Singlet Glueballs In Klebanov-Strassler Theory

A DISSERTATION
SUBMITTED TO THE FACULTY OF THE GRADUATE SCHOOL
OF THE UNIVERSITY OF MINNESOTA
BY

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IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
Doctor of Philosophy

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April, 2016

Acknowledgements

First of all I would like to thank my scientific adviser - Arkady Vainshtein for his incredible patience and support throughout the course of my Ph.D. program. I would also like to thank my committee members for taking time to read and review my thesis, namely Ronald Poling, Mikhail Shifman and Alexander Voronov. I am deeply grateful to Vasily Pestun for his support and motivation. Same applies to my collaborators Dmitry Melnikov and Anatoly Dymarsky who have suggested this research topic to me. I am thankful to my other collaborator - Peter Koroteev.

I would like to thank Emil Akhmedov, A.Yu. Morozov, Andrey Mironov, M.A. Olshanetsky, Antti Niemi, K.A. Ter-Martirosyan, M.B. Voloshin, Andrey Levin, Andrei Losev, Alexander Gorsky, S.M. Kozel, S.S. Gershtein, M. Vysotsky, Alexander Grosberg, Tony Gherghetta, R.B. Nevzorov, D.I. Kazakov, M.V. Danilov, A. Chervov and all other great teachers who have shaped everything I know about Theoretical Physics.

I am deeply grateful to all my friends and colleagues who have contributed to discussions and supported me throughout those years including A. Arbuzov, L. Kushnir, K. Kozlova, A. Shestov, V. Averina, A. Talkachova, A. Talkachou, A. Abyzov, V. Poberezhniy, A. Alexandrov, G. Nozadze, S. Solovyov, A. Zotov, Y. Chernyakov, N. Amburg, D. Vasiliev, A. Monin, A. Levchenko, R. Makhnenko, S. Paganelli, J. Magán, A. Lukyanov, D. Talalaev, G. Sharygin, J. Cembranos, W. Vinci, A. Moustafaev, A. Kobrinskiy, S. Dubinskiy, D. Ryzhikh, R. Lutchyn, M. Oleinikov, V. Dolgushev, S. Loktev, A. Kotov and many others.

Finally I would like to extend my gratitude to my family.

Dedication

To my family.

Abstract

In this thesis we complete the singlet glueball sector analysis of the $\mathcal{N} = 1$ supersymmetric Klebanov-Strassler gauge theory. Employing the string theory *holographic approach* we come up with a prediction of the spectrum of lightest glueballs in $SU(N)$ $\mathcal{N} = 1$ supersymmetric Yang-Mills theory at large N. Interestingly the spectrum of some of the glueballs is consistent with the lattice results for QCD glueballs.

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Chapter 1

Introduction

At strong coupling our main systematic tool – perturbation theory – does not apply. String theory offers another systematic tool for the analysis of strongly coupled physics. This tool is based on a certain weak-strong coupling duality: the large N gauge-gravity duality or holographic correspondence.

In this thesis we focus on the application of the gauge-gravity duality method to the calculation of the spectrum of glueballs in the $\mathcal{N} = 1$ supersymmetric Yang-Mills theory. For this purpose the holographic model of Klebanov and Strassler [5] will be employed. We make predictions about the masses of the glueballs with low spin (0 or 1). We also find that the available results, *e.g.* the ratios $m_{1^{+-}}/m_{2^{++}}$ and $m_{1^{--}}/m_{2^{++}}$, are also consistent with the lattice results beyond our expectations.

The thesis is based on a series of works [6]-[7], in which the glueball spectrum of the Klebanov-Strasser theory was studied. Some earlier works, see [8], [9], analyze the spectrum in the non-supersymmetric case. In the Klebanov-Strassler setup some issues of the non-supersymmetric holographic calculation are resolved, such as extra degeneracy of the 2^{++} and 0^{++} states.

The Klebanov-Strassler (KS) theory [5] is one of the most interesting examples of the holographic correspondence [10]. It provides the dual description of a $\mathcal{N} = 1$ supersymmetric non-conformal gauge theory with dynamical chiral symmetry breaking and confinement. A relatively simple construction from the point of view of the correspondence, it is often proposed to model the real world physics in such areas as viable string compactifications, inflation, supersymmetry breaking and more (e.g. in [11]).

Our motivation for working with the KS solution is the analysis of the low energy degrees of freedom (glueballs) of a theory that is a close relative of the pure gauge $\mathcal{N} = 1$ supersymmetric Yang-Mills theory. Holographic methods applied to such a class of theories could provide qualitative estimates for physical observables in the strong coupling regime. In particular such estimates would be useful in the setup discussed in [12], where we are interested in computing transition matrix elements of operators in the strongly coupled hidden sector [13]. In the conclusion 5 we speculate on consistency of our results with existing lattice calculations of glueball masses. For further motivation for calculation of glueball spectra using holographic correspondence see [14].

1.1 String theory approach

String theory was born for the purpose of describing the dynamics of strong interactions (of the QCD). At the present time it is the most promising candidate for the role of self-consistent fundamental quantum theory, that encompasses the ideas of Grand Unification, quantum gravity, supersymmetry, compactification of extra dimensions, topological theories, generalized sigma-models, duality and many more. String theory as the fundamental theory is far from being completed. However, it has proven itself as an incredibly rich framework in the course of its development. The methods that have been developed can be successfully applied to solve traditional problems of quantum field theory.

Non-abelian gauge theories form a foundation for the Standard Model. The important feature of those theories is that the coupling constant is decreasing with increasing energy and vise versa - a phenomenon known as “asymptotic freedom”. Thus at high energies the perturbation theory can be used to get asymptotically convergent series in coupling constant. On the other hand, at low energies the perturbation theory in QCD becomes less and less efficient until it stops working altogether at a characteristic scale Λ_{QCD}.

Thus, in order to understand the physics at low energies, to understand confinement as well as the spectrum of the hadron masses and their low-energy dynamics, it is required to develop non-perturbative methods. Thanks to the rich inherent structure of string theory many of such methods have been developed within its framework. The core instruments in those methods are the notions of supersymmetry and duality.

Like any symmetry, supersymmetry (the symmetry between bosonic and fermionic degrees of freedom) constrains the dynamics of the system. In some sense the physics of supersymmetric systems differs from the physics of non-supersymmetric systems in the same way as mathematical analysis of functions of complex variables from mathematical analysis of functions of real variables. The first case unlike the second one is strongly constrained, in particular the differentiability of a function in a vicinity is enough for it be analytically continued. The holomorphicity is the underlying reason for being able to get many of the exact solutions in supersymmetric field theories. However the theories applicable to nature do not display supersymmetry at low energies. Nevertheless, there are many indications that with growing energy scale the supersymmetry is being restored. Furthermore, many non-perturbative effects of non-supersymmetric theories can be modeled by equivalent effects in supersymmetric theories. For instance (and this example is most relevant for the subject of this work) a phenomenon of confinement is present in $\mathcal{N} = 1$ supersymmetric gauge theories.

Under duality one understands the existence of two alternative descriptions of one and the same physical system. Namely, it is the existence of one-to-one map between the spaces of states and the existence of a dual action, that generates equivalent dynamics. If it happens that the strong-coupling regime in one description corresponds to a weak-coupling regime in the other description, then such a dual description becomes a powerful tool to study the non-perturbative dynamics of the original system. Indeed, to calculate the non-perturbative spectrum the original theory it is enough to make a transformation of the states into the dual theory and do the calculations within its framework, using perturbation theory where it is rightfully applicable.

Historically string theory has been formulated as a first-quantized theory, i.e. through the set of rules that define the scattering amplitudes given by Feynman diagrams. The difference from the traditional field theory is that the Feynman diagrams are composed not from one dimensional lines but rather from two-dimensional surfaces. From the requirement that the quantum anomalies cancel each other the constraints on the number of dimensions of space-time have been derived as well as the type of auxiliary structures on string world sheets. In 80's it has been shown that there exists five perturbatively different types of superstrings in the critical space-time dimension 10: Type I, IIA, IIB, Het SO(32) and Het $E_8 \times E_8$. However, after the discovering of D-branes, it was observed that some of these string theories are dual to each other. It has been conjectured by Witten and eventually established that different string theories are simply different perturbative formulations of one and the same theory — the "M-theory". The low-energy limit of M-theory is 11 dimensional classical supergravity, but the precise formulation of the quantum M-theory remains an open problem.

The key objects in string theory that lead to formulation of the dualities are called Dp-branes: the dynamical objects that have p space-like dimensions. For instance, a D3 brane is a $3+1$ dimensional space-time object.

A particular interesting duality relevant to the description of QCD by gluonic strings is *open-closed* string duality which can be traced back to the 't Hooft large N expansion.

Therefore, in the context of gauge theory the *open-closed* string duality is called *large N* duality.

The first exactly solvable example of such dualities was observed in the late 80s and early 90s as the duality between $N \times N$ matrix models (open strings) and Liouville quantum gravity (closed strings). The $N \times N$ matrices can be thought as the fields of the 0-dimensional field theory on the stack of N branes of $SU(N)$ gauge theory.

The first exact example of the large N open-closed string duality for the full fledged $3+1$ quantum field theory is the famous Maldacena correspondence between an $\mathcal{N}=4$ supersymmetric Yang-Mills (SYM) in four dimensions and a type IIB theory of superstrings in the $AdS_5 \times S_5$ (AdS) background, which arises as a metric in the vicinity of the horizon of N D3-branes.

In this case the 't Hooft coupling constant $\lambda = Ng_{YM}^2$ corresponds to the string tension: $T = \frac{R^2}{2\pi\alpha'} = \frac{\sqrt{\lambda}}{2\pi}$, where $R = \sqrt{\alpha'}(g_{YM}^2 N)^{\frac{1}{4}} = l_s \lambda^{\frac{1}{4}}$ — is the curvature radius of the $AdS_5 \times S^5$ space. Here $l_s = \sqrt{\alpha'}$ is the string scale. Furthermore, the string coupling constant $g_s = e^{\Phi}$ and $g_{YM}^2 = 4\pi g_s$. It can be easily observed that the 't Hooft's planar limit ($N \to \infty$, λ constant) corresponds to non-interacting strings – genus 0 Riemann surfaces.

In the regime $\lambda \to \infty$ a perturbative description of the gauge theory becomes completely meaningless. However in the same limit $\lambda \to \infty$ the type IIB string theory in $AdS_5 \times S^5$ becomes a classical supergravity in $AdS_5 \times S^5$ background. Thus in this limit we have an example of duality between a gauge theory and gravity, which is an amazing fact in itself. It may seem that a gauge theory and gravity are two completely different descriptions that have nothing in come and can not be mapped into each other. However, it is exactly the fact that this duality interchanges the strong-coupling and weak-coupling regimes that makes this equivalence possible. Hence the first theory in strong-coupling regime may be replaced by the second theory in weak-coupling regime and vice versa. This, on one hand, makes it possible to make non-trivial predictions about non-perturbative dynamics of each one of them and on the other hand it makes it hard to rigorously prove of the duality itself. Of course in this duality differs crucially from the dual descriptions through gluonic strings of the real QCD, first of all because it has the $\mathcal{N} = 4$ supersymmetry and conformal symmetry. Nevertheless there are some common features.

Although by itself the $\mathcal{N} = 4$ model is very deep and interesting, *e.g.* at high energies it may qualitatively model QCD, it cannot serve to the purpose of describing the physics of confinement. For the latter one should find a way to break the excessive symmetries of the $\mathcal{N} = 4$ theory. There are few known way to do this in holography.

After Maldacena formulated gauge-gravity duality for 4d gauge $\mathcal{N} = 4$ superconformal Yang-Mills theory there was a sequence of works which gradually extended this duality for 4d gauge theories with less number of supersymmetries and finally for gauge theories without conformal symmetries and confinement such as Klebanov-Strassler duality.

The Klebanov-Strassler duality is the duality between $\mathcal{N} = 1$ supersymmetric gauge theory with gauge group $SU(N + M) \times SU(N)$ and certain chiral fields and superpotential on one side, and the type IIB string theory on a certain background called *warped deformed conifold* on the other side. In the regime when massive closed string excitations are decoupled the IIB string theory can be approximated by its low energy field theory: 10 dimensional IIB supergravity.

The confining $\mathcal{N} = 1$ KS gauge theory has massive glueball states. To determine the spectrum of these glueball states is a hard problem of the non-perturbative gauge theory.

On the other hand, using the gauge-gravity KS duality the hard question about the glueball spectrum in gauge theory can be mapped to much more accessible question about the spectrum of linearized excitations of the IIB classical supergravity on top of the KS background.

This thesis focuses on the computation of the glueball spectrum in KS gauge theory. We map the glueball spectrum problem from KS $\mathcal{N} = 1$ supersymmetric gauge theory to IIB supergravity. The linearization of II supergravity equations in KS background leads to a certain system of second order linear partial differential equations with variable coefficients that is analyzed in details.

1.2 Outline

- Chapter 2 the general principles of large N gauge-gravity correspondence are being explained. The simplest case of such a correspondence is the duality between the 4-dimensional $\mathcal{N} = 4$ supersymmetric Yang-Mills theory and type IIB string theory in the 10-dimensional space $AdS_5 \times S^5$ conjectured by Maldacena in [10]. However if one is interested in "modeling" QCD, then the $\mathcal{N} = 4$ supersymmetric Yang-Mills model has very limited relevance. In particular it is helpless if one attempts to model the phenomenon of confinement. In this Chapter the various attempts to find a model that would mimic certain features of QCD such as confinement, are being presented eventually leading to the main framework of the research presented in this thesis - the Klebanov-Strassler model.

- In Chapter 3 the details on Klebanov-Strassler model are being outlined. The correspondence between the Klebanov-Strassler $\mathcal{N} = 1$ supersymmetric non-conformal gauge theory with dynamical chiral symmetry breaking and confinement and its dual - the Klebanov-Strassler type IIB supergravity.

- Chapter 4 is devoted to glueballs in KS model — the main subject of this thesis. The method that is being used to study glueballs in KS-theory is being presented. The previous results are being summarized in this chapter as well as the new results obtained in the course of the research presented in this thesis.

- In the final Chapter 5 we summarize the new results and propose an outline for further application of the same methodology.

Chapter 2

Large N duality in string theory

In this section we are going to discuss a particular class of solutions to the low energy effective action of IIB supergravity described in section 2.1.

We start in section 2.2 from a maximally supersymmetric gauge theory in four dimensions, the $\mathcal{N} = 4$ super Yang-Mills theory, and describe its gravity dual appearing as the classical background of the IIB supergravity near the horizon of the stack of N extremal D3 branes. The correspondence between the gauge theory on the D3 branes and the IIB supergravity theory in the 10-dimensional bulk is the famous Maldacena correspondence[10].

Then in section 2.4 we proceed to the gauge theories with smaller than $\mathcal{N}$ number of supersymmetries which can be obtained from the maximal $\mathcal{N} = 4$ supersymmetric Yang-Mills by the orbifold construction after Douglas-Moore [15] for the orbifold $\mathbb{C}^2/\Gamma$, Douglas-Green-Morrison [16] for the orbifold $\mathbb{C}^3/\Gamma$, Green [17], Kachru-Silverstein [18], Bershadsky-Johansen [19], Lawrence-Nekrasov-Vafa [20], Ferrara-Zaffaroni [21], Hanany-Uranga [22].

Then in section 2.7.3 we continue to the conifold construction of Klebanov-Witten [23] and review Morrison-Plesser [24].

2.1 IIB supergravity

Fields and equations of IIB supergravity

The type IIB supergravity does not come directly from the dimensional reduction of the unique 11-dimensional supergravity. Instead one should use constraints coming from the supersymmetry which define the spectrum of fields and their supersymmetry transformations. The field content can be found from the spectrum of particles of the IIB superstring theory in 10 dimensions.

Bosonic fields of IIB supergravity

The massless bosonic spectrum of IIB supergravity consists of the NS-NS and R-R sector. In the NS-NS sector the IIB supergravity fields are the same as in IIA supergravity

- the dilaton Φ
- the metric g_{MN}
- the antisymmetric tensor $B_{\mu\nu}$

In the R-R sector the IIB supergravity fields are the

- the 0-form potential C
- the 2-form potential C_2
- the 4-form potential C_4 with the self-dual field strength $F_5 = \mathrm{d}C_4$ which satisfies $F_5 = \star F_5$

The IIB supergravity action

The bosonic part of the action of type IIB supergravity in Einstein frame [25]:

$$S_{10} = \frac{1}{2\kappa_{10}^2} \int d^{10}x \left(\sqrt{-g_{10}} \left[R - \frac{1}{2}(\partial\Phi)^2 - \frac{g_s}{12} \mathrm{e}^{-\Phi} (\partial B_2)^2 \right.\right.$$
$$\left.\left. - \frac{1}{2} \mathrm{e}^{2\Phi} (\partial C)^2 - \frac{g_s}{12} \mathrm{e}^{\Phi} (\partial C_2 - C \partial B_2)^2 - \frac{g_s^2}{4 \cdot 5!} F_5^2 \right] - \frac{g_s^2}{2 \cdot 4! \cdot (3!)^2} \epsilon_{10} C_4 \partial C_2 \partial B_2 + \cdots \right) \tag{2.1}$$

The dimension of the curvature R is [length^{-2}], and therefore the 10-dimensional gravitational constant κ_{10}^2 should have dimension [length^8].

The gravitational constant κ_{10}^2 can be related to the string tension $\alpha' = l_s^2$ by

$$\kappa_{10}^2 = \frac{1}{2}(2\pi)^7\alpha'^4 = \frac{1}{2}(2\pi)^7 l_s^8 \tag{2.2}$$

The equations of motions in type IIB supergravity in the Einstein frame [3]:

Variation of the metric g_{MN}:

$$\begin{aligned} R_{MN} = \frac{1}{2}\partial_M\Phi\partial_N\Phi + \frac{1}{2}\mathrm{e}^{2\Phi}\partial_M C\partial_N C + \frac{1}{96}g_s^2\tilde{F}_{MPQRS}\tilde{F}_N^{PQRS} \\ + \frac{g_s}{4}(\mathrm{e}^{-\Phi}H_{MPQ}H_N^{PQ} + \mathrm{e}^{\Phi}\tilde{F}_{MPQ}\tilde{F}_N^{PQ}) \\ - \frac{g_s}{48}g_{MN}(\mathrm{e}^{-\Phi}H_{PQR}H^{PQR} + \mathrm{e}^{\Phi}\tilde{F}_{PQR}\tilde{F}^{PQR}), \end{aligned} \tag{2.3}$$

Variation of the dilaton Φ

$$\mathrm{d}\star\mathrm{d}\Phi = \mathrm{e}^{2\Phi}\mathrm{d}C\wedge\star\mathrm{d}C - \frac{g_s}{2}\mathrm{e}^{-\Phi}H_3\wedge\star H_3 + \frac{g_s}{2}\mathrm{e}^{\Phi}\tilde{F}_3\wedge\star\tilde{F}_3, \tag{2.4}$$

Variation of the 0-form C

$$\mathrm{d}(\mathrm{e}^{2\Phi}\star\mathrm{d}C) = -g_s\mathrm{e}^{\Phi}H_3\wedge\star\tilde{F}_3, \tag{2.5}$$

Variation of the 2-form C_2

$$\mathrm{d}(\mathrm{e}^{\Phi}\star\tilde{F}_3) = g_s F_5\wedge H_3, \tag{2.6}$$

Variation of the B_{MN} field

$$\mathrm{d}\star(\mathrm{e}^{-\Phi}H_3 - C\mathrm{e}^{\Phi}\tilde{F}_3) = -g_s F_5\wedge F_3, \tag{2.7}$$

The self-duality constraint on the 4-form C_4 field strength F_5

$$\star\tilde{F}_5 = \tilde{F}_5, \tag{2.8}$$

Here we use notations

$$F_3 = \mathrm{d}C_2, \quad H_3 = \mathrm{d}B_2, \quad F_5 = \mathrm{d}C_4, \quad \tilde{F}_3 = F_3 - CH_3, \quad \tilde{F}_5 = F_5 + B_2\wedge F_3 \tag{2.9}$$

2.2 $\mathcal{N}=4$ super Yang-Mills and IIB in $AdS_5 \times S^5$

The basic model of the holographic correspondence is the duality conjectured by Maldacena [10] between the 4-dimensional $\mathcal{N}=4$ supersymmetric Yang-Mills theory and type IIB string theory living in the 10-dimensional space $AdS_5 \times S^5$.

2.2.1 Gauge theory / D-brane theory / open string

The $\mathcal{N}=4$ super Yang-Mills 4d gauge theory is the maximally supersymmetric gauge theory in 4 space-time dimensions. This is an exceptional gauge theory on several grounds.

We remind that in [26] Brink, Schwarz and Scherk have defined $\mathcal{N}=1$ supersymmetric extension of Yang-Mills gauge theory in arbitrary number dimensions as follows. Let M_d be the space-time. In addition to the gauge fields (connections on a G-principal bundle) we have the fermionic space of gluino fields. Mathematically, the gluino fields are the adjoint Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ valued sections Ψ of the spin bundle S on M_d. The Yang-Mills action for the supersymmetric theory is

$$S = \frac{1}{g_{\mathrm{YM}}^2} \int_{M_d} \langle F_A \wedge \star F_A \rangle + \star \langle \Psi \cdot \not{D}_A \Psi \rangle \tag{2.10}$$

Here we suppress the indices and use the form notations. The $\star$ denotes the Hodge star operation on differential forms and $\not{D}$ denotes the Dirac operator.

Brink, Schwarz and Scherk [26] obtained beautiful result that $\mathcal{N}=1$ supersymmetric Yang-Mills theory exists only for the four exceptional values of the dimension d of the space-time manifold M_d, namely for $d = 3, 4, 6, 10$.

Moreover, the $\mathcal{N}=1$ supersymmetry algebra in these dimensions corresponds to the algebraic structures of the four famous real division algebras $\mathbb{K}$ as follows:

- the algebra of real numbers $\mathbb{R}$: $\mathcal{N}=1$ SYM in $d=3$ with 2 supercharges
- the algebra of complex numbers $\mathbb{C}$: $\mathcal{N}=1$ SYM in $d=4$ with 4 supercharges
- the algebra of quaternionic numbers $\mathbb{H}$: $\mathcal{N}=1$ SYM in $d=6$ with 8 supercharges
- the algebra of octonionic numbers $\mathbb{O}$: $\mathcal{N}=1$ SYM in $d=10$ with 16 supercharges

The most exceptional $\mathcal{N} = 1$ supersymmetric Yang-Mills algebra is in dimension 10 with 16 supercharges. The fields of 10d SYM are gauge fields A_M, with $M = 0, \dots, 9$ and the Mayorana-Weyl 16-component fermions Ψ (we omit fermionic indices in the notations).

We can start from 10d $\mathcal{N} = 1$ SYM gauge theory and by the standard procedure of dimensional reduction from 10d to 4d define a gauge theory in 4d: namely split the 10d space into 4d and 6d directions and postulate that all fields are constant along the 6d directions. The 10d Lorentz group $SO(9,1)$ is broken into the 4d Lorentz group $SO(3,1)$ and a global group $SO(6)$, the so called R-symmetry group.

Under dimensional reduction from 10d to 4d the 6 components of the 10d gauge field $A_4, \dots A_9$ are converted into the 6 scalar fields in 4-dimensional theory. Those 6 scalar fields are transformed into each otherby the $SO(6)$ R-symmetry group.

Similarly, the 16 fermionic components of the 10-dimensional Mayorana-Weyl fermion Ψ after dimensional reduction to four dimensions give rise to 4 copies of the Mayorana fermions in 4d. The fermionic field of the usual $\mathcal{N} = 1$ 4-dimensional SYM is a single Mayorana fermion. Since the dimensional reduction of 10d $\mathcal{N} = 1$ SYM to 4d produces 4 copies of the fermionic fields for the 4d $\mathcal{N} = 1$ SYM, the resulting theory is called 4d $\mathcal{N} = 4$ supersymmetric Yang-Mills.

The generators $Q_\alpha|_{\alpha=1\dots16}$ of the 10d supersymmetry transform in the same way as the fermionic field Ψ_α under the 10d super-Poincaré group. The supersymmetry algebra is

$$\{Q_\alpha, Q_\beta\} = \Gamma^M_{\alpha\beta} P_M \tag{2.11}$$

where $\Gamma^M_{\alpha,\beta}$ are the 16×16 chiral 10d gamma-matrices.

After dimensional reduction from 10d to 4d, the $\mathcal{N} = 1$ 10d super-Poincaré algebra converts to $\mathcal{N} = 4$ 4d super-Poincaré algebra. The 16 components of fermionic generators of the supersymmetry Q_α are called *supercharges*.

Often it is convenient to label supersymmetric theories and their supersymmetric algebras not by the multiplicity of the minimal supersymmetry they possess (the $\mathcal{N}$-number), but by the total number of the fermionic components of the supersymmetry generators Q_α. Such labelling is invariant under dimensional reduction. In such conventions the $\mathcal{N} = 1$ SYM in 10d and its dimensional reduction, the $\mathcal{N} = 4$ SYM in 4d, are both theories with 16 supercharges.

In string theory the SYM with 16 supercharges in $p+1$-dimensional space-time and gauge group $U(N)$ arises as the low-energy theory on the stack of N parallel Dp-branes.

The $\mathcal{N}=1$ 10d SYM with $U(N)$ gauge group is the low-energy theory on the stack of N D9 branes in IIB superstring theory, and $\mathcal{N}=4$ 4d SYM with $U(N)$ gauge group is the low-energy theory on the stack of N D3 branes in IIB superstring theory.

The gauge theory with 16 supercharges becomes even more special in dimension $d=4$. Namely, 4d $\mathcal{N}=4$ SYM is invariant under conformal transformations of the 4d space-time. Such theory is called *conformal theory.* Moreover, the conformal symmetry of the classical $\mathcal{N}=4$ SYM is preserved on the quantum level: the β-function is exactly 0.

The conformal symmetry and $\mathcal{N}=4$ 4-dimensional supersymmetry together generate the $\mathcal{N}=4$ superconformal algebra. The $\mathcal{N}=4$ superconformal algebra has 32 fermionic generators: the original 16 fermionic charges Q_α and their images under the commutators with special conformal transformations, usually denoted as *special* conformal fermionic generators S_α. The full superconformal group of $\mathcal{N}=4$ SYM is the supergroup $PSU(2,2|4)$. It includes the bosonic factor $SU(2,2)$ locally isomorphic to the 4d conformal group $SO(4,2)$ and the bosonic factor $SU(4)$ locally isomorphic to $SO(6)$ R-symmetry.

We see that the low-energy limit of the theory on the D3 brane, the $\mathcal{N}=4$ 4d SYM, is in fact superconformal theory with 32 supercharges.

The D3 branes are massive solitonic objects in the IIB string theory on which the open strings end with Dirichlet (this is what the letter D stands for) boundary conditions. Large N number of D3 branes deforms the supergravity background of the IIB string theory.

Since the field theory limit of the theory on D3 branes is the superconformal theory with 32 supercharges, we expect to find such limit in the IIB supergravity in the background of large N number of D3 branes. This was constructed in the famous Maldacena paper [10] providing the first precise gauge-gravity duality.

2.2.2 Gravity theory / flux background / closed string

Here we describe the solution known as large N extremal supersymmetric D3-brane solution of supergravity by Gary Horowitz and Andrew Strominger [27], following for the most part the notations of [28]:

$$
\begin{aligned}
ds^2 &= h(r)^{-1/2}(-dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2) + h(r)^{1/2}(dr^2 + r^2 d\Omega_5^2) && (2.12)\\
F_5 &= (1+\star)dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3 \wedge dh^{-1}, && (2.13)\\
h &= 1 + \frac{R^4}{r^4}. && (2.14)
\end{aligned}
$$

where $\star$ is the Hodge operation on differential forms in the 10-dimensional space-time and $h(r)$ is the radial function in the directions orthogonal to the stack of N D3 branes.

This solution appears if one requires the Euclidean symmetry $ISO(3,1)$ and assumes the metric is spherically symmetric in $(10-4)$ dimensions with the source of Ramond-Ramond (R-R) electric charge at the origin:

$$\int_{S^5} \star F_5 = N, \tag{2.15}$$

where S^5 is the 5-sphere surrounding the source.

We have parametrized our space in the following way: the coordinates x_0, x_1, x_2, x_3 represent the $3+1$ dimensional Minkowski space, $d\Omega_5$ is the line element on a 5-sphere S^5 parametrized by some angles, r is an additional radial coordinate which plays a role of an energy scale in the AdS/CFT dictionary.

The parameter R^4 is given by the equation

$$R^4 \equiv 4\pi g_s N \alpha'^2 \tag{2.16}$$

where g_s is the IIB string coupling constant and $T = L^2/(2\pi\alpha')$ is the string tension.

The Yang-Mills coupling constant g_{YM} relates to g_s by the equation

$$4\pi g_s = g_{YM}^2 \tag{2.17}$$

In the large N limit of the gauge theory and in the large N expansion it is conventional to introduce the 't Hooft parameter

$$\lambda = g_{YM}^2 N \tag{2.18}$$

Then we have

$$R^4 = \lambda(\alpha')^2 \tag{2.19}$$

In general, for extremal Dp-brane the dilaton behaves as

$$e^{\Phi} = g_s h(r)^{3-p}/4 \tag{2.20}$$

and we see that precisely for $p = 3$ the dilaton field is constant

$$\mathrm{e}^{\Phi} = g_s \tag{2.21}$$

At large distance r from the D3 brane metric becomes flat Euclidean metric as the factor $h(r)$ approaches to 1 as $r \to \infty$.

On the other hand, the near horizon limit $r \to 0$ decouples the massive open string modes and reduces the open string theory on the stack of N D3 branes to the gauge theory. In this limit the metric becomes

$$ds^2 = \frac{r^2}{R^2}(-dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2) + R^2\frac{dr^2}{r^2} + R^2 d\Omega_5^2 \tag{2.22}$$

which is the $AdS_5 \times S^5$ metric with radius R defined by (2.19). Note the radius is the same for both the AdS_5 and the S^5.

The symmetries of the near horizon limit of the extremal D3 brane solution in supergravity solution match nicely to the superconformal symmetries of the $\mathcal{N} = 4$ super Yang-Mills gauge theory.

Indeed, the $AdS_5 \times S^5$ background has bosonic isometry $SO(4,2) \times SO(6)$ which matches to the product of the conformal group $SO(4,2)$ of the 4-dimensional $\mathcal{N} = 4$ gauge theory and its R-symmetry group $SO(6)$.

Moreover, the background $AdS_5 \times S^5$ is maximally supersymmetric background of the 10d IIB supergravity with 32 Killing spinors. These Killing spinors are matched with the fermionic generators of the superconformal symmetry of the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in four dimensions.

2.3 Spectral duality

The spectrum of the particles of a theory can be extracted from the poles of the 2-point correlation functions of the field theory operators. Holography gives a straightforward prescription of the calculation of any correlator at strong coupling: one needs to compute the classical gravity (bulk) action on the solutions to the gravity (bulk) field perturbations around the background solution with the following boundary conditions. The asymptotic values of the bulk fields are the sources of the field theory operators:

$$\delta\varphi_i(r, x^\mu|\varphi^{(0)}) \qquad \longleftrightarrow \qquad \delta\mathcal{L} = \int \mathrm{d}^4x\, \varphi_i^{(0)}\mathcal{O}_i\,, \tag{2.23}$$

where $\delta\varphi_i$ is a collective notation for the fluctuations of the bulk fields and $\varphi_i^{(0)}$ is their value on the UV boundary $r \to \infty$. In other words, fluctuations of the bulk fields correspond to deformations (2.23) of the dual theory by (single-trace) gauge invariant operators. The correlation functions are given by the variation of the on-shell bulk action with respect to the boundary values [29]:

$$\langle\mathcal{O}_{i_1}\cdots\mathcal{O}_{i_n}\rangle = \left.\frac{\delta^{(n)}S_{\text{bulk}}}{\delta\varphi_{i_1}^{(0)}\cdots\delta\varphi_{i_n}^{(0)}}\right|_{\varphi_i^{(0)}=0}. \tag{2.24}$$

Notice that for the 2-point functions it is enough to consider the action up to quadratic order in the fluctuations. Equivalently, it is just enough to solve the linearized gravity equations.

Boundary condition (2.23) sets a correspondence between the field theory operators and the gravity (bulk) fields. Therefore, a non-trivial check of the duality is the one-to-one correspondence between the bulk fields and possible single-trace gauge invariant operators. Notice however that the classical gravity approximation corresponds to the low-energy limit, which implies that higher spin operators, with large number of derivatives, are suppressed in the holographic consideration. In particular, only the states with the spin less than 2 can be derived in the lowest holographic approximation, with the exception of the spin 2 state related to the energy-momentum tensor operator.

Correspondence also establishes a relation between the mass of the state in the bulk and dimension of the dual operator. For the scalar field in the bulk this relation looks

$$\Delta = 2 + \sqrt{4 + m_5^2R^2}\,, \tag{2.25}$$

where m_5 is the eigenvalue of the Laplace-Beltrami operator on $T^{1,1}$, which is the same as the mass of the scalar in the reduced 5-dimensional equations. For the vector and tensor excitations one can find the mass-dimension relation in *e.g.* [29], however any tensor equation can be reduced to a Klein-Gordon one, for which (2.25) applies.

Practically, one does not need to compute the on-shell action and the variations (2.24) to find the spectrum. Instead one can write the linearized bulk equations corresponding to the operator of interest and solve the Sturm-Liouville problem for the resulting system. That is find the values of the 4-dimensional mass of the fluctuation, for which the solution is normalizable.

Scaling dimensions

In the AdS space the dimension of the operators is computed as follows. Let ϕ be a scalar field in the bulk, which has a 5-dimensional mass m_5, that is

$$\Box_5\phi - m_5^2\phi = 0. \tag{2.26}$$

In the $AdS_5 \times S^5$ metric,

$$\mathrm{d}s^2 = \frac{r^2}{R^2}\,\eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu + \frac{R^2}{r^2}\,\left(\mathrm{d}r^2 + r^2\mathrm{d}\Omega_5^2\right), \tag{2.27}$$

the above equation reads

$$r^2\partial_r^2\phi + 5r\partial_r\phi - m_5^2R^2\phi + m_4^2\,\frac{R^4}{r^2}\,\phi = 0. \tag{2.28}$$

Notice that 4-dimensional mass m_4 is not important at the boundary of AdS $r \to \infty$. We shall refer to the first two terms in the above equation as the canonical kinetic term. Denote k, the asymptotic exponent of the solution to (2.28),

$$\phi \sim r^k, \qquad k = -\,2 + \sqrt{4 + m_5^2R^2}. \tag{2.29}$$

Then the dimension of the operator, which couples to ϕ at the boundary ($\int \mathrm{d}^4x\,\phi\mathcal{O}$) is

$$\Delta = k + 4 = 2 + \sqrt{4 + m_5^2R^2}. \tag{2.30}$$

2.4 Reducing supersymmetry

The stack of N D3 branes in the flat 10-dimensional space $\mathbb{R}^{10}$ carries the maximal supersymmetric $\mathcal{N} = 4$ Yang-Mills theory.

Is it possible to reduce the number $\mathcal{N}$ of supersymmetries in the brane construction to approach more realistic theories? One possible way to do this is to consider the stack of N D3-branes in more general background of IIB theory.

Namely, we consider a stack of N D3-branes in IIB string theory with the 10-dimensional geometry of the form

$$M_{10} = M_4 \times X_6 \tag{2.31}$$

where M_4 is the 4-dimensional space-time and X_6 is the auxiliary space. We place N D3 branes along the 4d space-time M_4 and supported at a certain point in X_6.

If we take for the space X_6 the simplest possible flat space $X_6 = \mathbb{R}^6$ we find $\mathcal{N} = 4$ super Yang-Mills to be the low-energy theory on the stack of N D3 branes.

Indeed, the IIB string theory in 10 dimensions is invariant under the $(0, 2)$ space-time supersymmetry represented by a pair of Mayorana-Weyl 16 component supersymmetry generators with 32 supercharges in total. The D-branes can be thought as BPS boundary conditions for string theory worldsheets. The boundary condition relates the left-moving modes and the right-moving modes on the string worldsheet, and consequently reduce the number of supercharges by factor of two. We conclude from this counting that D3 brane in IIB string theory supports gauge theory with $\frac{1}{2} \times 32 = 16$ supercharges. Since the minimal supersymmetry $\mathcal{N} = 1$ in 4-dimensional gauge theory has 4 supercharges, we call the 4-dimensional gauge theory with 16 supercharges the $\mathcal{N} = 4$ super Yang-Mills theory.

To find out how we can reduce the number of supersymmetries on D3 brane in the background (2.31) let us compute the number of supercharges from the representation theory of spinors. The decomposition (2.31) breaks the 10d Lorentz group as follows

$$SO(3, 1) \times SO(6) \tag{2.32}$$

To be more precise, for the spinor representation theory we need to consider the $Spin(d)$ cover of the special orthogonal group $SO(d)$, so we consider decomposition of spinors with respect to

$$Spin(10) \supset Spin(3, 1) \times Spin(6) \simeq SL(2, \mathbb{C}) \times SU(4) \tag{2.33}$$

With respect to the $SL(2, \mathbb{C}) \times SU(4)$ subgroup the Mayorana-Weyl representation **16** of $Spin(10)$ decomposes as

$$\mathbf{16} \rightsquigarrow (\mathbf{2}, \mathbf{4}) \oplus (\bar{\mathbf{2}}, \bar{\mathbf{4}}) \tag{2.34}$$

which is exactly the representation of the supercharges in $\mathcal{N} = 4$ supersymmetry.

We observe that representation $\mathbf{4}$ and $\bar{\mathbf{4}}$ of the $SU(4)$ correspond to the 4 left and 4 right linear independent Killing spinors supported on the auxiliary space X_6. The 4 left spinors and 4 right spinors is the maximal number of linear independent Killing spinors that can be realized in 6-dimensions because 4 is precisely the dimension of the chiral spinor representation in 6-dimensional space. Indeed, for the present discussion we can think about Killing spinors as covariantly constant spinors. Since the space $\mathbb{C}^3$ is flat every spinor defined at a point can be covariantly extended to the whole space $\mathbb{C}^3$ without any obstructions.

$SU(1)$-holonomy X_6: the $\mathcal{N} = 4$ on D3 branes

What was just explained could be expressed by the fact that the holonomy group of the flat space $X_6 = \mathbb{C}^3$ is trivial $SU(1)$, and by tautological decomposition of the $\mathbf{4}$ of $SU(4)$ into the direct sum of 4 copies of the trivial representation of trivial group $SU(1) \subset SU(4)$

$$\mathbf{4}_{SU(4)} \rightsquigarrow (\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1})_{SU(1)} \tag{2.35}$$

We summarize that the trivial holonomy group $SU(1)$ of $X_6 = \mathbb{C}^3$ leads to the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory on the D3 branes in $M_4 \times X_6$. The number $\mathcal{N} = 4$ of supersymmetries is the number of singlets of the holonomy group of X_6 in the $\mathbf{4}$ of $SU(4)$.

$SU(2)$-holonomy X_6: the $\mathcal{N} = 2$ on D3 branes

Now consider the holonomy group of X_6 to be $SU(2) \subset SU(4)$. This is the case when X_6 is a product of flat 2-dimensional space, for example T^2 or $\mathbb{R}^2$ and 4-dimensional hyper-Kähler space like $K3$ or ALE space $\mathbb{C}^2/\Gamma$. In this case the representation $\mathbf{4}$ of $SU(4)$ decomposes with respect to $SU(4) \supset SU(2)$ as follows

$$\mathbf{4}_{SU(4)} = (\mathbf{2} \oplus \mathbf{1} \oplus \mathbf{1})_{SU(2)} \tag{2.36}$$

We observe $\mathcal{N} = 2$ singlets $\mathbf{1}$ of the holonomy group $SU(2)$ and consequently, the $SU(2)$-holonomy manifold X_6 leads to $\mathcal{N} = 2$ theory on the D3 branes on $M_4 \times X_6$.

$SU(3)$-holonomy X_6: the $\mathcal{N}=1$ on D3 branes

If the holonomy group of X_6 is $SU(3) \subset SU(4)$ it supports precisely one left and one right covariantly constant Killing spinor, and the $SU(3)$ holonomy group appears as the stabilizer of this Killing spinor. This is 1/4 of the case of maximal supersymmetry, hence $\mathcal{N}=1$ supersymmetry. Explicitly, we have 4 supercharges in representation

$$(\mathbf{2}, \mathbf{1}) \oplus (\bar{\mathbf{2}}, \bar{\mathbf{1}}) \tag{2.37}$$

where $\mathbf{1}$ is a single $\mathcal{N}=1$ singlet with respect to the $SU(4) \supset SU(3)$ group in the decomposition

$$\mathbf{4}_{SU(4)} \rightsquigarrow (\mathbf{3} \oplus \mathbf{1})_{SU(3)} \tag{2.38}$$

A manifold X_6 with holonomy group $SU(3)$ is Calabi-Yau manifold. We just rederived the well-known result that theory on D3 branes in the compactification of IIB superstring theory on CY manifold X_6 is $\mathcal{N}=1$ supersymmetric theory.

$SU(4)$-holonomy X_6: the $\mathcal{N}=0$ on D3 branes

Finally, if the holonomy group of the space X_6 is the maximal possible holonomy group $Spin(6) \simeq SU(4)$, there are no covariantly constant spinors, and the resulting theory is non-supersymmetric $\mathcal{N}=0$ Yang-Mills theory

$$\mathbf{4}_{SU(4)} \rightsquigarrow \mathbf{4}_{SU(4)} \tag{2.39}$$

because there are $\mathcal{N}=0$ singlets in the above decomposition.

Remark on Witten's model

In the case of modeling non-supersymmetric theories, like QCD, one can work with the so-called Witten's model [30]. Here the idea is to start with a gravity dual of a $4+1$ dimensional theory (type IIA in $AdS_6 \times S^4$) and compactify one of the field theory dimensions on a circle. The choice of antiperiodic boundary conditions on the circle for the fermions completely breaks SUSY and the theory gets an energy scale, related to the radius of the circle, which breaks conformal invariance. Witten's model is the most popular setup to study QCD holographically. The extension of the Witten's model suggested by Sakai and Sugimoto [31] includes flavor sector and is often called holographic

QCD. The Sakai-Sugimoto model reproduces many features of QCD qualitatively and to extent quantitatively. However, there exist several issues with the setups based on the Witten's model. For example, these models are not UV-complete. From the perspective of this discussion, Witten's model yields a spectrum of glueballs that possesses additional degeneracies [9] as compared to the lattice result [32].

2.5 Orbifold construction

A simple way to construct IIB string background space X_6 with various holonomies considered in the previous section is to consider the orbifold construction. We take X_6 to be the quotient

$$X_6 = X_\Gamma = \mathbb{R}^6/\Gamma \tag{2.40}$$

where Γ is a discrete subgroup of the $\mathbb{R}^6$ isometries $\Gamma \subset Spin(6) \simeq SU(4)$. The resulting space X_Γ is usually called the cone with singularity at the origin $0 \subset \mathbb{R}^6$, since the origin is fixed point of the action of Γ on $\mathbb{R}^6$. Let $1 \leq n \leq 4$ be the minimal integer such that discrete subgroup Γ is in fact a subgroup of $SU(n) \subset SU(4)$:

$$\Gamma \subset SU(n) \subset SU(4) \tag{2.41}$$

Then the holonomy group of X_Γ is $SU(n)$ and we can apply analysis of the previous section to determine the number $\mathcal{N}$ of supersymmetries on the D3 branes placed at the singularity of X_Γ.

The singularity in the orbifold $\mathbb{C}^3/\Gamma$ is an example of Gorenstein canonical singularity. A singularity of a complex n-dimensional space X is called Gorenstein if there exists non-vanishing holomorphic $(n, 0)$ form Ω near the singularity such for any blow-up resolved space $\tilde{X} \to X_\Gamma$ the holomorphic $(n, 0)$ form Ω can be extended to the resolved space $\tilde{X}$. In other words, a singular CY space X_Γ with Gorenstein singularity can be obtained in the limit of vanishing Kähler parameter of smoothed CY space $\tilde{X}$.

The Gorenstein singularity of the orbifold Calabi-Yau X_Γ is a toric CY singularity, which means that the complex n-dimensional space X_Γ can be realized as the symplectic reduction of the flat CY space $\mathbb{C}^k$ by such symplectic action of the torus $U(1)^{k-n}$ that reduced space $\mathbb{C}^k//U(1)^{k-n}$ is CY.

2.5.1 Toric singularity

As reviewed in [24] after [33] a complex CY toric singularity X_{2n}, with $\dim_{\mathbb{C}} X_{2n} = n$ can be parametrized by a convex polygon in $\mathbb{R}^{n-1}$ such that all vertices are in the integer lattice. The convex polygon defines the symplectic Kähler quotient of $\mathbb{C}^k$ by the rank $k-n$ torus $U(1)^{k-n}$ as follows.

1. Let $(v_j)_{j=1\dots k}$ be all integer points in the interior of the polygon including the boundary. To each integer point v_j in the polygon we associate a coordinate x_j in the flat complex space $\mathbb{C}^k$ in some standard basis.

2. To define the Kähler quotient $\mathbb{C}^k//U(1)^{k-n}$ we need to parametrize the $U(1)^{k-n}$ Hamiltonian action on $\mathbb{C}^k$ by the matrix of $U(1)^{k-n}$ charges $(Q_i^j)_{j=1\dots k, i=1\dots k-n}$ where j labels the complex coordinates in $\mathbb{C}^k$.

The matrix (Q_i^j) of the $U(1)$ charges is found from the system of $k-n$ linear relations that points $v_j \in \mathbb{R}^{n-1}$ satisfy

$$\sum_{j=1}^{k} Q_i^j v_j = 0, \qquad i = 1\dots k-n \tag{2.42}$$

subject to the constraints (CY conditions)

$$\sum_{j=1}^{k} Q_i^j = 0, \qquad i = 1\dots k-n \tag{2.43}$$

Then (Q_i^j) defines the matrix of weights for a representation of $U(1)^{k-n}$ on $\mathbb{C}^k$ to define the Kähler quotient.

To each $U(1)_i$ factor in $U(1)^{k-n}$ we associate a Hamiltonian moment map $\mu_i : \mathbb{C}^k \to \mathbb{R}$

$$\mu_i = \sum_{j=1}^{k} Q_i^j |x_k|^2, \qquad i = 1\dots k-n \tag{2.44}$$

The moment map μ_i for $i = 1\dots k-n$ generates $U(1)_i$ action on $\mathbb{C}^k$ as

$$x_j \mapsto x_j e^{\imath Q_i^j \theta^i} \tag{2.45}$$

where θ^i is the angular coordinate on $U(1)_i$.

The Kähler quotient $\mathbb{C}^k//U(1)^{k-n}$ is obtained by imposing the Hamiltonian constraints $\mu_i = \xi_i$, with real parameters $\xi_i \in \mathbb{R}$ called FI parameters, and then taking the quotient by the equivalence relation by the $U(1)^{k-n}$ action

$$X = \mathbb{C}^k // U(1)^{k-n} = \{\, x \in \mathbb{C}^k \,|\, \mu_i(x) = \xi_i \,\} / (x_j \sim x_j e^{\imath Q_i^j \theta^i}) \tag{2.46}$$

The number of FI parameters ξ_i is $k-n$. Variation of the FI parameters ξ_i changes the Kähler form in the reduced space X but not the complex structure. As a complex space the space X can be equivalently obtained as the complex quotient by the complexified group

$$X = (\mathbb{C}^k)_{\mathrm{stab}} / (\mathbb{C}^\times)^{k-n} \tag{2.47}$$

where $(\mathbb{C}^k)_{\mathrm{stab}}$ denotes the set of those points in $\mathbb{C}^k$ whose $(\mathbb{C}^\times)^{k-n}$ orbits intersect the moment map constraints $\mu_i = \xi_i$.

The Kähler quotients have been studied in the framework 2d $\mathcal{N} = (2,2)$ supersymmetric gauged linear sigma models [34]. Indeed, the 2d $\mathcal{N} = (2,2)$ supersymmetry can be obtained by the reduction of the 4d $\mathcal{N} = 1$ (both theories have 4 supercharges). The procedure of the Kähler reduction to analyze the classical moduli space of vacua is equivalent in both theories. In the language of the supersymmetric field theories with 4 supercharges, the moment map equations are the D-term equations.

For generic values of the FI parameters ξ_i the Kähler quotient space X is smooth, but if some $\xi \to 0$ the quotient X can develop singularity.

2.5.2 The $SU(2)$-holonomy orbifold and $\mathcal{N} = 2$ gauge theory

The simplest example of orbifold is for $X = \mathbb{C}^2/\mathbb{Z}_2$ of the complex dimension 2 which can be realized as the toric Kähler quotient of $\mathbb{C}^3$ by $U(1)$ action with weights $(1, 1, -2)$ and single real FI parameter ξ. The moment map is

$$\mu = |x_1|^2 + |x_2|^2 - 2|x_3|^2 \tag{2.48}$$

If $\xi > 0$ the moment map constraint $\mu = \xi$ implies that for any x_3 we have

$$|x_1|^2 + |x_2|^2 = \xi + 2|x_3|^2 \tag{2.49}$$

and consequently for any x_3 we have the two-sphere $S^2 = S^3/U(1)$ worth of $U(1)$-orbits for the complex (x_1, x_2) variables restricted to the three-sphere by the equation $|x_1|^2 + |x_2|^2 = \mathit{const}$. Topologically $X_{\xi>0}$ is the cotangent bundle of the two-sphere S^2

$$X_{\xi>0} = T^* S^2 \tag{2.50}$$

As a complex space space $X_{\xi>0}$ is a complex quotient of (x_1, x_2, x_3) by the $\lambda \in \mathbb{C}^\times$ action

$$(x_1, x_2, x_3) \sim (\lambda x_1, \lambda x_2, \lambda^{-2} x_3) \tag{2.51}$$

and this presentation implies that

$$X_{\xi>0} = \{\mathcal{O}(-2) \to \mathbb{CP}^1\} \tag{2.52}$$

where $\{\mathcal{O}(-2) \to \mathbb{CP}^1\}$ denotes the total space of the canonical $\mathcal{O}(-2)$ bundle over $\mathbb{CP}^1$.

From variables (x_1, x_2, x_3) we can form $\mathbb{C}^\times$-invariants

$$\begin{aligned} z_1 &= x_1^2 x_3 \\ z_2 &= x_2^2 x_3 \\ z_3 &= x_1 x_2 x_3 \end{aligned} \tag{2.53}$$

which satisfy one relation

$$z_1 z_2 = z_3^2 \tag{2.54}$$

Therefore, the singular complex space $\mathbb{C}^2/\mathbb{Z}_2$ can be described by the quadric cone (2.54) in $\mathbb{C}^3$.

If we consider $X_\Gamma = \mathbb{C}^3/\mathbb{Z}_2$ where $\mathbb{Z}_2$ acts only on $\mathbb{C}^2 \subset \mathbb{C}^3$ then the resulting quotient $\mathbb{C} \times (\mathbb{C}^2/\mathbb{Z}_2)$ is the $SU(2)$-holonomy orbifold which leads to the $\mathcal{N} = 2$ affine A_1-quiver theory on the D3 branes placed at the singularity [15].

The orbifold construction appeared in the works of Douglas and Moore [15] for the $\Gamma \subset SU(2)$ and the $SU(2)$-holonomy space $X_\Gamma = \mathbb{R}^2 \times (\mathbb{C}^2/\Gamma)$. The $\mathbb{Z}_2$ can be replaced by any discrete subgroup Γ of $SU(2)$. The resulting theory on D3 branes is $\mathcal{N} = 2$ affine ADE quiver gauge theory where the affine ADE diagram of the quiver is determined by McKay correspondence to the discrete $SU(2)$-subgroup Γ. See also Johnson-Myers [35].

2.5.3 The $SU(3)$-holonomy orbifolds and $\mathcal{N} = 1$ gauge theory

Later Douglas-Green-Morrison [16] considered the $SU(3)$-holonomy quotient $X_\Gamma = \mathbb{C}^3/\Gamma$ resulting in the $\mathcal{N} = 1$ supersymmetric quiver theory. For $\Gamma = \mathbb{Z}_n$ the $\mathcal{N} = 1$ quiver diagram has n nodes corresponding to the gauge groups and $\mathcal{N} = 1$ vector multiplets, and a number of arrows corresponding to the $\mathcal{N} = 1$ chiral multiplets in bifundamental representation. The gravity dual was analyzed in [18].

The matrix of $U(1)$ charges (Q_i^j) for the toric realization of the its toric presentation can be systematically determined by the group Γ. The singularity of the orbifold Calabi-Yau can be resolved by blow-ups. The resolving corresponds to the turning on the FI terms in the $\mathcal{N}=1$ gauge theory on D3 branes that probe the CY geometry.

Mukhopadhyay and Ray [36] and Morrison-Plesser [24] have considered $SU(3)$-holonomy orbifold X_Γ obtained for the group $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2$ where $\mathbb{Z}_2 \times \mathbb{Z}_2$ acts on $\mathbb{C}^3$ by generators $(-1,-1,1)$ and $(-1,1,-1)$. The quiver diagram of $\mathcal{N}=1$ gauge theory and the cubic superpotential can be systematically determined from the orbifold cation of Γ on $\mathbb{C}^3$ [18], [20], [24], [16]. The resulting gauge theory has $U(N)^4$ gauge group and 12 bifundamental chiral multiplets. The diagonal $U(1)$ is completely decoupled, so that we can consider the gauge theory with $SU(N)^4 \times U(1)^3$ gauge group.

The 12 chiral bifundamental fields which are coming from the Γ orbifold projection of the $U(4N)$ adjoint chiral fields (X, Y, Z) are

$$\begin{aligned} X_{14}, X_{23}, X_{41}, X_{32} \\ Y_{13}, Y_{31}, Y_{24}, Y_{42} \\ Z_{12}, Z_{21}, Z_{34}, Z_{43} \end{aligned} \tag{2.55}$$

where X_{ij}, Y_{ij}, Z_{ij} transforms in representation of $(N, \bar{N})$ of $U(N)_i \times U(N)_j$, or equivalently

$$X_{ij}, Y_{ij}, Z_{ij} \in \mathrm{Hom}(V_j, V_i) \tag{2.56}$$

where $V_i \simeq \mathbb{C}^N$ denotes the fundamental representation of $U(N)_i$. The cubic $\mathcal{N}=4$-theory orbifold superpotential is

$$\begin{aligned} W = \mathrm{tr}(Z_{12}(X_{23}Y_{31} - Y_{24}X_{41}) + Z_{21}(X_{14}Y_{42} - Y_{13}X_{32}) \\ + Z_{34}(X_{41}Y_{13} - Y_{42}X_{23}) + Z_{43}(X_{32}Y_{24} - Y_{31}X_{14})) \end{aligned} \tag{2.57}$$

The orbifold $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ gauge theory has gauge group $U(N)^4$ and the 4 associated FI parameters $\xi_1, \xi_2, \xi_3, \xi_4$, but since the diagonal $U(1)$ is completely decoupled $\sum_{i=1}^4 \xi_i = 0$, and we represent the 4 FI parameters $(\xi_1, \xi_2, \xi_3, \xi_4)$ in terms of 3 parameters ξ_1, ξ_2, ξ_3 by

$$(\xi_1, \xi_2, \xi_3, -\xi_1 - \xi_2 - \xi_3) \tag{2.58}$$

The three FI parameters (ξ_1, ξ_2, ξ_3) measure the Kähler deformations of the blow-up $\tilde{X}_\Gamma$ of the X_Γ. In [24] (appendix B) it is shown that the orbifold $X_{\mathbb{Z}_2 \times \mathbb{Z}_2}$ can be presented as the toric Kähler quotient of the $\mathbb{C}^6$ by $U(1)^3$. In the notations of (2.44) we have $n = 3$ and $k = 6$. The Reid toric data given by the following table

$$\begin{array}{ccccccc} & x_0 & x_1 & x_2 & y_0 & y_1 & y_2 \\ & (0,0) & (2,0) & (0,2) & (1,1) & (0,1) & (1,0) \\ \xi_1 & 1 & & & 1 & -1 & -1 \\ \xi_2 & & 1 & & -1 & 1 & -1 \\ \xi_3 & & & 1 & -1 & -1 & -1 \end{array} \tag{2.59}$$

Here the first line displays the names $(x_0, x_1, x_2, y_0, y_1, y_2)$ for the coordinates in $\mathbb{C}^6$ and the labels of the points in the integer lattice subspace of $\mathbb{R}^2$ bounded by a certain polygon. The second line displays the $\mathbb{R}^2$-coordinates of these points. The next three lines display the matrix (Q_i^j) with $i = 1 \dots 3$ and $j = 1 \dots 6$ of the weight charges.

2.6 Conifold singularity

As explained above, the singular CY space $X_\Gamma = \mathbb{C}^3/\Gamma$ for $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2$ has a family of Kahler deformed spaces $\tilde{X}_\Gamma$ with 3 FI parameters (ξ_1, ξ_2, ξ_3), so that if $\xi_1 = \xi_2 = \xi_3 = 0$ we recover the original orbifold $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$. However, we can less singular limit of the smooth $\tilde{X}_\Gamma$ and consider only one FI parameter approaching zero

$$\xi \equiv \xi_1 \to 0 \tag{2.60}$$

but keeping ξ_2, ξ_3 very large. In terms of the toric construction (2.59), if we are interested in the neighborhood of the singularity at $\xi_1 \to 0$, we can just discard the ξ_2 and ξ_3 together with x_1, x_2 columns because x_1, x_2 are invariant under the $U(1)_1$ action with moment map $\mu_1 = \xi_1$. Therefore, in the limit $\xi_2, \xi_3 \to \infty$ we find the toric singularity obtained from the $\mathbb{C}^4//U(1)$ Kähler quotient with the data

$$\begin{array}{ccccc} & x_0 & y_0 & y_1 & y_2 \\ & (0,0) & (1,1) & (0,1) & (1,0) \\ \xi_1 & 1 & 1 & -1 & -1 \end{array} \tag{2.61}$$

We denote the resulting singularity X_ξ.

In terms of the complexified quotient $\mathbb{C}^4//\mathbb{C}^\times$ we can present X_{ξ_1} as the space of $\lambda \in \mathbb{C}^\times$ orbits

$$X_\xi = \{(x_0, y_0, y_1, y_2) \sim (x_0\lambda, y_0\lambda, y_1\lambda^{-1}, y_2\lambda^{-1})\} \tag{2.62}$$

This implies that X_c is the total space of the holomorphic rank 2 bundle over $\mathbb{CP}^1$:

$$X_{\xi_1} = \{\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{CP}^1\} \tag{2.63}$$

The Kähler size of the $\mathbb{CP}^1$ is measured by the FI parameter ξ. The change of sign of ξ_1 causes the flop transition of X_{ξ_1} [34, 37, 36, 17]. Depending on the sign of ξ, the compact base $\mathbb{CP}^1$ in the X_{ξ_1} is obtained either from the projective ratio of $(x_0 : y_0)$ or $(y_1 : y_2)$.

The space of equivalence classes (2.62) can be realized as a complex hypersurface by forming the invariants under $\mathbb{C}^\times$ action on (x_0, y_0, y_1, y_2)

$$\begin{aligned} z_1 &= x_0 y_1 \\ z_2 &= y_0 y_2 \\ z_3 &= x_0 y_2 \\ z_4 &= y_0 y_1 \end{aligned} \tag{2.64}$$

and imposing the relation that these constraints satisfy:

$$(\xi\text{-}\textit{resolved})\ \text{conifold}\ X_\xi : \qquad z_1 z_2 - z_3 z_4 = 0, \qquad (z_1, z_2, z_3, z_4) \in \mathbb{C}^4 \tag{2.65}$$

The complex 3-dimensional space defined by the equation (2.65) is called *conifold*. The conifold is a non-compact CY space with singularity at the origin $z = 0$ called *ordinary double point*. Even though the conifold singularity (2.65) can be obtained locally in the certain degenerate limit of the resolved orbifold $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_3)$, the conifold singularity is not an orbifold singularity per se, i.e. the space (2.65) is not isomorphic to any quotient $\mathbb{C}^3/\Gamma$ for a discrete group Γ as a complete space. The conifold singularity is typical generic type singularity appearing at the boundary of moduli space of CY spaces, and hence was studied in greater details starting from [38].

The FI parameter ξ controls the *Kähler moduli* deformation of the CY conifold (2.65): the Kähler size of the $\mathbb{CP}^1$ obtained as the blow-up of the singularity at $\xi = 0$.

The ξ-*resolved conifold* topologically is obtained by blow-up of the singular conifold (2.65) at the origin and taking the Kähler size of the base $\mathbb{CP}^1_\xi$ to be given by ξ. As a complex space the resolved conifold is the total bundle of $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{CP}^1_\xi$.

The ϵ-*deformed conifold* is defined as a complex affine CY space by the ϵ deformation of the equation (2.65) in $\mathbb{C}^4$

$$\epsilon\text{-}\textit{deformed}\ \text{conifold} : \qquad z_1 z_2 - z_3 z_4 = \epsilon^2 \tag{2.66}$$

To summarize, we can smooth out the *singular conifold*, defined as the toric Kähler quotient (2.61) at $\xi = 0$, in two mutually excluding directions:

- ξ-*resolving* as a blow-up and variation of the Kahler moduli ξ
- ϵ-*deformation* as variation of the complex structure moduli ϵ

2.7 Klebanov-Witten model: D3 branes at singular conifold

The degenerate limit of the $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ orbifold $\mathcal{N} = 1$ theory can be used to derive the $\mathcal{N} = 1$ field theory associated to the $\xi = 0$ conifold (2.65) [24], [17], [36]. We start from the $U(N)^4$ theory with chiral bifundamental fields (2.55) and superpotential (2.57) and FI parameters $(\xi_1, \xi_2, \xi_3, -\xi_1 - \xi_2 - \xi_3)$ and send $\xi_2, \xi_3 \to \infty$. The (ξ_2, ξ_3) moment map equations (D-term constraints) are

$$\begin{aligned} X_{23}X_{23}^\dagger + Y_{24}Y_{24}^\dagger + Z_{21}Z_{21}^\dagger - X_{32}^\dagger X_{32} - Y_{42}^\dagger Y_{42} - Z_{12}^\dagger Z_{12} = \xi_2 \\ X_{32}X_{32}^\dagger + Y_{31}Y_{31}^\dagger + Z_{34}Z_{34}^\dagger - X_{23}^\dagger X_{23} - Y_{13}^\dagger Y_{13} - Z_{43}^\dagger Z_{43} = \xi_3 \end{aligned} \tag{2.67}$$

We pick the background

$$Y_{24} = \xi_2^{\frac{1}{2}} 1_{N\times N} \qquad Z_{34} = \xi_3^{\frac{1}{2}} 1_{N\times N} \tag{2.68}$$

which ensures the D-term constraints above.

We see the following effects.

1. The background effectively replaces the gauge group $U(N)_2 \times U(N)_3 \times U(N)_4$ by its diagonal subgroup

$$U(N)_{\tilde{2}} = \mathrm{diag}(U(N)_2 \times U(N)_3 \times U(N)_4) \tag{2.69}$$

which we will call from now simply $U(N)_2$. The remaining gauge group fields in $U(N)_3$ and $U(N)_4$ become massive by Higgs mechanism. Therefore, the gauge group is broken to

$$U(N)_1 \times U(N)_2 \tag{2.70}$$

2. We plug the background (2.68) in the superpotential (2.57) and integrate out the massive fields by resolving the constraints equations of motion

$$\frac{\partial W}{\partial \Phi} = 0 \tag{2.71}$$

for the fields $\Phi \in \{X_{41}, Z_{12}, Y_{13}, Y_{42}, X_{23}, Z_{43}, X_{32}\}$. This gives

$$\begin{aligned} X_{41} &= 0, & Y_{13} &= \xi_2^{\frac{1}{2}} \xi_3^{-\frac{1}{2}} Z_{12} \\ X_{23} &= \xi_3^{-\frac{1}{2}} Z_{21} X_{14}, & Y_{42} &= \xi_3^{-\frac{1}{2}} Y_{31} Z_{12} \\ X_{32} &= \xi_2^{-\frac{1}{2}} Y_{31} X_{14}, & Z_{43} &= \xi_2^{-\frac{1}{2}} Z_{21} Y_{13} \end{aligned} \tag{2.72}$$

We plug the solutions (2.72) back into the superpotential (2.57) and find superpotential as a function of remaining four chiral bifundamental fields $\{X_{14}, Y_{31}, Z_{12}, Z_{21}\}$

$$W = 2\xi_3^{-\frac{1}{2}} \operatorname{tr}(Z_{21} X_{14} Y_{31} Z_{12} - Y_{31} X_{14} Z_{21} Z_{12}) \tag{2.73}$$

To simplify notations we rename

$$\begin{aligned} A_1 &= Z_{21}, & A_2 &= Y_{31} \\ B_1 &= X_{14}, & B_2 &= Z_{12} \end{aligned} \tag{2.74}$$

so that $(A_i, B_i)_{i=1,2}$ are bi-fundamental fields for the gauge group $U(N)_1 \times U(N)_2$

$$\begin{aligned} A_1, A_2 &\in \bar{\mathbf{N}}_{\mathbf{1}} \otimes \mathbf{N}_{\mathbf{2}} \\ B_1, B_2 &\in \bar{\mathbf{N}}_{\mathbf{2}} \otimes \mathbf{N}_{\mathbf{1}} \end{aligned} \tag{2.75}$$

2.7.1 The IR Klebanov-Witten superconformal field theory

We conclude that gauge theory description of the field theory on N D3 branes placed on the singularity of the conifold $X_{\xi=0}$ and extended along the 4d space-time is the $\mathcal{N} = 1$ supersymmetric $U(N) \times U(N)$ gauge theory with bifundamental chiral multiplets $(A_i, B_i)_{i=1,2}$ and superpotential (2.73). The IR limit of this theory was analyzed in details by Klebanov-Witten in [23]. Klebanov and Witten in [23] argued that the IR limit of the theory on D3 branes on a singular conifold exists as a non-trivial interacting $\mathcal{N} = 1$ superconformal field theory (SCFT), which we will call in the following the Klebanov-Witten field theory.

Moreover, Klebanov-Witten argued that there are precisely two marginal deformations of the IR $\mathcal{N}=1$ SCFT: the superpotential (2.73) and the difference between the kinetic energies for two $SU(N)$ gauge group factors.

Klebanov-Witten argument is based on the exact β-function of Novikov-Shifman-Vainshtein-Zakharov (NSVZ) [39]. Let g_1, g_2 denote the coupling constants for the two gauge group factors, and let λ be the coefficient in front of the deformation by the superpotential (2.73). The NSVZ $\beta(g_i)$-function for $U(N)$ $\mathcal{N}=1$ theory with N_f fundamental and $N_{\bar{f}}$ anti-fundamental chiral multiplets is [40]

$$\beta(g_i) = -\frac{g_i^3}{16\pi^2}\frac{3N - N_f(\frac{1}{2}-\gamma_f) - N_{\bar{f}}(\frac{1}{2}-\gamma_{\bar{f}})}{1 - N\frac{g_i^2}{8\pi^2}} \tag{2.76}$$

where $\gamma_f, \gamma_{\bar{f}}$ is the anomalous dimension of the chiral and the anti-chiral multiplets.

In the Klebanov-Witten theory each gauge group factor $U(N)$ sees $N_f = 2N$ fundamental and $N_{\bar{f}} = 2N$ anti-fundamental fields. Therefore $N_f = N_{\bar{f}}$ is in the range $\frac{3}{2}N < N_f < 3N$ when the IR theory is non-trivial interacting superconformal field theory as was argued by Seiberg [40].

We impose the symmetry under the exchange of the $U(N)$ gauge group factors so that

$$g_1 = g_2 \tag{2.77}$$

To make sure that IR Klebanov-Witten theory is conformal we require vanishing of the exact β-function and we find that anomalous dimensions of the chiral multiplets A_i, B_i must satisfy

$$\gamma_A = \gamma_B = -\frac{1}{4} \tag{2.78}$$

The $\mathcal{N}=1$ superconformal group is the supergroup $SU(2,2|1)$ which includes the exact $U(1)_R$-symmetry factor. In $\mathcal{N}=1$ SCFT the chiral operators satisfy

$$D = \frac{3}{2}R \tag{2.79}$$

where $D = 1+\gamma$ is dimension of the operator and R is the $U(1)_R$-symmetry charge. Therefore we find the R-charges of the chiral fundamental multiplets A, B:

$$R_A = R_B = \frac{2}{3}(1-\frac{1}{4}) = \frac{1}{2} \tag{2.80}$$

which implies that the R-charge of the superpotential W operators is 2. Therefore, the superpotential W is a marginal perturbation of the IR fixed point.

2.7.2 The global symmetry group of the IR Klebanov-Witten SCFT

We list below the global symmetry group of the IR Klebanov-Witten SCFT.

1. The theory is $\mathcal{N} = 1$ SCFT with the superconformal symmetry group

$$\text{superconformal group}: \qquad SU(2,2|1) \tag{2.81}$$

which includes in particular exact $U(1)_R$-symmetry factor.

2. The bi-fundamental chiral multiplets $(A_i)_{i=1,\dots,2}$ form a doublet under global $SU(2)$, and the bi-fundamental chiral multiplet $(B_i)_{i=1,\dots,2}$ form doublet under another copy of global $SU(2)$. However we should quotient by the diagonal subgroup in the product of the two centers of the $SU(2)$ because the gauge group $U(N) \times U(N)$ contains $U(1)$ subgroup which acts, in particular, by $A \to -A$ and $B \to -B$. Therefore $A \to -A, B \to B$ and $A \to A, B \to -B$ are gauge equivalent transformations.

$$\text{global flavor symmetry}: \qquad (SU(2) \times SU(2))/\mathbb{Z}_2 \simeq SO(4) \tag{2.82}$$

3. Reflection $\mathbb{Z}_2$ symmetry $A \leftrightarrow B$. Consider the exchange of the bi-fundamental fields $A \leftrightarrow B$ accompanied by exchange of the two factors of the gauge group and accompanied by the R-symmetry $\pi/2$-rotation $e^{\frac{\imath\pi}{2}}$

4. Charge conjugation $\mathbb{Z}_2$ symmetry. Swap representations $\mathbf{N}$ with $\bar{\mathbf{N}}$ and simultaneously exchange the two gauge group factors. Under such operation $A \leftrightarrow A^t, B \leftrightarrow B^t$ where A^t, B^t denote the transposed matrices. To avoid the change of sign of superpotential the charge conjugation need to be accompanied by the R-symmetry $\pi/2$-rotation.

The product of the reflection $\mathbb{Z}_2$ symmetry $A \leftrightarrow B$ accompanied by the charge conjugation $\mathbb{Z}_2$ is called $\mathcal{I}$-symmetry [23, 24, 41].

2.7.3 Gravity dual of the IR Klebanov-Witten SCFT

Before Klebanov-Witten [23] and Morrison-Plesser [24] work, the large N gravity duals of the 4d superconformal gauge theories with non-maximal supersymmetries were originally analyzed in Bershadsky-Johansen [19], Lawrence-Nekrasov-Vafa [20], Ferrara-Zaffaroni [21], Hanany-Uranga [22].

The $\mathcal{N} = 1$ superconformal 4d gauge theory is invariant under the action of the $4 + 4 = 8$ fermionic generators: 4 Poincaré supersymmetry generators and 4 special superconformal symmetry generators.

Consequently, we are looking for the dual background of IIB supergravity which supports 8 Killing spinors.

In fact it is possible to show that for a conic 6d Calabi-Yau manifold (equivalently $SU(3)$-holonomy manifold) C_6 with conical metric

$$ds^2_{C_6} = dr^2 + r^2 ds^2_{X_5} \tag{2.83}$$

the following 10-dimensional warped product of the 4d-space time and the conic Calabi-Yau is the IIB supergravity background which supports 8 Killing spinors (see e.g. [28])

$$ds^2 = h^{-1/2}(-dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2) + h^{1/2} ds^2_{C_6}, \tag{2.84}$$

The manifold X_5 is the base of the Calabi-Yau cone Y_6. The holonomy consideration imply that X_5 is Sasaki-Einstein manifold.

We assume that the flux through the Sasaki-Einstein manifold X_5 is the same as in (2.15)

$$\int_{X_5} \star F_5 = N \tag{2.85}$$

Then the warping factor $h(r)$ given by equation (2.14) with R^4 in (2.16) replaced by more general formula

$$R^4 = g_s \frac{\sqrt{\pi}}{2} \frac{\kappa_{10} N}{\mathrm{Vol}(X_5)} = 4\pi g_s N \frac{\pi^3}{\mathrm{Vol}(X_5)} l_s^4 \tag{2.86}$$

Notice that for $X_5 = S^5$ we have $\mathrm{Vol}(S^5) = \pi^3$ and the equation (2.86) reduces to (2.16).

The gauge-gravity or open-closed string duality implies that the gravity dual metric is (2.84) where X_5 is the base of the Calabi-Yau conifold (2.65). In the near-horizon limit $r \ll R$ we can replace

$$h(r) = 1 + \frac{R^4}{r^4} \qquad \rightsquigarrow \qquad h(r) \simeq \frac{R^4}{r^4} \quad \text{for} \quad r \ll R \tag{2.87}$$

and in this limit the metric (2.84) becomes

$$ds^2 = \frac{r^2}{R^2}(-dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2) + R^2 \frac{dr^2}{r^2} + R^2 d\Omega_5^2 \tag{2.88}$$

where $d\Omega_5$ is the metric on the compact Sasaki-Einstein 5d manifold X_5 which is the base of the CY cone Y_6. Hence, the near-horizon limit is $AdS_5 \times X_5$. The isometry group $SO(4,2)$ of AdS_5 space reflects the corresponding IR gauge theory is conformal. The existence of 8 Killing spinors on X_5 enlarges the conformal algebra to the $\mathcal{N}=1$ superconformal algebra. The Sasaki condition (see for example appendix in [24]) implies existence of the isometric vector field on X_5. This vector field is constructed taking the radial vector field ∂_r on Y_6 and rotating it by the complex structure on Y_6. This vector field generates the $U(1)$ isometry of the Sasaki-Einstein manifold which can identified with the $U(1)_R$ symmetry of the IR superconformal gauge theory.

As we found in section 2.7.3 the world-volume theory on the stack of N D3 branes placed as the conifold Calabi-Yau singularity (2.65) flows in the IR to $U(N)\times U(N)$ gauge theory with four bi-fundamental chiral multiplets (A_i, B_i) and quartic superpotential (2.73). To find the IIB gravity dual explicitly we need to describe the Sasaki-Einstein manifold X_5 that lies in the base of the Calabi-Yau cone (2.65).

The conifold Calabi-Yau (2.65) has been analyzed by Candelas and de la Ossa in [38]. Their result is that the base of the CY conifold (2.65) is 5d Sasaki-Einstein manifold called $T^{1,1}$.

The 5d manifold $T^{1,1}$ is also known as

$$T^{1,1} = SO(4)/SO(2) = (SU(2)\times SU(2))/U(1) \tag{2.89}$$

or Stiefel manifold of the framed 2d planes in 4d space [24]. Here $SO(4) = (SU(2)\times SU(2))/\mathbb{Z}_2$ is the usual factorization of the $SO(4)$ into the rotations by the right (selfdual) and the left (anti-selfdual) generators, and the $U(1)$ acts diagonally. (The manifold $T^{1,1}$ and some other 5d Einstein manifolds isometries and some degree of supersymmetry were analyzed by Romans [42], see also Freund-Rubin ansatz [43])

The 5d Sasaki-Einstein space $T^{1,1}$ (2.89) can be thought as the total space of the S^1-bundle over the base $S^2\times S^2$ where the degree of the S^1 bundle over the two factors is $(1,1)$. If complexified, this circle bundles gives the CY conifold, the bundle $\mathcal{O}(-1)\oplus\mathcal{O}(-1)$ over $\mathbb{CP}^1\times\mathbb{CP}^1$.

We remark that more general $T^{p,q}$ Sasaki-Einstein spaces is defined space of the circle bundle over $S^2\times S^2$ with charges (p,q).

Assuming the representation of $SU(2)$ through Pauli matrices (see Appendix section A.3.2), the generator for $U(1)$ in the expression 2.89 may be written as $p\sigma_3\otimes\sigma_0+q\sigma_0\otimes\sigma_3$, where p and q are integers determining the space $T^{p,q}$.

One can show that for the case $p = q = 1$ the 5d space $T^{1,1}$ can be parametrized by the following angles $\psi, \theta_1, \phi_1, \theta_2, \phi_2$ where θ_i and ϕ_i are the Euler angles that parametrize each $SU(2)$, while ψ is the third Euler angle which becomes degenerate after factoring out the $U(1)$.

Concretely, the $T^{1,1}$ metric has the form

$$ds^2_{T^{1,1}} = \frac{1}{9}(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2)^2 + \frac{1}{6}\sum_{i=1}^{2}[d\theta_i^2 + \sin\theta_i{}^2 d\phi_i^2], \tag{2.90}$$

note that the angle ψ parametrizes the S^1 fiber of the S^1 fibration over $S^2 \times S^2$. The angle ψ is 4π-periodic

$$\psi \simeq \psi + 4\pi \tag{2.91}$$

Clearly, the $T^{1,1}$ is invariant under the $U(1)$ isometry which translates ψ. Because of the conventions (2.91) the mode

$$\exp(\frac{\imath}{2}R\psi) \tag{2.92}$$

carries $U(1)_R$ charge R.

We will be using the following basis [44] of 1−forms further on:

$$g^1 = \frac{e^1 - e^3}{\sqrt{2}}, \quad g^2 = \frac{e^2 - e^4}{\sqrt{2}}, \quad g^3 = \frac{e^1 + e^3}{\sqrt{2}}, \quad g^4 = \frac{e^2 + e^4}{\sqrt{2}}, \quad g^5 = e^5, \tag{2.93}$$

where

$$\begin{aligned} e^1 &\equiv -\sin\theta_1 d\phi_1, \quad e^2 \equiv d\theta_1, \quad e^3 \equiv \cos\psi \sin\theta_2 d\phi_2 - \sin\psi d\theta_2, \\ e^4 &\equiv \sin\psi \sin\theta_2 d\phi_2 + \cos\psi d\theta_2, \quad e^5 \equiv d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2. \end{aligned} \tag{2.94}$$

In this basis the metric on 5d Sasaki-Einstein $T^{1,1}$ (2.90) assumes the form:

$$ds^2_{T^{1,1}} = \frac{1}{9}(g^5)^2 + \frac{1}{6}\sum_{i=1}^{4}(g^i)^2, \tag{2.95}$$

And the metric on the 6d Calabi-Yau conifold over $T^{1,1}$ is

$$ds_6^2 = dr^2 + r^2 ds^2_{T^{1,1}} \tag{2.96}$$

Chapter 3

Klebanov-Strassler theory

The singular conifold as any cone is invariant under scaling transformation $r \to \lambda r$. This scale invariance leads to the conformal symmetry of the corresponding gauge theory. The IR limit is $r \to 0$ (the horizon of the AdS space) and the UV limit is $r \to \infty$ (the boundary of the AdS space). Notice that from the perspective of the 4d gauge thoery the coordinate r which characterizes distance from the horizon $r = 0$ (IR) has the meaning of energy scale.

However, we are also interested in the non-conformal gauge theories. The gravity dual of non-conformal gauge theories should have the scaling symmetry being broken and should not have the AdS_5 factors whose isometry can be associated with the conformal group.

As far as confinement is concerned it is also instructive to study $\mathcal{N} = 1$ theories as their low-energy physics resembles that of QCD, yet they are more tractable. One of the most popular $\mathcal{N} = 1$ holographic models in the market is the one derived by Klebanov and Strassler (KS) [5]. To arrive to the KS model one can start with the $\mathcal{N} = 1$ SCFT theory like Klebanov-Witten model described in the previous section and break the conformal symmetry.

To break the conformal invariance one has to break the isometries of the AdS_5-space. This can be achieved through the addition of fluxes of the 3-form fields on the gravity side. The fluxes will backreact on the geometry to produce log-factors in the metric, which can be related to the logarithmic running of the coupling constant(s).

The background metric can be written in the form

$$\mathrm{d}s^2 = h^{-1/2}\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + h^{1/2}\mathrm{d}s_6^2\,, \tag{3.1}$$

where h is the warp-factor, $\eta_{\mu\nu}$ is the Minkowski metric and $\mathrm{d}s_6^2 = \mathrm{d}r^2 + r^2\mathrm{d}\Omega^2_{T^{1,1}}$ is the metric on the space transverse to the Minkowski factor. This space is parameterized by the original AdS_5 radial coordinate r and the angles of $T^{1,1}$. It is a 6-dimensional cone over the $T^{1,1}$ called a conifold. In the conformal case the warp-factor has the form $h = R^4/r^4$, where R is the AdS radius. This ensures the 4-dimensional conformal group as the group of isometries. In the case of the KS solution the warp factor gets logarithmic corrections

$$h \propto \frac{(g_s M\alpha')^2}{r^4}\,\log\frac{r}{r_0} + O\left(\frac{R^4}{r^4}\right)\,, \qquad \text{UV limit} \quad r \to \infty\,. \tag{3.2}$$

Here g_s is the string coupling constant, α' is the square of string length (string tension) and M is parameter of the gravity solution, which is equal to the flux of a 3-form field. Equation (3.2) tells us that the metric (the dual theory) is almost conformal in the UV limit $r \to \infty$, as is the case with QCD. We will discuss more the RG behavior of the theory in the next section.

At $r = 0$ the conifold (2.65) used to describe conformal KW theory has a singularity. To make the geometry smooth and regular one can keep the size of S^3 in $T^{1,1}$ finite at the tip of the conifold. We can replace the singular conifold by the ϵ-deformed conifold that we discussed in (2.66).

The corresponding IR energy scale ϵ is a parameter of the KS theory. This way an IR scale is introduced, an analog of Λ_{QCD}. It is convenient to introduce another (dimensionless) radial coordinate τ:

$$r \propto \epsilon^{2/3}\mathrm{e}^{\tau/3}\,, \qquad r \to \infty\,. \tag{3.3}$$

In terms of τ the deformed conifold metric reads [45, 5]

$$ds_6^2 = \frac{1}{2}\epsilon^{4/3}K(\tau)\left[\frac{1}{3K(\tau)}(d\tau^2 + (g^5)^2) + \cosh^2\left(\frac{\tau}{2}\right)[(g^3)^2 + (g^4)^2] + \sinh^2\left(\frac{\tau}{2}\right)[(g^1)^2 + (g^2)^2]\right], \tag{3.4}$$

The remaining bosonic fields of the type IIB SUGRA have the form

$$
\begin{aligned}
\text{metric, } \mathrm{d}s^2 &= h^{-1/2}\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + h^{1/2}\mathrm{d}s_6^2 \\
\text{NS 3-form, } H_3 &= \mathrm{d}B_2 = \mathrm{d}\left(g_s M\,\mathrm{d}\tau \wedge (f\omega_{12} + k\omega_{34})\right) \\
\text{RR 3-form, } F_3 &= M\left(g^5 \wedge \omega_{34} + \mathrm{d}(F\omega_{13})\right) \\
\text{RR 5-form, } F_5 &= (1 + *_{10})B_2 \wedge F_3
\end{aligned}
\tag{3.5}
$$

while the dilaton Φ and the RR scalar C can be chosen to vanish. Here h is called the warp-factor, the $\eta_{\mu\nu}$ is the Minkowski metric, g_s is the string coupling, M is a parameter, encoding the rank of the gauge group in the IR, $*_{10}$ is the 10d Hodge operator.

The coefficients h, f, k and F are functions of the AdS_5 radial coordinate τ, which is related to the standard AdS coordinate $r \propto \epsilon^{2/3}\mathrm{e}^{\tau/3}$ (at large r). Explicit expressions for these functions and the conifold metric $\mathrm{d}s_6^2$ can be found in section 3.2 after [5]. The above solution is written in terms of the 1- and 2-forms invariant under the $SU(2) \times SU(2)$-symmetry of $T^{1,1}$. These include

$$
g^5\,, \qquad \omega_{12} = g^1 \wedge g^2\,, \qquad \omega_{34} = g^3 \wedge g^4\,, \qquad \omega_{13} = g^1 \wedge g^3 + g^2 \wedge g^4\,. \tag{3.6}
$$

The definition of the 1-forms $\{g^i\}$ is given by equation (4) in [5].

This geometry has the following interpretation in terms of the D-brane configuration in flat space. One places a large number N of D3-branes filling the Minkowski directions at the tip of the conifold. In the low energy limit the D3-branes curve the space and dissolve into the flux of the 5-form, *i.e.* they can be replaced by a curved metric supported by F_5. Next one wraps M D5-branes on the S^2-cycle of the $T^{1,1}$ $(0 \ll M \ll N)$. The D5-branes carry the 3-form (magnetic) charge. They source the flux of the F_3:

$$
\frac{1}{4\pi^2\alpha'}\int_{S^3} F_3 = M\,. \tag{3.7}
$$

In summary the geometry of the KS solution has several features similar to QCD. In the UV limit the solution respects conformal symmetry modulo logarithmic corrections. The geometry has an IR cutoff, which will later define the mass gap. One more property, which relates the geometry to the $\mathcal{N} = 1$ supersymmetric theory and QCD is the patern of breaking of chiral symmetry.

$\mathcal{I}$-symmetry

$\mathcal{I}$-symmetry is the $\mathbb{Z}_2$-symmetry of the KS solution. It interchanges the two spheres (θ_1, ϕ_1) and (θ_2, ϕ_2) simultaneously changing the sign of F_3 and H_3. Its action on the forms important for our calculations is as follows:

$$\begin{aligned} g^5 &\to g^5\,, & (3.8)\\ dg^5 &\to dg^5\,, & (3.9)\\ g^1\wedge g^2 &\to -g^1\wedge g^2\,, & (3.10)\\ g^3\wedge g^4 &\to -g^3\wedge g^4\,, & (3.11)\\ g^1\wedge g^3+g^2\wedge g^4 &\to -(g^1\wedge g^3+g^2\wedge g^4)\,. & (3.12)\end{aligned}$$

3.1 Gauge theory

Based one the properties of the gravity background, such as symmetries and geometrical features one can guess what the dual theory is. First of all, it must be a non-conformal $\mathcal{N}=1$ gauge theory. The number of D3-branes N must be related to the rank of the gauge group: in this case the group must be $SU(N)\times SU(N)$, however the presence of M D5-branes shifts the rank of one of the factors $\to SU(N)\times SU(N+M)$. The microscopic Lagrangian thus contains the gluon A_μ and gluino λ_α fields in the adjoint representation of the gauge group plus some extra stuff coming from massless open strings in such a geometry. In particular, one expects scalars modes that describe motion of the D3-branes along the conifold. Such scalars are a pair of doublets $A_{1,2}$ and $B_{1,2}$ in the bifundamental representations of the gauge group:

$$A_{1,2}\in(N,\overline{N+M})\,,\qquad B_{1,2}\in(\overline{N},N+M)\,. \tag{3.13}$$

Since the theory is supersymmetric the scalars are promoted to (chiral) superfields. The D-term of the scalars is precisely the algebraic definition of the conifold. The fact that there are 4 (complex) fields in total is related to the fact that the conifold is the complex co-dimension one surface in the 4-dimensional complex space $\mathbb{C}_4$.

The theory has a superpotential

$$W=\lambda\epsilon^{ik}\epsilon^{jl}\mathrm{tr}A_iB_jA_kB_l\,. \tag{3.14}$$

In the non-conformal case $M \neq 0$ the gauge couplings of the two gauge factors start to run in opposite directions. One expects then for the one of them to hit a Landau pole at a certain scale. One can see the effect of this in the metric (3.2): when $r < r_0$ the metric may appear negative definite. However the pole is replaced by a Seiberg type of duality transition.

Let us start from some UV limit of the theory and flow to the IR. One of the couplings become strong and we have to go to a Seiberg dual description of the same theory. Mesons of the original description become elementary particles of the dual description. In fact after the duality the theory maps to a same kind of theory with the only difference in the gauge group $SU(N) \times SU(N+M) \to SU(N) \times SU(N-M)$. Now the direction of the running of two couplings reverses and the other coupling flows towards the strong coupling regime. Eventually we end up with a cascade of Seiberg dualities which end in the IR fixed point once one of the ranks become smaller than M:

$$\begin{aligned} & SU(M) \longleftarrow SU(2M) \times SU(M) \longleftarrow \ldots \longleftarrow \\ & \qquad \longleftarrow SU(N) \times SU(N-M) \longleftarrow SU(N) \times SU(N+M) \longleftarrow \ldots \end{aligned} \tag{3.15}$$

There is no fixed point in the UV and the flow continue indefinitely.

In the original paper [5] the expectation of Klebanov and Strassler was that the IR fixed point is precisely the $\mathcal{N} = 1$ supersymmetric Yang-Mills theory, that is the KS theory is in the same universality class with the $\mathcal{N} = 1$ SYM. However, it was understood later that the IR theory is not the same. In particular, it was realized that there traces of the A and B fields in the IR. In the perspective of this review there are low mass states, which are the mesons of the A and B fields. The expectation values of the so-called baryonic operators spontaneously break the baryon $U(1)$ symmetry, which leads to the presence of massless states.

3.2 Gravity theory

In Klebanov-Strassler theory one introduces additional D5-branes and wrap them on the S^2 of the $T^{1,1}$. Alternatively these can be viewed as fractional D3-brranes [46, 47]:

$$\int_{S^3} F_3 = M \tag{3.16}$$

Specific solutions for the type IIB were obtained in [5]. The following ansatz satisfies the symmetries of the deformed conifold:

$$F_3 = M\left\{g^5\wedge g^3\wedge g^4 + d[F(\tau)(g^1\wedge g^3+g^2\wedge g^4)]\right\} = \tag{3.17}$$
$$= M\left\{g^5\wedge g^3\wedge g^4(1-F)) + g^5\wedge g^1\wedge g^2 F + F' d\tau\wedge(g^1\wedge g^3+g^2\wedge g^4)\right\}, \tag{3.18}$$

where F must satisfy the following boundary conditions: $F(0)=0$ and $F(\infty)=1/2$.

$$B_2 = g_s M\left[f(\tau)g^1\wedge g^2 + k(\tau)g^3\wedge g^4\right], \tag{3.19}$$
$$\begin{aligned} H_3 &= dB_2 = \\ &= g_s M\left[d\tau\wedge(f'g^1\wedge g^2+k'g^3\wedge g^4)+\frac{1}{2}(k-f)g^5\wedge(g^1\wedge g^3+g^2\wedge g^4)\right]. \end{aligned} \tag{3.20}$$

Since the five-form $\tilde{F}_5$ is self-dual, it can be decomposed in the following way:

$$\tilde{F}_5 = \mathcal{F}_5 + \star\mathcal{F}_5, \tag{3.21}$$

where

$$\mathcal{F}_5 = B_2\wedge F_3 = g_s M^2 l(\tau) g^1\wedge g^2\wedge g^3\wedge g^4\wedge g^5, \tag{3.22}$$
$$\star\mathcal{F}_5 = -g_s M^2\frac{16\epsilon^{-8/3}l(\tau)}{K^2h^2\sinh^2(\tau)}dx^0\wedge dx^1\wedge dx^2\wedge dx^3\wedge d\tau, \tag{3.23}$$

where

$$l = f(1-F)+kF. \tag{3.24}$$

The Hodge dual of the 3-forms are:

$$\star F_3 = Mh^{-1}dx^0\wedge dx^1\wedge dx^2\wedge dx^3\wedge\left[(1-F)\tanh^2\left(\frac{\tau}{2}\right)d\tau\wedge g^1\wedge g^2 + \right. \tag{3.25}$$
$$\left. + F\coth^2\left(\frac{\tau}{2}\right)d\tau\wedge g^3\wedge g^4 + F' g^5\wedge(g^1\wedge g^3+g^2\wedge g^4)\right] \tag{3.26}$$
$$\star H_3 = -g_s M h^{-1}dx^0\wedge dx^1\wedge dx^2\wedge dx^3\wedge\left[g^5\wedge(k'\tanh^2\left(\frac{\tau}{2}\right)g^1\wedge g^2 + \right. \tag{3.27}$$
$$\left. + f'\coth^2\left(\frac{\tau}{2}\right)g^3\wedge g^4) - \frac{1}{2}(f-k)d\tau\wedge(g^1\wedge g^3+g^2\wedge g^4)\right]. \tag{3.28}$$

The $10d$ metric is:

$$ds_{10}^2 = h^{-1/2}(\tau)\eta_{\mu\nu}dx^\mu dx^\nu + h^{1/2}(\tau)ds_6^2, \tag{3.29}$$

where ds_6^2 is the metric of the deformed conifold 3.4.

It is worth to mention a few other useful relations and facts.

Using the ansatz above one can find the solutions that satisfy the type IIB equations. Namely one needs to find the explicit expressions for the functions $F(\tau)$, $f(\tau)$, $k(\tau)$ and $h(\tau)$.

Three of these functions $F(\tau)$, $f(\tau)$ and $k(\tau)$ can be expressed algebraically:

$$F(\tau) = \frac{\sinh\tau - \tau}{2\sinh\tau}, \tag{3.30}$$

$$f(\tau) = \frac{\tau\coth\tau - 1}{2\sinh\tau}(\cosh\tau - 1), \tag{3.31}$$

$$k(\tau) = \frac{\tau\coth\tau - 1}{2\sinh\tau}(\cosh\tau + 1). \tag{3.32}$$

The fourth one - the warp factor $h(\tau)$ can be defined by the following integral expression:

$$h(\tau) = \alpha\frac{2^{2/3}}{4}\int_\tau^\infty dx\frac{x\coth x - 1}{\sinh^2 x}(\sinh(2\tau) - 2x)^{1/3}. \tag{3.33}$$

Note that

$$l(\tau) = f(1-F) + kF = \frac{\tau\coth\tau - 1}{4\sinh^2\tau}(\sinh 2\tau - 2\tau). \tag{3.34}$$

Details of KS background

In this section we give details on the Klebanov-Strassler IIB supergravity background [5].

We set $g_S = \alpha' = 1$ and $M = 2$. External differentials of some forms necessary for the calculation are

$$
\begin{aligned}
dg^5 = & \qquad -\big(g^1 \wedge g^4 + g^3 \wedge g^2\big)\,, \\
d(g^1 \wedge g^3 + g^2 \wedge g^4) = & \qquad (g^1 \wedge g^2 - g^3 \wedge g^4) \wedge g^5\,. \\
d(g^1 \wedge g^2) = & \; -\frac{1}{2}\big(g^1 \wedge g^3 + g^2 \wedge g^4\big) \wedge g^5\,, \\
d(g^3 \wedge g^4) = & \qquad -d(g^1 \wedge g^2)\,.
\end{aligned}
\tag{3.35}
$$

The auxiliary functions are

$$
\begin{aligned}
F(\tau) &= \frac{\sinh\tau - \tau}{2\sinh\tau}\,, \\
f(\tau) &= \frac{\tau\coth\tau - 1}{2\sinh\tau}(\cosh\tau - 1)\,, \\
k(\tau) &= \frac{\tau\coth\tau - 1}{2\sinh\tau}(\cosh\tau + 1)\,.
\end{aligned}
\tag{3.36}
$$

Some useful identities between them are

$$k - f = 2\,F'\,, \tag{3.37}$$

$$f' = (1 - F)\,\tanh^2(\tau/2)\,, \tag{3.38}$$

$$k' = F\,\coth^2(\tau/2)\,. \tag{3.39}$$

The NSNS two-form field and corresponding field strength are

$$B_2 = f(\tau) g^1 \wedge g^2 + k(\tau) g^3 \wedge g^4 \tag{3.40}$$

and

$$
\begin{aligned}
H_3 = dB_2 = d\tau \wedge (f' g^1 \wedge g^2 + k' g^3 \wedge g^4) \\
+ \frac{1}{2}\,(k - f) g^5 \wedge (g^1 \wedge g^3 + g^2 \wedge g^4)\,,
\end{aligned}
\tag{3.41}
$$

while the RR three-form field strength is

$$
\begin{aligned}
F_3 &= g^5 \wedge g^3 \wedge g^4 + d[F(\tau)(g^1 \wedge g^3 + g^2 \wedge g^4)] \\
&= g^5 \wedge g^3 \wedge g^4 (1 - F) + g^5 \wedge g^1 \wedge g^2 F + F' d\tau \wedge (g^1 \wedge g^3 + g^2 \wedge g^4)\,.
\end{aligned}
\tag{3.42}
$$

We also introduce the function $\ell(\tau)$ via

$$F_5 = (1 + *) B_2 \wedge F_3 = (1 + *)\,\ell(\tau) \omega_2 \wedge \omega_3\,, \tag{3.43}$$

where

$$\omega_2 = \frac{1}{2}\left(g^1 \wedge g^2 + g^3 \wedge g^4\right), \qquad \omega_3 = g^5 \wedge \omega_2 . \tag{3.44}$$

It may be convenient to write it as

$$\ell(\tau) \;=\; 2f + 4FF' \;\equiv\; 2f(1-F) + 2kF . \tag{3.45}$$

The metric may be written in the form

$$\mathrm{d}s^2 \;=\; G_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + \frac{\mathrm{d}\tau^2}{G^{\tau\tau}} + \frac{(g^1)^2}{G^{11}} + \frac{(g^2)^2}{G^{22}} + \frac{(g^3)^2}{G^{33}} + \frac{(g^4)^2}{G^{44}} + \frac{(g^5)^2}{G^{55}} , \tag{3.46}$$

where the metric components are

$$G^{\mu\nu} \;=\; h^{1/2}(\tau)\, \eta^{\mu\nu} , \tag{3.47}$$

$$G^{11} = G^{22} \;=\; \frac{2}{\epsilon^{4/3} K(\tau) \sinh^2(\tau/2) h^{1/2}(\tau)} , \tag{3.48}$$

$$G^{33} = G^{44} \;=\; \frac{2}{\epsilon^{4/3} K(\tau) \cosh^2(\tau/2) h^{1/2}(\tau)} , \tag{3.49}$$

$$G^{55} = G^{\tau\tau} \;=\; \frac{6\,K(\tau)^2}{\epsilon^{4/3} h^{1/2}} , \tag{3.50}$$

$$\sqrt{-G} \;=\; \frac{\epsilon^4}{96}\, h^{1/2} \sinh^2 \tau . \tag{3.51}$$

Here

$$K(\tau) = \frac{(\sinh(2\tau) - 2\tau)^{1/3}}{2^{1/3} \sinh\tau} , \tag{3.52}$$

and the warp factor is

$$h(\tau) = 4 \cdot 2^{2/3} \epsilon^{-8/3} I(\tau) , \tag{3.53}$$

where

$$I(\tau) \equiv \int_\tau^\infty dx \frac{x \coth x - 1}{\sinh^2 x} (\sinh(2x) - 2x)^{1/3} . \tag{3.54}$$

Then

$$h'(\tau) \;=\; -16 \epsilon^{-8/3} F'(\tau) K(\tau) . \tag{3.55}$$

Some useful relations between the metric components include:

$$h\sqrt{-G}(G^{11})^2G^{55} = \coth^2\frac{\tau}{2}\,, \tag{3.56}$$

$$h\sqrt{-G}(G^{33})^2G^{55} = \tanh^2\frac{\tau}{2}\,, \tag{3.57}$$

$$h\sqrt{-G}G^{11}G^{33}G^{55} = 1\,, \tag{3.58}$$

$$f'(G^{11})^2 = (1-F)G^{11}G^{33}\,, \tag{3.59}$$

$$k'(G^{33})^2 = FG^{11}G^{33} \tag{3.60}$$

$$h^{1/2}\sqrt{-G}(G^{55})^2 = \frac{3\epsilon^{4/3}}{8}\,K^4\sinh^2\tau\,, \tag{3.61}$$

$$h^{1/2}\sqrt{-G}G^{11}G^{33} = \frac{\epsilon^{4/3}}{6K^2}\,, \tag{3.62}$$

$$h\sqrt{-G}G^{55} = \frac{\epsilon^{8/3}h}{16}\,K^2\sinh^2\tau\,, \tag{3.63}$$

$$h^{3/2}\sqrt{-G} = \frac{\epsilon^4h^2}{96}\,\sinh^2\tau\,, \tag{3.64}$$

$$\tag{3.65}$$

Some useful expressions in terms of metric components read

$$F_3^2 = H_3^2 = 6G^{55}\left(2F'^2G^{11}G^{33} + F^2(G^{11})^2 + (1-F)^2(G^{33})^2\right) = \\ = \frac{36}{2^{1/3}}\,\frac{K^2}{I^{1/2}}\left(\frac{2}{3}\,\frac{I'}{IK^3\sinh\tau} - \frac{I''}{2I} - 2\,\frac{I'}{I}\,\frac{(K\sinh\tau)'}{K\sinh\tau}\right)\,, \tag{3.66}$$

$$F_5 = \ell\omega_2\wedge\omega_3 - \frac{\ell}{2}\,\sqrt{-G}(G^{11})^2(G^{33})^2G^{55}\mathrm{d}^4x\wedge\mathrm{d}\tau\,. \tag{3.67}$$

Asymptotic Behavior

Small τ

$$F \sim \tau^2/12 - 7\tau^4/720\,, \tag{3.68}$$

$$k \sim \tau/3 + \tau^3/180\,, \tag{3.69}$$

$$f \sim \tau^3/12 - \tau^5/80\,, \tag{3.70}$$

$$K \sim (2/3)^{1/3}\Big(1 - \tau^2/10\Big)\,, \tag{3.71}$$

$$\ell \sim 2\tau^3/9\Big(1 - \tau^2/5\Big)\,, \tag{3.72}$$

$$I \sim I(0)\,. \tag{3.73}$$

Large τ

$$F \sim \frac{1}{2}\,, \tag{3.74}$$

$$F' \sim (\tau - 1)\mathrm{e}^{-\tau}\,, \tag{3.75}$$

$$k \sim \frac{\tau - 1}{2}\,, \tag{3.76}$$

$$f \sim \frac{\tau - 1}{2}\,, \tag{3.77}$$

$$K \sim 2^{1/3}\mathrm{e}^{-\tau/3}\,, \tag{3.78}$$

$$\ell \sim \tau - 1\,, \tag{3.79}$$

$$I \sim 3 \cdot 2^{-1/3}\left(\tau - \frac{1}{4}\right)\mathrm{e}^{-4\tau/3}\,, \tag{3.80}$$

$$IK^2 \sinh^2 \tau \sim \frac{3}{4}\,2^{1/3}\left(\tau - \frac{1}{4}\right)\,. \tag{3.81}$$

Chapter 4

Glueballs in Klebanov-Strassler theory

This chapter is organized in the following way. First we discusse the global symmetries of the KS quantum field theory, the global $SU(2) \times SU(2)$ symmetry, the KS refletion symmetry $\mathcal{I} = \mathbb{Z}_2$, the space-time symmetries J, P, and the supersymmetry. It is convenient to label the quantum states of the theory by the quantum numbers that label the representations of global symmetries.

This thesis concerns only the singlet states with respect to the global $SU(2)\times SU(2)$ singlet states.

Klebanov-Strassler theory is not invariant under the usual charge conjugation C. However, the charge conjugation supplemented by the exchange $A_i \leftrightarrow B_i$ is a symmetry of the theory. This symmetry was named $\mathcal{I}$-symmetry [48]. Provided this, the C-numbers of the bulk fields can be fixed in a similar manner to the parity. One can infer that, while the charge conjugation flips the sign of C_2 and B_2 gauge potentials (of the F_3 and H_3-forms), the flip $A_i \leftrightarrow B_i$ interchanges the two S^2 in $T^{1,1}$. In the Euler coordinates the latter interchange reads $(\theta_1, \phi_1) \leftrightarrow (\theta_2, \phi_2)$. The pure gauge $\mathcal{N} = 1$ sector of the Klebanov and Strassler theory does not contain the A_i and B_i fields. Therefore, for this sector, the C and $\mathcal{I}$ quantum numbers coincide.

Since in the $SU(2)\times SU(2)$ singlet sector the $\mathcal{I}$-reflection symmetry ($\mathbb{Z}_2$) is equivalent to the charge conjugation symmetry C, and we organize the presentation with respect to this $\mathcal{I}$-symmetry.

The two main sections are called $\mathcal{I}$-odd section and $\mathcal{I}$-even section, with $\mathcal{I}$ odd sector presented in section 4.3, and $\mathcal{I}$ even sector presented in section 4.4.

Within each $\mathcal{I}$ even/odd section we organize the glueball states by their Lorentz quantum numbers: the spin J and the parity conjugation symmetry P, while the charge number is equal to the $\mathcal{I}$-number.

We consider perturbations over the background KS-background with respective quantum numbers, derive the corresponding linearized equations and find the spectrum and analyze the results.

4.1 Global symmetries

4.1.1 Global $SU(2) \times SU(2)$ of KS theory

One of the global symmetries that the Klebanov-Strassler theory possesses is the $SU(2)\times SU(2)$ group of isometries of the $T^{1,1}$. This fact is reflected in the spectrum of glueballs, which are organized in the representations of this group. This group is the reduction of the $SU(4)$ R-symmetry of the $\mathcal{N} = 4$ SYM and has no analog in QCD or $\mathcal{N} = 1$ SYM. Here we will only be interested in the $SU(2) \times SU(2)$-singlet sector of the theory. All the glueball states of the $\mathcal{N} = 1$ SYM are contained in this sector. For attempts to study the non-singlet sector see *e.g.* [49].

4.1.2 J^{PC}: spin, parity and charge

The glueballs states, that is the bound states of several gluons, or gluons and gluino in the supersymmetric case, are classified by the J^{PC} quantum numbers, which are the spin J, parity P and charge conjugation C. The quantum numbers are in correlation with the quantum numbers of the operator, which produces the glueball state. For the reference, see the classification in [50]. The quantum numbers of the operators are in turn in a correlation with the quantum numbers of the bulk fields. Let us discuss, how these are determined.

Spin J

The spin of the bulk field is determined by the representation of the 4-dimensional Lorentz group. Type IIB supergravity contains metric, 2-form and 4-form potentials, as well as scalar fields. It can be shown that any fluctuation of these fields can be parameterized in terms of either 0- (scalar) or 1-form (vector). The only possible spin 2 fluctuation is the fluctuation of the metric. This is again a restriction of the gravity limit of the holographic correspondence, which we mentioned above. The higher spin states are encoded in the string excitations, which are infinitely massive in the gravity limit $\alpha' \to 0$.

Parity P

Parity is a symmetry, which reflects the 3d spatial part of the 4d Minkowski space: $x^i \to -x^i$. The properties of the bulk fields under the parity transformation are determined through the interaction of the bulk fields with probe D3-branes [9]. The interaction of the fields on the D3-brane (A^a_μ, λ^a_α) with the bulk field is given by the DBI action with Chern-Simons terms. Since the parity of the D3-brane fields is known, the parity of the bulk fields is fixed from the invariance of the DBI action.

Charge C

As explained in the introduction to the chapter, for singlet glueball sector of the KS theory the charge conjugation C-parity and the $\mathcal{I}$-parity coincide.

4.1.3 Supersymmetry

The glueballs are also organized in the representations of $\mathcal{N} = 1$ supersymmetry. In our review of the representations of $\mathcal{N} = 1$ supersymmetry we follow for the most part the lectures by Matteo Bertolini [51].

In 1967 Coleman and Mandula have proven a theorem [52] which states that the only possible continuos symmetries of the S-matrix in a generic field theory are those generated by the Poincaré group generators, P_μ and $M_{\mu\nu}$, plus some internal symmetry group G that must commute with them, where G is a semi-simple group times abelian factors

$$[G, P_\mu] = [G, M_{\mu\nu}] = 0. \tag{4.1}$$

The assumptions of the theorem are very reasonable and physical: *locality*, *causality*, *positivity of energy*, *finiteness of number of particles*, etc ... One of the assumptions is that the symmetry algebra's generators are all *bosonic* i.e. the symmetry algebra only involves *commutators*. If one allows for fermionic generators, which satisfy anti-commutator relations, the Coleman-Mandula theorem can be generalized. Indeed in 1975 it has been proven by Haag, Lopuszanski and Sohnius [53] that such an extension allows for supersymmetry, in fact supersymmetry is the only way to add fermionic generators to a symmetry algebra for the theorem to stand.

In the following section we summarize the supersymmetry algebra.

Supersymmetry algebra

The list of commutators that define the full symmetry algebra (bar supersymmetry) reads

$$\begin{aligned}
[P_\mu, P_\nu] &= 0 \\
[M_{\mu\nu}, M_{\rho\sigma}] &= -\imath\eta_{\mu\rho}M_{\nu\sigma} - \imath\eta_{\nu\sigma}M_{\mu\rho} + \imath\eta_{\nu\rho}M_{\mu\sigma} \\
[M_{\mu\nu}, P_\rho] &= -\imath\eta_{\rho\mu}P_\nu + \imath\eta_{\rho\nu}P_\mu \\
[B_l, B_m] &= \imath f_{lm}{}^n B_n \\
[P_\mu, B_l] &= 0 \\
[M_{\mu\nu}, B_l] &= 0,
\end{aligned} \tag{4.2}$$

where P_μ and $M_{\mu\nu}$ are generators of the Poicaré symmetries, B_l are the generators of some internal symmetry group G and $f_{lm}{}^n$ are structure constants of the internal symmetry group G. B_l are Lorentz scalars and they are typically related to some conserved quantum number like electric charge, isospin, etc... Vanishing of the last two commutators is simply a reflection of the fact that the full algebra is a direct product of the Poincaré algebra and the algebra G spanned by the scalar bosonic generators B_l

$$ISO(1,3) \times G. \tag{4.3}$$

The supersymetry algebra is an extension of the Poincaré algebra. Is is a graded Lie algebra of grade one, namely

$$L = L_0 \oplus L_1, \tag{4.4}$$

where L_0 is the Poincaré algebra and $L_1 = \left(Q^I_\alpha, \bar{Q}^I_{\dot{\alpha}}\right)$ with $I = 1, \ldots, N$ ($Q^I_\alpha, \bar{Q}^I_{\dot{\alpha}}$ make a set of $N + N = 2N$ anticommuting fermionic generators which transform in the representations $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ of the Lorentz group, respectively).

The remaining commutator and anticommutator relations which make up the full supersymmetry algebra besides the ones given in 4.2 are:

$$\left[P_\mu, Q^I_\alpha\right] = 0 \tag{4.5}$$

$$\left[P_\mu, \bar{Q}^I_{\dot{\alpha}}\right] = 0 \tag{4.6}$$

$$\left[M_{\mu\nu}, Q^I_\alpha\right] = \imath(\sigma_{\mu\nu})_\alpha{}^\beta Q^I_\beta \tag{4.7}$$

$$\left[M_{\mu\nu}, \bar{Q}^{I\dot{\alpha}}\right] = \imath(\bar{\sigma}_{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} \bar{Q}^{I\dot{\beta}} \tag{4.8}$$

$$\left\{Q^I_\alpha, \bar{Q}^J_{\dot{\beta}}\right\} = 2\sigma^\mu_{\alpha\dot{\beta}} P_\mu \delta^{IJ} \tag{4.9}$$

$$\left\{Q^I_\alpha, Q^J_\beta\right\} = \epsilon_{\alpha\beta} Z^{IJ}, \qquad Z^{IJ} = -Z^{JI} \tag{4.10}$$

$$\left\{\bar{Q}^I_{\dot{\alpha}}, \bar{Q}^J_{\dot{\beta}}\right\} = \epsilon_{\dot{\alpha}\dot{\beta}} (Z^{IJ})^* \tag{4.11}$$

Before proceeding to the discussion of the supermultiplets it is worth explaining how do they arise. In what follows it will be briefly summarized how the representations of the supersymmetry algebra correspond to the representations of the Poincaré algebra (which correspond to "particles" of the theory).

The Poincaré algebra has two Casimir operators. (Casimir is an operator which commutes with all the generators).

$$P^2 = P_\mu P^\mu \quad \text{and} \quad W^2 = W_\mu W^\mu, \tag{4.12}$$

where $W^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} P_\nu M_{\rho\sigma}$ is the Pauli-Lubanski vector. One can classify irreducible representations of a group by the Casimir operators. For the Poincaré group the irreducible representations are simply what is being called particles. Let us summarize how this is being realized for the massless and massive particles.

For massless particles, $P^2 = 0$ and $W^2 = 0$. Thus in the rest frame $P_\mu = (E, 0, 0, E)$ and $W^\mu = M_{12} P^\mu$. Note that the Pauli-Lubanski vector is proportional to the 4-momentum and the proportionality constant is the helicity $M_{12} = \pm j$. So for massless representations the spin is fixed and the different states are distinguished by their energy and the sign of the helicity. For instance, photon is a massless particle with two helicity states, ± 1.

For massive particles with mass m, $P_\mu = (m, 0, 0, 0)$ in the rest frame. Thus the two Casimir operators reduce to $P^2 = m^2$ and $W^2 = -\vec{J}^2 = m^2 s(s+1)$, where s is the spin. In the rest frame $W_\mu = \left(0, \frac{1}{2}\epsilon_{i0jk} m M^{jk}\right)$. Thus the massive particles are being distinguished by their mass and their spin.

In the same way as a particle corresponds to an irreducible representation of the Poincaré algebra, a superparticle corresponds to an irreducible representation of the supersymmetry algebra. Since the Poincaré algebra is a subalgebra of the supersymmetry algebra, every irreducible respresentation of the supersymmetry algebra is automatically a representation of the Poincaré algebra (possibly a reducible one). This means that a superparticle corresponds to a set of particles, while the particles within a set are related by the action of the operators Q^I_α and $\bar{Q}^I_{\dot{\alpha}}$ and their spins differ by units of $\frac{1}{2}$. Such sets are called *supermultiplets*.

P^2 is still a Casimir in the supersymmetry algebra like it is in the Poincaré algebra, however W^2 is no longer a Casimir. This is due to the fact that $M_{\mu\nu}$ does not commute with the supersymmetry generators. Because of that the particles that belong to a supermultiplet do not have the same spin anymore, though they should still have the same mass. In Nature one does not observe the degeneracy between the masses of fermions and bosons and thus the supersymmetry must be broken, provided it exists at all.

Another important feature of the supermultiplet is that it must contain an equal number of bosonic and fermionic degrees of freedom (DOF), which can be easily demonstrated.

Since the mass is a conserved quantity for supermultiplets, it make sense to consider massless and massive supermultiplets separately. Below we start with the massless supermultiplets.

$U(1)_{\mathcal{R}}$-symmetry

The $U(1)_R$ symmetry is a classical automorphism of the $\mathcal{N}=1$ supersymmetry algebra. It acts on the supersymmetry generators as

$$\begin{aligned} Q_\alpha &\to e^{\imath\phi} Q_\alpha \\ Q_{\dot\alpha} &\to e^{-\imath\phi} Q_{\dot\alpha} \end{aligned} \tag{4.13}$$

In the quantum theory usually $U(1)_R$ symmetry is broken to its discrete subgroup [40].

In the superconformal $\mathcal{N}=1$ algebra the $U(1)_R$-symmetry forms enters into a supermultiplet with the scaling symmetry. Therefore in $\mathcal{N}=1$ SCFT the $U(1)_R$ symmetry is preserved even on the quantum level.

Consequently, $U(1)_R$-symmetry is a true quantum symmetry of the Klebanov-Witten superconformal $\mathcal{N}=1$ gauge theory associated to shifts of the ψ-coordinate in our conventions, but not in Klebanov-Strassler supersymmetry $\mathcal{N}=1$ but not conformal theory the $U(1)_R$ symmetry is broken to its discrete subgroup.

Massless supermultiplets

In this section it is described how the massless supermultiplets can be constructed and the special case of $\mathcal{N}$ is being given.

First of all for massless representations the central charges vanish $Z^{IJ}=0$, hence all Q's and $\bar{Q}$'s commute with each other. Here is how the irreducible representations can be constructed:

First of all we will be working in the rest frame, where $P_\mu=(E,0,0,E)$. From equation 4.9 one can see that

$$\{Q_1^I,\bar{Q}_{\dot 1}^J\}=0 \tag{4.14}$$

And hence, $Q_1^I=\bar{Q}_{\dot 1}^I=0$ due to the positiveness of the Hilbert space. Thus only half of the generators Q_α^I and $\bar{Q}_{\dot\alpha}^I$ survive. Now, one can define the ladder (raising and lowering) operators a_I and $a_I^\dagger$ in the following way:

$$a_I \equiv \frac{1}{\sqrt{4E}} Q_2^I, \qquad a_I^\dagger \equiv \frac{1}{\sqrt{4E}} \bar{Q}_{\dot 2}^I. \tag{4.15}$$

These ladder operators satisfy the following anticommutation relations:

$$\left\{a_I,a_J^\dagger\right\}=\delta^{IJ}, \quad \{a_I,a_J\}=0, \quad \left\{a_I^\dagger,a_J^\dagger\right\}=0. \tag{4.16}$$

Note that the operator a_I (and hence Q^I_2) lowers the helicity of a state by $\frac{1}{2}$, while the operator $a^\dagger_I$ (and hence $\bar{Q}^I_{\dot{2}}$) raises the helicity of the state by $\frac{1}{2}$, which can be seen from the equations 4.7. and 4.8.

To construct representation one starts from the state $|\lambda_0\rangle$ which is called the Clifford vacuum (i.e. the state that is annihilated by all a_I: $a_I|\lambda_0\rangle = 0$). Such state will carry some irreducible representation of the Poincaré algebra.

To obtain the full representation or supermultiplet, one has to act on the Clifford vacuum $|\lambda_0\rangle$ with the raising operators $a^\dagger_I$, so that the complete supermultiplet content is the following:

$$\begin{aligned} &|\lambda_0\rangle, \quad a^\dagger_I|\lambda_0\rangle \equiv |\lambda_0 + \frac{1}{2}\rangle_I, \quad a^\dagger_I a^\dagger_J|\lambda_0\rangle \equiv |\lambda_0 + 1\rangle_{IJ}, \\ &\ldots, \quad a^\dagger_1 a^\dagger_2 \ldots a^\dagger_N|\lambda_0\rangle \equiv |\lambda_0 + \frac{N}{2}\rangle. \end{aligned} \tag{4.17}$$

Note that generally a supermultiplet is not CPT invariant, since CPT transformation flips the helicity. One needs to complete the supermultiplet by the CPT-conjugate one in order to have a CPT-invariant theory. This is not needed only if the supermultiplet is self CPT-conjugate (this happens only when $\lambda_0 = -\frac{N}{4}$).

Finally we apply the general approach described above to the case of interest — the $\mathcal{N} = 1$ supersymmetry.

Chiral supermultiplet. Starting with $\lambda_0 = 0$ one arrives at the matter or *chiral* supermultiplet, historically known as Wess-Zumino multiplet:

$$\left(0, +\frac{1}{2}\right) \underset{CPT}{\oplus} \left(-\frac{1}{2}, 0\right). \tag{4.18}$$

The supermultiplet contains one Weyl fermion (two degrees of freedom) and one massless complex scalar (two degrees of freedom). This is the representation where matter sits in $\mathcal{N} = 1$ supersymmetric theory, this is why this supermultiplet is called matter supermultiplet.

Vector supermultiplet. Starting from $\lambda_0 = \frac{1}{2}$ one arrives at the gauge or vector supermultiplet:

$$\left(+\frac{1}{2}, +1\right) \underset{CPT}{\oplus} \left(-1, -\frac{1}{2}\right). \tag{4.19}$$

The supermultiplet contains one Weyl fermion (two degrees of freedom) and one massless vector (two degrees of freedom). The gauge fields of the supersymmetric theory are described by this representation.

Gravitino supermultiplet. Starting with $\lambda_0 = 1$ one constructs the gravitino supermultiplet:

$$\left(1, +\frac{3}{2}\right) \underset{CPT}{\oplus} \left(-\frac{3}{2}, -1\right). \tag{4.20}$$

The supermultiplet contains a massless spin $\frac{3}{2}$ particle (two degrees of freedom) and a massless vector (two degrees of freedom).

Graviton supermultiplet. Starting with $\lambda_0 = \frac{3}{2}$ one arrives at the graviton supermultiplet:

$$\left(+\frac{3}{2}, +2\right) \underset{CPT}{\oplus} \left(-2, -\frac{3}{2}\right). \tag{4.21}$$

This supermultiplet contains a graviton (two degrees of freedom) with helicity 2 and its supersymmetric partner - gravitino (two degrees of freedom) with helicity $\frac{3}{2}$

Massive supermultiplets

Once can construct the massive supermultiplets in the same way as the massless ones, however one thing is different: one can not get rid of half of the generators as one did for the massless case. Considering state with mass m in its rest frame, where $P_\mu = (m, 0, 0, 0)$ the equation 4.9 reduces to:

$$\left\{Q^I_\alpha, \bar{Q}^J_{\dot{\beta}}\right\} = 2m\delta_{\alpha\dot{\beta}}\delta^{IJ} \tag{4.22}$$

So we still have the full set of $2N + 2N = 4N$ ladder operators. This means that the massive supermultiplets generally contain more particles. Another important difference is that now one should speak of spin rather than helicity. The Clifford vacuum in this case is defined by mass m and spin s, so that the eigenvalue of $\vec{J}^2$ is $s(s+1)$. Thus the degeneracy of this vacuum will be $2s + 1$, since the j_3 runs from $-s$ to $+s$.

For the case of $N = 1$ supersymmetry one can define the ladder operators (which satisfy the usual oscillator algebra) in the following way

$$a_{1,2} \equiv \frac{1}{\sqrt{2m}} Q_{1,2}, \quad a^\dagger_{1,2} \equiv \frac{1}{\sqrt{2m}} \bar{Q}_{\dot{1},\dot{2}}. \tag{4.23}$$

There is twice as many operators as in the massless case as has been already mentioned above. Note that the ladder operator a_2 will lower the spin by $\frac{1}{2}$ while the operator $a_2^\dagger$ will raise it by $\frac{1}{2}$. For the ladder operators a_1 and $a_1^\dagger$ it will be the other way around: a_1 will raise the spin by $\frac{1}{2}$ while $a_1^\dagger$ will lower it by $\frac{1}{2}$. One can now define the Clifford vacuum as a state with mass m and lowest possible spin s_0 (i.e. a state that is annihilated by both a_2 and $a_1^\dagger$: $a_2|s_0\rangle = 0$ and $a_1^\dagger|s_0\rangle = 0$). Then as before one should act with the raising operators a_1 and $a_2^\dagger$ to construct the complete supermultiplets. Here is the general form of a supermultiplet that starts with spin s_0 for $\mathcal{N} = 1$ supersymmetric theory:

$$\begin{aligned}&|s_0\rangle, \quad a_1|s_0\rangle \equiv |s_0 + \frac{1}{2}\rangle_1, \quad a_2^\dagger|s_0\rangle \equiv |s_0 + \frac{1}{2}\rangle_2 \quad a_2^\dagger a_1|s_0\rangle \equiv |s_0 + 1\rangle \\ &+ \text{the } CPT\text{-conjugate}\end{aligned} \tag{4.24}$$

One can construct each supermultiplet now.

Matter supermultiplet. Starting with $s_0 = -\frac{1}{2}$ we construct the matter supermultiplet:

$$\left(-\frac{1}{2}, 0, 0, +\frac{1}{2}\right). \tag{4.25}$$

This coincides with the massless case. Note that the second scalar state 0 has the opposite parity to the first scalar. So the supermultiplet contains a massive complex scalar (two degrees of freedom) and a massive Majorana fermion (two degrees of freedom).

Vector supermultiplet. The second supermultiplet (called gauge or vector supermultiplet) can be constructed by starting with $s_0 = -1$. In this case one arrives at the following result (completing it with the CPT-conjugate:

$$\left(-1, -\frac{1}{2}, -\frac{1}{2}, 0, 0, +\frac{1}{2}, +\frac{1}{2}, 1\right) \tag{4.26}$$

This supermultiplet contains one massive vector (three degrees of freedom), one massive Dirac fermion (four degrees of freedom) and one massive real scalar (one degree of freedom). Note that the supermultiplet is longer than the massless vector supermultiplet as expected. Also note that the DOF of the massive vector supermultiplet are the same as of the massless vector supermultiplet plus one massless matter supermultiplet. This is consistent with the notion of Higgs-like mechanism to generate masses for vector fields. One can generate massive vector supermultiplets by the supersymmetric generalization of the Higgs mechanism (when a massless vector supermultiplet "absorbs" a chiral supermultiplet) in a renormalizable supersymmetric theory.

"Spin-$\frac{3}{2}$" supermultiplet. To construct this supermultiplet we start with spin $s_0 = \frac{1}{2}$. The content of the supermultiplet (extended with its CPT-conjugate) is given below:

$$\left(-\frac{3}{2}, -1, -1, -\frac{1}{2}, +\frac{1}{2}, 1, 1, +\frac{3}{2}\right). \tag{4.27}$$

"Spin-2" supermultiplet. The higher-spin supermultiplets are usually ignored as one does not expect to have massive spin-2 particles in a theory. However, we do not exclude them. In fact massive spin-2 glueballs are of special importance.

For the discussion of higher-spin supermultiplet see for example [54, 55, 56, 57].

In order to construct the spin-2 supermultiplet, one shall start from spin $s_0 = 1$, then the complete supermultiplet extended with its CPT-conjugate reads

$$\left(-2, -\frac{3}{2}, -\frac{3}{2}, -1, 1, +\frac{3}{2}, +\frac{3}{2}, 2\right). \tag{4.28}$$

In what follows we will ignore the fermionic components of the supermultiplets.

Supersymmetric Quantum Mechanics

Before we proceed to list all the glueball states, lets us describe in detail a certain technical trick we have employed a lot to find the spectrum of the states and their superpartner states.

We are going to start with a simple theorem. Consider a quantum mechanical hamiltonian $\mathcal{H}$

$$\mathcal{H} = -\partial_\tau^2 + V(\tau). \tag{4.29}$$

One can define operators $\mathcal{H}_{12}$ and $\mathcal{H}_{21}$ in the following way:

$$\begin{aligned} \mathcal{H}_{12} &= \partial_\tau + A(\tau) \\ \mathcal{H}_{21} &= \partial_\tau - A(\tau), \end{aligned} \tag{4.30}$$

where $A(\tau)$ is the solution of the following differential equation:

$$V = A_\tau + A^2 \tag{4.31}$$

Having defined the operators $\mathcal{H}_{12}$ and $\mathcal{H}_{21}$ as in 4.30, we see that

$$\mathcal{H} = -\mathcal{H}_{12}\mathcal{H}_{21}. \tag{4.32}$$

The statement of the theorem is that a different hamiltonian where $\mathcal{H}_{12}$ and $\mathcal{H}_{21}$ are interchanged in the following way:

$$\mathcal{H}_s = -\mathcal{H}_{21}\mathcal{H}_{12} \tag{4.33}$$

will have the same (other than the zero mode) spectrum as the original hamiltonian 4.29. This hamiltonian is usually called *partner hamiltonian* or *supersymmetric partner hamiltonian*. And the corresponding potential

$$V_s = -A_\tau + A^2 \tag{4.34}$$

is being called *partner potential*. The proof is straightforward. Consider the following ansatz for the eigenfunction of the partner hamiltonian $\psi_s \equiv \mathcal{H}_{21}\psi$, then acting on it with a partner hamiltonian we get:

$$\mathcal{H}_s\psi_s = -\mathcal{H}_{21}\mathcal{H}_{12}\mathcal{H}_{21}\psi = \mathcal{H}_{21}\mathcal{H}\psi. \tag{4.35}$$

Looking at the r.h.s. (right hand side) of this expression, we notice that

$$\mathcal{H}\psi = E\psi, \tag{4.36}$$

where E are the eigenvalues of the hamiltonian $\mathcal{H}$ and ψ are its eigenfunctions. Thus, continuing the line in 4.35 we arrive at

$$\mathcal{H}_s\psi_s = \mathcal{H}_{21}E\psi = E\mathcal{H}_{21}\psi = E\psi_s. \tag{4.37}$$

The proof is complete now. We also notice that the eigenfunctions of the hamiltonian $\mathcal{H}$ and its partner hamiltonian $\mathcal{H}_s$ are related through the action of the operator $\mathcal{H}_{21}$. Note, that we have never used the particular normalization of 4.29 and 4.30. What is important is that one can factorize the hamiltonian into two first order linear differential operators $\mathcal{H}_{12}$ and $\mathcal{H}_{21}$ as in 4.32.

In what follows we would like to reformulate the above in a different form to relate it to the supersymmetry transformations of the supersymmetry algebra for massive supermultiplets 4.24.

Let us define the following operator:

$$\mathcal{D} \equiv \begin{pmatrix} 0 & \mathcal{H}_{12} \\ \mathcal{H}_{21} & 0 \end{pmatrix}, \tag{4.38}$$

then we can see that

$$\hat{\mathrm{H}} \equiv -\mathcal{D}^2 \equiv -\begin{pmatrix} \mathcal{H}_{12}\mathcal{H}_{21} & 0 \\ 0 & \mathcal{H}_{21}\mathcal{H}_{12} \end{pmatrix}. \tag{4.39}$$

Our goal is to relate $\mathcal{H}_{12}$ and $\mathcal{H}_{21}$ to the supersymmetry transformations that relate the states within a supermultiplet 4.24. Let us consider the following ansatz for $\mathcal{H}_{12}$ and $\mathcal{H}_{21}$:

$$\begin{aligned} \mathcal{H}_{12} &= \frac{1}{2}Q_2\bar{Q}_{\dot{1}} \\ \mathcal{H}_{21} &= \frac{1}{2}\bar{Q}_{\dot{2}}Q_1, \end{aligned} \tag{4.40}$$

where Q_α and $\bar{Q}_{\dot{\beta}}$ are the standard supersymmetry operators.

Now, let us see how does the operator $\hat{\mathrm{H}}$ 4.39 act on the two-component subset of the massive supermultiplet 4.24

$$\hat{\mathrm{H}}\begin{pmatrix} |s_0\rangle \\ |s_0+1\rangle \end{pmatrix} \equiv -\begin{pmatrix} \mathcal{H}_{12}\mathcal{H}_{21}|s_0\rangle \\ \mathcal{H}_{21}\mathcal{H}_{12}|s_0+1\rangle \end{pmatrix} = -\frac{1}{4}\begin{pmatrix} Q_2\bar{Q}_{\dot{1}}\bar{Q}_{\dot{2}}Q_1|s_0\rangle \\ \bar{Q}_{\dot{2}}Q_1Q_2\bar{Q}_{\dot{1}}|s_0+1\rangle \end{pmatrix}, \tag{4.41}$$

where the ground state $|s_0\rangle$ of the massive supermultiplet 4.24 is defined in the same way it has been defined in section 4.1.3. Let us further expand the first component of the right hand side of the formula 4.41:

$$-\frac{1}{4}Q_2\bar{Q}_{\dot{1}}\bar{Q}_{\dot{2}}Q_1|s_0\rangle = -\frac{1}{4}Q_2\Big(\{\bar{Q}_{\dot{1}},\bar{Q}_{\dot{2}}\} - \bar{Q}_{\dot{2}}\bar{Q}_{\dot{1}}\Big)Q_1|s_0\rangle = \frac{1}{4}Q_2\bar{Q}_{\dot{2}}\bar{Q}_{\dot{1}}Q_1|s_0\rangle, \tag{4.42}$$

since by supersymmetry algebra anticommutators 4.11, the anticommutator $\{\bar{Q}_{\dot{1}},\bar{Q}_{\dot{2}}\}$ is equal to zero. Continuing the line from 4.42 we get:

$$\begin{aligned} -\frac{1}{4}Q_2\bar{Q}_{\dot{1}}\bar{Q}_{\dot{2}}Q_1|s_0\rangle = \frac{1}{4}Q_2\bar{Q}_{\dot{2}}\bar{Q}_{\dot{1}}Q_1|s_0\rangle = \frac{1}{4}Q_2\bar{Q}_{\dot{2}}\Big(\{Q_1,\bar{Q}_{\dot{1}}\} - Q_1\bar{Q}_{\dot{1}}\Big)|s_0\rangle = \\ = \frac{1}{4}Q_2\bar{Q}_{\dot{2}}\{Q_1,\bar{Q}_{\dot{1}}\}|s_0\rangle. \end{aligned} \tag{4.43}$$

The term proportional to $Q_1\bar{Q}_{\dot{1}}|s_0\rangle$ is zero, since by definition of the "vacuum" state $|s_0\rangle$ in section 4.1.3 $\bar{Q}_{\dot{1}}|s_0\rangle = 0$. Continuing the calculation from line 4.43 and using the fact that $Q_2|s_0\rangle$ is zero, since $|s_0\rangle$ is the "vacuum" state, one arrives at:

$$-\mathcal{H}_{12}\mathcal{H}_{21}|s_0\rangle \equiv -\frac{1}{4}Q_2\bar{Q}_{\dot{1}}\bar{Q}_{\dot{2}}Q_1|s_0\rangle = \frac{1}{4}\left\{Q_2,\bar{Q}_{\dot{2}}\right\}\left\{Q_1,\bar{Q}_{\dot{1}}\right\}|s_0\rangle. \tag{4.44}$$

Now using the anticommutation relations 4.22, we finally show that with the operators $\mathcal{H}_{12}$ and $\mathcal{H}_{21}$ defined by 4.40

$$\hat{\mathrm{H}}\begin{pmatrix}|s_0\rangle\\|s_0+1\rangle\end{pmatrix} \equiv \begin{pmatrix}-\mathcal{H}_{12}\mathcal{H}_{21} & 0\\ 0 & -\mathcal{H}_{21}\mathcal{H}_{12}\end{pmatrix}\begin{pmatrix}|s_0\rangle\\|s_0+1\rangle\end{pmatrix} = m^2\begin{pmatrix}|s_0\rangle\\|s_0+1\rangle\end{pmatrix}. \tag{4.45}$$

Note that the operator $\mathcal{H}_{21}$ transforms a state inside a massive supermultiplet 4.24 into its superpartner state with spin increased by 1, while the operator $\mathcal{H}_{12}$ transforms a state inside a massive supermultiplet 4.24 into its superpartner state with spin lower by 1. We only consider bosonic states in this thesis, so the operators $\mathcal{H}_{12}$ and $\mathcal{H}_{21}$ transform between the 2 bosonic states of the massive supermultiplet 4.24 with spins separated by 1. Thus, looking at the supermultiplet contents, we see that this approach will be applicable for the massive *vector supermultiplet* 4.26 and for the massive *spin*-2 *supermultiplet* 4.28. We will be able to relate the equations that describe the bosonic superpartners applying this trick.

4.1.4 Superconformal multiplets of the Klebanov-Witten theory

In the conformal limit of the Klebanov-Strassler theory, when it reduces to the Klebanov-Witten case, the spectrum of the gravity fluctuations can be classified by representations of the $SU(2,2|1)$ superconformal group. In other words the usual supersymmetry multiplets, discussed in the previous section will be extended to include the states obtained by the action of conformal generators. As a result various supersymmetry representations will be combined together to form longer superconformal multiplets.

The spectrum of the operators of the Klebanov-Witten model, that is of the type IIB supergravity on $AdS_5\times T^{1,1}$ was originally studied in [1] and [58], where a complete classification was presented. It was demonstrated that the operators of the dual CFT, whose conformal dimension is protected by supersymmetry, all have clear gravity dual representation in terms of fluctuations of the type IIB fields in the "shortened" multiplets of $SU(2,2|1)$. This correspondence provided a consistency check of the Klebanov-Witten model. Moreover, the gravity analysis predicts other light multiplets, whose dimension is not protected.

To summarize the results, $SU(2,2|1)$ symmetry gives rise to nine families of superconformal multiplets. The multiplets are typically labeled by the spin of its highest component. In particular, there is one graviton multiplet, four gravitino multiplets and four vector multiplets. In general these are "long" multiplets, summarized in the tables 2–10 on the pages 16–19 of [1]. In general these correspond to "long" off-shell multiplets. The type IIB gravity fields on $AdS_5 \times T^{1,1}$, of interest for this work, realize the versions of the multiplets that are shorter due to some "on-shell" conditions. There are three types of the shortened mutliplets that appear in the analysis on $AdS_5 \times T^{1,1}$. Those are "conserved", "chiral" and "semi-conserved" supermultiplets. Let us discuss how they come about.

In the AdS/CFT correspondence the determination of the spectrum of the theory reduces, in the supergravity limit, to the determination of the spectrum of the Kaluza-Klein (KK) modes of the linearized fluctuations of the supergravity fields over the "vacuum" background solution, in this case $AdS_5 \times T^{1,1}$. The linearized equations generally take the form

$$\left(\Box_x + \Box_y^{(\lambda_1,\lambda_2,R)}\right) \phi_{(\lambda_1,\lambda_2,R)}(x,y) = 0\,, \tag{4.46}$$

where $\phi(x,y)$ is a generalized notation for the supergravity fluctuation, which depends on the coordinates on AdS_5 (x) and on the coordinates on $T^{1,1}$ (y). Here the fluctuation is presented by a particular harmonic on $T^{1,1}$, that is by an irreducible representations $(\lambda_1, \lambda_2, R)$ of $SU(2) \times SU(2) \times U(1)_R$, which is the isometry group of $T^{1,1}$. For each irreducible representation the action of $\Box_y$, the Laplace-Beltrami operator on $T^{1,1}$, produces an eigenvalue, which serves as the mass term for the KK fluctuation in terms of the AdS_5 space:

$$\Box_y^{(\lambda_1,\lambda_2,R)} \phi_{(\lambda_1,\lambda_2,R)}(x,y) = m^2_{(\lambda_1,\lambda_2,R)} \phi_{(\lambda_1,\lambda_2,R)}(x,y)\,. \tag{4.47}$$

Similarly, Laplace-Beltrami operator $\Box_x$ acts in a specific way on the harmonics of the conformal group $SU(2,2)$, the isometry of the AdS_5 space. The 3+1-dimensional space-time dependence of the harmonics will be converted to the eigenvalue corresponding to the spin of the fluctuation. Together with the $SU(2) \times SU(2) \times U(1)_R$ eigenvalue these yield the AdS_5 mass m_5 of the fluctuation and the corresponding conformal dimension (weight) Δ of the operator.

The type IIB supergravity fields, as reviewed in section 2.1, are generically tensors with respect to the five-dimensional $T^{1,1}$ space. In other words they are classified as irreducible representations of the $SO(5)$ symmetry of the tangent space. In order to find the spectrum on the $T^{1,1}$ one has to reduce these $SO(5)$ harmonics to the ones of $SU(2) \times SU(2) \times U(1)_R$. For the purpose of this work we will only be interested in those harmonics that are singlets with respect to the $SU(2)\times SU(2)$ part. Corresponding reduced gravity fluctuations will be discussed in due course. Here we will just summarize how the $SU(2) \times SU(2) \times U(1)_R$ harmonics combine into the multiplets of $SU(2,2|1)$.

First, the type IIB fields have the following classification in terms of $SO(5)$ harmonics Y_Λ. Λ labels the irreducible representations of $SO(5)$, corresponding to symmetric, *e.g.* $Y_{(ab)}$, or antisymmetric, *e.g.* $Y_{[abcde]}$, tensors. Here a and b are indices along the $T^{1,1}$.

10 D 5 D	$h_{\mu\nu}$ $H_{\mu\nu}$	$h^a{}_a$ π	A_{abcd} b	B B	$A_{\mu\nu}$ $a_{\mu\nu}$	Y
10 D 5 D	$h_{a\mu}$ B_μ	$A_{\mu abc}$ ϕ_μ	$A_{\mu a}$ a_μ			Y_a
10 D 5 D	$A_{\mu\nu ab}$ $b^{\pm}_{\mu\nu}$	A_{ab} a				$Y_{[ab]}$
10 D 5 D	h_{ab} ϕ					$Y_{(ab)}$
10D 5D	λ λ	$\psi_{(a)}$ $\psi^{(L)}$	ψ_μ ψ_μ			Ξ
10D 5 D	ψ_a $\psi^{(T)}$					Ξ_a

Table 4.1: The table classifying the type IIB fields in terms of $SO(5)$ harmonics [1]. The five-dimensional notations for the gravity fields are the same that appear in [2]. The fields are grouped together according to the appropriate $SO(5)$ bosonic (Y) or fermionic (Ξ) harmonic.

Then, various $SU(2)\times SU(2)\times U(1)_R$ irreducible components of the $SO(5)$ harmonics will populate the aforementioned nine superconformal multiplets of $SU(2,2|1)$. The complete content of those multiplets was established in [1]: see pages 15-19 of this reference.

For each member of the superconformal multiplet the tables summarize its $SU(2,2)$ representation, *i.e.* the spin of the particle (s_1, s_2), its conformal dimension $E_0 \equiv \Delta$ and the corresponding AdS_5 mass m_5^2 in terms of the quantity H_0 defined below; and finally, its R-charge r. For the complete classification, one would also need the spins (j, l) under the global $SU(2) \times SU(2)$. These are provided by the relation

$$H_0 = 6\left(j(j+1) + l(l+1) - \frac{r^2}{8}\right). \tag{4.48}$$

In the following discussion of the spectrum of the Klebanov-Strassler theory, we will compare our results with the superconformal multiplets summarized in the tables. As we will see the study presented here shows, how those longer superconformal multiplets are decomposed in the sum of the massive representations of supersymmetry, when the conformal theory is broken. As mentioned before, we will be interested in those multiplets, invariant under the $SU(2) \times SU(2)$ global symmetry.

4.2 The reference map of glueball states and supermultiplets

Here, we give a reference to a section where the given supermultiplet is being discussed together with the citation to the papers where the sates of the supermultiplet were analyzed in the bottom line of the corresponding table (right after the operator that corresponds to the given supermultiplet). In the case when the states belonging to the supermultiplet have been analyzed in different papers, we do give appropriate citation following the given state. We do give a reference to the section of this thesis where the particular state of the supermultiplet is presented right after that state.

- $\mathcal{I}$-odd sector was analyzed in papers [59, 60] and is being presented in section 4.3 of this thesis.

0^{+-}, 1^{+-}	0^{+-}, 1^{+-}	0^{--},1^{--}	1^{+-},1^{--}	1^{+-},1^{--}
$\mathrm{tr}(Ae^V\bar{A}e^{-V} - Be^V\bar{B}e^{-V})$	$\mathrm{tr}e^V\bar{W}_{\dot{\alpha}}e^{-V}W^2$, $\mathrm{tr}e^V W_\alpha e^{-V}\bar{W}^2$			

- $\mathcal{I}$-even sector has been analyzed in papers [3, 4, 6, 61, 7] and is being presented in section 4.4 of this thesis

1^{++} (4.4.2), 2^{++} (4.4.3)	0^{++} (4.4.1) [3, 4], 1^{++} (4.4.2)
$\mathrm{tr} W_\alpha e^V \bar{W}_{\dot\alpha} e^{-V}$ (4.4.5) [61]	$\mathrm{tr} W^2 \bar{W}^2$ (4.4.4) [7]

0^{++}, 0^{-+}	0^{++}, 0^{-+}	0^{++}, 0^{-+}	0^{++}, 0^{-+}	0^{++}, 0^{-+}	0^{++}, 0^{-+}
(4.4.1) $\mathrm{tr} W^2$, $\mathrm{tr} W^2 e^{-V} \bar{W} e^V$					

4.3 I-odd sector

The glueball sector odd under the $\mathcal{I}$ conjugation was completely studied in [59, 60].

4.3.1 Massless spin 0 states

First of all this sector contains two massless bosonic states. The existence of these states was demonstrated in [62], where it was argued that one of them is a Goldstone mode of $U(1)_B$ symmetry, spontaneously broken by the expectation values of the special baryonic operators in the Klebanov-Strassler theory. The dual operators to the massless states have dimension $\Delta = 2$.

Massless pseudo-scalar state 0^{--}

The Goldstone boson is a pseudoscalar 0^{--} and is created by the baryonic number current operator.

The dual operator of the pseudoscalar is the longitudinal part of the baryon number current:

$$\partial_\mu J^{B\mu} = \mathrm{Im}\,\mathrm{Tr}(a_i^* \Box a_i - b_i^* \Box b_i) + \text{fermionic terms}\,, \tag{4.49}$$

Massless scalar state 0^{+-}

The second massless state is a scalar 0^{+-}. The scalar operator is the real part of the same complex operator as in (4.49).

4.3.2 Massive spin 0 states

The operators producing the massless states can also produce massive scalars. To find them, one should generalize the fluctuations of the bulk fields to include the dependence on space-time coordinates. More generally one may consider all possible $SU(2)\times SU(2)$ (pseudo-) scalar fluctuations [59]:

$$\delta B_2 = \chi\,\mathrm{d}g^5 + \partial_\mu\sigma\,\mathrm{d}x^\mu\cdot g^5\,,\qquad \delta g_{13}=\delta g_{24}=\psi\,; \tag{4.50}$$

$$\delta C_2 = \tilde{\chi}\,\mathrm{d}g^5 + \partial_\mu\tilde{\sigma}\,\mathrm{d}x^\mu\cdot g^5\,, \tag{4.51}$$

where the metric is excited along the $SU(2)\times SU(2)$ invariant direction specified by the interior product of the basis 1-forms $g^1\cdot g^3 + g^2\cdot g^4$. Notice that (4.50) describe a scalar 0^{+-}, while (4.51) – a pseudoscalar 0^{--} excitation.

Massive scalar state 0^{+-}

Analysis of the linearized equations show that out of the scalar functions χ, ψ and σ, only two are independent. As a result there two mixed excitations producing the 0^{+-} eigenstates with the dimensions of the dual operators $\Delta=2$ and $\Delta=5$. The equations describing the coupled system can be found in [59]. The $\Delta=2$ operator is the baryon number current (4.49). Through the analysis of the representations of the superconformal symmetry [1] one can find that the second operator is

$$\mathcal{O}^{0^{+-}} = \frac{1}{2}\,\mathrm{Re}\,\mathrm{Tr}D\{\lambda,\lambda\}\,, \tag{4.52}$$

where D is the auxiliary field of the pure gauge $\mathcal{N}=1$ sector.

Numerical analysis of the linearized equations gives the following result for the spectrum of the scalars O^{+-} [59]

$$m_n^2 = 0.277n^2 + 1.79n + 2.17 \tag{4.53}$$

in the units

$$\frac{3\epsilon^{4/3}}{2^{5/3}g_sM\alpha'}\,. \tag{4.54}$$

Massive pseudoscalar state 0^{--}

For the pseudoscalar fluctuation 4.51 the functions $\tilde{\chi}$ and $\tilde{\sigma}$ are not independent, so that there is only one pseudoscalar fluctuations in the $\mathcal{I}$-odd sector. The 0^{--} state will be given by the imaginary part of the same complex operator as (4.52)

$$\mathcal{O}^{0^{--}} = \frac{1}{2}\,\mathrm{Im}\,\mathrm{Tr}D\{\lambda,\lambda\}\,. \tag{4.55}$$

We remind that for the operators of the pure gauge sector, such as (4.52) and (4.55) the $\mathcal{I}$ and C numbers coincide.

The equation of motion that arises from the ansatz 4.51 were found in [59]:

$$\tilde{w}'' - \frac{\cosh^2\tau + 1}{\sinh^2\tau}\,\tilde{w} + \tilde{m}^2\frac{I(\tau)}{K(\tau)^2}\,\tilde{w} = 0 \tag{4.56}$$

Numerical analysis of the linearized equations gives the following result for the spectrum of the pseudoscalars 0^{--} [59]:

$$m_n^2 \;\; = \;\; 0.289n^2 + 1.15n + 0.996 \tag{4.57}$$

in the units

$$\frac{3\epsilon^{4/3}}{2^{5/3}g_s M\alpha'}\,. \tag{4.58}$$

It has been shown in [60] that this pseudoscalar lies in the same supermultiplet as the 1^{--} described in section 4.3.3. For further discussion of this supermultiplet see section 4.3.5

Computing Dimensions and $\mathcal{R}$-charges of the Operators

Let us find, for example, the dimension of the operator, corresponding to the 0^{--} scalar (4.56). Large τ asymptotic of the equation is

$$\tilde{w}'' - \tilde{w} = 0, \qquad \text{or,} \qquad r^2\tilde{w}'' + r\tilde{w}' - 9\tilde{w} = 0 \tag{4.59}$$

in terms of the conifold radial coordinate $r \propto e^{\tau/3}$. Coordinate r is equivalent to the AdS radial coordinate, mind the $\log r$ in the metric. Thus, to find the dimension of the operator we need to redefine function $\tilde{w}$ to obtain a canonical kinetic term. Such change is apparently $\tilde{w} = r^2 W$ and we have

$$r^2 W'' + 5rW' - 5W = 0. \tag{4.60}$$

Thus the dimension of the operator is

$$\Delta_{0^{--}} = 2 + \sqrt{5+4} = 5 \tag{4.61}$$

Massive vector field, described by the Lagrangian

$$\int \mathrm{d}^4x\,\mathrm{d}r\,\sqrt{g}\left(\frac{1}{4}\,g^{\mu\alpha}g^{\nu\beta}F_{\mu\nu}F_{\alpha\beta} + \frac{1}{2}\,m_5^2 g^{\mu\nu}A_\mu A_\nu\right), \tag{4.62}$$

has the following equation of motion

$$\frac{g_{\mu\rho}}{\sqrt{g}}\,\partial_\sigma\left(\sqrt{g}g^{\sigma\alpha}g^{\rho\beta}F_{\alpha\beta}\right) - m_5^2 A_\mu = 0. \tag{4.63}$$

For the four dimensional transverse field A_μ this takes form

$$r^2\partial_r^2 A_\mu + 5r\partial_r A_\mu - m_5^2 R^2 A_\mu + m_4^2\,\frac{R^4}{r^2}\,A_\mu = 0, \tag{4.64}$$

i.e. it has the same canonical kinetic term as the scalar.

In the warped geometry the $U(1)_{\mathcal{R}}$ symmetry acts as shifts of the ψ angle of T^{11}:

$$\psi \to \psi + \epsilon. \tag{4.65}$$

Let us study how the general ansatz considered in the section 2.7.1 transforms under $U(1)_{\mathcal{R}}$. Fluctuations of the 3-forms do not depend on ψ and thus carry zero charge under $U(1)_{\mathcal{R}}$. The general fluctuation of the 5-form is a mixture of the states with charges 0 and 2. Two forms, contained in F_5, have the following transformation properties:

$$g^1\wedge g^2 + g^3\wedge g^4 = -\sin\theta_1\mathrm{d}\phi_1\wedge\mathrm{d}\theta_1 + \sin\theta_2\mathrm{d}\phi_2\wedge\mathrm{d}\theta_2, \tag{4.66}$$

$$\begin{aligned} g^1\wedge g^2 - g^3\wedge g^4 = {} & \sin\psi\sin\theta_1\sin\theta_2\mathrm{d}\phi_1\wedge\mathrm{d}\phi_2 + \cos\psi\sin\theta_1\mathrm{d}\phi_1\wedge\mathrm{d}\theta_2 - \\ & -\cos\psi\sin\theta_2\mathrm{d}\phi_2\wedge\mathrm{d}\theta_1 - \sin\psi\mathrm{d}\theta_1\wedge\mathrm{d}\theta_2, \end{aligned} \tag{4.67}$$

$$\begin{aligned} g^1\wedge g^3 + g^2\wedge g^4 = {} & -\cos\psi\sin\theta_1\sin\theta_2\mathrm{d}\phi_1\wedge\mathrm{d}\phi_2 + \sin\psi\sin\theta_1\mathrm{d}\phi_1\wedge\mathrm{d}\theta_2 - \\ & -\sin\psi\sin\theta_2\mathrm{d}\phi_2\wedge\mathrm{d}\theta_1 + \cos\psi\mathrm{d}\theta_1\wedge\mathrm{d}\theta_2, \end{aligned} \tag{4.68}$$

$$g^1\wedge g^4 - g^2\wedge g^3 = -\sin\theta_1\mathrm{d}\phi_1\wedge\mathrm{d}\theta_1 - \sin\theta_2\mathrm{d}\phi_2\wedge\mathrm{d}\theta_2. \tag{4.69}$$

The expressions 4.66 and 4.69 do not depend on ψ and thus correspond to charge $\mathcal{R} = 0$ fluctuations. On the other hand the expressions 4.67 and 4.68 do contain ψ and thus we have to determine what their non-zero $\mathcal{R}$ charge is. We have to define the $\mathcal{R}$ charge in the following way: an expression will have charge $\mathcal{R}$ when it transforms as $e^{\imath\mathcal{R}\frac{\psi}{2}}$. The denominator 2 is needed to reflect the fact that ψ actually goes from 0 to 4π and thus the unity of the group must be $e^{\imath 2\pi} = e^{\imath\frac{4\pi}{2}}$. First of all, note that it makes no sense to discuss the $\mathcal{R}$-charge of a real expression, thus we have to construct complex expressions out of 4.67 and 4.68. A natural way is to look at Y_2 4.83 and its complex conjugate instead. It is easy to see that Y_2 and Y_2^* are proportional to $e^{\imath\psi}$ and $e^{-\imath\psi}$ and have $\mathcal{R}$ charge 2 and -2 respectively.

Metric fluctuation does not depend on ψ explicitly and is has no charge under $U(1)_{\mathcal{R}}$:

$$g^1 \cdot g^3 + g^2 \cdot g^4 = \frac{1}{2}\left(\mathrm{d}\theta_1^2 + \sin^2\theta_1 \mathrm{d}\phi_1^2 - \mathrm{d}\theta_2^2 - \sin^2\theta_2 \mathrm{d}\phi_2^2\right). \tag{4.70}$$

4.3.3 Massive spin 1 states

The $\mathcal{I}$-odd sector also contains 7 vector states [60]. The 4 states have the even parity $J^{PC} = 1^{+-}$ and 3 states have the odd parity $J^{PC} = 1^{--}$.

The P-even states pseudo-vectors 1^{+-}

The $P = +$ states can be derived from the following bulk fluctuations, written in terms of the $SU(2) \times SU(2)$-invariant forms on $T^{1,1}$,

$$\delta B_2 = \mathbf{J} \wedge \mathrm{d}\tau\,, \qquad \delta C_2 = *_4 \mathrm{d}_4 \mathbf{D} + \mathbf{C} \wedge g^5\,, \tag{4.71}$$

$$\begin{aligned}\delta F_5 = (1 + *_{10})\big(&\mathbf{F} \wedge \mathrm{d}\tau \wedge g^1 \wedge g_2 \wedge g^5 + \mathbf{G} \wedge \mathrm{d}\tau \wedge g^3 \wedge g^4 \wedge g^5 + \\ &+ (\mathrm{d}_4 P \wedge g^1 \wedge g^2 + \mathrm{d}_4 Q \wedge g^3 \wedge g^4) \wedge g^5 + \\ &+ \mathrm{d}_4\mathbf{R} \wedge \mathrm{d}\tau \wedge (g^1 \wedge g^3 + g^2 \wedge g^4)\big)\,,\end{aligned} \tag{4.72}$$

where $*_4$ and $*_{10}$ denote the 4-dimensional and 10-dimensional Hodge operators respectively, d_4 is the exterior derivative acting in Minkowski space. Bold face is used to denote 1-forms.

Analysis of the linearized equations show that 4 of the functions above, say $\mathbf{C}$, $\mathbf{D}$, $\mathbf{F}$ and $\mathbf{G}$ can be chosen as independent. The system of equations can be partially diagonalized analytically separating two eigenvectors and a system of 2 coupled equations. The eigenvalues of the latter can be found numerically and correspond to the 1^{+-}

states dual to the $\Delta = 3$ baryon number current operator and $\Delta = 6$ superpartner of the scalar (4.52). The two remaining eigenvectors form two new "gravitino" multiplets, which contain parity even and odd components of $\Delta = 5$ and $\Delta = 6$ operators:

$$\mathcal{O}^{(5)}_{\mu} = \mathrm{Tr} F_{\mu\nu} \lambda \sigma^{\nu} \bar{\lambda} + \dots\,, \qquad \mathcal{O}^{(6)}_{\mu\nu} = \mathrm{Tr} F_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} + \dots\,, \tag{4.73}$$

where elipses stand for higher order fermionic terms and terms containing auxiliary fields. Numerical calculation of the spectrum of the gravitino multiplets gives [60]

$$m_n^2 = 0.287 n^2 + 1.02 n + 0.633\,, \tag{4.74}$$

$$m_n^2 = 0.288 n^2 + 1.31 n + 1.44\,, \quad \text{both in the units of} \quad \frac{3\epsilon^{4/3}}{2^{5/3} g_s M \alpha'}\,. \tag{4.75}$$

The P-odd states vectors 1^{--}

The P-odd states are described by the ansatz

$$\delta B_2 = *_4 \mathrm{d}_4 \mathbf{H} + \mathbf{A} \wedge g^5\,, \qquad \delta C_2 = \mathbf{E} \wedge \mathrm{d}\tau\,, \tag{4.76}$$

$$\delta F_5 = (1 + *_{10})\big(\mathrm{d}_4 \mathbf{K} \wedge \mathrm{d}\tau \wedge g^1 \wedge g_2 + \mathrm{d}_4 \mathbf{L} \wedge \mathrm{d}\tau \wedge g^3 \wedge g^4 + \tag{4.77}$$

$$+ (\mathrm{d}_4 \mathbf{M} + \mathbf{N} \wedge \mathrm{d}\tau) \wedge (g^1 \wedge g^3 + g^2 \wedge g^4) \wedge g^5\big)\,, \tag{4.78}$$

Writing linearized equation one can express $\mathbf{E}$, $\mathbf{K}$, $\mathbf{L}$ and $\mathbf{M}$ in terms of $\mathbf{A}$, $\mathbf{H}$ and $\mathbf{N}$, leaving 3 independent functions.

4.3.4 Massless scalar supermultiplet

In (4.49) the a_i and b_i are the scalar components of the chiral superfields A_i and B_i. As expected, the two massless states form one CP-extended scalar multiplet and can be described by a single complex operator.

4.3.5 Massive vector supermultiplet

A linear combination of $\mathbf{A}$ and $\mathbf{H}$ describe a $\Delta = 6$ the 1^{--} vector state entering the same multiplet as 0^{--}.

4.4 I-even sector

In this section we study vector fluctuations over the Klebanov-Strassler type IIB supergravity solution that are even under the $\mathcal{I}$ conjugation. We are interested only in the states invariant under the global $SU(2)\times SU(2)$ symmetry, i.e. $SU(2)\times SU(2)$ singlets.

The spectrum of 0^{++} states was computed in [3, 4]. The seven equations for those states form a complicated coupled system [3], which has not been diagonalized so far. Therefore the mass eigenvalues are only known as elements of a single set without assignments to individual glueballs. The graviton supermultiplet containing 2^{++} and 1^{++} glueball states was studied in [61].

This section is organized as follows.

In section 4.4.1 we analyze the full spectrum of 0^{++} states following [4, 3]. Then in section 4.4.1 we continue with the discussion of I-even massive pseudoscalars (i.e. 0^{-+} states) which form six scalar multiplets together with six 0^{++} states.

In section 4.4.2 we study spin $1^{?+}$ states and the corresponding vector fluctuations over the Klebanov-Strassler type IIB supergravity solution. We identify the glueball 1^{++} state dual to the $U(1)_{\mathcal{R}}$ current that combines with the massive 2^{++} state (section 4.4.3) into massive graviton supermultiplet (section 4.4.5). We show that apart from the 1^{++} dual to the $U(1)_{\mathcal{R}}$ current there is only one more vector glueball. We compute its spectrum and find that it is degenerate with one of the seven scalars of [4]. The Supersymmetric Quantum Mechanics (SQM) is used to derive the equation for the 0^{++} superpartner, section 4.4.4. We identify another 1^{++} state that can be grouped into a massive vector supermultiplet together with the scalar 0^{++} state from see section 4.4.1. The spectrum of the new 1^{++} glueball allows to separate one of the seven eigenvalue subsets and find one of the eigenvectors of the system.

In section 4.4.3 we consider the 2^{++} states.

Under the assumption that the squared glueball mass fits well a quadratic dependence we try to extract in section 4.4.1 the remaining six 0^{++} states from the combined spectrum of [4]. The assumption is certainly true for other glueballs studied previously. Nevertheless we can confidently extract only two more sub-spectra, which correspond to heavier states. This signifies that either the quadraticity of the spectra is violated by lighter glueballs, or numerics does not work so well for small eigenvalues in the coupled system.

Combined with previous results, the results summarized in this section allow to complete the list of low energy singlet supermultiplets in the Klebanov-Strassler theory.

We conclude with a remark on dual operators and proposal of the complete spectrum of $SU(2) \times SU(2)$ singlet $\mathcal{N} = 1$ supermultiplets in the KS theory.

4.4.1 Massive spin 0 states

The $\mathcal{I}$-even scalar sector is the most difficult one to study. The reason is a heavy mixing among different states.

Massive scalar states 0^{++}

The original study of the spectrum of the 0^{++} states was performed in [3, 4].

The fluctuations were labeled $(f, q, y, \Phi, s, N_1, N_2)$ according to notations in the work of Apreda [63]. In the type IIB description these fluctuations correspond to

- fluctuation of the dilaton field Φ
- fluctuations f and y of the compact part of the metric

$$\delta(\mathrm{d}s^2_{T^{1,1}}) = \frac{\sqrt{2}}{3^{1/4}} \left[-\frac{8}{3} (g^5)^2 + (g^1)^2 + (g^2)^2 + (g^3)^2 + (g^4)^2 \right] f + \\ + \frac{1}{\sqrt{2}\, 3^{1/4}} \left[((g^1)^2 + (g^2)^2) - ((g^3)^2 + (g^4)^2) \right] y. \quad (4.79)$$

- "trace" fluctuation q of the full metric

$$\delta(\mathrm{d}s^2_{10}) = \sqrt{2}\, 3^{3/4} \left[-5(\mathrm{d}r^2 + e^{2A}\mathrm{d}x_\mu \mathrm{d}x^\mu) + \frac{1}{3} (g^5)^2 + \right. \\ \left. + \frac{1}{2} \left((g^1)^2 + (g^2)^2 + (g^3)^2 + (g^4)^2 \right) \right] q. \quad (4.80)$$

- fluctuation s of the NS-NS 3-form potential

$$\delta B_2 = \frac{1}{2} \left[g^1 \wedge g^2 + g^3 \wedge g^4 \right] s. \quad (4.81)$$

- real fluctuations N_1 and N_2 of the complex 3-form potential $C_2 + iB_2$

$$\delta(C_2 + iB_2) = \frac{1}{2}\, iY_2^*\, N_1 + \frac{1}{2}\, iY_2\, N_2, \tag{4.82}$$

where Y_2 is defined as

$$Y_2 = (g^1 \wedge g^2 - g^3 \wedge g^4) + i(g^1 \wedge g^3 + g^2 \wedge g^4). \tag{4.83}$$

These fluctuations are summarized in table 4.2, where the connection to the PT ansatz functions [64] and KRN modes on S^5 [2] are mentioned. Let us classify the scalars from the point of view superconformal representations and operators: see table 4.2.

Mode	PT	10d Fluctuation	KRN	Δ	$\mathcal{R}$
Φ	Φ	$\delta\Phi$	$\Re e\, B$	4	0
s	$-2h_1$	δB_2: $\omega_2 \propto g^1 \wedge g^2 + g^3 \wedge g^4$	$\Im m\, a$	4	0
y	a	$\delta\mathrm{d}s^2$: $(g^1)^2 + (g^2)^2 - (g^3)^2 - (g^4)^2$	ϕ	3	± 2
N_2	$h_2 + Pb$	$\delta(C_2 + iB_2)$: Y_2	a	3	2
f	$\frac{1}{5}(x + 3p)$	$\delta\mathrm{d}s^2$: $-\frac{10}{9}\,(g^5)^2 + 2\mathrm{d}s^2_{T^{1,1}}$	$\phi(\pi)$	6	0
N_1	$h_2 - Pb$	$\delta(C_2 + iB_2)$: Y_2^*	a	7	-2
q	$\frac{1}{5}(2p - x)$	$\delta\mathrm{d}s^2$: $-5\mathrm{d}s^2_5 + 3\mathrm{d}s^2_{T^{1,1}}$	$\phi(\pi)$	8	0

Table 4.2: C-even scalar fluctuations of the KS background found in [3, 4].

The system of 7 linearized equations obtained in [4] is strongly entangled. In fact, the system looks so complicated that is seems impossible to disentagle it and find the eigenvectors, at least in the direct approach. A tremendous breakthrough of [4] was the computation of the collective spectrum of all 7 states 0^{++} (see table 4.3). In the collective spectrum all the eigenvalues appear to be mixed together and naive attempts to identify any single tower of states may lead to wrong results.

n	m^2	n	m^2	n	m^2	n	m^2	n	m^2
1	0.185	16	5.63	31	12.09	46	21.33	61	32.30
2	0.428	17	5.63	32	12.99	47	21.58	62	33.04
3	0.835	18	6.59	33	13.02	48	22.10	63	34.82
4	1.28	19	6.66	34	13.31	49	23.53	64	35.21
5	1.63	20	6.77	35	14.23	50	23.95	65	35.54
6	1.94	21	7.14	36	15.03	51	24.24	66	37.65
7	2.34	22	8.08	37	15.09	52	25.94	67	38.17
8	2.61	23	8.25	38	16.16	53	26.32	68	38.47
9	3.32	24	8.57	39	16.89	54	26.67	69	39.32
10	3.54	25	9.54	40	17.03	55	27.30	70	41.15
11	4.12	26	9.62	41	17.44	56	28.95	71	41.79
12	4.18	27	9.72	42	18.61	57	29.25	72	42.01
13	4.43	28	10.40	43	19.22	58	29.62	73	44.22
14	4.43	29	11.32	44	19.40	59	31.57	74	45.01
15	5.36	30	11.38	45	20.79	60	31.93	75	45.19

Table 4.3: Klebanov-Strassler spin-0 spectrum, first 75 values. - Table 5 from page 45 of the paper [4]

One may observe from the study of the graviton multiplet and the $\mathcal{I}$-odd sector, that the spectrum of any individual tower may be excellently fit by a quadratic formula. However, there seems to be no way to separate 7 quadratic spectra from the collective spectrum. One would always fail to fit the lightest states. Perhaps, there is a deviation from the quadratic dependence at least for some eigenstates of the scalar spectrum.

In [7] it was shown how SUSY can help to resolve these problems (at least partially). It turns out that the 1^{++} state discussed in section 4.4.2 is degenerate with one of the 0^{++} states. Together they form a massive vector supermultiplet. For a more detailed discussion see 4.4.4. This way, one out of 7 towers can be extracted from the collective spectrum. The question is whether the remaining 6 towers can be disentagled.

We did attempt to disentangle the remaining 6 scalars numerically in [7] (only in the extended arxiv version of the paper). Below is the summary of that attempt:

Comparing the results on the spectrum of 1^{++} and the table of 0^{++} eigenvalues we have confirmed that the vector belongs to a massive supermultiplet together with one of the scalars (see section 4.4.4). We have therefore isolated a subset of eigenvalues from the full spectrum of 0^{++}. Moreover we have noticed that the subset fits very well a quadratic form. This is also true about other glueball spectra, e.g. the ones of the $\mathcal{I}$-odd states. Therefore one could try to disentangle the eigenvalue system of Berg et al [4] assuming that the remaining glueballs have a quadratic spectrum with a sufficient accuracy.[1]

As noticed in [4] for large values of m^2_{BHM} the spectrum shows some periodic pattern with expected periodicity seven. We employed the following approach. First we obtained an initial fit for few (3-4) points, which appeared periodically, and afterwards added the points predicted by the trial fit and checked the stability of the trial fit with respect to such an extension of the data set.

Despite the fact that this approach predicts several more points with a nice accuracy, in general it fails for light eigenvalues. Besides the known subset of the 1^{++} multiplet we have only found two more stable fits. Those correspond to the following subsets in table 5 of [4]:

- $\# = 7, 12, 18, 25, 33, 40, 47, 54, 61, 68, 75$ can be fitted with the parabola

$$m^2_{\mathrm{BHM}} = 0.269n^2 + 1.054n + 1.01\,, \tag{4.84}$$

- $\# = 10, 16, 23, 30, 37, 44, 51, 58, 65, 72$ can be fitted by

$$m^2_{\mathrm{BHM}} = 0.271n^2 + 0.787n + 0.619\,. \tag{4.85}$$

In figure 4.1 we show how the trial-fits work in several examples. In case of stable fits it is enough to fit by 3 points. We also show several unstable fits. One of them is accidentally good if one considers just a 3-point fit. However it gets distorted after adding more points. Moreover, the n^2 coefficient for this fit unusually deviates from the value 0.27, which should be more or less universal for all glueballs [61]. Attempts to fit

[1] In fact such an attempt was done in [4] (see page 28, set of expressions (5.44)), but apparently none of the seven towers identified there correspond to the mass tower of the 1^{++} multiplet. In what follows we will try to somewhat improve those results.

the rest of the eigenvalues lead to not so good fits (as one in figure 4.1), which fail to fit light states.

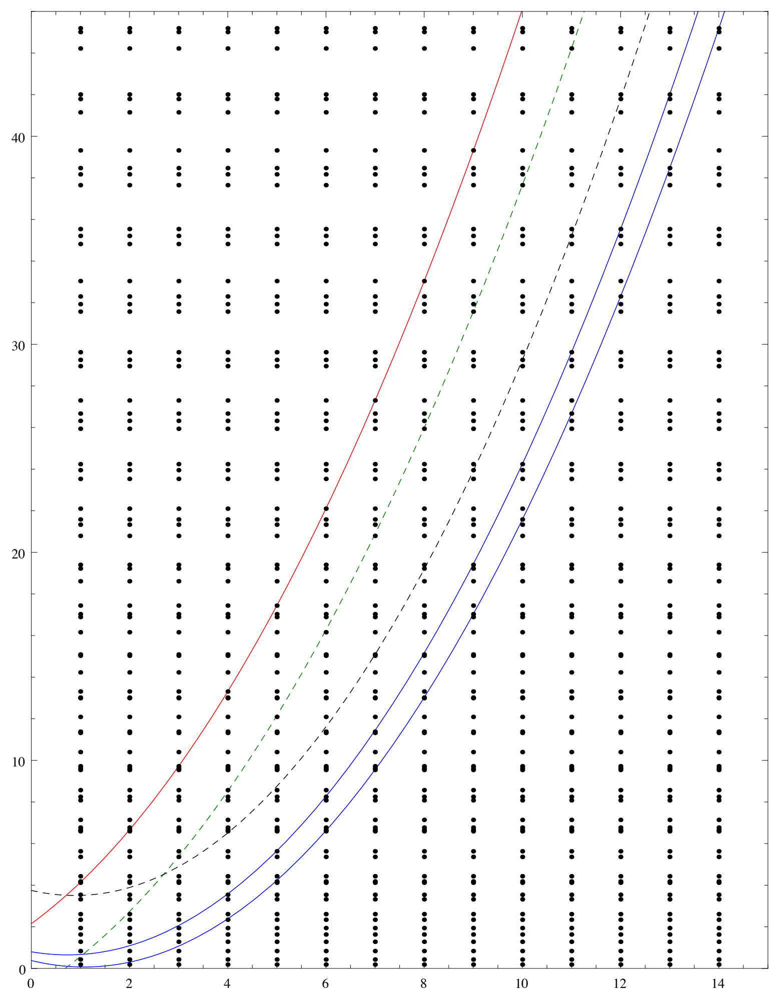

Figure 4.1: A fitting of the 0^{++} spectrum found in [4]. Above each value of n on the horizontal axis we put the whole spectrum of $m^2_{\rm BHM}$ (a vertical line of dots). The plot shows the known set from table 4.5 (red), two stable fits (blue), accidental good fit (green, dashed) and an example of not so good fit. The lightest eigenvalues in a particular fit do not necessarily correspond to $n = 1$.

The reason that two glueball towers (4.84) and (4.85) are special is perhaps due to the fact that they are heavier and correspond to operators of larger dimensions ($\Delta = 7, 8$). It was demonstrated in [3] that in the Klebanov-Tseytlin (KT) limit these glueballs decouple from the remaining system. The lighter four glueballs correspond to the dimension $\Delta = 3, 4$ operators such as the gluino bilinear and trace of energy-momentum tensor. These operators have large mixing and it is natural that corresponding glueballs also mix. Based on the result of the above analysis and KT limit result one might expect that equations for glueballs (4.84) and (4.85) can be decoupled from the full system.

Massive pseudoscalar states 0^{-+}.

This question could be addressed by looking at the 0^{-+} modes on top of the KS background. It could be shown that the pseudoscalar equations describe 6 independent modes necessary to complete 6 massive scalar multiplets with the remaining 0^{++}. The system is currently analyzed to reproduce the result of [3] for the spectrum and possibly to disentangle that result.

Here we will discuss pseudoscalar excitations of the KS background. We will establish connection to the scalars of [3, 4] discussed in the previous sections. A minimal consistent ansatz, which we find in the later sections, turns out to contain eight pseudoscalar fluctuations. The ansatz contains

- fluctuation of scalar RR potential C
- pseudoscalar fluctuation of the compact part of the metric

$$\delta(\mathrm{d}s^2_{T^{1,1}}) = \left[g^1 \cdot g^4 - g^2 \cdot g^3\right] B; \tag{4.86}$$

  Notice that this is the imaginary part of the complex fluctuation.

- fluctuations of the metric corresponding to the longitudinal part a of the vector particle $\mathbf{V}$ (4.114) found in [61] and 5th component A of the pseudovector $\mathbf{W}$ (4.115)

$$\delta(\mathrm{d}s^2) = \partial_\mu a \, \mathrm{d}x^\mu \cdot g^5 + A \, \mathrm{d}\tau \cdot g^5. \tag{4.87}$$

- fluctuation of the RR 3-form potential

$$\delta C_2 = \left[g^1 \wedge g^2 + g^3 \wedge g^4\right] C_2^+ \tag{4.88}$$

- real fluctuations G_1 and G_2 of the complex potential $C_2 + iB_2$[2]

$$\delta(C_2+iB_2) = -\frac{1}{2}\,Y_2^*\,G_1 - \frac{1}{2}\,Y_2\,G_2, \qquad Y_2 = (g^1\wedge g^2 - g^3\wedge g^4) + i(g^1\wedge g^3 + g^2\wedge g^4). \tag{4.89}$$

In practice however we will consider the fluctuations of the real and imaginary parts separately

$$\delta(C_2 + iB_2) = \left[g^1 \wedge g^2 - g^3 \wedge g^4\right] C_2^- + i\left[g^1 \wedge g^3 + g^2 \wedge g^4\right] B_2. \tag{4.90}$$

- There are also the following excitations of the 5-form

$$(1+*)(*_4 \mathrm{d}_4\phi_1 \wedge \mathrm{d}\tau \wedge g^5 + \phi_2\,\mathrm{d}^4x \wedge g^5 + *_4\mathrm{d}_4\phi_3 \wedge \mathrm{d}g^5). \tag{4.91}$$

As we will see below, only ϕ_3 is independent.

$$\phi_1 = a - \frac{(h^{1/2}\sqrt{-G}G^{11}G^{33}(G^{55})^2\phi_3)'}{\Box_4 h^{1/2}\sqrt{-G}(G^{11})^2(G^{33})^2}, \qquad \phi_2 = A + \frac{h^{1/2}G^{55}}{G^{11}G^{33}}\,\Box_4\phi_3.$$

Mode	10d Fluctuation	KRN	Δ	$\mathcal{R}$
C	δC_0	$\Im m\, B$	4	0
C_2^+	δC_2: $\omega_2 \propto g^1\wedge g^2 + g^3 \wedge g^4$	$\Re e\, a$	4	0
G_2	$\delta(C_2+iB_2)$: Y_2	a	3	2
B	$\delta\mathrm{d}s^2$: $g^1\cdot g^4 - g^2\cdot g^3$	ϕ	3	± 2
ϕ_3	δF_5: $*_4\mathrm{d}_4\phi_3 \wedge \mathrm{d}g^5$	ϕ_τ	3?	0
$a(\phi_1)$	$\delta\mathrm{d}s^2$: $\partial_\mu a\,\mathrm{d}x^\mu \cdot g^5$ (δF_5: $\partial_\mu a\,\mathrm{d}x^\mu\cdot g^5$)	B_τ	7?	0
$A(\phi_2)$	$\delta\mathrm{d}s^2$: $\mathrm{d}\tau\cdot g^5$ (δF_5: $\mathrm{d}^4x\wedge g^5$)	B_τ	7?	0
G_1	$\delta(C_2+iB_2)$: Y_2^*	a	7	-2

Table 4.4: C-even pseudoscalar fluctuations of the KS background. B_τ and ϕ_τ denote the fifth components of the KRN vectors B_μ and ϕ_μ

We summarize the pseudoscalar fluctuations in table 4.4.

[2] Defined this way G_1 and G_2 will have respectively the same $\mathcal{R}$-charges as scalars N_1 and N_2.

Detailed equations

$$
\begin{aligned}
&\frac{27}{32}\left(IK^4\sinh^2\tau\left(\frac{I'K^6\sinh^2\tau}{I^{3/2}}\,\phi_3\right)'\right)'+\\
&\qquad+\frac{3}{4}\,\frac{I'K^4\sinh^2\tau}{\sqrt{I}}\left(\frac{9IK^4\sinh^2\tau}{8}\tilde{m}^2-1\right)\phi_3-\frac{3}{4}\left(\frac{I'K^4\sinh^2\tau}{\sqrt{I}}\,a\right)'+\\
&\quad+\frac{3}{4}\,\frac{I'K^4\sinh^2\tau}{\sqrt{I}}\,A=-2^{1/3}\,\frac{I'}{K}\,B_2-2^{1/3}\left(\frac{I'}{K}\right)'C_2^--2^{1/3}\left(\frac{I'\cosh\tau}{K}\right)'C_2^+;
\end{aligned}
\tag{4.92}
$$

$$
\begin{aligned}
&\left(\frac{\cosh^2\tau+1}{I\sinh^2\tau}\,C_2^{-\prime}\right)'-\frac{C_2^-}{I}+\frac{\cosh^2\tau+1}{K^2\sinh^2\tau}\,\tilde{m}^2C_2^-+\left(\frac{2\cosh\tau}{I\sinh^2\tau}\,C_2^{+\prime}\right)'+\\
&\qquad+\frac{2\cosh\tau}{K^2\sinh^2\tau}\,\tilde{m}^2C_2^++\frac{K^2}{2^{1/3}I^{3/2}}\,B-\left(\frac{2^{1/3}I'}{2I^{3/2}K^2\sinh^2\tau}\,B\right)'-\\
&\quad-\frac{\tau}{2\sinh\tau}\,\left(\frac{C}{I}\right)'+\frac{I'}{I^2}\,B_2+2^{1/3}\frac{3I'K}{8I^{3/2}}\,A+2^{1/3}\,\frac{3}{8}\,\left(\frac{K^2}{I^{3/2}}\left(\frac{I'}{K}\right)'A\right)'+\\
&\qquad\qquad+\frac{3}{4}\left(\frac{I'}{K}\right)'\frac{\tilde{m}^2a}{2^{2/3}I^{1/2}}+\frac{27}{32}\,\frac{I'K^6\sinh^2\tau}{2^{2/3}I^{3/2}}\left(\frac{I'}{K}\right)'\tilde{m}^2\phi_3=0;
\end{aligned}
\tag{4.93}
$$

$$
\begin{aligned}
&\left(\frac{\cosh^2\tau+1}{I\sinh^2\tau}\,C_2^{+\prime}\right)'+\frac{(\cosh^2\tau+1)}{K^2\sinh^2\tau}\,\tilde{m}^2C_2^++\left(\frac{2\cosh\tau}{I\sinh^2\tau}\,C_2^{-\prime}\right)'+\\
&\qquad+\frac{2\cosh\tau}{K^2\sinh^2\tau}\,\tilde{m}^2C_2^--\left(\frac{2^{1/3}I'\cosh\tau}{2I^{3/2}K^2\sinh^2\tau}B\right)'-\left(\frac{C}{2I}\right)'+\\
&\quad+2^{1/3}\,\frac{3}{8}\,\left(\frac{K^2}{I^{3/2}}\left(\frac{I'}{K}\,\cosh\tau\right)'A\right)'+\frac{3}{4}\left(\frac{I'}{K}\,\cosh\tau\right)'\frac{\tilde{m}^2a}{2^{2/3}I^{1/2}}+\\
&\qquad\qquad+\frac{27}{32}\,\frac{I'K^6\sinh^2\tau}{2^{2/3}I^{3/2}}\,\left(\frac{I'}{K}\,\cosh\tau\right)'\tilde{m}^2\phi_3=0;
\end{aligned}
\tag{4.94}
$$

$$
\begin{aligned}
&B_2''-\frac{I'}{I}\,B_2'-\frac{\cosh^2\tau+1}{\sinh^2\tau}\,B_2+\tilde{m}^2\,\frac{I}{K^2}\,B_2+\frac{I\sqrt{K^3\sinh\tau}}{2^{1/3}}\left(\sqrt{\frac{K}{I^3\sinh\tau}}B\right)'+\\
&\quad+\frac{3I'I\sinh^2\tau}{2^{8/3}K}\,\left(\frac{K^2}{I^{3/2}\sinh^2\tau}\,A\right)'+\frac{3}{2^{8/3}}\,\frac{I^{1/2}I'}{K}\,\tilde{m}^2a+\frac{2^{2/3}I'I}{4K}\left(\frac{C}{I}\right)'+\\
&\qquad\qquad+\frac{I'}{I}\,C_2^-+\frac{2^{1/3}27}{64}\,\frac{I'^2K^5\sinh^2\tau}{I^{1/2}}\,\tilde{m}^2\phi_3=0;
\end{aligned}
\tag{4.95}
$$

$$C'' + 2\,\frac{(K\sinh\tau)'}{K\sinh\tau} C' + \tilde{m}^2 \frac{I}{K^2}\, C + \left(\frac{I''}{I} + 2\,\frac{I'}{I}\,\frac{(K\sinh\tau)'}{K\sinh\tau} \right) C -$$
$$- \frac{2^{5/3} I'}{I^{3/2} K \sinh^2\tau}\, B - \frac{3}{2}\,\frac{K^6}{I^{3/2}}\, A + \frac{2^{4/3}\tau}{IK^2 \sinh^3\tau}\, {C_2^-}' + \frac{2I'}{IK^3\sinh^2\tau}\, C_2^- +$$
$$- \frac{2^{4/3}}{IK^2\sinh^2\tau}\, {C_2^+}' - \frac{2}{IK^2\sinh^2\tau} \left(\frac{I'}{K}\, B_2 \right)' = 0; \quad (4.96)$$

$$B'' - \frac{I'}{I}\, B' + \tilde{m}^2\, \frac{I}{K^2}\, B + \frac{4B}{9K^6 \sinh^2\tau} + \frac{3}{4} \left(\frac{I'}{I} \right)^2 B + \frac{K'^2}{K^2}\, B + \frac{(K^2\coth\tau)'}{K^2}\, B +$$
$$+ \left(-\frac{2^{1/3} K^4}{I} + \frac{I''}{2I} + 2\,\frac{I'}{I}\,\frac{(K\sinh\tau)'}{K\sinh\tau} \right) B + \frac{3}{2}\, IK\tilde{m}^2 a + \frac{3}{2}\frac{I^{5/2}}{K} \left(\frac{K^4}{I^{5/2}}\, A \right)' -$$
$$- \frac{2^{4/3} I' K}{I^{1/2}}\, C - 2^{5/3} \sqrt{\frac{K}{I\sinh\tau}} \left(\sqrt{K^3 \sinh\tau} B_2 \right)' + \frac{2^{5/3} K^2}{I^{1/2}}\, C_2^- +$$
$$+ \frac{2^{4/3} I'}{I^{1/2} K^2 \sinh^2\tau}\, {C_2^-}' + \frac{2^{4/3} I' \cosh\tau}{I^{1/2} K^2 \sinh^2\tau}\, {C_2^+}' = 0; \quad (4.97)$$

$$a'' + 2\frac{(K^2\sinh\tau)'}{K^2\sinh\tau}\, a' - \frac{8a}{9K^6\sinh^2\tau} + \left(\frac{I''}{2I} + \frac{I'}{I}\,\frac{(K^2\sinh\tau)'}{K^2\sinh\tau} + \frac{1}{4} \left(\frac{I'}{I} \right)^2 \right) a -$$
$$- \frac{4B}{3K^3\sinh^2\tau} - \frac{(I^{1/2} K^2 \sinh^2\tau A)'}{I^{1/2} K^2 \sinh^2\tau} - \left(\frac{I'}{K} \right)' \frac{2^{7/3} C_2^-}{3 I^{1/2} K^4 \sinh^2\tau} -$$
$$- \left(\frac{I'}{K} \cosh\tau \right)' \frac{2^{7/3} C_2^+}{3 I^{1/2} K^4 \sinh^2\tau} - \frac{2^{7/3}\, I'}{3 I^{1/2} K^5 \sinh^2\tau}\, B_2 -$$
$$- \frac{9}{8}\,\frac{I'}{I^{1/2}} \left(\frac{I' K^6 \sinh^2\tau}{I^{3/2}}\, \phi_3 \right)' = 0; \quad (4.98)$$

$$- I\tilde{m}^2 A + K^2 \left(-\frac{I''}{I} + \frac{2}{\sinh^2\tau} - 2\frac{I'}{I}\,\frac{(K\sinh\tau)'}{K\sinh\tau} \right) A + 2I \left(\frac{K'}{K} - \frac{1}{4}\,\frac{I'}{I} \right) \tilde{m}^2 a +$$
$$+ I \left(\tilde{m}^2 a \right)' + \frac{4}{3 I^{3/2} \sinh^2\tau} \left(\frac{I^{3/2} B}{K} \right)' + \frac{2^{5/3} K^4}{3 I^{1/2}}\, C - \frac{2^{7/3} I'}{3 I^{1/2} K^3 \sinh^2\tau}\, C_2^- +$$
$$+ \frac{2^{7/3}}{3 I^{1/2} K^2 \sinh^2\tau} \left(\frac{I'}{K} \right)' {C_2^-}' + \frac{2^{7/3}}{3 I^{1/2} K^2 \sinh^2\tau} \left(\frac{I'}{K}\, \cosh\tau \right)' {C_2^+}' +$$
$$+ \frac{2^{7/3}}{3 I^{1/2} K^2 \sinh^4\tau} \left(\frac{I' \sinh^2\tau}{K}\, B_2 \right)' - \frac{9}{8}\, \frac{I'^2 K^6 \sinh^2\tau}{I}\, \tilde{m}^2 \phi_3 = 0. \quad (4.99)$$

4.4.2 Massive spin 1 states

In this section we are going to consider the general ansatz for the $\mathcal{I}$-even vector fluctuations invariant under the global $SU(2) \times SU(2)$ symmetry of the KS solution. To construct the ansatz we use the basis of $SU(2) \times SU(2)$ invariant one- and two-forms.[3]

We represent all 4-dimensional vector fluctuations by bold face 1-forms, e.g. $\mathcal{I} \equiv \mathcal{I}_\mu(x,\tau)\mathrm{d}x^\mu$; assuming the notation $\mathrm{d}\mathcal{I} \equiv \mathrm{d}_4\mathcal{I}$ and transversality $\mathrm{d} *_4 \mathcal{I} = 0$. In what follows x^μ are the 4-dimensional coordinates, τ is a radial coordinate on the deformed conifold, ϵ is the deformation parameter and g^5 is a $\mathcal{I}$-even $SU(2) \times SU(2)$-invariant 1-form on the base of the conifold. Explicit form of various background functions used in the derivation can be found in the appendix.

First of all, we notice that there are no $\mathcal{I}$-even vector fluctuations of the 3-forms F_3 and H_3. This follows from the fact that they should flip the sign under the $\mathcal{I}$ conjugation while the only $\mathcal{I}$-odd basis forms are certain two-forms supported on the conifold. The only singlet vector fluctuation of the metric has the form

$$\delta(\mathrm{d}s^2) = \frac{2}{G^{55}}\left(\tilde{\mathbf{V}} \cdot g^5 + \mathbf{Z} \cdot \mathrm{d}\tau\right) . \tag{4.100}$$

Taking into account this modification of the metric the general self-dual fluctuation of the 5-form reads

$$\begin{aligned}\delta F_5 = -\tilde{\mathbf{W}}\wedge\mathrm{d}g^5\wedge\mathrm{d}g^5+\mathbf{U}\wedge\mathrm{d}\tau\wedge\mathrm{d}g^5\wedge g^5+(\mathrm{d}\mathbf{S}+*_4\mathrm{d}\mathbf{T})\wedge\mathrm{d}g^5\wedge g^5+(*_4\mathrm{d}\mathbf{S}-\mathrm{d}\mathbf{T})\wedge\mathrm{d}\tau\wedge\mathrm{d}g^5-\\ - h^{1/2}\sqrt{-G}G^{11}G^{33}(G^{55})^2 *_4 \mathbf{U}\wedge\mathrm{d}g^5 + 2h^{1/2}\sqrt{-G}(G^{11})^2(G^{33})^2 *_4 \tilde{\mathbf{W}}\wedge\mathrm{d}\tau\wedge g^5-\\ -\frac{\ell}{2}h^{1/2}\sqrt{-G}(G^{11})^2(G^{33})^2 *_4 \tilde{\mathbf{V}}\wedge\mathrm{d}\tau\wedge g^5 .\end{aligned} \tag{4.101}$$

The ansatz (4.100-4.101) is the most general ansatz for singlet $\mathcal{I}$-even vector fluctuations. The equations that are modified by the ansatz at the linear level are the following:

- The Bianchi identity $\mathrm{d}F_5 = H_3 \wedge F_3$ gives the equations

$$\tilde{\mathbf{W}}' + \mathbf{U} = 0 , \tag{4.102}$$

$$-\mathrm{d}\tilde{\mathbf{W}} + \mathrm{d}\mathbf{S} + *_4\mathrm{d}\mathbf{T} = 0 , \tag{4.103}$$

[3] See [60] as well as [5, 44] for the background and coordinate conventions and the $\mathcal{I}$-values of the basis forms.

$$\mathrm{d}\mathbf{U} + \mathrm{d}\mathbf{S}' + *_4\mathrm{d}\mathbf{T}' = 0\,, \tag{4.104}$$

$$\mathrm{d} *_4 \mathrm{d}\mathbf{T} = 0\,, \tag{4.105}$$

$$\begin{aligned}&\left(h^{1/2}\sqrt{-G}G^{11}G^{33}(G^{55})^2 *_4 \mathbf{U}\right)' + \\ &\qquad + 2h^{1/2}\sqrt{-G}(G^{11})^2(G^{33})^2\left(*_4\tilde{\mathbf{W}} - \frac{\ell}{4} *_4 \tilde{\mathbf{V}}\right) + \mathrm{d} *_4 \mathrm{d}\mathbf{S} = 0\,.\end{aligned} \tag{4.106}$$

- Three equations come from the components of the Einstein equation:[4]

$$-\delta R_{\mu\nu} = -\frac{1}{2}\left(\partial_\mu \mathbf{Z}'_\nu + \partial_\nu \mathbf{Z}'_\mu\right) - (A' - 6p')(\partial_\mu \mathbf{Z}_\nu + \partial_\nu \mathbf{Z}_\mu) = 0\,, \tag{4.107}$$

$$\begin{aligned}-\delta R_{\mu\tau} &= \frac{1}{2}\left(2A'' + 4A'(2A' + x') + m^2 e^{-6p-x-2A}\right)\mathbf{Z}_\mu = \frac{\ell^2}{16}(G^{11})^2(G^{33})^2\mathbf{Z}_\mu + \\ &\quad + \frac{1}{8}\left(2F'^2G^{11}G^{33} + F^2(G^{11})^2 + (1-F)^2(G^{33})^2\right)\mathbf{Z}_\mu\,,\end{aligned} \tag{4.108}$$

$$\begin{aligned}-\delta R_{\mu 5} &= \frac{1}{2}\tilde{\mathbf{V}}''_\mu + \frac{1}{2}\left(2A' - 6p' + x'\right)\tilde{\mathbf{V}}'_\mu + \frac{1}{2}m^2 e^{-6p-x-2A}\tilde{\mathbf{V}}_\mu + \\ &\quad + \frac{1}{2}\left(-6p'' - x'' - 2\left(2A' + x'\right)\left(6p' + x'\right) - 2e^{-12p-4x}\right)\tilde{\mathbf{V}}_\mu = \\ &\quad = -\frac{\ell}{2}(G^{11})^2(G^{33})^2\left(\tilde{\mathbf{W}}_\mu - \frac{\ell}{8}\tilde{\mathbf{V}}_\mu\right) + \\ &\quad + \frac{1}{4}\left(2F'^2G^{11}G^{33} + F^2(G^{11})^2 + (1-F)^2(G^{33})^2\right)\tilde{\mathbf{V}}_\mu\,.\end{aligned} \tag{4.109}$$

The other supergravity equations remain unaltered at the linearized level.

Four of the five Bianchi identity equations can be solved to express everything in terms of the functions $\tilde{\mathbf{V}}$ and $\tilde{\mathbf{W}}$. The only one remaining equation is

$$\tilde{\mathbf{W}}''_\mu + \left(2\frac{K'}{K} - \frac{I'}{I}\right)\tilde{\mathbf{W}}'_\mu - \frac{8}{9}\frac{\tilde{\mathbf{W}}_\mu}{K^6\sinh^2\tau} + \tilde{m}^2\frac{I}{K^2}\tilde{\mathbf{W}}_\mu - \frac{2}{3}\frac{I'}{I}\frac{\mathbf{V}_\mu}{K^3\sinh\tau} = 0\,, \tag{4.110}$$

where we have redefined

$$\tilde{\mathbf{V}} = \frac{2^{1/3}3K}{I\sinh\tau}\mathbf{V}\,, \tag{4.111}$$

[4] Here it is convenient to parameterize the background by the functions p, x and A introduced in [64]. Expressions for them in the KS case can be found in the appendix.

and substituted for the 4-Laplacian, $*_4 \mathrm{d} *_4 \mathrm{d}\mathcal{I} = -\Box_4 \mathcal{I}$, its eigenvalue

$$\Box_4 \equiv m_4^2 = \frac{3\,\epsilon^{4/3}}{2 \cdot 2^{2/3}}\,\tilde{m}^2\,. \tag{4.112}$$

There is no non-trivial equation for $\mathbf{Z}$ from the Einstein equation. The only non-trivial equation that we obtain is the one involving $\mathbf{V}$ and $\tilde{\mathbf{W}}$,

$$\mathbf{V}_\mu'' + \left(2\,\frac{K'}{K} - \frac{I'}{I}\right)\mathbf{V}_\mu' - \frac{8}{9}\,\frac{\mathbf{V}_\mu}{K^6 \sinh^2\tau} + \tilde{m}^2\,\frac{I}{K^2}\,\mathbf{V}_\mu + \frac{2}{3}\,\frac{I'}{I}\,\frac{\mathbf{V}_\mu - 2\tilde{\mathbf{W}}_\mu}{K^3 \sinh\tau} = 0\,. \tag{4.113}$$

The general ansatz (4.100-4.101) leads to only two equations (4.110) and (4.113). In fact it reduces to the simplest generalization of the ansatz in [61]. It is straightforward to diagonalize the system of equations (4.110) and (4.113).

The massive 1^{++} states

1. Apparently there is a solution $\tilde{\mathbf{W}}_\mu = \mathbf{V}_\mu$. This is the solution found in [61], which corresponds to the 1^{++} glueball dual to the dimension $\Delta = 3$ operator of the $U(1)_\mathcal{R}$ current in the gauge theory.

$$\mathbf{V}_\mu'' + \left(2\,\frac{K'}{K} - \frac{I'}{I}\right)\mathbf{V}_\mu' - \frac{8}{9}\,\frac{\mathbf{V}_\mu}{K^6 \sinh^2\tau} + \tilde{m}^2\,\frac{I}{K^2}\,\mathbf{V}_\mu - \frac{2}{3}\,\frac{I'}{I}\,\frac{\mathbf{V}_\mu}{K^3 \sinh\tau} = 0\,. \tag{4.114}$$

This 1^{++} massive glueball state combines with the 2^{++} massive glueball state into the massive graviton supermultiplet 4.4.5

2. The other 1^{++} state can be found by subtracting the first equation from the second one. Introducing a new function $\mathbf{W}_\mu = \mathbf{V}_\mu - \tilde{\mathbf{W}}_\mu$ we obtain

$$\mathbf{W}_\mu'' + \left(2\,\frac{K'}{K} - \frac{I'}{I}\right)\mathbf{W}_\mu' - \frac{8}{9}\,\frac{\mathbf{W}_\mu}{K^6 \sinh^2\tau} + \tilde{m}^2\,\frac{I}{K^2}\,\mathbf{W}_\mu + 2\,\frac{I'}{I}\,\frac{(K\sinh\tau)'}{K\sinh\tau}\,\mathbf{W}_\mu = 0\,. \tag{4.115}$$

This vector mode corresponds to an operator of dimension $\Delta = 7$. In the limit of the singular Klebanov-Tseytlin (KT) solution [47] the equation (4.115) coincides with one of the vector equations found in [65].

Spectrum of vector (4.115) is presented in table 4.5. In the BHM [4] normalization first 9 eigenvalues coincide with given accuracy with a subset of BHM eigenvalues computed for $m^2_{\mathrm{BHM}} < 46$. In table 5 on page 45 of [4] these eigenvalues are at positions $\# = 11, 19, 27, 34, 41, 48, 55, 62, 69$. For large n this subset has periodicity 7 as expected. We conclude that pseudovector $\mathbf{W}$ is a superpartner of one of the scalars in [4]. For further discussion of the superpartner 0^{++} state see 4.4.4.

Table 4.5: First few m^2 eigenvalues and quadratic fit for the vector (4.115)

a) in normalization of [59]

4.38	7.08	10.3	14.2	18.5	23.5
29.0	35.1	41.8	49.1	56.9	65.3
74.3	83.9	94.0			

$\tilde{m}^2 = 2.32 + 1.80\,n + 0.287\,n^2$

b) in normalization of [4]

4.12	6.66	9.72	13.3	17.4	22.1
27.3	33.0	39.3	46.1	53.5	61.4
69.8	78.8	88.3			

$m^2_{\mathrm{BHM}} = 2.17 + 1.70\,n + 0.269\,n^2$

4.4.3 Massive spin 2 states

The spin 2 state is special as its produced by the energy-momentum operator $T_{\mu\nu}$, which is present in any theory. This is a 2^{++} state. The corresponding gravity fluctuation is the transverse traceless fluctuation of the metric along the Minkowski directions (graviton):[5]

$$T_{\mu\nu} \qquad \longleftrightarrow \qquad \delta g_{\mu\nu} = h_{\mu\nu}(x^\mu, \tau)\,. \tag{4.116}$$

There are no other operators with the same quantum numbers to mix with, correspondingly, there are no other bulk fields to be excited. The linearized equations for the graviton are particularly simple. They are the equation of the scalar coupled to gravity in a minimal way, that is massless Klein-Gordon equation:

$$\frac{1}{\sqrt{-g}}\partial_m\left(\sqrt{-g}g^{mn}\partial_n h_{\mu\nu}\right) = 0\,. \tag{4.117}$$

Let us consider small perturbations of the metric:

$$\tilde{g}_{MN} = g_{MN} + h_{MN}, \tag{4.118}$$

where g_{MN} is the Klebanov-Strassler solution. Note only the Minkowski part of the metric is varied here, i.e. only $h_{\mu\nu}$ is non zero.

Furthermore we consider an irreducible representation such that

$$h^\mu{}_\mu = 0 \tag{4.119}$$

Due to the gauge invariance we can pick such a $h_{\mu\nu}$ that the diagonal components are all zero.

[5] From now on we are going to use the radial coordinate τ from (3.3).

One can show in this case that only the Einstein equation 2.3 is modified in the first order in $h_{\mu\nu}$ [6] [6] .

$$-\nabla^\rho\nabla_\rho h_{\mu\nu} + \nabla^\rho\nabla_\mu h_{\nu\rho} + \nabla^\rho\nabla_\nu h_{\mu\rho} = -\frac{1}{24}\left(H_3^2 + g_s^2\tilde{F}_3^2\right) h_{\mu\nu} - \frac{g_s^2}{12} h_{\eta\rho}\tilde{F}_{\mu QRS}{}^{\eta}\tilde{F}_{\nu}{}^{QRS\rho} \tag{4.120}$$

Using 3.20 one can calculate H_3^2:

$$\begin{aligned} H_3^2 = & \frac{3g_s^2M^2}{4G_{\tau\tau}G_{11}^2\cosh^4\frac{\tau}{2}\sinh^2\tau}\big\{\tau^2(2\cosh^2\tau+1)- \\ & \qquad - 6\tau\cosh\tau\sinh\tau + \sinh^2\tau(\cosh^2\tau+2)\big\} = \\ = & \frac{288g_s^2M^2}{\epsilon^4\sinh^6\tau K^2}\left\{\tau^2(2\cosh^2\tau+1) - 6\tau\cosh\tau\sinh\tau + \sinh^2\tau(\cosh^2\tau+2)\right\} \end{aligned} \tag{4.121}$$

Here the indices m, n run over the 5-dimensional space parameterized by the space-time coordinates and τ, g_{mn} (g_{MN}) is the 5-dimensional (10-dimensional) background metric of Klebanov and Strassler. The spectrum of this equation was computed in [66] and can be approximated by a quadratic fit ($n = 1, 2, \ldots$)

$$m_n^2 = 0.290n^2 + 0.528n + 0.318 \qquad \text{in the units of} \quad \frac{3\epsilon^{4/3}}{2^{5/3}g_sM\alpha'} . \tag{4.122}$$

4.4.4 Massive vector supermultiplet

Here we discuss in detail how we complete the bosonic states from the massive vector multiplet that contains the 1^{++} (4.115) state. As we have mentioned before the spectrum of the 1^{++} found in [7] matches a subset of the spectrum that corresponds to the seven 0^{++} states calculated in [4]. This implies that the 1^{++} (4.115) state is part of the same supermultiplet as one of the 0^{++} states. The only such supermultiplet is massive vector supermultiplet 4.26.

Note, that at the very end of the section 4.1.3 we have listed all supermultiplets for which the SQM-trick described in detail in that section will be applicable and the massive vector supermultiplet is one such example.

In what follows we describe the technical details of the SQM-trick as it applies to the massive vector supermultiplet. The procedure allows us to find the equation for the

[6] Note: in reference [6] there were several typos in the equation 4.120, here it has been fixed

eigenfunctions and the spectrum of the 0^{++} and determine the properties of this 0^{++} state such us the dimension Δ and the R-charge.

The scalar superpartner should correspond to either dimension $\Delta = 6$ or $\Delta = 8$. To determine this precisely one can use the trick with Supersymmetric Quantum Mechanics (SQM) in the sense explained in [61] and derive the superpartner equation following [59]. That is, first write the equation (4.115) in the form

$$\psi'' - V\psi + \tilde{m}^2 \, \frac{I}{K^2} \, \psi = 0 \,, \tag{4.123}$$

where

$$V = -\frac{1}{2} \, \frac{(I/K^2)''}{I/K^2} + \frac{3}{4} \, \frac{(I/K^2)'^2}{(I/K^2)^2} + \frac{8}{9} \, \frac{1}{K^6 \sinh^2 \tau} - 2 \, \frac{I'}{I} \, \frac{(K \sinh \tau)'}{K \sinh \tau} \,. \tag{4.124}$$

Next solve the equation

$$V = A' + A^2 \,. \tag{4.125}$$

It is convenient to use the substitution

$$A = -B - \frac{1}{2} \, \frac{(I/K^2)'}{I/K^2} \,, \tag{4.126}$$

which leads to the equation

$$-B' + B^2 + B \left(\frac{I'}{I} - 2 \, \frac{K'}{K} \right) = \frac{8}{9} \, \frac{1}{K^6 \sinh^2 \tau} - 2 \, \frac{I'}{I} \, \frac{(K \sinh \tau)'}{K \sinh \tau} \,. \tag{4.127}$$

The equation for superpartner then is of the form (4.123) with the new potential given by $B' + B^2$.

To solve the equation (4.127) we use the following trick. We assume that the terms including function I cancel separately, i.e.

$$B_0 = -2 \, \frac{(K \sinh \tau)'}{K \sinh \tau} \,. \tag{4.128}$$

One can check that this is indeed a solution of the full equation. Thus we have found a particular solution to a Riccati equation (4.127). The general solution is given by

$$B = B_0 + \frac{1}{z} \,, \tag{4.129}$$

where z is the solution of the equation

$$z' + \left(\ln \frac{I}{K^6 \sinh^4 \tau} \right)' z = -1 \,. \tag{4.130}$$

The solution to this equation is given by

$$z = D \exp\left(-\ln \frac{I}{K^6 \sinh^4 \tau}\right) = D\, \frac{K^6 \sinh^4 \tau}{I}\,, \tag{4.131}$$

where D can be expressed through

$$D = \int_\tau^\infty \frac{I(x)}{K^6(x) \sinh^4 x} \underset{\cdot}{\mathrm{x}} + C\,, \tag{4.132}$$

where C is an arbitrary constant. Appropriate way to fix the integration constant is to ensure that the potential $B' + B^2 = B_0' + B_0^2 - 2B_0 \frac{D'}{D} + 2\frac{D'^2}{D^2} - \frac{D''}{D}$ is singular at the origin. Thus we have to take C to be infinite. Then the superpartner equation reads

$$\psi_s'' - V_s \psi_s + \tilde{m}_s^2 \frac{I}{K^2}\, \psi_s = 0\,, \tag{4.133}$$

and the potential V_s is

$$V_s = -\frac{8}{3} \frac{\sinh 2\tau}{\sinh 2\tau - 2\tau} + \frac{160}{9} \frac{\sinh^4 \tau}{(\sinh 2\tau - 2\tau)^2}\,. \tag{4.134}$$

The asymptotic form of the superpartner equation at the boundary $\tau \to \infty$ is

$$\psi_s'' - \frac{16}{9}\, \psi_s = 0\,, \tag{4.135}$$

which allows to establish the dimension of the dual operator to be $\Delta = 2 +$
$+ \sqrt{4 + (16 - 4)} = 6$.

It is also interesting to notice that with the help of this trick with SQM we find one of the seven eigenvectors of the complicated system in [4], which means that it can be further simplified. However, as we are going to explain below, it reasonable to expect that no more than two other eigenvectors could be decoupled.

4.4.5 Massive spin 2 supermultiplet

The simplest SUSY multiplet to analyze is the spin 2 supermultiplet, which contains massive spin 2 and spin 1 field together with two spin 3/2 particles.

It is a known result in SUSY theories that energy-momentum operator $T_{\mu\nu}$ enters the same supermultiplet as the current J_μ^R of the $U(1)_R$ symmetry [67]. In the Klebanov-Strassler geometry, the $U(1)_R$ symmetry is realized as shifts of the angle ψ on

$T^{1,1}$. One can find that this symmetry is anomalous, which has the following geometric realization [68]. The C_2 gauge potential has the following form asymptotically:

$$C_2 \propto M\alpha'\psi\,\omega_2\,, \qquad \tau\to\infty\,, \qquad F_3 = \mathrm{d}C_2. \tag{4.136}$$

where ω_2 is the volume form of the S^2 in $T^{1,1}$. The allowed gauge transformations can shift C_2 by an integer number of $\pi\alpha'\omega_2$. Since C_2 depends explicitly on ψ, $U(1)_R$ transformations do not leave it inavariant unless they coincide with the gauge transformations. On can see that, as $0\le\psi<4\pi$, the $\mathbb{Z}_{2M}$ subgroup of the $U(1)_R$ thus remain unbroken. The full solution for general τ breaks the $U(1)_R$ symmetry down to $\mathbb{Z}_2$. This is in fact the field theory pattern of the $U(1)_R$ symmetry breaking: anomaly breaks the symmetry down to $\mathbb{Z}_{2M}$, which is further spontaneously broken to $\mathbb{Z}_2$ by the choice of vacuum.

Associated to the $U(1)_R$ is a current operator in the gauge theory side. One can find the associated vector excitation $\mathbf{V}_\mu$ of the background from the following ansatz [61]:

$$\delta(\mathrm{d}s^2) = \mathbf{V}_\mu \mathrm{d}x^\mu g^5\,, \qquad g^5 = \mathrm{d}\psi + \cos\theta_1\mathrm{d}\phi_1 + \cos\theta_2\mathrm{d}\phi_2\,, \tag{4.137}$$

where g^5 is the $SU(2)\times SU(2)$ invariant (singlet) generalization of the 1-form $\mathrm{d}\psi$.

$U(1)_R$ current has the quantum numbers 1^{++} and conformal dimension $\Delta=3$. In fact, there is another 1^{++} vector excitation, which mixes with $\mathbf{V}_\mu$ [7]. One can find that the true eigenmodes of the system have $\Delta=3$ and $\Delta=7$, so that one of them is indeed dual to the $U(1)_R$ current, while the other one is dual to a hybrid operator

$$\mathcal{O}^{1^{++}}_\mu = \frac{1}{4}\,\mathrm{Tr}\{\lambda, F_{\alpha\beta}\}\sigma^\alpha\{\bar\lambda, F^\beta{}_\mu\} + \mathrm{Tr}\{\lambda,\tilde F_{\alpha\beta}\}\sigma^\alpha\{\bar\lambda,\tilde F^\beta{}_\mu\} + \ldots\,, \tag{4.138}$$

where elipses stand for the fermionic terms with derivatives and auxiliary fields. The superpartner of this operator will be discussed below.

As expected the spectrum of the dual $U(1)_R$ vector field coincides with the one of the graviton (4.122). The spectrum of the $\mathcal{O}^{1^{++}}_\mu$ is approximated by the fit

$$m_n^2 = 0.287n^2 + 1.80n + 2.32 \qquad \text{in the units of} \quad \frac{3\epsilon^{4/3}}{2^{5/3}g_sM\alpha'}\,. \tag{4.139}$$

Chapter 5

Conclusion and Discussion

We have summarized the results of the calculation of the spectrum of glueballs in the $SU(2) \times SU(2)$-singlet sector of the Klebanov-Strassler theory. Since one of our motivations was a prediction of the spectrum of the pure glue $\mathcal{N} = 1$ supersymmetric Yang-Mills theory, let us discuss the result from this perspective. First of all, let us recall that the gravity approximation only allows us to compute the spectrum of the states with spin less than or equal to one plus the 2^{++} state.

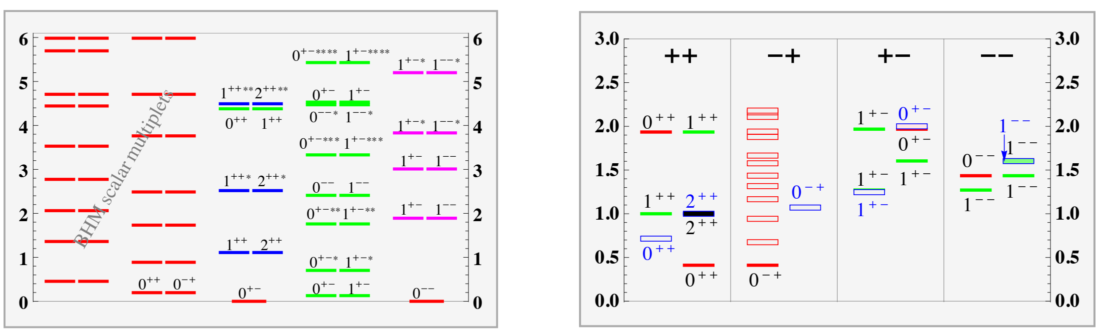

Figure 5.1: Left: $SU(2) \times SU(2)$-singlet (bosonic) spectrum (m^2) of the Klebanov-Strassler theory from holography in units (4.58). 2 columns on the left show the collective spectrum of 6 towers of the scalar multiplets. Right: Conjectured (bosonic) spectrum (m) of the $\mathcal{N} = 1$ SYM in units of the 2^{++} mass. Only the states with spin ≤ 1 and the 2^{++} can be computed in the gravity approximation. The collective spectrum of 6 towers of the scalar multiplets is shown in the $-+$ box (emtpy red boxes, no labels). Lattice prediction for certain states are shown (empty blue boxes, blue labels).

The total spectrum of the singlet sector is shown on figure 5.1(left). To obtain the spectrum of the $\mathcal{N}=1$ SYM theory, we need to throw away the states that contain the A_i and B_i fields. The adjusted spectrum is presented in figure 5.1(right), where we have left only the lightest states from each multiplet tower (with the exception of the scalar towers, for which we cannot identify the lightest states). All the masses are given in units of the 2^{++} mass. We have also included the positions of the glueballs in the pure glue non-supersymmetric $SU(3)$ Yang-Mills theory from the lattice calculation [69].

Before continuing with the comparison, let us remind what the lattice results status is.

Although QCD glueballs have not been identified unambiguously from experiment, their spectrum is believed to be known from the lattice calculation by Morningstar and Peardon [32]. (See a refinement of the original calculation in the paper of Chen *et al* [69].) In the absence of experimental data one may wonder whether the lattice prediction may be independently tested. As far as the supersymmetric extensions of QCD are concerned even the lattice results are absent due to the technical difficulties one faces extending the lattice methods to fermions.

The following table summarizes the results obtained for the spectrum of glueballs from lattice calculations [32, 69]

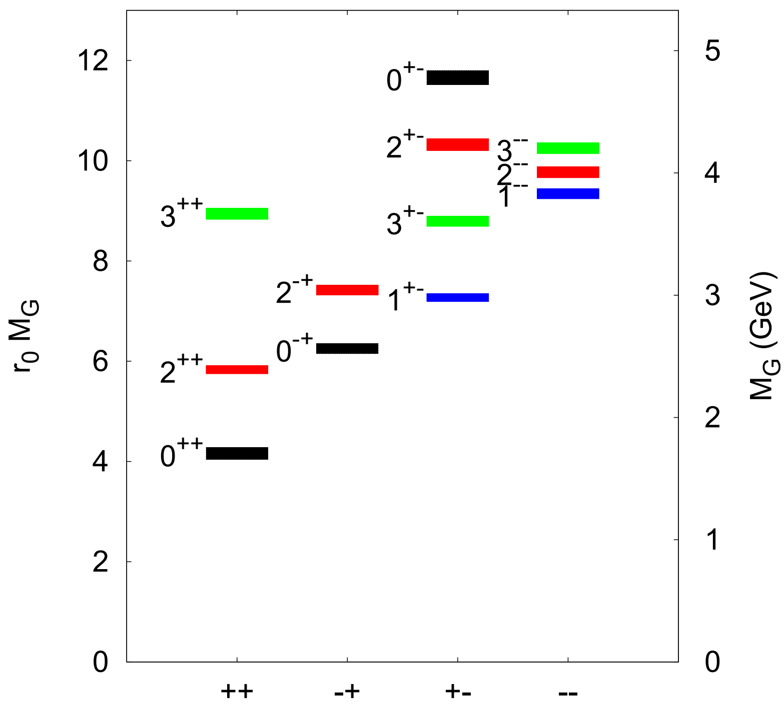

Figure 5.2: Glueball spectrum of $SU(3)$ pure gauge

Comparing the results presented in this thesis with the lattice computation, one can see a nice agreement for the 1^{+-} and 1^{--} states. The result for the 0^{+-} state looks tantalizing, but one should bear in mind that in the supersymmetric theory the 0^{+-} contains fermions (4.52), while in the non-supersymmetric case the underlying operator contains higher derivatives and should not be visible in the gravity approximation.

Thus one may speculate that despite the fact that the lattice simulations currently present the only way to access non-perturbative data in strongly coupled theories from a first principle calculation, in the case of glueball spectra a good quantitative estimate for the lightest states of low spin can be obtained by means of the gauge-gravity duality.

The scalar sector of the spectrum is still under the investigation. Although the mass eigenvalues have already been obtained, there is still a large ambiguity in the assignment of the operators to the eigenmodes. There are 6 massive scalar supermultiplets for which only collective spectrum is shown. The hope is that SUSY will help to identify each individual tower from this collective spectrum. It would be interesting to compare the lattice result for the 0^{++} and 0^{-+} states with the holographic prediction.

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Appendix A

Appendix

A.1 Notations

Table A.1: Notations

Notaion	Meaning
μ, ν	4d space-time indices
M, N	10d space-time indices

A.2 Acronyms

Table A.2: Acronyms

Acronym	Meaning
KS	Klebanov-Strassler [theory]
QFT	Quantum Field Theory
QCD	Quantum Chromodynamics
SYM	Supersymmetric Yang-Mills [theory]
NSVZ	Novikov Shifman Vainshtein Zakharov [exact β-function]
SCFT	Super Conformal Field Theory

Continued on next page

Table A.2 – continued from previous page

Acronym	Meaning
AdS	Anti de Sitter [space]
YM	Yang-Mills [theory]
CY	Calabi-Yau [space]
FI	Fayet-Iliopoulos [parameters]
DBI action	Dirac-Born-Infeld action
SISSA	Scuola Internazionale Superiore di Studi Avanzati (International School for Advanced Studies [Trieste, Italy]
DOF	degrees of freedom
C	charge conjugation
P	parity transformation
T	time reflection
CPT	a combination of charge conjugation, parity transformation and time reflection
r.h.s.	right hand side
SUSY	Supersymmetry
Het	Heterotic
KT	Klebanov-Tseytlin [limit/solution]

A.3 Conventions

A.3.1 p-forms

We define a p-form with purely bosonic components:

$$F_p = \frac{1}{p!} F_{\mu_1\mu_2\ldots\mu_p} \mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \ldots \wedge \mathrm{d}x^{\mu_p} \tag{A.1}$$

The Hodge-dual in 10 dimensions is defined as the following $10-p$-form

$$\star(F_p) = \frac{1}{(10-p)!} (\star F)_{\mu_1\mu_2\ldots\mu_{10-p}} \mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \ldots \wedge \mathrm{d}x^{\mu_{10-p}}, \tag{A.2}$$

where

$$(\star F)_{\mu_1\mu_2\ldots\mu_{10-p}} = \frac{\sqrt{-g}}{p!} \epsilon_{\nu_1\nu_2\ldots\nu_p\mu_1\mu_2\ldots\mu_{10-p}} F^{\nu_1\nu_2\ldots\nu_p} \tag{A.3}$$

A.3.2 SU(2)

Pauli matrices are defined in the following way:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\imath \\ \imath & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{A.4}$$

The identity 2×2 matrix is

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{A.5}$$

The Pauli matrices are generators of $SU(2)$. An arbitrary element of $SU(2)$ amy be written as

$$g = \mathrm{e}^{\imath \alpha^i \sigma_i}, \tag{A.6}$$

where we imply summation over $i = 1, 2, 3$.

Expanding the exponent it is easy to see that for any α

$$\mathrm{e}^{\imath \alpha \sigma_i} = \sigma_0 \cos\alpha + \imath \sigma_i \sin\alpha. \tag{A.7}$$

We can parametrize $SU(2)$ by Euler angles. Namely we can write an arbitrary element of $SU(2)$ in the following form:

$$g = \mathrm{e}^{\imath \frac{\psi}{2}\sigma_3} \mathrm{e}^{\imath \frac{\theta}{2}\sigma_2} \mathrm{e}^{\imath \frac{\phi}{2}\sigma_3} \tag{A.8}$$

Using A.7 we can expand this to get the matrix form for an arbitrary elelment of SU(2) parametrized by Euler angles [70]:

$$\begin{pmatrix} \cos\frac{\theta}{2}\mathrm{e}^{\imath\frac{\phi+\psi}{2}} & \sin\frac{\theta}{2}\mathrm{e}^{\imath\frac{\phi-\psi}{2}} \\ -\sin\frac{\theta}{2}\mathrm{e}^{-\imath\frac{\phi-\psi}{2}} & \cos\frac{\theta}{2}\mathrm{e}^{-\imath\frac{\phi+\psi}{2}} \end{pmatrix} \tag{A.9}$$

A.4 Linearized equations in the KW case

Linearized equations (A.95)-(A.99) were derived for a background given by equations (3.41)-(3.43) and (3.46) with generic functions F, f, k, ℓ and components of the metric only satisfying constraints $G^{33} = G^{11}$, $G^{44} = G^{22}$, $G^{\tau\tau} = G^{55}$ and $G_{\mu\nu} = h^{-1/2}\eta_{\mu\nu}$ Therefore those equations are also valid for the KW background. To apply them to the KW case one needs to set

$$\begin{aligned} f=k=F &= 0\,, & \text{(A.10)}\\ 1-F &\equiv 0\,, & \text{(A.11)}\\ \ell(\tau) &= \text{const}\,, & \text{(A.12)}\\ h\equiv H &= \left(1+\frac{L^4}{r^4(\tau)}\right)\,, & \text{(A.13)}\\ G^{11}=G^{33} &= \frac{6}{r^2(\tau)H^{1/2}}\,, & \text{(A.14)}\\ G^{55} &= \frac{9}{r^2(\tau)H^{1/2}}\,. & \text{(A.15)} \end{aligned}$$

Imposing these constraints recasts (A.95)-(A.99) in the form (we have to change the radial coordinate $\mathrm{d}/\mathrm{d}\tau=(r/3)\mathrm{d}/\mathrm{d}r$)

$$-\frac{r}{8}\left(r^3H\left(\frac{\phi_3}{r^4H^{3/2}}\right)'\right)'-\frac{1}{8}\,H^{1/2}\Box_4\phi_3+\frac{1}{r^2H^{1/2}}\,\phi_3+ \\ +\frac{r}{3}\left(\frac{1}{r^2H^{1/2}}\,a\right)'-\frac{1}{r^2H^{1/2}}\,A=0\,, \quad \text{(A.16)}$$

$$-\frac{2r}{3}\left(\frac{rC_2^{-\prime}}{3H}\right)'+\frac{2}{H}\,C_2^- -\frac{2}{9}\,r^2\Box_4C_2^- +\frac{36\ell}{r^4H^2}\,B_2=0; \quad \text{(A.17)}$$

$$\frac{2r}{3}\left(\frac{rC_2^{+\prime}}{3H}\right)'+\frac{2}{9}\,r^2\Box_4C_2^+=0; \quad \text{(A.18)}$$

$$-\frac{2r}{3}\left(\frac{r}{3H}B_2{}'\right)'-\frac{2}{H}\,B_2-\frac{2r^2}{9}\,\Box_4B_2+\frac{36\ell}{r^4H^2}\,C_2^-=0; \quad \text{(A.19)}$$

$$\frac{r}{3}\left(\frac{r^5}{3}\,C'\right)'+\frac{r^6H}{9}\,\Box_4C=0; \quad \text{(A.20)}$$

The components of the Ricci tensor take the following form with the above constraints on the metric (3.46) and after the substitutions (A.10)-(A.15):

$$-\,\delta R_{\mu5}=\partial_\mu\Bigg(\frac{a''}{2H^{1/2}}+\frac{3a'}{2H^{1/2}r}-\left(4+\frac{r}{2}\,\frac{H'}{H}+\frac{r^2}{8}\,\frac{H'^2}{H^2}\right)\frac{a}{r^2H^{1/2}}- \\ -\frac{3A'}{2H^{1/2}r}-\left(2+\frac{r}{4}\,\frac{H'}{H}\right)\frac{3A}{H^{1/2}r^2}\Bigg), \quad \text{(A.21)}$$

and

$$
\begin{aligned}
-\,\delta R_{\tau 5} = -\frac{m^2}{6}\, r H^{1/2} a' + \frac{m^2}{3}\left(H^{1/2} + \frac{r}{4}\,\frac{H'}{H^{1/2}}\right) a + \frac{m^2}{2}\, H^{1/2} A + \\
+ \frac{A}{4H^{1/2} r}\left(r\,\frac{H''}{H} + 5\,\frac{H'}{H} - r\,\frac{H'^2}{H^2}\right), \qquad \text{(A.22)}
\end{aligned}
$$

and

$$
\begin{aligned}
\delta R_{14} = -\delta R_{23} = -\frac{B''}{2H^{1/2}} - \left(1 - r\,\frac{H'}{H}\right)\frac{B'}{H^{1/2} r} - \frac{m^2}{2}\, H^{1/2} B + \\
+ \left(\frac{1}{2} - \frac{r^2}{8}\,\frac{H'^2}{H^2} - r\,\frac{H'}{H}\right)\frac{B}{H^{1/2} r^2}\,. \qquad \text{(A.23)}
\end{aligned}
$$

Taking the KW limit of the r.h.s part of the Einstein equation one obtains

$$
\begin{aligned}
-\frac{a''}{2H^{1/2}} - \frac{3a'}{2H^{1/2} r} + \left(4 + \frac{r}{2}\,\frac{H'}{H} + \frac{r^2}{8}\,\frac{H'^2}{H^2}\right)\frac{a}{r^2 H^{1/2}} + \frac{3A'}{2H^{1/2} r} + \\
+ \left(2 + \frac{r}{4}\,\frac{H'}{H}\right)\frac{3A}{H^{1/2} r^2} = \\
= \frac{729\ell^2}{2r^{10} H^{5/2}}\left(a - \frac{3}{4}\, r^5 H^{3/2}\left(\frac{\phi_3}{r^4 H^{3/2}}\right)'\right) \qquad \text{(A.24)}
\end{aligned}
$$

$$
\begin{aligned}
\frac{m^2}{6}\, r H^{1/2} a' - \frac{m^2}{3}\left(H^{1/2} + \frac{r}{4}\,\frac{H'}{H^{1/2}}\right) a - \frac{m^2}{2}\, H^{1/2} A - \\
- \frac{A}{4H^{1/2} r}\left(r\,\frac{H''}{H} + 5\,\frac{H'}{H} - r\,\frac{H'^2}{H^2}\right) = \\
= \frac{729\ell^2}{2r^{10} H^{5/2}}\left(A + \frac{1}{4}\, r^2 H \Box_4 \phi_3\right) \qquad \text{(A.25)}
\end{aligned}
$$

$$
\begin{aligned}
-\frac{B''}{2H^{1/2}} - \left(1 - r\,\frac{H'}{H}\right)\frac{B'}{H^{1/2} r} - \frac{m^2}{2}\, H^{1/2} B + \\
+ \left(\frac{1}{2} - \frac{r^2}{8}\,\frac{H'^2}{H^2} - r\,\frac{H'}{H}\right)\frac{B}{H^{1/2} r^2} = 0\,. \qquad \text{(A.26)}
\end{aligned}
$$

Now we will make a substitution

$$H(r) = \frac{L^4}{r^4}\,, \tag{A.27}$$

$$\ell = \frac{2}{27}\,L^4\,, \tag{A.28}$$

$$\phi_3 = r\hat{\phi}_3\,, \tag{A.29}$$

$$A = \frac{1}{3}\,r\hat{A}\,. \tag{A.30}$$

It leads to equations

$$-\frac{1}{8}\left(\frac{1}{r}\left(r^3\hat{\phi}_3\right)'\right)' - \frac{1}{8}\,\frac{L^4}{r^2}\,\Box_4\hat{\phi}_3 + \hat{\phi}_3 + \frac{1}{3}\,\left(a' - \hat{A}\right) = 0\,, \tag{A.31}$$

$$-\left(r^5 C_2^{-\prime}\right)' + 9r^3\,C_2^- - rL^4\Box_4 C_2^- + 12r^3\,B_2 = 0\,, \tag{A.32}$$

$$\left(r^5 C_2^{+\prime}\right)' + rL^4\Box_4 C_2^+ = 0\,, \tag{A.33}$$

$$-\left(r^5 B_2{}'\right)' - 9r^3\,B_2 - rL^4\,\Box_4 B_2 + 12r^3\,C_2^- = 0\,, \tag{A.34}$$

$$\left(r^5\,C'\right)' + rL^4\,\Box_4 C = 0\,, \tag{A.35}$$

$$-\frac{r^2}{2}\,a'' - \frac{3r}{2}\,a' + 4\,a + \frac{r^2}{2}\,\hat{A}' + \frac{3r}{2}\,\hat{A} = 2\,\left(a - \frac{3}{4}\,\frac{1}{r}\,\left(r^3\hat{\phi}_3\right)'\right)\,, \tag{A.36}$$

$$\frac{m^2L^2}{6r}\,a' - \frac{m^2L^2}{6r}\,\hat{A} + \frac{4r}{3L^2}\,\hat{A} = \frac{2}{L^2}\,\left(\frac{r}{3}\,\hat{A} + \frac{1}{4}\,\frac{L^4}{r}\,\Box_4\hat{\phi}_3\right) \tag{A.37}$$

$$r^2\,B'' + 10r\,B' + \frac{L^4\Box_4}{r^2}\,B - 5\,B = 0\,. \tag{A.38}$$

A.5 Details on 0^{-+} equations

A.5.1 Pseudo-scalar metric fluctuation

Consider the following (pseudoscalar) metric fluctuation.

$$\delta g = (e_1\epsilon_2 - e_2\epsilon_1)B = (g_1 g_4 - g_2 g_3)B. \tag{A.39}$$

Notice that this does not affect metric determinant at the linearized level,

$$\delta(\det g) = \det g \, \mathrm{tr}\, g^{-1}\delta g = O(B^2);$$

as well as it does not alter the self-duality of F_5.

The fluctuation gives the following contribution to the linearized equations:[1]

- RR 3-form e.o.m $\mathrm{d} * \tilde{F}_3 = F_5 \wedge H_3$

$$\begin{aligned}
\delta(\mathrm{d}*\tilde{F}_3) = \frac{1}{2}\,\mathrm{d}[&B\sqrt{-G}G^{55}G^{11}G^{33}(G^{33}(1-F)+G^{11}F)\mathrm{d}^4x\wedge\mathrm{d}\tau(g^1\wedge g^3+g^2\wedge g^4)+\\
&+2F'B\sqrt{-G}G^{55}G^{11}(G^{33})^2\mathrm{d}^4x\wedge g^1\wedge g^2\wedge g^5+\\
&+2F'B\sqrt{-G}G^{55}(G^{11})^2G^{33}\mathrm{d}^4x\wedge g^3\wedge g^4\wedge g^5] =\\
&= \Bigg[-\frac{1}{2}B\sqrt{-G}G^{55}G^{11}G^{33}(G^{33}(1-F)+G^{11}F)+\\
&+(F'B\sqrt{-G}G^{55}G^{11}(G^{33})^2)'\Bigg]\mathrm{d}^4x\wedge\mathrm{d}\tau\wedge g^1\wedge g^2\wedge g^5+\\
&+\Bigg[\frac{1}{2}B\sqrt{-G}G^{55}G^{11}G^{33}(G^{33}(1-F)+G^{11}F)+\\
&+(F'B\sqrt{-G}G^{55}(G^{11})^2G^{33})'\Bigg]\mathrm{d}^4x\wedge\mathrm{d}\tau\wedge g^3\wedge g^4\wedge g^5
\end{aligned} \tag{A.40}$$

- NS-NS 3-form e.o.m $\mathrm{d} * \tilde{H}_3 = -F_5 \wedge F_3$

$$\begin{aligned}
\delta(\mathrm{d}*H_3) = -\frac{1}{2}\,\mathrm{d}\Big[&B\sqrt{-G}G^{55}G^{11}G^{33}(G^{11}f'+G^{33}k')\mathrm{d}^4x\wedge(g^1\wedge g^3+g^2\wedge g^4)\wedge g^5+\\
&+(k-f)B\sqrt{-G}G^{55}G^{11}(G^{33})^2\mathrm{d}^4x\wedge\mathrm{d}\tau\wedge g^1\wedge g^2+\\
&+(k-f)B\sqrt{-G}G^{55}(G^{11})^2G^{33}\mathrm{d}^4x\wedge\mathrm{d}\tau\wedge g^3\wedge g^4\Big] =\\
= -\Bigg[&\frac{1}{4}(k-f)B\sqrt{-G}G^{55}G^{11}(G^{33})^2 - \frac{1}{4}(k-f)B\sqrt{-G}G^{55}(G^{11})^2G^{33}+\\
&+\left(\frac{1}{2}B\sqrt{-G}G^{55}G^{11}G^{33}(G^{11}f'+\right.\\
&\left.+G^{33}k')\right)'\Bigg]\mathrm{d}^4x\wedge\mathrm{d}\tau(g^1\wedge g^3+g^2\wedge g^4)g^5
\end{aligned} \tag{A.41}$$

[1] Here we explicitly set the background value of the dilaton Φ to zero.

- RR scalar e.o.m $\mathrm{d} * \mathrm{d}C = -H_3 \wedge * \tilde{F}_3$

$$-\delta(H_3 \wedge * \tilde{F}_3) = -B\sqrt{-G}G^{55}G^{11}G^{33}(2f'F'G^{11} + 2k'F'G^{33} + (k-f)(G^{33}(1-F) + G^{11}F))\mathrm{d}^4x \wedge \mathrm{d}\tau \wedge \omega_2 \wedge \omega_3. \quad \text{(A.42)}$$

- Einstein equation

Fluctuation B gives the following contribution to the r.h.s of the Einstein equation

$$\delta\left(\frac{1}{96}F_{MPQRS}F_N{}^{PQRS}\right) = \frac{1}{24}\delta g^{PP'}F_{MPQRS}F_{NP'}{}^{QRS},$$

and

$$\delta\left(\frac{1}{4}\,\tilde{F}_{MPQ}\tilde{F}_N^{\ PQ} + \frac{1}{4}\,H_{MPQ}H_N^{\ PQ} - \frac{1}{48}\,(\tilde{F}_3^2 + H_3^2)\right) = = \frac{1}{2}\,\delta G^{PR}(F_{MPQ}F_{NR}{}^Q + H_{MPQ}H_{NR}{}^Q) - \frac{1}{48}\delta G_{MN}(F_3^2 + H_3^2) \quad \text{(A.43)}$$

The following components are excited

$$\frac{1}{24}\delta g^{PP'}F_{MPQRS}F_{NP'}{}^{QRS} = \frac{\ell^2}{32}\,B(G^{11})^2(G^{33})^2G^{55}\,2(g^1g^4 - g^2g^3)_{MN}; \quad \text{(A.44)}$$

$$\frac{1}{4}\,\delta(\tilde{F}_{1PQ}\tilde{F}_4{}^{PQ} + H_{1PQ}H_4{}^{PQ}) = = -\frac{1}{4}\,BG^{11}G^{33}G^{55}\left(F(1-F) + 2F'^2 + f'k'\right); \quad \text{(A.45)}$$

$$\frac{1}{4}\,\delta(\tilde{F}_{2PQ}\tilde{F}_3{}^{PQ} + H_{2PQ}H_3{}^{PQ}) = = \frac{1}{4}\,BG^{11}G^{33}G^{55}\left(F(1-F) + 2F'^2 + f'k'\right); \quad \text{(A.46)}$$

$$\frac{1}{4}\,\delta(\tilde{F}_{\tau PQ}\tilde{F}_5{}^{PQ} + H_{\tau PQ}H_5{}^{PQ}) = = \frac{1}{4}\,BG^{11}G^{33}(k-f)\left(G^{11}(F+f') + G^{33}(1-F+k')\right); \quad \text{(A.47)}$$

The variation of Ricci-tensor components induced by the metric fluctuation are presented in the Appendix (A.111)-(A.119).

The fluctuation B can induce the following pseudoscalar fluctuations:

- $\delta g \propto \{\mathrm{d}x^\mu \cdot g^5, \mathrm{d}\tau \cdot g^5\}$
- δC;
- $\delta C_2 \propto \{g^1 \wedge g^2, g^3 \wedge g^4\}$;
- $\delta B_2 \propto g^1 \wedge g^3 + g^2 \wedge g^4$;
- $\delta C_4 \propto \{\mathrm{d}g^5 \wedge \mathrm{d}g^5, \mathrm{d}\tau \wedge g^5 \wedge \mathrm{d}g^5\}$.

A.5.2 Pseudo-vector metric fluctuation

Consider a pseudoscalar metric fluctuation that corresponds to the longitudinal part of the vector fluctuation considered in [61].

$$\delta g = \partial_\mu a \, \mathrm{d}x^\mu \cdot g^5 + A \, \mathrm{d}\tau \cdot g^5. \tag{A.48}$$

It does not affect metric determinant at the linearized level,

$$\delta(\det g) = \det g \, \mathrm{tr} \, g^{-1} \delta g = o(a, A);$$

but it alters the self-duality of F_5. In order to protect self-duality one has to shift F_5 by

$$\begin{aligned} \delta(*F_5) = \frac{1}{8} \, \ell G^{55} (\partial_\mu a \, \mathrm{d}x^\mu \wedge g^1 \wedge g^2 \wedge g^3 \wedge g^4 + A \, \mathrm{d}\tau \wedge g^1 \wedge g^2 \wedge g^3 \wedge g^4 - \\ - h^{1/2} \sqrt{-G} (G^{11})^2 (G^{33})^2 *_4 \mathrm{d}_4 a \wedge \mathrm{d}\tau \wedge g^5 - A\sqrt{-G} (G^{11})^2 (G^{33})^2 G^{55} \mathrm{d}^4 x \wedge g^5). \end{aligned} \tag{A.49}$$

This variation of F_5 will be considered separately in section A.5.6.

Such a fluctuation of the metric also modifies the following equations

- RR scalar e.o.m.

$$\begin{aligned} - \delta(H_3 \wedge *\tilde{F}_3) = A\sqrt{-G} (G^{55})^2 \Big(f' F (G^{11})^2 + \\ + k'(1 - F)(G^{33})^2 + 2F'^2 G^{11} G^{33} \Big) \mathrm{d}^4 x \wedge \mathrm{d}\tau \wedge \omega_2 \wedge \omega_3. \end{aligned} \tag{A.50}$$

- F_3 e.o.m.

$$\delta(\mathrm{d} * \tilde{F}_3) =$$
$$= \frac{1}{2}\,(AF'\sqrt{-G}G^{11}G^{33}(G^{55})^2 - (A(1-F)\sqrt{-G}(G^{33})^2(G^{55})^2)'$$
$$- \Box a(1-F)h^{1/2}\sqrt{-G}(G^{33})^2G^{55}) \wedge \mathrm{d}^4x \wedge \mathrm{d}\tau \wedge g^1 \wedge g^2 \wedge g^5 +$$
$$+ \frac{1}{2}\,(-AF'\sqrt{-G}G^{11}G^{33}(G^{55})^2 - (AF\sqrt{-G}(G^{11})^2(G^{55})^2)'$$
$$- \Box aFh^{1/2}\sqrt{-G}(G^{11})^2G^{55}) \wedge \mathrm{d}^4x \wedge \mathrm{d}\tau \wedge g^3 \wedge g^4 \wedge g^5 \quad \text{(A.51)}$$

- H_3 e.o.m.

$$\delta(\mathrm{d} * H_3) = \frac{1}{4}\Big(A\sqrt{-G}(G^{55})^2(k'(G^{33})^2 - f'(G^{11})^2) +$$
$$+ \Big(A(k-f)\sqrt{-G}G^{11}G^{33}(G^{55})^2\Big)' +$$
$$+ \Box a(k-f)h^{1/2}\sqrt{-G}G^{11}G^{33}G^{55}\Big) \wedge \mathrm{d}^4x \wedge \mathrm{d}\tau \wedge (g^1 \wedge g^3 + g^2 \wedge g^4) \wedge g^5 \quad \text{(A.52)}$$

- Einstein equation.

Metric fluctuation gives the following contribution to the r.h.s of the Einstein equation

$$\frac{1}{4}\,\delta\left(\tilde{F}_{1PQ}\tilde{F}_4{}^{PQ} + H_{1PQ}H_4{}^{PQ}\right) =$$
$$= \frac{1}{4}\,AF'(G^{55})^2\left((F+f')G^{11} + (1-F+k')G^{33}\right); \quad \text{(A.53)}$$

$$\frac{1}{4}\,\delta(\tilde{F}_{2PQ}\tilde{F}_3{}^{PQ} + H_{2PQ}H_3{}^{PQ}) =$$
$$= -\frac{1}{4}\,AF'(G^{55})^2\left((F+f')G^{11} + (1-F+k')G^{33}\right); \quad \text{(A.54)}$$

$$-\frac{1}{48}\,\delta(G_{\mu 5}\tilde{F}_{PQR}\tilde{F}^{PQR} + G_{\mu 5}H_{PQR}H^{PQR}) = -\frac{1}{96}\,\partial_\mu a(F_3^2 + H_3^2); \qquad \text{(A.55)}$$

$$-\frac{1}{48}\,\delta(G_{\tau 5}\tilde{F}_{PQR}\tilde{F}^{PQR} + G_{\mu 5}H_{PQR}H^{PQR}) = -\frac{1}{96}\,A(F_3^2 + H_3^2). \qquad \text{(A.56)}$$

The variation of Ricci-tensor components induced by the metric fluctuation are presented in the Appendix (A.111)-(A.119).

A.5.3 The RR 0-form fluctuation

We will denote the fluctuation of the RR scalar by the same letter C. At the linear level it modifies the following equations:

- RR scalar e.o.m

$$\delta(\mathrm{d} * \mathrm{d}C) = (2(\sqrt{-G}G^{55}C')' + 2h^{1/2}\sqrt{-G}\,\Box_4 C)\mathrm{d}^4x \wedge \mathrm{d}\tau \wedge \omega_2 \wedge \omega_3, \tag{A.57}$$

where $\Box C \mathrm{d}^4x \equiv \mathrm{d}_4 *_4 \mathrm{d}_4 C$;

$$\begin{aligned}\delta(CH_3 \wedge *H_3) = CH_3 \wedge *H_3 = 2C\sqrt{-G}G^{55}(f'^2(G^{11})^2 + k'^2(G^{33})^2 \\ + \frac{1}{2}(k-f)^2 G^{11}G^{33})\,\mathrm{d}^4x \wedge \mathrm{d}\tau \wedge \omega_2 \wedge \omega_3.\end{aligned} \tag{A.58}$$

- F_3 e.o.m

$$\begin{aligned}\delta(-\mathrm{d} * CH_3) = -\mathrm{d}(C * H_3) = -[(Ck'\sqrt{-G}(G^{33})^2G^{55})' \\ -\frac{1}{2}C(k-f)\sqrt{-G}G^{11}G^{33}G^{55}]\mathrm{d}^4x \wedge \mathrm{d}\tau \wedge g^1 \wedge g^2 \wedge g^5 - \\ -[(Cf'\sqrt{-G}(G^{11})^2G^{55})' + \\ +\frac{1}{2}C(k-f)\sqrt{-G}G^{11}G^{33}G^{55}]\mathrm{d}^4x \wedge \mathrm{d}\tau \wedge g^3 \wedge g^4 \wedge g^5,\end{aligned} \tag{A.59}$$

or

$$\begin{aligned}\delta(-\mathrm{d} * CH_3) = -\left[F\,\frac{hC' - Ch'}{h^2}\right]\mathrm{d}^4x \wedge \mathrm{d}\tau \wedge g^1 \wedge g^2 \wedge g^5 - \\ -\left[(1-F)\,\frac{hC' - Ch'}{h^2}\right]\mathrm{d}^4x \wedge \mathrm{d}\tau \wedge g^3 \wedge g^4 \wedge g^5;\end{aligned} \tag{A.60}$$

- H_3 e.o.m

$$\begin{aligned}\delta(-\mathrm{d} * C\tilde{F}_3) = -\mathrm{d}(C * F_3) = \Big[(CF'\sqrt{-G}G^{11}G^{33}G^{55})' + \\ +\frac{1}{2}C\sqrt{-G}G^{55}((1-F)(G^{33})^2 - \\ -F(G^{11})^2)\Big]\mathrm{d}^4x \wedge \mathrm{d}\tau \wedge (g^1 \wedge g^3 + g^2 \wedge g^4) \wedge g^5,\end{aligned} \tag{A.61}$$

or

$$\delta(-\mathrm{d} * C\tilde{F}_3) = \left[\frac{1}{2}(k-f)\,\frac{hC' - Ch'}{h^2}\right]\mathrm{d}^4x \wedge \mathrm{d}\tau \wedge (g^1 \wedge g^3 + g^2 \wedge g^4) \wedge g^5; \tag{A.62}$$

- Einstein equation.

$$\delta(\frac{1}{4}\,\tilde{F}_{MPQ}\tilde{F}_N^{\ PQ} - \frac{1}{48}\,g_{MN}\tilde{F}_{PQR}\tilde{F}^{PQR}) = -\frac{1}{4}\,C(F_{MPQ}H_N^{\ PQ} + H_{MPQ}F_N^{\ PQ}).$$

C contributes to the following components

$$-\frac{1}{4}\,C\,(F_{1ij}H_4^{\ ij}+H_{1ij}F_4^{\ ij}) = \frac{1}{4}\,CG^{55}(k-f)(G^{11}(F+f')+G^{33}(1-F+k')); \quad \text{(A.63)}$$

$$\begin{aligned}-\frac{1}{4}\,C\,(F_{2ij}H_3^{\ ij} + H_{2ij}F_3^{\ ij}) &= \\ &= -\frac{1}{4}\,CG^{55}(k-f)(G^{11}(F+f') + G^{33}(1-F+k')); \quad \text{(A.64)}\end{aligned}$$

$$\begin{aligned}-\frac{1}{4}\,C\,(F_{\tau ij}H_5^{\ ij} + H_{\tau ij}F_5^{\ ij}) &= \\ &= -\frac{1}{2}\,C((G^{11})^2Ff' + (G^{33})^2(1-F)k' + 2G^{11}G^{33}F'^2); \quad \text{(A.65)}\end{aligned}$$

A.5.4 The RR 2-form fluctuation

Consider a fluctuation of RR 2-potential proportional to $g^1 \wedge g^2 - g^3 \wedge g^4$. For the sake of generality consider first a fluctuation

$$C_2^-(g^1 \wedge g^2 - g^3 \wedge g^4) + C_2^+(g^1 \wedge g^2 + g^3 \wedge g^4),$$

which implies

$$\begin{aligned}\delta F_3 = \partial_\mu C_2^-\,\mathrm{d}x^\mu\wedge(g^1\wedge g^2-g^3\wedge g^4)+C_2^{-\prime}\,\mathrm{d}\tau\wedge(g^1\wedge g^2-g^3\wedge g^4)-C_2^-(g^1\wedge g^3+g^2\wedge g^4)\wedge g^5+ \\ + \partial_\mu C_2^+\,\mathrm{d}x^\mu \wedge (g^1 \wedge g^2 + g^3 \wedge g^4) + C_2^{+\prime}\,\mathrm{d}\tau \wedge (g^1 \wedge g^2 + g^3 \wedge g^4), \quad \text{(A.66)}\end{aligned}$$

Such a fluctuation gives the following contribution to the supergravity equations at linearized level

- F_3 e.o.m.

$$
\begin{aligned}
\mathrm{d} * \delta F_3 = \big[&- (C_2^{-\prime}\sqrt{-G}(G^{33})^2 G^{55})' + C_2^{-}\sqrt{-G}G^{11}G^{33}G^{55} \\
&- \Box_4 C_2^{-}\, h^{1/2}\sqrt{-G}(G^{33})^2\big]\mathrm{d}^4x \wedge \mathrm{d}\tau \wedge g^1 \wedge g^2 \wedge g^5 + \\
&+ \big[(C_2^{-\prime}\sqrt{-G}(G^{11})^2 G^{55})' - C_2^{-}\sqrt{-G}G^{11}G^{33}G^{55} + \\
&+ \Box_4 C_2^{-}\, h^{1/2}\sqrt{-G}(G^{11})^2\big]\mathrm{d}^4x \wedge \mathrm{d}\tau \wedge g^3 \wedge g^4 \wedge g^5 + \\
+ \big[(C_2^{+\prime}\sqrt{-G}(G^{33})^2 G^{55})' &+ \Box_4 C_2^{+}\, h^{1/2}\sqrt{-G}(G^{33})^2\big]\mathrm{d}^4x \wedge \mathrm{d}\tau \wedge g^1 \wedge g^2 \wedge g^5 + \\
&+ \big[(C_2^{+\prime}\sqrt{-G}(G^{11})^2 G^{55})' + \\
&+ \Box_4 C_2^{+}\, h^{1/2}\sqrt{-G}(G^{11})^2\big]\mathrm{d}^4x \wedge \mathrm{d}\tau \wedge g^3 \wedge g^4 \wedge g^5, \qquad \text{(A.67)}
\end{aligned}
$$

- H_3 e.o.m.

$$
\begin{aligned}
&- \delta(F_5 \wedge F_3) = \\
&\quad = -\frac{1}{2}\, C_2^{-}\,\ell\sqrt{-G}(G^{11})^2(G^{33})^2 G^{55}\, \mathrm{d}^4x \wedge \mathrm{d}\tau \wedge (g^1 \wedge g^3 + g^2 \wedge g^4) \wedge g^5. \qquad \text{(A.68)}
\end{aligned}
$$

- RR scalar e.o.m.

$$
\begin{aligned}
- H_3 \wedge *\delta F_3 = -2\sqrt{-G}G^{55}[&C_2^{-\prime}(f'(G^{11})^2 - k'(G^{33})^2) - C_2^{-}(k-f)G^{11}G^{33} + \\
&+ C_2^{+\prime}(f'(G^{11})^2 + k'(G^{33})^2)] \wedge \mathrm{d}^4x \wedge \mathrm{d}\tau \wedge \omega_2 \wedge \omega_3; \qquad \text{(A.69)}
\end{aligned}
$$

- Bianchi identity

$$
H_3 \wedge \delta F_3 = -\mathrm{d}[(C_2^{-}(k'-f') + C_2^{+}(k'+f'))\mathrm{d}\tau \wedge g^1 \wedge g^2 \wedge g^3 \wedge g^4]; \qquad \text{(A.70)}
$$

- Einstein equation

$$
\delta\Big(\frac{1}{4}\,\tilde{F}_{MPQ}\tilde{F}_N{}^{PQ} - \frac{1}{48}\,g_{MN}\tilde{F}_{PQR}\tilde{F}^{PQR}\Big) = \frac{1}{4}\,(\delta F_{MPQ}F_N{}^{PQ} + F_{MPQ}\delta F_N{}^{PQ}).
$$

C_2 contributes to the following components

$$
\begin{aligned}
&\frac{1}{4}\,\Big(\delta F_{1ij}F_4{}^{ij} + F_{1ij}\delta F_4{}^{ij}\Big) = \\
&\qquad\qquad = \frac{1}{4}\,G^{55}\Big(2C_2^{-}(FG^{11} + (1-F)G^{33}) - \\
&\qquad\qquad - C_2^{-\prime}(k-f)(G^{11} - G^{33}) - C_2^{+\prime}(k-f)(G^{11} + G^{33})\Big); \qquad \text{(A.71)}
\end{aligned}
$$

$$\frac{1}{4}\,(\delta F_{2ij}{F_3}^{ij}+F_{2ij}\delta {F_3}^{ij})=$$
$$=-\frac{1}{4}\,G^{55}\Big(2C_2^-\,(FG^{11}+(1-F)G^{33})-$$
$$-\,{C_2^-}'(k-f)(G^{11}-G^{33})-{C_2^+}'(k-f)(G^{11}+G^{33})\Big); \quad \text{(A.72)}$$

$$\frac{1}{4}\,(\delta F_{\tau ij}{F_5}^{ij}+F_{\tau ij}\delta {F_5}^{ij})=$$
$$=-\frac{1}{2}\,\Big(C_2^-\,(k-f)G^{11}G^{33}-{C_2^-}'(F(G^{11})^2-$$
$$-\,(1-F)(G^{33})^2)-{C_2^+}'(F(G^{11})^2+(1-F)(G^{33})^2)\Big); \quad \text{(A.73)}$$

$$\frac{1}{4}\,(\delta F_{\mu ij}{F_5}^{ij})=\frac{1}{2}\,\Big(\partial_\mu C_2^-\,(F(G^{11})^2-(1-F)(G^{33})^2)+$$
$$+\,\partial_\mu C_2^+\,(F(G^{11})^2+(1-F)(G^{33})^2)\Big); \quad \text{(A.74)}$$

A.5.5 The 2-form B fluctuation

Consider a fluctuation B_2 of the NS-NS 2-form potential, proportional to $g^1\wedge g^3+g^2\wedge g^4$, i.e.

$$\delta H_3=\partial_\mu B_2\,\mathrm{d}x^\mu\wedge(g^1\wedge g^3+g^2\wedge g^4)+$$
$$+\,B_2'\,\mathrm{d}\tau\wedge(g^1\wedge g^3+g^2\wedge g^4)+B_2(g^1\wedge g^2-g^3\wedge g^4)\wedge g^5. \quad \text{(A.75)}$$

The following equations will be modified:

- H_3 e.o.m.

$$\mathrm{d}*\delta H_3=-\Bigg[({B_2}'\sqrt{-G}G^{11}G^{33}G^{55})'-B_2\sqrt{-G}\,\frac{(G^{11})^2+(G^{33})^2}{2}\,G^{55}+$$
$$+\,\Box_4 B_2\,h^{1/2}\sqrt{-G}G^{11}G^{33}\Bigg]\wedge\wedge\mathrm{d}^4x\wedge\mathrm{d}\tau\wedge(g^1\wedge g^3+g^1\wedge g^3)\wedge g^5; \quad \text{(A.76)}$$

- F_3 e.o.m.

$$\delta(F_5\wedge H_3)=-\frac{1}{2}\,B_2\ell\sqrt{-G}(G^{11})^2(G^{33})^2G^{55}\,\mathrm{d}^4x\wedge\mathrm{d}\tau\wedge(g^1\wedge g^2-g^3\wedge g^4)\wedge g^5. \quad \text{(A.77)}$$

- RR scalar e.o.m.

$$
\begin{aligned}
-\delta H_3 \wedge *F_3 = -2\sqrt{-G}G^{55}\Big[&2{B_2}'F'G^{11}G^{33} + B_2F(G^{11})^2 - \\
&- B_2(1-F)(G^{33})^2\Big] \wedge \mathrm{d}^4x \wedge \mathrm{d}\tau \wedge \omega_2 \wedge \omega_3
\end{aligned} \tag{A.78}
$$

- Bianchi identity

$$
\delta H_3 \wedge F_3 = -[2B_2F'\mathrm{d}\tau \wedge g^1 \wedge g^2 \wedge g^3 \wedge g^4]; \tag{A.79}
$$

- Einstein equation

$$
\delta\left(\frac{1}{4}\,H_{MPQ}H_N^{\ PQ} - \frac{1}{48}\,g_{MN}H_{PQR}H^{PQR}\right) = \frac{1}{4}\,\left(\delta H_{MPQ}H_N^{\ PQ} + H_{MPQ}\delta H_N^{\ PQ}\right).
$$

B_2 contributes to the following components

$$
\begin{aligned}
&\frac{1}{4}\,\left(\delta H_{1ij}{H_4}^{ij} + H_{1ij}\delta {H_4}^{ij}\right) = \\
&\qquad = -\frac{1}{4}\,G^{55}(2B_2'(f'G^{11} + k'G^{33}) + B_2(k-f)(G^{11} - G^{33}));
\end{aligned} \tag{A.80}
$$

$$
\begin{aligned}
&\frac{1}{4}\,\left(\delta H_{2ij}{H_3}^{ij} + H_{2ij}\delta {H_3}^{ij}\right) = \\
&\qquad = \frac{1}{4}\,G^{55}(2B_2'(f'G^{11} + k'G^{33}) + B_2(k-f)(G^{11} - G^{33}));
\end{aligned} \tag{A.81}
$$

$$
\begin{aligned}
&\frac{1}{4}\,\left(\delta H_{\tau ij}{H_5}^{ij} + H_{\tau ij}\delta {H_5}^{ij}\right) = \\
&\qquad = \frac{1}{2}\,\left(B_2(f'(G^{11})^2 - k'(G^{33})^2) + B_2'(k-f)G^{11}G^{33}\right);
\end{aligned} \tag{A.82}
$$

$$
\frac{1}{4}\,\left(\delta H_{\mu ij}{H_5}^{ij}\right) = \frac{1}{2}\,\partial_\mu B_2(k-f)G^{11}G^{33}; \tag{A.83}
$$

A.5.6 The self-dual 5-form fluctuation

Metric fluctuation considered in section A.5.2 requires a shift of the 5-form of form (A.49). One can also add a self-dual fluctuation of the form

$$
(1+*)(*_4\mathrm{d}_4\phi_1 \wedge \mathrm{d}\tau \wedge g^5 + \phi_2\,\mathrm{d}^4x \wedge g^5 + *_4\mathrm{d}_4\phi_3 \wedge \mathrm{d}g^5)
$$

Let us choose the total fluctuation to be

$$\delta F_5 = \frac{1}{8}\,\ell G^{55}\Big(\partial_\mu(a+\phi_1)\,\mathrm{d}x^\mu\wedge g^1\wedge g^2\wedge g^3\wedge g^4 + (A+\phi_2)\,\mathrm{d}\tau\wedge g^1\wedge g^2\wedge g^3\wedge g^4 + \\ + \partial_\mu\phi_3\,\mathrm{d}x^\mu\wedge \mathrm{d}\tau\wedge g^5\wedge \mathrm{d}g^5 - h^{1/2}\sqrt{-G}(G^{11})^2(G^{33})^2 *_4 \mathrm{d}_4(a-\phi_1)\wedge \mathrm{d}\tau\wedge g^5 - \\ - (A-\phi_2)\sqrt{-G}(G^{11})^2(G^{33})^2G^{55}\mathrm{d}^4x\wedge g^5 - \\ - h^{1/2}\sqrt{-G}G^{11}G^{33}(G^{55})^2 *_4 \mathrm{d}_4\phi_3\wedge \mathrm{d}g^5\Big) \quad \text{(A.84)}$$

This fluctuation contributes to the following e.o.m.

- Bianchi identity

$$\mathrm{d}(\delta F_5) = -\frac{1}{4}\left[(\ell G^{55}\partial_\mu(a+\phi_1))' - \ell G^{55}\partial_\mu(A+\phi_2) + \\ + 2\ell G^{55}\partial_\mu\phi_3\right]\mathrm{d}x^\mu\wedge \mathrm{d}\tau\wedge\omega_2\wedge\omega_2 - \frac{1}{8}\Big[\ell h^{1/2}\sqrt{-G}(G^{11})^2(G^{33})^2G^{55}\Box_4(a-\phi_1) + \\ + \Big(\ell(A-\phi_2)\sqrt{-G}(G^{11})^2(G^{33})^2(G^{55})^2\Big)'\Big]\,\mathrm{d}^4x\wedge \mathrm{d}\tau\wedge g^5 - \\ - \frac{1}{8}\Big[\ell h^{1/2}\sqrt{-G}(G^{11})^2(G^{33})^2G^{55} *_4 \mathrm{d}_4(a-\phi_1) - \\ - \Big(\ell h^{1/2}\sqrt{-G}G^{11}G^{33}(G^{55})^3 *_4 \mathrm{d}_4\phi_3\Big)'\Big]\wedge \mathrm{d}\tau\wedge \mathrm{d}g^5 - \\ - \frac{1}{8}\,\ell\sqrt{-G}G^{11}G^{33}(G^{55})^2\Big[(A-\phi_2)G^{11}G^{33} + h^{1/2}G^{55}\Box_4\phi_3\Big]\,\mathrm{d}^4x\wedge \mathrm{d}g^5 \quad \text{(A.85)}$$

- F_3 e.o.m.

$$\delta F_5\wedge H_3 = \\ = \frac{\ell}{8}\,(A-\phi_2)\sqrt{-G}(G^{11})^2(G^{33})^2(G^{55})^2\,\mathrm{d}^4x\wedge \mathrm{d}\tau\wedge(f'g^1\wedge g^2 + k'g^3\wedge g^4)\wedge g^5; \quad \text{(A.86)}$$

- H_3 e.o.m.

$$-\delta F_5\wedge F_3 = \\ = -\frac{\ell}{8}\,(A-\phi_2)F'\sqrt{-G}(G^{11})^2(G^{33})^2(G^{55})^2\,\mathrm{d}^4x\wedge \mathrm{d}\tau\wedge(g^1\wedge g^3 + g^2\wedge g^4)\wedge g^5; \quad \text{(A.87)}$$

- Einstein equation. The following components receive contribution

$$\frac{1}{96}\,\delta\left(F_{\mu PQRS}{F_5}^{PQRS}\right)=\frac{1}{32}\,\ell^2(G^{11})^2(G^{33})^2G^{55}\partial_\mu\phi_1; \tag{A.88}$$

$$\frac{1}{96}\,\delta\left(F_{\tau PQRS}{F_5}^{PQRS}\right)=\frac{1}{32}\,\ell^2(G^{11})^2(G^{33})^2G^{55}\phi_2. \tag{A.89}$$

A.5.7 Linearized equations

Ten fluctuations a, A, B, C, $C_2^{\pm}$, B_2, ϕ_1, ϕ_2 and ϕ_3 considered in the previous section, lead to eleven linearized equations. Three of them are coming from Einstein equation, four from Bianchi identity, two from F_3 e.o.m.; H_3 and C e.o.m. give another pair of linearized equations.

Of the three following equations coming from Bianchi identity,

$$\Big[\ell h^{1/2}\sqrt{-G}(G^{11})^2(G^{33})^2G^{55}\Box_4(a-\phi_1) \\ +\left(\ell(A-\phi_2)\sqrt{-G}(G^{11})^2(G^{33})^2(G^{55})^2\right)'\Big]\,\mathrm{d}^4x\wedge\mathrm{d}\tau\wedge g^5=0, \tag{A.90}$$

and

$$\Big[\ell h^{1/2}\sqrt{-G}(G^{11})^2(G^{33})^2G^{55}\,*_4\,\mathrm{d}_4(a-\phi_1)- \\ -\left(\ell h^{1/2}\sqrt{-G}G^{11}G^{33}(G^{55})^3\,*_4\,\mathrm{d}_4\phi_3\right)'\Big]\wedge\mathrm{d}\tau\wedge\mathrm{d}g^5=0, \tag{A.91}$$

and

$$\Big[(A-\phi_2)G^{11}G^{33}+h^{1/2}G^{55}\Box_4\phi_3\Big]\,\mathrm{d}^4x\wedge\mathrm{d}g^5=0, \tag{A.92}$$

only two are independent. Those can be resolved to eliminate, say, ϕ_1 and ϕ_2;

$$\phi_1=a-\frac{(\ell h^{1/2}\sqrt{-G}G^{11}G^{33}(G^{55})^3\phi_3)'}{\ell h^{1/2}\sqrt{-G}(G^{11})^2(G^{33})^2G^{55}}, \tag{A.93}$$

$$\phi_2=A+\frac{h^{1/2}G^{55}}{G^{11}G^{33}}\,\Box_4\phi_3. \tag{A.94}$$

The remaining equations are as follows.

- The only equation that is left from the Bianchi identity is proportional to $\mathrm{d}x^\mu \wedge \mathrm{d}\tau \wedge \omega_2 \wedge \omega_2$;

$$-\frac{1}{4}\left(\frac{(\ell h^{1/2}\sqrt{-G}G^{11}G^{33}(G^{55})^3\phi_3)'}{h^{1/2}\sqrt{-G}(G^{11})^2(G^{33})^2}\right)' - \frac{\ell}{4}\,G^{55}\,\frac{h^{1/2}G^{55}}{G^{11}G^{33}}\,\Box_4\phi_3 + \frac{\ell}{2}\,G^{55}\phi_3 + \\ + \frac{1}{2}\,(\ell G^{55}a)' - \frac{\ell}{2}\,G^{55}A = 4B_2F' + 2C_2^-(k'-f') + 2C_2^+(k'+f'), \quad \text{(A.95)}$$

- F_3 e.o.m. gives two structures. The first one is proportional to $\mathrm{d}^4x \wedge \mathrm{d}\tau \wedge (g^1 \wedge g^2 - g^3 \wedge g^4) \wedge g^5$.

$$\begin{aligned} &- B\sqrt{-G}G^{55}G^{11}G^{33}(G^{33}(1-F)+G^{11}F) + (F'B\sqrt{-G}G^{55}G^{11}G^{33}(G^{33}-G^{11}))' \\ &+ AF'\sqrt{-G}G^{11}G^{33}(G^{55})^2 - \frac{1}{2}\left(A\sqrt{-G}(G^{55})^2((1-F)(G^{33})^2 - F(G^{11})^2)\right)' - \\ &- \frac{1}{2}\,\Box a h^{1/2}\sqrt{-G}G^{55}((1-F)(G^{33})^2 - F(G^{11})^2) - (C\sqrt{-G}G^{55}(k'(G^{33})^2 - \\ &- f'(G^{11})^2))' + C(k-f)\sqrt{-G}G^{11}G^{33}G^{55} - \\ &- (C_2^{-\prime}\sqrt{-G}G^{55}((G^{33})^2+(G^{11})^2))' + 2C_2^-\sqrt{-G}G^{11}G^{33}G^{55} - \\ &- \Box_4 C_2^-\,h^{1/2}\sqrt{-G}((G^{33})^2+(G^{11})^2) + \\ &+ (C_2^{+\prime}\sqrt{-G}G^{55}((G^{33})^2-(G^{11})^2))' + \Box_4 C_2^+\,h^{1/2}\sqrt{-G}((G^{33})^2-(G^{11})^2) + \\ &+ B_2\ell\sqrt{-G}(G^{11})^2(G^{33})^2G^{55} + \\ &+ \frac{\ell}{8}\,(f'-k')h^{1/2}\sqrt{-G}G^{11}G^{33}(G^{55})^3\Box_4\phi_3 = 0; \quad \text{(A.96)} \end{aligned}$$

- The second structure in F_3 e.o.m. is proportional to $\mathrm{d}^4x \wedge \mathrm{d}\tau \wedge (g^1 \wedge g^2 + g^3 \wedge g^4) \wedge g^5$.

$$\begin{aligned} &(F'B\sqrt{-G}G^{55}G^{11}G^{33}(G^{33}+G^{11}))' - (C\sqrt{-G}G^{55}(k'(G^{33})^2 + f'(G^{11})^2))' - \\ &- \frac{1}{2}\left(A\sqrt{-G}(G^{55})^2((1-F)(G^{33})^2 + F(G^{11})^2)\right)' - \\ &- \frac{1}{2}\Box a h^{1/2}\sqrt{-G}G^{55}((1-F)(G^{33})^2 + F(G^{11})^2) + \\ &+ (C_2^{-\prime}\sqrt{-G}G^{55}((G^{11})^2) - (G^{33})^2)' + \Box_4 C_2^-\,h^{1/2}\sqrt{-G}((G^{11})^2 - (G^{33})^2) + \\ &+ (C_2^{+\prime}\sqrt{-G}G^{55}((G^{33})^2+(G^{11})^2))' + \Box_4 C_2^+\,h^{1/2}\sqrt{-G}((G^{33})^2+(G^{11})^2) \\ &+ \frac{\ell}{8}\,(f'+k')h^{1/2}\sqrt{-G}G^{11}G^{33}(G^{55})^3\Box_4\phi_3 = 0; \quad \text{(A.97)} \end{aligned}$$

- H_3 e.o.m gives single equation proportional to $\mathrm{d}^4x \wedge \mathrm{d}\tau \wedge (g^1 \wedge g^3 + g^2 \wedge g^4) \wedge g^5$

$$
\begin{aligned}
&- BF'\sqrt{-G}G^{55}G^{11}G^{33}(G^{33} - G^{11}) \\
&\quad - (B\sqrt{-G}G^{55}G^{11}G^{33}(G^{11}f' + G^{33}k'))' + (AF'\sqrt{-G}G^{11}G^{33}(G^{55})^2)' + \\
&\quad + \frac{1}{2}A\sqrt{-G}(G^{55})^2(k'(G^{33})^2 - f'(G^{11})^2) + \Box a F' h^{1/2}\sqrt{-G}G^{11}G^{33}G^{55} + \\
&\quad + 2(CF'\sqrt{-G}G^{11}G^{33}G^{55})' + C\sqrt{-G}G^{55}((1-F)(G^{33})^2 - F(G^{11})^2) - \\
&\quad - 2({B_2}'\sqrt{-G}G^{11}G^{33}G^{55})' + B_2\sqrt{-G}\left((G^{11})^2 + (G^{33})^2\right)G^{55} - \\
&\quad - 2\Box_4 B_2\, h^{1/2}\sqrt{-G}G^{11}G^{33} + C_2^-\ell\sqrt{-G}(G^{11})^2(G^{33})^2G^{55} - \\
&\quad - \frac{\ell}{4}F'h^{1/2}\sqrt{-G}G^{11}G^{33}(G^{55})^3\Box_4\phi_3 = 0; \qquad \text{(A.98)}
\end{aligned}
$$

- The RR scalar gives an equation

$$
\begin{aligned}
&2(\sqrt{-G}G^{55}C')' + 2h^{1/2}\sqrt{-G}\,\Box_4 C - 2C\sqrt{-G}G^{55}\big({f'}^2(G^{11})^2 + {k'}^2(G^{33})^2 + \\
&\quad + 2{F'}^2G^{11}G^{33}\big) + 2B\sqrt{-G}G^{55}G^{11}G^{33}\Big(f'F'G^{11} + k'F'G^{33} + \\
&\quad + F'\left(G^{33}(1-F) + G^{11}F\right)\Big) - A\sqrt{-G}(G^{55})^2(f'F(G^{11})^2 + k'(1-F)(G^{33})^2 + \\
&\quad + 2{F'}^2G^{11}G^{33}) + 2{C_2^-}'\sqrt{-G}G^{55}(f'(G^{11})^2 - k'(G^{33})^2) - \\
&\quad - 2C_2^-(k-f)\sqrt{-G}G^{55}G^{11}G^{33} + 2{C_2^+}'\sqrt{-G}G^{55}(f'(G^{11})^2 + k'(G^{33})^2) + \\
&\quad + 4{B_2}'F'\sqrt{-G}G^{55}G^{11}G^{33} + \\
&\quad + 2B_2\sqrt{-G}G^{55}\left(F(G^{11})^2 - (1-F)(G^{33})^2\right) = 0; \qquad \text{(A.99)}
\end{aligned}
$$

- The above fluctuations excite three components of the Einstein equation. The first component is along the $g^1 \cdot g^4 - g^2 \cdot g^3$ direction

$$
\begin{aligned}
&-\frac{3\,K^2}{2\epsilon^{4/3}\sqrt{h}}\,B'' + \frac{3\,K^2}{2\epsilon^{4/3}\sqrt{h}}\,B'\,\frac{h'}{h} - \frac{1}{4}\,\sqrt{h}\Box_4 B - \frac{2B}{3\epsilon^{4/3}\sqrt{h}K^4\sinh^2\tau} - \\
&\quad - \frac{3\,K^2}{8\epsilon^{4/3}\sqrt{h}}\left(\frac{h'}{h}\right)^2 B - \frac{3\,(hK)'K'}{2\epsilon^{4/3}h^{3/2}}\,B - \frac{3\,(hK^2\coth\tau)'}{2\epsilon^{4/3}h^{3/2}}\,B - \frac{3}{8}\,\sqrt{h}K^3\Box_4 a + \\
&\quad + \frac{9\,K^5}{8\epsilon^{4/3}\sqrt{h}}\left(\frac{h'}{h}\,A - 8\frac{K'}{K}\,A - 2\,A'\right) = \frac{\ell^2}{32}\,B(G^{11})^2(G^{33})^2G^{55} \\
&\quad - \frac{1}{4}\,BG^{11}G^{33}G^{55}\left(F(1-F) + 2F'^2 + f'k'\right) + \frac{1}{4}\,AF'(G^{55})^2\big((F+f')G^{11} + \\
&\quad + (1-F+k')G^{33}\big) + \frac{1}{4}\,CG^{55}(k-f)(G^{11}(F+f') + G^{33}(1-F+k')) - \\
&\quad - \frac{1}{4}\,G^{55}(2B_2'(f'G^{11}+k'G^{33}) + B_2(k-f)(G^{11}-G^{33})) + \\
&\quad + \frac{1}{4}\,G^{55}\Big(2C_2^-\,(FG^{11}+(1-F)G^{33}) - C_2^{-\prime}(k-f)(G^{11}-G^{33}) - \\
&\quad - C_2^{+\prime}(k-f)(G^{11}+G^{33})\Big) - \frac{1}{96}\,B(H_3^2+F_3^2); \qquad \text{(A.100)}
\end{aligned}
$$

- The second component of Einstein equation is along $\mathrm{d}x^\mu \cdot g^5$ direction

$$
\begin{aligned}
&-\frac{3K^2\partial_\mu a''}{2\epsilon^{4/3}\sqrt{h}} - \frac{3\,K(2K'+K\coth\tau)\partial_\mu a'}{\epsilon^{4/3}\sqrt{h}} + \frac{3\,h'K^2}{8\epsilon^{4/3}h^{3/2}}\left(\frac{h'}{h} - 4\,\frac{K'}{K}\right)\partial_\mu a + \\
&\quad + \frac{4\,\partial_\mu a}{3\,\epsilon^{4/3}\sqrt{h}K^4\sinh^2\tau} + \frac{3\,(K^2\partial_\mu A)'}{2\epsilon^{4/3}\sqrt{h}} + \frac{3\,K^2}{4\epsilon^{4/3}\sqrt{h}}\left(4\,\coth\tau + \frac{h'}{h}\right)\partial_\mu A + \\
&+ \frac{2\,\partial_\mu B}{\epsilon^{4/3}\sqrt{h}K\sinh^2\tau} = -\frac{1}{96}\,\partial_\mu a(F_3^2+H_3^2) + \frac{1}{2}\left(\partial_\mu C_2^-\,(F(G^{11})^2 - (1-F)(G^{33})^2) + \right. \\
&\quad \left. + \partial_\mu C_2^+\,(F(G^{11})^2 + (1-F)(G^{33})^2)\right) + \\
&\quad + \frac{1}{2}\,\partial_\mu B_2(k-f)G^{11}G^{33} + \frac{1}{32}\,\ell^2(G^{11})^2(G^{33})^2G^{55}\partial_\mu a - \\
&\quad - \frac{\ell}{32}\,\frac{(\ell h^{1/2}\sqrt{-G}G^{11}G^{33}(G^{55})^3\partial_\mu\phi_3)'}{h^{1/2}\sqrt{-G}}; \qquad \text{(A.101)}
\end{aligned}
$$

- The last component is along $\mathrm{d}\tau \cdot g^5$

$$\frac{2}{\epsilon^{4/3}\sinh^2\tau}\left(\frac{B}{\sqrt{h}K}\right)' + \frac{\sqrt{h}}{2}\left(\frac{K'}{K} - \frac{1}{4}\frac{h'}{h}\right)\Box_4 a + \frac{\sqrt{h}}{4}(\Box_4 a)' - \frac{\sqrt{h}}{4}\Box_4 A +$$
$$+ \frac{4\,A}{3\epsilon^{4/3}\sqrt{h}K^4\sinh^2\tau} + \frac{3\,K^2}{4\,\epsilon^{4/3}\sqrt{h}}\Big[-2\frac{h'}{h}\coth\tau + \left(\frac{h'}{h}\right)^2 + 8\frac{K'}{K}\coth\tau - 2\frac{h'}{h}\frac{K'}{K} +$$
$$+4\left(\frac{K'}{K}\right)^2 - \frac{h''}{h} + 4\frac{K''}{K}\Big] A = \frac{1}{4}BG^{11}G^{33}(k-f)\left(G^{11}(F+f') + G^{33}(1-F+k')\right)$$
$$-\frac{1}{2}C((G^{11})^2Ff' + (G^{33})^2(1-F)k' + 2G^{11}G^{33}F'^2) - -\frac{1}{96}A(F_3^2 + H_3^2)$$
$$-\frac{1}{2}\Big(C_2^-(k-f)G^{11}G^{33} - C_2^{-\prime}(F(G^{11})^2 - (1-F)(G^{33})^2) - C_2^{+\prime}(F(G^{11})^2 +$$
$$+(1-F)(G^{33})^2)\Big) + \frac{1}{2}(B_2(f'(G^{11})^2 - k'(G^{33})^2) + B_2'(k-f)G^{11}G^{33})$$
$$+\frac{\ell^2}{32}(G^{11})^2(G^{33})^2G^{55}A + \frac{\ell^2}{32}h^{1/2}G^{11}G^{33}(G^{55})^2\Box_4\phi_3. \quad \text{(A.102)}$$

So far we have eight equations (A.95)-(A.102) for eight unknown functions a, A, C, B, C_2^+, C_2^-, B_2 and ϕ_3. Notice that second derivative of A does not appear in the equations. The last equation is first order and can be solved to completely eliminate A from all equations. Let us now rewrite all of equations in terms of functions I, K, $\sinh\tau$ and their derivatives.

Equation 1

$$\frac{2^{1/3}27}{32}\left(IK^4\sinh^2\tau\left(\frac{I'K^6\sinh^2\tau}{I^{3/2}}\phi_3\right)'\right)' +$$
$$+\frac{3}{2^{5/3}}\frac{I'K^4\sinh^2\tau}{\sqrt{I}}\left(\frac{3IK^4\sinh^2\tau}{2^{4/3}\epsilon^{4/3}}\Box_4 - 1\right)\phi_3 - \frac{3}{2^{5/3}}\left(\frac{I'K^4\sinh^2\tau}{\sqrt{I}}a\right)' +$$
$$+\frac{3}{2^{5/3}}\frac{I'K^4\sinh^2\tau}{\sqrt{I}}A =$$
$$= -2^{2/3}\frac{I'}{K}B_2 - 2^{2/3}\left(\frac{I'}{K}\right)'C_2^- - 2^{2/3}\left(\frac{I'\cosh\tau}{K}\right)'C_2^+; \quad \text{(A.103)}$$

Equation 2

$$
\begin{aligned}
&-\frac{\epsilon^{8/3}}{2^{2/3}}\left(\frac{\cosh^2\tau+1}{2I\sinh^2\tau}\,{C_2^-}'\right)' + \frac{\epsilon^{8/3}}{2^{2/3}}\,\frac{1}{2I}\,C_2^- - \frac{\epsilon^{4/3}(\cosh^2\tau+1)}{3K^2\sinh^2\tau}\,\Box_4 C_2^- \\
&\qquad -\frac{\epsilon^{8/3}}{2^{2/3}}\left(\frac{\cosh\tau}{I\sinh^2\tau}\,{C_2^+}'\right)' - \frac{2\epsilon^{4/3}\cosh\tau}{3K^2\sinh^2\tau}\,\Box_4 C_2^+ - \\
&\quad -\frac{\epsilon^{8/3}K^2}{4I^{3/2}}\,B + \left(\frac{2^{2/3}\epsilon^{8/3}I'}{8I^{3/2}K^2\sinh^2\tau}\,B\right)' - \frac{\epsilon^{8/3}}{2^{2/3}4}\,\left(\frac{K^3\sinh^2\tau-\cosh\tau}{I}\,C\right)' \\
&\qquad -\frac{\epsilon^{8/3}}{8}\,\frac{I'}{IK}\,C - \frac{\epsilon^{8/3}}{2^{2/3}}\,\frac{I'}{2I^2}\,B_2 - \frac{\epsilon^{8/3}}{2^{1/3}}\,\frac{3I'K}{16I^{3/2}}\,A + \\
&\qquad + \frac{\epsilon^{8/3}}{2^{2/3}}\,\frac{3}{16}\,\left(\frac{2^{1/3}K^5+I'K\coth\tau}{I^{3/2}}\,A\right)' \\
&\qquad + \frac{\epsilon^{4/3}}{2^{2/3}4}\,\frac{2^{1/3}K^4+I'\coth\tau}{I^{1/2}K}\,\Box_4 a + \\
&\qquad + \frac{\epsilon^{4/3}}{2^{2/3}}\,\frac{9}{32}\,I'K^5\sinh^2\tau\,\frac{\left(2^{1/3}K^4+I'\coth\tau\right)}{I^{3/2}}\,\Box_4\phi_3 = 0; \qquad \text{(A.104)}
\end{aligned}
$$

Equation 3

$$
\begin{aligned}
&-\left(\frac{2^{2/3}\epsilon^{8/3}I'\cosh\tau}{8I^{3/2}K^2\sinh^2\tau}B\right)' - \left(\frac{\epsilon^{8/3}C}{2^{2/3}4I}\right)' - \frac{3\epsilon^{8/3}}{2^{1/3}16}\,\left(\frac{2^{1/3}K^5\sinh\tau\cosh\tau+I'K}{I^{3/2}\sinh\tau}\,A\right)' \\
&\qquad - \frac{2^{1/3}K^4\sinh\tau\cosh\tau+I'}{2^{8/3}\epsilon^{-4/3}I^{1/2}K\sinh\tau}\,\Box_4 a + \\
&\qquad + \frac{\epsilon^{8/3}}{2^{5/3}}\left(\frac{\cosh^2\tau+1}{I\sinh^2\tau}\,{C_2^+}'\right)' + \frac{\epsilon^{4/3}(\cosh^2\tau+1)}{3K^2\sinh^2\tau}\,\Box_4 C_2^+ \\
&\qquad + \left(\frac{\epsilon^{8/3}\cosh\tau}{2^{2/3}I\sinh^2\tau}\,{C_2^-}'\right)' + \frac{2\epsilon^{4/3}\cosh\tau}{3K^2\sinh^2\tau}\,\Box_4 C_2^- - \\
&\qquad - \frac{\epsilon^{4/3}}{2^{2/3}}\,\frac{9}{32}\,I'K^5\sinh\tau\,\frac{(2^{1/3}K^4\sinh\tau\cosh\tau+I')}{I^{3/2}}\,\Box_4\phi_3 = 0; \qquad \text{(A.105)}
\end{aligned}
$$

Equation 4

$$
\begin{aligned}
-\frac{\epsilon^{8/3}I'}{2^{1/3}4I^{3/2}K^2\sinh^2\tau}\,B-\left(\frac{\epsilon^{8/3}K^2}{4I^{3/2}}\,B\right)'-\frac{3\epsilon^{8/3}}{2^{1/3}16}\,\left(\frac{I'K}{I^{3/2}}\,A\right)'\\
+\frac{3\epsilon^{8/3}K^2}{16I^{3/2}}\,\left(K^3\sinh^2\tau-\cosh\tau\right)A-\frac{\epsilon^{4/3}}{2^{2/3}4}\,\frac{I'}{I^{1/2}K}\,\Box_4 a-\\
-\left(\frac{\epsilon^{8/3}}{2^{5/3}I}\,{B_2}'\right)'+\frac{\epsilon^{8/3}}{2^{5/3}}\,\frac{\cosh^2\tau+1}{I\sinh^2\tau}\,B_2-\frac{\epsilon^{4/3}}{3K^2}\,\Box_4 B_2\\
-\left(\frac{\epsilon^{8/3}I'}{8IK}\,C\right)'-\frac{2^{1/3}K^4+I'\coth\tau}{8\epsilon^{-8/3}IK}\,C-\frac{\epsilon^{8/3}I'}{2^{5/3}I^2}\,C_2^- -\\
-\frac{\epsilon^{4/3}}{2^{2/3}}\,\frac{9}{32}\,\frac{{I'}^2K^5\sinh^2\tau}{I^{3/2}}\,\Box_4\phi_3=0; \quad \text{(A.106)}
\end{aligned}
$$

Equation 5

$$
\begin{aligned}
\frac{\epsilon^{8/3}}{8}\,(K^2\sinh^2\tau\,C')'+\frac{\epsilon^{4/3}I\sinh^2\tau}{2^{1/3}6}\,\Box_4 C\\
-\frac{\epsilon^{8/3}}{4}\,K^2\sinh^2\tau\left(\frac{2}{3}\,\frac{I'}{IK^3\sinh\tau}-\frac{I''}{2I}-2\,\frac{I'}{I}\,\frac{(K\sinh\tau)'}{K\sinh\tau}\right)C-\frac{\epsilon^{8/3}}{2^{4/3}}\,\frac{I'K}{I^{3/2}}\,B-\\
-\frac{3}{16}\,\frac{K^8\sinh^2\tau}{\epsilon^{-8/3}I^{3/2}}\,A-\frac{K^3\sinh^2\tau-\cosh\tau}{2^{5/3}\epsilon^{-8/3}I}\,{C_2^-}'+\frac{\epsilon^{8/3}}{4}\,\frac{I'}{IK}\,C_2^- +\\
+\frac{\epsilon^{8/3}}{2^{5/3}I}\,{C_2^+}'-\frac{\epsilon^{8/3}I'}{4IK}{B_2}'+\frac{2^{1/3}K^4+I'\coth\tau}{4\epsilon^{-8/3}IK}\,B_2=0; \quad \text{(A.107)}
\end{aligned}
$$

Equation 6

$$
\begin{aligned}
-\frac{3\,K^2}{2^{1/3}4I^{1/2}}\,B''+\frac{3\,K^2}{2^{1/3}4I^{1/2}}\,B'\,\frac{I'}{I}-\frac{I^{1/2}}{2^{2/3}\epsilon^{4/3}}\Box_4 B-\frac{2^{2/3}B}{6I^{1/2}K^4\sinh^2\tau}-\frac{3\,K^2}{2^{1/3}16I^{1/2}}\left(\frac{I'}{I}\right)^2 B\\
-\frac{3\,(IK)'K'}{2^{1/3}4I^{3/2}}\,B-\frac{3\,(IK^2\coth\tau)'}{2^{1/3}4I^{3/2}}\,B-\\
-\frac{3\epsilon^{-4/3}}{2^{5/3}}\,I^{1/2}K^3\Box_4 a+\frac{9\,K^5}{2^{1/3}16I^{1/2}}\left(\frac{I'}{I}\,A-8\frac{K'}{K}\,A-2\,A'\right)=\\
=-\frac{3}{2^{7/3}}\,\frac{K^2}{I^{1/2}}\left(\frac{2}{3}\,\frac{I'}{IK^3\sinh\tau}-\frac{I''}{2I}-2\,\frac{I'}{I}\,\frac{(K\sinh\tau)'}{K\sinh\tau}\right)B+\\
+\frac{3K^2}{2^{1/3}8I^{1/2}}\left(\frac{I'}{I}\right)^2 B-\frac{3K^6}{4I^{3/2}}\,B-\frac{9I'K^5}{2^{1/3}4I^{3/2}}\,A-\frac{3I'K^3}{2I}\,C-\\
-\frac{3K^4}{2^{2/3}I}\,B_2'+\frac{3I'}{2I\sinh^2\tau}\,B_2+\frac{3K^4}{2^{2/3}I}\,C_2^-+\frac{3I'}{2I\sinh^2\tau}\,{C_2^-}'+\frac{3I'\cosh\tau}{2I\sinh^2\tau}\,{C_2^+}'; \quad \text{(A.108)}
\end{aligned}
$$

Equation 7

$$
\begin{aligned}
&\frac{2^{2/3}\,B}{2I^{1/2}K\sinh^2\tau} - \frac{3K^2a''}{2^{1/3}4I^{1/2}} - \frac{3\,K(2K'+K\coth\tau)a'}{2^{4/3}I^{1/2}} + \frac{3\,I'K^2}{2^{1/3}16I^{3/2}}\left(\frac{I'}{I} - 4\,\frac{K'}{K}\right)a+\\
&\quad + \frac{2^{2/3}\,a}{3\,I^{1/2}K^4\sinh^2\tau} + \frac{3\,(K^2A)'}{2^{1/3}4I^{1/2}} + \frac{3\,K^2}{2^{1/3}8I^{1/2}}\left(4\,\coth\tau + \frac{I'}{I}\right)A\\
&\quad = -\frac{3}{2^{1/3}4}\,\frac{K^2}{I^{1/2}}\left(\frac{2}{3}\,\frac{I'}{IK^3\sinh\tau} - \frac{I''}{2I} - 2\,\frac{I'}{I}\,\frac{(K\sinh\tau)'}{K\sinh\tau}\right)a+\\
&\quad + \frac{2^{1/3}K^4 + I'\coth\tau}{IK^3\sinh^2\tau}\,C_2^- + \frac{2^{1/3}K^4\sinh\tau\cosh\tau + I'}{IK^3\sinh^3\tau}\,C_2^+ - \frac{I'}{IK^3\sinh^2\tau}\,B_2\\
&\quad + \frac{3I'^2K^2}{2^{1/3}8I^{5/2}}\,a - \frac{27}{2^{1/3}32}\,\frac{I'K^2}{I}\left(\frac{I'K^6\sinh^2\tau}{I^{3/2}}\,\phi_3\right)'; \qquad \text{(A.109)}
\end{aligned}
$$

Equation 8

$$
\begin{aligned}
&\frac{2^{2/3}}{2\sinh^2\tau}\left(\frac{B}{I^{1/2}K}\right)' + \frac{2^{1/3}I^{1/2}}{\epsilon^{4/3}}\left(\frac{K'}{K} - \frac{1}{4}\,\frac{I'}{I}\right)\Box a + \frac{2^{1/3}I^{1/2}}{2\epsilon^{4/3}}\,(\Box a)' - \frac{2^{1/3}I^{1/2}}{2\epsilon^{4/3}}\,\Box A+\\
&\quad + \frac{2^{2/3}\,A}{3I^{1/2}K^4\sinh^2\tau} + \frac{3\,K^2}{2^{1/3}8I^{1/2}}\Big[-2\frac{I'}{I}\coth\tau + \left(\frac{I'}{I}\right)^2 + 8\frac{K'}{K}\coth\tau - 2\frac{I'}{I}\frac{K'}{K}+\\
&\quad + 4\left(\frac{K'}{K}\right)^2 - \frac{I''}{I} + 4\frac{K''}{K}\Big]A = -\frac{2^{2/3}I'}{I^{3/2}K\sinh^2\tau}\,B - \frac{K^4}{2^{2/3}I}\,C + \frac{I'}{IK^3\sinh^2\tau}\,C_2^-\\
&\quad + \frac{2^{1/3}K^4 + I'\coth\tau}{IK^3\sinh^2\tau}\,C_2^{-\prime} + \frac{2^{1/3}K^4\sinh\tau\cosh\tau + I'}{IK^3\sinh^3\tau}\,C_2^{+\prime}-\\
&- \frac{2^{1/3}(K^3\sinh^2\tau - \cosh\tau)}{IK^2\sinh^2\tau}\,B_2 - \frac{I'}{IK^3\sinh^2\tau}\,B_2' + \frac{3I'^2K^2}{2^{1/3}8I^{5/2}}\,A + \frac{9}{8}\,\frac{I'^2K^6\sinh^2\tau}{2^{2/3}\epsilon^{4/3}I^{3/2}}\,\Box_4\phi_3-\\
&\quad - \frac{3}{2^{1/3}4}\,\frac{K^2}{I^{1/2}}\left(\frac{2}{3}\,\frac{I'}{IK^3\sinh\tau} - \frac{I''}{2I} - 2\,\frac{I'}{I}\,\frac{(K\sinh\tau)'}{K\sinh\tau}\right)A. \qquad \text{(A.110)}
\end{aligned}
$$

A.5.8 Pseudoscalar metric and Ricci tensor deformations

The metric fluctuation $\delta g = (g^1 g^4 - g^2 g^3)B$:

$$
\delta R_{\mu 5} = \frac{4\,\partial_\mu B}{\epsilon^{4/3}\sqrt{h}K\sinh^2\tau}\,, \qquad \text{(A.111)}
$$

$$
\delta R_{\tau 5} = \frac{4\,(KB)'}{\epsilon^{4/3}\sqrt{h}K^2\sinh^2\tau} - \frac{2\,B}{\epsilon^{4/3}\sqrt{h}K\sinh^2\tau}\,\frac{h'}{h}\,, \qquad \text{(A.112)}
$$

and

$$\delta R_{14} = -\delta R_{23} = -\frac{3\,K^2}{\epsilon^{4/3}\sqrt{h}}\,B'' + \frac{3\,K^2}{\epsilon^{4/3}\sqrt{h}}\,B'\,\frac{h'}{h} - \frac{1}{2}\sqrt{h}\Box_4 B - \frac{4B}{3\epsilon^{4/3}\sqrt{h}K^4\sinh^2\tau}$$
$$- \frac{3\,K^2}{4\epsilon^{4/3}\sqrt{h}}\left(\frac{h'}{h}\right)^2 B - \frac{3\,(hK)'K'}{\epsilon^{4/3}h^{3/2}}\,B - \frac{3\,(hK^2\coth\tau)'}{\epsilon^{4/3}h^{3/2}}\,B\,. \quad \text{(A.113)}$$

The metric fluctuation $\delta g = A_\mu \mathrm{d}x^\mu g^5 \equiv \partial_\mu a \mathrm{d}x^\mu g^5$:

$$\delta R_{\mu 5} = -\frac{3K^2 A''_\mu}{\epsilon^{4/3}\sqrt{h}} - \frac{6\,K(2K' + \coth\tau)A'_\mu}{\epsilon^{4/3}\sqrt{h}} + \frac{1}{2}\sqrt{h}(\partial_\mu\partial^\nu A_\nu - \partial^\nu\partial_\nu A_\mu)$$
$$+ \frac{3\,h'K^2}{4\epsilon^{4/3}h^{3/2}}\left(\frac{h'}{h} - 4\,\frac{K'}{K}\right)A_\mu + \frac{8\,A_\mu}{3\,\epsilon^{4/3}\sqrt{h}K^4\sinh^2\tau}\,, \quad \text{(A.114)}$$

and

$$\delta R_{\tau 5} = \sqrt{h}\left(\frac{K'}{K} - \frac{1}{4}\,\frac{h'}{h}\right)\partial^\mu A_\mu + \frac{1}{2}\sqrt{h}\,(\partial^\mu A_\mu)'\,, \quad \text{(A.115)}$$

$$\delta R_{14} = -\delta R_{23} = -\frac{3}{4}\,\sqrt{h}K^3\partial^\mu A_\mu\,. \quad \text{(A.116)}$$

The metric fluctuation $\delta g = A\,\mathrm{d}\tau g^5$:

$$\delta R_{\mu 5} = \frac{3\,(K^2\partial_\mu A)'}{\epsilon^{4/3}\sqrt{h}} + \frac{3\,K^2}{2\epsilon^{4/3}\sqrt{h}}\left(4\,\coth\tau + \frac{h'}{h}\right)\partial_\mu A\,, \quad \text{(A.117)}$$

and

$$\delta R_{\tau 5} = -\frac{1}{2}\sqrt{h}\Box_4 A + \frac{3\,K^2}{2\,\epsilon^{4/3}\sqrt{h}}\Big[-2\frac{h'}{h}\coth\tau + \left(\frac{h'}{h}\right)^2 + 8\frac{K'}{K}\coth\tau - 2\frac{h'}{h}\frac{K'}{K}$$
$$+ 4\left(\frac{K'}{K}\right)^2 - \frac{h''}{h} + 4\frac{K''}{K}\Big]A + \frac{8\,A}{3\epsilon^{4/3}\sqrt{h}K^4}\,, \quad \text{(A.118)}$$

and

$$\delta R_{14} = -\delta R_{23} = \frac{9\,K^5}{4\epsilon^{4/3}\sqrt{h}}\left(\frac{h'}{h}\,A - 8\frac{K'}{K}\,A - 2\,A'\right)\,. \quad \text{(A.119)}$$

The Ricci tensor components:

$$\delta R = \Bigg(- \frac{3K^2\epsilon^{4/3}B''}{2^{4/3}I^{1/2}} + \frac{3\epsilon^{4/3}K^2I'}{2^{4/3}I^{3/2}}B' - \frac{3\epsilon^{4/3}K^2I'^2}{2^{1/3}8I^{5/2}}B - \frac{2^{1/3}I^{1/2}}{\epsilon^{4/3}}\Box_4 B -$$
$$- \frac{3\epsilon^{4/3}K'\left(IK\right)'}{2^{4/3}I^{3/2}}B - \frac{3\epsilon^{4/3}\left(IK^2\coth\tau\right)'}{2^{4/3}I^{3/2}}B - \frac{2^{2/3}\epsilon^{4/3}B}{3I^{1/2}K^4\sinh^2\tau} - \frac{3I^{1/2}K^3}{2^{2/3}\epsilon^{4/3}}\Box_4 a -$$
$$- \frac{9}{8}\frac{K^5\epsilon^{4/3}}{2^{1/3}I^{1/2}}\left(2A' - \left(\frac{I'}{I} - 8\frac{K'}{K}\right)A\right)\Bigg)\left(g^1\cdot g^4 - g^2\cdot g^3\right) +$$
$$+ \partial_\mu\Bigg(- \frac{3K^2\epsilon^{4/3}a''}{2^{4/3}I^{1/2}} - \frac{3\epsilon^{4/3}K\left(2K' + K\coth\tau\right)a'}{2^{1/3}I^{1/2}} + \frac{6\epsilon^{4/3}I'K^2}{2^{1/3}16I^{3/2}}\left(\frac{I'}{I} - 4\frac{K'}{K}\right)a$$
$$+ \frac{2^{5/3}\epsilon^{4/3}a}{3I^{1/2}K^4\sinh^2\tau} + \frac{6\epsilon^{4/3}\left(K^2A\right)'}{2^{1/3}4I^{1/2}} + \frac{3\epsilon^{4/3}K^2}{2^{1/3}4I^{1/2}}\left(4\coth\tau + \frac{I'}{I}\right)A + \frac{2^{2/3}\epsilon^{4/3}B}{I^{1/2}K\sinh^2\tau}\Bigg)\mathrm{d}x^\mu\cdot g^5 +$$
$$+ \Bigg(\frac{2^{2/3}\epsilon^{4/3}}{\sinh^2\tau}\left(\frac{B}{I^{1/2}K}\right)' + \frac{2^{1/3}I}{\epsilon^{4/3}K^2}\left(\frac{K^2\Box_4 a}{I^{1/2}}\right)' - \frac{2^{1/3}I^{1/2}}{\epsilon^{4/3}}\Box_4 A + \frac{2^{5/3}\epsilon^{4/3}A}{3I^{1/2}K^4\sinh^2\tau} +$$
$$+ \frac{2^{2/3}\epsilon^{4/3}3K^2}{8I^{1/2}}\Bigg(- 2\frac{I'}{I}\coth\tau + \left(\frac{I'}{I}\right)^2 + 8\frac{K'}{K}\coth\tau -$$
$$- 2\frac{I'}{I}\frac{K'}{K} + 4\left(\frac{K'}{K}\right)^2 - \frac{I''}{I} + 4\frac{K''}{K}\Bigg)A\Bigg)\mathrm{d}\tau\cdot g^5\,. \quad \text{(A.120)}$$

A.6 Hodge duals of the KS forms

Here F_k is some four-dimensional k-form.

$$* \left(F_k \wedge g^1 \wedge g^2 \wedge g^5\right) = (-)^{k+1}h^{k/2}\sqrt{-G}(G^{11})^2G^{55}(*_4F_k)\wedge d\tau\wedge g^3\wedge g^4\,, \quad \text{(A.121)}$$
$$* \left(F_k \wedge g^3 \wedge g^4 \wedge g^5\right) = (-)^{k+1}h^{k/2}\sqrt{-G}(G^{33})^2G^{55}(*_4F_k)\wedge d\tau\wedge g^1\wedge g^2\,, \quad \text{(A.122)}$$
$$* \left(F_k \wedge \omega_3\right) = (-)^{k+1}\frac{1}{2}h^{k/2}\sqrt{-G}G^{55}(*_4F_k)\wedge d\tau\wedge\left((G^{11})^2g^3\wedge g^4 + (G^{33})^2g^1\wedge g^2\right)\,, \quad \text{(A.123)}$$

$$* \left(F_k \wedge d\tau \wedge g^1 \wedge g^2 \wedge g^5\right) = h^{k/2}\sqrt{-G}(G^{11})^2(G^{55})^2(*_4F_k)\wedge g^3\wedge g^4\,, \quad \text{(A.124)}$$
$$* \left(F_k \wedge d\tau \wedge g^3 \wedge g^4 \wedge g^5\right) = h^{k/2}\sqrt{-G}(G^{33})^2(G^{55})^2(*_4F_k)\wedge g^1\wedge g^2\,, \quad \text{(A.125)}$$
$$* \left(F_k \wedge d\tau \wedge \omega_3\right) = \frac{1}{2}h^{k/2}\sqrt{-G}(G^{55})^2(*_4F_k)\wedge\left((G^{11})^2g^3\wedge g^4 + (G^{33})^2g^1\wedge g^2\right)\,, \quad \text{(A.126)}$$

$$* \big(F_k \wedge g^1 \wedge g^2\big) = h^{k/2}\sqrt{-G}(G^{11})^2(*_4F_k) \wedge d\tau \wedge g^3 \wedge g^4 \wedge g^5\,, \tag{A.127}$$

$$* \big(F_k \wedge g^3 \wedge g^4\big) = h^{k/2}\sqrt{-G}(G^{33})^2(*_4F_k) \wedge d\tau \wedge g^1 \wedge g^2 \wedge g^5\,, \tag{A.128}$$

$$* \big(F_k \wedge \omega_2\big) = \frac{1}{2}h^{k/2}\sqrt{-G}(*_4F_k) \wedge d\tau \wedge \big((G^{11})^2 g^3 \wedge g^4 + (G^{33})^2 g^1 \wedge g^2\big) \wedge g^5\,, \tag{A.129}$$

$$* \big(F_k \wedge d\tau \wedge g^1 \wedge g^2\big) = h^{k/2}(-)^k\sqrt{-G}(G^{11})^2 G^{55}(*_4F_k) \wedge g^3 \wedge g^4 \wedge g^5\,, \tag{A.130}$$

$$* \big(F_k \wedge d\tau \wedge g^3 \wedge g^4\big) = h^{k/2}(-)^k\sqrt{-G}(G^{33})^2 G^{55}(*_4F_k) \wedge g^1 \wedge g^2 \wedge g^5\,, \tag{A.131}$$

$$* \big(F_k \wedge d\tau \wedge \omega_2\big) = \frac{1}{2}h^{k/2}(-)^k\sqrt{-G}G^{55}(*_4F_k) \wedge \big((G^{11})^2 g^3 \wedge g^4 + (G^{33})^2 g^1 \wedge g^2\big) \wedge g^5\,, \tag{A.132}$$

$$*\big(F_k\big) = 2h^{k/2}\sqrt{-G}(*_4F_k) \wedge d\tau \wedge \omega_2 \wedge \omega_3\,, \tag{A.133}$$

$$*\big(F_k \wedge d\tau\big) = 2h^{k/2}(-)^k\sqrt{-G}G^{55}(*_4F_k) \wedge \omega_2 \wedge \omega_3\,, \tag{A.134}$$

$$*\big(F_k \wedge g^5\big) = 2h^{k/2}(-)^{k+1}\sqrt{-G}G^{55}(*_4F_k) \wedge d\tau \wedge \omega_2 \wedge \omega_2\,, \tag{A.135}$$

$$*\big(F_k \wedge d\tau \wedge g^5\big) = 2h^{k/2}\sqrt{-G}(G^{55})^2(*_4F_k) \wedge \omega_2 \wedge \omega_2\,, \tag{A.136}$$

$$*\big(F_k \wedge dg^5\big) = -h^{k/2}\sqrt{-G}G^{11}G^{33}(*_4F_k) \wedge d\tau \wedge dg^5 \wedge g^5\,, \tag{A.137}$$

$$*\big(F_k \wedge d\tau \wedge dg^5\big) = h^{k/2}(-)^{k+1}\sqrt{-G}G^{11}G^{33}G^{55}(*_4F_k) \wedge dg^5 \wedge g^5\,, \tag{A.138}$$

$$*\big(F_k \wedge dg^5 \wedge g^5\big) = h^{k/2}(-)^k\sqrt{-G}G^{11}G^{33}G^{55}(*_4F_k) \wedge d\tau \wedge dg^5\,, \tag{A.139}$$

$$*\big(F_k \wedge (g^1 \wedge g^3 + g^2 \wedge g^4)\big) = -h^{k/2}\sqrt{-G}G^{11}G^{33}(*_4F_k) \wedge d\tau \wedge (g^1 \wedge g^3 + g^2 \wedge g^4) \wedge g^5\,, \tag{A.140}$$

$$\begin{aligned} * \big(F_k \wedge d\tau \wedge (g^1 \wedge g^3 + g^2 \wedge g^4)\big) = \\ = (-)^{k+1}h^{k/2}\sqrt{-G}G^{11}G^{33}G^{55}(*_4F_k) \wedge (g^1 \wedge g^3 + g^2 \wedge g^4) \wedge g^5\,, \end{aligned} \tag{A.141}$$

$$\begin{aligned} * \big(F_k \wedge (g^1 \wedge g^3 + g^2 \wedge g^4) \wedge g^5\big) = \\ = (-)^k h^{k/2}\sqrt{-G}G^{11}G^{33}G^{55}(*_4F_k) \wedge d\tau \wedge (g^1 \wedge g^3 + g^2 \wedge g^4)\,, \end{aligned} \tag{A.142}$$

$$* \left(F_k \wedge d\tau \wedge (g^1 \wedge g^3 + g^2 \wedge g^4) \wedge g^5\right) =$$
$$= -h^{k/2}\sqrt{-G}G^{11}G^{33}(G^{55})^2(*_4F_k) \wedge (g^1 \wedge g^3 + g^2 \wedge g^4)\,. \quad \text{(A.143)}$$

$$*\big(F_k \wedge F_3\big) = (-)^{k+1}h^{k/2-1}(*_4F_k) \wedge H_3\,, \quad \text{(A.144)}$$
$$*\big(F_k \wedge H_3\big) = (-)^{k}h^{k/2-1}(*_4F_k) \wedge F_3\,, \quad \text{(A.145)}$$

$$*\big(F_k \wedge \mathrm{d}\tau \wedge \mathrm{d}g^5 \wedge g^5\big) = -h^{k/2}\sqrt{-G}G^{11}G^{33}(G^{55})^2(*_4F_k) \wedge \mathrm{d}g^5\,, \quad \text{(A.146)}$$
$$*\big(F_k \wedge \mathrm{d}g^5 \wedge \mathrm{d}g^5\big) = -2h^{k/2}\sqrt{-G}(G^{11})^2(G^{33})^2(*_4F_k) \wedge d\tau \wedge g^5\,, \quad \text{(A.147)}$$
$$*_4 *_4 F_k = (-)^{k+1}F_k \quad \text{(A.148)}$$
$$*_4\mathrm{d}_4 *_4 \mathrm{d}_4F_1 = -\Box_4F_1\,. \quad \text{(A.149)}$$

$$m_4^2 = \frac{3\,\epsilon^{4/3}}{2\cdot 2^{2/3}}\,\tilde{m}^2\,. \quad \text{(A.150)}$$