Relationship between the Learning Hierarchy and Academic Achievement on Strategies Used by Third-Grade Students when Solving Multiplication Word Problems

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Dedication

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Abstract

Distinguishing between sources of variability in mathematics performance may contribute to a more comprehensive theory of mathematics skills. Research has frequently examined student differences based upon scores on achievement tests, which provide overall student proficiency, but may not provide the detailed information for identifying and remediating student difficulties. Additionally, studies have reported, students with mathematics difficulties often struggle with problem solving but specific differences in problem solving strategies have not been thoroughly identified and as a result of the complexity of problem solving an incorrect response may be due to a number of potential reasons.

Previous studies have also examined assessing various types of knowledge for the purposes of understanding student skills and identifying appropriate intervention. The Learning Hierarchy (Haring & Eaton, 1978) considers how students learn different academic skills as they progress through a four phase learning sequence and has previous support as an intervention heuristic (Daly & Ardoin, 1997), but there is limited research for mathematics (Burns, Codding, Boice, & Lukito, 2010).

The purpose of this study is to extend previous research by examining the Learning Hierarchy conceptual model as a framework for intervention design based on student performance on a computation mathematics fluency measure and broaden the research base around the development and characteristics of problem solving for students with various levels of achievement and extending it by examining differences of strategy use of students in specific phases in the Learning Hierarchy.

Participants were 492 third grade students attending elementary schools within one school district in the upper Midwestern United States. Students were administered measures of computation fluency and application. Students were classified into four categorical phases based on accuracy and fluency scores (Burns, 2004; Burns, VanDerHeyden, & Jiban, 2006; VanDerHeyden & Burns, 2008). To examine strategic development and competence, student responses were scored for overall accuracy and coded for strategy used to solve the problem (Zhang, Ding, Barrett, Xin, 2014).

The results support previous research findings in strategy development suggesting that mathematics achievement significantly predicts accuracy of strategy used (Zhang et al., 2014). When student performance was compared based upon the phases of the Learning Hierarchy, students in initial phases displayed more variation in strategy selection than students in fluent groups but, were less accurate and demonstrated consistent use of lower quality strategies. The current findings are also promising for consideration of the Learning Hierarchy as a potential conceptual heuristic model in mathematics given that the observed and expected profiles were not significantly different.

The current results were contextualized within previous research and potential implications for theory and future research in mathematics were discussed. Specifically, the need for further research supporting the validity of the Learning Hierarchy framework in other areas of mathematics as well as the potential of understanding strategic development on instructional and interventions practices for proficiency in multiplication. Lastly, strengths and limitations to the study were outlined.

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CHAPTER 1

INTRODUCTION

Developing proficiency with mathematical skills and concepts is a primary instructional goal. However, the percentage of students in 4th grade who performed at or above proficient level on the National Assessment of Educational Progress (NAEP) was only 42%, which although was an increase from 2011 to 2013, is still concerning (U.S. Department of Education, 2013). The NAEP assessment framework classifies questions to measure one of the five mathematics content areas; number properties and operations, measurement, geometry, data analysis, statistics, and probability, and algebra. The distribution of items within the content areas reflects the importance given to each area. At the 4th grade level, emphasis on number properties and operations is evident by the largest percentage of questions allotted to that content area (National Governors Association for Best Practices & Council of Chief State School Officers, 2010). The National Research Council (2001; NRC) also considered number properties and operations to be important by defining mathematical proficiency as the interweaving of conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition (NRC, 2001).

Mathematics proficiency has become increasingly important due to the growing emphasis on technology in the workplace. Mathematical achievement predicts graduation in high school, college, and early-career earnings (National Mathematics Advisory Panel, 2008). Given the importance of mathematics proficiency it is critical that educators understand how these skills develop, so that they can intervene early to remediate mathematics skills deficits.

Statement of the Problem

Intervention selection is more likely to be successful when it is based on an evidence-based heuristic that can correctly target and predict intervention effectiveness to meet students need based on student performance. Assessing student academic skills to select intervention (e.g., skill-by-treatment Interaction) has been investigated in previous research with promising results (Burns, 2011; Burns, Codding, Lukito, & Boice, 2010; Codding et al., 2007). Considering the factors that contribute to achievement, specifically in the area of mathematics is essential for educational practitioners to understand if we are to provide suitable interventions for low achieving students.

Low mathematics achievement is associated with the inability to understand and apply advanced problem solving strategies when completing mathematics problems (Geary, Hoard, & Byrd-Craven, 2004). Word problem solving is commonly used within content domains such as numbers and operations to measure mathematics knowledge because it also represents the simultaneous application of several areas of mathematics proficiency requiring students to demonstrate mastery of skills to make decisions as to the pertinent information needed to solve the problem, selection of appropriate strategies needed to help solve problems, and lastly derive the solution (Acosta-Tello, 2010; Jitendra, Sczesniak, & Deatline-Buchman, 2005), and is closely linked to overall mathematics achievement (Foegen & Deno 2001; Fuchs et al., 1994; Helwig et al., 2002).

Word problem solving is also a critical aspect of mathematics instruction and an important indicator of overall mathematics skill (National Council of Teachers of Mathematics [NCTM], 2000). However, word problem solving involves multiple skills and the inability to successfully complete word problems does not suggest which skill should be targeted for intervention. Mathematical content is often a balanced combination of procedure and understanding (CCSS, 2010; National Mathematics Advisory Panel [NMAP], 2008). If a student is successful with various word problem-solving mathematics tasks, then they likely are proficient in the underlying skills, but difficulty with word problem-solving mathematics tasks does not identify in which underlying skill(s) the student is deficient. Moreover, word problem solving involves the application of skills after they are initially learned and practiced to fluency, and the progress from acquire, to fluent, to application is a useful heuristic, called the learning hierarchy (Haring & Eaton, 1978), that can be used for intervention design for mathematics (Burns et al., 2010).

Study Purpose

The purpose of this study is to extend previous research by examining the Learning Hierarchy (LH) conceptual model as a framework for intervention design based on student performance and broaden the research base around the development and characteristics of problem solving for students with various levels of achievement. Specifically, the study examined differences of strategy use in solving word problems at various levels of achievement as well as in specific phases in the LH. First, students completed a battery of mathematics assessments targeting conceptual understanding, computational fluency, and application and word problem solving skills. Next, student performance on the measure of computational fluency was used to assign students to specific phase in the LH. Word problem solving responses were coded to allow further examination of accuracy and quality of strategy use among students.

Significance of the Study

The current study seeks to contribute to the current understanding of problem-solving strategies utilized by students at varying skill levels. Word-problem solving is an important component of mathematics instruction as well as a significant indicator of overall mathematics performance. Students are required to demonstrate word-problem solving skills in order to meet proficiency standards both in the classroom and on state accountability tests. For this reason, understanding and identification of strategy use while solving may be beneficial for identification and intervention of mathematical skills. Thus, this study seeks to examine performance of students at various levels in the LH to understand characteristics and development of word problem skills.

Research Questions

The following research questions guided the current study:

1. How well do data from measures of multiplication fact fluency fit the Learning Hierarchy?
2. To what extent does academic achievement predict accuracy of strategy used?
3. To what extent does academic achievement predict quality of strategy used?
4. To what extent does phase in Learning Hierarchy effect accuracy of strategy?
5. To what extent does phase in Learning Hierarchy effect quality of strategy?

Definitions

Learning Hierarchy (LH): A conceptual model developed by Haring and Eaton (1978) that suggests stages through which students advance when learning a new concept or skill, (a) acquisition, (b) fluency, (c) generalization, and (d) adaptation.

Acquisition Phase: The first phase in Learning Hierarchy, students acquiring a novel skill but are not yet accurate and task completion is slow.

Computational Fluency: Familiarity with symbols, rules, knowledge of procedures and skills, and having and executing methods flexibly, accurately, and efficiently to solve problems (Hiebert & LeFevre, 1986; Kilpatrick, Swafford, & Finell, 2001: National Council of Teachers of Mathematics (NCTM), 2003; Rittle-Johnson, Siegler, & Alibali, 2001). Computational fluency is also referred to as procedural knowledge or procedural fluency.

Fluency Phase: The second phase in the Learning Hierarchy, students acquired skill demonstrating accuracy but task completion remains slow.

Generalization Phase: The third phase in the Learning Hierarchy, students have acquired skill demonstrating accuracy and speed in task completion.

Mathematics Difficulty (MD): A broad construct that represents students identified as having difficulty in an area(s) of mathematics based upon average performance (e.g.,below the 35th percentile) on a standardized math assessment (Gersten, Jordan, & Flojo, 2005; Hanich, Kaplan, Jordan, & Dick, 2001; Mazzocco, 2007).

Strategy: Knowledge and familiarity with patterns of steps taking, including rules and reasoning processes within procedures and methods utilized to product solutions (Anghiler, 1989; Kilpatrick et al., 2001; Sherin & Fuson, 2003).

Word Problem Solving (WPS): Within the content domain of numbers and operations, tasks representing integration of computation and application of knowledge (i.e., both understanding and use of math concepts) requiring students to understand information, select and monitor solution plan, and derive procedural calculation (Acosta-Tello, 2010; Leh, Jitendra, Caskie, & Griffin, 2007; Mayer, 1999; Nathan, Long, & Alibali, 2002).

Delimitations

The following limitations were placed on the study:

(a) Study participants were limited to 3rd grade students from one suburban school district in the Midwestern United States. I chose third grade because it is the grade level at which number sense should be firmly established for most students, students are expected to fluently compute multiplication, and the instructional focus is on skills such as multiplication and division (Common Core State Standards Initiative, 2010).

(b) The study focused solely on multiplication, excluding addition, subtraction, and division operations. Previous studies have focused on computational fluency in addition and subtraction, however; there is less research on multiplication, specifically strategy use when solving problems.

(c) Selected strategies were coded based upon written responses only without additional prompting or questions to understand student strategy selection .

Organization of the Dissertation

This dissertation is organized around four additional chapters. Chapter 2 provides an overview of the literature relevant to (a) mathematical and strategic development, (b) mathematical problem solving and strategies, and (c) the Learning Hierarchy (Haring & Eaton, 1978) as a potential intervention heuristic. Chapter 3 outlines the methodology used in the current study, including description of participant characteristics, measures, procedures, and data analysis. Chapter 4 presents the results for each research question including several tables to aid in data interpretation. Chapter 5 includes a discussion of the study findings within the context of previous research, discusses results in terms of potential implications for theory, possible considerations for practice and future research as well as limitations for interpreting data.

CHAPTER 2

LITERATURE REVIEW

Chapter 2 presents relevant literature in mathematical development and is organized into three sections. The first section provides a discussion of the research base around mathematical development with a focus on computational fluency contributing to mathematical proficiency. The next section examines strategic development as it relates to computational fluency with multiplication and word problem solving. Finally, the last section discusses computation and strategies within mathematics word problem solving. The chapter concludes with a brief review of mathematics intervention heuristics, with specific focus on the Learning Hierarchy as a conceptual model and potential framework for identifying student skill levels and informing instructional decisions in the area of computational fluency strategy use for word problem solving.

Mathematical Development

Mathematical Proficiency in mathematics is fundamental for employment and higher educational goals (Ketterlin-Geller, Chard, & Fien, 2008). The development of mathematics competence during elementary school influences success in middle and high school, which determines higher education opportunities and employment (Fuchs, Fuchs, & Courey, 2005). Unfortunately, a majority of students are still not proficient in mathematics with minimal growth toward the mathematics proficiency goal (National Assessment of Educational Progress, 2011; Ketterlin-Geller et al., 2008), which is concerning because comprehension of higher level mathematics material may be hindered by failure in basic mathematics skills (Gersten & Chard, 1999; Codding, Shiyko, Russo, Birch, Fanning & Jaspen, 2007).

The National Research Council (NRC; Kilpatrick, Swafford, & Findell, 2001) identified the five interwoven and interdependent strands of mathematical proficiency as (a) conceptual understanding, (b) procedural fluency (i.e., computation fluency), (c) strategic competence, (d) adaptive reasoning, and (e) productive disposition. The importance of two critical components; conceptual and procedural understanding in mathematics competence has been firmly established within developmental and cognitive research (Canobi, 2004; Hiebert & Wearne, 1996; Rittle- Johnson, Siegler, & Alibali, 2001).

Theoretical viewpoints on the causal interrelations of conceptual and procedural knowledge have added complexity to understanding development of mathematics proficiency. Many theories of the development of conceptual and procedural knowledge have focused on which type of knowledge develops first (Rittle-Johnson, Siegler, & Alibali, 2001). The developmental precedence of conceptual knowledge, suggested by a *concepts-first* model proposes children are either born with or initially develop conceptual knowledge within a given domain and then use this knowledge to develop procedural skills (Geary, 1994; Gelman & Williams, 1998). Theories opposing this view suggest conceptual knowledge develops after acquisition of procedural knowledge. *Procedures-first* theories purport children learn problem solving procedures within a domain which assists with development of key concepts and conceptual knowledge (Karmiloff-Smith, 1992; Siegler & Stern, 1998).In contrast to these theories, an iterative model of development has been proposed which suggests conceptual and procedural knowledge develop simultaneously with increases in one knowledge resulting in increases in the other type of knowledge (Rittle-Johnson et al., 2001; Rittle-Johnson & Koedinger, 2009). Competence in mathematics requires children developing and connecting knowledge of concepts and procedures (Silver, 1986). In the following section, I will define conceptual knowledge and procedural/ computational fluency.

Conceptual knowledge. Conceptual knowledge in mathematics consists of the understanding of relationships between pieces of information and building relationships between existing pieces of knowledge (Hiebert & LeFevre, 1986; Hiebert & Carpenter, 1992). VanDeWalle and Lovin (2006) considered conceptual knowledge to be part of a network of ideas consisting of logical relationships constructed internally, which Goldman and Hasslebring (1997) called a "connect web" (p.4). Furthermore, understanding the principles and interrelations between units of knowledge of a domain whether implicit or explicit has been used to define the deep understanding of the meaning of mathematics considered to be conceptual knowledge (Rittle-Johnson et al., 2001).

There is less consensus regarding how to assess conceptual knowledge. For example, some researchers have developed conceptual-based mathematics CBM (Helwig, Anderson, & Tindal, 2002), but those measures examined application more than understanding the underlying concepts (Zaslofsky & Burns, 2014). Assessing knowledge of underlying mathematics principles (e.g., inversion and commutative property) has been used to measure conceptual knowledge (Canobi et al., 1998; Geary, 2006), but students may be able to employ and continuously use strategies related to mathematics principles without a developed conceptual understanding (Canobi, 2009). Finally, other approaches that involved interviewing students (e.g., asking them to think aloud as they work or to draw and explain the problem) have been used to assess conceptual understanding, but those approaches were difficult to implement and often resulted in unreliable data (Ginsburg, 2009).

Computation fluency. Computational fluency sometimes referred to as procedural fluency or procedural knowledge in mathematics appears as a seemingly straightforward construct, but it is defined differently by researchers. Hiebert and Lefevre (1986) were likely the first to define it by calling it a familiarity with symbols and the rules for writing symbols, which was perceived to be a surface awareness rather than knowledge of meaning. Rittle-Johnson and colleagues (2001) indicated that computational fluency consists of executing action sequences to solve problems, is tied to specific problems types and not generalizable. As a strand of mathematical proficiency, procedural fluency concerns " the knowledge of procedures, knowledge of when and how to use them appropriately, and skill in performing them flexibly, accurately, and efficiently (Kilpatrick et al., 2001; p. 118) . Definitions vary somewhat, they all involve knowledge of rules and procedures required for mathematics processes, and all encompass familiarity with symbols and rules or procedures (Zamarian, Lopez-Rolon, & Delazer, 2007).

Computational fluency is considered to be not generalizable due to ties to specific problem types (Rittle-Johnson et al., 2001). However, computational fluency is easily observable and measurable as a discrete skill and therefore frequently used in assessment and intervention. The National Council of Teachers of Mathematics (NCTM) suggests computational fluency, "requires a balance and connection between conceptual understanding and computational proficiency" (NCTM, 2000, p. 35) and includes a deeper understanding.

Previous research has often equated procedural knowledge with rote knowledge or computation drill and practice and considered it an impediment for acquiring conceptual knowledge, especially for students struggling with mathematics (Baroody & Hume, 1991; Jitendra, DiPipi, Perron-Jones, 2002). However, Price, Mazzocco, & Anasari (2013) examined early mathematics skills as a predictor of later academic achievement and data suggest mechanisms associated with procedural calculation of arithmetic problems are related to high school level mathematics competence. In addition to conflicting results and opinions, a "uniform and clear cut" (Baroody et al., 2007, p.119) definition or operationalization of both conceptual and procedure knowledge has yet to be established. Despite disagreement, there is strong empirical support for conceptual and procedural knowledge to be considered interdependent (Canobi & Bethune, 2008; Rittle-Johnson & Alibali, 1999), following a pattern of iterative development (Rittle-Johnson, Siegler, & Alibali, 2001), and critical to mathematic learning and proficiency (Kilpatrick et al., 2001).

Computational fluency has been a focus in instruction and research of various areas of mathematics content; basic and multi-digit arithmetic computation (Foegen, Olson, & Impecoven-Lind, 2008; Rittle-Johnson et al., 2001; Woodward, 2006), fractions

(Bailey, Hoard, Nugent, & Geary, 2012). Automaticity with basic skills is considered as a critical component of academic success within mathematics and directly influences computational fluency (Woodward, 2006). Students need a level of automaticity and fluency that allows them to progress toward achieving proficiency and goals such as benchmarks articulated in the standards. If a student is able to quickly and accurately retrieve a mathematics fact, he or she is better able to access working memory resources, reducing cognitive load (Delazer et al., 2005). Further, this automaticity is particularly important in mental computation, estimation, and approximation skills (Woodward, 2006).

The nature of procedural fluency/computational fluency is more than simple recall and instead consists of knowledge and skill representing the ability to act upon and transform information (Anderson, 1983; Gagne, 1985; Star, 2007). When procedures are well learned they can be accessed and performed rapidly, with little conscious effort, later appearing automatic (Derry, 1990). Acquiring computational fluency is a process that begins slowly, characterized by low accuracy and speed and dependent on practice for improvement. Fluency requires more than memorization, however; often students memorize a response without actually understanding the concept or procedure leading to a correct answer. Computational fluency involves thinking and requires students to know when not just how to use a procedure as well as not just accuracy but flexibility and efficiency (Star, 2007). Computational fluency is important and a component of mathematical proficiency, however; effort to understand development of computational fluency is necessary to inform instruction and intervention.

Fluency through strategies. Computation fluency is frequently evaluated by the accuracy and speed with which problems are solved, however; increased attention to the solution procedure as an indicator of competence has been evident in the literature (LeFevre & Bisanz, 1996; Mabbot & Bisanz, 2003; Siegler, 1988). Multiple procedures to solve multiplication problems are commonly used by children (Siegler, 1988) as well as by adults (LeFevre & Bisanz, 1996) and influence accuracy and fluency of problem solving (LeFevre, Bisanz, Daley, Buffone, Greenahm, & Sadesky, 1996). A practice approach emphasizing “thinking strategies” (Kilpatrick et al., 2003, pg. 193) may improve student achievement with single-digit calculations by supporting a concurrent development of computation and understanding (Rittle-Johnson & Siegler, 1998). Moreover, “the integration of conceptual and procedural knowledge permits greater flexibility in the intervention and use of procedures and strategies” (Baroody, 2003; pg. 15).

Mastery of basic operation facts, such as single digit multiplication are essential and the basis for applications related to other content areas (e.g., money and time) as well as problem solving and abstract thinking (Codding, Shiyko, Russo, Birch, Fanning, & Jaspen, 2007; Shaprio, 2006), and much like the automatic process theory of reading (Samuels, 1987), computing basic mathematics fact operations without devoting significant cognitive resources to the task allows students to dedicate resources to more advanced applications within problems. Fluency and proficiency of mathematics facts are consistent difficulties for students struggling to learn mathematics (National Mathematics Advisory Panel, 2008). Compared to decoding in reading, proficiency with mathematics facts, reduces the cognitive load allowing attention to higher cognitive demands (Clarke, Doabler, & Nelson, 2014)

The National Council of Teachers of Mathematics (NCTM, 2006) as well as National Research Council (NCR, 2001) have drawn attention to the importance of basic fact mastery based upon research suggesting students who have automaticity of basic facts are better able to comprehend mathematical concepts as well as problem solving approaches presented within mathematics curriculum. A majority of research has focused on the conceptual and procedural relationship and although important, the framework of mathematical proficiency presents as five interdependent strands. Therefore, attention and further examination of the iterative development of other strands such as procedural fluency and strategic competence may have implications for identification of difficulty and interventions to assist students in acquiring mathematical proficiency.

Synthesis

The NRC (2001) proposed a framework of mathematics proficiency comprised of five intertwined strands which has been cited extensively within current research. Although presented as a multi-dimensional construct, research has consistently emphasized only the constructs of conceptual understanding and procedural fluency as critical for mathematics achievement with a paucity of research on the development of the remaining three strands. Additionally, common characterizations of procedural fluency are limiting and may impact how to define, instruct, and evaluate the construct individually and within the mathematics proficiency framework (Star, 2007; Rittle-

Johnson & Star, 2007; Baroody, 2007). Procedural fluency evaluated by the accuracy and speed with which problems is evident in research, however; attention to and further examination of solution procedures as an indicator of competence may provide a more comprehensive understanding of the development of procedural fluency (Mabbot & Bisanz, 2003). Moreover, continuing to examine the iterative development of each strand within the framework, may assist educators in supporting student development of mathematics proficiency.

Strategic Development

Strategic competence has been described as " the ability to formulate mathematical problems, represent them, and solve them" (Kilpatrick et al., 2003, pg. 124). As stated previously, conceptual and procedural knowledge influence understanding and competence in mathematics. Procedural fluency in computation is increasingly intertwined with strategic development; progressing from effortful procedures to increasingly efficient procedures while maintaining accuracy when solving basic arithmetic problems (Lemaire & Siegler, 1995).

Development of computation fluency with strategic competence has been demonstrated to follow a process where basic computational calculations are solved using procedural strategies with a gradual shift toward being solved by retrieval, a more efficient strategy (Ashcraft, 1992, Geary, 1991; Price et al., 2013; Siegler, 1988). Additionally, changes in efficiency are suggested to occur due to children using memory based procedures more frequently and with practice, reduced execution time of strategy (e.g., counting) and retrieval of information from memory (Delaney et. al., 1998; Geary et al., 1996; Lemarie & Siegler, 1995). Kilpatrick et al (2001) state,“ When instruction emphasizes thinking strategies, children are able to develop the strands of proficiency in a unified manner” (p 7).

Previous research findings (e.g., Geary, 2004; Jordan, Kaplan, & Hanich, 2002) found that students with mathematics difficulties demonstrated developmentally different characteristics, including persistent deficits in mastering arithmetic combinations and retrieval combinations due to immature strategies resulting in difficulties developing computational fluency and problem solving (Fuchs et al., 2005; Jordan et al., 2002). Teaching effective strategies (e.g., mature and efficient) to struggling students to improve mastery and fluency is an important contributor to a student’s ability to problem solve (Baker, Gersten, & Lee, 2002; Gersten, Jordan, & Flojo, 2005).

Research in the areas of addition and subtraction suggests strategy instruction may assist students with organizing facts thus facilitating retention and direct recall (Baroody & Ginsbury, 1986; Fuson, 2003; Issacs & Carroll, 1999) with recent work in multiplication (Sherin & Fuson, 2005). Although the process of learning multiplication is comparable to some aspects of learning addition, multiplicative thinking has been suggested as clearly distinguishable from additive thinking because the meaning of the numbers is different (Clark & Kamii, 1996). A brief description of models of multiplication that may contribute to strategy development and computational fluency will be provided.

Models of Strategy Development

A number of models for strategy development with both adult and children have been proposed, all of which represented knowledge of multiplication as an associative network that links problem and answer (Ashcraft, 1987; Campbell, 1987; Siegler, 1988; Sigler & Shipley, 1988; Siegler & Shrager, 1984). A Multiple-procedure model, also referred to as a strategy choice model (Siegler, 1986), is a common model of multiplication and suggests a variety of process are used in addition to direct retrieval to solve cognitive tasks (Geary et al., 1993; LeFevre et al., 1996; Siegler, 2007). Adaptive use of alternative strategies when problem solving has been suggested to contribute to mathematical skills. Geary and Burlingham-Dubree (1989) found an adaptive strategy choice, as defined by Lemaire and Siegler (1995) as choosing the strategy that leads fastest to an accurate answer, on addition problems by students moderately correlated ($r = .71$, $p < .01$) with performance on increasingly complex mathematical achievement measures.

Sigeler and Shrager (1984) developed the Distribution of Associations model to account for strategy choices in addition and subtraction facts and hypothesized application to many tasks including multiplication. This model suggests the processes by which a strategy is chosen, for examplean immature strategy such as counting instead of a more mature strategy of retrieval depends on the association of problem, strategy used, errors, and correct responses. Following the Distribution of Association model, Sigler and Shipley (1995) proposed a multiple route model of simple addition, called Adaptive Strategy Choice Model (ASCM), that they suggested was applicable to multiplication.

The ASCM was proposed as a computer simulation of strategic development suggesting strategies use problems to generate information related to the production, speed, and accuracy of an answer. The model is an extension of the previously the proposed Distribution of Associations model (Siegler, 1988; Sigler & Shradger, 1984) in which the selection of procedure depended on three types of associative strengths: (a) global strength in which success of each procedure across problems with same operands; (b) featural strength in which effectiveness of procedure on problems with same features (e.g., same operand); and (c) problem specifics in which success of each procedure on each problem. All three strengths are assumed to change with development and practice and influence problem solving.

Understanding variability beyond general features of learning (i.e., performance increases in speed and accuracy with experience) calls for consideration of underlying changes such as strategy preference, frequency, performance, and selection that may be of importance for understanding how children learn (Siegler, 2007). Lemaire and Seigler (1995) conducted a longitudinal investigation of second grade students' acquisition of single digit multiplication skill by assessing speed, accuracy, and strategy use. Early accuracy of back up strategies (e.g., repeated addition) and retrieval strategies in first session moderately correlated with later accuracy of retrieval strategy (r= .69, .71, p < .05). Additionally, early incorrect use of back up strategy was moderately correlated with later frequency of use of back up strategy (r = .63, p < .05). Results supported the adaptive strategy choice model and suggested increases in accuracy and speed that influence learning also reflects changes in strategic competence (i.e., acquisition of novel strategies, efficiency, execution, and adaptive flexibility in strategy use). The study also reported use of multiple strategies aligned with problem features and individual differences in student strategy competence.

Evidence on multiple solution routes in simple arithmetic including addition and multiplication suggests children not only acquire but demonstrate variation in procedures to solve simple problems (Cooney & Ladd, 1992; Lemaire & Siegler, 1995). Siegler (1988) investigated strategy use by children solving simple arithmetic problems and found that children in Grades 2 and 3 used retrieval on 68% of trials, repeated addition on 22% of trials, problem decomposition on 5% of trials, and counting sets or objects on 4% of trials. Cooney and Ladd (1992) found similar results for students in grades 3 and 4 used direct retrieval to solve 55% and 74% of problems respectively and used repeated addition and a form of decomposition (e.g., derived facts; solving 8 x 9 as [(8 x 10) – 8]) on all other problems.

The notion of multiple solution routes within children's multiplication performance is further supported by research in which adults were found to use retrieval on only 71% of single digit addition problems and counting procedures or other decomposition procedures on the remaining 19% of problems (LeFevre et. al., 1996). Additionally, students with varying levels of ability and skill demonstrate individual differences in strategy use. For example, Mabbot and Bisanz (2003) classified students into three distinct groups based on performance of multiplication computation and found group difference on items direct retrieval and back up strategies were used. The most prevalent strategy types included and studied within research will be discussed.

Strategy Types

Research on development of strategies for single-digit addition is in general agreement on types of strategies and the terminology for describing these types (Fuson, 1992; Sherin & Fuson, 2005; Sigeler, 2007). Research on development of strategies for multiplication of single-digit numbers is less widely studied and although there is a growing body of research, there is inconsistency of types and terminology of strategies used within the literature (Anghileri, 1989; Siegler, 1988; Sherin & Fuson, 2005). Researchers generally agree that students progress through the use of various computation methods, which may or may not include strategies such as counting, repeated addition, decomposition, or a hybrid of strategies when learning multiplication (Anghileri, 1989, Mulligan & Mitchelmore, 1997; Sherin & Fuson, 2005, Siegler, 2007).

In an attempt to construct a more consistent and standardized strategy list, Sherin and Fuson (2005) proposed a taxonomy of strategies for single-digit multiplication based upon a synthesis of existing research at the time as well as their own data and analyses. Siegler (1988; 1995; 2007) synthesized previous research and proposed a taxonomy of strategies that has been utilized in research frequently to account for the performance of children (Baroody, 1994, Geary Lemaire & Siegler, 1995; Xin et al., 2014) and is used within the current study.

Acquisition of multiplication skills is dependent on knowledge of addition and counting (Geary, 2000; Siegler, 1988) and proficiency with multiplication is developed over time with students learning facts using more basic methods progressing toward more advanced methods (Cooney et al., 1988 & Kilpatrick et al., 2001). When first learning to multiply, strategies used by many children include unitary counting, repeated addition/counting by with later progression to decomposition and direct retrieval. A brief description of each strategy will be provided.

Counting. Counting strategies are considered the slowest and least accurate of strategies that children use to solve arithmetic problems. Barrouillet, Mignon, & Thevenot (2008) examined strategies in subtraction problem solving in children and suggested the algorithmic strategy of counting used may prevent reinforcement of facts. Students directly model the problem, sometimes by representing the multiplicand by sets of tally marks and then counting each individual tally mark or with pictures representing sets and objects in each set. Although basic, this modeling of a problem demonstrates a connection between conceptual understanding, procedural knowledge, and strategic competence and allows an opportunity for practice while building understanding and fluency (Kilpatrick et al., 2001).

Repeated addition. Repeated addition involves representing the multiplicand (first number) the number of times indicated by the multiplier (second number) and then adding these numbers. For example to solve 4 x 6, a student would set up the problem as 4 + 4 + 4 + 4 + 4 + 4 = and then add the digits to find the sum of 24, which is the product of 4 x 6. Finding the product may rely on a student's competency with counting by digits such as 2s or 5s. Although multiplication is commonly thought of as repeated addition, research shows that multiplication requires higher-order multiplicative thinking, which develops out of but not equal to knowledge of addition (Clark & Kamii, 1996). A common error in the execution of the repeated addition strategy involves adding the multiplicand too many or too few times.

Decomposition. More mature strategies such as decomposition involve use of rules and/or derived facts to solve problems. This strategy relies on a student's retrieval of specific multiplication facts from memory, including doubles or facts with factors of 2 or 5. Siegler (1988) suggests doubles facts (e.g. 2 x 2) are memorized before other combination, similar to as in addition and that answers to doubles facts are the initial step in decomposition strategies. For example, if attempting to solve a problem such as 6 x 7, a student may know the doubles fact of 6 x 6 = 36 and then add an additional 6 to that product to find the answer of 42. Errors using this strategy often occur when students retrieve an incorrect answer to a supposedly known fact or incorrectly add to the product.

Direct retrieval. Learned associations of pairs of factors with their product, where response is rapid with no visible computation is identified within strategy taxonomies as direct retrieval (Lefevre et al., 1996; Lemaire & Siegler, 1995; Siegler, 1988), known fact (Anghileri, 1989; Mulligan & Mitchelmore, 1997), or learned product (Sherin & Fuson, 2005). This strategy is suggested to become more prevalent during development due the strength of association with problem and answer which increases with practice (Lemaire & Siegler, 1995; Siegler, 1988).

Sherin and Fuson (2005) propose that the terms "fact" and "retrieval" are misleading and suggest an overly simplistic description of this strategy. This may be of concern regarding interpretation of findings in that, strategies reported by children as direct retrieval may actually just be other effectively and efficiently used strategies and as

Siegler (1987) suggests may not be an accurate reflection of retrieval of the association between problems and answers.

Multiple studies suggest that students and even adults are using a variety of strategies beyond memorization when providing accurate and quick responses of multiplication facts, even within drill instruction (Baroody, 2006; Cooney et al., 1988). Furthermore, students use different strategies throughout their development of computational fluency as well as during other tasks (e.g., word problem solving) and strategies will vary within and across classrooms dependent on instruction and student development (Sherin & Fuson, 2005). Proficiency with multiplication involves more than stating a procedure and students practicing for memorization. Students need opportunity to practice methods as well as learn and use concepts to increase their mathematical proficiency (Kilpatrick et al., 2001).

Synthesis

The development of computational fluency and strategy competency is suggested to progress from effortful procedures gradually shifting toward increasingly efficient procedures (Lemaire & Siegler, 1995). Models of strategy development in addition have been proposed to account for performance of students learning with general agreement on type and terminology (Sherin & Fuson, 2005; Siegler, 1988; Siegler & Shrager, 1986). Despite, initial agreement that students progress through the use of various computation methods, the development of strategies for multiplication is less widely studied and further complicated by inconsistency of strategy types and terminology within the literature (Sherin & Fuson, 2005). Additionally, the extent to which previous findings apply to students with math difficulties is not clear. Examining variability in student strategy use in multiplication may be of importance for understanding student development of proficiency in multiplication.

Mathematic Problem Solving and Strategies

Word problem-solving measures have been suggested to be useful indicators of mathematics proficiency (Jitendra,Sczesniak, & Deatline-Buchman, 2005) and require application of mathematic concepts and skills. Below I will present and discuss strategy use and development as a potential factor contributing to mathematics word problem solving proficiency.

Application

Application is broadly defined as a means of utilizing a procedure in a context, a mechanism for exhibiting conceptual understanding and as the utilization of knowledge necessary for using skills in multiple settings (Howell &Nolet, 2000; Krathwohl, 2002; Hudosn & Miller, 2006; Kelley, 2008). Application and problem-solving are often paired because the skills and knowledge appear to have an iterative relation, with inclusion of both in the definition of each. Problem solving has been described as having two specific aspects; deciding what to do and how to do it (Kelley, 2008), but problem solving has also been described as the application of strategies, skills, and knowledge or as the amalgamation of application and computation knowledge (Fuchs et al, 2004; Hudson & Nolet, 2000).

Development of early mathematics proficiency is associated with ability to solve a variety of increasingly complex mathematical problems, educational outcomes such as graduation, and successful independent living (Kilpatrick et al., 2001; Patton, Cronin, Bassett, & Koppel, 1997). The ability to solve mathematical word problems is considered an essential component of mathematics competency and requires the ability to utilize basic mathematical skills.

Word problem solving. Mathematical content is often a balanced combination of procedure and understanding (CCSS, 2010; National Mathematics Advisory Panel [NMAP], 2008). If a student is successful with various word problem-solving mathematics tasks, then it is suggested that the student likely be proficient in the underlying skills, but difficulty with word problem-solving mathematics tasks does not identify in which underlying skill(s) the student is deficient. When students are struggling in mathematics, areas of difficulty within problem solving can be ambiguous. Kingsdorf and Krawec (2014) used Mayer's (1985) model of the problem solving to examine word problem solving across students with and without learning disabilities. Mayer's model suggests there are four phases within problem solving: (a) problem translation, (b) problem integration, (c) solution planning, and (d) solution execution.

The first two phases are categorized under *problem representation*. There has been a significant increase in research focusing on *problem translation* which requires comprehension of what the problem is saying as well as *problem integration* which requires ability to mathematically interpret the problem components to form a representation (Fuchs & Fuchs, 2002, Jitendra et al., 2007; Montague & Applegate, 1993). Although word problems require interpretation and analysis, solution accuracy requires computational proficiency. The second general phase includes the *planning phase* which is related to knowledge of strategic planning demonstrated by skill of determining operations to use and the *execution phase* which is related to algorithmic understanding demonstrated by skill of completing the computation.

Mayer's model applied within research has illustrated how each phase of the problem solving process is complex and the correct answer is dependent on the accuracy of each preceding phase (Jitendra et al., 2005;2007). Students struggling with skills associated with word problems solving has been suggested to be reflected in achievement test performance, however; due to the broad skill sets within achievement tests, the component skills are rarely examined therefore only providing general areas of weakness versus specific skill difficulties (Cirno & Berch, 2010).

Application through word problem solving. Low mathematics achievement is associated with the inability to understand and apply advanced problem solving strategies to solve mathematics problems (Geary, Hoard, & Byrd-Craven, 2004). Word problem solving is commonly used within content domains such as numbers and operations to measure mathematics knowledge because it also represents the simultaneous application of several areas of mathematics proficiency requiring students to demonstrate mastery of skills to make decisions as to the pertinent information needed to solve the problem, selection of appropriate strategies needed to help solve problems, and lastly derive the solution (Acosta-Tello, 2010), and is closely linked to overall mathematics achievement (Foegen & Deno 2001; Fuchs et al., 1994; Helwig et al., 2002).

Word problem solving is a critical aspect of mathematics instruction and an important indicator of overall mathematics skill (National Council of Teachers of Mathematics [NCTM], 2000). However, word problem solving involves multiple skills including concepts and practiced skills (e.g., foundational skills) and the inability to successfully complete word problems does not suggest which skill should be targeted for intervention.

The National Mathematics Advisory Panel Report (U.S. Department of Education, 2008) emphasizes the importance of evaluating child learning outcomes. Identifying critical skills and their expected time of development can assist with development of benchmarks which can be used to evaluate student learning to ensure instruction is contributing to continued growth and mastery of fundamental skills and concepts (VanDerHeyden, 2010). In the area of mathematics, development of computational fluency skills is considered critical and generative, meaning that mastery of these skills is highly related to improved functioning across numerous contexts and these skills have been suggested as useful indicators for learning within mathematics (Burns & Klingbeil, 2010; VanDerHeyden, 2010). Development and learning outcomes are frequently evaluated relative to trajectories of students who are not at risk for mathematical difficulties or poor learning outcomes, however; development and learning outcomes of students who are struggling is important to understand in order to inform intervention.

Strategies and Mathematics Word Problem Solving Proficiency

Strategy use may be an important factor in explaining differences in problem solving among various levels of mathematics achievement. Previous studies have examined how multiplication problem solving strategies develop among average achieving students (Anghileri, 1989; Lemaire & Siegler, 1995; Mulligan & Michelmore, 1997; Park & Nunes, 2000; Sherin & Fuson, 2005; Zhang et al., 2011; 2014) and students with mathematics difficulties (Woodward, 2006). Students use several multiplication strategies initially relying on basic counting strategies, but then progress to strategies considered more mature and requiring fewer steps than counting and finally to direct use/learned product of known facts (Lemaire & Siegler, 1995; Mabbott & Bisanz, 2003; Van der Ven, Boom, Krosenbergen, & Leseman, 2012). Studies on mathematical problem solving strategies have predominately focused on accuracy, with later focus on flexibility and quality of strategies. I will discuss these three concepts below.

Accuracy. Accuracy is freedom from mistake or error (i.e., correctness), the ability to work or perform without making mistake, and according to Haring and Eaton (1978), accuracy is the first goal of instruction. Research has demonstrated increasing opportunities for accurate academic responding can lead to increases in skill development (Skinner, 1998). Accuracy has been included as a criterion to determine instructional level in mathematics (Gickling & Thompson, 1985; Burns et al., 2006), previous research found a large effect size (d = 1.19, SD = .37) for academic tasks completed with 90% accuracy, and it has been used as a criterion in previous research to judge mastery of skills (Burns, 2004; 2011; Burns et al., 2015). However; the way accuracy is measured has frequently included a fluency component and previous research found fluency data were psychometrically preferable to the accuracy data (Burns, 2004).

Flexibility. Many mathematics curricula reforms have flexibility as a central goal with expectations that students develop, understand, and use multiple strategies for solving problems. However, research on flexibility with mathematical procedures is limited despite its inclusion in the definition of procedural fluency. In relation to the use of mathematic procedures, flexibility is knowledge of and ability to use multiple solution methods across a set of problems in a domain (Rittle-Johnson & Star, 2007; Star, 2005). Flexibility in strategy use rather than consistent use of a single procedure to solve a class of problems has been suggested as a potential indicator of mathematical competence (Dowker, 1992). Rittle-Johnson and Star (2007) found students engaged in comparing different solutions for the same equation made greater gains in flexibility than students who used the same solution. In fact students who compared multiple methods were more capable of using multiple methods to solve the same equation as well as exhibit more efficient methods for solving equations.

Quality. Problems can be solved in a variety of ways, but not all ways are considered equal with regards to efficiency or appropriateness (Star & Newton, 2009). Quality refers to the ability to solve a problem efficiently and effectively, using more mature methods and choosing strategically from among methods so as to reduce computation demands (Berk, Taber, Gorowara & Poetzl, 2009; Star & Siefert, 2006). Empirical evidence suggests less mature strategies require more computational demands (e.g., steps and numbers to keep track of) than the more mature strategies increasing difficulty for students with poor working memory (Hecht, 2002) and low mathematics achievement (Geary & Hoard, 2005).

Although fluency and automaticity through direct retrieval of facts are important when applying computational skills to other areas of mathematics, back up strategies within intervention and instruction need to be considered. If a student has only memorized without progress through a continuum of strategies and forgets the fact, the student is left with no way to solve the problem. Due to difference in development of computational fluency the instruction and intervention children require may differ in order to become fluent at retrieving answers to basic mathematics facts and correctly solving word problems.

In multiplication, students frequently use repeated addition or skip counting as an initial procedural strategies for solving facts and with additional practice, typically developing students establish an association with each fact, recalling it automatically instead of calculating (Siegler, 1988) .This association with facts contributes to automaticity, for example, Kirby and Becker (1988) found reduced automaticity in basic operations and strategy use, including use of inefficient or inappropriate strategy resulted in increased difficulties in mathematics problem solving, therefore requiring extensive practice in the correct strategies.

Strategy Intervention

Interventions for students struggling are needed for as students with mathematics difficulties increase in age, the discrepancy in ability to recall basic facts increases when compared to students without mathematics difficulties (Hasselbring et al., 1988).

Additionally, students with mathematics difficulties do not spontaneously produce task appropriate strategies necessary for adequate performance leading to the need for direct and explicit instruction before they show signs of performing strategically. Torbeyns, Verschafell, and Ghesiquiere (2005) examined application of school taught strategies by first grade students of different mathematical achievement levels and found high-achieving students applied strategies more efficiently than lower achieving students. Moreover, Geary et al. (1993) suggest students with mathematics difficulties do not develop sophisticated fact strategies naturally and empirical research on strategy instruction in mathematics facts for students with mathematics difficulties is limited with varied outcomes with regard to the development of fluency (Morin & Miller, 1998; Tournaki, 2003).

Cummings and Elkins (1999) indicated that teaching facts to academically low achieving students should consist of strategy instruction integrated with timed practice drills and that teaching strategies helps to increase a student's flexible use of numbers but does not necessarily lead to fluency. Students may vary in computational proficiency skills and/or have difficulty for different underlying reasons and so it may also follow that they respond differently to intervention (Powell, Fuchs, Fuchs, Cirino, & Fletcher, 2009), therefore "additional research should be undertaken on the nature, development, and assessment of mathematical proficiency" (Kilpatrick et al., 2001, p.14).

Basic facts are often taught through practice, drills, and memorization (Brownell & Chazal, 1935; Kilpatrick et al., 2001). Drill and practice interventions have demonstrated promise in improving recall of basic facts (Burns, 2005; Van Houten &

Thompson, 1976; Skinner, McLaughlin, & Logan, 1997). Students at risk for or with mathematics difficulties demonstrate a reduced ability to automatically retrieve number fact answers in early elementary, as well as use immature counting procedures, are slower to produce answers, and demonstrate reduce accuracy in provided responses while continuing to demonstrate reduce efficiency in retrieving number facts in middle and late elementary grades (Geary, Hamson, & Hoard, 2000; Jordan & Hanish, 2000; Mabbott & Bisanz, 2008). Some research suggests students often use strategies other than “just knowing”/direct retrieval when solving basic facts. For example, a study of 4-12 year old students’ understanding of multiplication reported only above average children use direct retrieval, whereas 81% of test items solved showed evidence of students using multiple strategies (Anghileri, 1989; Stell & Funnell, 2001).

Empirical evidence suggests practice will have its greatest effect when basic facts are continually linked to meaningful examination of patterns and strategies instead of in isolation (Sherin & Fuson, 2005). Additionally, the National Council of Teachers of Mathematics (NCTM, 1989) emphasized importance of the “right conditions” when using practice designed to improve accuracy or fluency. They suggested “practice with facts should only be used after children have developed an efficient way to derive the answer” (NCTM, p. 47).

Synthesis

Word problem solving is an aspect of mathematics instruction frequently used within number and operation application tasks and is associated with overall mathematics proficiency and achievement (Acosta-Telo, 2010; Helwig et al., 2002; Jitendra et al., 2007; NCTM, 2000). Students employing meaningfully acquired strategies to solve mathematics tasks is associated with higher mathematics achievement (Baroody & Dowker, 2003; Geary et al., 2004; Woodward, 2006; Siegler, 2007; Zhang et al., 2013). Further examination of strategy development with students at different levels of instruction and achievement may improve understanding and be an important factor in explaining differences in mathematics proficiency in various domains. A more comprehensive understanding of student strategy development may possibly aid in improving instruction and intervention to assist students reaching mathematics proficiency.

Mathematics Intervention Heuristic

Intervention selection is more likely to be successful when it is based on an evidence-based heuristic that can correctly target and predict intervention effectiveness to meet students need based on student performance (Kavale & Forness, 2001). Assessing student academic skills to select intervention (e.g., skill-by-treatment Interaction) has been investigated in previous research with promising results (Burns, 2011; Burns et al., 2010; Codding et al., 2007). Consideration of factors that contribute to achievement, specifically in the area of mathematics is essential for educational practitioners to understand if we are to provide suitable interventions for low achieving students.

The five strands of mathematics proficiency include; conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition with the later three dependent on conceptual understanding and procedural fluency (National Research Council, 2001).The importance of two critical components;

conceptual and procedural understanding in mathematics has been firmly established (Canobi, 2004; Hiebert & Wearne, 1996; Rittle- Johnson, Siegler, & Alibali, 2001). Burns & Klingbeil (2010) suggest that the distinction between conceptual and procedural understanding could provide the basis for a skill by treatment interaction.

Numerous research studies have classified students into groups based upon performance on measures of achievement, however; standardized norm referenced measures may lack instructional relevance (Ysseldyke & Bolt, 2007). Mathematical difficulty is frequently operationalized as low mathematical performance on standardized tests of mathematics (Fuchs et al., 2010; Geary, Hoard, Byrd-Craven, Nugent, & Numtee, 2007; Zhang, Xin, Harris, & Ding, 2014), but cutoff scores for low performance used in research ranged from the 40th percentile to the 5th percentile (Geary et al., 2007). This broad range of scores classified as “low performance” may not be an accurate representation of student knowledge and skills as well as impact intervention outcomes.

Kanive and Burns (2015) examined the relationship between conceptual knowledge, computational fluency, and word problem solving with third-grade students. Data in the study were divided into group based on achievement level score (below or above the 25th percentile).

Results indicated conceptual and procedural measures were significantly related to word problem solving skills, however; mathematical achievement seemed to affect the relationship. The correlation with word problem solving was $r = .36$, $p < .001$ for a conceptual understanding measure and $r = .50$, $p < .001$, for a computational fluency measure.

Data were further analyzed with a regression model that also included mathematics skills levels as determined by a measure of achievement and the interaction of the skill level with both measures. The overall model was significant and accounted for a large percentage of the variance with all variables significantly related to word problem solving ($R^2 = .47$). Skill group accounted for a significant amount of variance of the word problem solving score and interaction between skill group and conceptual and procedural measures was significant as well. For students scoring below the 25th percentile, the conceptual measure accounted for 40 % ($r = .63$, $p < .05$) of the variance in word problem solving. For students above the 25th percentile, computational fluency accounted for 18 % of the variance of the word problem solving score, compared to the conceptual measure accounted for 6 % of the variance of the word problem solving score.

Siegler (1988) examined individual differences in first graders' strategy choices in addition and subtraction based on knowledge of problems and thresholds for stating retrieved answers. A cluster analysis classified students into three groups (not-so-good, good, and perfectionist). Students in the perfectionist group had good knowledge of problems and set very high thresholds resulting in 92% correct retrieval, whereas students in the good and not-so-good groups had good knowledge of problems but set lower thresholds or limited knowledge of problems but set low thresholds respectively. Educational implications from this study suggest it may be useful to teach children, particularly those in the not-so-good group to more accurately execute backup strategies. Children using back up strategies such as counting on fingers are often discouraged by parents and teachers (Siegler & Sharager, 1984), however; teaching children back up strategies resulting in a correct answer affords them increased opportunities to learn the correct answer and build association of problem and answer.

Mabbot and Bisanz (2003) examined developmental change and individual differences in children's multiplication. One finding in the study from a cluster analysis revealed within a sample of fourth grade students, three groups of students (really good, good, and not-so-good) consistently performed well or poorly across tasks of conceptual knowledge, computational skill, and working memory measures. However; another group of students constituting 10% of the sample did not fit the pattern. Specifically, this group of students performed well on memory span tasks but did poorly on computational measures of multiplication and achievement. This was of particular interest, due to the discrepancy with previous research finding working memory facilitates development of problem-answer associations (Geary, 1994).

Synthesis

If differences of learning and knowing exist between students at various levels of achievement or other criterion then there may be significant implications regarding instruction and intervention. Examining different patterns of performance and sources of variability in mathematics performance may have theoretical implications by reconciling contradictory findings from prior research and contributing to a more comprehensive theory of mathematics skills and proficiency (Hecht et al., 2013; Hecht & Vagi, 2012). However within research, the methods and/or data to characterize students into groups or profiles are not consistent or use achievement levels which are vague with regard to skill deficits, therefore making it difficult to interpret and generalize findings. The Learning

Hierarchy has been suggested as a potential heuristic to better understand student mathematics difficulties in order to develop interventions and contribute to future research.

Learning Hierarchy

The Learning Hierarchy (Haring & Eaton, 1978) is conceptual model used for better understanding and targeting of academic interventions (Ardoin & Daly, 2007) by identifying interventions that are matched to student skill within one of four phases of student learning. The Learning Hierarchy has been utilized as heuristic to understand and guide intervention development and selection for reading (Daly, Lentz, & Boyer, 1996; Parker & Burns, 2014), mathematics (Burns, 2011; Burns et al., 2010; Codding et al., 2007), and writing (Parker, McMaster & Burns, 2011). Moreover, the Learning Hierarchy has been suggested as a potential model for correctly targeting academic interventions (Burns, VanDerHyden, & Zaslofsky, 2014).

Haring and Eaton's (1978) Learning Hierarchy suggests four stages through which students advance when learning a new skill or concept, (a) acquisition, (b) fluency (proficiency), (c) generalization, and (d) adaptation. Skills are first executed in a slow and inaccurate manner (acquisition phase), then accuracy increases but task completion remains slow (fluency phase). Once speed and accuracy are maintained, learning can be applied to responding to new stimuli and solving problems (generalization phase). Finally, students are able to generalize newly learned information to solve a problem (adaptation).

Instructional tasks should match the phase in which the Learning Hierarchy that the student is functioning. For example, the primary instructional goal for the acquisition phase is to increase accuracy of responding, but additional practice is needed in the fluency phase so that students can complete the task with both accuracy and sufficient speed (Burns, Codding, Boice, & Lukito, 2010). During the generalization stage of learning, students must transfer basic mathematics skills into novel mathematics tasks (Rivera & Byrant, 1992)

Using the Learning Hierarchy to identify instructional activities may be one way to determine interventions that are most likely to be successful with individual students. Burns and Klingbeil (2010) applied the phases of the Learning Hierarchy to both conceptual and procedural knowledge as a potential heuristic for intervention selection and guide for learning. Additionally, (Burns 2011; 2012) and Burns et al.(2015) examined the effectiveness of using a conceptual and procedural knowledge heuristic by matching mathematics interventions to student skill deficits and found promising results. Codding et al., (2007) compared the effects of two empirically supported interventions on mathematics fluency of second and third grade students and results suggested initial level of fluency impacted intervention effectiveness.

Computational fluency is considered a significant goal for overall mathematics understanding because students must be fluent with basic mathematics skills in order to complete more advanced mathematics tasks (National Council of Teachers of Mathematics, NCTM, 2010) Haring and Eaton’s (1978) multi-stage hierarchy of skill development may assist with understanding how differences in learning progress among children with various levels of skill and knowledge. Initially, a student is inaccurate and slow when first introduced to a novel skill or mathematics task. As students progress in skill, rate of accuracy and speed when completing a mathematics task increases. With adequate accuracy and speed (e.g., fluency), students are ready to apply their knowledge and later use their knowledge to solve novel or more complex mathematics tasks. The process of learning a skill or concept can be partitioned into the four phases of the Learning Hierarchy (e.g., accuracy, fluency, generalization, and application) and are suggested as critical in the development of mathematics proficiency (Rivera & Bryant, 1992).

Use of mature and efficient strategies contributes to the development of computational fluency and improves word problem solving, therefore it has been hypothesized that teaching effective strategies to enhance mastery and fluency should be incorporated into instruction and intervention (Bryant et al., 2008). Strategic development of low-achieving students in addition problem solving has been examined within cognitive research, however; the progression low-achieving students make from less mature/inefficient to more mature/efficient strategies has not been well established, especially with multiplication (Geary & Hoard, 2005).

Strategy development among average students has also been widely examined whereas the strategic development of low achieving students has received less attention (Downton, 2008; Geary & Hoard, 2005; Park & Nunes, 2000; Siegler, 1988; 2005; Zhang et al., 2011; 2014). Traditional instruction for average achieving students includes presentation of multiple strategies in order to facilitate flexible strategy use (NRC, 2001).

However, the research base on teaching problem-solving skills to students with mathematics difficulties suggests instruction that prioritizes strategies and explicitly teaches one strategy at a time as most effective for lower achieving students struggling in problem solving (Gersten et al., 2009; Krosenbergen & VanLuit, 2002).

Synthesis

The relationship between various measures of computational fluency and problem solving has been investigated; however, mathematics achievement seems to affect the relationship (Jitendra et al.,2013; Kanive & Burns, 2015). Examining development and sources of variable in mathematics performance of students at various levels, including but not limited to mathematics achievement, may have implications regarding understanding and instruction. Mathematics conceptual models are frequently used within research. The Learning Hierarchy, suggesting four stages students advance through when learning, has been used as heuristic to direct intervention development and selection with promising results.

Summary and Research Questions

Despite promising research regarding the use of the Learning Hierarchy as conceptual model for intervention development and selection for reading (Daly et al., 1996; Parker & Burns, 2014), mathematics (Burns et al., 2010; Codding et al., 2007), and writing (Parker et al., 2011) the validity of the model for intervention selection has yet to be examined. Further investigation of the development and characteristics of computational fluency strategy use in problem solving for students with various levels of achievement, and how they fit within a mathematics intervention heuristic that has been suggested as a model for targeting interventions (Burns et al., 2014) and may potentially contribute to improved intervention development, selection, and effectiveness. Identifying student strategy use at various levels of learning may contribute to practice, specifically how instruction is most effective when it addresses specific areas of need of students.

The following research questions guided the current study:

1. How well do data from the measure of multiplication fact fluency fit the Learning Hierarchy?
2. To what extent does academic achievement predict accuracy of strategy used?
3. To what extent does academic achievement predict quality of strategy used?
4. To what extent does phase in Learning Hierarchy effect accuracy of strategy?
5. To what extent does phase in Learning Hierarchy effect quality of strategy?

CHAPTER 3

METHOD

The present study utilized a cross-sectional design to examine the Learning Hierarchy conceptual model as a framework for intervention design based on student performance. In addition, using this design to examine the relationship between and investigate differences of strategies of multiplication used for word problem solving by third-grade students with various achievement levels, specifically within the four phases of the Learning Hierarchy. The current study's participants, materials, and procedures will be described below.

Setting and Participants

The participants for the study were 492 students from 22 third-grade classrooms in six elementary schools from one suburban school district. The school district consisted of 11 schools serving 7,412 students, 87% of which were Caucasian, 12% of which students received free or reduced price lunch, 3% were classified as English Language Learners (ELL), and 12% were students with Individualized Education Plans. A total of 85% of the third-grade students in the participating district were considered proficient on the state accountability test for mathematics on the most recent assessment.

Third grade was targeted for participation because it is the grade level at which number sense should be firmly established for most students, students are expected to fluently compute multiplication, and the instructional focus is on skills such as multiplication and division (Common Core State Standards Initiative, 2010; NCTM, 2000). The mean age of the sample was 8.44 (SD = .50) years. The sampled included

249 (51%) male and 244 (49%) female students, 83% of which were Caucasian, 4% were Asian-American, 3% were African American, 5% were Hispanic, 3% were Native American, and 2% were other of other ethnicities. Students with IEPs were included because so few students at each school were not proficient on state accountability tests for mathematics.

All participating students received mathematics instruction in their classrooms for 90 mins each day. The mathematics curriculum used in the third grade classrooms was called Math Expressions (Fuson, 2006), which is published by Houghton Mifflin Harcourt (2009). The objective of Math Expressions is to enhance students' mathematics proficiency by providing in-depth grade level topics through teacher-led lectures, creative exploration, and practice in small groups or individually with a focus on problem solving, drawings, and algorithms. The curriculum includes extensive unit reviews providing various ways to teach concepts and guide instruction so that teachers can better differientate instruction for students. The Institute for Education Sciences suggests, Math Expresssions as having potentially positive effects on students' math achievement based on a study examining effecitiveness of several elementary school mathematics curricula (U.S Department of Education, 2010).

Measures

Participants completed three measures, two of which were administered specifically for the study. Each measure is described below.

Measure of Academic Progress-Math (MAP-Math). Students were screened with the Measure of Academic Progress for Math (MAP-Math), a norm-referenced, untimed, group administered, computer adaptive multiple choice test used as a benchmark assessment to predict student statewide assessment performance (Northwest Evaluation Association; NWEA, 2003). Classroom teachers administered the MAP-Math tests in fall, winter, and spring of each academic year as part of a district-wide evaluation program. Computerized scoring generates a Rausch Unit score (RIT) upon completion of the assessment that can be compared across grades. The MAP-Math assesses number sense, computation and estimation, measurement, algebra, geometry and statistics and probability. The MAP-Math technical manual provides evidence of sound reliability and concurrent validity (NWEA, 2003), and independent research found reliability coefficients that ranged from .92 to .95 (Brown & Coughlin, 2007).

Curriculum-Based Measure-Math Computation (CBM-M). Students were administered a single-skill (single-digit multiplication) curriculum-based measure of math (CBM-M) to examine math computation (Appendix A). The probe consisted of math problems with single digit factors 2 through 9 (e.g., 2 x 2 to 9 x 9) written with a vertical orientation in 10 rows of 6 each on two sheets which was created by the researcher. To create this measure, all possible combinations for digits 2 through 9 were recorded and placed in order randomly. Standard CBM-M assessment procedures (Shinn, 1989) were followed, providing 2 minutes to complete as many problems as possible. Probes were scored for both accuracy and fluency. Students responses were converted to a digits correct per minute (dcpm) metric to assess fluency with single- digit multiplication facts, and an accuracy score was created by dividing the total number of items correctly completed by the total number of items completed.

Data obtained from single-skill mathematics assessments have demonstrated high reliability (Burns, VanDerHeyden, & Jiban, 2006) and are dependable for criterion-referenced decisions regarding that skill (Hintze, Christ, & Keller, 2002). CBM-M data have been shown to be sufficiently reliable, with alternate-form reliability estimates that ranged from .72 to .93 and with internal consistency coefficients that were above .90 (Foegen, Jiban, & Deno, 2007). Moreover, validity estimates that used a variety of criterion measures ranged from moderate (r = .35) to strong (r = .87) (Foegen et al, 2007).

Word problem-solving measure. Students completed a word problem solving assessment to examine application of math knowledge, strategy use, and competence on word problem solving (Appendix B). The assessment included 19 word problems that involved single-skill multiplication. Items for the assessment were selected from several standardized assessment subtests categorized as measures of application skills including, (a) the Math Concepts and Applications subtests of the Kaufman Test of Educational Achievement, Second Edition (KTEA-II; Kaufman & Kaufman, 2004), (b) Application subtest of the Key Math Revised (Connolly, 2000), (c) Math Reasoning subtest of the Weschler Individual Achievement Test, Second Edition (WIAT-II; Psychological Corporation, 2001), (d) Practical Applications subtest of the Comprehensive Math Abilities Test (CMAT; Hresko, Schlieve, Herron, & Sherbenau, 2003), € Diagnostic Achievement Battery, Third Edition (DAB-III; Newcomer, 2001), and (f) Applied Problems subtest of the Woodcock Johnson Achievement Test, Third Edition (WJ-Ach III; Woodcock, McGrew, & Mather,2001). Items were selected from the application subtest of the respective achievement test if the item required single-digit multiplication. Internal consistency for the present sample was *r=.90.*

Each question was read aloud to the group and the measure was untimed. Students were asked to apply simple computation skills to solve single-digit multiplication including only digits 2 through 9. Students were instructed to provide a response and show their work. Responses were scored for overall accuracy and coded for strategy use to solve the problem.

Coding strategies. The strategies that students used to solve the problems were coded into five categories according to the coding scheme adapted from the literature on solving multiplicative problems (Lemaire & Siegler, 1995; Siegler, 1988; Zhang, Ding, Barrett, Xin, & Liu, 2014). Strategies were coded into five categories; (a) no response, (b) incorrect operations, (c) counting, (d) repeated addition, (e) decomposition, and (f) product response receiving score from zero to five indicating; 0 as least efficient through 5 as most efficient (Appendix C). For the purposes of this study, the term efficient instead of mature as previously referenced, will be used to describe strategy quality. Siegler, (1988; 2006) reported higher level strategies result in increased accuracy and shorter problem solving speed. Furthermore, accuracy of strategy is influenced by the required amount of time as well as cognitive resources. This study examines accuracy of strategy use separately, therefore level of efficiency will represent the amount of time required, with shorter time requirements considered as more efficient. No response, included items with no answer or student indicating “I don’t know” and received a quality score of zero. Incorrect operations included item responses or procedures reflecting erroneous selection and/or application of operation (e.g., 4 x 5 = performs addition operation 4 + 5 = 9) and received a quality score of one. Counting strategies identified as tally marks drawn to represent factors with student counting each individual tally received a quality score of two. Repeated addition consisted of items solved by adding one factor a number of times as the other factor (e.g., 4 x 5 = 5 + 5 + 5 + 5 or 4 + 4 + 4 + 4 + 4) and assigned a quality score of three. Decomposition category identified strategies to create a series of simpler multiplication problems to solve (e.g., 7 x 8 = (5 x 8) + (2 x 8) = 40 + 16 = 56). The final category, product response, included numerical responses that may or may not have an accurate number sentence received the highest score of five, considered for the purposes of this study as the most efficient strategy. These categories were then used to measure strategic development and word problem-solving competence. The four areas of focus are presented below.

Accuracy of problem solving. The accuracy of problem solving was calculated as the total points awarded for each item. One point was assigned for the correct answer. For example, if a child correctly answered 10 out of the 19 items, his/her accuracy was 10. The maximum accuracy score was 19 when a student correctly answered all problems of a probe.

Frequency of each type of strategy used. Frequency of strategy use was calculated as the number of items attempted with a specific strategy among the total 19 items in a probe. Providing no response did not count as a strategy type, but incorrect operations were considered a strategy type due to displaying an understanding of the processes by which one arrives at an answer (Zhang et al., 2014). For an example of frequency scoring, if out of the nineteen items on a probe, a student used incorrect operation four times, attempted to use repeated addition for five items, and used product response for the other ten items, then his/her frequency for using incorrect operations was four, his/her frequency of repeated addition was five, and frequency of using product response was ten.

Accuracy of strategy. The accuracy with which strategies were applied to solve word problem was calculated for each student using the percent of occasions executing the attempted strategy resulted in a correct response. Providing no response and incorrect operations were not considered strategy types due to strategy use not resulting in correct responses (Zhang et al., 2011; 2014). Accuracy of performing the strategy was calculated as the number of items solved accurately with a specific strategy divided by the number attempted with a specific strategy. For example, if a student attempted to complete eight problems with product response but performed this strategy with correct answers on only four of the eight problems, a score of 50% for his/her accuracy would be given.

Quality of strategy use. Quality of the strategies used was calculated as a frequency of the number of items attempted with a specific strategy among the total 19 items in a probe and as a mean score. Strategies were coded from 0 to 5, with 0 as the least efficient (e.g., stating I don't know/no strategy) and 5 most efficient strategy (e.g., product response) The maximum quality score average was 5 when a student utilizes product response strategy to solve all 19 multiplication word problems. For example, if out of the 19 items on a probe, a student used incorrect operation 4 times, attempted to use repeated addition for 5 items, and used product response for the other 10 items, then his/her mean score based upon above responses would be 3.37 ((4 x 1) + (5 x 2) + (10 x 5) = 64/19=3.37).

Procedure

Passive consent forms were sent home to students' guardians and active assent forms were distributed and read to the students. The consent and assent forms provided an explanation of purpose and procedures along with possible risks and benefits of participating in the study.

Nine school psychology graduate students from an educational psychology program with previous training and experience as data collectors administered a battery of assessments. Assessors were provided with scripted directions. All measures except the standardized academic achievement measure were administered to groups of students in their own classroom during a single session that lasted approximately 40 to 60 minutes. To minimize the possible effects of differences in readings skill, the word problems on the application measure were read aloud by the data collector. The order of the assessments was counterbalanced to reduce any potential order effects. Data collection was completed in 22 sessions across 5 days. The standardized testing was completed by the district testing coordinator in the winter and results were provided to the researcher.

Fidelity

Implementation fidelity was directly observed with the use of a checklist designed to assess whether the graduate student administering measures followed scripted directions and data collection procedures. Seven implementation fidelity observations were conducted across the 22 sessions. Fidelity was calculated by dividing the number of items observed by the total number of items and multiplying by 100. The average implementation fidelity rating across seven observations was 100%.

Inter-scorer Agreement

Several graduate school student researchers independently scored 30% of each math measure to assess inter-scorer agreement. Inter-scorer agreement was calculated by dividing the number of agreements by the number of agreements plus disagreements and multiplying the result by 100 to obtain a percentage. The inter-scorer agreements for the CBM-Math and Conceptual measures were 96% and 99% respectively. Inter-rater reliability on the Word Problem-Solving Measure for problem-solving accuracy was 98% and for coding the strategy type was 96%.

Data Analysis

The data from the present study were analyzed in several stages. First, multiplication fluency data were used to determine students' phase within the learning hierarchy framework. For the purpose of this study, students were classified into four categorical phases (i.e., inaccurate and slow, accurate and slow, accurate and fast, and inaccurate and fast) based on accuracy and fluency scores on a measure of single digit multiplication fact computation. Score cutoffs of 90% accuracy (Burns, 2004) on all problems attempted and 17 DCPM were used for CBM-M fluency measure based on instructional level recommendations (Burns, VanDerHeyden, & Jiban, 2006; VanDerHeyden & Burns, 2008).

Students were classified as accurate if their scores on the CBM-M fluency assessment was 90% and students were classified as fluent if their score on the CBM-M fluency assessment was at or above 17 DCPM. Students were classified as inaccurate and slow (i.e., acquisition phase) if their scores were below 90% accuracy and 17 DCPM. Students were classified as accurate and slow (i.e., proficiency phase) if their scores were 90% and above but below 17 DCPM. Students were classified as accurate and fast (i.e., generalization phase) if their scores were at or above 90% accuracy and at or above 17 DCPM (n = 187). Students were classified as inaccurate and fast if their scores were below 90% accuracy but at or above 17 DCPM).

Next, several analyses were used to answer the five research questions. The first research question regarding the fit of data from measures of multiplication fact fluency was examined by analyzing phase of learning hierarchy with a chi-square test. Chi-square analyses provide an indication of how well the frequency of categorical variables fall as expected. Expected frequency of students within phases of the learning hierarchy were calculated as percentages based upon categories of instructional level, achievement and reported end of year benchmark.

The National Assessment of Education Progress (NAEP), assess student performance in mathematics and NAEP achievement levels define what students know and should be able to do. In 2013, 83% of 4th grade students performed at or above the Basic achievement level indicating partial mastery of fundamental skills, with 42% performing at or above the Proficient level indicating a demonstration of competence in mathematic (NAEP, 2013). Multiplication facts are foundational for further advancement in mathematics and form the basis for more advanced skills and concepts across many domain in mathematics (Kilpatrick et al., 2001). Statistics on math-skill mastery derived from Renaissance Learning's online database (N=436,627) examined math facts mastery by grade level and showed that most students score below benchmark in their grade for the four operations (Renaissance Learning, 2012). Additionally, only 38% of 3rd graders and 29% of 4th graders reached mastery benchmark. The results do not suggest that others students do not know how to perform operations, it instead suggests that they have not reached mastery of mathematics facts.

Burns et al., (2006) compared mathematics performance of second through fifth-grade students to previously reported fluency and accuracy criteria using categories of performance (frustration, instructional, and mastery). Accuracy and fluency data were converted to three possible categories and the study reported, 22% of sample of second and third-grade student were considered to be at a frustration level, 69% were considered to be at an instructional level, and 8% at the mastery level. Due to difficulty determining proficiency within the Learning Hierarchy phases previous research findings were used to derive expected values. The percentage of students in an acquisition phase was proposed to be similar to the percentage of students at a frustration level from the Burns et al., (2006) which categorized levels using accuracy and fluency data. The percentage of students in the fluency and generalization phases were derived based upon the percentage of students proficient and reaching mastery from the Renaissance Learning database. The expected values derived for this study were, 25% of students fall in the Acquisition phase, 35% of students fall in the Fluency/Proficiency phase, 35 % of students fall in the

Generalization phase, and 5% of students in an "other" phase considering error in addition to students not falling within a specific phase according to accuracy and fluency data.

The second and third research questions, addressed the extent to which academic achievement predicts accuracy and quality of strategies used on a measure of word problem solving respectively and a series of regression models were fitted to answer to research questions. The following linear regression equations were used:

$$\textit{accuracy of product response strategy used on word problem solving measure} = b_0 + b_1 x_{\textit{MAP RIT Reading}} + b_2 x_{\textit{MAP Math Number and Operation}} + e$$

$$\textit{accuracy of decomposition strategy used on word problem solving measure} = b_0 + b_1 x_{\textit{MAP RIT Reading}} + b_2 x_{\textit{MAP Math Number and Operation}} + e$$

$$\textit{accuracy of repeated addition strategy used on word problem solving measure} = b_0 + b_1 x_{\textit{MAP RIT Reading}} + b_2 x_{\textit{MAP Math Number and Operation}} + e$$

$$\textit{accuracy of counting strategy used on word problem solving measure} = b_0 + b_1 x_{\textit{MAP RIT Reading}} + b_2 x_{\textit{MAP Math Number and Operation}} + e$$

$$\textit{quality of strategy} = b_0 + b_1 x_{\textit{MAP Reading RIT}} + b_2 x_{\textit{MAP Math Numbers and Operations}} + e$$

A one-way ANOVA was used to answer the fourth and fifth research questions regarding the effect of the phase in the learning hierarchy on accuracy and quality of strategy use. The ANOVAs were conducted to compare whether there was a significant difference between phase groups on accuracy of each strategy and quality of strategies used.

CHAPTER 4

RESULTS

Purpose and Research Questions

The purpose of this chapter is to present the results from the current study. The Learning Hierarchy (Haring & Eaton, 1978) considers how students learn different academic skills. This framework proposes that students develop skills by progressing through a four stage learning sequence. The Learning Hierarchy model may provide a useful framework for identifying student need and indicating components of effect intervention. Therefore, this study sought to examine whether data from single digit multiplication fluency measures reflected the learning sequence providing support for the hierarchy framework.

In order for mathematics instruction and intervention to be effective, a clear understanding of student problem solving is needed, however; as a result of the complexity of problem solving, a student's correct or incorrect answer may be due to a number of reasons. Previous studies have reported children with mathematics difficulties have lower scores on mathematics achievement and problem solving compared to other children but specific differences in problem solving strategies have not been thoroughly identified. Thus, by examining the extent to which achievement level predicts accuracy and quality of strategy, deficits may be able to be identified in problem solving leading to implementation of the most appropriate intervention.

Distinguishing between sources of variability in mathematics performance may contribute to creating a more comprehensive theory of mathematics skills. Studies frequently examine student differences in problem solving by examining levels of achievement, but standardized mathematics achievement tests combine many different types of items and therefore may not provide detailed information for identifying and remediating specific deficits. Therefore, examining the effect that stage in the Learning Hierarchy has on strategy accuracy and quality may have implications for instruction and intervention in problem solving. A review of the research questions will be provided followed by a presentation of descriptive statistics relevant to the current study and lastly a review of data related to each research question. The research questions that guided the current study were:

1. How well do data from measures of multiplication fact fluency fit the Learning Hierarchy?
2. To what extent does academic achievement predict accuracy of strategy used?
3. To what extent does academic achievement predict quality of strategy used?
4. To what extent does phase in Learning Hierarchy effect accuracy of strategy?
5. To what extent does phase in Learning Hierarchy effect quality of strategy?

Descriptive Analysis

Table 1 reports descriptive characteristics of measures. Distributions for each measure were acceptably normally distributed with the exception of Reading RIT, which demonstrated negative skew and normal kurtosis, and Math Facts digits correct per minute (DCPM), which demonstrated positive skew and a larger kurtosis. Although the skew and kurtosis values suggest a non-normal distribution, Math Facts DCPM was only used to classify students into groups and not used in any parametric analyses.

Descriptive characteristics of frequency and accuracy of strategies used by students to solve items on a measure of word problem solving are reported in Table 2. Descriptive characteristics of overall quality of strategies used by students to solve items on a measure of word problem solving are reported in Table 3.Correlations among measures are reported in Table 4. Statically significant bivariate correlations were found among all measures.

Coding and Categorical Coding

The type and occurrence of strategies used by students to solve single-digit multiplication word problems was of primary interest to this study. Student multiplication problem solving was reviewed and mean number of strategy uses of particpants are presented in Table 5. Within the overall sample, 53% (*n* = 259) of students used the product response strategy most frequently on a measure of word problem solving and 81% (*n* = 400) used the decomposition strategy least on a measure of word problem solving.

Students were classified with Learning Hierarchy categories according to CBM-M data. Fluency was judged with instructional level criteria (Burns et al., 2006) and accuracy was compared to a criterion of 90% (Burns, 2004). A total of 117 (24% of sample) students were inaccurate (<90% correct) and slow (< 17 DCPM) and were classified as within the acquisition phase of learning. Data for 173 (35%) students were accurate (≥ 90%) and slow, which was classified as the proficiency phase. Data for 187 (38%) of the students were both accurate and fast (≥ 17 DCPM), which was classified as the generalization phase. Finally, only 15 (3%) of the students had data that were inaccurate and fast, which is not a category in the Learning Hierarchy, but a possible categorical outcome.

Students in the Acquisition Phase (inaccurate and slow) and Proficiency Phase (accurate and slow) displayed more strategy variation on a measure of word problem solving than students in Generalization phase (accurate and fast) and Inaccurate and Fast groups. Students in the Inaccurate and fast and Generalization phase groups used one strategy 60% (n=9) and 81% (n=152) on 16-19 items respectively compared to students in Acquisition or Proficiency phases using one strategy 24% (n=28) and 40% (n=70) respectively.

Table 1

Descriptive Characteristics of Measures

Measures	N	Mean (*SD*)	Min	Max	Skew (*SE*)	Kurtosis (*SE*)
Math RIT	492	205.78	176	260	.42	2.03
		(10.28)			(.110)	(.220)
Number and Operations	492	204.13	166	258	.29	1.14
		(11.57)			(.110)	(.220)
Reading RIT	492	200.35	149	236	-.59	1.14
		(12.36)			(.110)	(.220)
Math Facts DCPM	492	16.47	0.50	71.85	1.81	4.48
		(11.03)			(.110)	(.220)
Word Problem-Solving	492	15.05	0	19	-1.27	1.24
		(3.92)			(.110)	(.220)

Note. RIT = Measures of Academic Progress-Raush Unit scores; DCPM = math fact computational fluency task digits correct per minute.

Table 2

Descriptive Characteristics of Frequency of Strategies Used and Accuracy by Students on Word Problem Solving Measure Items

	N	Mean Frequency of Use	SD	N	Mean Accuracy	SD
Product Response	492	14.15	5.07	487	.87	.18
Decomposition	492	0.47	1.29	92	.71	.37
Repeated Addition	492	0.67	1.43	147	.73	.38
Counting	492	2.10	3.32	224	.67	.33
Incorrect Operation	492	1.12	2.58			
No Response	492	0.49	1.19			

Note. Frequency of strategy use was calculated as number of times strategy was used on 19 items from a measure of word problem solving Accuracy of strategy use refers to percentage of items answered correctly using a specific strategy.

Table 3

Descriptive Characteristics of Quality of Strategies Used on Word Problem Solving

	N	*Mean*	*SD*	*Min*	*Max*
Quality	492	*4.20*	*.80*	*1.00*	*5.00*

Table 4

Correlations among Measures

	1	*2*	*3*	*4*
1. Math RIT- Number & Operation		*.52**	*.42**	*.46**
2. Reading RIT	*.52**		*.32**	*.43**
3. Math Facts DCPM	*.42**	*.32**		*.50**
4. WPS	*.46**	*.43**	*.50**	

* $p < .05$
Note. RIT = Measures of Academic Progress-Raush Unit scores; DCPM = math fact computational fluency task digits correct per minute; WPS = word problem solving task

Table 5

Mean Number of Uses of Strategies to Solve Items among Students in Four Phases of the Learning Hierarchy on Word Problem Solving.

	Incorrect Operation		Counting		Repeated Addition		Decomposition		Product Response	
	M	SD	M	SD	M	SD	M	SD	M	SD
IS (117)	1.86	3.06	3.29	3.79	1.22	1.97	.74	1.88	11.47	5.06
AS (173)	1.36	2.67	2.92	3.77	.72	1.46	.63	1.39	12.77	5.21
AF (187)	.66	2.15	.66	1.74	.29	.80	.22	.74	16.94	3.40
IF (15)	.60	.83	1.33	2.35	.53	.99	.33	.89	15.87	3.52
Total (492)	11.12	2.58	2.10	3.32	.66	1.43	.47	1.28	14.14	5.07

Note. IS= Acquisition (i.e., inaccurate and slow), AS= Proficiency (i.e., accurate and slow), AF= Generalization (i.e., accurate and fast), and IF= Inaccurate and fast

Alignment of Fluency Data with Learning Hierarchy

The extent to which data from the current study fit the sequence of phases of the Learning Hierarchy was examined by first calculating the percentage of students within each phase of the Learning Hierarchy (a) Accuracy (i.e., inaccurate and slow), (b) Fluency (i.e., accurate and slow), (c) Generalization (i.e., accurate and fast). A chi-square statistic was then computed to examine differences between expected and observed across phases of the Learning Hierarchy. Expected frequency of students within phases of the Learning Hierarchy were calculated as percentages based upon achievement levels from a standardized measure of mathematics achievement as well as reported end of year benchmark (NAEP, 2013; Renaissance Learning, 2012).. Based upon, percentage of students proficient and reaching mastery the following percentages of students were derived. The expected values were 25% of students fall in the Acquisition phase, 35% of students fall in the Fluency/Proficiency phase, 35 % of students fall in the Generalization phase, and 5% of students in an "other" phase. Findings suggest that observed data were consistent with the phases of the Learning Hierarchy and differences resulted in a non-significant effect $X^2(df = 3, N = 492) = 5.31$, p=.249.

Academic Achievement and Word Problem Solving Strategy Accuracy

The second research question examined the predictability of an academic achievement measure on accuracy of strategy for word problem solving. In order to analyze this question, a linear regression model was fitted where student's performance on the number and operations strand of a measure of academic achievement predicted accuracy of strategy used on a measure of word problem solving. Each predictor variable was entered simultaneously. The following linear regression equations were used:

$$accuracy\ of\ \text{product response}\ strategy\ used\ on\ \text{word problem solving}\ measure = b_0 + b_1 x_{MAP\ RIT\ Reading} + b_2 x_{MAP\ Math\ Number\ and\ Operation} + e$$

$$accuracy\ of\ \text{decomposition}\ strategy\ used\ on\ \text{word problem solving}\ measure = b_0 + b_1 x_{MAP\ RIT\ Reading} + b_2 x_{MAP\ Math\ Number\ and\ Operation} + e$$

$$accuracy\ of\ \text{repeated addition}\ strategy\ used\ on\ \text{word problem solving}\ measure = b_0 + b_1 x_{MAP\ RIT\ Reading} + b_2 x_{MAP\ Math\ Number\ and\ Operation} + e$$

$$accuracy\ of\ \text{counting strategy}\ used\ on\ \text{word problem solving}\ measure = b_0 + b_1 x_{MAP\ RIT\ Reading} + b_2 x_{MAP\ Math\ Number\ and\ Operation} + e$$

The results (Table 6) of the linear regression analysis indicated that, together, students' performance on MAP Reading and MAP Math Numbers and Operation strand explained a significant 12% of the variance in students' accuracy of product response strategy used on a measure of word problem solving F (2, 490) = 33.22, $p < .001$, R^2 = .120. The results (Table 7) of the linear regression analysis indicated that student performance on MAP Reading and MAP Math Numbers and Operation strand explained a significant 9.90% of the variance in students' accuracy of decomposition strategy used on a measure of word problem solving F (2, 91) = 4.86, p= .010, R^2 = .099. The results (Table 8) of the linear regression analysis indicated that student performance on MAP Reading and MAP Math Numbers and Operation strand did explain a significant 6.10% of the variance in students' accuracy of repeated addition strategy used on a measure of word problem solving F (2, 144) = 4.65, p = .011, R^2 = .061. The results of the linear regression analysis indicated that student performance on MAP Reading and MAP Math Numbers and Operation strand explained a nonsignificant 2.6% of the variance in students' accuracy of counting strategy used on a measure of word problem solving F (2, 220) = 2.91, p= .057, R^2 = .026.

Further review of results revealed that students' performance on the MAP Math Numbers and Operation strand alone significantly predicted the accuracy of product response strategy (β=.22 , p<.001), accuracy of decomposition strategy (β = .33 , p <.01), and accuracy of repeated addition strategy (β = .26, p < .01) but did not significantly predict accuracy of counting strategy (β = .11, p = .18).

Table 6

Predicting Accuracy of Product Response on a Measure of Word Problem Solving

	B	SE B	β	t	R^2
Constant	-.333	1.48			
MAP Reading Rausch Unit	.003	.001	0.18	3.58*	.09
MAP Number & Operation	.003	.001	0.22	4.40*	.04

*p<.001

Table 7

Predicting Accuracy of Decomposition Strategy on a Measure of Word Problem Solving

	B	SE B	β	t	R^2
Constant	-1.57	.843			
MAP Reading Rausch Unit	-.001	.004	-.03	-.30	.01
MAP Number & Operation	.012	.004	.33	2.90*	.09

*p<.01

Table 8

Predicting Accuracy of Repeated Addition on a Measure of Word Problem Solving

	B	SE B	β	t	R^2
Constant	- 1.12	.67			
MAP Reading Rausch Unit	- .001	.003	-.02	-.22	.01
MAP Number & Operation	.010	.004	.26	2.74*	.06

*p<.01

Academic Achievement and Quality of Strategy Selection

The third research question examined the predictability of an academic achievement measure on quality of strategy use for word problem solving. In order to analyze this question, a linear regression model was fitted where student's performance on the number and operations strand of a measure of academic achievement predicted quality of strategy used on a measure of word problem solving. Each predictor variable was entered simultaneously. The following linear regression equation was used:

$$\textit{Strategy quality used on word problem solving measure} = b_0 + b_1 x_{\textit{MAP RIT Reading}} + b_2 x_{\textit{MAP Math Number and Operation}} + e$$

The results (Table 9) of the linear regression analysis indicated that, together, students' performance on MAP Reading and MAP Math Numbers and Operation strand explained a significant 21% of the variance in students' quality of strategy used on a measure of word problem solving $F(2, 490) = 66.10$, p <.001, $R^2 = .213$. Further review of results revealed that students' performance on the MAP Math Numbers and Operation strand alone significantly predicted the quality of strategies (*β=.36 , p<.001)*.

Table 9

Predicting Quality of Strategy on Measure of Word Problem Solving

	B	SE B	β	t	R^2
Constant	- 3.58	.69			
MAP Reading Rausch Unit	- .01	.003	.16	3.38*	.12
MAP Number & Operation	.03	.004	.36	7.65*	.09

**p<.01*

Learning Hierarchy Phase and Strategy Accuracy

The fourth research question examined the extent to which phase of the Learning Hierarchy effects accuracy of strategy used. A series of one way ANOVAs were conducted to examine phase in the Learning Hierarchy group equivalency on accuracy of direct retrieval, decomposition, repeated addition, and counting strategy used. Results (Table10) indicated significant differences between phase groups on accuracy of product response, F (3, 483) = 21.68, p <.05, and counting strategy, F (3, 220) = 3.90, p = .010. No significant difference were indicated between phase groups on accuracy of decomposition, F (3, 89) = 1.19, p = .320, or repeated addition strategy, F (3, 143) = 1.95, p =.124. Planned post-hoc analyses (Table11) indicated students in the later phase of the Learning Hierarchy (i.e., accurate and fast) using the product response strategy were significantly more accurate than students in the inaccurate and slow phase (mean difference = .15, SD = .02, *p < .05*) as well as students in the accurate and slow phase (mean difference = .10, SD = .02, *p < .05*). Students in the accurate and slow phase were significantly more accurate than students in the inaccurate and slow phase (mean difference = .05, SD = .02, *p* = .036). On the counting strategy accuracy, post hoc analyses (Table12) suggested, students in the later phase of the Learning Hierarchy (i.e., accurate and fast) using the counting strategy were significantly more accurate than students in the inaccurate and slow phase (mean difference = .21, SD = .06, *p* = .007) but no other significant differences between students in the later phase or first phase with students in the accurate and slow or inaccurate and fast phases were indicated.

Table 10

One way ANOVAs results for difference in strategy accuracy between Learning Hierarchy phases

	IS		AS		AF		IF					
	M	SD	M	SD	M	SD	M	SD	*df*	Sums of Squares	Mean Square	F
Product	.79	.21	.85	.15	.95	.09	.85	.16	3	1.85	.62	21.68*
Response												
Decomposition	.60	.42	.76	.34	.76	.36	.84	.23	3	.49	.16	1.19
Repeated Add	.63	.43	.78	.35	.80	.36	.67	.24	3	.85	.28	1.95
Counting	.59	.34	.69	.30	.80	.31	.59	.35	3	1.20	.40	3.89*

Note. IS= Acquisition (i.e., inaccurate and slow), AS= Proficiency (i.e., accurate and slow), AF= Generalization (i.e., accurate and fast), and IF= Inaccurate and fast.

**p*<.01

Table 11.

Tukey's post hoc analysis for differences between phases on accuracy of product response strategy

	IS		AS		AF		IF	
	Mean Difference	*p*	Mean Difference	*p*	Mean Difference	*p.*	Mean Difference	*P*
IS			**.05**	**.04**	**.15**	**.<.01**	.05	.68
AS	-.05	.04			**.09**	**<.01**	-.01	1.00
AF	-.15	<.01	-.09	<.01			-.10	.12
IF	-.05	.68	.01	1.00	.10	.12		

Note. IS= Acquisition (i.e., inaccurate and slow), AS= Proficiency (i.e., accurate and slow), AF= Generalization (i.e., accurate and fast), and IF= Inaccurate and fast.

Table12.

Tukey's post hoc analysis for differences between phases on accuracy of counting strategy

	IS		AS		AF		IF	
	Mean Difference	*p*	Mean Difference	*p*	Mean Difference	*p*	Mean Difference	*p*
IS			.10	.15	.21	.007	-.01	1.00
AS	-.10	.15			.10	.32	-.10	.84
AF	-.21	.007	-.10	.32			-.21	.39
IF	.01	1.00	.10	.84	.21	.39		

Note. IS= Acquisition (i.e., inaccurate and slow), AS= Proficiency (i.e., accurate and slow), AF= Generalization (i.e., accurate and fast), and IF= Inaccurate and fast.

Learning Hierarchy Phase and Strategy Quality

The fifth research question examined the extent to which phase of the Learning Hierarchy effects quality of strategies used. A one way ANOVA was conducted to examine phase in the Learning Hierarchy group on quality of strategy used. Results (Table13) indicated significant differences between phase groups on quality of strategies used, F (3, 489) = 39.87, p <.001. Post hoc analysis (Table14) suggested students in the later phase of the Learning Hierarchy (i.e., accurate and fast) used significantly more higher quality strategies than students in the earlier phases of the Learning Hierarchy, specifically students in the inaccurate and slow phase (mean difference = .89, SD= .09, p < .001) and students in the accurate but slow phase (mean difference = .71, SD = .08, p <.001). Students in the inaccurate and fast phase used significantly more higher quality strategies than students in the inaccurate and slow phase (mean difference = .72, SD = .22, p <.01).

Table 13

One way ANOVA Results for Difference on Strategy Quality Between Learning Hierarchy Phases

	Mean	SD	*df*	Sums of Squares	Mean Square	F
Strategy Quality			3	74.15	24.72	39.87*
IS	3.78	.88				
AS	3.97	.93				
AF	4.68	.56				
IF	4.50	.56				

Note. IS= Acquisition (i.e., inaccurate and slow), AS= Proficiency (i.e., accurate and slow), AF= Generalization (i.e., accurate and fast), and IF= Inaccurate and fast.
*p<.01

Table 14

Tukey's post hoc Analysis for Differences Between Phases on Strategy Quality

	IS		AS		AF		IF	
	Mean Difference	*p*	Mean Difference	*p*	Mean Difference	*p*	Mean Difference	*p*
IS			.19	.20	.89	<.01	.72	<.01
AS	.-19	.20			.71	<.01	.54	.06
AF	.89	<.01	-.71	<.01			-.18	.84
IF	-.72	<.01	-.54	.06	.18	.84		

Note. IS= Acquisition (i.e., inaccurate and slow), AS= Proficiency (i.e., accurate and slow), AF= Generalization (i.e., accurate and fast), and IF= Inaccurate and fast.

CHAPTER 5

DISCUSSION

Organization of the Chapter

The objective of this chapter is to contextualize the results from the current study within previous research, theory and practice. This chapter starts with a review of the study purpose. Then, a discussion of each research question is included with results presented within the context of previous research. Next, potential implications for future practice and research are included as well as implications for theory. Lastly, the chapter concludes with directions for future research and strengths and limitations of the current study.

Study Purpose Review

Word problem solving is a critical aspect of mathematics instruction, an important indicator of overall mathematics skill (National Council of Teachers of Mathematics [NCTM], 2000) and is closely linked to overall mathematics achievement (Foegen & Deno 2001; Fuchs et al., 1994; Helwig et al., 2002). However, word problem solving reflects the simultaneous application of several areas of mathematics proficiency requiring students to demonstrate mastery of skills to make decisions as to the pertinent information needed to solve the problem, selection of appropriate strategies needed to help solve problems, and lastly derive the solution (Acosta-Tello, 2010). For students who struggle to complete word problems, the conglomerate of skills makes is difficult to determine which skill should be targeted for intervention.

Low mathematics achievement is associated with the inability to understand and apply advanced problem solving strategies to solve mathematics problems (Geary et al., 2004; Geary & Hoard, 2005) with students often struggling to retrieve mathematics facts or procedural steps as well as with understanding mathematics concepts by acquiring and utilizing developmentally immature strategies. Understanding how different achieving students develop problem solving strategies may be an important factor in explaining differences and providing suitable interventions for students.

Selecting mathematics interventions with an evidence-based heuristic can help correctly target and predict intervention effectiveness in order to better meet student needs. Previous research found promising results for using student data to identify mathematics intervention (Burns, 2011; Burns et al., 2010; Codding et al., 2007). Haring and Eaton's (1978) Learning Hierarchy has been utilized as heuristic to understand and guide intervention development and selection in areas of reading (Daly, Lentz, & Boyer, 1996; Parker & Burns, 2014), mathematics (Burns, 2011; Codding et al., 2007), and writing (Parker, McMaster & Burns, 2011) and suggested as a potential model for correctly targeting interventions (Burns, VanDerHyden, & Zaslofsky, 2014) by identifying interventions that are matched to skill within four phases of learning (Ardoin & Daly, 2007). Using the Learning Hierarchy to identify instructional activities may be one way to determine interventions that are most likely to be successful with individual students. However, despite promising research regarding the use of the conceptual model, the validity of the model for intervention selection has yet to be examined.

The purpose of the present study was to extend previous research by examining the Learning Hierarchy (LH) conceptual model as a framework for instruction and intervention design based on student performance. Additionally, the study sought to contribute the research base around the development and characteristics of problem solving for students at varying levels of skill in mathematics. Specifically, the study examined differences of strategy use in solving single digit multiplication word problems of third grade students demonstrating various levels of achievement as well as within specific phases in the LH.

Learning Hierarchy

The first question examined the extent to which fluency data from the current study fit the sequence of phases (a) Accuracy (i.e., inaccurate and slow), (b) Fluency (i.e., accurate and slow), and (c) Generalization (i.e., accurate and fast) of the Learning Hierarchy. The percentage of students within each phase of the Learning Hierarchy was calculated, with expected frequency based upon frustration, instructional, and mastery level categories from Burns et al., (2006), achievement level, and reported end of the year benchmarks (NAEP, 2013; RL, 2010). Chi-square statistic revealed differences resulted in a non-significant effect, suggesting that observed data were consistent with what would be predicted given the phases of the Learning Hierarchy as described by Haring and Eaton (1978).

The fourth group consisting of 3% of the sample could be postulated to be due to error, but students who demonstrate a profile of inaccurate yet fast performance may actually require additional problem analysis. Kling and Bay Williams (2015) suggest fluency is more than speed and finding an answer efficiently involves flexibility and accuracy using appropriate strategies aligning with definition provided by Common Core State Standards in Mathematics (National Governor's Association, 2010). Therefore, students who are within the inaccurate but fast phase may reach fluency criteria based upon digits correct per minute yet not demonstrate accuracy. Skill versus performance deficits can be determined with a brief experimental analysis of instructional components to increase positive outcomes (Daly et al., 1997, Martens, Witt, Daly & Vollmer, 1999), which may be an appropriate plan of action for students who were fluent but inaccurate .

The current findings are promising for considering the Learning Hierarchy as a potential conceptual heuristic model in mathematics given that the observed and expected profiles were not significantly different despite a relatively large sample (*n* = 492). Previous research has also supported the Learning Hierarchy as an intervention heuristic (Daly & Ardoin, 1997), but there is limited research for mathematics (Burns et al., 2010).

Criteria referenced fluency data are useful for instructional planning and evaluating academic interventions (Salvia &Ysseldyke, 2004). Empirically derived criteria for mathematics fluency identified by Burns et al., (2006) were utilized in this study and accuracy of 90 percent has been used as criterion to judge whether students demonstrated sufficient understanding (Burns, 2004) and demonstrated a significant effect on the placement decision (Burns & VanDerHyden, 2008; Burns, 2011; Burns et al., 2015; Codding et al., 2007). Within the context of assessment, the Learning Hierarchy (Haring & Eaton, 1978) proposes four phases of skill development starting with acquisition and ending with adaptation. This framework suggests students gain competence and proficiency with skills as they progress through phases. Although the goal is for adaptation of skill within learning tasks, it may useful to examine student foundational skills (e.g., basic math operation facts) to better understand student skill level within tasks to increase student competence through instruction and intervention.

Academic Achievement and Accuracy

The second question examined the relationship between student performance on the number and operation strand from a measure of mathematics achievement on student accuracy of single digit multiplication word problem solving strategy. Regression analyses revealed that, overall, achievement measures (MAP Reading and MAP Math Number and Operation Strand) did explain a statistically significant proportion of variance in student word problem solving accuracy on all strategies except for counting. Additionally, further analysis revealed number and operation strand alone significantly predicted word problem solving accuracy of all strategies except counting.

Results from this study are consistent with previous research findings that students with low performance on various measures of mathematical achievement are less accurate, specifically when using direct retrieval strategies to solve computation problems (Geary & Hoard, 2005; Geary, Hoard, & Bailey, 2011). Mathematics difficulty is operationalized in the literature as low mathematics performance on standardized tests of mathematics (Fuchs et al., 2010). Students with mathematic difficulties compared to peers perform more poorly on various mathematics skills including calculation and problem solving (Powell et al., 2011). More specifically, due to skills difficulties students with mathematics difficulties struggle on word problems which is reflected in performance on standardized achievement tests (Kingsdorf & Krawee, 2014; Powell et al., 2008).

Many studies have had a narrow focus on accuracy reflected by direct retrieval strategies on computational problems. Zhang et al., (2014) examined multiple strategies used by low and high achieving students and found groups differed significantly in problem solving accuracy on multi digit computation problems. Research has suggested difficulty with problem solving accuracy using direct retrieval strategies by lower achieving students and/or students with mathematics difficulties may be due to retrieval and retention difficulties, specifically long term and working memory deficits (Geary & Hoard, 2005; Swanson et al., 2004). The quality of strategy used (direct retrieval vs non retrieval) may impact accuracy of problem solving, without the ability to retrieve facts students face increased processing demands using inefficient methods often leading to increased errors (Woodward, 2006). However, even when students are required to only use retrieval to solve simple problems, students with mathematics difficulties had significantly lower accuracy (Geary et al., 2000).

The findings that the MAP-Math Number and Operation strand alone significantly predicted accuracy of problem solving is of interest. Geary, Hoard, and Hamson (1999; 2000) assessed performance of students with Math Difficulty only (MD-only) or Math Difficulty and Reading Difficulty (MDRD) on various aspects of mathematical cognition and reported both groups demonstrated deficits in computation (addition) skills, however; there were significant group differences in percentage of errors compared to MD-only students, MDRD students were less accurate, making more procedural and retrieval errors. Previous research focused on fact retrieval suggests under untimed conditions on calculation measures, accuracy of MDRD students was significantly lower than MD-only and control students whereas MD-only student performed similar to control students (Hanich, Jordan, Kaplan, & Dick, 2001) and on timed measures, MD-only students performed significantly better than MDRD across both simple and complex problems (Jordan & Montani, 1997).

Prior research suggests the possibility mathematics retrieval skills may differ as a function of MD subtype based upon distinctive mathematical deficits between MD-only and MDRD students, supporting the need for studies that compare the responses of these subtypes to mathematics instruction and intervention (Powell, Fuchs, Fuchs, Cirino, & Fletcher, 2009) The study conducted by Powell et al., (2009) proposed MD subtypes differ in their skills and have difficulty for various underlying reasons, and therefore respond differently to interventions. Results were suggested to provide evidence of an interaction between intervention condition and MD subtype with no significant difference among intervention conditions for MDRD but MD-only data suggest improved performance. Students in the fact retrieval tutoring significantly outperformed students in the procedural ($d=1.11$) and control conditions ($d=1.50$). However, previous studies did not examine the strategies used to retrieve answers and acknowledge possible differences in terms of strategy type and efficiency of strategy used by MD-only and MDRD students.

Academic Achievement and Quality

The third research question examined the relationship between student performance on a measure of academic achievement on quality of strategy in multiplication word problem solving. Regression analysis revealed that achievement measures did explain a significant proportion of variance in quality of strategy used. Moreover, further analysis revealed performance on MAP Math-Number and Operation strand alone significantly predicted quality of strategy used on a measure of word problem solving.

Results from this study are consistent with previous research findings on quality of strategy use. Anghileri (1989) examined children's understanding of multiplication and results suggested students demonstrating above average skills most often used a direct retrieval method, however; for all other students on a majority of test items solved utilization of other strategies such as modeling and counting were more frequently observed. Previous research suggests children with mathematics difficulties tend to use more developmentally immature and less efficient strategies more commonly used by younger students (Baroody, 2003;Geary et al., 2011; Geary & Hoard, 2005; Siegler, 1988;Tournaki, 2003;Woodward, 2006).

Geary, Brown, Samaranayake (1991) studied strategy choice in addition procedures of students who test scores indicated a continued need for remedial education in mathematics and found children in this group tended to use the retrieval strategy less frequently and counting strategies more frequently Results from the study were also consistent within the context of the strategy choice model (Siegler, 1988; 1995). When examining the pattern of development, children without mathematics difficulties across time increased reliance on direct retrieval, demonstrated fewer errors and decreased usage of less efficient strategies, whereas students with difficulties across time relied on the less efficient strategies, rarely used direct retrieval and when used often resulted in errors. Similar results were reported by Zhang (2011; 2014), suggesting low achieving students used advanced strategies less frequently and were more likely to not provide a response indicating lower strategy quality. Thus, research has consistently found that children with mathematics difficulties with and without reading difficulties differ from peers in ability to use retrieval strategies to solve both simple computations and word problems (Geary, 2005; Temple & Sherwood, 2002, Zhang, 2013).

Similar to the previous research question, the MAP Math Number and Operation strand alone significantly predicted quality of strategy on word problem solving. Students with lower achievement scores and mathematics difficulties perform significantly below other students on word problem solving tasks (Jordan & Hanich, 2001). However, when students with MDRD and MD-only are compared, MDRD students use sophisticated strategies in problem solving less frequently, are slower to progress to using more efficient procedures consistently across time and grades, and rarely catch up to average performing peers than MD-only students.

The mean strategy quality score was 4.20 out of a possible 5 points. Lower achieving students may have used a higher quality of strategy such as product response, however; use of this strategy may have resulted in either an incorrect response or even with a correct response, demonstrated only memorization not conceptual understanding of multiplication. In one study, a sample of 4th through 6th graders demonstrated they could solve a multiplication problem, however; almost 75% of fourth graders showed no multiplicative context when asked to write a story problem for a single digit fact (O'Brien & Casey, 1983). Another study reported less than 30% of fourth graders and only half of fifth graders consistently demonstrated a deeper understanding of multiplication (Clark & Kamii, 1996). The use of developmentally immature and inefficient strategies has been suggested to be a result of poor conceptual understanding (Geary et al., 2000; Sherin & Fuson, 2005; Woodward, 2006; Zhang, 2014) or poor working memory (Bryd-Craven, Hoard, & Geary, 2002; Woodward, 2006).

Learning Hierarchy and Accuracy

The fourth research question examined the extent to which phases of Learning Hierarchy effects accuracy of word problem solving strategy used. Results indicated significant difference between group on accuracy of word problem solving with product responseand counting strategies. Specifically students in the later phases of the hierarchy AF (Accurate and Fast) were more accurate in word problem solving using product responsethan students in earlier phases IS (Inaccurate and Slow) and AS (Accurate and Slow). Additionally, students in the AS phase were more accurate in word problem solving using product response than students in the earliest phase (Inaccurate and Slow) of the hierarchy. Using the counting strategy, similar results were found with students in the later phase more accurate in word problem solving than students in the earliest phase. Many studies have examined student differences and response to intervention using board measures of standardized achievement with only recent focus on student level of learning (Burns, 2011; Cates & Rhymer, 2003; Codding et al., 2007).

Results from the current study align with previous findings suggesting that many students with poor mathematical performance attempt to use direct retrieval strategies but cannot retrieve the facts accurately (Geary & Hoard, 2005; Siegler, 1988). Students in the earliest phase of the Learning Hierarchy are inaccurate and slow, therefore using a more efficient strategy that exceeds understanding and instead is rote memorization may have resulted in an inaccurate response. Additionally, it has been stressed that conceptual understanding and procedural knowledge should be developed prior to memorization, therefore students in the inaccurate and slow phase may not understand the process enough to begin memorization (Miller, Mercer, & Dillon, 1992). It is also worth noting that previous research found that higher level strategies usually generated higher accuracy and faster problem solving speed (Siegler, 2006;2007; Zhang et al., 2014). The current results do not contradict these because it examines the question from the reverse perspective (accuracy and speed on strategy use as opposed to effect of strategy use on accuracy and speed), but does suggest an area in need of future research.

This difference between students' strategy use and accuracy on word problem solving reflects a common challenge of determining instruction that will contribute to students all progressing at the same rate (Woodward, 2006). Significant differences in word problem solving using the product response strategy between students in the two earliest phases (Inaccurate & Slow and Accurate & Slow) may be of interest. Due to the complexity of mathematics problem solving, an incorrect answer may be due to a number of reasons (Kingsdorf & Krawec, 2014). Adding to the complexity of identifying struggles in problem solving, there are conflicting approaches on how to teach strategies, especially for students struggling. Students acquiring the flexible use of multiple strategies have been identified by the National Research Council (2001), however; instruction for students with mathematic difficulties seems contradictory, suggesting explicitly teaching a single strategy that will be most effective, using a variety of instructional methods (Gersten et al., 2009; Krosenberg & VanLuit, 2002).

Although considered a less efficient strategy, research suggests counting strategies elicit initial acquisition of multiplication, keep computational fluency connected to conceptual understanding and give students opportunity to practice to support understanding and fluency (Cooney et al., 1998; Kilpatrick et al., 2001). Students in the inaccurate and slow phase may not have a firm understanding of multiplicative thinking and require additional review before moving to fluency and application tasks, therefore utilization of a strategy requiring memorization may not be appropriate due to frequent errors in recall (Geary et al., 2011). Direct retrieval strategies considered a faster approach to solving problems are often observed with students who are confident in their memory of facts, whereas, slower approaches, often referred to as back up strategies may be more time consuming, however; may increase accuracy of problem solving when students are uncertain of their memory (e.g., direct retrieval) skills (Siegler, 1988; 1995; Zhang et al., 2013). Scoring an answer to a problem as simply incorrect provides no specific information, whereas examining strategy use and accuracy of problem solving within the context of the Learning Hierarchy model may be beneficial to guide instruction or intervention and increase likelihood of effectiveness

Learning Hierarchy & Strategy Quality

The fifth research question examined the extent to which phases of Learning Hierarchy effect quality of word problem solving strategy used. Results indicated significant difference between groups on strategy quality on word problem solving. Specifically students in the later phases of the hierarchy AF (Accurate and Fast) utilized higher quality strategies in word problem solving than students in earlier phases, Inaccurate and Slow and Accurate and Slow. Results from the current study align with models suggesting students initially learning a skill tend to use less efficient strategies more frequently than when students have more experience with the skill (Siegler, 2007; Geary, 2011; Zhang, 2011; 2014). This is not surprising given that proficiency with mathematical operations such as multiplication is considered to be developed over time and students learn facts progressing from simple to more advanced methods (Anghileri, 1989; Issacs & Carroll, 1999; Kilpatrick et al., 2001; Sherin & Fuson, 2005).

Results also showed students in the IF (Inaccurate and Fast) phase utilized significantly more higher quality strategies in word problem solving than students in the earliest phase (Inaccurate and Slow) of the hierarchy. Zhang et al., (2014) found low achieving students use efficient strategies less frequently, but similar to this study did not consider accuracy when coding for quality of strategy utilized, therefore, students may be using procedures considered to be more efficient yet not reaching correct response so consideration of quality should be interpreted with caution. Efficient strategies contribute to the development of computational fluency, however; direct retrieval strategies considered rote memorization by some is not fluency. Wallace and Gurganus (2005) suggest that mathematical fluency consists of a deeper understanding of concepts and flexibility. The National Council of Teachers of Mathematics emphasizes the importance of computational fluency stating, “Developing fluency requires a balance and connection between conceptual understanding and computational proficiency” (NCTM, 2000, p.35). Therefore, students in the Inaccurate and Fast group may be using a strategy considered to be more efficient or higher quality, however; not demonstrating fluency in multiplication problem solving because the strategy results in an incorrect response.

Using less efficient strategies is often associated with low achievement (Zhang et al., 2014) and has been demonstrated in several studies (Geary, 2011; Siegler, 1995; 2007; Woodward, 2006). However, further examination of lower quality strategies may be of use in identifying level of student understanding. Non-retrieval procedures which were coded as a lower quality compared to direct retrieval procedures are stipulated to be more likely to be executed incorrectly (Siegler, 2007). For example, solving problems using the decomposition procedure also referred to as derived fact, requires students to solve a multiplication problem different than the one presented and then has an addition process to reach the final answer (LeFevre et al., 1996), therefore students may attempt to use a more efficient strategy to avoid additional procedures even if the strategy results in an inaccurate response. Siegler’s models (1988; 1995; 2007) on selection of procedures are suggested to be shaped by the familiarity of problem and the results of performance. Lower quality strategies may be used more due to resulting in the correct response and potentially highlighting consideration of strategy quality and accuracy versus consideration of frequency of higher quality strategies.

Implications

The results of the current study are mostly consistent with previous research and have potential implications for practice, research and theory. I will discuss the potential implications below. Potential implications for research and practice are presented as suggestions for future discussion or consideration within research and practice, therefore implications based upon study data should be interpreted with caution.

Potential Implications for Future Practice

Previous studies have examined assessing various types of knowledge for the purposes of understanding student skills and identifying appropriate intervention. The Learning Hierarchy embedded within these frameworks may provide additional insight into student understanding and proficiency. Within initial support from the current study for the Learning Hierarchy as a validated conceptual framework for multiplication, future research and practice may want to consider phases of learning within a specific concept or skill to analyze the deficits of students who struggle with multiplication.

Mastery of multiplication based upon immediate retrieval of correct responses may only demonstrates student ability to complete a timed task but not necessarily understand it (Kling & Bay-Williams,2015). Thus, practitioners could consider assessing and intervening for strategies beyond direct retrieval in multiplication problem solving to better understand student difficulty. Extensive practice learning rules in a rote manner and memorizing in absence of the corresponding concepts implicit to the procedures may results in students completing mathematical computation without understanding (Carpenter et al., 1983). The current data suggest that it is possible to gain some insight into how well a student understands a problem by examining the strategies used. Previous studies report students can carry out routines while lacking understanding of concepts (Carpenter et al., 1983; Hiebert & Wearne, 1996). Previous studies have also demonstrated the effectiveness of using a conceptual knowledge and procedural fluency heuristic to identify skill deficits and target interventions (Burns, 2011; 2012) and future research may consider incorporating assessment of strategy into various measures. Although the current data are consistent with the previous research regarding conceptual understanding of mathematics, it was not directly research in the current study and is an area for future research.

Implications for Theory

Theoretical and conceptual frameworks provide a structure to guide practices, identify problems and solutions, and advance intervention research (Burns, 2011; Hughes, 2015; Tharinger, 2000; Tilly, 2008). Haring and Eaton's (1978) Learning Hierarchy has frequently been used as a guide for instruction and intervention development for reading (Ardoin & Daly, 2007), but much less so for mathematics. The Learning Hierarchy is a framework that differentiates levels of learning that are testable through observations of skills therefore suggesting a potential theoretical basis for learning within instruction and implementation of intervention. This study provides initial support for the validity of the Learning Hierarchy for mathematics, at least for multiplication. Additional research may lead to a strong intervention heuristic in which practitioners examined accuracy and fluency of multiplication skills to suggest interventions with high likelihoods for success.

Directions for Future Research

Future research could examine the Learning Hierarchy model within a conceptual and procedural knowledge framework as a potential heuristic for identifying student difficulties and targeting interventions in mathematics as both frameworks have demonstrated promising results within instruction and intervention research (Burns, 2011; Burns et al., 2015). Moreover, when using the Learning Hierarchy as a conceptual framework for other heuristics, the operational definitions of accuracy and fluency used to determine student level of learning may be of interest within future research. Kling and Bay-Williams (2015) address fluency development suggesting the current understanding of fluency with multiplication is not consistent in literature or practice, ranging from being considered a measure of speed to “skill in carrying out procedures flexibly, accurately, efficiently, and appropriately” (CCSSM, 2010, p.6). Multiplication fluency occurs as students progress through three developmental phases (Baroody, 2006) suggesting the effectiveness of approaches to learning multiplication facts are dependent on phases and emphasizing explicit development of reasoning strategies results in increased mastery as well as a way to regenerate a response if it cannot be recalled . Considering fluency beyond a measure of speed within the Learning Hierarchy framework of conceptual and procedural knowledge may provide addition insight into student understanding and assist intervention development. However, considerable additional research is still needed.

Although the current study supports the use of the Learning Hierarchy as an intervention heuristic, it only supported the structure and did not examine intervention utility. Future researchers could use the fluency and accuracy criteria used here to identify interventions for multiplication based on the Learning Hierarchy to determine if students differentially respond as predicted.

Future research may extend the work of the current study to further examine strategy use by students within various phases of the Learning Hierarchy using different measures of strategy use. Perhaps other assessment approaches that are used for measures of conceptual knowledge such as semi-structured interviews (Hiebert & Wearne,1996; Kling & Bay-Williams, 2014) or computational fluency such as CBM-M fluency probes used in combination with error pattern analysis (Dennis, Calhoon, Olson, & Williams, 2014; Kingsdorf & Krawec, 2014) may support the current study or lead to a better understanding of student understanding and specific skill deficits.

The current study could also be replicated and extended to various age groups and mathematical skills. For example, the study design could be replicated with multi-digit addition or division problems, or even with algebraic concepts with older students. In addition to extensions, future researchers could replicate the results in a line of inquiry to better support the validity of the Learning Hierarchy as a framework to better understand multiplication or mathematics in general.

Strengths and Limitations

It is important to consider the findings of the current study within the context of its limitations as well as acknowledge strengths. This study sought to validate the

Learning Hierarchy as well as examine the strategy use during multiplication word problem solving within achievement and learning level phases. As stated above, the Learning Hierarchy has been used within intervention and practice demonstrating promising results, but this is one of few studies that focused on the model for mathematics and perhaps the only one that examined the validity of the underlying structure. First, the current study included 492 participants which provided a large sample yet may have limited assessment measures and methods chosen for the study. However, the total sample size may highlight study results validating the Learning Hierarchy conceptual framework by finding a non-significant difference between the sample of observed and expected data often more difficult to indicate with larger sample size. The current study occurred over several days across school sites within the district, however; data were obtained from a single assessment session which may have not been sufficient enough to examine strategy use within word problem solving. Additionally, the current study participants were from a single school district limiting possible implications that can be drawn from the study.

Another limitation to the current study includes the method of assessment for strategy use relied on students showing work during problem solving on the word problem measure and not on the CBM-M computation. Sherin & Fuson (2005) suggest students may use different strategies throughout their development as well as during other tasks, therefore different strategies may have been seen if the computation measure in addition to the word problem solving measure was used for coding strategies. Due to sample size and resource restrictions some assessment methods such as student interviews or think alouds were not possible. Many academic strategies are cognitive and covert, therefore demonstrations and examples may need to be assessed and provided via verbal mediation or self-talk (Baker, Gersten, & Scanlon, 2002; Howell & Schumann, 2011). Furthermore, the current study's word problem assessment consisted of one step single digit multiplication problems. If the word problem solving measure include more complex probems, more complex strategies may been seen. The current study utilized a taxonomy of strategies adapted from previous research (Siegler, 1988; 2007; Zhang et al., 2014) and although this list of strategies is consistent with other suggested models, the list should not be considered exhaustive, specifically noting the exclusion of hybrid strategies. When coding strategies,a product reponse strategy within this study as in previous studies including direct retrevial type strategies was inferred from answer only responses, therefore a different or hybrid strategy may have been utilized to reach response but not demonstrated by student in written response. Additionally, a higher level of inference used during coding by considering an answer only response as most efficient compared to other strategies may have been misleading and suggested an overly simplistic description of this strategy.

Previous studies have included 3rd, 4th, and 5th graders in their sample usually grouped by achievement level, therefore students were mostly likely have had significant classroom instruction in multiplication and could be using more mature strategies due to experience with problems. The current study sample included only 3rd graders recently introduced to multiplication therefore this sample may have provided novice learners as well as more advanced/expert learners unlike previous studies. Additionally, previous research has examined strategy development and difference of students based upon performance on measures of achievement, whereas this current study extended previous research within a Learning Hierarchy framework considering student phase in learning within a specific skill instead of overall performance.

Conclusion

Developing proficiency with mathematical skills and concepts is a primary instructional goal; however, the low percentage of students performing at or above proficient levels is still concerning (U.S. Department of Education, 2013). Given the importance of mathematics proficiency it is critical to understand how concepts and skills develop, so educators can intervene early to remediate mathematics skills deficits. Intervention selection is more likely to be successful when it is based on an evidence-based heuristic that can correctly target and predict intervention effectiveness to meet students need based on student performance (Burns, 2011; Burns et al., 2010; Codding et al., 2007) and the Learning Hierarchy is a conceptual framework that has demonstrated promising results.

Low mathematics achievement is associated with low proficiency in basic operations and the inability to understand and apply advanced strategies to solve mathematics problems (Geary, Hoard, & Byrd-Craven, 2004). Word problem solving is a critical aspect of mathematics instruction and an important indicator of overall mathematics skill (NCTM, 2000). However, word problem solving involves multiple skills and the inability to successfully complete word problems does not identify in which underlying skill(s) the student is deficient.

Mathematical content is often a balanced combination of procedure and understanding (CCSS, 2010; National Mathematics Advisory Panel [NMAP], 2008). Identifying interventions to meet the needs of students is a challenge. The current study sought to examine the validity of the commonly used Learning Hierarchy as a conceptual framework as well as extend previous research examining strategy use in solving multiplication word problems. A better understanding of strategic development may assist with deciding the most effective instructional and intervention practices for proficiency in multiplication, but additional research is needed. The interesting results found here suggest that the additional research is warranted.

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Appendices

Appendix A.

Single skill Curriculum Based Measure-Math Computation (CBM-M)

2	5	4	7	3	6
x 3	x 6	x 7	x 3	x 8	x 7
9	3	6	2	2	8
x 3	x 7	x 2	x 5	x 2	x 2
7	5	4	5	7	4
x 9	x 4	x 8	x 9	x 5	x 4
4	3	7	8	9	7
x 6	x 6	x 4	x 6	x 4	x 8
9	4	3	4	4	7
x 5	x 9	x 4	x 2	x 3	x 6
8	5	5	2	8	9
x 7	x 8	x 7	x 8	x 3	x 2

Appendix B

Word Problem Solving Measure

ID#____

1. A man is selling peanuts at the zoo. At $2 for a bag of peanuts how much will two bags cost?

2. Juana counted all the boxes of pencils in the supply room. There were seven full boxes. If each box holds 8 pencils, how many pencils were there all together?

3. A zoo has two times as many lions as goats. The zoo has five goats. How many lions does the zoo have?

4. A classroom has 7 boxes of pencils. There are 6 pencils in each box. How many pencils are there in all?

5. There are four rows of stamps on a page. Each row has five stamps. In all how many stamps are there?

6. Kate has 3 bags. Each bag holds six marbles. How many marbles are there in all?

7. Fruit trees are growing in six rows. There are four trees in each row. How many trees are there in all?

8. There are 9 bundles of fishing poles. There are 8 fishing poles in each bundle. How many fishing poles are there in all?

9. The airplane cabin has 8 rows of 4 seats. How many seats are in the cabin?

10. Diane chooses from 3 kinds of ice cream and 4 toppings. How many combinations of ice cream and toppings are there?

11. If there are 7 days in a week, how many days are in two weeks?

12. A parking garage charges $5 for each hour. How much will it cost Maria to park for 3 hours?

13. Lee is paid for each piece of furniture he builds. He is paid $8.00 for each table he assembles. This week he assembled 6 tables. What is his pay for the week?

14. A train station has 4 rows of seats. There are 9 seats in each row. How many seats are there in the station?

15. Art drove 8 miles an hour for 3 hours. How far did he drive?

16. Four people each have six dollars. How much money do they have together?

17. Highway mileage varies from car to car. Judy's car gets five miles to a gallon of gas and Bob's car gets six miles to a gallon of gas. How many miles can Judy drive on 6 gallons of gas?

18. If you have 3 pencils in each hand, how many pencils do you have all together?

19. A fair has 9 different contests. If each contest awards 4 ribbons, how many ribbons are awarded at the fair?

Appendix C

Single- digit Multiplication Word Problem Solving Measure Coding Rubric

Strategy Type	Definition	Example	Score
Incorrect Response	Numbers and procedures put together reflecting erroneous selection and/or application of operation.	4x8=12, 4x8=48, or 4x8=84	1
Counting	Representations and pictures drawn to model the problem and counting how many in total.	Model the problem with representation (tallies, pictures, graphs) and count representation	2
Repeated addition	Repeatedly adding one factor for as many times as the other factor	3x4= 4+4+4 or 3+3+3+3	3
Decomposition	Breaking down one or both factors and referring to known problems to create simpler problems to solve.	7 x 8= 7 x 7 + 7 or 6 x 7= 3 x 7 x 2 or 8 x 7= 2 x 7 x 2 x 2 (e.g., Doubles, doubles plus one)	4
Product Response	Numerical responses with or without an accurate number sentence.	Problem 3 x 8= Student response 24	5

Adapted from Multiplication Strategies and Appropriation of Computational Resources (Sherin & Fuson, 2005), Zhang et al, 2011;2014, and Strategy Choice Procedures and the Development of Multiplication Skill (Siegler, 1988),