span with respect to the topology induced by the filtration. A *continuous basis* for A allows elements to be written as arbitrary linear combinations of the basis elements. We say that a linear function $f : A \to A$ is *continuous* if it respects arbitrary linear combinations of elements in A.

3.4.1 (P, θ)-set-valued partitions

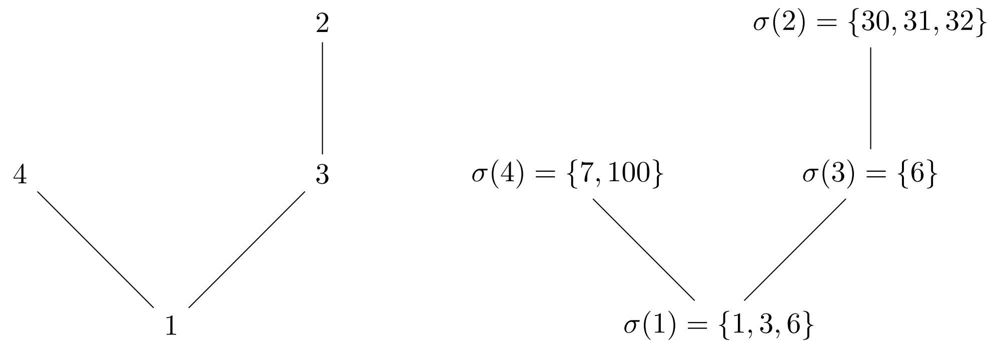
Following [LP07], we define $\mathfrak{m}$QSym, the *Hopf algebra of multi-quasisymmetric functions*, by defining the continuous basis of multi-fundamental quasisymmetric functions, $\tilde{L}_\alpha$. We start with a finite poset P with n elements and a bijective labeling $\theta : P \to [n]$. Let $\tilde{\mathbb{P}}$ denote the set of nonempty, finite subsets of the positive integers. If $a \in \tilde{\mathbb{P}}$ and $b \in \tilde{\mathbb{P}}$ are two such subsets, we say that $a < b$ if $max(a) < min(b)$. Similarly, $a \leq b$ if $max(a) \leq min(b)$.

We next define the (P, θ)-set-valued partitions. The definition is almost identical to that of the more well-known (P, θ)-set partitions except that σ now assigns a nonempty, finite subset of positive integers to each element of the poset instead of assigning a single positive integer.

Definition 3.4.1. *Let* (P, θ) *be a poset with a bijective labeling. A* (P, θ)*-set-valued partition is a map* $\sigma : P \to \tilde{\mathbb{P}}$ *such that for each covering relation* $s \lessdot t$ *in* P*,* *1.* $\sigma(s) \leq \sigma(t)$ *if* $\theta(s) \leq \theta(t)$,

2. $\sigma(s) < \sigma(t)$ *if* $\theta(s) > \theta(t)$.

Example 3.4.2. The diagram on the left shows an example of a poset P with a bijective labeling θ. We identify elements of P with their labeling. The diagram on the right shows a valid (P, θ)-set-valued partition σ. Note that since $3 < 2$ in the poset, we must have the strict inequality $max(\sigma(3)) = 6 < min(\sigma(2)) = 30$.



We denote the set of all (P, θ)-set-valued partitions for given poset P by $\tilde{\mathcal{A}}(P, \theta)$. For each element i in P, let $\sigma^{-1}(i) = \{x \in P \mid i \in \sigma(x)\}$. Now define $\tilde{K}_{P,\theta} \in \mathbb{Z}[[x_1, x_2, \ldots]]$ by

$$\tilde{K}_{P,\theta} = \sum_{\sigma \in \tilde{\mathcal{A}}(P,\theta)} x_1^{\#\sigma^{-1}(1)} x_2^{\#\sigma^{-1}(2)} \cdots.$$

For example, the (P, θ)-set-valued partition in the previous example contributes

$$x_1 x_3 x_6^2 x_7 x_{30} x_{31} x_{32} x_{100}$$

to $\tilde{K}_{P,\theta}$. Note that $\tilde{K}_{P,\theta}$ will be of unbounded degree for any nonempty poset P.

3.4.2 The multi-fundamental quasisymmetric functions

Given a composition α of n, we write w_α to denote any permutation in $\mathfrak{S}_n$ with $\mathcal{C}(w_\alpha) = \alpha$. We may now define the multi-fundamental quasisymmetric function $\tilde{L}_\alpha$ indexed by composition α.

Definition 3.4.3. *Let P be a finite chain $p_1 < p_2 < \ldots < p_k$, $w \in \mathfrak{S}_k$ a permutation, and $\mathcal{C}(w) = \alpha$ the composition of n associated to the descent set of w. We label P using* *w with* $\theta(p_i) = w_i$. *Then*

$$\tilde{L}_\alpha = \tilde{K}_{(P,w)} = \sum_{\sigma \in \tilde{\mathcal{A}}(P,w)} x_1^{\#\sigma^{-1}(1)} x_2^{\#\sigma^{-1}(2)} \cdots .$$

It is easy to see that $\tilde{K}_{(P,w)}$ depends only on α. Note that this is an infinite sum of unbounded degree. The sum of the lowest degree terms in $\tilde{L}_\alpha$ gives L_α, the fundamental quasisymmetric function in QSym.

Example 3.4.4. Let $\alpha = (2,1)$ and $w_\alpha = 231$. We consider all (P, w_α)-set-valued partitions on the chain shown below on the far left. The seven images to its right show examples of images of the map σ. shown.

$$\begin{array}{cccccccc}
1 & \{2\} & \{3\} & \{3\} & \{3,4\} & \{4\} & \{101\} & \{100,101\} \\
| & | & | & | & | & | & | & | \\
3 & \{1\} & \{1\} & \{2\} & \{2\} & \{2,3\} & \{7,100\} & \{7\} \\
| & | & | & | & | & | & | & | \\
2 & \{1\} & \{1\} & \{1\} & \{1,2\} & \{1,2\} & \{5,6,7\} & \{5,6,7\}
\end{array}$$

Using the examples above, we see that

$$\tilde{L}_{(2,1)} = x_1^2 x_2 + x_1^2 x_3 + x_1 x_2 x_3 + 2x_1 x_2^2 x_3 x_4 + 2x_5 x_6 x_7^2 x_{100} x_{101} + \ldots,$$

an infinite sum of unbounded degree.

Definition 3.4.5. *Given a poset* P *with* n *elements, a linear multi-extension of* P *by* $[N]$ *is a map* $e : P \to 2^{[N]}$ *for* $N \geq n$ *such that*

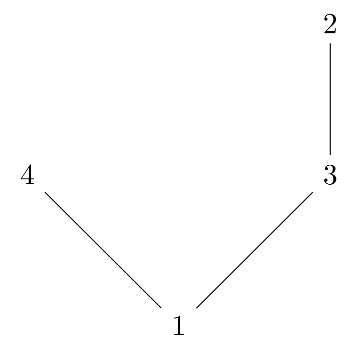
1. $e(x) < e(y)$ *if* $x < y$ *in* P,
2. *each* $i \in [N]$ *is in* $e(x)$ *for exactly one* $x \in P$, *and*
3. *no set* $e(x)$ *contains both* i *and* $i+1$ *for any* i.

We then define the *multi-Jordan-Holder set* $\tilde{\mathcal{J}}(P,\theta)$ to be the union of the sets

$$\tilde{\mathcal{J}}_N(P,\theta) = \{\theta(e^{-1}(1))\theta(e^{-1}(2)) \cdots \theta(e^{-1}(N))\}.$$

Note that elements in $\tilde{\mathcal{J}}_N(P,\theta)$ are $\mathfrak{m}$-permutations of $[n]$ with N letters, where we define an $\mathfrak{m}$-permutation of $[n]$ to be a word in the alphabet $1, 2, \ldots, n$ such that no two consecutive letters are equal.

Example 3.4.6. Consider again the labeled poset below.



We can define a linear multi-extention of P by $[7]$ by $e(1) = 1$, $e(3) = \{3,5\}$ $e(4) = \{2,4,6\}$, and $e(2) = 7$. This linear multi-extension contributes the $\mathfrak{m}$-permutation 1434342 to $\tilde{\mathcal{J}}_7(P,\theta)$.

The following result is proven in [LP07] by giving an explicit weight-preserving bijection between $\tilde{\mathcal{A}}(P,\theta)$ and the set of pairs (w,σ') where $w \in \tilde{\mathcal{J}}_N(P,\theta)$ and $\sigma' \in \tilde{\mathcal{A}}(C,w)$, where $C = (c_1 < c_2 < \ldots < c_r)$ is a chain with r elements. One can easily recover this bijection from the bijection given in the proof of Theorem 3.7.4 by restricting to $\tilde{\mathcal{A}}(P,\theta)$.

Theorem 3.4.7. *[LP07, Theorem 5.6] We can write*

$$\tilde{K}_{(P,\theta)} = \sum_{n \geq N} \sum_{w \in \tilde{\mathcal{J}}_N(P,\theta)} \tilde{L}_{\mathcal{C}(w)}.$$

We now describe how to express $\tilde{L}_\alpha$ as an infinite linear combination of L_α's, where L_α is the fundamental quasisymmetric function in QSym. Let $L_\alpha^{(i)}$ denote the homogeneous component of $\tilde{L}_\alpha$ of degree $|\alpha| + i$.

Given $D \subset [n-1]$ and $E \subset [n+i-1]$, an injective, order-preserving map $t : [n-1] \to [n+i-1]$ is an *i-extension* of D to E if $t(D) \subset E$ and $(E \backslash t(D)) = ([n+i-1]) \backslash t([n-1])$. In other words, E is the union of the image of D and the elements not in the image of

t. Thus $|E| = |D| + i$. Let $T(D, E)$ denote the set of i-extensions from D to E. For example, if $D = \{1, 2\} \subset [2]$ and $E = \{1, 2, 3\} \subset [3]$, then $|T(D, E)| = 3$. On the other hand, if we have $D' = \{1, 2\} \subset [3]$ and $E' = \{1, 3, 4\} \subset [4]$, then $|T(D', E')| = 0$. The proof of the following theorem is similar to that of Theorem 3.7.1.

Theorem 3.4.8. *[LP07, Theorem 5.12] Let α be a composition of n and $D = D(\alpha)$ be the corresponding descent set. Then for each $i \geq 0$, we have*

$$L_\alpha^{(i)} = \sum_{E \subset [n+i-1]} |T(D, E)| L_{\mathcal{C}(E)}.$$

3.4.3 Hopf structure

Next we describe the bialgebra structure of $\mathfrak{m}$QSym using the continuous basis of multi-fundamental quasisymmetric functions. The first step is to define the *multishuffle* of two words in a fixed alphabet. To that end, we give the following definition.

Definition 3.4.9. *Let $a = a_1 a_2 \cdots a_k$ be a word. We call $w = w_1 w_2 \cdots w_r$ a multiword of a if there exists non-decreasing, surjective map $t : [k] \to [r]$ such that $w_j = a_{t(j)}$.*

As an example, consider the permutation 1342 as a word in $\mathbb{N}$. Then 11333422 and 1342 are both multiwords of 1342, while 34442 and 1133244 are not multiwords of 1342.

Definition 3.4.10. *Let $a = a_1 a_2 \cdots a_k$ and $b = b_1 b_2 \cdots b_n$ be words with distinct letters. We say that $w = w_1 w_2 \cdots w_m$ is a multishuffle of a and b if the following conditions are satisfied:*

1. *$w_i \neq w_{i+1}$ for all i*
2. *when restricted to $\{a_i\}$, w is a multiword of a*
3. *when restricted to $\{b_j\}$, w is a multiword of b.*

Eventually we would like to multishuffle two permutations, so they will not have distinct letters. We adjust our definition as follows. Given a permutation $w = w_1 w_2 \cdots w_k$, define $w[n] = (w_1 + n)(w_2 + n) \cdots (w_k + n)$ to be the word obtained by adding n to each digit entry of w. For example, for $w = 21$, $w[4] = 65$. We then define the multishuffle of two permutations $u \in \mathfrak{S}_n$ and w by declaring it to be the multishuffle of u and $w[n]$.

Starting with permutations $u = 1342$ and $w = 21$, we see that $v = 161613346252$ is a multishuffle of $u = 1342$ and $w[4] = 65$, where we shift w by 4 since 4 is the largest letter in u. If we restrict to the letters in u, $v|_u = 11133422$ is a multiword of u, and similarly $v|_w[4] = 6665$ is a multiword of $w[4]$.

Proposition 3.4.11. *[LP07, Proposition 5.9] Let α be a composition of n and β be a composition of m. Then*

$$\tilde{L}_\alpha \tilde{L}_\beta = \sum_{u \in \mathrm{Sh}^{\mathfrak{m}}(\mathrm{w}_\alpha, \mathrm{w}_\beta[\mathrm{n}])} \tilde{L}_{\mathcal{C}(u)},$$

where the sum is over all multishuffles of w_α and $w_\beta[n]$.

Note that this is an infinite sum whose lowest degree terms are exactly those of $L_\alpha L_\beta$, the product of the two corresponding fundamental quasisymmetric functions.

To define the coproduct, we need the following definition.

Definition 3.4.12. *Let $w = w_1 w_2 \cdots w_k$ be a permutation. Then Cuut(w) is the set of terms of the form $w_1 w_2 \cdots w_i \otimes w_{i+1} w_{i+2} \cdots w_k$ for $i \in [0, k]$ or of the form $w_1 w_2 \cdots w_i \otimes w_i w_{i+1} \cdots w_k$ for $i \in [1, k]$.*

For example, $\mathrm{Cuut}(132) = \{\emptyset \otimes 132, 1 \otimes 132, 1 \otimes 32, 13 \otimes 32, 13 \otimes 2, 132 \otimes 2, 132 \otimes \emptyset\}$. Notice how this compares to the cut coproduct of the shuffle algebra described in Section 3.2 to understand the strange spelling.

Proposition 3.4.13. *[LP07, Proposition 5.10] We have that*

$$\Delta(\tilde{L}_\alpha) = \tilde{L}_\alpha(x, y) = \sum_{u \otimes u' \in \mathrm{Cuut}(w_\alpha)} \tilde{L}_{\mathcal{C}(u)}(x) \otimes \tilde{L}_{\mathcal{C}(u')}(y).$$

Example 3.4.14. Let $\alpha = (1)$ and $\beta = (2, 1)$ with $w_\alpha = 1$ and $w_\beta = 231$. Then

$$\tilde{L}_\alpha \tilde{L}_\beta = \tilde{L}_{(3,1)} + \tilde{L}_{(1,2,1)} + \tilde{L}_{(2,2)} + \tilde{L}_{(2,1,1)} + \tilde{L}_{(3,1,1)} + \tilde{L}_{(2,2,1,1)} + \tilde{L}_{(2,2,1,2)} + \dots,$$

where the terms listed correspond to the multishuffles 1342, 3142, 3412, 3421, 13421, 131421, and 3414212 of w_α and $w_\beta[1]$. We also compute

$$\begin{aligned} \Delta(\tilde{L}_\beta) &= \emptyset \otimes \tilde{L}_{(2,1)} + \tilde{L}_{(1)} \otimes \tilde{L}_{(2,1)} + \tilde{L}_{(1)} \otimes \tilde{L}_{(1,1)} + \tilde{L}_{(2)} \otimes \tilde{L}_{(1,1)} \\ &\quad + \tilde{L}_{(2)} \otimes \tilde{L}_{(1)} + \tilde{L}_{(2,1)} \otimes \tilde{L}_{(1)} + \tilde{L}_{(2,1)} \otimes \emptyset. \end{aligned}$$

We give a combinatorial formula for the antipode map in $\mathfrak{m}$QSym in Theorem 3.6.8. In Section 3.7, we give an antipode map in terms of a new basis introduced within the section.

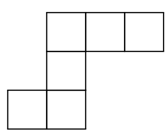
3.5 The Hopf algebra of noncommutative symmetric functions

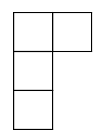
3.5.1 Noncommutative ribbon functions

Since QSym $= \oplus_{n\geq 0} V_n$ is graded and of finite type, we may consider its restricted dual, NSym $:= QSym^0 = \oplus_{n\geq 0}(V_n^*)$, the Hopf algebra of *noncommutative symmetric functions*. NSym has two distinguished bases, $\{H_\alpha\}$ dual to $\{M_\alpha\}$ and $\{R_\alpha\}$ dual to $\{L_\alpha\}$. We will focus on $\{R_\alpha\}$, the *noncommutative ribbon functions*, for the remainder of this section.

There is a bijection between compositions and ribbon diagrams sending $\alpha = (\alpha_1, \ldots, \alpha_k)$ to the skew diagram λ/μ with k rows where row $k - i$ has α_{i+1} squares and there is exactly one column of overlap between adjacent rows. Thinking of $\{R_\alpha\}$ as being indexed by ribbon diagrams will often be useful.

Example 3.5.1. The ribbon diagram on the left corresponds to $\alpha = (2, 1, 3)$, and the diagram on the right corresponds to $\beta = (1, 1, 2)$.





3.5.2 Hopf structure

Proposition 3.5.2. *Let $\alpha = (\alpha_1, \ldots, \alpha_k)$ and $\beta = (\beta_1, \ldots, \beta_m)$ be compositions. Then*

$$R_\alpha R_\beta = R_{\alpha \triangleleft \beta} + R_{\alpha \triangleright \beta},$$

where $\alpha \triangleleft \beta = (\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_m)$ and $\alpha \triangleright \beta = (\alpha_1, \ldots, \alpha_k + \beta_1, \beta_2, \ldots, \beta_m)$.

Example 3.5.3. It is helpful to think of multiplication of noncommutative ribbon functions in terms of their ribbon diagrams. From the theorem above, we see that

$$R_{(2,2)} R_{(1,2)} = R_{(2,2,1,2)} + R_{(2,3,2)}.$$

This is shown in Figure 3.3.

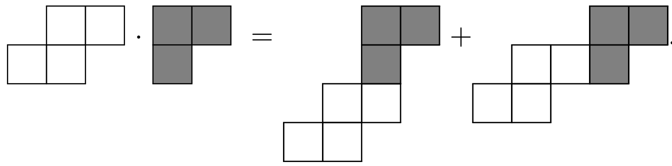


Figure 3.3: Multiplying $R_{(2,2)}$ and $R_{(1,2)}$

Proposition 3.5.4. *Let α be a composition of n and $w_\alpha \in \mathfrak{S}_n$ be a permutation with $\mathcal{C}(w) = \alpha$. Then*

$$\Delta(R_\alpha) = \sum_{w_\alpha \in \mathrm{Sh}(w_\beta, w_\delta + i)} R_\beta \otimes R_\delta,$$

where $i \in \mathbb{N}$, $w_\beta \in \mathfrak{S}_i$ and $\mathrm{Sh}(w_\beta, w_\delta + i)$ denotes the set of all shuffles of w_β and $w_\delta + i$.

Example 3.5.5. We compute $\Delta(R_{(1,2,1)})$. Let $w_{(1,2,1)} = 3142$. We see this 3142 a shuffle of $\emptyset$ and 3142, 1 and 342, 12 and 34, 312 and 4, and 3142 and $\emptyset$. Thus

$$\Delta(R_{(1,2,1)}) = R_\emptyset \otimes R_{(1,2,1)} + R_{(1)} \otimes R_{(2,1)} + R_{(2)} \otimes R_{(2)} + R_{(1,2)} \otimes R_{(1)} + R_{(1,2,1)} \otimes R_\emptyset.$$

Theorem 3.5.6. *We have that*

$$S(R_\alpha) = (-1)^{|\alpha|} R_{\omega(\alpha)}.$$

Since the basis $\{R_\alpha\}$ is dual to the basis $\{L_\alpha\}$, we can prove the product, coproduct, and antipode formulas for $\{R_\alpha\}$ by duality.

There is a Hopf morphism from the noncommutative symmetric functions onto the symmetric functions, $\pi : \mathrm{NSym} \twoheadrightarrow \Lambda$, that takes the basis element R_α to s_α, the Schur function on the ribbon shape α. Thus we can think of the symmetric functions as a quotient of the noncommutative symmetric functions.

3.6 The Hopf algebra of Multi-noncommutative symmetric functions

We next describe a K-theoretic analogue of the noncommutative symmetric functions called the Multi-noncommutative symmetric functions or $\mathfrak{M}\mathrm{NSym}$. We recall its bialgebra structure as given in [LP07] and develop a combinatorial formula for its antipode map.

3.6.1 Multi-noncommutative ribbon functions and bialgebra strucure

$\mathfrak{M}$NSym has a basis $\{\tilde{R}_\alpha\}$ of Multi-noncommutative ribbon functions indexed by compositions, which is an analogue to the basis of noncommutative ribbon functions $\{R_\alpha\}$ for NSym. As in the previous section, thinking of $\{\tilde{R}_\alpha\}$ as being indexed by ribbon diagrams will be useful.

We first introduce a product structure on $\{\tilde{R_\alpha}\}$ as given in [LP07].

Proposition 3.6.1. *[LP07, Proposition 8.1] Let $\alpha = (\alpha_1, \ldots, \alpha_k)$ and $\beta = (\beta_1, \ldots, \beta_m)$ be compositions. Then*

$$\tilde{R}_\alpha \bullet \tilde{R}_\beta = \tilde{R}_{\alpha \lhd \beta} + \tilde{R}_{\alpha \cdot \beta} + \tilde{R}_{\alpha \rhd \beta},$$

where $\alpha \lhd \beta = (\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_m)$, $\alpha \cdot \beta = (\alpha_1, \ldots, \alpha_{k-1}, \alpha_k + \beta_1 - 1, \beta_2, \ldots, \beta_m)$, and $\alpha \rhd \beta = (\alpha_1, \ldots, \alpha_k + \beta_1, \beta_2, \ldots, \beta_m)$.

Example 3.6.2. It is helpful to think of the product using ribbon diagrams. From the statement above, we have

$$\tilde{R}_{(2,2)} \bullet \tilde{R}_{(1,2)} = \tilde{R}_{(2,2,1,2)} + \tilde{R}_{(2,2,2)} + \tilde{R}_{(2,3,2)}.$$

In pictures, this is in Figure 3.4.

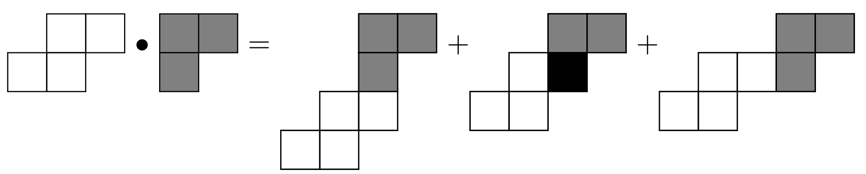


Figure 3.4: Multiplying $\tilde{R}_{(2,2)}$ and $\tilde{R}_{(1,2)}$

In contrast to the product in $\mathfrak{m}$QSym, the product in $\mathfrak{M}$NSym is a finite sum whose highest degree terms are those of the corresponding product $R_\alpha R_\beta$ in NSym.

Proposition 3.6.3. *The coproduct of a basis element is*

$$\Delta(\tilde{R}_\alpha) = \sum_{w_\alpha \in \mathrm{Sh}^{\mathfrak{m}}(w_\beta, w_\delta[i])} \tilde{R}_\beta \otimes \tilde{R}_\delta,$$

where $i \in \mathbb{N}$ and $w_\beta \in \mathfrak{S}_i$.

Note that since multishuffles of w_β and $w_\delta[i]$ may not have adjacent letters that are equal, we may define the descent set of a multishuffle of w_β and $w_\delta[i]$ in the usual way.

Example 3.6.4. In general, computing the coproduct in $\mathfrak{M}$NSym is not an easy task. However, for compositions with only one part, we have

$$\Delta(\tilde{R}_{(n)}) = \tilde{R}_{(n)} \otimes 1 + \tilde{R}_{(n-1)} \otimes \tilde{R}_{(1)} + \tilde{R}_{(n-2)} \otimes \tilde{R}_{(2)} + \dots \tilde{R}_{(1)} \otimes \tilde{R}_{(n-1)} + 1 \otimes \tilde{R}_{(n)}$$

because the only way a multishuffle of two permutations results in an increasing sequence is for it to be the concatenation of two increasing permutations. We use this fact in the proof of the antipode in $\mathfrak{M}$NSym.

3.6.2 Antipode map for $\mathfrak{M}$NSym

Suppose we have a ribbon shape corresponding to α, a composition of n. We say that ribbon shape β is a *merging* of ribbon shape α if we can obtain shape β from shape α by merging pairs of boxes that share an edge. The order in which the pairs are merged does not matter, only set of boxes that were merged. Let $M_{\alpha,\beta}$ be the number of ways to obtain shape β from shape α by merging. We will label each box in the ribbon shape to keep track of our actions.

Example 3.6.5. Let $\alpha = (2,2,1)$ and $\beta = (2,1)$. Then $M_{\alpha,\beta} = 3$. The labeled ribbon shape α and the three mergings resulting in shape β are shown in Figure 3.6.5.

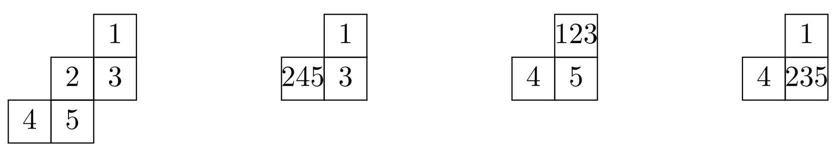


Figure 3.5: Ribbon shape $(2,2,1)$ and its three mergings of ribbon shape $(2,1)$

Theorem 3.6.6. *Let α be a composition of n. Then*

$$S(\tilde{R}_\alpha) = (-1)^n \sum_\beta M_{\omega(\alpha),\beta} \tilde{R}_\beta,$$

where we sum of over all compositions β.

Note that only finitely many terms will be nonzero because $M_{\omega(\alpha),\beta} = 0$ if $|\beta| > |\alpha|$.

Proof. We prove this by induction on the number of parts of the composition α.

We compute directly that

$$0 = S(\tilde{R}_{(1)}) = S(\tilde{R}_{(1)}) \cdot 1 + 1 \cdot S(\tilde{R}_{(1)}) = S(\tilde{R}_{(1)}) + \tilde{R}_{(1)}$$

by Definition 3.2.11, so $S(\tilde{R}_{(1)}) = -\tilde{R}_{(1)}$.

Now assume that $S(\tilde{R}_{(k)}) = (-1)^k \sum_{i=0}^{k-1} \binom{k-1}{i} \tilde{R}_{1^{i+1}}$ for all $k < n$. Then, using Example 3.6.4 and Definition 3.2.9, we see that

$$\begin{aligned} 0 &= \tilde{R}_{(n)} + S(\tilde{R}_{(n)}) + \sum_{i=1}^{n-1} S(\tilde{R}_{(i)}) \bullet \tilde{R}_{(n-i)} \\ &= \tilde{R}_{(n)} + S(\tilde{R}_{(n)}) + \sum_{i=1}^{n-1} \left((-1)^i \sum_{j=0}^{i-1} \binom{i-1}{j} \tilde{R}_{(1^{j+1})} \right) \bullet \tilde{R}_{(n-i)} \\ &= \tilde{R}_{(n)} + S(\tilde{R}_{(n)}) + \sum_{i=1}^{n-1} (-1)^i \sum_{j=0}^{i-1} \binom{i-1}{j} (\tilde{R}_{(1^{j+1},n-i)} + \tilde{R}_{(1^j,n-i+1)} + \tilde{R}_{(1^j,n-i)}). \end{aligned}$$

There are five types of terms that appear in this sum.

1. $\tilde{R}_{(1^s,m)}$, where $s = n - m$. The coefficient of this term is

$$(-1)^{n-m} \binom{n-m-1}{k-1} + (-1)^{n-m-1} \binom{n-m}{k} = 0.$$

2. $\tilde{R}_{(m)}$, where $1 < m < n$. The coefficient of this term is

$$(-1)^{n-m} \binom{n-m-1}{0} + (-1)^{n-m+1} \binom{n+m}{0} = 0.$$

3. $\tilde{R}_{(1^s,m)}$, where $s < n - m$, and $m > 0$. The coefficient of this term

$$(-1)^{n-m} \binom{n-m-1}{0} + (-1)^{n-m-1} 1 + (-1)^{n-m+1} \binom{n-m}{1} = 0.$$

4. $\tilde{R}_{(1^k)}$, where $k \leq n$. The coefficient of this term is

$$(-1)^{n-1} \binom{n-2}{k-2} + (-1)^{n-1} \binom{n-2}{k-1} = (-1)^{n-1} \binom{n-1}{k-1}.$$

5. $\tilde{R}_{(n)}$. The coefficient of this term is

$$(-1)^1\binom{0}{0} = -1.$$

Thus

$$0 = S(\tilde{R}_{(n)}) + (-1)^{n-1}\sum_{s=1}^{n}\binom{n-1}{s-1}\tilde{R}_{1^s},$$

and so

$$S(\tilde{R}_{(n)}) = (-1)^n\sum_{s=0}^{n-1}\binom{n-1}{s}\tilde{R}_{(1^{s+1})}.$$

It is clear that there are $\binom{n-1}{s}$ mergings of $\omega(\alpha) = (1^n)$ that result in shape (1^{s+1}) since we are choosing s of the $n-1$ border edges to remain intact.

Now suppose $S(\tilde{R}_\alpha) = (-1)^n\sum_\beta M_{\omega(\alpha),\beta}\tilde{R}_\beta$ holds for all compositions α with up to $k-1$ parts, and let $\beta = (\beta_1, \beta_2, \ldots, \beta_k)$ be a composition with k parts. We know that

$$\tilde{R}_\beta = \tilde{R}_{(\beta_1,\beta_2,\ldots,\beta_{k-2},\beta_{k-1})} \bullet \tilde{R}_{(\beta_k)} - \tilde{R}_{(\beta_1,\beta_2,\ldots,\beta_{k-2},\beta_{k-1}+\beta_k)} - R_{(\beta_1,\beta_2,\ldots,\beta_{k-2},\beta_{k-1}+\beta_k-1)},$$

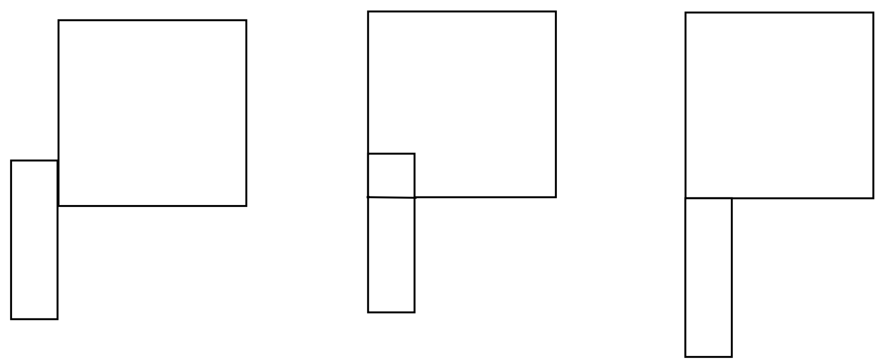
and so

$$\begin{aligned} S(\tilde{R}_\beta) &= S(\tilde{R}_{(\beta_k)}) \bullet S(\tilde{R}_{(\beta_1,\beta_2,\ldots,\beta_{k-2},\beta_{k-1})}) - S(\tilde{R}_{(\beta_1,\beta_2,\ldots,\beta_{k-2},\beta_{k-1}+\beta_k)}) \\ &\quad - S(\tilde{R}_{(\beta_1,\beta_2,\ldots,\beta_{k-2},\beta_{k-1}+\beta_k-1)}). \end{aligned}$$

In Figure 3.6, let the thin rectangle represent all mergings of $\omega(\beta_k)$ and the square represent all mergings of $\omega(\beta_1, \ldots, \beta_{k-1})$.

Then the image labeled (1) represents all mergings obtained by adding the last part of a merging of $\omega(\beta_k)$ to the first part of a merging of $\omega(\beta_1, \ldots, \beta_{k-1})$. The image labeled (2) represents all mergings obtained by merging the topmost box in a merging of $\omega(\beta_k)$ with the bottom leftmost box of a merging of $\omega(\beta_1, \ldots, \beta_k)$. These two mergings with multiplicities are exactly the shapes we want in $S(\tilde{R}_\beta)$.

The image labeled (3) represents all mergings obtained by concatenating a merging of $\omega(\beta_k)$ with a merging of $\omega(\beta_1, \ldots, \beta_{k-1})$. We do not want these mergings to appear in $S(\tilde{R}_\beta)$ because it is impossible for boxes that are side by side in $\omega(\beta)$ to be stacked one on top of the other in a merging of $\omega(\beta)$.

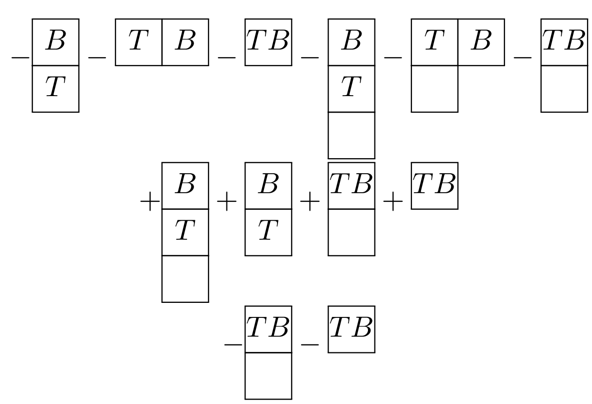


(1) (2) (3)

Figure 3.6: NSym antipode merging schematic

We use the fact that $S(\tilde{R}_{\beta_k}) \bullet S(\tilde{R}_{(\beta_1,\ldots,\beta_{(k-1)})})$ results in all mergings of type (1), (2), and (3), $S(\tilde{R}_{(\beta_1,\ldots,\beta_{k-1}+\beta_k)})$ gives all mergings of type (2) and (3), and $S(\tilde{R}_{(\beta_1,\ldots,\beta_{k-1}+\beta_k-1)})$ contains exactly those mergings of type (2). The parity of the sizes of the compositions provides the necessary cancellation and leaves us with all mergings of type (1) and (2), as desired. $\square$

Example 3.6.7. Consider $S(\tilde{R}_{(1,2)}) = S(\tilde{R}_{(1,1)}) \bullet S(\tilde{R}_{(1)}) - S(\tilde{R}_{(1,1,1)}) - S(\tilde{R}_{(1,1)})$. The image below shows all of the mergings in $S(\tilde{R}_{(1,1)}) \bullet S(\tilde{R}_{(1)})$ in the first line with the proper sign, subtracts mergings of $S(\tilde{R}_{(1,1,1)})$ in the second line, and subtracts mergings of $S(\tilde{R}_{(1,1)})$ in the third line. The box labeled T represents the top box in mergings of $\omega(\beta_k) = \omega(2) = (1,1)$, and the box labeled B represents the bottom leftmost box in mergings of $\omega(\beta_1,\ldots,\beta_{k-1}) = \omega(1) = (1)$.



$$= -\begin{array}{|c|c|}\hline T & B \\ \hline \multicolumn{1}{|c|}{} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} - \begin{array}{|c|c|}\hline T & B \\ \hline\end{array} - \begin{array}{|c|}\hline TB \\ \hline \\ \hline\end{array} - \begin{array}{|c|}\hline TB \\ \hline\end{array}$$

3.6.3 Antipode map for $\mathfrak{m}$QSym

We know from [LP07, Theorem 8.4] that the bases $\{\tilde{L}_\alpha\}$ and $\{\tilde{R}_\alpha\}$ satisfy the criteria in Lemma 3.2.12. Extending the definition below by continuity gives the following antipode formula in $\mathfrak{m}$QSym.

Theorem 3.6.8. *Let α be a composition of n. Then*

$$S(\tilde{L}_\alpha) = \sum_{\beta} (-1)^{|\beta|} M_{\beta,\omega(\alpha)} \tilde{L}_\beta,$$

where the sum is over all compositions β.

Note that while $S(\tilde{R}_\alpha)$ is a finite sum of Multi-noncommutative ribbon functions for any α, $S(\tilde{L}_\alpha)$ is an infinite sum of multi-fundamental quasisymmetric functions for any α. Since any arbitrary linear combination of multi-fundamental quasisymmetric functions is in $\mathfrak{m}$QSym, this is an admissible antipode formula.

3.7 A new basis for $\mathfrak{m}$QSym

3.7.1 (P, θ)-multiset-valued partitions

To create a new basis for $mQSym$, which will be useful in finding antipode formulas, we extend the definition of a (P, θ)-set-valued partition to what we call a (P, θ)-multiset-valued partition in the natural way. In a (P, θ)-multiset-valued partition σ, we allow $\sigma(p)$ to be a finite multiset of $\tilde{\mathbb{P}}$, keeping all other definitions the same. An example of a (P, θ)-multiset-valued partition is shown in Figure 3.7.

Now define $\hat{\mathcal{A}}(P, \theta)$ to be the set of all (P, θ)-multiset-valued partitions. For each element $i \in \mathbb{P}$, let $\sigma^{-1}(i)$ be the multiset $\{x \in P \mid i = \sigma(x)\}$. In the example shown above, $\sigma^{-1}(1) = \{1, 1, 3\}$. Now define $\hat{K}_{P,\theta} \in \mathbb{Z}[[x_1, x_2, \ldots]]$ by

$$\hat{K}_{P,\theta} = \sum_{\sigma \in \hat{\mathcal{A}}(P,\theta)} x_1^{\#\sigma^{-1}(1)} x_2^{\#\sigma^{-1}(2)} \cdots .$$

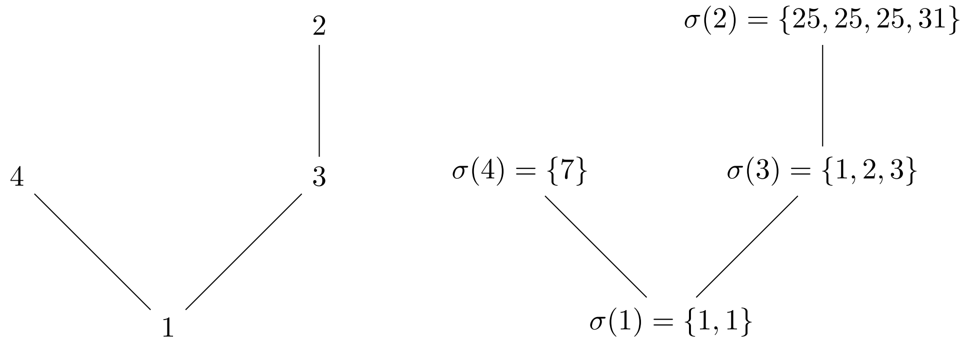


Figure 3.7: A (P, θ)-multiset-valued partition

Using this multiset analogue of our definitions, we define

$$\hat{L}_\alpha = \hat{K}_{(P,w)} = \sum_{\sigma \in \hat{\mathcal{A}}(P,w)} x_1^{\#\sigma^{-1}(1)} x_2^{\#\sigma^{-1}(2)} \cdots,$$

where $P = p_1 < \ldots < p_k$ is a finite linear order and $w \in \mathfrak{S}_k$.

3.7.2 Properties

Recall the definition of $T(D(\alpha), E)$ from Section 3.4.

Theorem 3.7.1. *We have that*

$$\hat{L}_\alpha^{(i)} = \sum_{E \subset [n+i-1]} |T(D(\omega(\alpha)), E)| L_{\omega(\mathcal{C}(E))}.$$

Proof. Let $\omega = \omega_\alpha$ and consider the subset $\hat{\mathcal{A}}_i(C, \omega) \subset \hat{\mathcal{A}}(C, \omega)$ consisting of multiset-valued (C, ω)-partitions σ of size $|\sigma| = n+i$. We must show that the generating function of $\hat{\mathcal{A}}_i(C, \omega)$ is equal to

$$\sum_{E \subset [n+i-1]} |T(D(\omega(\alpha)), E)| L_{\omega(\mathcal{C}(E))}.$$

Here, we define L_α by

$$L_\alpha = \sum_{\sigma \in \mathcal{A}(P,w)} x_1^{\#\sigma^{-1}(1)} x_2^{\#\sigma^{-1}(2)} \cdots,$$

where $\mathcal{A}(P,w)$ is the set of all (P,w)-partitions as in [LP07]. Indeed, for each pair $t \in T(D(\omega(\alpha)),E)$ for some E, the function $L_{\omega(\mathcal{C}(E))}$ is the generating function of all $\sigma \in \hat{\mathcal{A}}_i(C,\omega)$ satisfying $|\sigma(c_i)| = t(n+i-j) - t(n+i-(j+1))$, where $t(0) = 0$ and $t(n) = n+i$. Letting $C' = c'_1 < c'_2, \ldots, c'_{n+i}$ be a chain with $n+i$ elements, we obtain a $(\mathcal{C}',\omega(\mathcal{C}(E)))$-partition $\sigma' \in \mathcal{A}(C',\omega(\mathcal{C}(E)))$ by assigning the elements of $\sigma(c_i)$ in increasing order to $c'_{t(i-1)+1}, \ldots, c'_{t(i)}$. $\square$

Example 3.7.2. Letting $\alpha = (1,2,1,2)$, we see that

$$\hat{L}_\alpha^{(1)} = L_{(2,2,1,2)} + 2L_{(1,3,1,2)} + L_{(1,2,2,2)} + 2L_{(1,2,1,3)}.$$

In this example, the coefficient of $L_{(1,3,1,2)}$ is 2 because

$$\begin{aligned} |T(D(\omega(1,2,1,2)),\{1,4,5\} \subset [n+1-1=6])| &= |T(D(1,3,2),\{1,4,5\})| \\ &= |T(\{2,4\},\{1,4,5\})| \\ &= 2. \end{aligned}$$

Given the basis of multi-quasisymmetric functions, $\{\tilde{L}_\alpha\}$, the set $\{\hat{L}_\alpha\}$ is natural to consider because of the next proposition.

In the following sections, ω will denote the fundamental involution of the symmetric functions definied by $\omega(e_n) = h_n$ for all for elementary symmetric function e_n and complete homogeneous symmetric function h_n and for all n. It is known that $\omega(L_\alpha) = L_{\omega(\alpha)}$ in QSym, where $\omega(\alpha)$ was defined in Section 3.3.4.

Proposition 3.7.3. *We have $\omega(\tilde{L}_\alpha) = \hat{L}_{\omega(\alpha)}$, and the set of $\hat{L}_\alpha$'s form a basis for $\mathfrak{m}$QSym.*

Proof. Using Proposition 3.7.1,

$$\omega(\tilde{L}_\alpha) = \omega\left(\sum_{E \subset [n+i-1]} |T(D(\alpha),E)| L_{\mathcal{C}(E)}\right) = \sum_{E \subset [n+i-1]} |T(D(\alpha),E)| L_{\omega(\mathcal{C}(E))} = \hat{L}_{\omega(\alpha)}.$$

$\square$

We have an analogue of Stanley's Fundamental Theorem of P-partitions for our new basis of $\hat{L}_\alpha$'s. The proof of this result follows closely that of Theorem 3.4.7 given in [LP07].

Theorem 3.7.4. *We have*

$$\hat{K}_{P,\theta} = \sum_{N \geq n} \sum_{w \in \tilde{\mathcal{J}}_{N(P,\theta)}} \hat{L}_{\mathcal{C}(w)}.$$

Proof. We prove this result by giving an explicit weight-preserving bijection between $\hat{\mathcal{A}}(P,\theta)$ and the set of pairs (w,σ') where $w \in \tilde{\mathcal{J}}_N(P,\theta)$ and $\sigma' \in \hat{\mathcal{A}}(C,w)$ where $C = (c_1 < c_2 < \cdots < c_l)$ is a chain with $l = \ell(w)$ elements. Let $\sigma \in \hat{\mathcal{A}}(P,\theta)$. For each i, let $\sigma^{-1}(i)$ denote the submultiset of $[n]$ via θ, and let $w_\sigma^{(i)}$ denote the word of length $|\sigma^{-1}(i)|$ obtained by writing the elements of $\sigma^{-1}(i)$ in increasing order. Note that it is possible for $w_j^{(i)} = w_{j+1}^{(i)}$. This will occur when the letter i appears more than once in some $\sigma(s)$ for $s \in P$.

Let w denote the unique $\mathfrak{m}$-permutation such that $w_\sigma := w_\sigma^{(1)} w_\sigma^{(2)} \cdots$ is a multiword of w and $t : \ell(w_\sigma) \to \ell(w)$ be the associated function as in Definition 3.4.9. We know that w_σ is a finite word because $\sigma^{(-1)}(r) = \emptyset$ for sufficiently large r. Note that w_σ uses all letters $[n]$. Now define $\sigma' \in \hat{\mathcal{A}}(C,w)$ by

$$\sigma'(c_i) = \{r^k \mid r \in \mathbb{P} \text{ and } w_\sigma^{(r)} \text{ contributes } k \text{ letters to } w_\sigma|_{t^{-1}(i)}\}$$

where $w_\sigma|_{t^{-1}(i)}$ is the set of letters in w_σ at the positions in the interval $t^{-1}(i)$. We will show that this defines a map $\alpha : \sigma \mapsto (w,\sigma')$ with the required properties.

First, w is the multi-permutation associated to the linear multi-extension e_w of P by $\ell(w)$ defined by the condition that $e_w(x)$ contains j if and only if $w_j = \theta(x)$. It follows from the definition that this $e_w : P \to 2^{[1,\ell(w)]}$ is a linear multi-extension. To check that σ' is a multiset-valued (C,w) partition, we note that $\sigma'(c_i) \leq \sigma'(c_{i+1})$ because the function t is non-decreasing. Moreover, if $w_i > w_{i+1}$, then $\sigma'(c_i) < \sigma'(c_{i+1})$ because each $w_\sigma^{(r)}$ is increasing.

We define the inverse map $\beta : (w,\sigma') \mapsto \sigma$ by the formula

$$\sigma(x) = \bigcup_{j \in e_w(x)} \sigma'(c_j).$$

The (P,θ)-multiset-valued partition σ respects θ because e_w is a linear multi-extension. Thus if $x < y$ in P and $\theta(x) > \theta(y)$, then $\sigma(x) < \sigma(y)$ since $e_w(x) < e_w(y)$ and there is a descent in w between the corresponding entries of $\theta(x)$ and $\theta(y)$.

Then $\beta \circ \alpha = \mathrm{id}$ follows immediately. For $\alpha \circ \beta = \mathrm{id}$, consider a subset $\sigma'(c_j) \subset \sigma(x)$. One checks that this subset gives rise to $|\sigma'(c_j)|$ consecutive letters all equal to $\theta(x)$ in w_σ and that this is a maximal set of consecutive repeated letters. This shows that one can recover σ'. To see that w is recovered correctly, one notes that if $\sigma'(c_j)$ and $\sigma'(c_{j+1})$ contain the same letter r then $w_j < w_{j+1}$ so by definition w_j is placed correctly before w_{j+1} in $w_\sigma^{(r)}$.

$\square$

Example 3.7.5. Let θ be the labeling

$$\begin{array}{|c|c|c|}\hline 3 & 4 & 5 \\ \hline 1 & 2 & \\ \cline{1-2}\end{array}$$

of the shape $\lambda = (3, 2)$. Take the (λ, θ)-partition

$$\begin{array}{|c|c|c|}\hline 112 & 23 & 345 \\ \hline 45 & 667 & \\ \cline{1-2}\end{array}$$

in $\hat{\mathcal{A}}(\lambda, \theta)$. Then we have

$$w_\sigma = (3, 3; 3, 4; 4, 5; 1, 5; 1, 5; 2, 2; 2),$$

where for example, $w_\sigma^{(1)} = (3, 3)$ since the cells labeled 3 contains two copies of the number 1 in σ. Therefore

$$w = (3, 4, 5, 1, 5, 1, 5, 2)$$

and the corresponding composition $\mathcal{C}(w)$ is $(3, 2, 2, 1)$. Then σ' written as sequence is

$$\{1, 1, 2\}, \{2, 3\}, \{3\}, \{4\}, \{4\}, \{5\}, \{5\}, \{6, 6, 7\}.$$

For example $\sigma'(c_1) = \{1, 1, 2\}$ since $w_\sigma^{(1)}$ contributes two 3's and $w_\sigma^{(2)}$ contributes one 3 to the beginning of w_σ.

To obtain the inverse map, β, read w and σ' in parallel and place $\sigma'(c_i)$ into cell $\theta_s^{-1}(w_i)$. For example, we put $\{1, 1, 2\}$ into the cell labeled 3, and we put $\{2, 3\}$ into the cell labeled 4.

The linear multi-extension, e_w in this example can be represented by the filling below.

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 357 \\ \hline 46 & 8 & \\ \cline{1-2}\end{array}$$

3.7.3 Antipode

Recall that in QSym, $S(L_\alpha) = (-1)^{|\alpha|}L_{\omega(\alpha)} = (-1)^{|\alpha|}\omega(L_\alpha) = \omega(L_\alpha(-x_1, -x_2, \ldots))$. Using the set $\{\hat{L}_\alpha\}$, we have a similar result in $\mathfrak{m}$QSym.

Theorem 3.7.6. *In* $\mathfrak{m}$QSym*,*

$$S(\tilde{L}_\alpha) = \hat{L}_{\omega(\alpha)}(-x_1, -x_2, \ldots).$$

Proof. Using Theorem 3.4.7 and the antipode in QSym, we see that

$$\begin{aligned} S(\tilde{L}_\alpha) &= S\left(\sum_{E\subset[n+i-1]} |T(D,E)|L_{\mathcal{C}(E)}\right) \\ &= \sum_{E\subset[n+i-1]} |T(D,E)|S(L_{\mathcal{C}(E)}) \\ &= \sum_{E\subset[n+i-1]} |T(D,E)|(-1)^{|\mathcal{C}(E)|}L_{\omega(\mathcal{C}(E))} \\ &= \hat{L}_{\omega(\alpha)}(-x_1, -x_2, \ldots). \end{aligned}$$

□

3.8 The Hopf algebra of symmetric functions

We briefly review the Hopf structure of the ring of symmetric functions over $\mathbb{Z}$, Sym, focusing on the basis of Schur functions. Recall the Schur function, s_λ, is defined to be

$$s_\lambda := \sum_T \mathbf{x}^{\mathrm{cont}(T)},$$

where T runs through all semistandard (i.e. column-strict) tableaux of shape λ. The following theorem is given without proof.

Theorem 3.8.1. *The Hopf algebra of symmetric functions has the following structure.*

(a) We have

$$\Delta(s_\lambda)(\boldsymbol{x}) = s_\lambda(\boldsymbol{x}, \boldsymbol{y}) = \sum_{\mu,\nu} c^\lambda_{\mu,\nu} s_\mu(\boldsymbol{x}) s_\nu(\boldsymbol{y}) = \sum_{\mu\subseteq\lambda} s_\mu \otimes s_{\lambda/\mu}.$$

(b) Multiplication of Schur functions is as follows:

$$s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu,\nu} s_\lambda.$$

(c) Let ω denote the fundamental involution, $|\lambda/\mu|$ the number of squares in the skew diagram λ/μ, and λ^t the transpose of the partition λ. Then

$$S(s_{\lambda/\mu}) = (-1)^{|\lambda/\mu|}\omega(s_{\lambda/\mu}) = (-1)^{|\lambda/\mu|} s_{\lambda^t/\mu^t}.$$

3.9 The Hopf algebra of multi-symmetric functions

We next describe the space of multi-symmetric functions, $\mathfrak{m}$Sym. We refer the reader to [LP07] for details.

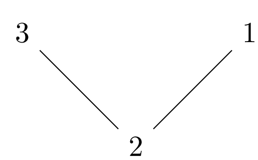
3.9.1 Set-valued tableaux

Let P be the poset of squares in the Young diagram of a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t)$ and θ_s be the bijective labeling of P obtained from labeling P in row reading order, i.e. from left to right the bottom row of λ is labeled $1, 2, \ldots, \lambda_t$, the next row up is labeled $\lambda_t + 1, \ldots, \lambda_t + \lambda_{t-1}$ and so on. Note that K_{λ,θ_s} is the Schur function s_λ. We define

$$\mathfrak{m}\mathrm{Sym} = \prod_\lambda \mathbb{Z}\tilde{K}_{\lambda,\theta_s}$$

to be the subspace of $\mathfrak{m}$QSym continuously spanned by the $\tilde{K}_{\lambda,\theta_s}$, where λ varies over all partitions. From this point forward, we will write $\tilde{K}_\lambda$ in place of $\tilde{K}_{\lambda,\theta_s}$ and call a $(\lambda/\mu, \theta_s)$-set-valued partition a *set-valued tableau* of shape λ/μ. Set-valued tableaux will be discussed in more detail in Section 4.3.2.

Example 3.9.1. For $\lambda = (2,1)$, we have $\tilde{K}_\lambda = x_1^2 x_2 + 2x_1x_2x_3 + x_1^2x_2^2 + 3x_1^2x_2x_3 + 8x_1x_2x_3x_4 + \ldots$, corresponding to the following labeled poset:



3.9.2 Basis of stable Grothendieck polynomials

We now introduce another (continuous) basis for $\mathfrak{m}$Sym, the *stable Grothendieck polynomials.* Stable Grothendieck polynomials originated from the Grothendieck polynomials of Lascoux and Schützenberger [LSb], which served as representatives of K-theory classes of structure sheaves of Schubert varieties. Through the work of Fomin and Kirillov [FK96] and Buch [Buc02], the stable Grothendieck polynomials, a limit of the Grothendieck polynomials, were discovered and given the combinatorial interpretation in the theorem below. These symmetric functions play the role of Schur functions in the K-theory of Grassmannians. They will be discussed in more detail in Section 4.6.1.

Theorem 3.9.2. *[Buc02, Theorem3.1] The stable Grothendieck polynomial $G_{\lambda/\mu}$ is given by the formula*

$$G_{\lambda/\mu} = \sum_T (-1)^{|T|-|\lambda/\mu|} x^T,$$

where the sum is taken over all set-valued tableaux of shape λ/μ.

The stable Grothendieck polynomials are related to the $\tilde{K}_\lambda$ by

$$\tilde{K}_\lambda(x_1, x_1, \ldots) = (-1)^{|\lambda|} G_\lambda(-x_1, -x_2, \ldots).$$

Remark 3.9.3. In [Buc02], Buch studied a bialgebra $\Gamma = \oplus_\lambda \mathbb{Z} G_\lambda$ spanned by the set of stable Grothendieck polynomials. Note that the bialgebra Γ is not the same as $\mathfrak{m}$Sym. In particular, the antipode formula given in Theorem 3.11.2 is valid in $\mathfrak{m}$Sym but not in Γ as only finite linear combinations of stable Grothendieck polynomials are allowed in Γ.

3.9.3 Weak set-valued tableaux

The following definition is needed to introduce one final basis for $\mathfrak{m}$Sym, $\{J_\lambda\}$.

Definition 3.9.4. *A weak set-valued tableau T of shape λ/ν is a filling of the boxes of the skew shape λ/ν with finite, non-empty multisets of positive integers so that*

(1) the largest number in each box is strictcy smaller than the smallest number in the box directly to the right of it, and

(2) the largest number in each box is less than or equal to the smallest number in the box directly below it.

In other words, we fill the boxes with multisets so that rows are strictly increasing and columns are weakly increasing. For example, the filling of shape $(3,2,1)$ shown below gives a weak set-valued tableau, T, of weight $x^T = x_1x_2^3x_3^3x_4^2x_5x_6x_7$.

12	33	46
223	4	
57		

Let $J_{\lambda/\nu} = \sum_T x^T$ be the weight generating function of weak set-valued tableaux T of shape λ/ν.

Theorem 3.9.5. *[LP07, Proposition 9.22] For any skew shape λ/ν, we have*

$$\omega(\tilde{K}_{\lambda/\nu}) = J_{\lambda/\nu}.$$

3.10 The Hopf algebra of Multi-symmetric functions

3.10.1 Reverse plane partitions

We next introduce the big Hopf algebra of Multi-symmetric function, $\mathfrak{M}$Sym, with basis $\{g_\lambda\}$. $\mathfrak{M}$Sym is isomorphic to *Sym* as a Hopf algebra, but the basis $\{g_\lambda\}$ is distinct from the basis of Schur functions, $\{s_\lambda\}$ for *Sym*.

Definition 3.10.1. *A reverse plane partition T of shape λ is a filling of the Young diagram of shape λ with positive integers such that the numbers are weakly increasing in rows and columns.*

Given a reverse plane partition T, let $T(i)$ denote the numbers of columns of T which contain the number i. Then $x^T := \prod_{i\in\mathbb{P}} x_i^{T(i)}$. Now we may define the *dual stable Grothendieck polynomial*

$$g_\lambda = \sum_{sh(T)=\lambda} x^T,$$

where we sum over all reverse plane partitions of shape λ. For a skew shape λ/μ, we may define g_λ/μ analogously, summing over reverse plane partitions of shape λ/μ.

Example 3.10.2. We use the definition of g_λ to compute

$$g_{(2,1)} = 2x_1x_2x_3 + 2x_1x_3x_4 + \ldots + x_1^2x_2 + x_1^2x_3 + \ldots + x_1^3 + x_2^3 + \ldots + x_1x_2 + x_1x_3 + \ldots$$

corresponding to fillings of type

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 1 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 1 \\ \cline{1-1}\end{array}.$$

3.10.2 Valued-set tableaux

We introduce one more basis for $\mathfrak{M}$Sym, $\{j_\lambda\}$, which is the continuous Hopf dual of $\{\tilde{J}_\lambda\} = \{(-1)^{|\lambda|}J_\lambda(-x_1, -x_2, \ldots)\}$.

Definition 3.10.3. *A valued-set tableau T of shape λ/μ is a filling of the boxes of λ/μ with positive integers so that*

(1) the transpose of the filling of T is a semistandard tableau, and

(2) we have a decomposition of the shape into a disjoint union of groups of boxes, $\lambda/\mu = \bigsqcup A_j$, so that each A_i is connected, contained in a single column, and each box in A_i contains the same number.

Given such a valued-set tableau, T, let a_i be the number of groups A_j that contain the number i. Then $x^T := \Pi_{i\geq 1}x_i^{a_i}$. Finally, let $j_{\lambda/\mu} := \sum_T x^T$, where the sum is over all valued-set tableaux of shape λ/μ.

Example 3.10.4. The image below shows an example of a valued-set tableau. This tableau contributes the monomial $x_1x_2x_3x_5x_6^2$ to $j_{(4,3,1,1)}$.

$$\begin{array}{|c|c|c|c|}\hline 1 & 3 & 5 & 6 \\ \hline 2 & 3 & 6 \\ \cline{1-3} 2 \\ 2 \\ \cline{1-1}\end{array}$$

Proposition 3.10.5. *[LP07, Proposition 9.25] We have*

$$\omega(g_{\lambda/\mu}) = j_{\lambda/\mu}.$$

3.11 Antipode results for $\mathfrak{m}$Sym and $\mathfrak{M}$Sym

As with $\mathfrak{m}$QSym and $\mathfrak{M}$NSym, there is a pairing $\langle g_\lambda, G_\mu \rangle = \delta_{\lambda,\mu}$ with the usual Hall inner product for Sym defined by $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda,\mu}$ and the structure constants satisfy the conditions of Lemma 3.2.12. See Theorem 9.15 in [LP07] for details. It follows that

$$\langle \omega(g_\lambda), \omega(G_\mu) \rangle = \langle j_\lambda, \tilde{J}_\mu \rangle = \delta_{\lambda,\mu}$$

and

$$\langle \tilde{j}_\lambda, \tilde{K}_\mu \rangle = \langle (-1)^{|\lambda|} g_\lambda(-x_1, -x_2, \ldots), (-1)^{|\mu|} G_\mu(-x_1, -x_2, \ldots) \rangle = \delta_{\lambda,\mu}.$$

We will use these facts to translate antipode results between $\mathfrak{m}$Sym and $\mathfrak{M}$Sym.

Using results from Section 3.7, the following lemma will allow us to easily prove results regarding the antipode map in $\mathfrak{m}$Sym.

Lemma 3.11.1. *We can expand* $J_\lambda = \sum_{n \geq N} \sum_{w \in \tilde{\mathcal{J}}_N(P,\theta)} \hat{L}_{\omega(\mathcal{C}(w))}$.

Proof. We know from Theorem 3.4.7 that

$$\tilde{K}_{(P,\theta)} = \sum_{n \geq N} \sum_{w \in \tilde{\mathcal{J}}_N(P,\theta)} \tilde{L}_{\mathcal{C}(w)},$$

so

$$J_\lambda = \omega(\tilde{K}_\lambda) = \sum_{n \geq N} \sum_{w \in \tilde{\mathcal{J}}_N(P,\theta)} \omega(\tilde{L}_{\mathcal{C}(w)}) = \sum_{n \geq N} \sum_{w \in \tilde{\mathcal{J}}_N(P,\theta)} \hat{L}_{\omega(\mathcal{C}(w))}.$$

□

Recall that in Sym, $S(s_\lambda) = (-1)^{|\lambda|}\omega(s_\lambda)$, so one may expect similar behavior from $\tilde{K}_\lambda$ and G_λ. Indeed, we obtain the theorem below.

Theorem 3.11.2. *In* $\mathfrak{m}$Sym*, the antipode map acts as follows.*

(a) $S(\tilde{K}_\lambda) = J_\lambda(-x_1, -x_2, \ldots) = (-1)^{|\lambda|}\omega(G_\lambda)$*, and*

(b) $S(G_\lambda) = (-1)^{|\lambda|} J_\lambda = (-1)^{|\lambda|}\omega(\tilde{K}_\lambda)$.

Proof. For the first assertion, we have that

$$\begin{aligned}
S(\tilde{K}_\lambda) &= S(\sum_{n\geq N}\sum_{w\in\tilde{\mathcal{J}}_N(P,\theta)}\tilde{L}_{\mathcal{C}(w)}) \\
&= \sum_{n\geq N}\sum_{w\in\tilde{\mathcal{J}}_N(P,\theta)} S(\tilde{L}_{\mathcal{C}(w)}) \\
&= \sum_{n\geq N}\sum_{w\in\tilde{\mathcal{J}}_N(P,\theta)} \hat{L}_{\omega(\mathcal{C}(w))}(-x_1,-x_2,\ldots) \\
&= J_\lambda(-x_1,-x_2,\ldots).
\end{aligned}$$

And for the second assertion,

$$\begin{aligned}
S(G_\lambda) &= S((-1)^{|\lambda|}\tilde{K}_\lambda(-x_1,-x_2,\ldots)) \\
&= (-1)^{|\lambda|}S(\tilde{K}_\lambda(-x_1,-x_2,\ldots) \\
&= (-1)^{|\lambda|}J_\lambda.
\end{aligned}$$

□

By Lemma 3.2.12, we immediately have the following results in $\mathfrak{M}$Sym.

Theorem 3.11.3. *We have*

(a) $S(\tilde{j}_\lambda) = (-1)^{|\lambda|}g_\lambda$*, where* $\tilde{j}_\lambda = (-1)^{|\lambda|}j_\lambda(-x_1,-x_2,\ldots)$*, and*

(b) $S(j_\lambda) = g_\lambda(-x_1,-x_2,\ldots)$.

Next, we work toward expanding $S(G_\lambda)$ and $S(\tilde{j}_\lambda)$ in terms of $\{G_\mu\}$ and $\{\tilde{j}_\mu\}$, respectively. We introduce two theorems of Lenart as well as the notion of a hook-restricted plane partitions.

Given partitions λ and μ with $\mu \subset \lambda$, define an *elegant filling* of the skew shape λ/μ to be a semistandard filling such that the numbers in row i lie in $[1, i-1]$. Now let f_λ^μ denote the number of elegant fillings of λ/μ for $\mu \subset \lambda$ and set $f_\lambda^\mu = 0$ otherwise.

Theorem 3.11.4. *[Len00, Theorem 2.7] For a partition* λ*, we have*

$$s_\lambda = \sum_{\mu\supset\lambda} f_\mu^\lambda G_\mu,$$

where f_μ^λ *is the number of elegant fillings of* λ/μ.

For the second theorem, let $r_{\lambda\mu}$ be the number of elegant fillings of λ/μ such that both rows and columns are strictly increasing. We will refer to such fillings as *strictly elegant*.

Theorem 3.11.5. *[Len00, Theorem 2.2] We can expand the stable Grothendieck polynomial G_λ in terms of Schur functions as follows*

$$G_\lambda = \sum_{\mu \supset \lambda} (-1)^{|\mu/\lambda|} r_{\lambda\mu} s_\mu.$$

Given two partitions, λ and μ, we now define the number P^μ_λ. First, $P^\mu_\lambda = 0$ if $\mu \not\subseteq \lambda$, and $P^\mu_\lambda = 1$ if $\lambda = \mu$. If $\mu \subset \lambda$, then P^μ_λ is equal to the number of *hook restricted plane partitions* of the skew shape λ/μ. A hook restricted plane partition is a filling of the boxes of λ/μ with positive integers such that the numbers are weakly decreasing along rows and columns with the following restrictions.

(1) If box b in shape λ/μ shares an edge with a box in shape μ, then the number in box b must lie in $[1, h(b)]$, where $h(b)$ is the number of boxes in μ lying above b in the same column as b or lying to the left of box b in the same row as b. We may think of $h(b)$ as being a reflected hook length of b.

(2) If box b in shape λ/μ does not share an edge with a box in shape μ, let a_1 and a_2 denote the boxes in λ/μ directly above and directly to the left of b. It is possible that one of these two boxes does not exist. Define $h(b)$ to be the minimum of $h(a_1)$ and $h(a_2)$, where $h(a_i) = \infty$ if box a_i does not exist. Then the number in box b must lie in $[1, h(b)]$.

Example 3.11.6. The diagram on the left shows $h(b)$ for each box b in the shape $(5, 5, 5)/(4, 2)$ and is also an example of a hook restricted plane partition on $(5, 5, 5)/(4, 2)$. The diagram on the right shows another hook restricted plane partition on $(5, 5, 5)/(4, 2)$.

				4
		3	3	3
2	2	2	2	2

				3
		3	3	3
2	2	2	1	1

Theorem 3.11.7. *Let λ and μ be partitions. Then*

(a) $S(G_\mu) = (-1)^{|\mu|} \sum_\lambda P_\lambda^{\mu^t} G_\lambda$, *and*

(b) $S(\tilde{j}_\lambda) = (-1)^{|\lambda|} \sum_\mu P^\mu_{\lambda^t} \tilde{j}_\mu$.

Proof. We will focus on part (a), and part (b) will follow from Lemma 3.2.12.

From Theorem 3.11.2, we know that

$$S(G_\lambda) = (-1)^{|\lambda|} J_\lambda,$$

so it remains to expand J_λ in terms of stable Grothendieck polynomials.

From Theorem 3.11.5, it easily follows that we can write

$$\tilde{K}_\lambda = \sum_{\mu \supset \lambda} r_{\lambda\mu} s_\mu.$$

Applying ω to both sides, we have

$$\tilde{J}_\lambda = \sum_{\mu \supset \lambda} r_{\lambda\mu} s_{\mu^t}.$$

Now we can use Theorem 3.11.4 to write

$$\tilde{J}_\lambda = \sum_{\substack{\mu \supset \lambda \\ \nu \supset \mu^t}} r_{\lambda\mu} f_\nu^{\mu^t} G_\nu.$$

Thus the coefficient of G_ν in $\tilde{J}_\lambda$ is $\displaystyle\sum_{\substack{\mu \text{ such that} \\ \mu \supset \lambda \text{ and} \\ \mu^t \subset \nu}} r_{\lambda\mu} f_\nu^{\mu^t}$.

We describe a bijection between partitions of shape ν^t which contain some $\mu \supset \lambda$ such that the filling of μ/λ is stricly elegant and boxes in ν^t/μ are filled such that the transpose is an elegant filling of ν/μ^t and hook-restricted plane partitions of ν/λ^t. Note that if we have a hook-restricted plane partition of ν^t/λ, then its transpose is a hook-restricted plane partition of ν/λ^t.

We first define a map ϕ from pairs consisting of a strictly elegant filling and the transpose of an elegant filling to a hook restricted plane partitions. Suppose we have such a filling of shape ν^t and some μ with $\lambda \subset \mu \subset \nu^t$. For any box b in ν^t, let $d(b)$ denote the southwest to northeast diagonal that contains box b. If box b is in row i and column j, then $d(b) = i + j - 1$. Let $c(b)$ denote the column that contains box b and e_b denote the integer in box b. To obtain a hook-restricted plane partition follow these steps:

(1) if box b is in μ, fill the corresponding box in the hook-restriced plane partition with $\phi(b) = d(b) - e_b$, and

(2) if box b is in ν^t/μ, fill the corresponding box in the hook-restricted plane partition with $\phi(b) = c(b) - e_b$.

It is easy to see that the parts of the hook-restricted plane partition correponding to shape μ and to ν^t/μ are weakly decreasing in rows and columns. We now check that entries are weakly decreasing along the seams. If box b is in μ, then $e_b \leq i_b - 1$, where b is in row i_b and column j_b. Therefore

$$\phi(b) = d(b) - e_b \geq i_b + j_b - 1 - (i_b - 1) = j_b.$$

If box a is in ν^t/μ, then $1 \leq \phi(a) = e_a \leq j_a - 1$, so

$$1 \leq c(a) - e_a = j_a - e_a \leq j_a - 1$$

If b and a are adjacent, then $j_b \leq j_a$, so $\phi(b) \geq \phi(a)$.

Next, we check that for all boxes in ν^t/λ that are adjacent to λ, $\phi(b) \in [1, h(b)]$, so the resulting filling is indeed a hook-restricted plane partition. Let box b be in μ in row i and column j with l boxes above b contributing to $h(b)$ and k boxes to the left of b contributing to $h(b)$. (See Example 3.11.8 below.) Then since rows and columns are strictly increasing, $e_b \geq (j - k) + (i - l - 1)$. If follows that

$$\phi(b) = d(b) - e_b \leq d(e) - (j - k) - (i - l - 1) = l + k = h(b),$$

as desired.

Example 3.11.8. In the figure below, boxes in λ are marked with a dot. For box b, we have $i = 2$, $j = 4$, $l = 1$, and $k = 2$.

$\cdot$	$\cdot$	$\cdot$	$\cdot$
$\cdot$	$\cdot$		b
$\cdot$	$\cdot$		

Next suppose box b described above is in ν^t/μ. Because the transpose of the filling of ν^t/μ is an elegent filling, $e_b \geq j - k$. Then we have that

$$\phi(b) = c(b) - e_b \leq k \leq k + l = h(b).$$

Note that since rows and columns of the image of ϕ are weakly decreasing, we have shown that $\phi(b) \in [1, h(b)]$ for all boxes b.

Beginning with a hook-restricted plane partition of ν^t/μ, we define a map, ψ, to recover μ and the fillings of μ/λ and ν^t/μ as follows. If the integer in the box in row i and column j is greater than or equal to j, then that box is in μ and $\psi(b) = d(b) - e_b$. Note that since $e_b \geq j$, $\psi(b) = (i + j - 1) - e_b \leq i - 1$, as is required to be strictly elegant. If the entry is less than j, that box is in ν^t/μ, and $\psi(b) = c(b) - e_b$. Note here that $e_b \leq j$ implies that $\psi(b) = c(b) - e_b \leq j - 1$, which is necessary to have an elegant filling. It is easy to see that rows and columns in μ will be strictly increasing in the image of ψ and that in ν^t/μ, rows will be stricly increasing and columns will be weakly increasing. Thus the image of ψ is a strictly elegant filling of $\mu \supset \lambda$ and an elegant filling of ν/μ^t. Clearly the composition of ϕ and ψ is the identity, so they are indeed inverses.

□

Note that the antipode applied to G_λ gives an infinite sum of stable Grothendieck polynomials (see Remark 3.9.3) while applying S to $\tilde{j}_\lambda$ can be written as a finite sum of $\tilde{j}$'s. This implies that while the space spanned by stable Grothendieck polynomials, Γ, is not a Hopf algebra, the space spanned by $\tilde{j}$'s is a Hopf algebra.

Example 3.11.9. To illustrate the bijection described above, consider $\lambda = (3, 2, 1)$, $\mu = (3, 3, 2, 2)$, and $\nu^t = (5, 4, 4, 3)$. The figure on the left is a filling such that μ/λ is strictly elegant and the transpose of ν^t/μ is elegant. The entries in μ/λ are in bold. The figure on the right is the corresponding hook-restricted plane partition of ν^t/λ.

			2	4
		1	3	
	1	1	3	
2	**3**	2		

			2	1
		3	1	
	3	2	1	
2	2	1		

If b is the box in the bottom left corner of the partition on the left, we see that $\phi(b) = d(b) - e_b = 4 - 2 = 2$. If a is the box in the upper right corner of the partition on the left, we have $\phi(a) = c(a) - e_a = 5 - 4 = 1$. In the hook-restricted plane partition on the left, we can see that the boxes in positions $(4, 1)$, $(3, 2)$, $(4, 2)$, and $(2, 3)$ are in μ/λ in the image of ψ since in these boxes $e_b \geq j_b$.

Chapter 4

K-theoretic Poirier-Reutenauer bialgebra

4.1 Introduction

In [PR95], Poirier and Reutenauer defined a Hopf algebra structure on the $\mathbb{Z}$-span of all standard Young tableaux, called the *Poirier-Reutenauer Hopf algebra* and denoted PR, using the Knuth equivalence relations on words and Robinson-Schensted-Knuth insertion as follows.

For a standard Young tableau T, define **T** to be the formal sum of all words Knuth equivalent to the reading word of T, or equivalently, the sum of all words that insert into T using the Robinson-Schensted-Knuth algorithm:

$$\mathbf{T} = \sum_{w \approx \mathfrak{row}(T)} w.$$

The product of **T** and **T'** is defined to be a shuffle product of the corresponding words, and the coproduct of **T** is defined using the standard cut coproduct on the corresponding words: $\Delta(w_1 \cdots w_k) = \sum_{i=0}^{k} w_1 \cdots w_i \otimes w_{i+1} \cdots w_k$. The antipode map will not be discussed here.

While being interesting in its own right, the Poirier-Reutenauer Hopf algebra allows us to obtain a version of the Littlewood-Richardson rule for the cohomology rings of

Grassmannians. In other words, it yields an explicitly positive description for the structure constants of the cohomology ring in the basis of Schubert classes. As previously discussed, Schubert classes can be represented by Schur functions of partitions that fit inside a rectangle. Thus, an essentially equivalent formulation of the problem is to describe structure constants of the ring of symmetric functions in terms of the basis of Schur functions. We refer the reader to [MS05] for a great introduction to the subject.

To obtain this version of Littlewood-Richardson rule, define a bialgebra morphism ϕ from PR to the ring of symmetric functions, Sym, by $\phi(\mathbf{T}) = s_{\lambda(T)}$, where $s_{\lambda(T)}$ is the Schur function indexed by the shape of tableau T. Then, using the product and coproduct formulas in PR, we arrive at the product and coproduct formulas in Sym given in Corollaries 4.2.4 and 4.2.5.

4.1.1 K-Poirier-Reutenauer bialgebra and Littlewood-Richardson rule

The combinatorics of the K-theory of Grassmannians has been developed in [FG06, FK96, LS83]. In [Buc02] Buch gave an explicit description of the *stable Grothendieck polynomials*, which represent Schubert classes in the K-theory ring. Such a description was already implicit in [FG06]. Buch then proceeded to give a Littlewood-Richardson rule, which describes the structure constants of the ring with respect to the basis of those classes. An alternative description of those structure constants was obtained by Thomas and Yong in [TY11b, TY11a].

In [BKS$^+$08], a natural analogue of Robinson-Schensted-Knuth insertion called *Hecke insertion* is defined. The result of such insertion is an *increasing tableau*, which is a natural analogue of a standard Young tableau.

A question arises then: can one use Hecke insertion to define a K-theoretic analogue of the Poirier-Reutenauer Hopf algebra? Can one then proceed to obtain a version of the Littlewood-Richardson rule analogous to Corollary 4.2.4 and Corollary 4.2.5? It turns out the answer is yes, although there are additional obstacles to overcome. This is the goal of this chapter. Our construction is described in detail in the future sections. Here we give a brief overview with emphasis on the nature of obstacles that appear and on methods used to overcome them.

It turns out that there is no *local* way to describe equivalence between words that Hecke insert into the same tableau. This was, of course, already known in [BKS$^+$08].

The consequence is that the verbatim definition of the Poirier-Reutenauer bialgebra simply does not work. Indeed, let $P_H(w)$ be the increasing tableau obtained by Hecke inserting word w. If for an increasing tableau T we define

$$\mathbf{T} = \sum_{P_H(w)=T} w,$$

where the sum is over all words that Hecke insert into T, the resulting sums are not closed under the natural product and coproduct, see Remark 4.4.8 and Remark 4.4.14.

Instead, we use classes defined by the *K-Knuth equivalence relation* of [BS14], a combination of the Hecke equivalence of [Buc02] and Knuth equivalence. The relation is defined by the following three local rules:

$$\begin{aligned} pp &\equiv p && \text{for all } p \\ pqp &\equiv qpq && \text{for all } p \text{ and } q \\ pqs \equiv qps \text{ and } sqp &\equiv spq && \text{whenever } p < s < q. \end{aligned}$$

It is important to note that the K-Knuth classes combine some classes of increasing tableaux, as seen in [BS14]. In other words, there are K-Knuth equivalence classes of words that have more than one corresponding tableau. For example, the K-Knuth equivalence class of 3124 contains the *two* increasing tableaux shown below.

1	2	4
3	4	

1	2	4
3		

We invite the reader to verify that the row reading words of those tableaux can be indeed connected to each other by K-Knuth equivalence relations.

In order to get a working version of the Littlewood-Richardson rule, such tableaux need to be avoided. We use the notion of a *unique rectification target* of Buch and Samuel [BS14], which are increasing tableaux with the property of being the only increasing tableau in their K-Knuth equivalence class. We will refer to a unique rectification target as a URT.

Finally, armed with this notion of unique rectification targets, we can state and prove the following versions of the Littlewood-Richardson rule, similar to those of Corollary 4.2.4 and Corollary 4.2.5. The first was proven previously in [BS14, Corollary 3.19] and in less generality in [TY11b, Theorem 1.2] using a K-theoretic analogue of jeu de taquin.

Theorem (Theorem 4.7.1)**.** *Let T be a URT of shape μ. Then the coefficient $c^{\nu}_{\lambda,\mu}$ in the decomposition*

$$G_\lambda G_\mu = \sum_{\nu} (-1)^{|\nu|-|\lambda|-|\mu|} c^{\nu}_{\lambda,\mu} G_\nu$$

is equal to the number of increasing tableaux R of skew shape ν/λ such that $P_H(\mathfrak{row}(R)) = T$.

While we obtain the next result only for unique rectification targets, [TY11a, Theorem 1.4] proves it for arbitrary increasing tableaux.

Theorem (Theorem 4.7.4)**.** *Let T_0 be a URT of shape ν. Then the coefficient $d^{\nu}_{\lambda,\mu}$ in the decomposition*

$$\Delta(G_\nu) = \sum_{\lambda,\mu} (-1)^{|\nu|-|\lambda|-|\mu|} d^{\nu}_{\lambda,\mu} G_\lambda \otimes G_\mu$$

is equal to the number of increasing tableaux R of skew shape $\lambda \oplus \mu$ such that $P_H(\mathfrak{row}(R)) = T_0$.

Remark 4.1.1. Let us clarify the relationship between the two previous theorems, the two theorems of Thomas and Yong: Theorem [TY11a, Theorem 1.4] and [TY11b, Theorem 1.4], and the result of Buch and Samuel [BS14, Corollary 3.19]. Their theorems are stated in terms of K-theoretic jeu de taquin, which is introduced by Thomas and Yong. In [BS14], Buch and Samuel prove that K-Knuth equivalence is equivalent to K-theoretic jeu de taquin equivalence in [BS14, Theorem 6.2], thus explaining the connection.

Therefore, our Theorem 4.7.1 is a corollary of [TY11a, Theorem 1.4], where our theorem is more specialized since we require T to be a URT. On the other hand, Theorem 4.7.4 is the same as [BS14, Corollary 3.10], which both generalize [TY11b, Theorem 1.4], as we allow S to be an arbitrary URT rather than fixing a particular (*superstandard*) choice.

In our proof of an analogue of Theorem 4.2.3, it is more natural to work with the *weak set-valued tableaux* defined in [LP07] than with the set-valued tableaux of Buch [Buc02]. However, as we show in Corollary 4.6.12, the two languages are equivalent.

4.1.2 Chapter overview

In Section 5.7.2, we describe the Poirier-Reutenauer Hopf algebra in detail and give all relevant definitions. We define the bialgebra morphism from PR to the ring of symmetric functions and derive a version of the Littlewood-Richardson rule.

In Section 4.3, we describe Hecke insertion and reverse Hecke insertion. We define the insertion tableau, $P_H(w)$, and the recording tableau, $Q_H(w)$, for a word w. We review several relevant results regarding Hecke insertion. We then review the *K-Knuth equivalence* [BS14] of finite words on the alphabet $\{1, 2, 3, \ldots\}$ and discuss certain characteristics of this equivalence.

In Section 4.4, we define $[[h]]$ to be the sum of all words in the Hecke equivalence class of a word h. We define a vector space, KPR, spanned by all such sums. We introduce a bialgebra structure on KPR and show that KPR has no antipode for this bialgebra structure. Thus, we obtain the K-theoretic Poirier-Reutenauer bialgebra.

In Section 4.5, we recall from [BS14] the notion of a *unique rectification target* (URT), a tableau that is the unique tableau in its K-Knuth equivalence class. We rephrase the product and coproduct formulas from the previous section for K-Knuth equivalence classes that correspond to URTs.

In Section 4.6, we draw a connection between the material in the previous sections and the ring of symmetric functions. We again define the *stable Grothendieck polynomials*, G_λ, as in [Buc02] by using set-valued tableaux and discuss their structure constants. We then recall how to use *weak set-valued tableaux* to define *weak stable Grothendieck polynomials*, J_λ. We show that the bialgebra structure constants of the G_λ and the J_λ coincide up to a sign. Using the fundamental quasisymmetric functions, we define a bialgebra morphism, ϕ, with the property that $\phi([[h]])$ can be written as a sum of weak stable Grothendieck polynomials.

In Section 4.7, we use the bialgebra morphism from Section 5 to state and prove a Littlewood-Richardson rule for the product and coproduct of the stable Grothendieck polynomials.

4.2 Poirier-Reutenauer Hopf algebra

In [PR95], Poirier and Reutenauer defined a Hopf algebra structure on the $\mathbb{Z}$-span of all standard Young tableaux. This Hopf algebra structure was later studied in papers such as [DHNT11] and [Tas06] and also appears implicitly in [LLT02]. Let us recall the definition and illustrate it with few examples.

Given a (possibly skew) Young tableau T, its *row reading word*, $\mathfrak{row}(T)$, is obtained by reading the entries in the rows of T from left to right starting with the bottom row and ending with the top row. For the first standard Young tableau shown below, $\mathfrak{row}(T) = 38257146$. For the standard Young tableau of skew shape, $\mathfrak{row}(T) = 2143$.

$$\begin{array}{ccc} 1 & 4 & 6 \\ 2 & 5 & 7 \\ 3 & 8 & \end{array} \qquad\qquad \begin{array}{ccc} & & 3 \\ & 1 & 4 \\ 2 & & \end{array}$$

Next consider words with distinct letters on some ordered alphabet A. We define the *Knuth relations* by the relations

$$pqs \approx qps \text{ and } sqp \approx spq \quad \text{whenever } p < s < q.$$

Given two words, w_1 and w_2, we say that they are *Knuth equivalent*, denoted $w_1 \approx w_2$, if w_2 can be obtained from w_1 by a finite sequence of Knuth relations. For example, $52143 \approx 25143$ because

$$52143 \approx 52413 \approx 25413 \approx 25143.$$

We can extend Knuth equivalence to tableaux by saying that tableaux T_1 and T_2 are equivalent, denoted $T_1 \approx T_2$, if $\mathfrak{row}(T_1) \approx \mathfrak{row}(T_2)$. For example,

$$T_1 = \begin{array}{cc} 1 & 2 \\ 3 & \end{array} \qquad \approx \qquad T_2 = \begin{array}{ccc} & & 2 \\ & 3 & \\ 1 & & \end{array} .$$

From Theorem 5.2.5 of [LLT02], any word with letters exactly $[k]$ is Knuth equivalent to $\mathfrak{row}(T)$ for a unique standard Young tableau T of straight shape. This unique standard Young tableau may be obtained via RSK insertion of the word, see [Sta99]. For example, $52143 \approx \mathfrak{row}(T)$ for

$$T = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}\,.$$

For a standard Young tableau T, let $\mathbf{T} = \sum_{w \approx \mathfrak{row}(T)} w$. In other words, $\mathbf{T}$ is the sum of words that are Knuth equivalent to $\mathfrak{row}(T)$. Let PR be the $\mathbb{R}$-vector space generated by the set of $\mathbf{T}$ for all standard Young tableaux.

Following [PR95], we next describe a bialgebra structure on PR. Start with two words, w_1 and w_2, in PR, where w_1 has letters exactly $[n]$ for some positive integer n. Recall that $w_2[n]$ is the word obtained by adding n to each letter of w_2. Now define the product $w_1 * w_2$ to be $w_1 \sqcup\!\sqcup\, w_2[n]$, the shuffle product of w_1 and $w_2[n]$. For example, $12 * 1 = 12 \sqcup\!\sqcup\, 3 = 123 + 132 + 312$.

For a word w without repeated letters, define $st(w)$ to be the unique word on $\{1, 2, \ldots, |w|\}$ obtained by applying the unique order-preserving injective mapping from the letters of w onto $\{1, 2, \ldots, |w|\}$ to the letters of w. For example, $st(1426) = 1324$. Then define the coproduct on PR by defining

$$\Delta(w) = \sum st(u) \otimes st(v),$$

where the sum is over all words u and v such that w is the concatenation of u and v. For example, $\Delta(312) = \emptyset \otimes 312 + 1 \otimes 12 + 21 \otimes 1 + 312 \otimes \emptyset$, where $\emptyset$ denotes the empty word. If we extend $*$ and Δ by linearity, PR forms a bialgebra [PR95].

4.2.1 Two versions of the Littlewood-Richardson rule

To see how the Poirier-Reutenauer Hopf algebra helps us obtain a Littlewood-Richardson rule for the cohomology rings of Grassmannians, let us state the following theorems. We use $T|_{[n]}$ to denote the tableau obtained from tableau T by deleting boxes with labels outside of $[n]$.

Theorem 4.2.1. *[LLT02, Theorem 5.4.3] Let T_1 and T_2 be standard Young tableaux. We have*

$$\mathbf{T_1} * \mathbf{T_2} = \sum_{T \in T(T_1 \sqcup\!\sqcup T_2)} \mathbf{T},$$

where $T(T_1 \sqcup\!\sqcup T_2)$ is the set of standard tableaux T such that $T|_{[n]} = T_1$ and $T|_{[n+1,n+m]} \approx T_2$.

Given a tableau T, define $\overline{T}$ to be the tableau of the same shape as T with reading word $st(\mathfrak{row}(T))$. The following theorem is analogous to Theorem 4.2.1 and is not hard to prove using the methods of [LLT02].

Theorem 4.2.2. *Let S be a standard Young tableau. We have*

$$\Delta(\mathbf{S}) = \sum_{(T',T'')\in T(S)} \overline{\mathbf{T}}' \otimes \overline{\mathbf{T}}'',$$

where $T(S)$ is the set of pairs of tableaux, T' and T'', such that $\mathfrak{row}(T')\mathfrak{row}(T'') \approx \mathfrak{row}(S)$.

Let Sym denote the ring of symmetric functions, and let s_λ denote its basis of Schur functions as in Section 3.8.

We are interested in a combinatorial rule for the coefficients $c^\nu_{\lambda,\mu}$ in the decompositions

$$s_\lambda s_\mu = \sum_\nu c^\nu_{\lambda,\mu} s_\nu.$$

To this end, we next define a bialgebra morphism from PR to Sym. We may then use the fact that we know how to multiply in PR to find the desired coefficients.

Define $\psi : PR \longrightarrow \mathrm{Sym}$ by

$$\psi(\mathbf{T}) = s_{\lambda(T)},$$

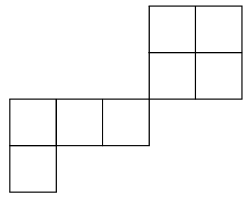
where $\lambda(T)$ denotes the shape of T.

Theorem 4.2.3. *[LLT02, Theorem 5.4.5] The map ψ is a bialgebra morphism.*

Applying ψ to the equalities in Theorem 4.2.1 and Theorem 4.2.2, we obtain the following two versions of the Littlewood-Richardson rule.

Corollary 4.2.4. *[Sta11, Theorem A1.3.1] Let T be a standard Young tableau of shape μ. The coefficient $c^\nu_{\lambda,\mu}$ is equal to the number of standard Young tableaux R of skew shape ν/λ such that $\mathfrak{row}(R) \approx \mathfrak{row}(T)$.*

Given two Young diagrams, λ and μ, define skew shape $\lambda \oplus \mu$ to be the skew shape obtained by putting λ and μ corner to corner. For example, the figure below shows $(3,1) \oplus (2,2)$.



Corollary 4.2.5. *[LLT02, Theorem 5.4.5] Let S be a standard Young tableau of shape ν. Then the coefficient $c^{\nu}_{\lambda,\mu}$ in the decomposition is equal to the number of standard Young tableaux R of skew shape $\lambda \oplus \mu$ such that $\mathfrak{row}(R) \approx S$.*

4.3 Hecke insertion and the K-Knuth monoid

4.3.1 Hecke insertion

An *increasing tableau* is a filling of a Young diagram with positive integers such that the entries in rows are strictly increasing from left to right and the entries in columns are strictly increasing from top to bottom.

Example 4.3.1. The tableau shown on the left is an increasing tableau. The tableau on the right is not an increasing tableau because the entries in the first row are not strictly increasing.

1	2	4	5
2	3	5	7
6	7		
8			

1	2	2	4
3	4		
5			

Lemma 4.3.2. *There are only finitely many increasing tableaux filled with a given finite alphabet.*

Proof. If the alphabet used has n letters, each row and each column cannot be longer than n. $\square$

Recall that $[k]$ denotes $\{1, \ldots, k\}$. Of particular importance in what follows will be increasing tableaux on exactly the alphabet $[k]$ for some positive integer k. We call such increasing tableaux *initial*. For example, the tableau below on the left is initial but the tableau on the right is not because its alphabet is $\{1, 2, 4, 5\}$, which is not $[k]$ for any k.

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 2 & 5 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array} \qquad\qquad \begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 5 \\ \hline 2 & 5 \\ \cline{1-2}\end{array}$$

We follow [BKS+08] to give a description of Hecke (row) insertion of a positive integer x into an increasing tableau Y resulting in an increasing tableau Z. We also record a box c where this insertion terminates. The shape of Z is obtained from the shape of Y by adding at most one box. If a box is added in position (i, j), then we set $c = (i, j)$. In the case where no box is added, then $c = (i, j)$, where (i, j) is a special corner indicating where the insertion process terminated. We will use a parameter $\alpha \in \{0, 1\}$ to keep track of whether or not a box is added to Y after inserting x by setting $\alpha = 0$ if $c \in Y$ and $\alpha = 1$ if $c \notin Y$. We use the notation $Z = (Y \xleftarrow{H} x)$ to denote the resulting tableau, and we denote the outcome of the insertion by (Z, c, α).

We now describe how to insert x into increasing tableau Y by describing how to insert x into R, a row of Y. This insertion may modify the row and may produce an output integer, which we will insert into the next row. To begin the insertion process, insert x into the first row of Y. The process stops when there is no output integer. The rules for insertion of x into R are as follows:

(H1) If x is weakly larger than all integers in R and adjoining x to the end of row R results in an increasing tableau, then Z is the resulting tableau and c is the new corner box where x was added.

(H2) If x is weakly larger than all integers in R and adjoining x to the end of row R does not result in an increasing tableau, then $Z = Y$, and c is the box at the bottom of the column of Z containing the rightmost box of the row R.

For the next two rules, assume R contains boxes strictly larger than x, and let y be the smallest such box.

(H3) If replacing y with x results in an increasing tableau, then replace y with x. In this case, y is the output integer to be inserted into the next row

(H4) If replacing y with x does not result in an increasing tableau, then do not change row R. In this case, y is the output integer to be inserted into the next row.

Example 4.3.3.

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 5 \\ \hline 2 & 3 & 4 & 6 \\ \hline 6 \\ \cline{1-1} 7 \\ \cline{1-1}\end{array} \xleftarrow{H} 3 \;=\; \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 5 \\ \hline 2 & 3 & 4 & 6 \\ \hline 6 \\ \cline{1-1} 7 \\ \cline{1-1}\end{array}$$

We use rule (H4) in the first row to obtain output integer 5. Notice that the 5 cannot replace the 6 in the second row since it would be directly below the 5 in the first row. Thus we use (H4) again and get output integer 6. Since we cannot add this 6 to the end of the third row, we use (H2) and get $c = (4, 1)$. Notice that the shape did not change in this insertion, so $\alpha = 0$.

Example 4.3.4.

$$\begin{array}{|c|c|c|}\hline 2 & 4 & 6 \\ \hline 3 & 6 & 8 \\ \hline 7 \\ \cline{1-1}\end{array} \xleftarrow{H} 5 \;=\; \begin{array}{|c|c|c|}\hline 2 & 4 & 5 \\ \hline 3 & 6 & 8 \\ \hline 7 & 8 \\ \cline{1-2}\end{array}$$

The integer 5 bumps the 6 from the first row using (H3). The 6 is inserted into the second row, which already contains a 6. Using (H4), the second row remains unchanged and we insert 8 into the third row. Since 8 is larger than everything in the third row, we use (H1) to adjoin it to the end of the row. Thus $\alpha = 1$ and $c = (3, 2)$.

In [BKS$^+$08], Buch, Kresch, Shimozono, Tamvakis, and Yong give the following algorithm for reverse Hecke insertion starting with the triple (Z, c, α) as described above and ending with a pair (Y, x) consisting of an increasing tableau and a positive integer.

(rH1) If y is the cell in square c of Z and $\alpha = 1$, then remove y and reverse insert y into the row above.

(rH2) If $\alpha = 0$, do not remove y, but still reverse insert it into the row above.

In the row above, let x be the largest integer such that $x < y$.

(rH3) If replacing x with y results in an increasing tableau, then we replace x with y and reverse insert x into the row above.

(rH4) If replacing x with y does not result in an increasing tableau, leave the row unchanged and reverse insert x into the row above.

(rH5) If R is the first row of the tableau, the final output consists of x and the modified tableau.

Theorem 4.3.5. *[BKS+ 08, Theorem 4] Hecke insertion $(Y, x) \mapsto (Z, c, \alpha)$ and reverse Hecke insertion $(Z, c, \alpha) \mapsto (Y, x)$ define mutually inverse bijections between the set of pairs consisting of an increasing tableau and a positive integer and the set of triples consisting of an increasing tableau, a corner cell of the increasing tableau, and $\alpha \in \{0, 1\}$.*

Buch, Kresch, Shimozono, Tamvakis, and Yong prove the following lemma about Hecke insertion, which will be useful later.

Lemma 4.3.6. *[BKS+ 08, Lemma 2] Let Y be an increasing tableau and let x_1 and x_2 be positive integers. Suppose that Hecke insertion of x_1 into Y results in (Z, c_1) and Hecke insertion of x_2 into Z results in (T, c_2). Then c_2 is strictly below c_1 if and only if $x_1 > x_2$.*

Define the *row reading* word of an increasing tableau T, $\mathfrak{row}(T)$, to be its content read left to right in each row, starting from the bottom row and ending with the top row.

Example 4.3.7. The second tableau in Example 4.3.4 has the reading word 78368245.

Suppose $w = w_1 \dots w_n$ is a word. Its *Hecke insertion tableau* is

$$P_H(w) = (\dots((\emptyset \xleftarrow{H} w_1) \xleftarrow{H} w_2) \dots \xleftarrow{H} w_n).$$

We shall also need the following two lemmas. For the first, recall that $T|_{[k]}$ denotes the tableau obtained from T by deleting all boxes with labels outside of $[k]$. We similarly define $w|_{[k]}$ to be the word obtained from w by deleting all letters outside of $[k]$.

Lemma 4.3.8. *If $P_H(w) = T$ then $P_H(w)|_{[k]} = P_H(w|_{[k]}) = T|_{[k]}$.*

Proof. This follows from the insertion rules; letters greater than k never affect letters in $[k]$. $\square$

Lemma 4.3.9. *For any tableau T, $P_H(\mathfrak{row}(T)) = T$.*

Proof. It is easy to see that when each next row is inserted, it pushes down the previous rows. $\square$

4.3.2 Recording tableaux

Recall the definition of set-valued tableaux from Section 3.9.1. Next, we present this same definition in a way that is independent of the material in Chapter 3.

A *set-valued tableau T* of shape λ is a filling of the boxes with finite, non-empty subsets of positive integers so that

1. the smallest number in each box is greater than or equal to the largest number in the box directly to the left of it (if that box is present), and

2. the smallest number in each box is strictly greater than the largest number in the box directly above it (if that box is present).

Given a word $h = h_1 \dots h_l$, we can associate a pair of tableaux $(P_H(h), Q_H(h))$, where $P_H(h)$ is the insertion tableau described previously and $Q_H(h)$ is a set-valued tableau called the *recording tableau* obtained as follows. Start with $Q_H(\emptyset) = \emptyset$. At each step of the insertion of h, let $Q_H(h_1 \dots h_k)$ be obtained from $Q_H(h_1 \dots h_{k-1})$ by labeling the special corner, c, in the insertion of h_k into $P_H(h_1 \dots h_{k-1})$ with the positive integer k. Then $Q_H(h) = Q_H(h_1 h_2 \dots h_l)$ is the resulting strictly increasing set-valued tableau.

Example 4.3.10. Let h be 15133. We obtain $(P_H(h), Q_H(h))$ with the following sequence, where in column k, $Q_H(h_1 \dots h_k)$ is shown below $P_H(h_1 \dots h_k)$.

$$\begin{array}{|c|}\hline 1 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 5 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 5 \\ \hline 5 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 5 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 5 \\ \cline{1-1}\end{array} = P_H(h)$$

$$\begin{array}{|c|}\hline 1 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 34 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 25 \\ \hline 34 \\ \cline{1-1}\end{array} = Q_H(h)$$

Call a word h *initial* if the letters appearing in it are exactly the numbers in $[k]$ for some positive integer k. For example, 16624315 is initial, but 153125 is not initial.

Example 4.3.11. The word 13422 is initial since the letters appearing in it are the numbers from 1 to 4. On the other hand, the word 1422 is not initial because the letters appearing in it are 1, 2, and 4 and do not form a set $[k]$ for any k.

Theorem 4.3.12. *The map sending $h = h_1 \ldots h_n$ to $(P_H(h), Q_H(h))$ is a bijection between words and ordered pairs of tableaux of the same shape (P, Q), where P is an increasing tableau and Q is a set-valued tableau with entries $\{1, 2, \ldots, n\}$. It is also a bijection if there is an extra condition of being initial imposed both on h and P.*

Proof. It is clear from the definition of $Q_H(h)$ that $P_H(h)$ and $Q_H(h)$ have the same shape, and it is clear from the insertion algorithm that $P_H(h)$ is an increasing tableau and $Q_H(h)$ is an increasing set-valued tableau. Thus, we must show that given (P, Q), one can uniquely recover h.

To recover h, perform reverse Hecke insertion in P multiple times as follows. Let l be the the largest entry in Q and call its cell $c(l)$. If l is the only entry in $c(l)$ inside in Q, perform reverse Hecke insertion with the triple $(P, c(l), \alpha = 1)$. If the l is not the only entry in its cell in Q, perform reverse Hecke insertion with the triple $(P, c(l), \alpha = 0)$. This reverse Hecke insertion will end with output (P_2, x_l). Set $Q_2 = Q - \{l\}$, and follow the same procedure described above replacing Q with Q_2 and P with P_2. The reverse insertion will end with output (P_3, x_{l-1}). Set $Q_3 = Q_2 - \{l-1\}$. Continue this process until the output tableau is empty. By Theorem 4.3.5, $h = x_1 \ldots x_l$, $P_H(h) = P$, and $Q_H(h) = Q$. $\square$

Example 4.3.13. We start with the pair (P, Q) from the previous example and recover h.

$$P = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 5 \\ \cline{1-1}\end{array} \qquad\qquad Q = \begin{array}{|c|c|}\hline 1 & 25 \\ \hline 34 \\ \cline{1-1}\end{array}$$

We first notice the largest entry of Q is in cell $(1, 2)$ and is not the smallest entry in cell $(1, 2)$, so we perform the reverse Hecke insertion determined by the triple $(P, (1, 2), 0)$. The output of this reverse insertion is $(P_2, 3)$, so $h_5 = 3$.

$$P_2 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 5 \\ \cline{1-1}\end{array} \qquad\qquad Q_2 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 34 \\ \cline{1-1}\end{array}$$

The largest entry in Q_2 is in cell $(2,1)$ and is not the smallest entry in cell $(2,1)$, so we perform the reverse Hecke insertion determined by $(P_2,(2,1),0)$ and obtain output $(P_3,3)$. Thus $h_4=3$.

$$P_3 = \begin{array}{|c|c|}\hline 1 & 5 \\ \hline 5 \\ \cline{1-1}\end{array} \qquad\qquad Q_3 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$$

The largest entry in Q_3 is in cell $(2,1)$ and is the smallest entry in its cell. We perform reverse insertion $(P_3,(2,1),1)$, obtain output $(P_4,1)$, and set $h_3=1$.

$$P_4 = \begin{array}{|c|c|}\hline 1 & 5 \\ \hline\end{array} \qquad\qquad Q_4 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}$$

In the last two steps, we recover $h_2=5$ and $h_1=1$.

4.3.3 K-Knuth equivalence

We next introduce the *K-Knuth monoid* of [BS14] as the quotient of the free monoid of all finite words on the alphabet $\{1,2,3,\ldots\}$ by the following relations:

$$\begin{array}{lcl} (1) & pp \equiv p & \text{for all } p \\ (2) & pqp \equiv qpq & \text{for all } p \text{ and } q \\ (3) & pqs \equiv qps \text{ and } sqp \equiv spq & \text{whenever } p<s<q. \end{array}$$

This monoid is better suited for our purposes than Hecke monoid of [BKS+08]; see Remark 4.5.4.

We shall say two words are *K-Knuth equivalent* if they are equal in the K-Knuth monoid. We denote K-Knuth equivalence by $\equiv$. We shall also say two words are *insertion equivalent* if they Hecke insert into the same tableau. We shall denote insertion equivalence by $\sim$.

Example 4.3.14. The words 34124 and 3124 are K-Knuth equivalent, since

$$34124 \equiv 31424 \equiv 31242 \equiv 13242 \equiv 13422 \equiv 1342 \equiv 1324 \equiv 3124.$$

They are not insertion equivalent, however, since they insert into the following two distinct tableaux.

$$P_H(34124) = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 4 \\ \cline{1-2}\end{array} \qquad\qquad P_H(3124) = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array}\ .$$

Example 4.3.15. As we soon shall see, $13524 \not\equiv 15324$.

4.3.4 Properties of K-Knuth equivalence

We will need three additional properties of Hecke insertion and K-Knuth equivalence. The first follows from [BS14, Theorem 6.2].

Theorem 4.3.16. *Insertion equivalence implies K-Knuth equivalence: if $w_1 \sim w_2$ for words w_1 and w_2, then $w_1 \equiv w_2$.*

As we saw in Example 4.3.14, the converse of this result is not true.

We now examine the length of the longest strictly increasing subsequence of a word w, denoted by $\mathfrak{lis}(w)$, and length of the longest strictly decreasing subsequence of w, $\mathfrak{lds}(w)$. To illustrate, for $w = 13152435$, $\mathfrak{lis}(w) = 4$, which is obtained by strictly increasing subsequence 1245, and $\mathfrak{lds}(w) = 3$, which is obtained by strictly decreasing subsequence 543. The next result follows from the K-Knuth equivalence relations.

Lemma 4.3.17. *If $w_1 \equiv w_2$, then $\mathfrak{lis}(w_1) = \mathfrak{lis}(w_2)$ and $\mathfrak{lds}(w_1) = \mathfrak{lds}(w_2)$.*

Proof. It is enough to assume the two words differ by one equivalence relation.

Suppose $w_1 = upv$ and $w_2 = uppv$ for some possibly empty words u and v. Then if $u'pv'$ is a strictly increasing sequence in w_1, for some possibly empty u' subword of u and v' subword of v, it is also a strictly increasing sequence in w_2. And since a strictly increasing sequence can only use one occurrence of p, any strictly increasing sequence in w_2 is also strictly increasing in w_1.

Next, consider the case where $w_1 = upqpv$ and $w_2 = uqpqv$, and assume $p < q$. If $u'pqv'$ is a strictly increasing sequence in w_1, notice that we have the same sequence in w_2. Similarly, if a strictly increasing sequence of w_1 is of the form $u'pv'$ or $u'qv'$, we have the same sequence in w_2. Since strictly increasing sequences of w_2 involving the p or q that are outside of u and v have the same form as those described above, any strictly increasing sequence of w_2 is appears as a strictly increasing sequence in w_1.

Lastly, suppose $w_1 = upqsv$ and $w_2 = uqpsv$ for $p < s < q$. If a strictly increasing sequence in w_1 (resp. w_2) uses only one of the p and q outside of u and v, then clearly this is still a strictly increasing sequence in w_2 (resp. w_1). If a strictly increasing sequence in w_1 is $u'pqv'$, then $u'psv'$ is a strictly increasing sequence in w_2 of the same length and vice versa.

A similar argument applies for $\mathfrak{lds}(w_1)$ and $\mathfrak{lds}(w_2)$. $\square$

We can use this result to verify that 13524 is not K-Knuth equivalent to 15324, as promised in Example 4.3.15. Indeed, $\mathfrak{lds}(13524) = 2$ and $\mathfrak{lds}(15324) = 3$.

Remark 4.3.18. We do not know any analogue of the other Greene-Kleitman invariants; see [Gre76] and [TY09].

We shall need the following lemma.

Lemma 4.3.19. *[BS14, Lemma 5.5] Let I be an interval in the alphabet A. If $w \equiv w'$, then $w|_I \equiv w'|_I$.*

The last result in this section was proved by H. Thomas and A. Yong in [TY09]. It gives information about the shape of $P_H(w)$ and of $P_H(h)$ for any $h \equiv w$.

Theorem 4.3.20. *[TY09, Theorem 1.3] For any word w, the size of the first row and first column of its insertion tableau are given by $\mathfrak{lis}(w)$ and $\mathfrak{lds}(w)$, respectively.*

4.4 K-theoretic Poirier-Reutenauer

Let $[[h]]$ denote the sum of all words in the K-Knuth equivalence class of an initial word h:

$$[[h]] = \sum_{h \equiv w} w.$$

This is an infinite sum. The number of terms in $[[h]]$ of length l is finite, however, for every positive integer l.

Let KPR denote the vector space spanned by all sums of the form $[[h]]$ for some initial word h. We will endow KPR with a product and a coproduct structure, which are compatible with each other. We will refer to the resulting bialgebra as the *K-theoretic Poirier-Reutenauer bialgebra* and denote it by KPR.

4.4.1 K-Knuth equivalence of tableaux

Suppose we have increasing tableaux T and T'. Recall that $\mathfrak{row}(T)$ denotes the row reading word of T. As in [BS14], we say that T is *K-Knuth equivalent to T'*, denoted $T \equiv T'$, if $\mathfrak{row}(T) \equiv \mathfrak{row}(T')$.

Example 4.4.1. For the T and T' shown below, we have that $T \equiv T'$ because

$$\mathfrak{row}(T) = 34124 \equiv \mathfrak{row}(T') = 3124$$

as shown in Example 4.3.14.

$$T = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 4 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array} \qquad \equiv \qquad T' = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}$$

Note that by Lemma 4.3.17 and Theorem 4.3.20, if two tableaux are equivalent, their first rows have the same size and their first columns have the same size.

The following lemma says that each element of KPR splits into insertion classes of words.

Lemma 4.4.2. *We have*

$$[[h]] = \sum_T \left(\sum_{P_H(w)=T} w \right)$$

where the sum is over all increasing tableaux T whose reading word is in the K-Knuth equivalence class of h.

Proof. The result follows from Theorem 4.3.16. $\square$

This expansion is always finite by Lemma 4.3.2.

4.4.2 Product structure

Let $\sqcup\!\sqcup$ denote the usual shuffle product of words as defined in Example 3.2.7: the sum of words obtained by interleaving the two words while maintaining their relative order. Let h be a word in the alphabet $[n]$, and let h' be a word in the alphabet $[m]$. Let $w[n]$ denote the word obtained from w by increasing each letter by n. Define the product $[[h]] \cdot [[h']]$ by

$$[[h]] \cdot [[h']] = \sum_{w \equiv h, w' \equiv h'} w \sqcup\!\sqcup w'[n].$$

Theorem 4.4.3. *For any two initial words h and h', their product can be written as*

$$[[h]] \cdot [[h']] = \sum_{h''} [[h'']],$$

where the sum is over a certain set of initial words h''.

Proof. From Lemma 4.3.19, we know that if a word appears in the right-hand sum, the entire equivalence class of this word appears as well. The claim follows. $\square$

Example 4.4.4. Let $h = 12$, $h' = 312$. Then

$$\begin{aligned}
[[12]] \cdot [[312]] &= 12 \sqcup\!\sqcup 534 + 12 \sqcup\!\sqcup 354 + 112 \sqcup\!\sqcup 534 + 112 \sqcup\!\sqcup 354 + \dots \\
&= [[53124]] + [[51234]] + [[35124]] + [[351234]] + [[53412]] + [[5351234]].
\end{aligned}$$

Theorem 4.4.5. *Let h be a word in alphabet $[n]$, and let h' be a word in alphabet $[m]$. Suppose $\mathcal{T} = \{P_H(h), T_1', T_2', \dots, T_s'\}$ is the equivalence class containing $P_H(h)$. We have*

$$[[h]] \cdot [[h']] = \sum_{T \in T(h \sqcup\!\sqcup h')} \sum_{P_H(w) = T} w,$$

where $T(h \sqcup\!\sqcup h')$ is the finite set of tableaux T such that $T|_{[n]} \in \mathcal{T}$ and $\mathfrak{row}(T)|_{[n+1,n+m]} \equiv h'[n]$.

Proof. If w is a shuffle of some $w_1 \equiv h$ and $w_2 \equiv h'[n]$, then by Lemma 4.3.8 $P_H(w)|_{[n]} = P(w_1) \in \mathcal{T}$. By Lemma 4.4.2 and Theorem 4.4.3, we get the desired expansion. Its finiteness follows from Lemma 4.3.2. $\square$

Example 4.4.6. If we take $h = 12$ and $h' = 312$, then

$$P_H(h) = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array} \quad \text{and } P_H(h') = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}\ .$$

The insertion tableaux appearing in their product are those shown below.

$$\begin{array}{lll} 1&2&4\\ 3\\ 5 \end{array} \qquad \begin{array}{lll} 1&2&4\\ 3&5\\ 5 \end{array} \qquad \begin{array}{lll} 1&2&4\\ 3&4\\ 5 \end{array} \qquad \begin{array}{llll} 1&2&3&4\\ 5 \end{array} \qquad \begin{array}{lll} 1&2&4\\ 3&5 \end{array} \qquad \begin{array}{llll} 1&2&3&4\\ 3&5 \end{array}$$

$$\begin{array}{ll} 1&2\\ 3&4\\ 5 \end{array} \qquad \begin{array}{llll} 1&2&3&4\\ 3\\ 5 \end{array} \qquad \begin{array}{llll} 1&2&3&4\\ 3&5\\ 5 \end{array}$$

Each of them restricted to $[2]$ is clearly $P_H(12)$. One can check that each of the row reading words restricted to the alphabet $\{3, 4, 5\}$ is K-Knuth equivalent to 534. For example, in case of the last tableau

$$53534 \equiv 53354 \equiv 5354 \equiv 5534 \equiv 534.$$

Note that the first three tableaux listed are equivalent to each other and the last two tableaux listed are equivalent to each other. We will see in the next section that the fourth and sixth tableaux are not equivalent. With this in mind, we can see that there are no other equivalent pairs by examining the sizes of the first rows and first columns. The six classes of tableaux in this example correspond to the six equivalence classes in Example 4.4.4.

Corollary 4.4.7. *The vector space KPR is closed under the product operation. That is, the sum appearing on the right hand side in Theorem 4.4.3 is always finite.*

Proof. We know from Lemma 4.4.2 that K-Knuth classes are coarser than insertion classes. Thus finiteness of right hand side in Theorem 4.4.3 follows from that in Theorem 4.4.5. $\square$

Remark 4.4.8. The product of insertion classes is not necessarily a linear combination of insertion classes. For example, consider the following tableaux, T and T'.

$$T = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array} \qquad\qquad T' = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array}$$

Then 12 and 1342 are in the insertion classes of T and T', respectively, and we get 315642 as a term in their shuffle product. The insertion tableau of 315642 is shown below.

$$P_H(315642) = \begin{array}{|c|c|c|}\hline 1 & 2 & 6 \\ \hline 3 & 4 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array}$$

Notice that $P_H(3156442) = P_H(315642)$, but 3156442 will not appear in the shuffle product of the insertion classes of 12 and 1342 since $314562|_{[3,6]} = 35644 \not\sim 3564$.

4.4.3 Coproduct structure

For any word w, let $\overline{w}$ denote the *standardization* of w: if a letter a is kth smallest among the letters of w, it becomes k in $\overline{w}$. For example, $\overline{42254} = 21132$. Note that the standardization of a word is always an initial word.

Let $w = a_1 \ldots a_n$ be an initial word. Define the coproduct of a word w, $\Delta(w)$, by

$$\Delta(w) = \sum_{i=0}^{n} \overline{a_1 \ldots a_i} \otimes \overline{a_{i+1} \ldots a_n}.$$

Similarly, define the coproduct of $[[h]]$ by

$$\Delta([[h]]) = \sum_{h \equiv w} \Delta(w).$$

Example 4.4.9. We have

$$\Delta(34124) = \emptyset \otimes 34124 + 1 \otimes 3123 + 12 \otimes 123 + 231 \otimes 12 + 3412 \otimes 1 + 34124 \otimes \emptyset,$$

and

$$\Delta([[34124]]) = \Delta(34124) + \Delta(31424) + \Delta(31242) + \dots .$$

Here, $\emptyset$ should be understood to be the identity element of the ground field. We denote it by $\emptyset$ so as to avoid confusion with the word 1.

Theorem 4.4.10. *For any initial word h, its coproduct can be written as*

$$\Delta([[h]]) = \sum_{h',h''} [[h']] \otimes [[h'']],$$

where the sum is over a certain set of pairs of initial words h', h''.

Proof. For $w = a_1 \dots a_n$ let $\blacktriangle(w) = \sum_{i=0}^{n} a_1 \dots a_i \otimes a_{i+1} \dots a_n$, and $\blacktriangle([[h]]) = \sum_{h \equiv w} \blacktriangle(w)$. It is clear that $\blacktriangle([[h]]) = \sum_{h',h''} [[h']] \otimes [[h'']]$ for some collection of pairs of words h' and h''. This is because K-Knuth equivalence relations are local and thus can be applied on both sides of $\otimes$ in parallel with applying the same relation to the corresponding word on the left. It remains to standardize every term on the right and to use the fact that K-Knuth equivalence relations commute with standardization. $\square$

Example 4.4.11. If we take $h = 12$ in the previous theorem, we have

$$\Delta([[12]]) = [[\emptyset]] \otimes [[12]] + [[1]] \otimes [[1]] + [[12]] \otimes [[1]] + [[1]] \otimes [[12]] + [[12]] \otimes [[\emptyset]].$$

Theorem 4.4.12. *Let h be a word. We have*

$$\Delta([[h]]) = \sum_{(T',T'') \in T(h)} \left(\sum_{P_H(w) = \overline{T'}} w \right) \otimes \left(\sum_{P_H(w) = \overline{T''}} w \right),$$

where $T(h)$ is the finite set of pairs of tableaux, T' and T'', such that $\mathfrak{row}(T')\mathfrak{row}(T'') \equiv h$.

Proof. As we have seen in the proof of Theorem 4.4.10, $\blacktriangle([[h]]) = \sum_{h',h''}[[h']] \otimes [[h'']]$. Note that the sum on the right is multiplicity-free and that if we split each of the $[[h']]$ and $[[h'']]$ into insertion classes, we get exactly

$$\sum_{(T',T'')\in T(h)} \left(\sum_{P_H(w)=T'} w \right) \otimes \left(\sum_{P_H(w)=T''} w \right).$$

It remains to apply standardization to get the desired result. The finiteness follows from Lemma 4.3.2. □

Corollary 4.4.13. *The vector space KPR is closed under the coproduct operation. That is, the sums appearing on the right hand side in Theorem 4.4.10 are always finite.*

Proof. Entries in the tableaux T and T' are a subset of letters in the word h. The statement follows from finiteness in Theorem 4.4.12 and the fact that K-Knuth classes are coarser than insertion classes. □

Remark 4.4.14. It is not true that insertion classes are closed under the coproduct. For example, $123 \otimes 1$ is a term in $\Delta(1342)$ and thus in the coproduct of its insertion class, but $123 \otimes 11$ is not. To see this, consider all words h containing only 1, 2, 3, and 4 such that $123 \otimes 11$ is in $\Delta(h)$. These words are 12344, 12433, 13422, and 23411, none of which are insertion equivalent to 1342.

4.4.4 Compatibility and antipode

Recall that a product $\cdot$ and a coproduct Δ are *compatible* if the coproduct is an algebra morphism:

$$\Delta(X \cdot Y) = \Delta(X) \cdot \Delta(Y).$$

A vector space endowed with a compatible product and coproduct is called a *bialgebra*. We refer the reader to [SS93] for details on bialgebras.

Theorem 4.4.15. *The product and coproduct structures on KPR defined above are compatible, thus giving KPR a bialgebra structure.*

Proof. The result follows from the fact that the same is true for initial words. Indeed, if $u = a_1 \dots a_n$ and $w = b_1 \dots b_m$ are two initial words, then

$$\Delta(w \cdot u) = \Delta(w \sqcup\!\sqcup\, u[n]) = \sum_{v=c_1 \dots c_{n+m}} \sum_{i=1}^{n+m} \overline{c_1 \dots c_i} \otimes \overline{c_{i+1} \dots c_{n+m}},$$

where v ranges over shuffles of w and $u[n]$. On the other hand,

$$\Delta(w) \cdot \Delta(u) = \left(\sum_{i=1^n} \overline{a_1 \dots a_i} \otimes \overline{a_{i+1} \dots a_n} \right) \cdot \left(\sum_{j=1^m} \overline{b_1 \dots b_j} \otimes \overline{b_{j+1} \dots b_m} \right) =$$

$$= \sum_{i,j} (\overline{a_1 \dots a_i} \sqcup\!\sqcup\, \overline{b_1 \dots b_j}[i]) \otimes (\overline{a_{i+1} \dots a_n} \sqcup\!\sqcup\, \overline{b_{j+1} \dots b_m}[n-i]).$$

The two expressions are easily seen to be equal. □

We also remark that KPR has no antipode map. Indeed, assume S is an antipode. Then since

$$\Delta([[1]]) = \emptyset \otimes [[1]] + [[1]] \otimes [[1]] + [[1]] \otimes \emptyset,$$

we solve

$$S([[1]]) = -\frac{[[1]]}{\emptyset + [[1]]}.$$

This final expression is not a finite linear combination of basis elements of KPR, and thus does not lie in the bialgebra.

4.5 Unique Rectification Targets

As we have seen, K-Knuth equivalence classes may have several corresponding insertion tableaux. The following is an open problem.

Problem 4.5.1. *Describe K-Knuth equivalence classes of increasing tableaux.*

Of special importance are the K-Knuth equivalence classes with only one element.

4.5.1 Definition and examples

We call T a *unique rectification target* or a URT if it is the only tableau in its K-Knuth equivalence class [BS14, Definition 3.5]. In other words, T is a URT if for every $w \equiv \mathfrak{row}(T)$ we have $P_H(w) = T$. The terminology is natural in the context of the K-theoretic jeu de taquin of Thomas and Yong [TY09]. If $P_H(w)$ is a URT, we call the equivalence class of w a *unique rectification class.*

For example, $P_H(1342)$ is not a unique rectification target because $3124 \equiv 34124$, as shown in Example 4.3.14, and $P_H(3124) \neq P_H(34124)$ as shown below.

$$P_H(3124) = \begin{ytableau} 1 & 2 & 4 \\ 3 \end{ytableau} \qquad\qquad P_H(34124) = \begin{ytableau} 1 & 2 & 4 \\ 3 & 4 \end{ytableau}$$

It follows that [[3124]] is not a unique rectification class.

In [BS14], Buch and Samuel give a uniform construction of unique rectification targets of any shape as follows. Define the *minimal increasing tableau* M_λ of shape λ by filling the boxes of λ with the smallest possible values allowed in an increasing tableau. In other words, M_λ is the tableau obtained by filling all boxes in the kth southwest to northeast diagonal of λ with positive integer k.

Example 4.5.2. The tableaux below are minimal increasing tableaux.

$$M_{(3,2,1)} = \begin{ytableau} 1 & 2 & 3 \\ 2 & 3 \\ 3 \end{ytableau} \qquad\qquad M_{(5,2,1,1)} = \begin{ytableau} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 \\ 3 \\ 4 \end{ytableau}$$

In [TY11b, Theorem 1.2], Thomas and Yong prove that the *superstandard tableau* of shape λ, S_λ, is a URT, where S_λ is defined to be the standard Young tableau with $1, 2, \ldots, \lambda_1$ in the first row, $\lambda_1 + 1, \lambda_1 + 2, \ldots, \lambda_1 + \lambda_2$ in the second row, etc.

Example 4.5.3. The following are superstandard tableaux.

$$S_{(3,2,1)} = \begin{ytableau} 1 & 2 & 3 \\ 4 & 5 \\ 6 \end{ytableau} \qquad\qquad S_{(5,2,1,1)} = \begin{ytableau} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 \\ 8 \\ 9 \end{ytableau}$$

The interested reader should consult [GMP+16] for the latest work on URTs and K-Knuth equivalence of increasing tableaux. In particular, this paper gives an efficient algorithm to determine if a given tableau is a URT, which was stated as an open problem in the first version of this chapter.

Remark 4.5.4. Note that if one uses the less restrictive Hecke equivalence of [BKS+08] instead of the K-Knuth equivalence of [BS14], URT are extremely scarce. For example, the tableau with reading word 3412 is equivalent to the tableau with reading word 3124. In fact, with this definition there is no standard URT of shape $(2, 2)$.

4.5.2 Product and coproduct of unique rectification classes

As we have seen before, the product and coproduct of insertion classes do not necessarily decompose into insertion classes. However, the story is different if the classes are unique rectification classes, as seen in the following theorems.

Theorem 4.5.5. *Let T_1 and T_2 be URTs. We have*

$$\left(\sum_{P_H(w)=T_1} w\right) \cdot \left(\sum_{P_H(w)=T_2} w\right) = \sum_{T \in T(T_1 \sqcup\!\sqcup T_2)} \sum_{P_H(w)=T} w,$$

where $T(T_1 \sqcup\!\sqcup T_2)$ is the finite set of tableaux T such that $T|_{[n]} = T_1$ and $P_H(\mathfrak{row}(T)|_{[n+1,n+m]}) = T_2$.

Proof. Since T_1 and T_2 are URTs, the left hand side is $([[\mathfrak{row}(T_1)]]) \cdot ([[\mathfrak{row}(T_2)]])$ and $T(T_1 \sqcup\!\sqcup T_2)$ is $T(\mathfrak{row}(T_1) \sqcup\!\sqcup \mathfrak{row}(T_2))$ as in Theorem 4.4.5. $\square$

Theorem 4.5.6. *Let T_0 be a URT. We have*

$$\Delta\left(\sum_{P_H(w)=T_0} w\right) = \sum_{(T',T'') \in T(T_0)} \left(\sum_{P_H(w)=\overline{T'}} w\right) \otimes \left(\sum_{P_H(w)=\overline{T''}} w\right),$$

where $T(T_0)$ is the finite set of pairs of tableaux, T' and T'', such that $P_H(\mathfrak{row}(T')\mathfrak{row}(T'')) = T_0$.

Proof. Since T_0 is a URT, the left hand term is $\Delta([[\mathfrak{row}(T_0)]])$ and $T(T_0) = T(\mathfrak{row}(T_0))$ as described in Theorem 4.4.12. $\square$

Remark 4.5.7. Note that a product of unique rectification classes is not necessarily a sum of unique rectification classes. For example, if we let $w' = 12$ and $w''[2] = 34$, then 13422 appears in the shuffle product. One checks $P_H(13422)$ is one of the tableaux in Example 4.3.14 and thus is not a URT.

Similarly, the coproduct of a unique rectification class does not necessarily decompose into unique rectification classes. Consider T_0, T', T'', and T''' below.

$$T_0 = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2}\end{array} \qquad T' = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 \\ \cline{1-1}\end{array} \qquad T'' = \begin{array}{|c|c|}\hline 2 & 4 \\ \hline\end{array} \qquad T''' = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 5 \\ \cline{1-2}\end{array}$$

One can check that T_0 is a URT and $P_H(\mathfrak{row}(T')\mathfrak{row}(T'')) = P(312524) = T_0$, but T' is not a URT since it is equivalent to T'''.

4.6 Connection to symmetric functions

4.6.1 Symmetric functions and stable Grothendieck polynomials

As in the previous chapter, we denote the ring of symmetric functions in an infinite number of variables x_i by

$$\mathrm{Sym} = \mathrm{Sym}(x_1, x_2, \ldots) = \oplus_{n \geq 0} \mathrm{Sym}_n.$$

The nth graded component, Sym_n, consists of homogeneous elements of degree n with $\mathrm{Sym}_0 = \mathbb{R}$. There are several important bases of Sym indexed by partitions λ of integers. The two most notable such bases are the monomial symmetric functions, m_λ, and Schur functions, s_λ. We refer the reader to [Sta99] for definitions and further details on the ring Sym.

For $f(x_1, x_2, \ldots) \in \mathrm{Sym}$ let

$$\Delta(f) = f(y_1, \ldots; z_1, \ldots) \in \mathrm{Sym}(y_1, \ldots) \otimes \mathrm{Sym}(z_1, \ldots)$$

be the result of splitting the alphabet $\{x_1, x_2, \ldots\}$ into two disjoint alphabets, $\{y_1, y_2, \ldots\}$ and $\{z_1, z_2, \ldots\}$. Sym is known to be a bialgebra with this coproduct, see [Zel81].

Let us denote by $\hat{\mathrm{Sym}}$ the completion of Sym, which consists of possibly infinite linear combinations of the monomial symmetric functions, m_λ. Each element of $\hat{\mathrm{Sym}}$ can be split into graded components, each being a finite linear combination of monomial symmetric functions. Also, let $\mathrm{Sym} \hat{\otimes} \mathrm{Sym}$ denote the completion of $\mathrm{Sym} \otimes \mathrm{Sym}$, consisting of possibly infinite linear combinations of terms of the form $m_\lambda \otimes m_\mu$. It is not hard to see that the completion $\hat{\mathrm{Sym}}$ inherits a bialgebra structure from Sym in the following sense.

Theorem 4.6.1. *If $f, g \in \hat{\mathrm{Sym}}$ then*

$$f \cdot g = \sum_\mu c_\mu m_\mu \in \hat{\mathrm{Sym}}.$$

If $f \in \hat{\mathrm{Sym}}$, then

$$\Delta(f) = \sum_{\mu,\nu} c_{\mu,\nu} m_\mu \otimes m_\nu \in \mathrm{Sym} \hat{\otimes} \mathrm{Sym}.$$

Furthermore, the coefficients c_μ and $c_{\mu,\nu}$ in both expressions are unique.

Proof. It is easy to see that each m_μ in the first case and each $m_\mu \otimes m_\nu$ in the second case can appear only in finitely many terms on the left. The claim follows. □

We next expand upon the discussion of stable Grothendieck polynomials in Section 3.9.2.

Recall the definition of a set-valued tableau given in Section 4.3.2. Given a set-valued tableau T, let x^T be the monomial in which the exponent of x_i is the number of occurrences of the letter i in T. Let $|T|$ be the degree of this monomial.

Example 4.6.2. The tableau shown below has $x^T = x_1 x_3 x_4 x_5^2 x_6^3 x_8 x_9$ and $|T| = 11$.

134	4	568
56	6	
9		

Buch [Buc02] proves a combinatorial interpretation of the single stable Grothendieck polynomials indexed by partitions, G_λ, which we again present as the definition. This interpretation is implicitly present in the earlier paper [FG06].

Theorem 4.6.3. *[Buc02, Theorem 3.1] The single stable Grothendieck polynomial G_λ is given by the formula*

$$G_\lambda = \sum_T (-1)^{|T|-|\lambda|} x^T,$$

where the sum is over all set-valued tableaux T of shape λ.

Example 4.6.4. We have

$$G_{(2,1)} = x_1^2x_2 + 2x_1x_2x_3 - x_1^2x_2^2 - 2x_1^2x_2x_3 - 8x_1x_2x_3x_4 + \ldots,$$

where, for example the coefficient of $x_1^2x_2x_3$ is -2 because of the tableaux shown below and the fact that for each of them, $|T| - |\lambda| = 1$.

$$\begin{array}{|c|c|}\hline 1 & 12 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \qquad\qquad \begin{array}{|c|c|}\hline 1 & 13 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

The following claim is not surprising.

Lemma 4.6.5. *Each element $f \in \hat{\mathrm{Sym}}$ can uniquely be written as*

$$f = \sum_{\mu} c_\mu G_\mu.$$

Similarly, each element of $\mathrm{Sym} \hat{\otimes} \mathrm{Sym}$ can uniquely be written as

$$\sum_{\mu,\nu} c_{\mu,\nu} G_\mu \otimes G_\nu.$$

Proof. Fix any complete order on monomials m_λ that agrees with the reverse dominance order for a fixed size $|\lambda|$ and satisfies $m_\mu < m_\lambda$ for $|\mu| < |\lambda|$. See, for example, [MS05] for details. Then m_λ is the minimal term of G_λ, and we can uniquely recover coefficients c_μ by using the set $\{G_\lambda\}$ to eliminate minimal terms in f. The proof of the second claim is similar. □

What is surprising, however, is the following two theorems proven by Buch in [Buc02]. A priori, the products and the coproducts of stable Grothendieck polynomials do not have to decompose into *finite* linear combinations.

Theorem 4.6.6. *[Buc02, Corollary 5.5] We have*

$$G_\lambda G_\mu = \sum_{\nu} c^{\nu}_{\lambda,\mu} G_\nu,$$

where the sum on the right is over a finite set of partition shapes ν.

Theorem 4.6.7. *[Buc02, Corollary 6.7] We have*

$$\Delta(G_\nu) = \sum_{\nu} d^{\nu}_{\lambda,\mu} G_\lambda \otimes G_\mu,$$

where the sum on the right is over a finite set of pairs of partitions, λ and μ.

4.6.2 Weak set-valued tableaux

Recall the discussion weak set-valued tableaux and the basis $\{J_\lambda\}$ in Section 3.9.3. We now present a basis that differs from the basis $\{J_\lambda\}$ by shape transposition and relabeling. To emphasize this similarity, we call the new basis $\{J^t_\lambda\}$.

A *row weak set-valued tableau* T of shape λ is a filling of the boxes with finite, non-empty multisets of positive integers so that

1. the smallest number in each box is greater than or equal to the largest number in the box directly to the left of it (if that box is present), and
2. the smallest number in each box is strictly greater than the largest number in the box directly above it (if that box is present).

Note that the numbers in each box are not necessarily distinct.

For a row weak set-valued tableau T, define x^T to be $\prod_{i\geq 1} x_i^{a_i}$, where a_i is the number of occurrences of the letter i in T.

Example 4.6.8. The following row weak set-valued tableau has $x^T = x_1^3x_2^4x_3^2x_4^2x_5x_6x_8$.

$$T = \begin{array}{|c|c|c|c|}\hline 11 & 12 & 2 & 346 \\ \hline 223 & 45 & 8 \\ \cline{1-3}\end{array}$$

Let $J^t_\lambda = \sum_T x^T$ denote the weight generating function of all row weak set-valued tableaux T of shape λ. We will call J^t_λ the *weak stable Grothendieck polynomial* indexed by λ.

Example 4.6.9. We have that

$$J^t_{(2,1)} = x_1^2x_2 + 2x_1x_2x_3 + 2x_1^3x_2 + 3x_1^2x_2^2 + 2x_1x_2^3 + 8x_1x_2x_3x_4 + \ldots,$$

where, for example, the coefficient of $x_1^2x_2^2$ is 3 because of the following weak set-valued tableaux.

$$\begin{array}{|c|c|}\hline 11 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 12 \\ \hline 2 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 22 \\ \cline{1-1}\end{array}$$

Remark 4.6.10. In [LP07], weak stable Grothendieck polynomials J_λ were introduced when studying the effect of standard ring automorphism ω on the stable Grothendieck polynomials G_λ. In particular, it was shown in [LP07, Theorem 9.21] that J_λ are symmetric functions. It follows directly from this that J^t_λ are symmetric.

Theorem 4.6.11. *We have*

$$J^t_\lambda(x_1, x_2, \ldots) = (-1)^{|\lambda|} G_\lambda\left(\frac{-x_1}{1-x_1}, \frac{-x_2}{1-x_2}, \ldots\right).$$

Proof. There is a correspondence between set-valued tableaux and weak set-valued tableaux as follows. For each set-valued tableau T, we can obtain a family of weak set-valued tableaux of the same shape, call the family $\mathcal{T}$, by saying that $W \in \mathcal{T}$ if and only if W can be constructed from T by turning subsets in boxes of T into multisets. Conversely, given any weak set-valued tableau, we can find the set-valued tableau it corresponds to by transforming its multisets into subsets containing the same positive integers. For example, if we have the T shown below, then W_1 and W_2 are in $\mathcal{T}$.

$$T = \begin{array}{|c|c|c|}\hline 13 & 4 & 57 \\ \hline 4 & 6 \\ \cline{1-2}\end{array} \qquad W_1 = \begin{array}{|c|c|c|}\hline 133 & 4 & 57 \\ \hline 4 & 666 \\ \cline{1-2}\end{array} \qquad W_2 = \begin{array}{|c|c|c|}\hline 13 & 4 & 557 \\ \hline 44 & 66 \\ \cline{1-2}\end{array}$$

Thus if $x^T = x_1^{a_1} x_2^{a_2} x_3^{a_3} \ldots$, we have

$$\sum_{W \in \mathcal{T}} x^W = \left(\frac{x_1}{1-x_1}\right)^{a_1} \left(\frac{x_2}{1-x_2}\right)^{a_2} \left(\frac{x_3}{1-x_3}\right)^{a_3} \cdots$$

since we can choose to repeat any of the a_i occurrences of i any positive number of times.

Therefore

$$\begin{aligned}(-1)^{|\lambda|} G_\lambda\left(\frac{-x_1}{1-x_1}, \frac{-x_2}{1-x_2}, \ldots\right) &= (-1)^{|\lambda|} \sum_T (-1)^{|T|-|\lambda|} \left(\frac{-x}{1-x}\right)^T \\ &= \sum_T \left(\frac{x}{1-x}\right)^T \\ &= J^t_\lambda(x_1, x_2, \ldots),\end{aligned}$$

where the sum is over set-valued tableaux T.

□

Corollary 4.6.12. *The structure constants of the rings with bases G_λ and J^t_λ coincide up to sign. In other words,*

$$J^t_\lambda J^t_\mu = \sum_\nu c^\nu_{\lambda,\mu} J^t_\nu$$

if and only if

$$G_\lambda G_\mu = \sum_\nu (-1)^{|\nu|-|\lambda|-|\mu|} c^\nu_{\lambda,\mu} G_\nu.$$

Proof. In one direction it is clear, in the other follows from

$$G_\lambda(x_1, x_2, \dots) = (-1)^{|\nu|-|\lambda|-|\mu|} J^t_\lambda\left(\frac{-x_1}{1-x_1}, \frac{-x_2}{1-x_2}, \dots\right).$$

□

4.6.3 Fundamental quasisymmetric functions revisited

Recall the correspondence between compositions of n and subsets of $[n-1]$ described in Section 2.0.2. We recall that the *descent set* of a word $h = h_1h_2\dots h_l$ to be the set $\mathcal{D}(H) = \{i \mid h_i > h_{i+1}\}$. Given that $\mathcal{D}(h) = \{\alpha_1, \alpha_2, \dots, \alpha_m\}$, we have the associated composition

$$\mathcal{C}(h) = \mathcal{C}(\mathcal{D}(h)) = (\alpha_1, \alpha_2 - \alpha_1, \alpha_3 - \alpha_2, \dots, \alpha_m - \alpha_{m-1}, l - \alpha_m).$$

We call $\mathcal{C}(h)$ the *descent composition* for h.

Example 4.6.13. If $h = 11423532$, the descent set of h is $\{3, 6, 7\}$ and $\mathcal{C}(h) = (3, 3, 1, 1)$.

We shall now define the fundamental quasisymmetric function, L_α, in a way that does not depend on (P, θ)-partitions. Given α, a composition of n, define

$$L_\alpha = \sum_{\substack{i_1 \le \dots \le i_n \\ i_j < i_{j+1} \text{ if } j \in S_\alpha}} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

Example 4.6.14. The fundamental quasisymmetric function indexed by the composition $(1, 3)$ is

$$L_{(1,3)} = x_1x_2^3 + x_1x_3^3 + x_2x_3^3 + x_1x_2^2x_3 + x_1x_2x_3^2 + x_1x_2x_3x_4 + \dots,$$

an infinite sum where all terms have degree 4. Note that every term must have $i_1 < i_2$ since $S_{(1,3)} = \{2\}$, so $x_1^2x_2^2$ will never appear in $L_{(1,3)}$.

Given a weak set-valued tableau T filled with elements of $[n]$, each appearing once, we say that there is a *descent* at entry i if $i+1$ is strictly below i. We may then find the descent set of T and determine the composition corresponding to its descent set, $\mathcal{C}(T)$, by listing the entries in increasing order and marking the entries at which there was a descent in the tableau.

Example 4.6.15. The descent set of T shown below is $\{3, 5, 9\}$ and the corresponding composition is $\mathcal{C}(T) = (3, 2, 4, 2)$.

$$\begin{array}{|c|c|c|c|}\hline 123 & 5 & 89 & 11 \\ \hline 46 & 7 & 10 \\ \cline{1-3} \end{array}$$

Given any weak set-valued tableau T, we determine $\mathcal{C}(T)$ by first standardizing tableau T. To standardize T, first find all c_1 occurrences of 1 in T and replace them from southwest to northeast with $1, 2, \ldots, c_1$. Next, replace the c_2 occurrences of 2 from southwest to northeast with $c_1 + 1, c_1 + 2, \ldots, c_1 + c_2$. Continue this process, replacing the c_i occurrences of i from southwest to northeast with the next available consecutive integer. The resulting tableau is the standardization of T. We may then find the descent set of the standardization and let $\mathcal{C}(T)$ be the associated composition.

Example 4.6.16. The standardization of the weak set-valued tableau shown below is the tableau T of Example 4.6.15. Thus $\mathcal{C}(T') = \mathcal{C}(T) = (3, 2, 4, 2)$.

$$T' = \begin{array}{|c|c|c|c|}\hline 122 & 3 & 56 & 8 \\ \hline 34 & 4 & 7 \\ \cline{1-3} \end{array}$$

Recall from Section 4.3.2 that $Q_H(h)$ denotes the recording tableau of Hecke insertion.

Theorem 4.6.17. *Given a word h and its recording tableau $Q_H(h)$, $\mathcal{C}(h) = \mathcal{C}(Q_H(h))$.*

Proof. According to Lemma 4.3.6, there is a descent at position i of word h if and only if the entry $i + 1$ is strictly below entry i in $Q_H(h)$. □

Example 4.6.18. Consider $h = 13324535$ with $P_H(h)$ and $Q_H(h)$ shown below.

$$P_H(h) = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 5 \\ \hline 3 & 4 \\ \cline{1-2} \end{array} \qquad Q_H(h) = \begin{array}{|c|c|c|c|}\hline 1 & 23 & 5 & 68 \\ \hline 4 & 7 \\ \cline{1-2} \end{array}$$

One easily checks that $\mathcal{C}(h) = (3, 3, 2) = \mathcal{C}(Q_H(h))$.

4.6.4 Decomposition into fundamental quasisymmetric functions

Theorem 4.6.19. *For any fixed increasing tableau T of shape λ we have*

$$J^t_\lambda = \sum_{P_H(h)=T} L_{\mathcal{C}(h)}.$$

Proof. We give an explicit weight-preserving bijection between the set of weak set-valued tableaux of shape λ and the set of pairs (h, σ') where $h = h_1h_2 \dots h_l$ is a word with $P_H(h) = T$ and σ' is a sequence of positive integers $(i_1 \leq i_2 \leq \dots \leq i_l)$, where $i_j < i_{j+1}$ if $j \in \mathcal{D}(h)$.

Suppose we have a weak set-valued tableau W of shape λ. To obtain h, first standardize W. Next, using T as $P_H(h)$ and the standardization of W as $Q_H(h)$, perform reverse Hecke insertion. Let the entries of W in increasing order be $i_1, i_2, i_3, \dots, i_l$, where each i_j is a positive integer, and denote $\sigma' = (i_1, i_2, \dots, i_l)$. We then have $i_j \leq i_{j+1}$ for all $j \in \{1, 2, \dots, l-1\}$, and $i_j < i_{j+1}$ if $j \in \mathcal{D}(h)$ by Theorem 4.6.17.

For the reverse map, suppose we have $h = h_1h_2 \dots h_l$ with $P_H(h) = T$ and some

$$\sigma' = (i_1 \leq i_2 \leq \dots \leq i_l).$$

Then let W be the recording tableau of the insertion of h, which uses the positive integers of σ'. In other words, i_j is used to label the special corner c of the insertion $P_H(h_1h_2 \dots h_{j-1}) \xleftarrow{H} h_j$. According to Lemma 4.3.6, the result is a valid weak set-valued tableau. Using Theorem 4.3.12 we conclude we indeed have a bijection.

It remains to note that for a fixed h, we have

$$\sum_{\sigma'=(i_1 \leq i_2 \leq \dots \leq i_l)} \prod_{j=1}^{l} x_{i_j} = L_{\mathcal{C}(h)},$$

where the sum is over σ' such that $i_j < i_{j+1}$ if $j \in \mathcal{D}(h)$. $\square$

Example 4.6.20. Suppose we start with increasing tableau T and the weak set-valued tableau W shown below.

$$T = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} \end{array} \qquad\qquad W = \begin{array}{|c|c|c|} \hline 122 & 235 & 58 \\ \hline 45 & 677 \\ \cline{1-2} \end{array}$$

Performing reverse Hecke insertion with the standardization of W recording the special box c at each step, we obtain

$$h = 1114412252233.$$

The corresponding composition is $(5, 4, 4)$, and

$$\sigma' = (1, 2, 2, 2, 3, 4, 5, 5, 5, 6, 7, 7, 8).$$

To understand the inverse map, simply let $T = P_H(h)$ and record the special box c at each step with the positive integers of σ' to obtain the corresponding weak set-valued tableau.

Remark 4.6.21. Pairs (h, σ') as above are an analogue of *biwords* of [LLT02].

Remark 4.6.22. The decomposition of weak stable Grothendieck polynomials into fundamental quasisymmetric functions is similar to Stanley's theory of P-partitions, see [Sta99]. A different K-theoretic analogue of such a decomposition appears in [LP07].

4.6.5 Map from the KPR to symmetric functions

Consider the map $\phi : KPR \longrightarrow Sym$ given by

$$[[h]] \mapsto \sum_{w \equiv h} L_{\mathcal{C}(w)}.$$

Theorem 4.6.23. *Map ϕ is a bialgebra morphism.*

Proof. First we show the map preserves the product. Recall that

$$L_{\mathcal{C}(w')} \cdot L_{\mathcal{C}(w'')} = \sum_{w \in \ \mathrm{Sh}(w', w''[n])} L_{\mathcal{C}(w)},$$

where the sum is over all shuffles of w' and $w''[n]$ (see [LP07]). Thus,

$$\begin{aligned}
\phi([[h_1]] \cdot [[h_2]]) &= \phi\left(\sum_{w' \equiv h_1, w'' \equiv h_2} w' \sqcup\!\sqcup\, w''\right) \\
&= \sum_{w' \equiv h_1, w'' \equiv h_2} L_{\mathcal{C}(w' \sqcup\sqcup w'')} \\
&= \left(\sum_{w' \equiv h_1} L_{\mathcal{C}(w')}\right)\left(\sum_{w'' \equiv h_2} L_{\mathcal{C}(w'')}\right) = \phi([[h_1]])\phi([[h_2]]).
\end{aligned}$$

For the coproduct, we show the result for ϕ applied to $w \equiv h$ with $\phi(w) = L_{\mathcal{C}(w)}$ using that $\Delta(L_{\mathcal{C}(w)}) = \sum L_\beta \otimes L_\gamma$, where we sum over all $\beta = (\beta_1, \ldots, \beta_k)$ and $\gamma = (\gamma_1, \ldots, \gamma_n)$ such that $(\beta_1, \ldots, \beta_k, \gamma_1, \ldots, \gamma_n) = \mathcal{C}(w)$ or $(\beta_1, \ldots, \beta_{k-1}, \beta_k + \gamma_1, \gamma_2, \ldots, \gamma_n) = \mathcal{C}(w)$, see [LP07]. Then, for any $w \equiv h$,

$$\Delta(\phi(w)) = \Delta(L_{\mathcal{C}(w)}) = \sum L_\beta \otimes L_\gamma = \phi \otimes \phi(\Delta(w))$$

because the terms in $\Delta(w) = \sum_{i=0}^{|w|} w_1 \cdots w_i \otimes w_{i+1} \cdots w_{|w|}$ with $i \in \mathcal{D}(w)$ give exactly the terms $L_\beta \otimes L_\gamma$ where $(\beta_1, \ldots, \beta_k, \gamma_1, \ldots, \gamma_n) = \mathcal{C}(w)$ and all terms with $i \notin \mathcal{D}(w)$ give exactly the terms $L_\beta \otimes L_\alpha$ where $(\beta_1, \ldots, \beta_{k-1}, \beta_k + \gamma_1, \gamma_2, \ldots, \gamma_n) = \mathcal{C}(w)$. Thus $\Delta(\phi([[h]])) = \phi \otimes \phi(\Delta([[h]])$. □

Theorem 4.6.24. *We have*

$$\phi([[h]]) = \sum_{\mathfrak{row}(T) \equiv h} J^t_{\lambda(T)},$$

where the sum is over all tableaux K-Knuth equivalent to $[[h]]$, and $\lambda(T)$ denotes the shape of T.

Proof. Using Theorem 4.6.19, we have

$$\phi([[h]]) = \sum_{w \equiv h} L_{\mathcal{C}(w)} = \sum_{T \equiv P_H(h)} \sum_{P_H(w)=T} L_{\mathcal{C}(w)} = \sum_{T \equiv P_H(h)} J^t_{\lambda(T)} = \sum_{\mathfrak{row}(T) \equiv h} J^t_{\lambda(T)}.$$

□

4.7 Littlewood-Richardson rule

4.7.1 LR rule for Grothendieck polynomials

Theorem 4.7.1. *Let T be a URT of shape μ. Then the coefficient $c^\nu_{\lambda,\mu}$ in the decomposition*

$$G_\lambda G_\mu = \sum_\nu (-1)^{|\nu|-|\lambda|-|\mu|} c^\nu_{\lambda,\mu} G_\nu$$

is equal to the number of increasing tableaux R of skew shape ν/λ such that $P_H(\mathfrak{row}(R)) = T$.

Proof. In addition to the URT T of shape μ, fix a URT T' of shape λ. Then by Theorems 4.5.5, 4.6.24, and 4.6.23,

$$\begin{aligned}
J^t_\lambda J^t_\mu &= \phi([[\mathfrak{row}(T')]])\phi([[\mathfrak{row}(T)]]) \\
&= \phi\left([[\mathfrak{row}(T')]] \cdot [[\mathfrak{row}(T)]]\right) \\
&= \phi\Big(\sum_{Y \in T(T' \sqcup T)} \sum_{P_H(w)=Y} w\Big) \\
&= \sum_{Y \in T(T' \sqcup T)} J^t_{\lambda(Y)},
\end{aligned}$$

where $T(T' \sqcup\!\sqcup T)$ is the finite set of tableaux Y such that $T|_{[|\lambda|]} = T'$ and $P_H(\mathfrak{row}(Y)|_{[|\lambda|+1,|\lambda|+|\mu|]}) = T$. Thus the coefficient of J_ν in the product is the number of increasing tableaux R of skew shape ν/λ such that $P_H(\mathfrak{row}(R)) = T$. The desired result follows from Corollary 4.6.12. $\square$

Example 4.7.2. The coefficient of $G_{(4,3,1)}$ in $G_{(3,1)}G_{(2,1)}$ is -3. To see this, fix T to be the tableau with reading word 312 as in the previous example, note $(-1)^{|\nu|-|\lambda|-|\mu|} = -1$, and notice the tableaux shown below are the only tableaux of shape $(4,3,1)/(3,1)$ with $P_H(\mathfrak{row}(R)) = T$.

$$\begin{array}{|c|c|c|c|} \cline{4-4} \multicolumn{3}{c|}{} & 2 \\ \cline{2-4} \multicolumn{1}{c|}{} & 1 & 2 & \multicolumn{1}{c}{} \\ \cline{1-3} 3 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|c|} \cline{4-4} \multicolumn{3}{c|}{} & 2 \\ \cline{2-4} \multicolumn{1}{c|}{} & 1 & 3 & \multicolumn{1}{c}{} \\ \cline{1-3} 3 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|c|} \cline{4-4} \multicolumn{3}{c|}{} & 2 \\ \cline{2-4} \multicolumn{1}{c|}{} & 1 & 3 & \multicolumn{1}{c}{} \\ \cline{1-3} 1 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}$$

Note that the claim may be false if T is not a URT.

Example 4.7.3. Suppose we want to find the coefficient of $(4,2)$ in the product of $(4,3,2)$ and $(2,1)$. Using Buch's rule [Buc02], we compute that the coefficient is 3, corresponding to the following fillings of $(2,1)$:

$$\begin{array}{|c|c|} \hline 12 & 3 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|} \hline 1 & 23 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 23 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}\ .$$

However, if we choose the filling of $(3,2)$ with row reading word 34124, one can easily check that there are only two ways to fill $(4,3,2)/(2,1)$ with words equivalent to 34124 that insert into the chosen filling of $(3,2)$. The fillings are shown below.

$$\begin{array}{|c|c|c|c|} \hline \ & \ & 2 & 4 \\ \hline \ & 2 & 4 & \multicolumn{1}{c}{} \\ \cline{1-3} 1 & 3 & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array} \qquad \begin{array}{|c|c|c|c|} \hline \ & \ & 2 & 4 \\ \hline \ & 1 & 4 & \multicolumn{1}{c}{} \\ \cline{1-3} 3 & 4 & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array}$$

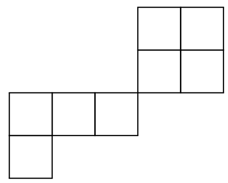
Now we can give our own proof of Theorem 4.6.6.

Proof. Combine Theorem 4.6.23 with Corollary 4.4.7.

Alternatively, the argument can be made directly from Theorem 4.7.1. Note that the set of shapes ν such that there exists an increasing tableau R of skew shape ν/λ such that $P_H(\mathfrak{row}(R)) = T$ is finite. This is because each cell in ν/λ can be filled only with letters occurring in T, and thus size of each row and column in ν/λ is bounded. $\square$

4.7.2 Dual LR rule for Grothendieck polynomials

Given two Young diagrams, λ and μ, define skew shape $\lambda \oplus \mu$ to be the skew shape obtained by putting λ and μ corner to corner. For example, the figure below shows $(3,1) \oplus (2,2)$.



Theorem 4.7.4. *Let T_0 be a URT of shape ν. Then the coefficient $d^{\nu}_{\lambda,\mu}$ in the decomposition*

$$\Delta(G_\nu) = \sum_{\lambda,\mu} (-1)^{|\nu|-|\lambda|-|\mu|} d^{\nu}_{\lambda,\mu} G_\lambda \otimes G_\mu$$

is equal to the number of increasing tableaux R of skew shape $\lambda \oplus \mu$ such that $P_H(\mathfrak{row}(R)) = T_0$.

Proof. We have that

$$\begin{aligned}
\Delta(J^t_\nu) &= \Delta(\phi([[\mathfrak{row}(T_0)]])) \\
&= \phi \otimes \phi(\Delta([[\mathfrak{row}(T_0)]])) \\
&= \phi \otimes \phi \left(\sum_{(T',T'')\in T(T_0)} \sum_{P_H(w)=\overline{T'}} w \otimes \sum_{P_H(w)=\overline{T''}} w \right) \\
&= \sum_{(T',T'')\in T(T_0)} \phi([[\mathfrak{row}(\overline{T'})]]) \otimes \phi([[\mathfrak{row}(\overline{T''})]]) \\
&= \sum_{(T',T'')\in T(T_0)} J^t_{\lambda(T')} \otimes J^t_{\lambda(T'')},
\end{aligned}$$

where $T(T_0)$ is the finite set of pairs of tableaux T', T'' such that $P_H(\mathfrak{row}(T')\mathfrak{row}(T'')) = T_0$. Letting $R = T' \oplus T''$, the coefficient of $J^t_\lambda \otimes J^t_\mu$ is exactly the number of increasing tableaux R of skew shape $\lambda \oplus \mu$ such that $P_H(\mathfrak{row}(R)) = T_0$. The desired result follows from Corollary 4.6.12. $\square$

Example 4.7.5. Fix T_0 to be the URT of shape $(3,2)$ with reading word 45123. The coefficient of $G_{(2,1)} \otimes G_{(2,1)}$ in $G_{(3,2)}$ is -3 because of the following three tableaux of shape $(2,1) \oplus (2,1)$.

$$\begin{array}{cccc} & & 2 & 3 \\ & & 5 & \\ 1 & 2 & & \\ 4 & & & \end{array} \qquad \begin{array}{cccc} & & 2 & 3 \\ & & 5 & \\ 1 & 4 & & \\ 4 & & & \end{array} \qquad \begin{array}{cccc} & & 2 & 3 \\ & & 5 & \\ 1 & 5 & & \\ 4 & & & \end{array}$$

Note that the claim may be false if T_0 is not a URT.

Example 4.7.6. Suppose we have

$$T_0 = \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 4 & \\ \cline{1-2} \end{array} \quad .$$

We saw in Example 4.3.14 that T_0 is not a URT. Now let $\lambda = (2,1)$ and $\mu = (3,1)$. According to Buch's rule in [Buc02], the coefficient of $G_\lambda \otimes G_\mu$ in $\Delta(G_{(3,2)})$ is at least 1 due to the following set-valued tableau:

$$\begin{array}{|c|c|c|} \hline 1 & 1 & 3 \\ \hline 23 & 34 & \\ \cline{1-2} \end{array} \quad .$$

However, one can check that there is no skew tableau R of shape $(2,1) \oplus (3,1)$ with $P_H(\mathfrak{row}(R)) = T_0$.

Now we can give our own proof of Theorem 4.6.7.

Proof. Combine Theorem 4.6.23 with Corollary 4.4.13.

Alternatively, the argument can be made directly from Theorem 4.7.4. Note that the number of pairs of partitions, λ and μ, such that there exists an increasing tableaux R of skew shape $\lambda \oplus \mu$ such that $P_H(\mathfrak{row}(R)) = T_0$ is finite. This is because each λ and μ has to be filled with alphabet of T_0 only, hence we can apply Lemma 4.3.2. $\square$

Chapter 5

Dual filtered graphs

5.1 Introduction

Fomin's *dual graded graphs* [Fom94], as well as their predecessors, Stanley's *differential posets* [Sta88], were invented as a tool to better understand the Robinson-Schensted insertion algorithm. Dual graded graphs are significant in the areas where the Robinson-Schensted correspondence appears, for example in Schubert calculus or in the study of representations of towers of algebras [BLL12, BS05]. A recent work of Lam and Shimozono [LS07] associates a dual graded graph to any Kac-Moody algebra, bringing their study to a new level of generality.

5.1.1 Weyl algebra and its deformations

One way to view the theory is as a study of certain *combinatorial representations* of the *first Weyl algebra*. Let us briefly recall the definitions. The *Weyl algebra*, or the first Weyl algebra, is an algebra over some field K (usually $K = \mathbb{R}$) generated by two generators U and D with a single relation $DU - UD = 1$. It was originally introduced by Hermann Weyl in his study of quantum mechanics. We refer the reader to [Bjo79], for example, for more background on the Weyl algebra.

A *graded graph* is a triple $G = (P, \rho, E)$, where P is a discrete set of vertices, $\rho : P \to \mathbb{Z}$ is a rank function, and E is a multiset of edges/arcs (x, y), where $\rho(y) = \rho(x) + 1$. In other words, each vertex is assigned a rank, and edges can only join vertices in successive ranks. For the set of vertices P, let P_n denote the subset of vertices of rank n. For any field K of characteristic zero, the formal linear combinations of vertices of G form the vector space KP.

Let $G_1 = (P, \rho, E_1)$ and $G_2 = (P, \rho, E_2)$ be a pair of graded graphs with a common vertex set and rank function. From this pair, define an *oriented graded graph* $G = (P, \rho, E_1, E_2)$ by orienting the edges of G_1 in the direction of increasing rank and the edges of G_2 in the direction of decreasing rank. Let $a_i(x, y)$ denote the number of edges of E_i joining x and y or the multiplicity of edge (x, y) in E_i. We define the *up* and *down operators* $U, D \in End(KP)$ associated with graph G by

$$Ux = \sum_y a_1(x,y)y$$

and

$$Dy = \sum_x a_2(x,y)x.$$

Graded graphs G_1 and G_2 with a common vertex set and rank function are said to be *dual* if

$$DU - UD = I,$$

where I is an identity operator acting on KP. Thus, we see that KP is a representation of the Weyl algebra. Furthermore, it is a representation of a very special kind, where U and D act in a particularly nice combinatorial way on a fixed basis.

One would then expect that variations of the Weyl algebra would correspond to some variations of the theory of dual graded graphs. One such variation is the *q-Weyl algebra* defined by the relation

$$DU - qUD = 1.$$

The corresponding theory of *quantum dual graded graphs* was pursued by Lam in [Lam10].

In this chapter, we shall study the theory arising from an analogue W of the Weyl algebra with defining relation

$$DU - UD = D + 1.$$

We shall refer to it as the Ore algebra [Ore33], and we shall see that it is a very natural object. In particular, the corresponding theory of *dual filtered graphs* fits naturally with the existing body of work on the K-theory of Grassmannians.

Remark 5.1.1. It is easy to see that by rescaling U and D, we can pass from arbitrary $DU - UD = \alpha D + \beta$ to $DU - UD = D + I$. We thus suffer no loss in generality by writing down the relation in the latter form.

5.1.2 Differential vs difference operators

The following provides some intuitive explanation for the relation between representations of Weyl and Ore algebras. The simplest representation of the Weyl algebra is that acting on a polynomial ring. Indeed, consider polynomial ring $R = \mathbb{R}[x]$, and for $f \in R$ let

$$U(f) = xf, \quad D(f) = \frac{\partial f}{\partial x}.$$

It is an easy exercise to verify that this indeed produces a representation of the Weyl algebra. In fact, this is how it is often defined.

Let us now redefine the down operator to be the *difference operator*:

$$U(f) = xf, \quad D(f) = f(x+1) - f(x).$$

Lemma 5.1.2. *This gives a representation of the Ore algebra W. In other words, those operators satisfy*

$$DU - UD = D + I,$$

where I is the identity map on R.

Proof. We have

$$(DU - UD)(f) = ((x+1)f(x+1) - xf(x)) - x(f(x+1) - f(x))$$

$$= f(x+1) = f(x) + (f(x+1) - f(x)) = (I + D)(f).$$

$\square$

It can also be noted that differential and difference operators can be related as follows:

$$D = e^{\frac{\partial}{\partial x}} - 1.$$

We omit the easy proof for brevity. As we shall see in Example 5.7.10, this representation of W corresponds to a very basic dual filtered graph.

5.1.3 Pieri and Möbius constructions

We make use of two conceptual ways to build examples of dual filtered graphs. The first construction starts with an algebra A, a derivation D on A, and an element $f \in A$ such that $D(f) = f + 1$. We often build the desired derivation D using a *bialgebra* structure on A. This construction is very close to an existing one in the literature [BLL12, Nze06, LSa], where dual graded graphs are constructed from graded Hopf algebras. In fact, if we were to require $D(f) = 1$, we would get dual graded graphs instead of dual filtered graphs in our construction. We refer to this method as the *Pieri construction* or sometimes as the *Pieri deformation*.

Instead, we can also start with an existing dual graded graph $G = (P, \rho, E_1, E_2)$ (see definition below), composed of graphs $G_1 = (P, \rho, E_1)$ and $G_2 = (P, \rho, e_2)$, and adjust E_1 and E_2 in the following manner. To obtain G_1', we add $\#\{x|y \text{ covers } x \text{ in } G_1\}$ loops at each vertex $y \in P$ to E_1. As for G_2', we create a new edge set E_2' by forming

$$a_2'(x, y) = |\mu(x, y)|$$

edges between vertices x and y, where μ denotes the Möbius function in $G_2 = (P, \rho, E_2)$. We refer to this construction as the *Möbius construction* or *Möbius deformation.* Note that this does not always produce a pair of dual filtered graphs, and it is mysterious to determine when it does. In some major examples, however, it is the result of this construction that relates to Robinson-Schensted-like algorithms, for example to *Hecke insertion* of [BKS$^+$08].

The following observation seems remarkable, and we call it the *Möbius via Pieri phenomenon.* Let A be a graded Hopf algebra, and let bialgebra $\tilde{A}$ be its *K-theoretic deformation* in some appropriate sense which we do not know how to formalize. Let G be a natural dual graded graph associated with A. What we observe is that applying the Möbius construction to G yields *the same result* as a natural Pieri construction applied to $\tilde{A}$. In other words, the diagram in Figure 5.1 commutes.

This happens for Young's lattice (see Section 5.4.4) and for the binary tree dual graded graph, see Section 5.7.1. We expect this phenomenon to also occur for the shifted Young's lattice.

The crucial condition necessary for this phenomenon to be observed is for A to be the associated graded algebra of $\tilde{A}$, but this is not always the case. For example, this

$$\begin{tikzcd}
A \arrow[r, "K \text{ deformation}"] \arrow[d, "\substack{\text{Pieri}\\ \text{construction}}"'] & \tilde{A} \arrow[d, "\substack{\text{Pieri}\\ \text{construction}}"] \\
G \arrow[r, "\text{Möbius construction}"] & \tilde{G}
\end{tikzcd}$$

Figure 5.1: Möbius via Pieri phenomenon

is not the case for K-theoretic analogues of the Poirier-Reutenauer and Malvenuto-Reutenauer Hopf algebras, as described in Sections 5.7.2 and 5.7.3. To put it simply, the numbers of basis elements for the filtered components of A and $\tilde{A}$ are distinct in those cases, thus there is no hope to obtain corresponding graphs one from the other via Möbius construction. Interestingly, those are exactly the examples we found where the Möbius construction fails to produce a dual filtered graph.

5.1.4 Chapter overview

In Section 5.2, we recall the definition of dual graded graphs from [Fom94] as well as three major examples: Young's lattice, Young-Fibonacci lattice, and shifted Young's lattice. We then explain how one can view the RSK algorithm locally via the machinary of *growth diagrams*, also introduced by Fomin in [Fom94, Fom95].

In Section 5.3, we formulate our definition of dual filtered graphs. We then introduce the trivial, Pieri and Möbius constructions.

In Section 5.4, we build the Pieri and Möbius deformations of Young's lattice. We observe that Hecke insertion is a map into a pair of paths in the Möbius deformation of Young's lattice. We provide growth rules that realize Hecke insertion. We also show how the Pieri construction applied to the ring generated by *Grothendieck polynomials* yields the same result as the Möbius construction applied to Young's lattice. Thus we demonstrate an instance of the Möbius via Pieri phenomenon.

In Section 5.5, we build Pieri and Möbius deformations of the shifted Young's lattice. We introduce *shifted Hecke insertion* and remark that its result always coincides with that of K-theoretic jeu de taquin rectification as defined in [CTY14]. We also provide the corresponding growth rules.

In Section 5.6, we build Pieri and Möbius deformations of the Young-Fibonacci lattice. We define K-Young-Fibonacci tableaux and suggest the corresponding insertion algorithm and growth rules.

In Section 5.7, we consider some other examples of dual filtered graphs. Of special interest are the Pieri constructions associated to the quasisymmetric functions, to the Poirier-Reutenauer and Malvenuto-Reutenauer Hopf algebras, as well as to their K-theoretic analogues, which we draw from [LP07, PP16]. The Hopf algebra of quasisymmetric functions and its K-theoretic analogue provide another instance of the Möbius via Pieri phenomenon.

In Section 5.8, we use the calculus of up and down operators to prove some identities similar to the ones known for dual graded graphs. In particular, we formulate and prove a K-theoretic analogue of the Frobenius-Young identity.

Finally, in Section 5.9, we discuss progress and conjectures related to constructing dual filtered graphs in the affine Grassmannian setting.

5.2 Dual graded graphs

5.2.1 Dual graded graphs

This section follows [Fom94], and we refer the reader to this source for further reading on dual graded graphs.

A graded graph is a triple $G = (P, \rho, E)$, where P is a discrete set of vertices, $\rho : P \to \mathbb{Z}$ is a rank function, and E is a multiset of edges/arcs (x, y), where $\rho(y) = \rho(x) + 1$. In other words, each vertex is assigned a rank, and edges can only join vertices in successive ranks. For the set of vertices P, let P_n denote the subset of vertices of rank n. For any field K of characteristic zero, the formal linear combinations of vertices of G form the vector space KP.

In future examples, the idea of *inner corners* and *outer corners* of certain configurations of boxes or cells will be necessary. We will call a cell an inner corner if it can be removed from the configuration and the resulting configuration is still valid. A cell is an outer corner of a configuration if it can be added to the configuration, and the resulting configuration is still valid.

Example 5.2.1. Young's lattice is an example of a graded graph where for any partition λ, $\rho(\lambda) = |\lambda|$, and there is an edge from λ to μ if μ can be obtained from λ by adding one box. Ranks zero through five are shown in Figure 5.2. We see that $(3, 31) \in E$ and $\rho(3) + 1 = 3 + 1 = \rho(31) = 4$.

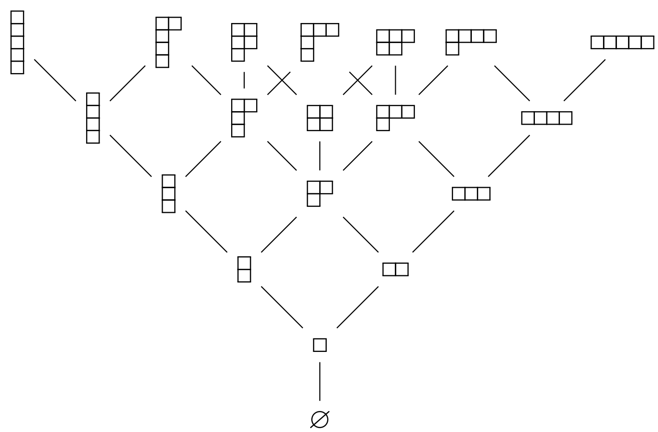
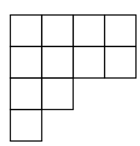


Figure 5.2: Young's lattice

If we look at the partition (4,4,2,1) below, we see that the inner corners are in positions (2,4), (3,2), and (4,1), and the outer corners are in positions (1,5), (3,3), (4,2), and (5,1).



Let $G_1 = (P, \rho, E_1)$ and $G_2 = (P, \rho, E_2)$ be a pair of graded graphs with a common vertex set and rank function. From this pair, define an *oriented graded graph* $G = (P, \rho, E_1, E_2)$ by orienting the edges of G_1 in the direction of increasing rank and the edges of G_2 in the direction of decreasing rank. Let $a_i(x, y)$ denote the number of edges of E_i joining x and y or the multiplicity of edge (x, y) in E_i. We define the *up* and *down operators* $U, D \in End(KP)$ associated with graph G by

$$Ux = \sum_y a_1(x, y)y$$

and

$$Dy = \sum_x a_2(x, y)x.$$

For example, in Young's lattice shown above, $U(21) = 31 + 22 + 211$ and $D(21) = 2 + 11$.

The restrictions of U and D to "homogeneous" subspaces KP_n are denoted by U_n and D_n, respectively. Graded graphs G_1 and G_2 with a common vertex set and rank function are said to be *r-dual* if

$$D_{n+1}U_n = U_{n-1}D_n + rI_n$$

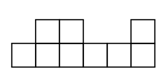
for some $r \in \mathbb{R}$ and simply *dual* if

$$D_{n+1}U_n = U_{n-1}D_n + I_n.$$

We focus on the latter.

Example 5.2.2. Young's lattice is an example of a self-dual graded graph.

Example 5.2.3. Another well-known example of a self-dual graded graph is the *Young-Fibonacci lattice*, $\mathbb{YF}$. The first six ranks are shown in Figure 5.3. The vertices of the graph are all finite words in the alphabet $\{1, 2\}$, where the rank of a word is the sum of its entries. The covering relations are as follows: a word w' covers w if and only if either $w' = 1w$ or $w' = 2v$ for some v covered by w. For example, the word 121 covers the word 21 since 121 is obtained by concatentating 1 and 21, and 121 is covered by 221 because 121 covers 21. The words in the alphabet $\{1, 2\}$ are sometimes called snakes and can be represented as a collection of boxes whose heights correspond to the entries of the word called snakeshapes. For example, the word 122112 can be pictured as

 .

Thought of this way, the rank of a word is the number of boxes in its corresponding snakeshape.

Young's lattice and the Young-Fibonacci lattice discussed above are the most interesting examples of self-dual graded graphs. We close this section by describing the graph of shifted shapes, $\mathbb{SY}$, and its dual, which together form a dual graded graph.

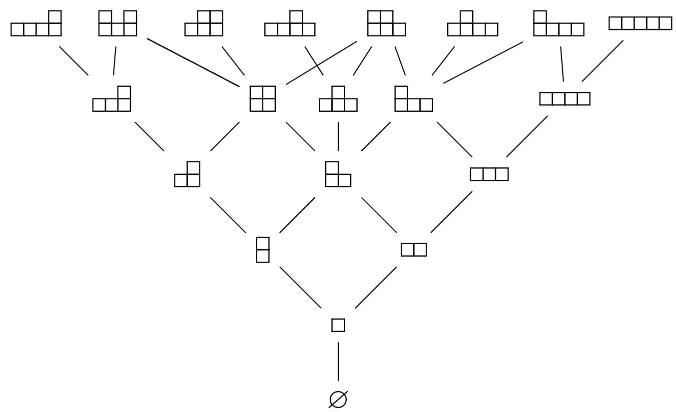
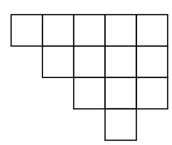


Figure 5.3: Young-Fibonacci lattice

Example 5.2.4. Given a strict partition λ (i.e. $\lambda = (\lambda_1 > \lambda_2 > \ldots > \lambda_k)$), the corresponding shifted shape λ is an arrangement of cells in k rows where row i contains λ_i cells and is indented $i-1$ spaces. For example, the strict partition $(5, 4, 3, 1)$ corresponds to the shifted shape shown below.



We say that cells in row i and column i are *diagonal* cells and all other cells are *off-diagonal*.

The graph of shifted shapes, $\mathbb{SY}$, has shifted shapes as vertices, the rank function counts the number of cells in a shape, and shifted shape λ covers μ if λ is obtained from μ by adding one cell. The first six ranks of the graph $\mathbb{SY}$ are shown on the left in Figure 5.4. The graph shown on the right is its dual. In the dual graph, there is one edge between λ and μ if λ is obtained from μ by adding a diagonal cell, and there are two edges between λ and μ if λ is obtained from μ by adding an off-diagonal cell. To form the dual graded graph, the edges of $\mathbb{SY}$ are oriented upward and those of its dual are oriented downward.

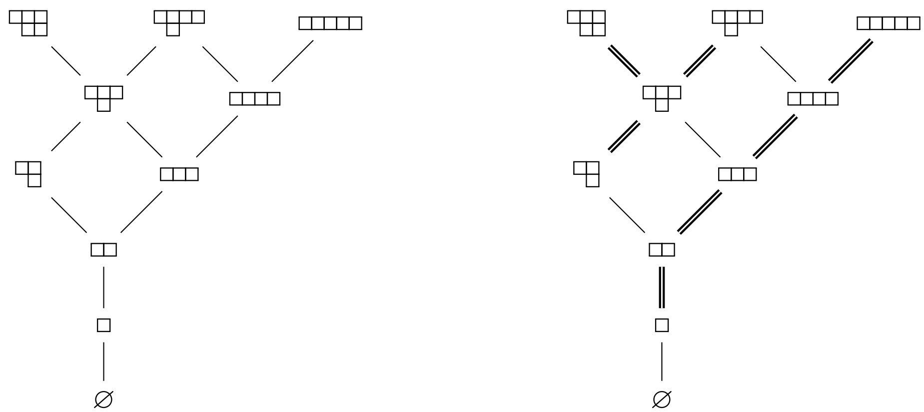


Figure 5.4: Shifted Young's lattice

5.2.2 Growth rules as generalized RSK

Recall the background on Robinson-Schensted-Knuth given in Section 2.0.3, and recall that $P(w)$ and $Q(w)$ denote the insertion and recording tableaux, respectively.

If $w = w_1 w_2 \cdots w_k$ is a permutation, we can obtain $(P(w), Q(w))$ from w by using growth diagrams. First, we create a $k \times k$ array with an X in the w_i^{th} square from the bottom of column i. For example, if $w = 14253$, we have

			X	
	X			
				X
		X		
X				

.

We will label the corners of each square with a partition. We begin by labeling all corners along the bottom row and left side of the diagram with the empty shape, $\varnothing$.

To complete the labeling of the corners, suppose the corners μ, λ, and ν are labeled, where μ, λ, and ν are as in the picture below. We label γ according to the following rules.

$$\begin{tikzcd} \mu \arrow[r, dash] \arrow[d, dash] & \gamma \arrow[d, dash] \\ \lambda \arrow[r, dash] & \nu \end{tikzcd}$$

(L1) If the square does not contain an X and $\lambda = \mu = \nu$, then $\gamma = \lambda$.

(L2) If the square does not contain an X and $\lambda \subset \mu = \nu$, then μ/λ is one box in some row i. In this case, γ is obtained from μ by adding one box in row $i = 1$.

(L3) If the square does not contain an X and $\mu \neq \nu$, define $\gamma = \mu \cup \nu$.

(L4) If the square contains an X, then $\lambda = \nu = \mu$ and γ is obtained from μ by adding one box in the first row.

Following these rules, there is a unique way to label the corners of the diagram. The resulting array is called the *growth diagram* of w, and rules (L1)-(L4) are called *growth rules.* For the remainder of this paper, we will use the word "square" when referring to a square in the growth diagram and the word "box" or "cell" when referring to a box in a partition, shifted partition, or snakeshape.

The chains of partitions read across the top of the diagram and up the right side of the diagram determine $Q(w)$ and $P(w)$, respectively. To obtain a tableau of shape λ from a chain of partitions $\varnothing \subset \lambda^1 \subset \lambda^2 \subset \ldots \subset \lambda^k = \lambda$, label the box of λ^i/λ^{i-1} by i.

Example 5.2.5. The growth diagram for the word 14253 is shown below.

$$\begin{array}{ccccccccccc}
\varnothing & \text{—} & 1 & \text{—} & 2 & \text{—} & 21 & \text{—} & 31 & \text{—} & 32 \\
| & & | & & | & & | & X & | & & | \\
\varnothing & \text{—} & 1 & \text{—} & 2 & \text{—} & 21 & \text{—} & 21 & \text{—} & 31 \\
| & & | & X & | & & | & & | & & | \\
\varnothing & \text{—} & 1 & \text{—} & 1 & \text{—} & 2 & \text{—} & 2 & \text{—} & 3 \\
| & & | & & | & & | & & | & X & | \\
\varnothing & \text{—} & 1 & \text{—} & 1 & \text{—} & 2 & \text{—} & 2 & \text{—} & 2 \\
| & & | & & | & X & | & & | & & | \\
\varnothing & \text{—} & 1 & \text{—} & 1 & \text{—} & 1 & \text{—} & 1 & \text{—} & 1 \\
| & X & | & & | & & | & & | & & | \\
\varnothing & \text{—} & \varnothing & \text{—} & \varnothing & \text{—} & \varnothing & \text{—} & \varnothing & \text{—} & \varnothing
\end{array}$$

From the chains of partitions on the rightmost edge and uppermost edge of the diagram, we can read the insertion tableau and recording tableau, respectively.

$$P(14253) = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} \end{array} \qquad\qquad Q(14253) = \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2} \end{array}$$

We next draw a connection between the growth rules described and an explicit cancellation in Young's lattice that shows that $DU - UD = I$. We shall start by giving explicit cancellation rules to pair all down-up paths from μ to ν with up-down paths from μ to ν, which will leave one up-down path unpaired in $\mu = \nu$. (Note that if $\rho(\mu) \neq \rho(\nu)$, then there are no up-down or down-up paths from μ to ν.)

(C1) If $\mu \neq \nu$, there is only one down-up path from μ to ν, which passes through $\mu \cap \nu$. Pair this with the only up-down path from μ to ν, which passes through $\mu \cup \nu$.

(C2) If $\mu = \nu$, then any down-up path can be represented by the inner corner of μ that is deleted in the downward step, and any up-down path can be represented by the outer corner of μ that is added in the upward step. Pair each down-up path with the up-down path that adds the first outer corner of μ strictly below the inner corner of the down-up path.

Now, the only up-down path that has not been paired with a down-up path is the path corresponding to adding the outer corner of μ in the first row. This gives $(DU - UD)\mu = \mu$.

It is easy to see that this cancellation yields exactly the growth rules described above. Cancellation rule (C1) determines growth rule (L3), cancellation rule (C2) determines growth rule (L2), and the up-down path left unmatched determines growth rule (L4). If we started with a different explicit cancellation, we could modify (L2), (L3), and (L4) accordingly to obtain new growth rules. Note that growth rule (L1) is simply a way to transfer information and will remain the same for any explicit cancellation.

Example 5.2.6. Starting with $\mu = (3, 1)$, cancellation rule (C2) says that the down-up path from μ to itself passing through $(2, 1)$ is paired with the up-down path passing through $(3, 2)$. The corresponding growth rule in this situation, (L2), is illustrated on the right.

$$\begin{tikzcd} 32 \arrow[d, dash] \\ 31 \arrow[d, dash] \\ 21 \end{tikzcd} \qquad\qquad \begin{tikzcd} 31 \arrow[r, dash] \arrow[d, dash] & 32 \arrow[d, dash] \\ 21 \arrow[r, dash] & 31 \end{tikzcd}$$

5.3 Dual filtered graphs

5.3.1 The definition

A weak filtered graph is a triple $G = (P, \rho, E)$, where P is a discrete set of vertices, $\rho : P \to \mathbb{Z}$ is a rank function, and E is a multiset of edges/arcs (x, y), where $\rho(y) \geq \rho(x)$. A strict filtered graph is a triple $G = (P, \rho, E)$, where P is a discrete set of vertices, $\rho : P \to \mathbb{Z}$ is a rank function, and E is a multiset of edges/arcs (x, y), where $\rho(y) > \rho(x)$. Let P_n as before denote the subset of vertices of rank n.

Let $G_1 = (P, \rho, E_1)$ and $G_2 = (P, \rho, E_2)$ be a pair of filtered graphs with a common vertex set, G_1 weak while G_2 strict. From this pair, define an *oriented filtered graph* $G = (P, E_1, E_2)$ by orienting the edges of G_1 in the positive filtration direction and the edges of G_2 in the negative filtration direction. Recall that $a_i(x, y)$ denotes the number of edges of E_i joining x and y or the multiplicity of edge (x, y) in E_i. Using the *up* and *down operators* associated with graph G as previously defined, we say that G_1 and G_2 are *dual filtered graphs* if for any $x \in G$,

$$(DU - UD)x = \alpha x + \beta Dx$$

for some $\alpha, \beta \in \mathbb{R}$. We focus on examples where $DU - UD = D + I$; see Remark 5.1.1.

In the next sections, we describe three constructions of dual filtered graphs.

5.3.2 Trivial construction

Every dual graded graph has an easy deformation that makes it a dual filtered graph. To construct it, simply add $\rho(x)$ "upward" loops to each element x of the dual graded graph, where ρ is the rank function. We will call this the *trivial construction*.

Theorem 5.3.1. *For any dual graded graph G, the trivial construction gives a dual filtered graph G'.*

Proof. It suffices to show that if q covers p in G, then $[p](DU - UD)(q) = [p](D)(q)$. If we restrict to all paths that do not contain a loop, the coefficient of p in $(DU - UD)(q)$ is 0 since without loops, we have a pair of dual graded graphs. Hence we may restrict our attention to up-down paths and down-up paths that contain a loop. The up-down paths beginning at q and ending at p are exactly the paths consisting of a loop at q

followed by an edge from q to p. There are $\rho(q)e(q \to p)$ such paths. The down-up paths containing a loop are the paths consisting of an edge from q to p followed by a loop at p. There are $e(q \to p)\rho(p)$ such paths. Thus

$$[p](DU - UD)(q) = e(q \to p)(\rho(q) - \rho(p)) = e(q \to p) = [p]D(q).$$

$\square$

Example 5.3.2. The trivial construction for the first four ranks of Young's lattice is shown in Figure 5.5. The edges in the graph on the left are oriented upward, and the edges in the graph on the right are oriented downward.

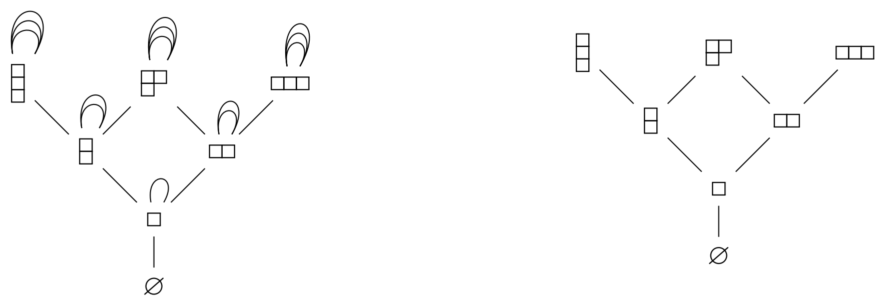


Figure 5.5: Trivial construction on Young's lattice

5.3.3 Pieri construction

The following construction is close to the one for dual graded graphs described in the works of Bergeron-Lam-Li [BLL12], Nzeutchap [Nze06], and Lam-Shimozono [LSa].

Let

$$A = \oplus_{i \geq 0} A_i$$

be a filtered, commutative algebra over a field $\mathbb{F}$ with $A_0 = \mathbb{F}$ and a linear basis $a_\lambda \in A_{|\lambda|}$. Here, the λ's belong to some infinite indexing set P, each graded component of which

$$P_i = \{\lambda \mid |\lambda| = i\}$$

is finite. The property of being filtered means that $a_\lambda a_\mu = \sum c_\nu a_\nu$, where $|\lambda| + |\mu| \leq |\nu|$.

Let $f = f_1 + f_2 + \ldots \in \hat{A}$ be an element of the completion $\hat{A}$ of A such that $f_i \in A_i$. Assume $D \in End(A)$ is a derivation on A satisfying $D(f) = f + 1$.

To form a graph, we define E_1 by defining $a_1(\lambda, \mu)$ to be the coefficient of a_λ in $D(a_\mu)$. We also define E_2 by defining $a_2(\lambda, \mu)$ to be the coefficient of a_μ in $a_\lambda f$.

Theorem 5.3.3. *The resulting graph is a dual filtered graph.*

Proof. It follows from

$$D(fa_\lambda) = D(f)a_\lambda + fD(a_\lambda),$$

or

$$D(fa_\lambda) - fD(a_\lambda) = a_\lambda + fa_\lambda.$$

$\square$

Assume A_1 is one-dimensional, and let g be its generator. We shall often find a derivation D from a *bialgebra* structure on A. Indeed, assume Δ is a coproduct on A such that $\Delta(a) = 1 \otimes a + a \otimes 1 + \ldots$, where the rest of the terms lie in $(A_1 \oplus A_2 \oplus \ldots) \otimes (A_1 \oplus A_2 \oplus \ldots)$. Assume also ξ is a map $A \otimes A \longrightarrow A$ such that $\xi(p \otimes g) = p$ and for any element $q \in A_i$, $i \neq 1$, we have $\xi(p \otimes q) = 0$.

Lemma 5.3.4. *The map $D(a) = \xi(\Delta(a))$ is a derivation.*

Proof. We have $\xi(\Delta(ab)) = \xi(\Delta(a)\Delta(b)))$. Pick a linear basis for A that contains 1 and g. Express $\Delta(a)$ and $\Delta(b)$ so that the right side of tensors consists of the basis elements. The only terms that are not killed by ξ are the ones of the form $p \otimes g$. The only way we can get such terms is from $(p_1 \otimes 1) \cdot (p_2 \otimes g)$ or $(p_1 \otimes g) \cdot (p_2 \otimes 1)$. Those are exactly the terms that contribute to

$$\xi(\Delta(a)) \cdot b + a \cdot \xi(\Delta(b)).$$

$\square$

5.3.4 Möbius construction

Let $G = (P, \rho, E_1, E_2)$ be a dual graded graph. From G, form a pair of filtered graphs as follows. Let $G'_1 = (P, \rho, E'_1)$ have the set of edges E'_1 consisting of the same edges as E_1 plus $\#\{x \mid y \text{ covers } x \text{ in } G_1\}$ loops at each vertex $y \in P$. Let $G'_2 = (P, \rho, E'_2)$ have the set of edges E'_2 consisting of

$$a'_2(x, y) = |\mu(x, y)|$$

edges between vertices x and y, where μ denotes the Möbius function in (P, ρ, E_2). Compose G_1' and G_2' into an oriented filtered graph $\tilde{G}$.

As we shall see, in all three of our major examples, the Möbius deformation forms a dual filtered graph. Unlike with the Pieri construction, we do not have the algebraic machinary explaining why the construction yields dual filtered graphs. Instead, we provide the proofs on a case-by-case basis. Nevertheless, the Möbius construction is the most important one, as it is this construction that relates to insertion algorithms and thus to K-theory of Grassmannians and orthogonal Grassmannians.

It is not the case that every dual graded graph's Möbius deformation makes it a dual filtered graph. For example, it is easy to see that the Möbius deformation of the graphs in Example 5.7.7 and Example 5.7.4 are not dual filtered graphs.

Problem 5.3.5. *Determine for which dual graded graphs G the Möbius construction $\tilde{G}$ yields a dual filtered graph.*

5.4 Major examples: Young's lattice

5.4.1 Pieri deformations of Young's lattice

Let $A = \Lambda$ be the ring of symmetric functions. Let s_λ, p_λ, and h_λ be its bases of Schur functions, power sum symmetric functions, and complete homogeneous symmetric functions. Let

$$f = h_1 + h_2 + \dots .$$

Define up and down edges of a filtered graph $G = (P, \rho, E_1, E_2)$ by letting $a_1(\mu, \nu)$ be the coefficient of s_ν in $p_1 s_\mu$, and $a_2(\mu, \nu)$ be the coefficient of s_ν in $f s_\mu$.

Recall that given two partitions, μ and ν, μ/ν forms a *horizontal strip* if no two boxes of μ/ν lie in the same column. For example, $(4, 2, 1)/(2, 1)$ forms a horizontal strip while $(4, 3, 1)/(2, 1)$ does not.

Lemma 5.4.1. *The up edges E_1 coincide with those of Young's lattice, while the down edges E_2 connect shapes that differ by a horizontal strip.*

Proof. The statement follows from the Pieri rule and the fact that $p_1 = h_1$, see [Sta99]. $\square$

In Figure 5.6, the first six ranks are shown. The edges in the graph on the left are upward-oriented, and the edges on the graph on the right are downward-oriented.

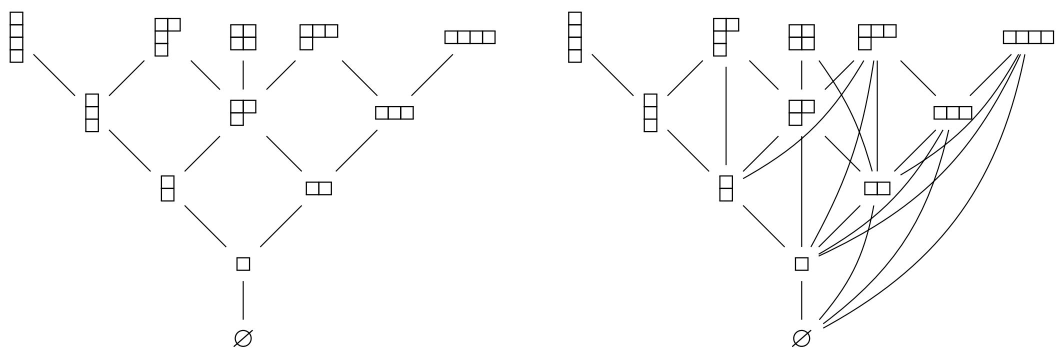


Figure 5.6: Pieri construction from the Schur function basis

Theorem 5.4.2. *The Pieri deformation of Young's lattice is a dual filtered graph with*

$$DU - UD = D + I.$$

Recall the following facts regarding the ring of symmetric functions.

Theorem 5.4.3. *[Sta99, Mac98]*

(a) Λ *is a free polynomial ring in* $p_1, p_2, \ldots$.

(b) Λ *can be endowed with a standard bilinear inner product determined by* $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda,\mu}$.

Because of the first property, we can differentiate elements of Λ with respect to p_1 by expressing them first as a polynomial in the p_i's. We shall need the following two properties of f and p_1.

Lemma 5.4.4. *(a) For any* $h, g \in \Lambda$ *we have*

$$\langle g, p_1 h \rangle = \langle \frac{d}{dp_1} g, h \rangle.$$

(b) We have

$$\frac{d}{dp_1} f = f + 1.$$

Proof. By bilinearity, it suffices to prove the first claim for g, h in some basis of Λ. Let us choose the power sum basis. Then we want to argue that

$$\langle p_\lambda, p_1 p_\mu \rangle = \langle \frac{d}{dp_1} p_\lambda, p_\mu \rangle.$$

This follows from the fact that p_λ's form an orthogonal basis and [Sta99, Proposition 7.9.3].

For the second claim, using [Sta99, Proposition 7.7.4] we have

$$1 + f = e^{p_1 + \frac{p_2}{2} + \cdots},$$

which of course implies $\frac{d}{dp_1}(1 + f) = 1 + f$. $\square$

Now we are ready for the proof of Theorem 5.4.2.

Proof. Applying Lemma 5.4.4 we see that $A = \Lambda$, $a_\lambda = s_\lambda$, $D = \frac{d}{dp_1}$ and $f = h_1 + h_2 + \ldots$ satisfy the conditions of Theorem 5.3.3. The claim follows. $\square$

Remark 5.4.5. We can make a similar argument for the analogous construction where the edges in E_1 are again those of Young's lattice and the edges in E_2 connect shapes that differ by a vertical strip. In this case we use $f = e_1 + e_2 + \ldots$, where the e_i are the elementary symmetric functions.

5.4.2 Möbius deformation of Young's lattice

Given two partitions, λ and ν, λ/ν forms a *rook strip* if no two boxes of λ/ν lie in the same row and no two boxes of λ/ν lie in the same column. In other words, λ/ν is a rook strip if it is a disconnected union of boxes. We state the following well-known result about the Möbius function on Young's lattice.

Proposition 5.4.6. *We have*

$$|\mu(\lambda, \nu)| = \begin{cases} 1 & \text{if } \lambda/\nu \text{ is a rook strip;} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The statement follows from the fact that Young's lattice is a distributive lattice, and the interval $[\lambda, \mu]$ is isomorphic to a Boolean algebra if and only if μ/λ is a rook strip. See [Sta11, Example 3.9.6]. $\square$

The Möbius deformation of Young's lattice is shown in Figure 5.7 with upward-oriented edges shown on the left and downward-oriented edges shown on the right. Notice that loops at a partition λ may be indexed by inner corners of λ.

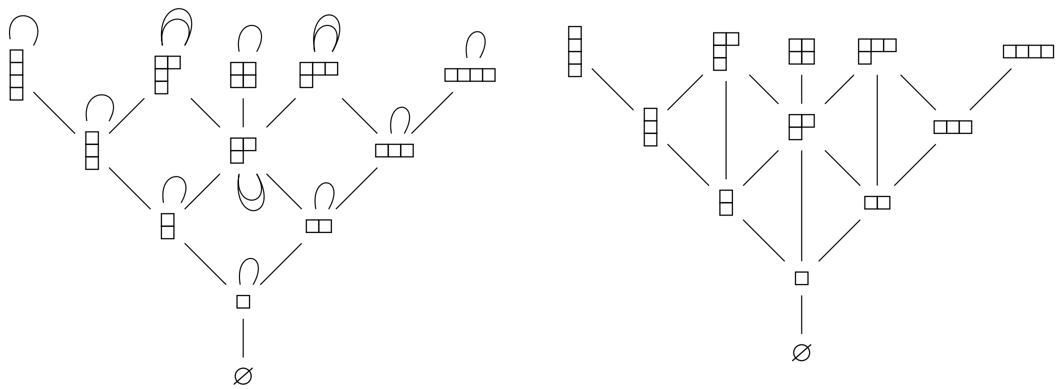


Figure 5.7: Möbius construction applied to Young's lattice

Theorem 5.4.7. *The Möbius deformation of Young's lattice forms a dual filtered graph with*

$$DU - UD = D + I.$$

Proof. It suffices to show that $[\lambda](DU-UD)\mu = [\lambda]D\mu$ when μ covers λ since we started with a dual graded graph.

Suppose shape λ is covered by shape μ. Each inner corner of μ contributes one to the coefficient of λ in $DU\mu$ via taking the loop corresponding to that inner corner of μ as an upward move and then going down to λ.

Next, consider shape μ and mark the outer corners of λ contained in μ. The marked boxes will be inner corners of μ. The shapes with an upward arrow from μ and a downward arrow to λ are exactly those obtained by adding one box to an outer corner of μ which is not adjacent to one of the marked boxes. There are $\#\{\text{outer corners of } \lambda\} - |\mu/\lambda|$ possible places to add such a box. Thus the coefficient of λ in $DU\mu$ is

$$\#\{\text{inner corners of } \mu\} + \#\{\text{outer corners of } \lambda\} - |\mu/\lambda|.$$

For down-up paths that involve three shapes, consider the inner corners of μ that are also inner corners of λ. Every shape obtained by removing one of these inner corners determines one down-up path from μ to λ, and there are $\#\{\text{inner corners of } \mu\} - |\mu/\lambda|$ such shapes. The remaining down-up paths are given by going from μ to λ and then taking a loop at λ. Hence the coefficient of λ in $UD\mu$ is

$$\#\{\text{inner corners of } \mu\} - |\mu/\lambda| + \#\{\text{inner corners of } \lambda\}.$$

Since all shapes have one more outer corner than inner corner, the coefficient of λ in $(DU - UD)\mu$ is 1.

$\square$

5.4.3 Hecke growth and decay

Recall the definition of increasing tableaux, set-valued tableaux, Hecke insertion, and reverse Hecke insertion from Section 4.3. These procedures correspond to the Möbius deformation of Young's lattice in the same way that RSK corresponds to Young's lattice. In other words, if we insert a word w of length n using Hecke insertion, the construction of the insertion tableau will be represented as a path of length n downward that ends at $\varnothing$ in the Möbius deformation of Young's lattice, and the construction of the recording tableau is represented as a path of length n upward starting at $\varnothing$.

Given any word, $h = h_1 h_2 \dots h_k$ containing $n \leq k$ distinct numbers, we can create an $n \times k$ array with an X in the h_i^{th} square from the bottom of column i. Note that there can be multiple X's in the same row but is at most one X per column. For example, if $h = 121331$, we have

			X	X	
	X				
X		X			X

We will label each of the four corners of each square with a partition and some of the horizontal edges of the squares with a specific inner corner of the partition on the left vertex of the edge, which will indicate the box where the insertion terminates. We do this by recording the row of the inner corner at which the insertion terminates. We begin by labeling all corners along the bottom row and left side of the diagram with the partition $\varnothing$.

$$\begin{array}{ccccccccccccc}
\varnothing & \text{---} & & \text{---} & & \text{---} & & \text{---} & & \text{---} & & \text{---} & \\
\vert & & \vert & & \vert & & \vert & X & \vert & X & \vert & & \vert \\
\varnothing & \text{---} & & \text{---} & & \text{---} & & \text{---} & & \text{---} & & \text{---} & \\
\vert & & \vert & X & \vert & & \vert & & \vert & & \vert & & \vert \\
\varnothing & \text{---} & & \text{---} & & \text{---} & & \text{---} & & \text{---} & & \text{---} & \\
\vert & X & \vert & & \vert & X & \vert & & \vert & & \vert & X & \vert \\
\varnothing & \text{---} & \varnothing & \text{---} & \varnothing & \text{---} & \varnothing & \text{---} & \varnothing & \text{---} & \varnothing & \text{---} & \varnothing
\end{array}$$

To complete the labeling of the corners, suppose the corners μ, λ, and ν are labeled, where μ, λ, and ν are as in the picture below. We label γ according to the following rules.

$$\begin{tikzcd}
\mu \arrow[r, dash] \arrow[d, dash] & \gamma \arrow[d, dash] \\
\lambda \arrow[r, dash] & \nu
\end{tikzcd}$$

If the square contains an X:

(1) If $\mu_1 = \nu_1$, then γ/μ consists of one box in row 1.

(2) If $\mu_1 \neq \nu_1$, then $\gamma = \mu$ and the edge between then is labeled by the row of the highest inner corner of μ.

If the square does not contain an X and if either $\mu = \lambda$ or $\nu = \lambda$ with no label between λ and ν:

(3) If $\mu = \lambda$, then set $\gamma = \nu$. If $\nu = \lambda$, then $\gamma = \mu$.

If $\nu \not\subseteq \mu$ and the square does not contain an X:

(4) In the case where $\nu \not\subseteq \mu$, $\gamma = \nu \cup \mu$.

If $\nu \subseteq \mu$ and the square does not contain an X:

(5) If ν/λ is one box in row i and μ/ν has no boxes in row $i+1$, then γ/μ is one box in row $i+1$.

(6) If ν/λ is one box in row i and μ/ν has a box in row $i+1$, then $\gamma = \mu$ and the edge between then is labeled $i+1$.

(7) If $\nu = \lambda$, the edge between them is labeld i, and there are no boxes of μ/ν immediately to the right or immediately below this inner corner of ν in row i, then $\gamma = \mu$ with the edge between them labeled i.

(8) If $\nu = \lambda$, the edge between them is labeled i, and there is a box of μ/ν directly below this inner corner of ν in row i, then $\gamma = \mu$ with the edge between them labled $i + 1$.

(9) If $\nu = \lambda$, the edge between them is labeled i, and there is a box of μ/ν immediately to the right of this inner corner of ν in row i but no box of μ/ν in row $i + 1$, then γ/μ is one box in row $i + 1$.

(10) If $\nu = \lambda$, the edge between them is labeled i, and there is a box of μ/ν immediately to the right of this inner corner of ν in row i and a box of μ/ν in row $i + 1$, then $\gamma = \mu$ with the edge between them labeled $i + 1$.

We call the resulting array the *Hecke growth diagram* of h. In our previous example with $h = 121342$, we would have the diagram in Figure 5.8.

$$
\begin{array}{ccccccccccccc}
\varnothing & \text{—} & 1 & \text{—} & 2 & \text{—} & 21 & \text{—} & 31 & \overset{1}{\text{—}} & 31 & \overset{2}{\text{—}} & 31 \\
| & & | & & | & & | & X & | & X & | & & | \\
\varnothing & \text{—} & 1 & \text{—} & 2 & \text{—} & 21 & \text{—} & 21 & \text{—} & 21 & \overset{2}{\text{—}} & 21 \\
| & & | & X & | & & | & & | & & | & & | \\
\varnothing & \text{—} & 1 & \text{—} & 1 & \overset{1}{\text{—}} & 1 & \text{—} & 1 & \text{—} & 1 & \overset{1}{\text{—}} & 1 \\
| & X & | & & | & X & | & & | & & | & X & | \\
\varnothing & \text{—} & \varnothing & \text{—} & \varnothing & \text{—} & \varnothing & \text{—} & \varnothing & \text{—} & \varnothing & \text{—} & \varnothing
\end{array}
$$

Figure 5.8: Hecke growth diagram

Let $\mu_0 = \varnothing \subseteq \mu_1 \subseteq \ldots \mu_k$ be the seqence of partitions across the top of the Hecke growth diagram, and let $\nu_0 = \varnothing \subseteq \nu_1 \subseteq \ldots \nu_n$ be the sequence of partitions on the right side of the Hecke growth diagram. These sequences correspond to increasing tableaux $Q(h)$ and $P(h)$, respectively. If the edge between μ_i and μ_{i+1} is labeled j, then $\mu_i = \mu_{i+1}$, and the label $i + 1$ of $Q(h_1 \cdots h_{i+1})$ is placed in the box at the end of row j of $Q(h_1 \cdots h_i)$. In the example above, we have

$$
P(h) = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 2 \\ \cline{1-1} \end{array}
\qquad\qquad
Q(h) = \begin{array}{|c|c|c|} \hline 1 & 2 & 45 \\ \hline 36 \\ \cline{1-1} \end{array}
$$

Theorem 5.4.8. *For any word h, the increasing tableau $P(h)$ and set-valued tableau $Q(h)$ obtained from the sequence of partitions across the right side of the Hecke growth diagram for h and across the top of the Hecke growth diagram for h, respectively, are $P_H(h)$ and $Q_H(h)$, the Hecke insertion and recording tableau for h.*

For the following proof, again note that the loops at shape λ may be indexed by inner corners of λ.

Proof. Suppose that the square described in rules 1 through 10 is in row t and column s. We will argue the result by induction.

Lemma 5.4.9. *Oriented as in the square above, $|\nu/\lambda| \leq 1$.*

Proof. By the induction hypothesis, λ is the recording tableau after inserting the numbers in columns 1 through $s-1$, and ν is the insertion tableau after inserting the number in column s. Since we are only inserting one number, $|\nu/\lambda|$ is at most 1. □

Lemma 5.4.10. *Oriented as in the square above, μ/λ is a rook strip, that is, no two boxes in μ/λ are in the same row or column.*

Proof. By induction, μ/λ represents the positions of boxes filled with t at some point in the insertion. Since insertion results in an increasing tableau, μ/λ must be a rook strip. □

(1) First note that in any square with an X, $\lambda = \nu$ since there is exactly one square in each column and that an X in the square we are considering corresponds to inserting a t into the tableau of shape λ. Since, by the induction hypothesis, shapes along columns represent the insertion tableau, $\mu_1 = \nu_1 = \lambda_1$ means that before adding the t, there are no t's in the first row of the insertion tableau. Since t is weakly the largest number inserted to this point, inserting the t will result in a t being added to the first row of the insertion tableau. Thus $\gamma = (\mu_1+1, \mu_2, \mu_3, \ldots)$.

(2) If $\mu_1 \neq \nu_1 = \lambda_1$, then after inserting the numbers in columns 1 through $s-1$, there is a t at the end of the first row of the insertion tableau. It follows that inserting another t will result in this t merging with the t at the end of the first row, and the special corner in this case becomes the bottom box of the last column, or in other words, the highest inner corner of μ.

(3) If $\mu = \lambda$, then there is no X in row t in columns 1 to s. Thus there is no t to add, and so the insertion tableau ν should not change. Similarly for if $\nu = \lambda$.

(4) Suppose that ν/λ is one box in row i and μ/λ is a rook strip containing boxes in rows $j_1, j_2, \ldots, j_k$. Then $\nu \not\subseteq \mu$ implies that $i \notin \{j_1, j_2, \ldots, j_k\}$. Since ν/λ is one box, the last action in the insertion sequence of r into the insertion tableau of shape λ is a box being added in row i. Since there is not a t in row i, the bumping sequence when inserting r into the insertion tableau of shape μ will not disturb the t's.

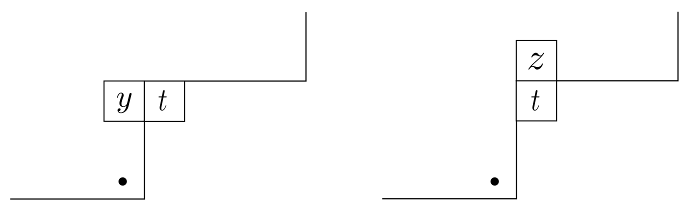
(5) Suppose ν is λ plus one box in position (i, j). Then since $\nu \subseteq \mu$, μ/λ contains the box at (i, j). It follows that inserting r into the insertion tableau of shape μ will result in bumping the t in position (i, j) since the bumping path of r inserting in the insertion tableau of shape λ ends by adding a box at (i, j). Since there is no box in row $i + 1$ of μ/λ, everything in row $i + 1$ of the insertion tableau of shape μ is strictly less than t. It follows that the t bumped from (i, j) can be added to the end of row $i + 1$.

(6) This is almost identical to the proof of Rule 4. Since there is now a box in row $i + 1$ of μ/λ, there is a t at the end of row $i + 1$ of the insertion tableau of shape μ. Inserting r will bump the t in row i as above, but now the t will merge with the t in row $i + 1$. Thus the special corner is now at the inner corner of μ in row $i + 1$.

(7) There are two ways that inserting r into the insertion tableau of shape μ will involve the t's. Assume ν is obtained from λ by taking a loop in row i and column j. Assume rows k through i all have length j.

Case 1: A box containing y was bumped from row $k-1$, merged with itself in row k, and there was a t to the right of the y. Note that $k \neq i$. Then the y duplicates and bumps the t in row k, but this t cannot be added to row $k + 1$ as it would be directly below the original t. Thus, by the rules of Hecke insertion, the special corner is the box at the bottom of column j, box (i, j). See the left-hand figure below where the dot indicates the box (i, j).

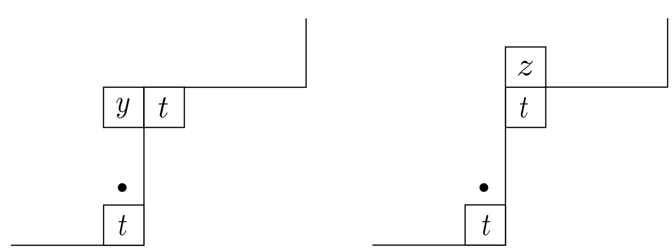
Case 2: Consider the situation in the right-hand figure below where the dot indicates the box (i, j). Assume that during the insertion of r, some number merged with itself to the left of z in row $k-1$ and z is duplicated bumped to the next row. Following the rules of Hecke insertion, z bumps the t in row k, but as it cannot replace the t, the t remains at the end of row k. The t that was bumped cannot be added to end of row $k+1$, so no new boxes are created in the insertion and the special corner is the box at the bottom of column j, box (i, j).



(8) There are three ways that inserting r into the insertion tableau of shape μ will involve the t's. Assume ν is obtained from λ by taking a loop in row i and column j. Assume rows k through i all have length j.

Case 1: If there is a t in square $(i, j+1)$, we are in the situation described by Rule 9.

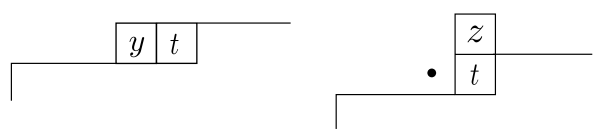
Cases 2 and 3 are described in the proof of Rule 7. The difference is that the bottom of column j after we insert r is now the box $(i+1, j)$, so the special corner in each case will be $(i+1, j)$, as desired.



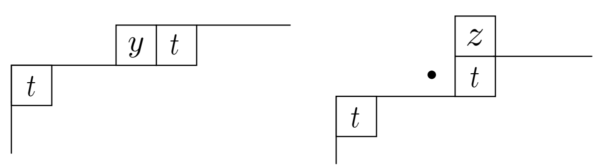
(9) Note that a loop in row i with a box of μ/ν to its right implies that $\nu_{i-1} > \nu_i > \nu_{i+1}$. There are two ways for the special corner of the insertion of r into the insertion tableau of shape λ could have been (i, j).

Case 1: Some y was bumped from row $i-1$ and merged with itself in box (i, j). Since there is a t to the right of box (i, j) in μ, this will result in duplicating and bumping t. Since there is nothing in row $i+1$ of μ/ν, everything in row $i+1$ of μ is strictly less than t. Hence the t bumped from row i can be added to the end of row $i+1$.

Case 2: As in the figure below, z was duplicated and bumped from box $(i-1, j+1)$. This z cannot replace the t in box $(i, j+1)$ since it would be directly below the orginal z, but it bumps the t to row $i+1$. Since there is nothing in row $i+1$ of μ/ν, everything in row $i+1$ of μ is strictly less than t. Hence the t bumped from row i can be added to the end of row $i+1$.



(10) This is similar to the situation described in the proof of Rule 9. The difference is that a box in row $i+1$ of μ/ν means there is already a t at the end of row $i+1$ of μ, so each insertion will end with the bumped t merging with itself at the end of row $i+1$.



$\square$

5.4.4 Möbius via Pieri

The following shows that we have an instance of the Möbius via Pieri phenomenon in the case of Young's lattice. Namely, the Möbius deformation of Young's lattice is obtained by the Pieri construction from an appropriate K-theoretic deformation of the underlying Hopf algebra.

Recall the definition of the stable Grothendieck polynomials as in Theorem 3.9.2, and define the *signless stable Grothendieck polynomials* $\tilde{G}_\lambda$ by omitting the sign. Note that these coincide with certain of the $\{\tilde{K}\}$ as defined in Chapter 3. The structure constants will coincide with those of the stable Grothendieck polynomials up to sign. More precisely,

$$\tilde{G}_\lambda \tilde{G}_\mu = \sum c^\nu_{\lambda,\mu} \tilde{G}_\nu,$$

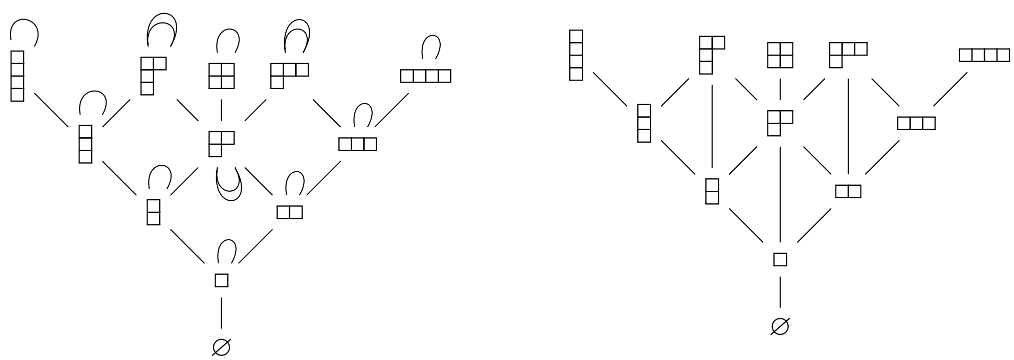
where $c^\nu_{\lambda,\mu}$ is the number of increasing tableaux R of skew shape ν/λ such that $P_H(\mathfrak{row}(R)) = S_\mu$, and

$$\Delta(\tilde{G}_\nu) = \sum_{\lambda,\mu} d^\nu_{\lambda,\mu} \tilde{G}_\lambda \otimes \tilde{G}_\mu,$$

where $d^\nu_{\lambda,\mu}$ is the number of increasing tableaux R of skew shape $\lambda \oplus \mu$ such that $P_H(\mathfrak{row}(R)) = S_\nu$; see Chapter 4. Let A be the subring of the completion $\hat{\Lambda}$ of the ring of symmetric functions with basis $\{\tilde{G}_\lambda\}$.

We consider the following Pieri construction in A. Let $g = \tilde{G}_1$, and define an operator D by $D(\tilde{G}_\nu) = \xi(\Delta(\tilde{G}_\nu))$. We define a graph G, where elements are partitions, $a_1(\mu, \lambda)$ is the coefficient of $\tilde{G}_\mu$ in $D(\tilde{G}_\lambda)$, and $a_2(\mu, \lambda)$ is the coefficient of $\tilde{G}_\lambda$ in $\tilde{G}_\mu \tilde{G}_1$.

Proposition 5.4.11. *The resulting graph coincides with the one obtained via Möbius construction in Section 5.4.2.*



Proof. To verify this claim, we first show why the absolute value of the coefficient of $\tilde{G}_\nu$ in $\tilde{G}_\lambda \tilde{G}_1$ is 1 if ν/λ is a rook strip and is 0 otherwise. This follows from that fact that the only words that Hecke insert into S_1, the superstandard tableau of shape (1), are words consisting only of the letter 1. Thus, $\mathfrak{row}(R) = 11 \cdots 1$ for any R contributing to $c^\nu_{\lambda,1}$. Increasing tableaux of shape ν/λ with row word $11 \cdots 1$ are precisely rook strips.

Secondly, it must be true that $d^{\lambda}_{\lambda,1}$ is the number of inner corners of λ and if $\nu \neq \lambda$, $d^{\nu}_{\lambda,1}$ is 1 if ν is obtained from λ by adding one box and 0 otherwise. Suppose $\nu \neq \lambda$ and consider all skew tableaux R of shape $\lambda \oplus 1$ such that $P_H(\mathfrak{row}(R)) = S_\nu$. To obtain such λ, simply reverse insert an entry of S_ν with $\alpha = 1$ and use the output integer to fill the single box in $\lambda \oplus 1$. The resulting shapes are the shapes obtained from ν by deleting one inner corner of μ. Similarly, if $\nu = \lambda$, we reverse insert each inner corner of S_λ using $\alpha = 0$ and then let R be $S_\lambda \oplus x$, where x is the result of the reverse insertion. $\square$

5.5 Major examples: shifted Young's lattice

5.5.1 Pieri deformation of shifted Young's lattice

Let $A = \Lambda'$ be the subring of the ring of symmetric functions generated by the odd power sums $p_1, p_3, \ldots$. Let Q_λ and P_λ be the bases of Schur Q and Schur P functions for this ring; see [Mac98, Sag87]. Here λ varies over the set P of partitions with distinct parts. We refer the reader to [Mac98] for background. Let q_i be defined by

$$q_i = \sum_{0 \leq j \leq i} e_j h_{i-j},$$

and let $f = q_1 + q_2 + \ldots$.

Define up and down edges of a filtered graph $G = (P, \rho, E_1, E_2)$ by letting $a_2(\mu, \nu)$ be the coefficient of Q_μ in fQ_ν and $a_1(\mu, \nu)$ be the coefficient of P_μ in $p_1 P_\nu$.

We will fill a shifted partition λ with ordered alphabet $1' < 1 < 2' < 2 < 3' < 3 \ldots$ according to the usual rules:

- The filling must be weakly increasing in rows and columns.
- There can be at most one instance of k' in any row.
- There can be at most one instance of k in any column.
- There can be no primed entries on the main diagonal.

For example, the shifted partition below is filled according to the rules stated above.

$$\begin{array}{|c|c|c|c|c|}
\hline
1 & 2' & 2 & 2 & 4' \\
\hline
\multicolumn{1}{c|}{} & 2 & 3' & 5' & 5 \\
\cline{2-5}
\multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 4 & 5' & \multicolumn{1}{|c}{} \\
\cline{3-4}
\multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 6 & \multicolumn{1}{|c}{} \\
\cline{4-4}
\end{array}$$

Given two shifted shapes, μ and ν, we say that μ/ν forms a *border strip* if it contains no 2-by-2 square.

Lemma 5.5.1. *The Pieri deformation of the shifted Young's lattice is formed by adding downward-oriented edges from μ to ν whenever μ/ν forms a border strip. The number of such edges added is the same as the number of ways to fill the boxes of the border strip with k and k' according to the usual rules.*

Proof. It follows from formula (8.15) in [Mac98], which is an analogue of the Pieri rule for Schur P functions, and the fact that $q_1 = 2p_1$. $\square$

The first six ranks of the Pieri deformation are shown in Figure 5.9.

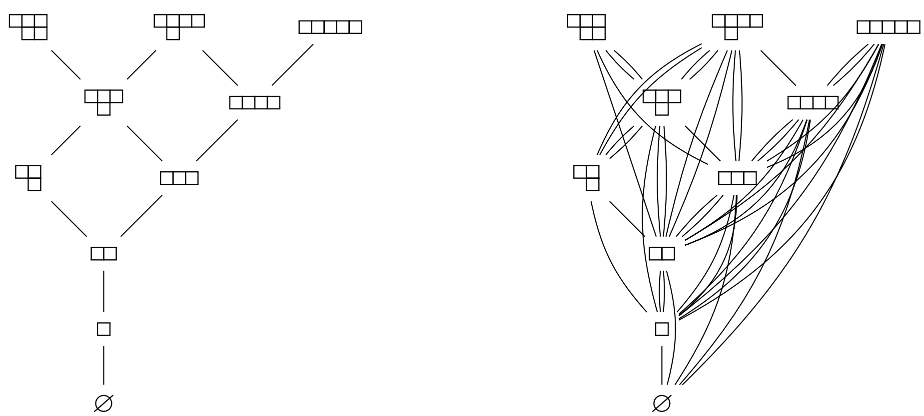


Figure 5.9: Pieri deformation of shifted Young's lattice

Notice that there are two edges from 41 to 2 corresponding to the following fillings of the border strip $(4, 1)/(2)$.

	k′	k
k		

	k	k
k		

Theorem 5.5.2. *The Pieri deformation of the dual graded graph of shifted shapes satisfies*

$$DU - UD = D + I.$$

We shall need the following two properties of Λ'.

Theorem 5.5.3. *[Mac98]*

- *Λ' is a free polynomial ring in odd power sums $p_1, p_3, \ldots$.*

- *Λ' inherits the standard bilinear inner product from Λ, which satisfies $\langle Q_\lambda, P_\mu\rangle = 2^{l(\mu)}\delta_{\lambda,\mu}$.*

Because of the first property, we can again differentiate elements of Λ' with respect to p_1, by expressing them first as a polynomial in the p_i's. We shall need the following property of f.

Lemma 5.5.4. *We have*

$$\frac{d}{dp_1} f = 2f + 2.$$

Proof. From [Mac98, Ex. 6 (a), III, 8] we have

$$1 + f = e^{2p_1 + \frac{2p_3}{3} + \cdots},$$

which of course implies $\frac{d}{dp_1}(1+f) = 2 + 2f$. □

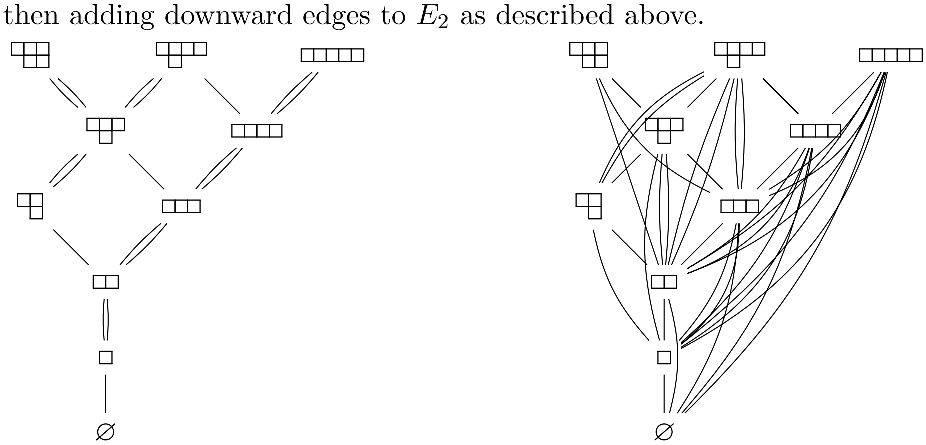
Now we are ready for the proof of Theorem 5.5.2.

Proof. Applying Lemma 5.4.4 we see that $A = \Lambda'$, $a_\lambda = Q_\lambda$, $D = \frac{1}{2}\frac{d}{dp_1}$ and $f = q_1 + q_2 + \ldots$ satisfy the conditions of Theorem 5.3.3. The claim follows. □

Remark 5.5.5. If we instead take $a_1(\mu,\nu)$ to be two times the coefficient of Q_ν in $p_1 Q_\mu$, and $a_2(\mu,\nu)$ to be $\frac{1}{2}$ times the coefficient of P_μ in fP_ν, we get a dual filtered graph with

$$DU - UD = 2D + I.$$

This choice corresponds to first swapping E_1 and E_2 in the original dual graded graph of $\mathbb{SY}$ and then adding downward edges to E_2 as described above.



5.5.2 Möbius deformation of shifted Young's lattice

The Mobius deformation of the shifted Young's lattice is shown in Figure 5.10 with upward-oriented edges shown on the left and downward-oriented edges on the right. Note that we swapped the E_1 and E_2 of Example 5.2.4.

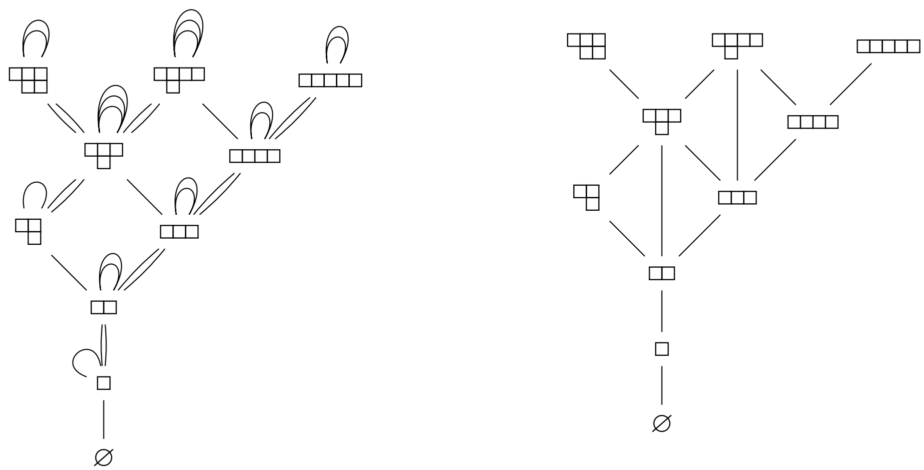


Figure 5.10: Möbius deformation of shifted Young's lattice

Lemma 5.5.6. *For the lattice of shifted shapes, the Mobius function is given by the following formula:*

$$\mu(p,q) = \begin{cases} (-1)^{|p|-|q|} & \textit{if } q/p \textit{ is a disjoint union of boxes} \\ 0 & \textit{otherwise} \end{cases}$$

Proof. It is known that $\mathbb{SY}$ is a distributive lattice [Fom94, Example 2.2.8], and so it follows that $\mu(p,q)$ is nonzero if and only if the interval $[p,q]$ is a Boolean lattice, in which case $\mu(p,q)$ is as described above, [Sta99, Example 3.9.6]. And $[p,q]$ is a Boolean lattice exactly when q/p is a disjoint union of boxes, i.e. no two boxes of q/p share an edge. □

Theorem 5.5.7. *The Möbius deformation of the shifted Young's lattice forms a dual filtered graph with*

$$DU - UD = D + I.$$

Proof. Given that the non-deformed pair form a pair of dual graded graphs, it suffices to show that

$$[\lambda](DU - UD)(\mu) = [\lambda](D)(\mu)$$

when μ covers λ.

For a shifted shape x, let $I_{diag}(x)$ denote the size of the set of inner corners of x that are on the diagonal, $I_{odiag}(x)$ denote the size of the set of inner corners of x that are not on the diagonal, $O_{diag}(x)$ denote the size of the set of outer corners of x that are on the diagonal, and $O_{odiag}(x)$ denote the size of the set of outer corners of x that are not on the diagonal.

Lemma 5.5.8. *For any shifted shape λ,*

$$O_{diag}(\lambda) + 2O_{odiag}(\lambda) - I_{diag}(\lambda) - 2I_{odiag}(\lambda) = 1.$$

Proof. Consider some element $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$. Suppose $\lambda_k = 1$. We can add a box at some subset of rows $\{i_1 = 1, i_2, \ldots, i_t\}$ and fill each with either k or k'. Thus $O_{diag}(\lambda) + 2O_{odiag}(\lambda) = 2t$. We can delete a box in rows $i_2 - 1, i_3 - 1, \ldots, i_t - 1, k\}$, where each can be filled with k or k' except the box in row k, which is on the diagonal. Thus $I_{diag}(\lambda) - 2I_{odiag}(\lambda) = 2(t-1) + 1$.

Now suppose $\lambda_k \geq 2$. We can add a box at some subset of rows $\{i_1 = 1, i_2, \ldots, i_t\, k + 1\}$ and fill each with either k or k' except for the box in row $k+1$, which is on the diagonal. This gives $O_{diag}(\lambda) + 2O_{odiag}(\lambda) = 2t+1$. We can delete boxes in rows $\{i_2 - 1, \ldots, i_t - 1\, k\}$, where each can be filled with k or k', giving $I_{diag}(\lambda) - 2I_{odiag}(\lambda) = 2t$. □

First, consider all up-down paths from μ to λ that begin with a loop at μ. There are

$$2I_{odiag}(\mu) + I_{diag}(\mu)$$

such paths since there are the same number of loops at μ. Up-down paths that do not start with a loop can be counted by the number of outer corners of λ that are not inner corners of q counted with multiplicity two if they are off the main diagonal since each of these corners may be added to q for the step up and then removed in addition to the boxes of μ/λ for the step down. There are

$$2(O_{odiag}(\lambda) - I_{odiag}(\mu/\lambda)) + (O_{diag}(\lambda) - I_{diag}(\mu/\lambda))$$

such corners. Thus there are

$$2I_{odiag}(\mu)+I_{diag}(\mu)+2(O_{odiag}(\lambda)-I_{odiag}(\mu/\lambda))+(O_{diag}(\lambda)-I_{diag}(\mu/\lambda))$$

up-down paths from μ to λ.

Next, consider all down-up paths from μ to λ that end with a loop at λ. There are exactly

$$2I_{odiag}(\lambda)+I_{diag}(\lambda)$$

such paths. Down-up paths that do not end with a loop can be counted by the inner corners of q that are also inner corners of λ counted twice if they are off the main diagonal since removing this type of inner corner in addition to the boxes of λ/μ will result in a shifted shape covered by λ. There are

$$2(I_{odiag}(\mu)-I_{odiag}(\mu/\lambda))+(I_{diag}(\mu)-I_{diag}(\mu/\lambda))$$

such corners. Thus there are

$$2I_{odiag}(\lambda)+I_{diag}(\lambda)+2(I_{odiag}(\mu)-I_{odiag}(\mu/\lambda))+(I_{diag}(\mu)-I_{diag}(\mu/\lambda))$$

down-up paths from μ to λ.

Hence

$$\begin{aligned}[\lambda](DU-UD)(\mu) &= 2I_{odiag}(\mu)+I_{diag}(\mu)+2(O_{odiag}(\lambda)-I_{odiag}(\mu/\lambda))\\ &\quad +(O_{diag}(\lambda)-I_{diag}(\mu/\lambda))\\ &\quad -(2I_{odiag}(\lambda)+I_{diag}(\lambda)+2(I_{odiag}(\mu)-I_{odiag}(\mu/\lambda))\\ &\quad +(I_{diag}(\mu)-I_{diag}(\mu/\lambda))))\\ &= 2O_{0diag}(\lambda)+O_{diag}(\lambda)-2I_{odiag}(\lambda)-I_{diag}(\lambda)\\ &= 1\end{aligned}$$

by Lemma 5.5.8. □

5.5.3 Shifted Hecke insertion

Definition 5.5.9. *An increasing shifted tableau is a filling of a shifted shape with non-negative integers such that entries are strictly increasing across rows and columns.*

The following are examples of increasing shifted tableaux.

$$\begin{array}{ccc} 1 & 2 & 4 \\ & 4 & \end{array} \qquad\qquad \begin{array}{ccccc} 2 & 3 & 5 & 7 & 8 \\ & 5 & 8 & & \\ & & 9 & & \end{array}$$

We now describe an insertion procedure for increasing shifted shapes that is similar to Hecke insertion, which we call *shifted Hecke insertion.* This procedure is a natural analogue of that of Sagan and Worley, [Sag87, Wor84]. As before, any insertion tableau will correspond to a walk downward ending at $\varnothing$ in the dual filtered graph of shifted shapes from the Möbius deformation and any recording tableau corresponds to a walk upward starting at $\varnothing$.

We start by describing how to insert x into an increasing shifted tableau Y to obtain increasing shifted tableau Z. We begin by inserting $x = y_1$ into the first row of Y. This insertion may modify the first row of Y and may produce an output integer y_2. Following the rules below, we then insert y_2 into the second row of Y and continue in this manner until one of two things happens. Either at some stage the insertion will not create an output integer, in which case the insertion of x into Y terminates, or y_k will replace some diagonal element of Y. In the latter case, we continue the insertion process along the columns: y_{k+1} is inserted into the column to the right of its original position and so on until some step in this process does not create an output integer.

For each insertion, we designate a specific box of the resulting tableau, Z, as being the box where the insertion terminated. We will later use this notion to define recording tableaux.

The rules for inserting any positive integer x into a row or column are as follows:

(S1) If x is weakly larger than all integers in the row (resp. column) and adjoining x to the end of the row (resp. column) results in an increasing tableau, then Z is the resulting tableau. We say that the insertion terminated at this new box.

Example 5.5.10. Inserting 4 into the first row of the tableau on the left gives the tableau on the right. This insertion terminated in position $(1, 3)$.

$$\begin{array}{cc} 1 & 2 \\ & 4 \end{array} \qquad\qquad \begin{array}{ccc} 1 & 2 & 4 \\ & 4 & \end{array}$$

Inserting 5 into the third column of the resulting tableau gives the tableau shown below. This insertion terminated in position $(2, 3)$.

1	2	4
	4	5

(S2) If x is weakly larger than all integers in the row (resp. column) and adjoining x to the end of the row (resp. column) does not result in an increasing tableau, then $Z = Y$. If x was row inserted into a nonempty row, we say that the insertion terminated at the box at the bottom of the column containing the rightmost box of this row. If x was row inserted into an empty row, we say the insertion terminated at the rightmost box of the previous row. If x was column inserted, we say the insertion terminated at the rightmost box of the row containing the bottom box of the column x could not be added to.

Example 5.5.11. Inserting 4 into the first row of the tableau on the left does not change the row and does not give an output integer, and so the insertion does not change the tableau. The insertion terminated in position $(2, 3)$

1	2	4
	3	5

Inserting 4 into the third column of the tableau below does not change the column and does not produce an output integer, and so the insertion does not change the tableau. The insertion terminated in position $(1, 4)$.

1	2	4	5
	3		

Inserting 2 into the (empty) second row of the tableau below does not change the row. The insertion terminated in position (1,2)

1	2

For the last two rules, suppose the row (resp. column) contains a box with label strictly larger than x, and let y be the smallest such box.

(S3) If replacing y with x results in an increasing tableau, then replace y with x. In this case, y is the output integer. If x was inserted into a column or if y was on the main diagonal, proceed to insert all future output integers into the next column to the right. If x was inserted into a row and y was not on the main diagonal, then insert y into the row below.

Example 5.5.12. Inserting 6 into the second row of the tableau on the left results in the tableau on the right with output integer 7 to be inserted into the third row.

1	2	3	4
	4	7	8
		8	

1	2	3	4
	4	6	8
		8	

Inserting 6 into the third column of the tableau on the left also results in the tableau on the right with output integer 7, but this time, 7 is to be inserted into the fourth column.

Inserting 3 into the second row of the tableau on the left results in the tableau shown below with output integer 4 to be inserted into the third column.

1	2	3	4
	3	7	8
		8	

(S4) If replacing y with x does not result in an increasing tableau, then do not change the row (resp. column). In this case, y is the output integer. If x was inserted into a column or if y was on the main diagonal, proceed to insert all future output integers into the next column to the right. If x was inserted into a row, then insert y into the row below.

Example 5.5.13. Inserting 2 into the first row of the tableau on the left does not change the tableau and produces output integer 5 to be inserted into the second row.

1	2	5	8
	3	6	

Inserting 5 into the third column does not change the tableau and gives output integer 6 to be inserted into the fourth column.

Inserting 2 into the second row does not change the tableau. In this case, the output integer is 3 and is to be inserted into the third column.

Using this insertion algorithm, we define the *shifted Hecke insertion tableau* of a word $w = w_1 w_2 \cdots w_n$ to be

$$P_S(w) = (\ldots((\varnothing \leftarrow w_1) \leftarrow w_2)\ldots) \leftarrow w_n.$$

Example 5.5.14. We show the sequence of shifted tableaux obtained while computing $P_S(4211232)$. We start by inserting 4 into the first row.

$$\begin{array}{|c|}\hline 4\\\hline\end{array}\quad \begin{array}{|c|c|}\hline 2&4\\\hline\end{array}\quad \begin{array}{|c|c|c|}\hline 1&2&4\\\hline\end{array}\quad \begin{array}{|c|c|c|}\hline 1&2&4\\\hline\end{array}\quad \begin{array}{|c|c|c|}\hline 1&2&4\\\hline \multicolumn{1}{c|}{}&4&\multicolumn{1}{c}{}\\\cline{2-2}\end{array}\quad \begin{array}{|c|c|c|}\hline 1&2&3\\\hline \multicolumn{1}{c|}{}&4&\multicolumn{1}{c}{}\\\cline{2-2}\end{array}\quad \begin{array}{|c|c|c|}\hline 1&2&3\\\hline \multicolumn{1}{c|}{}&3&4\\\cline{2-3}\end{array}$$

The following result was proven in [HKP$^+$15] by Hamaker, Keilthy, Patrias, Webster, Zhang, and Zhou, and was stated as a conjecture in the first version of this chapter.

Proposition 5.5.15. *[HKP$^+$15] The result of such insertion agrees with that of K-theoretic jeu de taquin rectification described in [CTY14].*

In this setting, recording tableaux will be set-valued shifted tableaux.

Definition 5.5.16. *A set-valued shifted tableau T of shifted shape λ is a filling of the boxes with finite, nonempty subsets of primed and unprimed positive integers so that*

1. *the smallest number in each box is greater than or equal to the largest number in the box directly to the left of it (if that box is present),*

2. *the smallest number in each box is greater than or equal to the largest number in the box directly to the above it (if that box is present),*

3. *any positive integer appears at most once, either primed or unprimed, but not both, and*

4. *there are no primed entries on the main diagonal.*

A set-valued shifted tableau is called *standard* if the set of labels consists of $1, 2, \ldots, n$, each appearing either primed or unprimed exactly once, for some n.

A recording tableau for a word $w = w_1 w_2 \ldots w_n$ is a standard shifted set-valued tableau and is obtained as follows. Begin with $Q_S(\varnothing) = \varnothing$. If the insertion of w_k into $P_S(w_1 \cdots w_{k-1})$ resulted in adding a new box to $P_S(w_1 \cdots w_{k-1})$, add this same box with label k if the box was added via row insertion and k' if the box was added via column insertion to $Q_S(w_1 \cdots w_{k-1})$ to obtain $Q_S(w_1 \cdots w_k)$. If the insertion of k into $P_S(w_1 \cdots w_{k-1})$ did not change the shape of $P_S(w_1 \cdots w_{k-1})$, obtain $Q_S(w_1 \cdots w_k)$ from $Q_S(w_1 \cdots w_{k-1})$ by adding the label k to the box where the insertion terminated if the last move was a row insertion into a nonempty row and k' if the last move was a column insertion. If the last move was row insertion into an empty row, label the box where the insertion terminated k'.

Example 5.5.17. The top row of tableaux shows the sequence of tableaux obtained from inserting $w = 4211232$ as in the previous example, and the bottom row shows the corresponding steps to form $Q_S(w)$.

$$\begin{array}{|c|}\hline 4\\\hline\end{array}\quad \begin{array}{|c|c|}\hline 2&4\\\hline\end{array}\quad \begin{array}{|c|c|c|}\hline 1&2&4\\\hline\end{array}\quad \begin{array}{|c|c|c|}\hline 1&2&4\\\hline\end{array}\quad \begin{array}{|c|c|c|}\hline 1&2&4\\\hline \multicolumn{1}{c|}{}&4&\multicolumn{1}{c}{}\\\cline{2-2}\end{array}\quad \begin{array}{|c|c|c|}\hline 1&2&3\\\hline \multicolumn{1}{c|}{}&4&\multicolumn{1}{c}{}\\\cline{2-2}\end{array}\quad \begin{array}{|c|c|c|}\hline 1&2&3\\\hline \multicolumn{1}{c|}{}&3&4\\\cline{2-3}\end{array} = P_S(w)$$

$$\begin{array}{|c|}\hline 1\\\hline\end{array}\quad \begin{array}{|c|c|}\hline 1&2'\\\hline\end{array}\quad \begin{array}{|c|c|c|}\hline 1&2'&3'\\\hline\end{array}\quad \begin{array}{|c|c|c|}\hline 1&2'&3'4'\\\hline\end{array}\quad \begin{array}{|c|c|c|}\hline 1&2'&3'4'\\\hline \multicolumn{1}{c|}{}&5&\multicolumn{1}{c}{}\\\cline{2-2}\end{array}\quad \begin{array}{|c|c|c|}\hline 1&2'&3'4'\\\hline \multicolumn{1}{c|}{}&56&\multicolumn{1}{c}{}\\\cline{2-2}\end{array}\quad \begin{array}{|c|c|c|}\hline 1&2'&3'4'\\\hline \multicolumn{1}{c|}{}&56&7'\\\cline{2-3}\end{array} = Q_S(w)$$

We next define a reverse insertion procedure so that given a pair $(P_S(w), Q_S(w))$, we can recover w.

First locate the box containing the largest label of $Q_S(w)$, call the label n and the position of the box (i_n, j_n), and find the corresponding box in $P_S(w)$. Say the integer in positioin (i_n, j_n) of $P_S(w)$ is y_n. We then perform reverse insertion on cell (i_n, j_n) of $P_S(w)$ by following the rules below.

(rS1) If n is the only label in cell (i_n, j_n) of $Q(w)$, remove box (i_n, j_n) from $P_S(w)$ and reverse insert y_n into the row above if n is unprimed and into the column to the left if n is primed.

(rS2) If n is not the only label in cell (i_n, j_n) of $Q_S(w)$, do not remove box (i_n, j_n) from $P_S(w)$, but still reverse insert y_n into the row above if n is unprimed and into the column to the left if n is primed.

In the row above if y_n is reverse inserted into a row or the column to the left if it is reverse inserted into a column, let x be the largest label with $x < y_n$.

(rS3) If replacing x with y_n results in an increasing shifted tableau, replace x with y_n. If y_n was reverse column inserted and x was not on the main diagonal, reverse insert x into the column to the left. Otherwise, reverse insert x into the row above.

(rS4) If replacing x with y_n does not result in an increasing shifted tableau, leave the row or column unchanged. If y_n was reverse column inserted and x was not on the main diagonal, reverse insert x into the column to the left. Otherwise, reverse insert x into the row above.

If we are in the first row of the tableau and the last step was a reverse row insertion or we are in the first column and the last step was a reverse column insertion, then x and the modified tableau, which we will call $P_{S,1}(w)$, are the final output value. Define $x_1 = x$.

Repeat this process using the pair $(P_{S,1}(w), Q_{S,n-1}(w))$, where $Q_{S,n-1}(w)$ is the result of removing the entry n or n' from $Q_S(w)$. Define x_2 to be the output value and $P_{S,2}(w)$ to be the modified tableau. Continue this process with pairs $(P_{S,2}(w), Q_{S,n-2}(w)), \ldots, (P_{S,n-1}(w), Q_{S,1}(w))$. Then we claim $w = x_1 x_2 \cdots x_n$.

Theorem 5.5.18. *There is a bijection between pairs consisting of an increasing shifted tableau and a standard set-valued shifted tableau of the same shape, (P, Q), and words, where the word w corresponds to the pair $(P_S(w), Q_S(w))$.*

Proof. We show that given a pair of tableaux of the same shape, (P, Q), where P is an increasing shifted tableau on some $[n]$ and Q is standard shifted set-valued tableau on $1 < 2' < 2 < 3' < \ldots < n' < n$, we can recover w so that $P = P_S(w)$ and $Q = Q_S(w)$. Assume first that (P, Q) has been obtained by inserting some integer h into a pair (Y, Z). We must show that the reverse insertion procedure defined above recovers h and (Y, Z) from (P, Q).

It is clear that if the insertion of h ended by adding a new box to Y and Z using (S1), then reverse insertion steps (rS1), (rS2), (rS3), and (rS4) undo insertion steps (S1), (S2), (S3), and (S4), respectively. Therefore, it suffices to show that if the insertion of h does not result in adding a new box to Y and Z using (S2), then the reverse insertion procedure can recover h from (Y, Z).

Suppose the last step of the insertion of h into Y is inserting some x into nonempty row i using (S2) and the insertion terminates in row $i + j$. We first use $(rS2)$ and reverse insert the entry at the end of row $i + j$ into row $i + j - 1$, which will not change row $i + j - 1$ because the entry being reverse inserted is strictly larger than all entries in row $i + j - 1$ but cannot replace the entry at the end of row $i + j - 1$ since this would put it directly above the occurence of the entry in row $i+j$. Next, we reverse insert the entry at the end of row $i+j-1$ into $i+j-2$, and in the same manner, it will not change row $i+j-2$. This process continues until we reverse insert the entry at the end of row $i+1$ into i and obtain output integer x. From this point onward, the reverse insertion rules will exactly undo the corresponding insertion rules to recover h. We can use the transpose of this argument if x is column inserted into column j using (S2) and the insertion terminates in column $j+i$.

For example, inserting 4 into the first row of the tableau below does not change the tableau, and the insertion terminates at $(3,4)$. Starting reverse insertion at $(3,4)$, we reverse insert 6 into the second row. It cannot replace the 5, so the third row doesn't change, and 5 is reverse inserted into the first row. The 5 cannot replace the 4, so the first row is unchanged, and we end with the original tableau and the integer 4.

1	2	3	4
	3	4	5
		5	6

If x is inserted into empty row i using (S2). Then our original tableau looked like the one shown below, where the bottom row shown is row $i-1$ and the dot indicates where the insertion terminated. In this case, x must have been bumped by a row insertion of a into row $i-1$. We show reverse insertion starting from the box where the insertion terminated recovers the step of inserting a into row $i-1$.

	a	x				$\cdot$	

Using (rS4), we reverse insert the entry at the end of row $i-1$ into the column to the left. The column will not change since the entry being reverse inserted is strictly larger than all entries in the column and cannot replace the entry at the end of the column. We then reverse insert the second rightmost entry of row $i-1$ into the column to its left. Continuing in this manner, we eventually reverse column insert x into the main diagonal, which results in reverse row inserting a into row $i-2$. From this point on, the reverse insertion rules will exactly undo the corresponding insertion rules to recover h.

For example, suppose some insertion ends with inserting 2 into the empty second row of the tableau below. The insertion terminates in position $(1,4)$, and we see that the 2 must have been bumped by a row insertion of 1 into the first row. We start reverse column insertion in position $(1,4)$ and reverse column insert 4 into the third column. The integer 4 cannot replace the 3 in the third column since it would be directly to the left of the 4 in the fourth column, so the third column is left unchanged. We reverse column insert 3 into the second column. Again, the column remains unchanged, and we reverse insert 2 into the first column. The 2 cannot replace the 1, so the column is unchanged. Since the 1 was on the main diagonal, we next reverse row insert 1 into the row above.

1	2	3	4

□

5.5.4 Shifted Hecke growth and decay

As before, given any word $w = w_1 w_2 \cdots w_k$, containing $n \leq k$ distinct numbers, we can create an $n \times k$ array with an X in the w_i^{th} square from the bottom of column i. Note that there can be multiple X's in the same row but is at most one X per column.

We will label the corners of each square with a shifted partition and label some of the horizontal edges of the squares with a specific inner corner where the insertion terminated and/or a 'c' to designate column insertion. To denote a specific inner corner, we will give either its row or column. For example, the edge label 2 denotes the inner corner at the end of the second row of the shifted diagram, and an edge labeled 2c denotes the inner corner at the end of the second column of the shifted diagram. We begin by labeling all corners along the bottom row and left side of the diagram with the empty shape, $\varnothing$, as before.

To complete the labeling of the corners, suppose the corners μ, λ, and ν are labeled, where μ, λ, and ν are as in the picture below. We label γ according to the following rules.

$$\begin{tikzcd}
\mu \arrow[r, dash] \arrow[d, dash] & \gamma \arrow[d, dash] \\
\lambda \arrow[r, dash] & \nu
\end{tikzcd}$$

If the square contains an X:

(1) If $\lambda_1 = \mu_1$, then γ/μ consists of one box in the first row.

(2) If $\lambda_1 + 1 = \mu_1$, then $\gamma = \mu$ and the edge between them is labeled with a 1 to signify the inner corner in the first row of γ.

$$\begin{tikzcd} 1 \arrow[r, dash] \arrow[d, dash] \arrow[dr, phantom, "X"] & 2 \arrow[d, dash] \\ 1 \arrow[r, dash] & 1 \end{tikzcd} \qquad \begin{tikzcd} 2 \arrow[r, dash, "1"] \arrow[d, dash] \arrow[dr, phantom, "X"] & 2 \arrow[d, dash] \\ 1 \arrow[r, dash] & 1 \end{tikzcd}$$

If the square does not contain an X and $\mu = \lambda$ or if $\nu = \lambda$ with no edge label on the bottom edge

(3) If $\mu = \lambda$, then set $\gamma = \nu$ and label the top edge label as the bottom edge if one exists. If $\nu = \lambda$, then $\gamma = \mu$.

$$\begin{tikzcd} 1 \arrow[r, dash, "\mathrm{c}"] \arrow[d, dash] & 2 \arrow[d, dash] \\ 1 \arrow[r, dash, "\mathrm{c}"] & 2 \end{tikzcd} \qquad \begin{tikzcd} 2 \arrow[r, dash] \arrow[d, dash] & 2 \arrow[d, dash] \\ 1 \arrow[r, dash] & 1 \end{tikzcd}$$

If $\nu \not\subseteq \mu$ and the square does not contain an X:

(4) In the case where $\nu \not\subseteq \mu$, $\gamma = \nu \cup \mu$, and the top edge label is the same as the bottom edge label.

$$\begin{tikzcd} 21 \arrow[r, dash] \arrow[d, dash] & 31 \arrow[d, dash] \\ 2 \arrow[r, dash] & 3 \end{tikzcd}$$

If $\nu \subseteq \mu$ and the square does not contain an X:

Suppose the square of ν/λ is one box in position (i, j). If (i, j) is not on the diagonal and the edge between λ and ν is not labeled c:

(5) If there is no box of μ/λ in row $i+1$ of μ, then γ is obtained from μ by adding a box to the end of row $i+1$.

(6) If there is a box of μ/λ in row $i+1$ of μ, then $\gamma = \mu$ and the top edge is labeled $i+1$.

$$
\begin{tikzcd}
2 \arrow[r, dash] \arrow[d, dash] & 21 \arrow[d, dash] \\
1 \arrow[r, dash] & 2
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
31 \arrow[r, dash, "2"] \arrow[d, dash] & 31 \arrow[d, dash] \\
2 \arrow[r, dash] & 3
\end{tikzcd}
$$

If (i,j) is on the main diagonal or the bottom edge is labeled c:

(7) If there is no box of μ/λ in column $j+1$ of μ, then γ is obtained from μ by adding a box to the end of column $j+1$. The top edge is labeled c.

(8) If there is a box of μ/λ in column $j+1$ of μ, then $\gamma = \mu$. The top edge is labeled with c and $j+1$.

$$
\begin{tikzcd}
2 \arrow[r, dash, "\mathrm{c}"] \arrow[d, dash] & 3 \arrow[d, dash] \\
1 \arrow[r, dash, "\mathrm{c}"] & 2
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
31 \arrow[r, dash, "3\mathrm{c}"] \arrow[d, dash] & 31 \arrow[d, dash] \\
2 \arrow[r, dash, "\mathrm{c}"] & 21
\end{tikzcd}
$$

If $\lambda = \nu$ and the bottom edge is labeled with i but not with c:

(9) If there is no box of μ/λ immediately to the right of or immediately below the box at the end of row i of ν, then $\gamma = \mu$ and the top edge is labeled by i.

(10) If there is a box of μ/λ directly below the box at the end of row i of ν, then $\gamma = \mu$ and the edge between them is labeled by $i+1$.

(11) If there is a box of μ/λ immediately to the right of the box at the end of row i of ν, say in position $(i, j+1)$, no box of μ/λ in row $i+1$, and the outer corner of row $i+1$ is not in column j, then γ/μ is one box in row $i+1$.

(12) If there is a box of μ/λ immediately to the right of the box at the end of row i of ν, say in position $(i, j+1)$, no box of μ/λ in row $i+1$, and the outer corner of row $i+1$ is in column $j+1$, then the outer corner of row $i+1$ is on the main diagonal. In this case, $\gamma = \mu$, and the top edge is labeled with $j+1$ and c.

$$\begin{tikzcd} 31 \arrow[r, dash, "1"] \arrow[d, dash] & 31 \arrow[d, dash] \\ 3 \arrow[r, dash, "1"] & 3 \end{tikzcd} \quad \begin{tikzcd} 21 \arrow[r, dash, "2"] \arrow[d, dash] & 21 \arrow[d, dash] \\ 2 \arrow[r, dash, "1"] & 2 \end{tikzcd} \quad \begin{tikzcd} 3 \arrow[r, dash] \arrow[d, dash] & 31 \arrow[d, dash] \\ 2 \arrow[r, dash, "1"] & 2 \end{tikzcd} \quad \begin{tikzcd} 2 \arrow[r, dash, "2c"] \arrow[d, dash] & 2 \arrow[d, dash] \\ 1 \arrow[r, dash, "1"] & 1 \end{tikzcd}$$

If the $\lambda = \nu$ and the bottom edge is labeled with j and with c:

(13) If there is no box of μ/λ immediately to the right of or immediately below the box at the end of column j of ν, then $\gamma = \mu$ and the edge between them is labeled by j and c.

(14) If there is a box of μ/λ immediately to the right of the box at the end of column j of ν, then $\gamma = \mu$ and the top edge is labeled by $j+1$ and c.

(15) If there is a box of μ/λ directly below the box at the end of column j of ν but no box of μ/λ in column $j+1$, then γ/μ is one box in column $j+1$. The top edge is labeled c.

$$\begin{tikzcd} 42 \arrow[r, dash, "4c"] \arrow[d, dash] & 42 \arrow[d, dash] \\ 41 \arrow[r, dash, "4c"] & 41 \end{tikzcd} \quad \begin{tikzcd} 41 \arrow[r, dash, "4c"] \arrow[d, dash] & 41 \arrow[d, dash] \\ 31 \arrow[r, dash, "3c"] & 31 \end{tikzcd} \quad \begin{tikzcd} 32 \arrow[r, dash] \arrow[d, dash] & 42 \arrow[d, dash] \\ 31 \arrow[r, dash, "3c"] & 31 \end{tikzcd}$$

We call the resulting array the *shifted Hecke growth diagram* of w. In our previous example with $w = 4211232$, we would have the diagram in Figure 5.11.

Let $\mu_0 = \varnothing \subseteq \mu_1 \subseteq \ldots \mu_k$ be the seqence of shifted partitions across the top of the growth diagram, and let $\nu_0 = \varnothing \subseteq \nu_1 \subseteq \ldots \nu_n$ be the sequence of shifted partitions on the right side of the growth diagram. These sequences correspond to increasing shifted tableaux $Q(w)$ and $P(w)$, respectively. If the edge between μ_i and μ_{i+1} is labeled j,

$$
\begin{array}{ccccccccccccccc}
\varnothing & \text{—} & 1 & \overset{c}{\text{—}} & 2 & \overset{c}{\text{—}} & 3 & \overset{3c}{\text{—}} & 3 & \text{—} & 31 & \overset{2}{\text{—}} & 31 & \overset{c}{\text{—}} & 32 \\
| & X & | & & | & & | & & | & & | & & | & & | \\
\varnothing & \text{—} & \varnothing & \text{—} & 1 & \overset{c}{\text{—}} & 2 & \overset{2c}{\text{—}} & 2 & \overset{1}{\text{—}} & 2 & \text{—} & 3 & \text{—} & 31 \\
| & & | & & | & & | & & | & & | & X & | & & | \\
\varnothing & \text{—} & \varnothing & \text{—} & 1 & \overset{c}{\text{—}} & 2 & \overset{2c}{\text{—}} & 2 & \overset{1}{\text{—}} & 2 & \text{—} & 2 & \overset{1}{\text{—}} & 2 \\
| & & | & X & | & & | & & | & X & | & & | & X & | \\
\varnothing & \text{—} & \varnothing & \text{—} & \varnothing & \text{—} & 1 & \overset{1}{\text{—}} & 1 & \text{—} & 1 & \text{—} & 1 & \text{—} & 1 \\
| & & | & & | & X & | & X & | & & | & & | & & | \\
\varnothing & \text{—} & \varnothing & \text{—} & \varnothing & \text{—} & \varnothing & \text{—} & \varnothing & \text{—} & \varnothing & \text{—} & \varnothing & \text{—} & \varnothing
\end{array}
$$

Figure 5.11: Shifted Hecke growth diagram

then $\mu_i = \mu_{i+1}$, and the label $i+1$ of $Q(w_1 \cdots w_{i+1})$ is placed in the box at the end of row j of $Q(w_1 \cdots w_i)$. If the edge between μ_i and μ_{i+1} is labeled jc, then $\mu_i = \mu_{i+1}$, and the label $i+1'$ of $Q(w_1 \cdots w_{i+1})$ is placed in the box at the end of column j of $Q(w_1 \cdots w_i)$. For the shifted Hecke growth diagram above, we have:

$$
P(w) = \begin{array}{|c|c|c|}
\hline
1 & 2 & 3 \\
\hline
\multicolumn{1}{c|}{} & 3 & 4 \\
\cline{2-3}
\end{array}
\qquad\qquad
Q(w) = \begin{array}{|c|c|c|}
\hline
1 & 2' & 3'4' \\
\hline
\multicolumn{1}{c|}{} & 56 & 7' \\
\cline{2-3}
\end{array}\ .
$$

We see that this P and Q agree with $P_S(w)$ and $Q_S(w)$, which is not a coincidence.

Theorem 5.5.19. *For any word w, the increasing shifted tableaux P and Q obtained from the sequence of shifted partitions across the right side of the shifted Hecke growth diagram for w and across the top of the shifted Hecke growth diagram for w, respectively, are $P_S(w)$ and $Q_S(w)$, the shifted Hecke insertion and recording tableau for w.*

Proof. Suppose the square we are examining is in row s and column t. We argue by induction. We will use the same two lemmas from the Hecke growth diagram proof, which have analogous proofs.

Lemma 5.5.20. *Oriented as in the square above, $|\nu/\lambda| \leq 1$.*

Lemma 5.5.21. *Oriented as in the square above, μ/λ is a rook strip, that is, no two boxes in μ/λ are in the same row or column.*

(1-2) Note that if the square contains an X, then $\lambda = \nu$ because there are no other X's in column t. If $\lambda_1 = \mu_1$, then there are no s's in the first row of insertion tableau

μ. Thus the inserted s will be added to the end of the first row, creating γ. If $\lambda_1 = \mu_1$, then there is already an s at the end ofthe first row of insertion tableau μ. The inserted s cannot be added to the first row, so the $\mu = \gamma$. In this case, the insertion terminated at the end of the first row. We know this is an inner corner because s is the largest number inserted so far, so there can not be anything directly below the box labeled s in the first row.

(3) If $\mu = \lambda$, there is no X to the left of square (s,t) in row s. Thus nothing changes between ν and γ as we consider occurences of entry s. If $\lambda = \nu$, there is no X below square (s,t) in column t and so no new insertion.Thus nothing changes between μ and γ.

(4) Suppose ν/λ is one box in row n and μ/λ contains boxes in rows exactly $\{j_1, j_2, \ldots, j_k\}$. Suppose the box in ν/λ corresponds to inserting $r < s$. Then $\nu \nsubseteq \mu$ implies that $n \notin \{j_1, j_2, \ldots, j_k\}$. Since ν/λ is one box, the last action in the insertion sequence of r into the insertion tableau of shape λ is a box being inserted in row n. Since there is no s in row n, the bumping sequence when inserting r into the insertion tableau of shape μ will not disturb the s's. The edge between λ and ν is either labeled c or not, corresponding to whether or not inserting the number in column t involves column insertions. This remains unchanged.

(5-6) Since ν/λ is one box in position (i,j), there is some X in position (r,t) for $r < s$. It follows that inserting r into the insertion tableau μ will result in bumping the s in position (i,j) of μ. If there is no box in row $i+1$ of μ/λ, there is not an s in row $i+1$ of insertion tableau μ. Thus the s in position (i,j) is bumped from row i by r and added to the end of row $i+1$. If there is a box in row $i+1$ of μ/λ, there is already an s in row $i+1$ of insertion tableau μ. It follows that the s bumped from row i cannot be added to row $i+1$, and so $\gamma = \mu$. The insertion terminates at the end of row $i+1$ since there cannot be any boxes directly below the s in row $i+1$.

(7-8) If (i,j) is on the main diagonal or the edge between λ and ν is labeled c, any bumped entries will be column inserted into the column j. The rules are the transpose of rules 5 and 6, and the new edge is labeled c since all subsequent bumping will be column inserted after the first column insertion.

(9) Since there is no box directly to the right of (i,j), nothing is bumped when inserting $r < s$ into insertion tableau μ. Since there is no box directly below (i,j), the

insertion terminates at inner corner (i, j).

(10) In the situation described above, if there is a box directly below (i, j), the insertion now terminates at the inner corner in row $i + 1$.

(11) When inserting r into μ, the s in position $(i, j+1)$ is bumped but not replaced and inserted into row $i+1$. Since there is no s in row $i+1$, this s can be added to the row as long as it is not directly below the s in position (i, j).

(12) When inserting r into μ, the s in position $(i, j+1)$ is bumped but not replaced and inserted into row $i+1$. Since there is no s in row $i+1$, we attempt to add this s to the end of row $i+1$. However, this s would end directly below the s in row i, and so cannot be added. If the outer corner of row $i+1$ is not on the main diagonal, the insertion terminates at the end of row $i+1$. If it is on the diagonal, the insertion terminates at the end of row i, and we know the last box in row i must be in column j since there are no entries larger than s. This edge is labeled with c since this action may bump an entry from the main diagonal in column j in future steps.

(13-15) Rules 13-15 are the tranpose of rules 10-12, and may be explained in an analogous way. $\square$

We can also formulate the rules for the reverse insertion, as follows. The proof is omitted for brevity.

Reverse Rules

(R1) If γ/μ is one box in the first row, then the square has a X and $\lambda = \nu$.

(R2) If $\gamma = \mu$ and the edge between them is labeled 1, then the square has an X and $\lambda = \nu$.

(R3) If $\gamma = \mu$ with no edge label, then $\lambda = \mu$. If $\gamma = \nu$, then $\lambda = \mu$ and the edge label between λ and ν is the same as the label between γ and μ.

(R4) If $\nu \not\subseteq \mu$, then $\lambda = \mu \cap \nu$.

(R5) If γ/μ is one box in row $i+1$ for some $i > 0$, the edge between γ and μ has no label, and μ/ν has no box in row i, then ν/λ is one box in row i.

(R11) If γ/μ is one box in row $i+1$ for some $i > 0$, the edge between γ and μ has no label, and μ/ν has a box in row i, then $\lambda = \nu$ and the edge between them is labeled i.

(R7) If γ/μ is one box in column $j+1$ for some $j>0$, the edge between γ and μ is labeled c, and μ/ν has no box in column j, then ν/λ is one box in column j and the edge between ν and λ is labeled c.

(R15) If γ/μ is one box in column $j+1$ for some $j>0$, the edge between γ and μ is labeled c, and μ/ν has a box in column j, then $\nu=\lambda$ and the edge between them is labeled j and c.

(R6) If $\gamma=\mu$, the edge between them is labeled $i+1$ for some $i>0$, μ/ν has a box in row $i+1$, and $\mu_i \neq \mu_{i+1}+1$, then ν/λ is one box in row i.

(R9) If $\gamma=\mu$, the edge between them is labeled $i+1$ for some $i>0$, μ/ν has no box in row $i+1$, then $\lambda=\nu$ and the edge between them is labeled $i+1$.

(R10) If $\gamma=\mu$, the edge between them is labeled $i+1$ for some $i>0$, μ/ν has a box in row $i+1$, and $\mu_i=\mu_{i+1}+1$, then $\lambda=\nu$ and the edge between them is labeled i.

(R8) If $\gamma=\mu$, the edge between them is labeled $j+1$ and c for some $j>0$, μ/ν has a box in column $j+1$, and the j^{th} and $j+1^{th}$ columns of μ are not the same size, then ν/λ is one box in column j.

(R12) If $\gamma=\mu$, the edge between them is labeled $j+1$ and c for some $j>0$, and the bottom box of column $j+1$ is in the last row of μ, then $\lambda=\nu$ and the edge between them is labeled with the bottom row of ν.

(R13) If $\gamma=\mu$, the edge between them is labeled $j+1$ and c for some $j>0$, μ/μ has no box in column $j+1$, then $\lambda=\nu$ and the edge between them is labeled $j+1$ and c.

(R14) If $\gamma=\mu$, the edge between them is labeled $j+1$ and c for some $j>0$, μ/μ has a box in column $j+1$, and and the j^{th} and $j+1^{th}$ columns of μ are the same size, then $\lambda=\nu$ and the edge between them is labeled j and c.

5.6 Major examples: Young-Fibonacci lattice

5.6.1 Möbius deformation of the Young-Fibonacci lattice

In order to describe the Möbius deformation of the Young-Fibonacci lattice, we first need to determine its Möbius function.

Theorem 5.6.1. *Let $X > Y$ be elements in the Young-Fibonacci lattice and suppose X has n edges below it. Then*

$$\mu(Y,X) = \begin{cases} n-1 & \text{if } 2Y = X \\ -1 & \text{if } X \text{ covers } Y \\ 0 & \text{otherwise} \end{cases}$$

Proof. Suppose $X > Y$. If $\rho(X) = \rho(Y) + 1$, then clearly, $\mu(Y,X) = -1$.

If $\rho(X) = \rho(Y) + 2$ and $X = 2Y$, then $\mu(Y,X) = n-1$ since $2Y$ covers exactly the elements that cover Y. If $X \neq 2Y$, then it can be checked that X covers exactly one element that covers Y. Therefore $\mu(Y,X) = 0$.

Assume now $\rho(X) \geq \rho(Y)+2$. We will argue that $\mu(Y,X) = 0$ by induction on $\rho(X)$. The base case $\rho(X) = \rho(Y) + 2$ was just proved. Let us argue the step of induction. There are two cases.

- We have $2Y < X$. In this case, consider $\sum_{Y \leq W < X} \mu(Y,W)$, we argue it is 0. Indeed, $\sum_{Y \leq W \leq 2Y} \mu(Y,W) = 0$. For the $2Y \not\leq W < X$ in the interval, we use induction assumption to conclude $\mu(Y,W) = 0$.

- We have $2Y \not< X$. We claim there is only one Z which covers Y such that $Z < X$. Indeed, since Young-Fibonacci lattice is a lattice, and since join of any two distinct such Z's is $2Y$, we would have a contradiction. Then $\mu(Y,Y) + \mu(Y,Z) = 1 + (-1) = 0$. For the rest of $Y < W < X$ we have $\mu(Y,W) = 0$ by induction assumption.

□

By the previous result, for each element X in the Young-Fibonacci lattice, the Möbius deformation is formed by adding an upward-oriented loop for each of the n

elements X covers and adding $n-1$ downward-oriented edges from X to Y if $X = 2Y$. For example, there are two edges from 221 to 21. The first few ranks are shown in Figure 5.12.

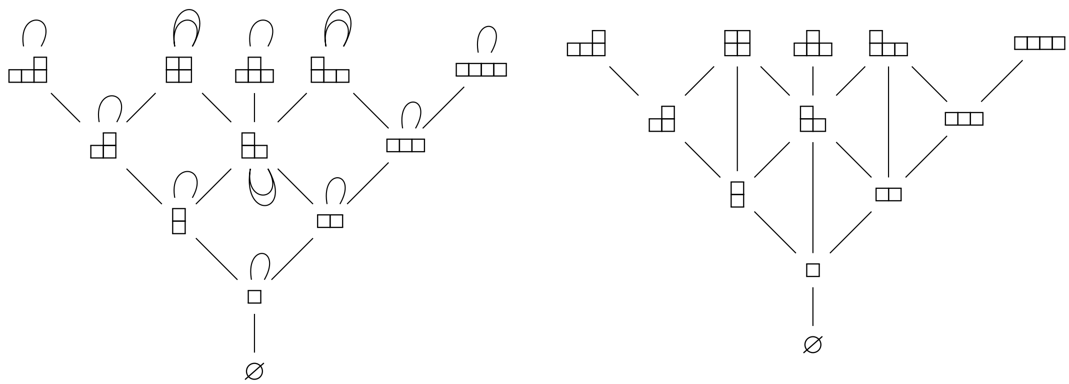


Figure 5.12: Möbius deformation of the Young-Fibonacci lattice

Theorem 5.6.2. *The Möbius deformation of the Young-Fibonacci lattice forms a dual graded graph with*

$$(DU - UD) = D + I.$$

Proof. Suppose that shape X covers shape Y, and first assume that $\rho(Y) = \rho(X) - 1$. The up-down paths from X to Y that consist of a loop at X followed by following the edge from X to Y can be counted by the number of loops at X, i.e. the number of edges below X in the Young-Fibonacci lattice. In the second type of up-down path, we start at X, move up to a distinct shape W, and then move down to Y. The only W with $\rho(W) = \rho(Y) + 2$ that cover Y is the W defined by $W = 2Y$. This W covers X by definition of the Young-Fibonacci lattice. To count the number of paths through W, we need to count the number of edges from W to Y. This number is the same as one fewer than the number edges below W in $\mathbb{YF}$, which is the same as one fewer than the number of edges above Y in $\mathbb{YF}$.

To count down-up paths, we first count the paths that consist of the edge from X to Y followed by a loop at Y. These can be counted by the number of loops at Y or, in other words, the one fewer than the number of edges above Y in $\mathbb{YF}$. The second type of down-up paths involve traveling from X down to some Z with $\rho(Z) + 2 = \rho(X)$ and then going up to Y. There is a unique such Z defined by $2Z = X$, and Y covers this Z. We thus need to count the number of ways to get from X to Z, which is the same as one fewer than the number of edges below X in $\mathbb{YF}$.

Putting the previous two arguments together, $[Y](DU - UD)(X) = 1$ in this case, as desired.

Now suppose $\rho(Y) = \rho(X) - 2$. The only up-down paths from X to Y consist of a loop at X followed by an edge from X to Y. There are

$$(\#\{\text{edges below } X \text{ in } \mathbb{YF}\})(\#\{\text{edges below } X \text{ in } \mathbb{YF}\} - 1)$$

such paths.

To count down-up paths, we must count the number of ways to choose an edge from X to Y and then choose a loop of Y. There are

$$(\#\{\text{edges below } X \text{ in } \mathbb{YF}\} - 1)(\#\{\text{edges above } Y \text{ in } \mathbb{YF}\} - 1)$$

such paths. Since

$$\#\{\text{edges down from } X \text{ in } \mathbb{YF}\} = \#\{\text{edges above } Y \text{ in } \mathbb{YF}\},$$

we have that

$$[Y](DU - UD)(X) = \#\{\text{edges below } X \text{ in } \mathbb{YF}\} - 1 = [Y]D(X).$$

□

5.6.2 K-Young-Fibonacci insertion

As with the previous examples of Möbius construction, we describe an insertion procedure that corresponds to the Möbius deformation of the Young-Fibonacci lattice, which is an analogue of the Young-Fibonacci insertion of Fomin. We recommend [Fom94] for details about Young-Fibonacci insertion. In this analogue, any walk upward from $\varnothing$ will correspond to a recording tableau and any walk downward to $\varnothing$ will correspond to an insertion tableau. We begin by describing the insertion tableaux in this setting.

Definition 5.6.3. *A K-Young-Fibonacci (KYF) tableau is a filling of a snakeshape with positive integers such that*

(i) *for any pair* $\begin{array}{|c|}\hline B\\ \hline A\\ \hline\end{array}$ *the inequality* $A \leq B$ *holds;*

(ii) *to the right of any* $\begin{array}{|c|}\hline B\\ \hline A\\ \hline\end{array}$*, there are no numbers from the interval* $[A, B]$ *in either the upper or lower rows;*

(iii) *if the position above* $\boxed{A}$ *is not occupied yet, then to the right of* $\boxed{A}$ *there are no numbers greater than or equal to* A *in either the lower or upper rows;*

(iv) *any pair* $\begin{array}{|c|}\hline A\\ \hline A\\ \hline\end{array}$ *must come before any single box* $\boxed{B}$ *and must not be in rightmost column of the snakeshape. In addition,* A *may not be the smallest entry in the snakeshape.*

Example 5.6.4. The three snakeshapes below are valid KYF tableaux.

$$\begin{array}{|c|c|c|c|c|c|} \cline{2-3}\cline{5-5} \multicolumn{1}{c|}{} & 3 & 4 & \multicolumn{1}{c|}{} & 7 & \multicolumn{1}{c}{} \\ \hline 9 & 2 & 1 & 8 & 6 & 5 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|c|c|} \cline{1-2}\cline{5-5} 2 & 3 & \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 5 \\ \hline 2 & 1 & 7 & 6 & 4 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|c|c|} \cline{1-2}\cline{5-5} 4 & 5 & \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 2 \\ \hline 4 & 5 & 6 & 3 & 1 \\ \hline \end{array}$$

The snakeshapes below are not valid KYF tableaux. The first violates condition (iii), the second violates condition (ii), and the third. fourth, and fifth violate condition (iv).

$$\begin{array}{|c|c|c|} \cline{2-3} \multicolumn{1}{c|}{} & 3 & 4 \\ \hline 4 & 2 & 1 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|} \cline{1-2} 3 & 4 & \multicolumn{1}{c}{} \\ \hline 2 & 1 & 4 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|c|c|} \cline{1-2}\cline{5-5} 4 & 6 & \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 2 \\ \hline 4 & 5 & 7 & 3 & 2 \\ \hline \end{array} \qquad \begin{array}{|c|c|} \hline 3 & 2 \\ \hline 1 & 2 \\ \hline \end{array} \qquad \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 1 & 2 \\ \hline \end{array}$$

Let τ be a valid KYF tableau. We show how to insert positive integer x into τ to obtain a new valid KYF tableau that may or may not be different than τ. Notice that this insertion procedure is different than Hecke insertion and shifted Hecke insertion in that the algorithm does not proceed row-by-row or column-by-column. As before, we designate a specific box of the resulting tableau, τ', as being the box where the insertion terminated. We will later use this notion to define recording tableaux.

The rules for inserting any positive integer x into τ are as follows:

(YF1) If x is equal to the smallest entry in τ, then $\tau = \tau'$. In this case, we say the insertion terminated in the top cell of the first column of $\tau = \tau'$. If this is not the case, continue to step 2.

Example 5.6.5. Inserting 1 into the tableau on the left or 2 into the tableau on the right will not change the tableaux.

$$\begin{array}{|c|c|c|} \cline{2-2} \multicolumn{1}{c|}{} & 2 & \multicolumn{1}{c}{} \\ \hline 4 & 1 & 3 \\ \hline \end{array} \qquad\qquad \begin{array}{|c|c|c|} \cline{1-1} 3 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \hline 2 & 5 & 4 \\ \hline \end{array}$$

(YF2) Attach a new box $\boxed{x}$ just to the left of τ in the lower row.

(YF3) Find all the entries of τ that are greater than or equal to x and sort them:

$$x \leq a_1 \leq a_2 \leq \ldots \leq a_k.$$

If in τ we have $\begin{array}{|c|}\hline A \\ \hline A \\ \hline \end{array}$, then we consider the A in the upper row to be the larger of the two.

(YF4) If $a_i = a_{i+1}$, then replace a_{i+1} with $*$.

(YF5) Now put a box $\boxed{\bullet}$ just above $\boxed{x}$ and move the a_i chainwise according to the following rule: a_1 moves into the new box, a_2 moves to a_1's original position, a_3 moves to a_2's original position, etc. The box that was occupied by a_k disappears. If it was located in the lower row, then the left and right parts of the snake are concatenated.

(YF6) If there is a box with $*$ and no box directly above it, delete this box and concatenate the left and right parts of the snake. If this happens, we say the insertion terminated at the box in the upper row of the column directly to the right of this concatenation. Otherwise, a box was added in the insertion process, and we say the insertion terminated at this new box.

(YF7) Replace any pair $\begin{array}{|c|}\hline A \\ \hline * \\ \hline \end{array}$ with $\begin{array}{|c|}\hline A \\ \hline A \\ \hline \end{array}$.

Example 5.6.6. Let's insert 3 into $\tau = \begin{array}{|c|c|c|} \cline{1-2} 3 & 4 & \multicolumn{1}{c}{} \\ \hline 2 & 4 & 1 \\ \hline \end{array}$. We first attach $\boxed{3}$ as shown below.

$$\begin{array}{|c|c|c|c|} \cline{2-3} \multicolumn{1}{c|}{} & 3 & 4 & \multicolumn{1}{c}{} \\ \hline 3 & 2 & 4 & 1 \\ \hline \end{array}$$

Next, we locate and order

$$3 \leq (a_1 = 3) \leq (a_2 = 4) \leq (a_3 = 4),$$

and we replace a_3 with $*$. After shifting the boxes as in YF5, we have

$$\begin{array}{|c|c|c|c|}\cline{1-2} 3&4&\multicolumn{2}{c}{}\\\hline 3&2&*&1\\\hline\end{array}\ .$$

We then delete the box with $*$ to obtain the final KYF tableau below.

$$\begin{array}{|c|c|c|}\cline{1-2} 3&4&\multicolumn{1}{c}{}\\\hline 3&2&1\\\hline\end{array}\ .$$

This insertion terminates at the box at the top of the third column.

Notice also that inserting 1 into τ does not change the tableau, and this insertion terminates at the box at the top of the first column.

Example 5.6.7. We insert 2 into KYF tableau $\begin{array}{|c|c|c|c|}\cline{3-3} \multicolumn{1}{c}{}&\multicolumn{1}{c|}{}&2&\multicolumn{1}{c}{}\\\hline 4&3&2&1\\\hline\end{array}$. We first attach a box containing 2 to the left of the original tableau. We then locate and order

$$2 \leq (a_1 = 2) \leq (a_2 = 2) \leq (a_3 = 3) \leq (a_4 = 4).$$

We replace a_2 with $*$ and shift the boxes to obtain

$$\begin{array}{|c|c|c|c|}\cline{1-1}\cline{3-3} 2&\multicolumn{1}{c|}{}&3&\multicolumn{1}{c}{}\\\hline 2&4&*&1\\\hline\end{array}\ .$$

After performing the last step of the insertion procedure, we end with the tableau shown below.

$$\begin{array}{|c|c|c|c|}\cline{1-1}\cline{3-3} 2&\multicolumn{1}{c|}{}&3&\multicolumn{1}{c}{}\\\hline 2&4&3&1\\\hline\end{array}$$

This insertion terminated at the box at the top of the first column of the resulting tableau.

As usual, for a word $w = w_1 w_2 \dots w_n$, we define $P_{YF}(w)$ by setting

$$P_{YF}(w_1 \dots w_k) = P_{YF}(w_1 \dots w_{k-1}) \leftarrow w_k.$$

Example 5.6.8. The sequence of KYF tableaux below shows the intermediate tableaux obtained in computing $P_{YF}(1334241)$, which is shown on the right.

$$\begin{array}{|c|}\hline 1\\\hline\end{array}\qquad
\begin{array}{|c|c|}\hline 3&1\\\hline\end{array}\qquad
\begin{array}{|c|c|}\cline{1-1} 3&\multicolumn{1}{c}{}\\\hline 3&1\\\hline\end{array}\qquad
\begin{array}{|c|c|c|}\cline{2-2} \multicolumn{1}{c|}{}&3&\multicolumn{1}{c}{}\\\hline 4&3&1\\\hline\end{array}\qquad
\begin{array}{|c|c|c|}\cline{1-2} 3&4&\multicolumn{1}{c}{}\\\hline 2&4&1\\\hline\end{array}\qquad
\begin{array}{|c|c|c|}\cline{1-2} 4&3&\multicolumn{1}{c}{}\\\hline 4&2&1\\\hline\end{array}\qquad
\begin{array}{|c|c|c|}\cline{1-2} 4&3&\multicolumn{1}{c}{}\\\hline 4&2&1\\\hline\end{array}$$

We need the next definition to define the recording tableaux for KYF insertion.

Definition 5.6.9. *A standard set-valued KYF tableau is a snakeshape whose boxes are filled with finite, nonempty subsets of positive integers that satisfy the following conditions, where $\bar{A}, \bar{B}, \bar{C}$ are subsets, $\bar{A} < \bar{B}$ when max$(\bar{A})$ <min$(\bar{B})$, and the letters $[n]$ are used exactly once for some n:*

(i) for any pair $\begin{array}{|c|}\hline \bar{B} \\ \hline \bar{A} \\ \hline \end{array}$ the inequality $\bar{A} < \bar{B}$ holds;

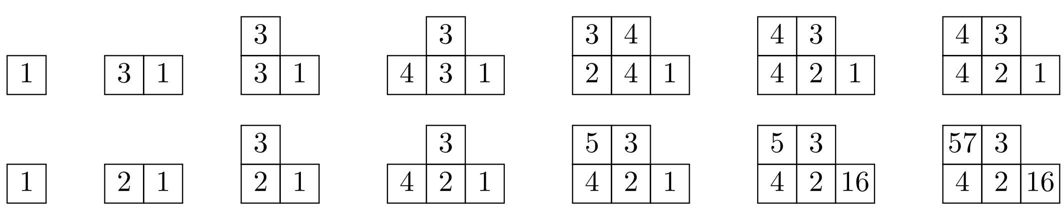
(ii) to the right of any $\begin{array}{|c|}\hline \bar{B} \\ \hline \bar{A} \\ \hline \end{array}$, there are no numbers from the interval $[max(\bar{A}), min(\bar{B})]$ in either the upper or lower rows;

(iii) if the position above $\boxed{\bar{A}}$ is not occupied yet, then to the right of $\boxed{\bar{A}}$ there are no entries greater than min$(\bar{A})$ in either the lower or upper rows;

A recording tableau for a word $w = w_1 w_2 \ldots w_n$ is a standard set-valued KYF tableau and is obtained as follows. Begin with $Q_{YF}(\varnothing) = \varnothing$. If the insertion of w_k into $P_{YF}(w_1 \cdots w_{k-1})$ resulted in adding a new box to $P_{YF}(w_1 \cdots w_{k-1})$, add this same box with label k to $Q_{YF}(w_1 \ldots w_{k-1})$ to obtain $Q_{YF}(w_1 \ldots w_k)$.

If the insertion of w_k into $P_{YF}(w_1 \cdots w_{k-1})$ did not change the shape of $P_{YF}(w_1 \cdots w_{k-1})$, obtain $Q_{YF}(w_1 \ldots w_k)$ from $Q_{YF}(w_1 \ldots w_{k-1})$ by adding the label k to the box where the insertion terminated.

Example 5.6.10. In Example 5.6.8, we computed $P_{YF}(1334241)$. We repeat this computation on the top row and show the corresponding steps of building $Q_{YF}(1334241)$ on the bottom row.



We next define a reverse insertion procedure so that given a pair $(P_{YF}(w), Q_{YF}(w))$, we can recover $w = w_1 \ldots w_n$.

First locate the box containing the largest label of $Q_{YF}(w)$, call the label n and its column c, and find the corresponding box in $P_{YF}(w)$. Let x denote the label in the leftmost box of the bottom row of $P_{YF}(w)$.

(rYF1) If the label n of Q was not the only label in its box and was located in the upper row of the first column, then $P_{YF}(w) = P_{YF}(w_1 \dots w_{n-1}) \leftarrow s$, where s is the smallest entry in $P_{YF}(w_1 \dots w_{n-1})$. Finally, $Q_{YF}(w_1 \dots w_{n-1})$ is obtained from $Q_{YF}(w)$ by removing the label n.

In all remaining cases, $P_{YF}(w) = P_{YF}(w_1 \dots w_{n-1}) \leftarrow x$, and we describe how to constuct $P_{YF}(w_1 \dots w_{n-1})$. In each case, $Q_{YF}(w_1 \dots w_{n-1})$ is obtained from $Q_{YF}(w)$ by removing the label n. If n is the only label in its box, the box is removed.

(rYF2) If the label n of Q was the only label in its box:

(a) Delete the leftmost square in the bottom row.

(b) Let k denote the largest entry in $P_{YF}(w)$, and sort the entries of $P_{YF}(w)$ as shown below:

$$x \leq b_1 \leq b_2 \leq \dots \leq b_t = k.$$

If we have $\begin{array}{|c|}\hline A \\ \hline A \\ \hline\end{array}$, then we consider the A in the upper row to be the larger of the two.

(c) If $b_i = b_{i+1}$, replace b_i with $*$.

(d) Move the b_i chainwise according the following rule: b_1 moves into b_2's position, b_2 moves into b_3's position, etc until b_{t-1} moves into b_t's position. The box that was occupied by b_1 disappears.

(e) Place b_t in the position determined by the shape of $Q_{S,n-1}$.

(f) Replace any pair $\begin{array}{|c|}\hline * \\ \hline A \\ \hline\end{array}$ with $\begin{array}{|c|}\hline A \\ \hline A \\ \hline\end{array}$.

(rYF3) If the label n of Q was not the only label in its box and the largest entry in $P_{YF}(w)$ is k:

(a) Add the column $\begin{array}{|c|}\hline \bullet \\ \hline k \\ \hline\end{array}$ directly to the left of column c to $P_{YF}(w)$.

(b) Sort the entries of $P_{YF}(w)$ as shown below:

$$k = b_1 \geq b_2 \geq b_2 \dots \geq x = b_t.$$

If we have $\begin{array}{|c|}\hline A \\ \hline A \\ \hline\end{array}$, then we consider the A in the upper row to be the larger of the two.

(c) If $b_i = b_{i+1}$, replace b_{i+1} with $*$.

(d) Move the b_i chainwise according to the following rule: b_1 moves into the new box $\boxed{\bullet}$, b_2 moves to b_1's original position, b_3 moves to b_2's original position, etc. The box that was occupied by $b_t = x$ disappears.

(e) Replace any pair $\begin{array}{|c|}\hline * \\ \hline A \\ \hline\end{array}$ with $\begin{array}{|c|}\hline A \\ \hline A \\ \hline\end{array}$.

The steps above clearly reverse the KYF insertion steps, giving us the result below. The proof is omitted for brevity.

Theorem 5.6.11. *KYF insertion and reverse KYF insertion define mutually inverse bijections between the set of words on $\mathbb{N}$ and the set of pairs (P_{YF}, Q_{YF}) consisting of a KYF tableau and a set-valued KYF tableau of the same shape.*

5.6.3 KYF growth and decay

As before, given any word $w = w_1 w_2 \cdots w_k$, containing $n \leq k$ distinct numbers, we can create an $n \times k$ array with an X in the w_i^{th} square from the bottom of column i. Note that there can be multiple X's in the same row but is at most one X per column.

We will label the corners of each square with a snakeshape, label some of the horizontal edges of the squares with a specific inner corner by writing the column of the inner corner, and label some vertical edges corresponding to where a 2 was added to the bottom shape to obtain the top shape. For example, if the corner at the bottom of the veritcal edge is labeled with 221, the corner at the top is labeled 2221, and the vertical edge has label 3, this means that the third column of 2's in 2221 is the column that was added when going from 221 to 2221. We begin by labeling all corners in the bottom row and left side of the diagram with the empty shape, $\varnothing$.

To complete the labeling of the corners, suppose the corners μ, λ, and ν are labeled, where μ, λ, and ν are as in the picture below. We label γ according to the following rules.

$$\begin{tikzcd}
\mu \arrow[r, dash] \arrow[d, dash] & \gamma \arrow[d, dash] \\
\lambda \arrow[r, dash] & \nu
\end{tikzcd}$$

If the square contains an X:

(1) If $\lambda = \nu = \varnothing$, then $\gamma = \mu = 1$ and the top edge is labeled 1.

(2) If $\lambda = \mu = \nu$, then $\gamma = 1\lambda$.

(3) If $\mu = 2\lambda$ and the left edge is labeled i, then $\gamma = 2\lambda$, the top edge is labeled $i + 1$, and the right edge is labeled 1.

(4) If $\mu \neq \lambda$, $\mu \neq 2\lambda$, and $\mu \neq 1$, then $\gamma = 2\lambda$ and the right edge is labeled 1.

$$\begin{tikzcd} 1 \arrow[r, dash, "1"] \arrow[d, dash] \arrow[dr, phantom, "X"] & 1 \arrow[d, dash] \\ \varnothing \arrow[r, dash] & \varnothing \end{tikzcd} \qquad \begin{tikzcd} \lambda \arrow[r, dash] \arrow[d, dash] \arrow[dr, phantom, "X"] & 1\lambda \arrow[d, dash] \\ \lambda \arrow[r, dash] & \lambda \end{tikzcd} \qquad \begin{tikzcd} 2\lambda \arrow[r, dash, "i+1"] \arrow[d, dash, "i"'] \arrow[dr, phantom, "X"] & 2\lambda \arrow[d, dash, "1"] \\ \lambda \arrow[r, dash] & \lambda \end{tikzcd}$$

$$\begin{tikzcd} \mu \arrow[r, dash] \arrow[d, dash] \arrow[dr, phantom, "X"] & 2\lambda \arrow[d, dash, "1"] \\ \lambda \arrow[r, dash] & \lambda \end{tikzcd}$$

If the square does not contain an X and $\mu = \lambda$ or $\nu = \lambda$ with no edge label between λ and ν:

(5) If $\mu = \lambda$, then set $\gamma = \nu$ and label the top edge with the same label as the bottom edge if one exists. If $\nu = \lambda$, then $\gamma = \mu$ with the right edge labeled with the same label as the left edge if such a label exists.

$$\begin{tikzcd} \lambda \arrow[r, dash, "\mathrm{i}"] \arrow[d, dash] & \nu \arrow[d, dash] \\ \lambda \arrow[r, dash, "\mathrm{i}"] & \nu \end{tikzcd} \qquad \begin{tikzcd} \mu \arrow[r, dash] \arrow[d, dash, "i"'] & \mu \arrow[d, dash, "i"] \\ \lambda \arrow[r, dash] & \lambda \end{tikzcd}$$

If $\nu = \lambda$ and the bottom edge is labeled 1:

(6) If $\nu = \lambda$ and the bottom edge is labeled 1, then $\gamma = \mu$, the top edge is labeled 1, and the label on the left edge (if one exists) is the same as the label on the right edge.

$$\begin{tikzcd}
\mu \arrow[r, dash, "1"] & \mu \\
\lambda \arrow[u, dash, "j"] \arrow[r, dash, "1"] & \lambda \arrow[u, dash, "j"']
\end{tikzcd}$$

If no previous cases apply, then set $\gamma = 2\lambda$ and follow the rule below:

(7) If the bottom edge is labled i, then label the right edge i. If the left edge is labled j, then label the top edge $j+1$.

$$\begin{tikzcd}
\mu \arrow[r, dash] & 2\lambda \\
\lambda \arrow[u, dash] \arrow[r, dash, "i"] & \nu \arrow[u, dash, "i"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mu \arrow[r, dash, "j+1"] & 2\lambda \\
\lambda \arrow[u, dash, "j"] \arrow[r, dash] & \nu \arrow[u, dash]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mu \arrow[r, dash, "j+1"] & 2\lambda \\
\lambda \arrow[u, dash, "j"] \arrow[r, dash, "i"] & \nu \arrow[u, dash, "i"']
\end{tikzcd}$$

We call the resulting array the KYF growth diagram of w. For example, continuing with the word from Example 5.6.8, $w = 1334241$, we would have the diagram below.

Example 5.6.12. Figure 5.13 KYF growth diagram for the word 1334241.

Let $\mu_0 = \varnothing \subseteq \mu_1 \subseteq \ldots \mu_k$ be the seqence of snakeshapes across the top of the growth diagram, and let $\nu_0 = \varnothing \subseteq \nu_1 \subseteq \ldots \nu_n$ be the sequence of snakeshapes on the right side of the KYF growth diagram. These sequences correspond to KYF tableaux $Q(w)$ and $P(w)$, respectively.

If the edge between ν_i and ν_{i+1} is labeled j, then $2\nu_i = \nu_{i+1}$, and a column of i's is added in the j^{th} column of ν_i to obtain ν_{i+1}. If the edge between μ_i and μ_{i+1} is labeled j, then $\mu_i = \mu_{i+1}$ and the label $i+1$ of $Q(w_1 \cdots w_{i+1})$ is placed in the box at the top of the the j^{th} column of $Q(w_1 \cdots w_i)$.

In the example above, we have

$$
\begin{array}{ccccccccccccccc}
\varnothing & — & 1 & — & 11 & — & 21 & — & 121 & - & 221 & \overset{3}{-} & 221 & \overset{1}{-} & 221 \\
| & & | & & | & & | & X & | & & {}_{2}| & X & {}_{1}| & & {}_{1}| \\
\varnothing & — & 1 & — & 11 & — & 21 & — & 21 & \overset{2}{—} & 21 & — & 21 & \overset{1}{—} & 21 \\
| & & | & X & | & X & {}_{1}| & & {}_{1}| & & | & & | & & | \\
\varnothing & — & 1 & — & 1 & — & 1 & — & 1 & — & 11 & — & 11 & \overset{1}{—} & 11 \\
| & & | & & | & & | & & | & X & | & & | & & | \\
\varnothing & — & 1 & — & 1 & — & 1 & — & 1 & — & 1 & — & 1 & \overset{1}{—} & 1 \\
| & X & | & & | & & | & & | & & | & & | & X & | \\
\varnothing & — & \varnothing & — & \varnothing & — & \varnothing & — & \varnothing & — & \varnothing & — & \varnothing & — & \varnothing
\end{array}
$$

Figure 5.13: K-Young-Fibonacci growth diagram

$$
P = \begin{array}{|c|c|c}
\hline
4 & 3 & \multicolumn{1}{|c}{} \\
\hline
4 & 2 & \multicolumn{1}{c|}{1} \\
\hline
\end{array}
\qquad\qquad
Q = \begin{array}{|c|c|c}
\hline
57 & 3 & \multicolumn{1}{|c}{} \\
\hline
4 & 2 & \multicolumn{1}{c|}{16} \\
\hline
\end{array}\ ,
$$

which agrees with the insertion and recording tableau from Example 5.6.8. As for Hecke insertion and shifted Hecke insertion, this is not a coincidence.

Theorem 5.6.13. *For any word w, the KYF tableau and set-valued KYF tableau P and Q obtained from the sequence of snakeshapes across the right side of the KYF growth diagram for w and across the top of the KYF growth diagram for w, respectively, are $P_{YF}(w)$ and $Q_{YF}(w)$, the KYF insertion and recording tableau for w.*

Proof. Suppose the square under consideration is in row t and column s. Assume also the X in column s is in row r. We may or may not have $r = t$. We will argue the result by induction on $t+s$. The base case is trivial. As in the previous growth diagram proofs, we have the following lemmas.

Lemma 5.6.14. *Oriented as in the square above, $|\nu/\lambda| \leq 1$.*

Lemma 5.6.15. *Oriented as in the square above, to obtain μ from λ, we add either one box or a column of two boxes.*

Rule 1 follows from the fact that inserting t into the tableau $\boxed{t}$ does not change the tableau. Rules 2 applies when inserting positive integer t that is strictly larger than all other entries of the KYF tableau. Thus a box with t will be added directly to the left of the tableau. In Rule 3, we are adding integer t to a tableau that already has 2 boxes labeled t in the i^{th} column, and t is the largest integer in the tableau. Following the insertion rules, we end with a column of t's in the first column, hence the right edge is labeled 1. The special corner of this insertion is in the $(i+1)^{th}$ column, the column where the t on the top row of the original KYF tableau was deleted in the insertion procedure.

In Rule 4, there is exactly one t in insertion tableau μ. Thus, inserting a t will result in a tableau where the first column has two boxes filled with a t.

If $\lambda = \nu$ as in Rule 5, then $r > t$. Thus no integer is being inserted as we move from μ to γ, so $\mu = \gamma$ and the edge labeles are unchanged. Similarly, if if $\lambda = \mu$, then there is no X to the left of the square we are considering, and so no boxes labeled t in the KYF tableau γ. Thus there is no change in the insertion and recording tableaux.

Rule 6 follows directly from insertion rule YF1.

In considering Rule 7, note that $r < t$ because if not, we would be in the situation described in Rule 5. In Rule 7, $\gamma = 2\lambda$ because no matter what r is inserted, since $\mu \neq \lambda$, there is at least one entry of μ that is larger than r. This means that when we insert r into insertion tableau μ, a box with r will be added to the first column, and after the shifting of YF5, there will be a box directly above this new box with r. An entry t will shift into the position of any box that was "lost" when shifting during the insertion of r into λ, and if there were two boxes labeled t in μ, the second will be deleted in YF6. It follows that $\gamma = 2\lambda$.

If the bottom edge is labeled i, then insertion of r into insertion tableau λ included the situation described by YF6 in column i of λ. When we insert r into insertion tableau μ, column i now becomes $\begin{array}{|c|}\hline t \\ \hline * \\ \hline\end{array}$ and thus $\begin{array}{|c|}\hline t \\ \hline t \\ \hline\end{array}$ in γ. This means that the edge between ν and γ should be labeled i, indicating where the t's are in insertion tableau γ.

If the left edge is labeled j, then the difference between insertion tableaux λ and μ is a column $\begin{array}{|c|}\hline t \\ \hline t \\ \hline\end{array}$ in column j of μ. The process of inserting r into insertion tableau μ will match that of inserting r into insertion tableau λ until we get to the point where we are shifting the entries t and $*$. Since t is the largest entry in μ, the insertion of r into μ will end with deleting the unmatched $*$ from column $j+1$ of the insertion tableau γ, and therefore the top edge is labeled $j+1$. $\square$

We can also formulate the rules for the reverse insertion as follows. The proof is omitted for brevity.

Reverse Rules

(R1) If $\mu = \gamma = 1$, $\nu = \varnothing$, at the top edge is labeled 1, then the square contains an X and $\lambda = \varnothing$.

(R2) If $\mu = \nu$ and $\gamma = 1\mu = 1\nu$, then the square contains an X and $\lambda = \mu = \nu$.

(R3) $\mu = \gamma = 2\nu$, the top edge is labeled i and the right edge is labeled 1, then there is an X in the square, $\lambda = \nu$ and the left edge is labeled $i - 1$.

(R4) If $\gamma = 2\nu$ and the right edge is labeled 1, then the square contains an X and $\lambda = \nu$.

(R5) If $\gamma = \nu$, then set $\lambda = \mu$ and label the bottom edge with the same label as the top edge if one exists. If $\gamma = \mu$, then $\lambda = \nu$ and the left edge has the same label as the right edge if such a label exists.

(R6) If $\gamma = \mu$ and the top edge is labeled 1, then $\lambda = \nu$, the bottom edge is labeled 1, and the left edge has the same label as the right edge.

(R7) If no previous cases apply, then λ is obtained from γ by removing the first 2. If the right edge is labeled i, then the bottom edge is labeled i. If the top edge is labeled j, then the left edge is labeled $j - 1$.

5.7 Other examples

5.7.1 Binary tree deformations

Example 5.7.1. The lifted binary tree is shown on the left in Figure 5.14, where vertices can be naturally labeled by bit strings: 0, 1, 10, 11, 100, 101,. . .. The graph *BinWord* with the same set of vertices and rank function is shown on the right. In *BinWord*, an element x covers y if y can be obtained from x by deleting a single digit from x, and in addition, 1 covers 0. The lifted binary tree and *BinWord* form a dual graded graph, see [Fom94, Example 2.4.1].

We form a Pieri deformation by interpreting the graph in the context of the ring of quasisymmetric functions, QSym. If we interpret "1" as L_1, "10" as L_2, "100" as L_3,

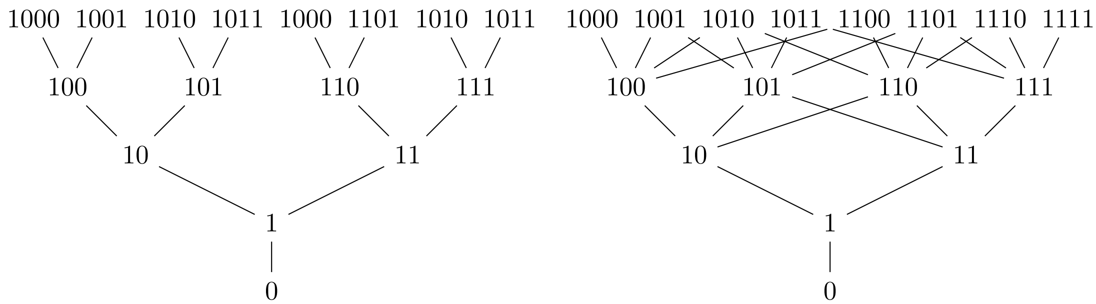


Figure 5.14: Lifted binary tree and *BinWord*

11001 as L_{131}, and so on, we see that x covers y in the graph on the right exactly when x appears in the product $L_y \cdot L_1$, where L_α is the fundamental quasisymmetric function with the usual product from Section 3.3. Let D be the operator that subtracts 1 from the rightmost number in the subscript or deletes the rightmost number if it is 1. For example, $D(L_{14}) = L_{13}$ and $D(L_{1421}) = L_{142}$. Then x covers y in the graph on the left exactly when $y = D(x)$. To form the Pieri deformation, we can let

$$f = L_1 + L_{11} + L_{111} + \dots.$$

In other words, the multiplicity of the edge from x down to y is the coefficient of x in $y \cdot (L_1 + L_{11} + \dots)$.

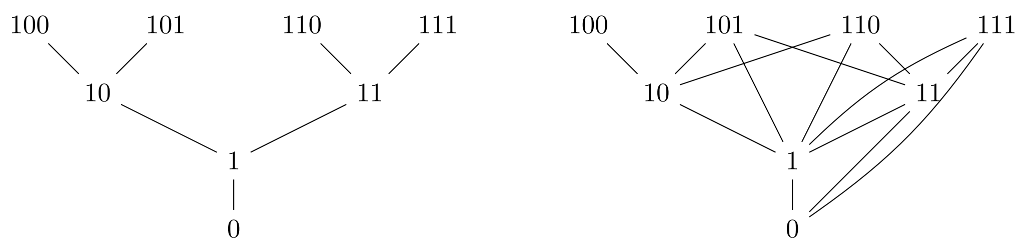


Figure 5.15: Lifted binary tree and *BinWord* Pieri Deformation

Using $A = \mathrm{QSym}$ and D and f as defined above, it is easy to see that $D(f) = L_\varnothing + f = id + f$. Using the Hopf algebra structure of QSym, it is clear that D is a derivation by Lemma 5.3.4. Thus the resulting graph is a dual filtered graph by Theorem 5.3.3.

Example 5.7.2. Consider the Hopf algebra of multi-quasisymmetric functions, $\mathfrak{m}$QSym, defined in [LP07] and described in Section 3.4.

Taking A to be $\mathfrak{m}$QSym with the basis $\{\tilde{L}_\alpha\}$ indexed by compositions, we create a dual filtered graph by taking $f = \tilde{L}_{(1)}$ and $D = \xi \circ \Delta$ with $g = \tilde{L}_{(1)}$. The first five ranks are shown in Figure 5.16.

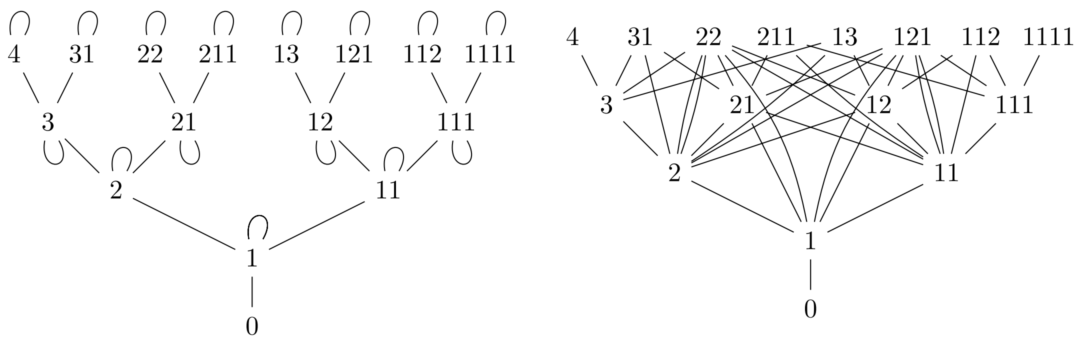


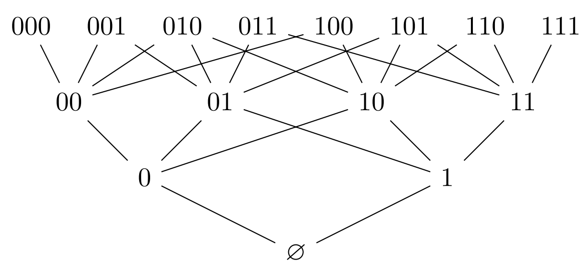
Figure 5.16: $\mathfrak{m}$QSym Pieri construction

This is a dual filtered graph by Theorem 5.3.3.

The following proposition shows that we have another instance of Möbius via Pieri phenomenon, just like in the case of Young's lattice.

Proposition 5.7.3. *The Pieri construction above is a Möbius deformation of the dual graded graph in Example 5.7.1.*

Proof. We can transform the poset *BinWord* from Example 5.7.1 into the poset of subwords shown below by ignoring the first 1 in every word. In doing this, the element $\varnothing$ in *BinWord* disappears, 1 becomes $\varnothing$ in the subword poset, 10 becomes 0, 11001 becomes 1001 and so on. With this change, $x < y$ in *BinWord* exactly when x can be found as a subword of y, ignoring the inital 1's in each.



Given a word $y = y_1 y_2 \cdots y_n$, define its *repetition set* $\mathcal{R}(y) = \{i : y_{i-1} = y_i\}$. An embedding of x in y is a sequence $1 \leq i_1 < i_2 < \ldots < i_k \leq n$ such that $x = y_{i_1} y_{i_2} \cdots y_{i_k}$. It is a *normal embedding* if $\mathcal{R}(y) \subseteq \{i_1, i_2, \ldots, i_k\}$. For two words, x and y, let $\binom{y}{x}_n$ denote the number of normal embeddings of x in y. For example, $\binom{10110}{10}_n = 1$ and $\binom{1010}{10}_n = 3$. Now, from Theorem 1 in [Bjö90],

$$\mu(x, y) = (-1)^{|y|-|x|} \binom{y}{x}_n .$$

We show that the coefficient of $\tilde{L}_\beta$ in $\tilde{L}_\alpha \tilde{L}_{(1)}$ is the number of normal embeddings of α into β after viewing α and β as sequences of 0's and 1's and deleting the first 1 in each by showing the unique way to insert strings of alternating 1's and 0's into α at a specific location to obtain β in the multishuffle. So, for example, the coefficient of $\tilde{L}_{(1111)}$ in $\tilde{L}_{(11)} \tilde{L}_{(1)}$ is zero since $\binom{111}{1}_n = 0$ and the coefficient of $\tilde{L}_{(122)}$ in $\tilde{L}_{(12)} \tilde{L}_{(1)}$ is 3 since $\binom{1010}{10}_n = 3$.

Choose word w_α on $[n]$ and multishuffle it with $n+1$. We note that adding two 0's to α that are not separated by a 1 in α in the multishuffle means increasing the length of an increasing segment of w_α by two, which is impossible as adding $n+1$ to the end of an increasing segment is the only way to increase its length, and the letters of w_α must stay in the same relative order in the multishuffle. Similarly for adding two 1's to w_α that are not separated by a 0 in α. This agrees with the Möbius function since there are no normal embeddings in this case, e.g. $\binom{1010001}{101}_n = 0$.

Suppose we want to insert a segment of alternating 1's and 0's in α in a place with a 0 (if anything) on the left and a 1 (if anything) on the right. Let w_α be $w_1 w_2 \cdots w_{k-1} w_k w_{k+1} \cdots w_n$, where the w_k and w_{k+1} correspond to the 1 and 0 of α mentioned above. The only way to add an alternating sequence beginning with 0 is to insert the string $(n+1) w_k (n+1) w_k (n+1) \ldots$ between w_k and w_{k+1}. The resulting word,

$$w_1 \cdots w_{k-1} w_k (n+1) w_k (n+1) w_k \cdots w_{k+1} \cdots w_n$$

is clearly a multishuffle of w_α and $n+1$. Similarly, to add an alternating sequence beginning with 1, "delete" the w_k from w and replace it with $(n+1) w_k (n+1) w_k (n+1) \cdots$. The resulting word,

$$w_1 \cdots w_{k-1} (n+1) w_k (n+1) w_k \cdots w_{k+1} \cdots w_n$$

is a multishuffle of w_α and $n+1$.

If instead we insert the alternating string in a place of α with a 1 on the left and a 0 on the right, we only need to consider strings beginning with 0 as the 1 at the beginning of the string can be added as in the previous case. To insert an alternating string beginning with 0, we must insert the segment $(n+1)w_k(n+1)w_k\cdots$ between w_k and w_{k+1} of w_α.

If we insert the alternating string in place of α with 0's to the left and right, we only need to consider strings that begin with 1. To do this, delete the w_k from w and insert the segment $(n+1)w_k(n+1)w_k\cdots$ between w_{k-1} and w_{k+1}

If we insert the alternating string in place of α with 1's to the left and right, we only need to consider strings that begin with 0. To do this, insert the segment $(n+1)w_k(n+1)w_k\cdots$ between w_k and w_{k+1}. $\square$

5.7.2 Poirer-Reutenauer deformations

Example 5.7.4. The *SYT-tree* has as vertices standard Young tableau, and standard Young tableau T_1 covers standard Young tableau T_2 if T_2 is obtained from T_1 by deleting the box with largest entry. It is shown in Figure 5.17 on the left. It is dual to the *Schensted graph*, which is shown on the right.

The *Schensted graph* is constructed using RSK insertion. A standard Young tableau T_1 covers T_2 if T_1 appears in the product of T_2 with the single-box standard tableau, where multiplication of standard Young tableaux is as follows. Suppose T_1 and T_2 are standard Young tableaux with row words $\mathfrak{row}(T_1)$ on $[n]$ and $\mathfrak{row}(T_2)$ on $[m]$, respectively. Then let $T_1 \cdot T_2$ be the sum of tableaux obtained by first lifting $\mathfrak{row}(T_1)$ to $[n+m]$ in all ways that preserve relative order and then RSK inserting the corresponding lifting of $\mathfrak{row}(T_2)$. For example, in multiplying $\boxed{1}\boxed{2}$ by itself, we would get the sum of $(P(12) \leftarrow 3) \leftarrow 4$, $(P(13) \leftarrow 2) \leftarrow 4$, $P((14) \leftarrow 2) \leftarrow 3$, $(P(23) \leftarrow 1) \leftarrow 4$, $(P(24) \leftarrow 1) \leftarrow 3$, and $(P(34) \leftarrow 1) \leftarrow 2$.

Let $f = T_1 + T_2 + T_3 + \ldots$, where T_i is the one-row tableau with row reading word $123\ldots i$, and create a Pieri deformation by adding edges to the *Schensted graph* so that $a_2(T_i, T_j)$ is the coefficient of T_j in $T_i \cdot f$.

To see that this gives a dual filtered graph, take A to be the algebra of standard

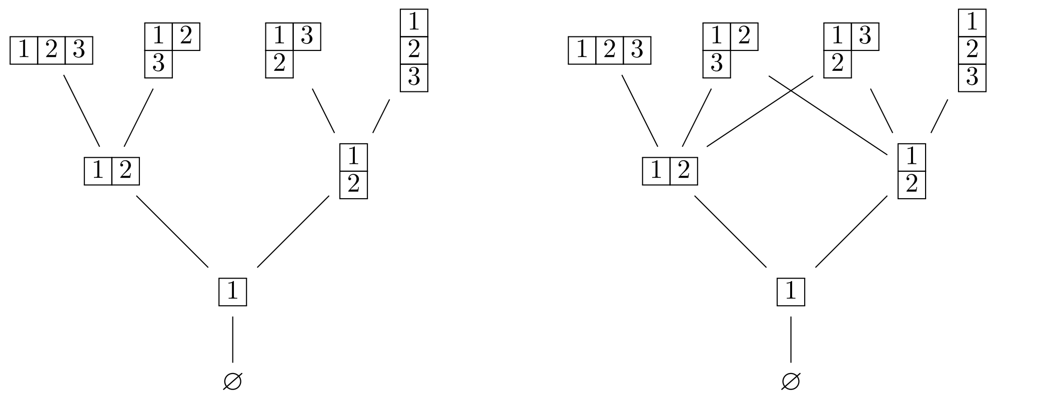


Figure 5.17: The SYT-tree and the Schensted graph

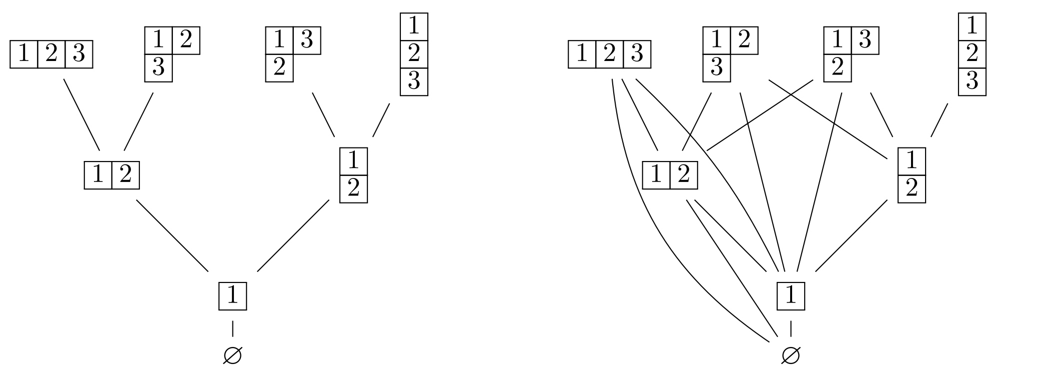


Figure 5.18: Pieri deformation of SYT-Schensted pair

Young tableaux with multiplication as defined above. Note that A is dual to the Poirier-Reutenauer Hopf algebra [PR95]. Take D to be the operator that deletes the box with the largest entry, and notice that T_1 covers T_2 in *SYT-tree* exactly when $D(T_1) = T_2$. The operator D is a derivation by Lemma 5.3.4 using the fact that $\Delta(T) = \sum(u \otimes v)$, where we sum over words u and v such that $\mathfrak{row}(T)$ is a shuffle of u and v. Lastly, it is easy to see that $D(f) = \varnothing + f$. Thus by Theorem 5.3.3, we have a dual filtered graph.

Example 5.7.5. Take A to be the K-theoretic Poierier-Reuntenauer bialgebra (KPR) defined in Chapter 4 with a basis of K-Knuth classes of initial words (i.e. words containing only $1, 2, \ldots, n$ for some n) on $\mathbb{N}$.

Letting $g = [[1]]$ and $f = [[1]]$, we create a Pieri construction using Theorem 5.3.3.

The partial dual filtered graph is shown below. The first five ranks are shown completely, and there are a few additional elements shown in the second graph to illustrate all elements that cover [[212]]. To see that there is an upward edge from [[3412]] to [[3124]], we first notice that $34124 \in [[3124]]$ and that $3412 \otimes 1$ appears in the coproduct of 34124.

In the Figure 5.19, K-Knuth equivalence classes of words are represented by an increasing tableau. Note that there may be more than one increasing tableau in any given class.

5.7.3 Malvenuto-Reutenauer deformations

Example 5.7.6. Below we have dual graphs where vertices are permutations. On the left in Figure 5.20, a permutation σ covers π if π is obtained from σ by deleting the the rightmost number (in terms of position in the permutation). On the right in Figure 5.20, a permutation σ covers π if π is obtained from σ by deleting the largest number in the permutation. There are other similar constructions, which can be found in [Fom94, Example 2.6.8].

Consider the Hopf algebra of permutations with the shuffle product and coproduct defined by $\Delta(w) = \sum std(u) \otimes std(v)$, where the concatenation of u and v is w and $std(w)$ sends w to the unique permutation with the same relative order. For example, $std(1375) = 1243$. Take D to be the operator that deletes the rightmost letter of the permutation. Then we see that $a_1(v, w)$ is the coefficient of v in $D(w)$. We create a Pieri deformation by letting $f = 1 + 12 + 123 + \dots$ and $a_2(v, w)$ be the coefficient of w in $f \cdot v$. Clearly $D(f) = \varnothing + f$, and we use Lemma 5.3.4 to see tha D is a derivation. It then follows from Theorem 5.3.3 that the resulting graph is a dual filtered graph. It is shown in Figure 5.21.

Example 5.7.7. We next describe a K-theoretic analogue of the Malvenuto-Reutenauer Hopf algebra. For details, see [LP07]. A small multi-permutation or $\mathfrak{m}$-permutation of $[n]$ is a word w in the alphabet $1, 2, \dots, n$ such that no two consecutive letters in w are equal. Now let the small multi-Malvenuto-Reutenauer Hopf algebra, $\mathfrak{m}$MR be the free $\mathbb{Z}$-module of arbitrary $\mathbb{Z}$-linear combinations of multi-permutations. Recall the definition of the multishuffle product from Example 5.7.2. Given two $\mathfrak{m}$-permutations $w = w_1 \cdots w_k$ and $u = u_1 \cdots u_l$, define their product to be the multishuffle product of

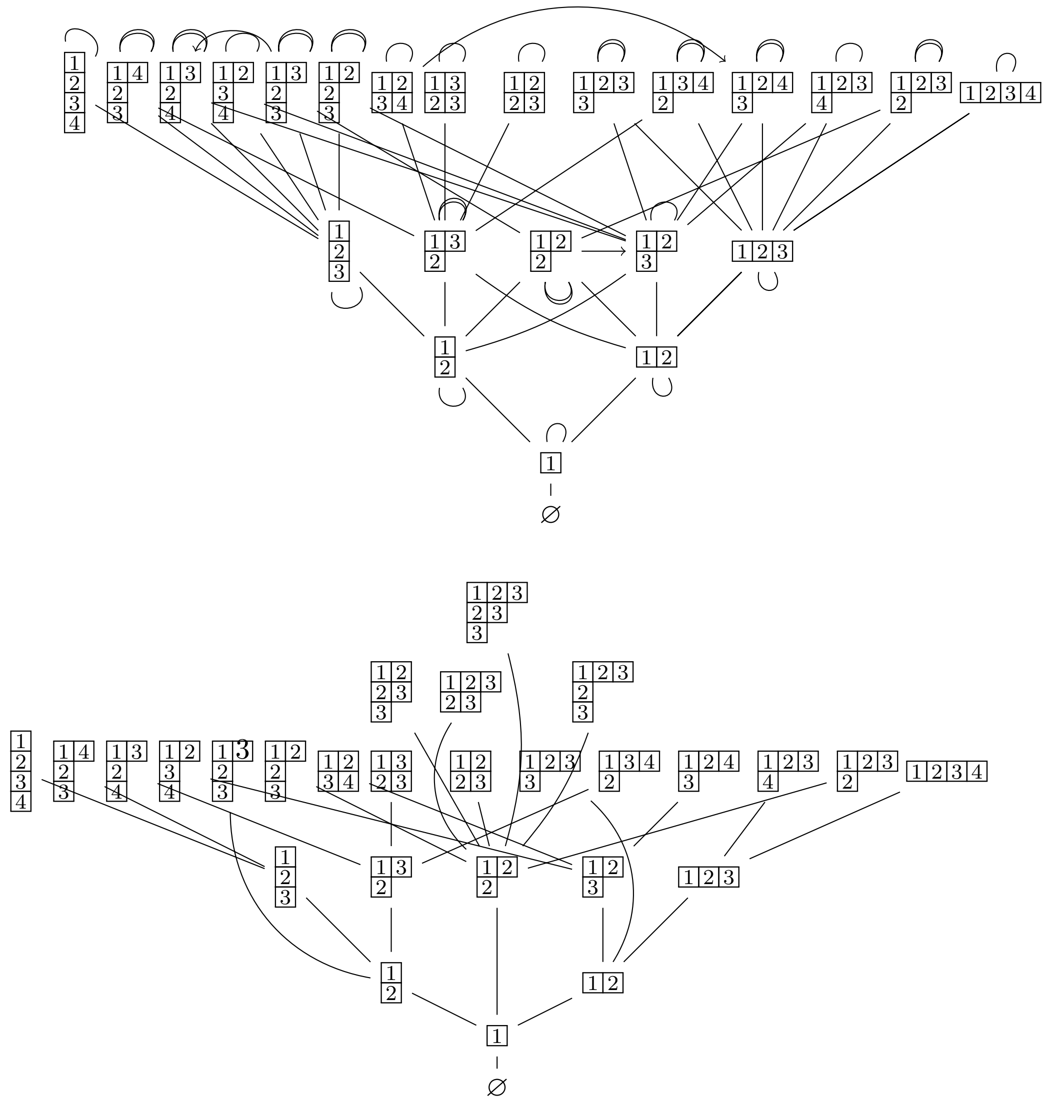


Figure 5.19: Pieri construction on the K-Poirier Reutenauer bialgebra

w with $u[n]$, where w contains exactly the numbers $1, 2, \ldots, n$ and $u[n] = (u_1 + n)(u_2 + n) \ldots (u_l + n)$.

To define the coproduct, recall the *cuut coproduct* from Definition 3.4.12. Then for any word w, let

$$\blacktriangle(w) = \sum_{h \in \mathrm{Cuut}(w)} h$$

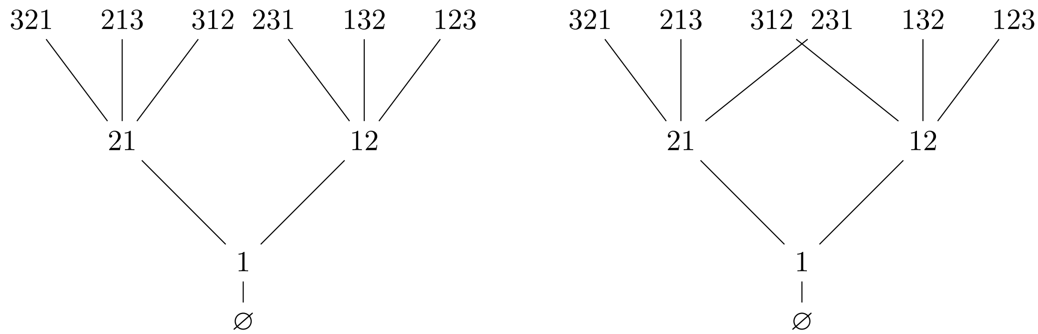


Figure 5.20: Dual graded graph on permutations

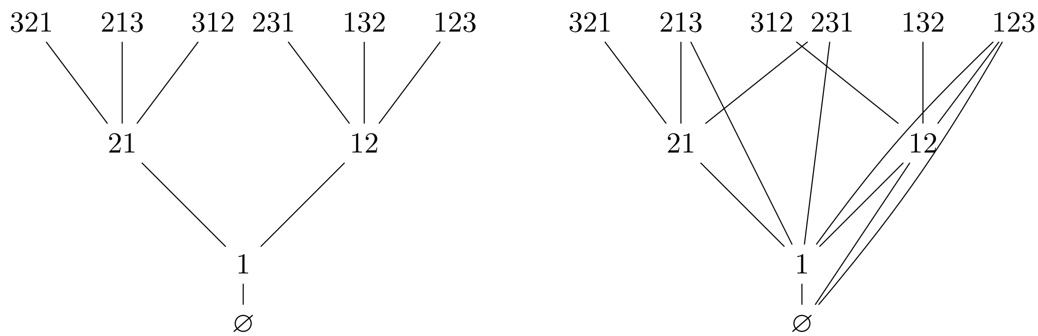


Figure 5.21: Pieri construction on the Malvenuto-Reutenauer Hopf algebra

Let $st(w)$ send a word w to the unique $\mathfrak{m}$-permutation u of the same length such that $w_i \leq w_j$ if and only if $u_i \leq u_j$ for each $1 \leq i, j \leq l(w)$. Finally, define the coproduct in $\mathfrak{m}$MR by $\Delta(w) = st(\blacktriangle(w))$.

Using $g = 1$, $D = \xi \circ \Delta$, and $f = 1$, we form the dual filtered graph partially shown in Figure 5.22. Notice that $D(w_1 \dots w_k) = w_1 \cdots w_k + w_1 \cdots w_{k-1}$, so $D(f) = \emptyset + f$.

5.7.4 Stand-alone examples

Example 5.7.8. There is another graph with the same set of vertices as the Young-Fibonacci lattice called the Fibonacci graph. The vertices of the Fibonacci graph are words on the alphabet $\{1, 2\}$, or snakeshapes, and a word w is covered by w' if w' is obtained from w by adding a 1 at the end or by changing any 1 into a 2. In the graph that is dual to the Fibonacci graph, w is covered by w' if w is obtained from w' by

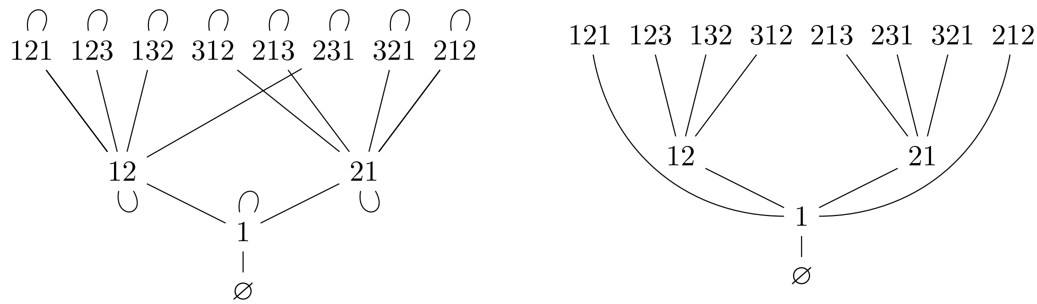


Figure 5.22: Pieri construction on $\mathfrak{m}$MR

deleting a 1, and the multiplicity of the edge between w and w' is the number of ways to delete a 1 from w' to get w. The pair form a dual graded graph shown in Figure 5.23, see [Fom94, Example 2.3.7], where the numbers next to the edges denote multiplicity.

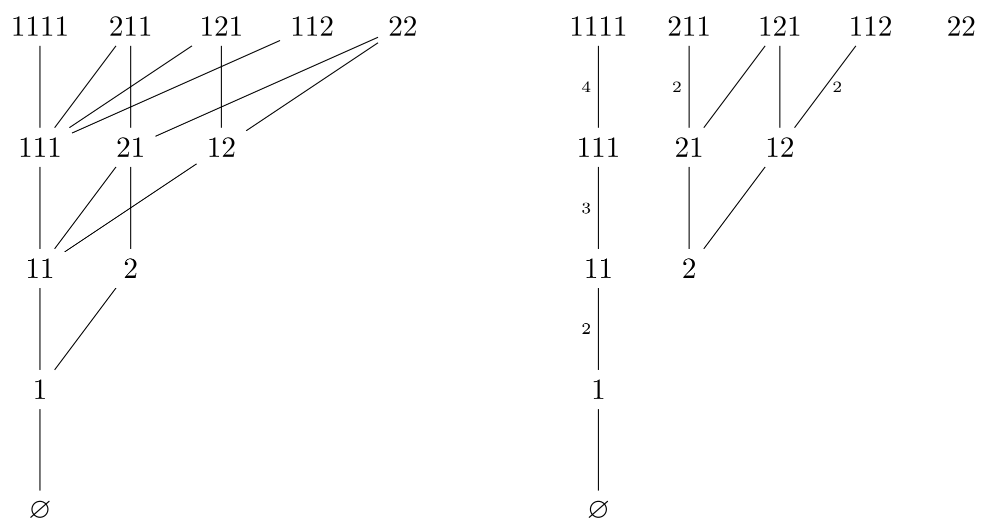


Figure 5.23: The Fibonacci graph

We construct a Pieri deformation of this dual graded graph by adding downward-oriented edges so that in the resulting set of downward-oriented edges, there is an edge from w' to w for every way w can be obtained from w' by deleting at least one 1. For example, there are six edges from 1111 to 11 since there are six ways to delete two 1's from 1111 to obtain 11.

Theorem 5.7.9. *The resulting graph is a dual filtered graph.*

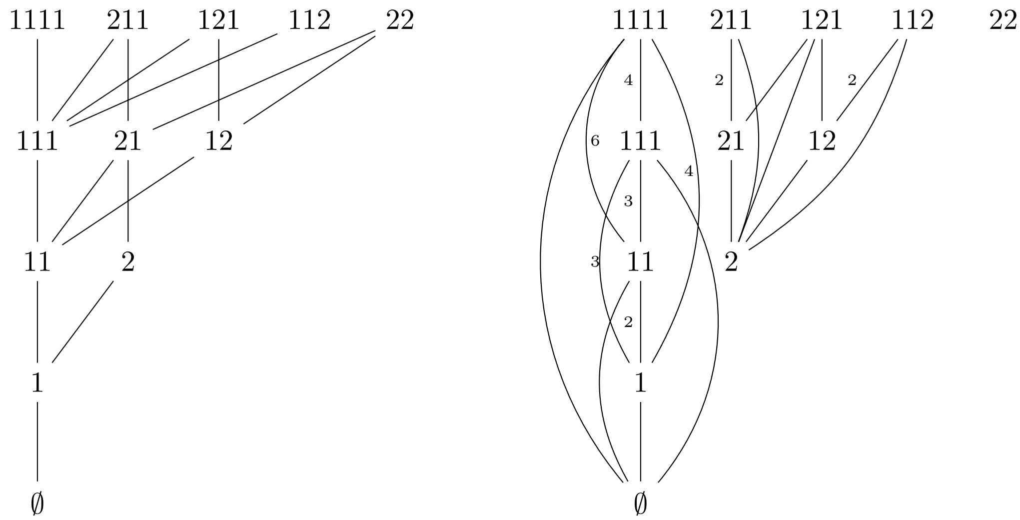


Figure 5.24: Pieri deformation of the Fibonacci graph

Proof. Consider an algebra structure on words in alphabet $\{1, 2\}$ with shuffle product, as in [Fom94]. Consider D to be the operator that changes any 2 into a 1 or deletes the last digit if it is 1. According to [Fom94, Lemma 2.3.9], D is a derivation such that the coefficient of v in $D(w)$ is the multiplicity $a_1(v, w)$ for the Fibonacci graph. Take $f = 1 + 11 + 111 + \ldots$. It is easy to see that $D(f) = id + f$ and that the coefficient of w in fv is the multiplicity $a_2(v, w)$. Thus, Theorem 5.3.3 applies, and the statement follows. □

Example 5.7.10. The dual filtered graph on the polynomial ring shown in Figure 5.25 is that mentioned in Section 5.1.2. We take U to be multiplication by x and $D = e^{\frac{\partial}{\partial x}} - 1$. It is easy to check that $DU - UD = D + I$, so the result is a dual filtered graph.

5.8 Enumerative theorems via up-down calculus

Let $\varnothing$ be the minimal element of a dual filtered graph satisfying $DU - UD = 1 + D$. Let $T(n, k)$ be the number of ways n labeled objects can be distributed into k nonempty parcels. We have

$$T(n, k) = k! \cdot S(n, k),$$

where $S(n, k)$ is the Stirling number of the second kind.

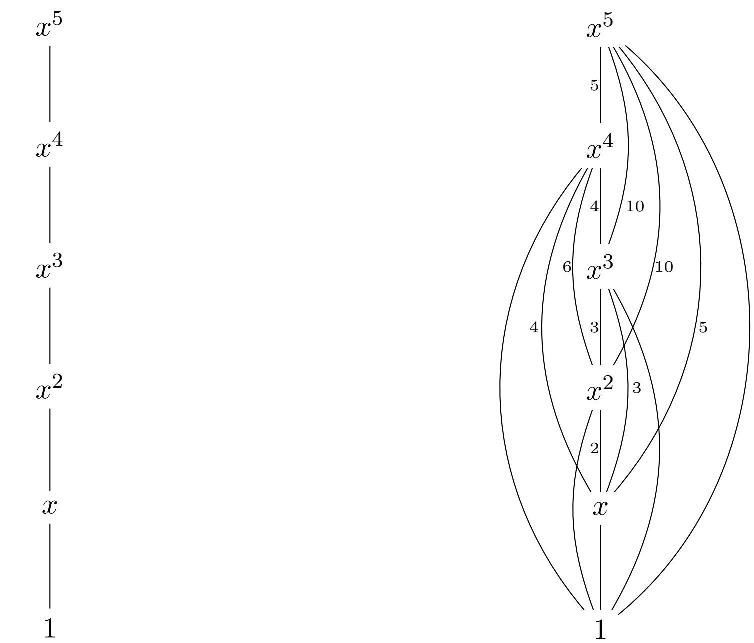


Figure 5.25: Dual filtered graph on polynomial ring

Theorem 5.8.1. *For any dual filtered graph, the coefficient of* $\varnothing$ *in* $D^k U^n(\varnothing)$ *is* $T(n,k)$.

Proof. We think of replacing the fragments DU inside the word by either UD, D, or 1. This way the initial word gets rewritten until there is no DU in any of the terms:

$$D^k U^n = D^{k-1}UDU^{n-1} + D^k U^{n-1} + D^{k-1}U^{n-1} = \dots.$$

Only the terms of the form U^t that appear at the end can contribute to the coefficient we are looking for, since $D(\varnothing) = 0$. Among those, only the terms $U^0 = 1$ can contribute. Thus, we are looking for all the terms where D's and U's eliminated each other.

It is easy to see that each D eliminates at least one U, and each U must be eliminated by some D. The number of ways to match D's with U's in this way is exactly $T(n,k)$. $\square$

Let f^λ be the number of increasing tabeleaux of shape λ. Let g_n^λ be the number of set-valued tableaux of shape λ and content $1,\dots,n$. Let $F(n)$ denote the *Fubini number*, or *ordered Bell number*—the number of ordered set partitions of $[n]$.

Corollary 5.8.2. *We have*

$$\sum_{|\lambda|\leq n} f^{\lambda} g_n^{\lambda} = F(n).$$

Proof. The left side is clearly the coefficient of $\varnothing$ in $(D + D^2 + \ldots + D^n)U^n(\varnothing)$. It remains to note that $\sum_{k=1}^{n} T(n,k) = F(n)$. □

This is the analogue of the famous Frobenius-Young identity. Of course, this also follows from bijectivity of Hecke insertion. The advantage of our proof is that a similar result exists for any dual filtered graph.

The following result is analogous to counting oscillating tableaux.

Theorem 5.8.3. *For any dual filtered graph the coefficient of $\varnothing$ in $(D + U)^n(\varnothing)$ is equal to the number of set partitions of $[n]$ with parts of size at least 2.*

Proof. As before, the desired coefficient is equal to the number of ways for all D's to eliminate all U's via the commutation relation. Each factor in the product

$$(D + U)(D + U) \ldots (D + U)$$

thus either eliminates one of the factors to the right of it, or is eliminated by a factor to the left. Grouping together such factors, we get a set partition with parts of size at least 2. On the other hand, any such set partition corresponds to a choice of D in the first factor of each part and of U's in the rest. Thus, it corresponds to term 1 after such D's eliminate such U's. The statement follows. □

5.9 Affine Grassmannian

One natural question to ask is if we can apply the machinery of dual filtered graphs to the affine Schubert calculus setting of Lam, Lapointe, Morse, Schilling, Shimozono, and Zabrocki [LLM$^+$14]. In this section, we give a brief overview of the necessary combinatorial objects. We then discuss our progress, complications, and conjectures in this area. Many details are omitted as the aim is simply to give a reader a sense of the problem.

5.9.1 Dual k-Schur functions

For the following definitions, let $k \in \mathbb{N}$. We follow [Mor12] closely.

A *k-core* is a partition that does not contain any cells with hook-length k. A *k-bounded partition* is a partition with largest part at most k. The k-residue of a cell (i, j) is $j - i$ mod p. For example, the 5-residues of a 5-core are shown below.

$$\begin{array}{|c|c|c|c|c|c|c|}\hline 0&1&2&3&4&0&1\\\hline 4&0&1&2\\\cline{1-4} 3&4&0&1\\\cline{1-4} 2\\\cline{1-1} 1\\\cline{1-1}\end{array}$$

There is a bijection between the set of $(k+1)$-cores κ with $|\kappa| = m$ and k-bounded partitions of m, where $|\kappa|$ is the number of cells with hook length at most $k + 1$. For example, the shapes in Figure 5.26 are 3-cores, and the corresponding 2-bounded partitions can be obtained by deleting the boxes labeled with $\bullet$. In this picture, $|\kappa|$ is equal to the number of cells with no $\bullet$. For brevity, we will not discuss the details of this bijection but instead refer the interested reader to [LLM+14] for further discussion. For future definitions, we write $\mathfrak{c}(\lambda)$ to denote the $(k + 1)$-core corresponding to the k-bounded partition λ and $\mathfrak{p}(\mu)$ to denote the k-bounded partition corresponding to $(k + 1)$-core μ.

We next define k-tableaux on our way to defining dual k-Schur functions.

Definition 5.9.1. *Let λ be a $k + 1$-core and let $\alpha = (\alpha_1, \ldots, \alpha_r)$ be a composition of $|\mathfrak{p}(\lambda)|$. A k-tableau of shape λ and weight α is a semistandard filling of λ with integers 1,2,...,r such that the collection of boxes filled with letter i are labeled by exactly α_i distinct $k + 1$-residues.*

For example, the 3-tableaux of weight $(1, 3, 1, 2, 1, 1)$ and shape $(8, 5, 2, 1)$ are shown below.

$$\begin{array}{|c|c|c|c|c|c|c|c|}\hline 1&2&2&2&3&4&4&6\\\hline 2&3&4&4&6\\\cline{1-5} 4&6\\\cline{1-2} 5\\\cline{1-1}\end{array}\qquad \begin{array}{|c|c|c|c|c|c|c|c|}\hline 1&2&2&2&3&4&4&5\\\hline 2&3&4&4&5\\\cline{1-5} 4&5\\\cline{1-2} 6\\\cline{1-1}\end{array}\qquad \begin{array}{|c|c|c|c|c|c|c|c|}\hline 1&2&2&2&4&4&5&6\\\hline 2&4&4&5&6\\\cline{1-5} 3&6\\\cline{1-2} 4\\\cline{1-1}\end{array}$$

We can now define the *dual k-Schur function* [LM08] $\mathfrak{S}_\lambda^{(k)}$ for any k-bounded partition by

$$\mathfrak{S}_\lambda^{(k)} = \sum_T x^{w(T)},$$

were we sum over k-tableaux T of shape $\mathfrak{c}(\lambda)$.

5.9.2 Affine stable Grothendieck polymonials

We next work toward defining the affine version of set-valued tableaux, which generalize k-tableaux.

Let $T_{\leq x}$ denote the tableaux obtained from tableau T by deleting all entries greater than x. See the example below.

$$T = \begin{array}{|c|c|c|}\hline 1,2 & 3 & 4 \\ \hline 4 \\ \cline{1-1} 5 \\ \cline{1-1}\end{array} \qquad T_{\leq 4} = \begin{array}{|c|c|c|}\hline 1,2 & 3 & 4 \\ \hline 4 \\ \cline{1-1}\end{array} \qquad T_{\leq 3} = \begin{array}{|c|c|}\hline 1,2 & 3 \\ \hline\end{array}$$

Definition 5.9.2. *A standard affine set-valued tableau T of degree n is a set-valued filling such that for each $1 \leq x \leq n$, shape$(T_{\leq x})$ is a core and the cells containing an x form the set of all removable corners of $T_{\leq x}$ with the same residue.*

Example 5.9.3. [Mor12, Example 5.14] Set $k = 2$. The standard affine set-valued tableaux of degree 5 with shape $(3, 1, 1)$ are shown below.

$$\begin{array}{|c|c|c|}\hline 1 & 3,4 & 5 \\ \hline 2 \\ \cline{1-1} 3,4 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1} 3,5 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|c|}\hline 1,2 & 3 & 4 \\ \hline 4 \\ \cline{1-1} 5 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|c|}\hline 1,2 & 4 & 5 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 4 & 5 \\ \hline 2,3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 2,3 & 4 \\ \hline 4 \\ \cline{1-1} 5 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 3,4 \\ \hline 3,4 \\ \cline{1-1} 5 \\ \cline{1-1}\end{array}$$

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3,5 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}$$

Before defining the semistandard version, it will help to recall a fact about semistandard (non-affine) tableau: semistandard tableaux are in bijection with standard tableaux that have reading words that increase in the alphabets

$$\mathcal{A}_{\alpha,x} = [1 + \sum^{x-1} \alpha, \sum^{x} \alpha],$$

where $\sum^x \alpha = \sum_{i=1}^x \alpha_i$ and $x = 1, \ldots, \ell(\alpha)$. We can realize this correspondence by starting with the reading word of a semistandard tableaux, say of content $(\alpha_1, \ldots, \alpha_r)$, and replacing the 1's with integers $1, 2, \ldots, \alpha_1$ from right to left in the reading word, the 2's with $\alpha_1 + 1, \ldots, \alpha_1 + \alpha_2$ from right to left, etc. The result is a reading word of a standard tableaux.

Define the *reading word of a set-valued tableau* to be the word obtained by reading the columns bottom to top, left to right, and reading subsets in decreasing order. Define the weight of the reading word in the usual way. For example, the reading word of the first tableau in Example 5.9.3 is 4321435 and has weight (1,1,2,2,1). To prevent multiplicity, we define the *highest reading word* to be the word obtained by reading the highest occurrence of the letters in the reading word. The highest reading words of the seven tableaux in Example 5.9.3 are 21435, 52134, 52134, 32145, 51324, 51243, and 41253.

Definition 5.9.4. *For any k-bounded composition α, an affine set-valued tableau of weight α is a standard affine set-valued tableau of degree $|\alpha|$ where, for each $1 \leq x \leq \ell(\alpha)$,*

1. *the highest reading word in $\mathcal{A}_{\alpha,x}$ is increasing,*
2. *the letters of $\mathcal{A}_{\alpha,x}$ occupy α_x distinct residues, and*
3. *the letters of $\mathcal{A}_{\alpha,x}$ form a horiztonal strip.*

The following results help make sense of this definition. Recall that the hook of a partition $\lambda = (\lambda_1, \ldots, \lambda_r)$, $h(\lambda)$, is $\lambda_1 + \lambda'_1 - 1$.

Proposition 5.9.5. *[Mor12, Proposition 9] For any k-bounded partition λ with $h(\lambda) \leq k$, the affine set-valued tableaux of shape λ are the set-valued tableaux of shape λ.*

Proposition 5.9.6. *[Mor12, Proposition 13] Fix k. The set of affine set-valued tableaux of weight α and shape $\mathfrak{c}(\lambda)$ is the same as the set of k-tableaux with weight λ and shape $\mathfrak{c}(\lambda)$ when $|\alpha| = |\lambda|$.*

Using affine set-valued tableaux, we can define the *affine stable Grothendieck polynomial* [Lam06] $G_\lambda^{(k)}$ for any k-bounded partition λ to be

$$G_\lambda^{(k)} = \sum_T (-1)^{|\lambda|-|T|} x^{w(T)},$$

where we sum over the set of affine set-valued tableaux of shape $\mathfrak{c}(\lambda)$.

5.9.3 Dual graded and dual filtered graphs

There is an affine version of Young's lattice for each $k \in \mathbb{N}$, and it is no longer self dual in the dual graded graph sense. The analogue of Young's lattice for $k = 2$ and its dual are shown in Figure 5.26. Single and double edges are drawn as such, and edges with multiplicity three are labeled with a 3. The fact that the pair shown is a dual graded graph comes from the affine insertion algorithm that the graph models.

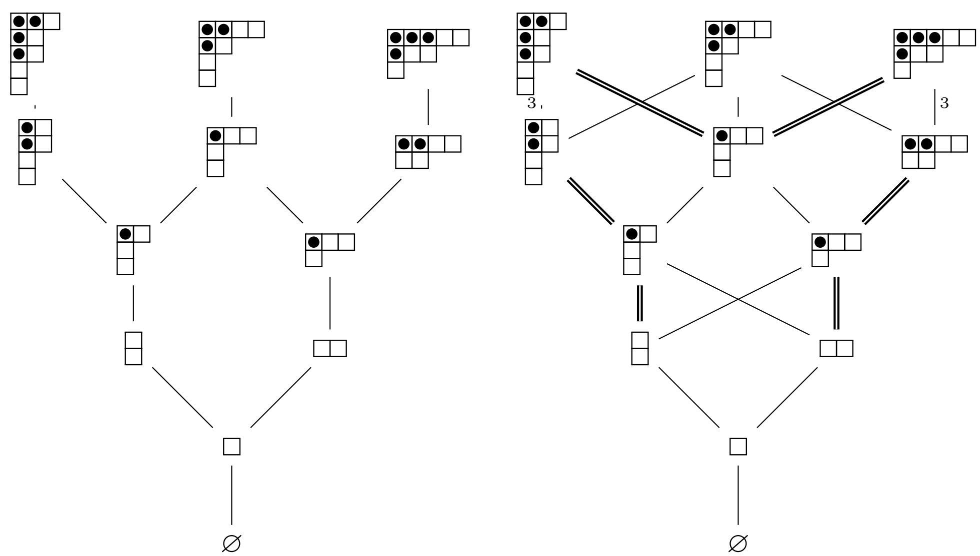


Figure 5.26: The affine version of Young's lattice for $k = 2$

Not much is known about the combinatorics of K-theory of the affine Grassmannian. For example, there is no Pieri formula for the affine stable Grothendieck polynomials, there is no K-theoretic affine insertion algorithm, and there is no combinatorial description for the dual affine stable Grothendieck polynomials. The lack of Pieri rule provides an obstacle in forming the dual graded graph using the Pieri construction and the bialgebra of affine stable Grothendieck polynomials. However, the lack of knowledge in this area also makes it more desirable to build the related dual graded graphs as we may be able to learn new information. In the remainder of this section, we mention one promising direction we have explored and hope to come back to in the future.

One may compute products of affine stable Grothendieck polynomials using Sage software described in [LLM+14]. In order to use the Pieri construction, we need to be able to multiply any affine stable Grothendieck polynomial by $G^{(k)}_{\square}$. If one uses Sage to do some computations, one can construct graphs like the one below for $k = 2$.

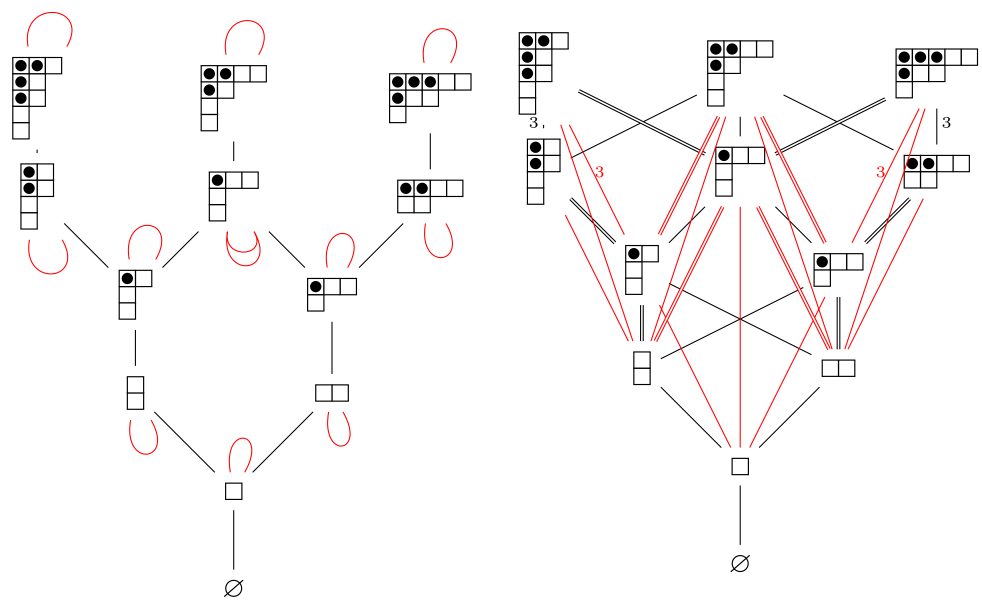


Figure 5.27: "Möbius" construction on affine Young's lattice for $k = 2$

It looks like one should be able to define a generalized Möbius function in this setting to realize the graph in Figure 5.27 as the Möbius construction of that in Figure 5.26, and this appears to be true in general. In support of this idea, after computing many examples for different k, it seems that the generalized Möbius function depends only on the graph. In other words, when the same interval appears in different settings, this interval has the same generalized Möbius function.

Moreover, another example of a dual graded graph that has edge multiplicity in downward oriented edges is the graph $\mathbb{SY}$ as in Section 5.2.1 and shown again in Figure 5.28.

One can play the same game as in Section 5.5.3 using increasing shifted tableaux where primes are allowed as insertion tableaux and set-valued shifted tableaux where primes are not allowed as recording tableaux. This will correspond to the graph $\mathbb{SY}$.

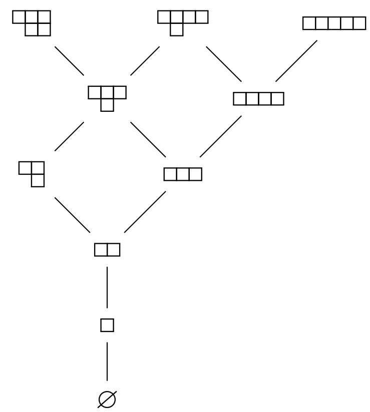


Figure 5.28: The graph $\mathbb{SY}$

One can then create a K-theoretic analogue of this insertion algorithm and create the corresponding dual filtered graph. One can then use this dual filtered graph as data in constructing a generalized Möbius function.

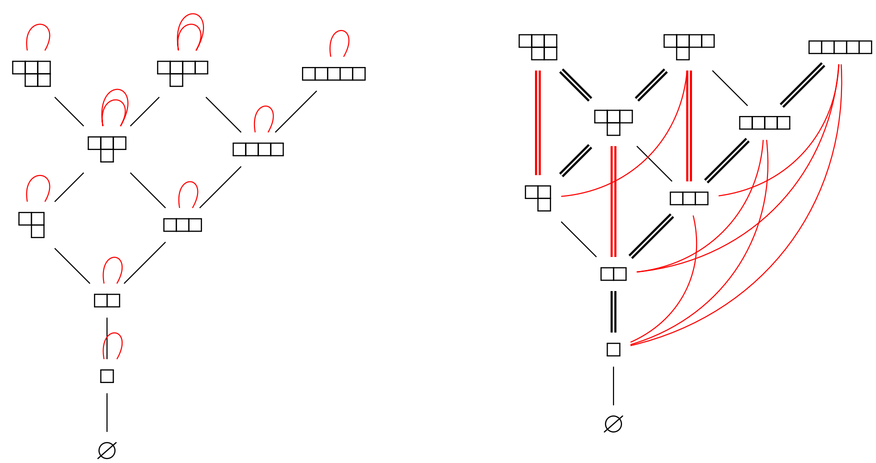
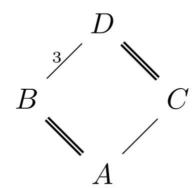


Figure 5.29: Generalized Möbius function on $\mathbb{SY}$

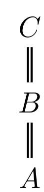
Problem 5.9.7. *Define a generalized Möbius function $\tilde{\mu}$ valid on posets where edge multiplicity is allowed that coincides with the Pieri construction using affine stable Grothendieck polynomials.*

Conjecture 5.9.8. *The generalized Möbius function on an interval of a graph depends only on the structure of the interval and not on the graph in which it is contained.*

For example, every time the interval below has appeared in an affine Young diagram for some k, it is true that $\tilde{\mu}(A, D) = 3$. Here, we determine the values of $\tilde{\mu}$ using the computations in Sage mentioned above.



Or every time the interval below has appeared in an affine Young diagram or in Figure 5.29, $\tilde{\mu}(A, C) = 1$.



Conjecture 5.9.9. *The affine Young's lattice is an example of the Möbius via Pieri phenomenon using the generalized Möbius function.*

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