Topics in Two-dimensional Heterotic and Minimal Supersymmetric Sigma Models

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Dedication

To those who held me up over the years

Abstract

Two-dimensional $\mathcal{N} = (0,1)$ and $(0,2)$ supersymmetric sigma models can be mainly obtained in two ways: non-minimal heterotic deformation of $\mathcal{N} = (1,1)$ and $(2,2)$ sigma models, and minimal construction which contains only $(0,1)$ or $(0,2)$ supermultiplets. The former deformed models with $\mathcal{N} = (0,2)$ supersymmetries emerge as low-energy world sheet theories on non-Abelian strings supported in some $\mathcal{N} = 1$ four-dimensional Yang-Mills theories. The latter, on the other hand, can be regarded as the elementary building blocks to construct generic $\mathcal{N} = (0,1)$ or $(0,2)$ chiral models.

In the thesis, we will study both types of sigma models. We start with the deformed heterotic sigma models with $\mathcal{N} = (0,2)$ supersymmetries. Our investigation is around the calculation of NSVZ exact β-function of the heterotic models through instanton technique, and also verifies it by straightforward two-loop calculation and the "Konishi anomaly" of the hypercurrent. Finally, we also consider isometries on their target spaces, and show that the heterotic deformation is free of isometry and holonomy anomalies.

Then we turn to analysis of a more fundamental minimal construction of chiral sigma models with $\mathcal{N} = (0,1)$ and $(0,2)$ supersymmetries. These minimal models with only (left) chiral fermions may intrinsically suffer from chiral anomalies that will render the theories mathematically inconsistent. We focus on two important examples, the minimal $\mathrm{O}(N)$ and $\mathrm{CP}(N-1)$ models, and calculate their isometry anomalies. We show that the $\mathrm{CP}(N-1)$ models with $N > 2$ has non-removable chiral anomalies, while the $\mathrm{O}(N)$ models are anomaly free and thus exist quantum mechanically. We also disclose a relation between isometry anomalies in these non-linear sigma models ($NL\sigma M$) and gauge anomalies in gauged linear sigma models ($GL\sigma M$).

Finally, we reveal a relation on anomaly correspondence between $NL\sigma M$ and $GL\sigma M$ to minimal models on homogeneous spaces. We interpret these anomalies more from geometric perspectives and relate them to the characteristic classes of the target spaces. Through explicit calculation of anomalous fermionic effective action, we show how to add a series of local counterterms to remove the anomalies. We eventually reach a result that the remedy

procedure is equivalent to require the target spaces of theories with trivial first Pontryagin class, and thus demonstrate Moore and Nelson's consistency condition in the case of homogeneous spaces. More importantly, we find that local counterterms further constrain "curable" models and make some of them flow to non-trivial infrared superconformal fixed point. We also discuss a interesting relation between $\mathcal{N} = (0,1)$ and $(0,2)$ supersymmetric sigma model and gauge theories in the spirit of 't Hooft anomaly matching condition.

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Chapter 1

Introduction

1.1 Background

It is well known that, in the context of quantum field theory, when the system has a continuous symmetry group G but with the vacuum being invariant only under a subgroup H of G, the spontaneous symmetry breaking from group G to H will result in massless modes, known as Nambu-Goldstone bosons, corresponding to those broken symmetries around the vacuum [1,2]. The interactions among these light scalar fields are described by sigma models whose Lagrangians are completely determined by the geometry of the target space manifold G/H. In this sense, sigma models are originally considered as low-energy effective theories of certain UV theories in four-dimensional spacetime.

In contrast to the story in four dimensions, two-dimensional sigma models can be regarded as underlying theories themselves. When the target spaces G/H are symmetric spaces, the sigma models are renormalizable in the usual sense due to isometries of G/H. The renormalizability of two-dimensional sigma models on arbitrary Riemann manifold is also further generalized by Friedan [3] when the models are considered as effective theories on string world sheet. Historically the β-function of two-dimensional $\mathrm{O}(N)$ sigma model was first calculated by Polyakov and the model was shown to be an asymptotic free theory. More interestingly, due to the wild quantum effects in the IR regime, it is shown that there is actually no Goldstone bosons in two-dimensional spacetime! Therefore many two-dimensional sigma models, although starting from massless theories, will eventually generate a mass gap and restore the full G symmetries in deep infra-red. Moreover, some types of sigma models, $\mathrm{CP}(N-1)$ for example, also bear non-trivial topological solutions of classical field equations. These instanton or soliton solutions in turn qualitatively shed light on the mechanism of mass gap generated in strong coupling regime [4]. Remarkably, all these features, asymptotic freedom, dynamical mass gap, instanton effect, etc., are also shared with quantum chromodynamics (QCD), the physics theory in our real world, while the four-dimensional QCD as well as other non-Abelian gauge theories are far from completely understood due to their complicated vacuum structures [5, 6]. Therefore two-dimensional sigma models appear to be a quite natural and ideal "theoretical laboratory" to model four-dimensional gauge theories, and test ideas and methods to obtain better understandings of non-perturbative phenomena in four-dimensional gauge theories such as confinement of color, spontaneous breaking of chiral symmetry, etc.

1.2 Supersymmetry and $2d/4d$ correspondence

A powerful tool to study non-perturbative physics of $2d/4d$ quantum field theories is the use of supersymmetry. In the early time, the invention of supersymmetry in the 1970's last century was aimed at solving a series of important phenomenological physics questions, the hierarchy problem, gauge coupling unification, dark matter candidate and so forth. However experimental attempts to look for signals of supersymmetry in these years were not successful. The mass of Higgs boson discovered in Large Hadron Collider (LHC) was around 125 GeV [7,8], which was not favored by supersymmetry. More recently (December 2015) the Run 2 result at LHC further pushed up the limits on superpartner masses and thus rendered the possibility of supersymmetry [9], at least in the form we currently think about, notably smaller. Nevertheless, the theoretical role of supersymmetry in understanding physics at strong coupling regime is significantly important, for in many examples the Bogomolny-Prassad-Somerfield (BPS) sectors [10, 11] of supersymmetric theories are protected by supersymmetries from correction down to strong coupling regime. We therefore can investigate these BPS states in strong coupling regime through calculations in the weak coupling limit by dualities [12–15]. The close relation mentioned in the previous paragraph between $2d$ sigma models and $4d$ non-Abelian gauge theories, after enhanced by supersymmetries, expresses the $2d/4d$ correspondence in a more transparent and quantitative way.

As an instructive example of the $2d/4d$ correspondence as well as one of the motivations for this thesis, we consider the four-dimensional $\mathcal{N}=2$ $U(N)$ SQCD with N hypermultiplets [16–18]. The theory supports solitonic string vortex solutions in the color-flavor locked phase, which has internal degree of freedoms forming a CP$(N-1)$ moduli space. To describe the low-energy dynamics of vortex string, one needs to promote these associated degree of freedom to oscillate and thus give rise to massless modes classically described as a two-dimensional CP$(N-1)$ sigma model. Further the string solution is 1/2-BPS saturated and inherits half of supersymmetries from the bulk $4d$ gauge theories. Therefore, accounting for fermionic zero modes as well, we obtain a two-dimensional $\mathcal{N}=(2,2)$ sigma model with the target space CP$(N-1)$. The vortex string solution plays the role of the mapping between the supersymmetric $4d$ gauge theories to $2d$ sigma models. Furthermore the BPS kinks in $2d$ sigma models are nothing but confined monopoles in the BPS spectrum of the $4d$ gauge theories. The identification of these BPS spectra keeps not only at classical but even quantum level for they are both protected by supersymmetries. One can try to extend the correspondence from four-dimensional $\mathcal{N}=2$ down to $\mathcal{N}=1$ gauge theories. This work is initiated and further explored by Edalati-Tong and Shifman-Yung [19, 20]. Roughly speaking, the solitonic solutions in $4d$ $\mathcal{N}=1$ gauge theories, called heterotic vortex strings, analogously map the $4d$ bulk theories to $2d$ $\mathcal{N}=(0,2)$ heterotic sigma models deformed from original $\mathcal{N}=(2,2)$ ones. The two-dimensional heterotic sigma models with $\mathcal{N}=(0,2)$ supersymmetries is thus one of the starting points of the author's research.

Another advantage benefited from supersymmetries reflects on the loop calculations under instanton background in supersymmetric field theories. The instanton background, similar to solitonic solutions above, also preserves half of supersymmetries. These residue supersymmetries guarantee cancellation between bosonic and fermionic modes for two and higher loops in instanton background. Therefore only zero modes and the one-loop contribution need to be considered. This observation amounts to the exact Novikov-Shifman-Vainshtein-Zakharov (NSVZ) β-functions in both $4d$ $\mathcal{N}=1$ supersymmetric gauge theories and $2d$ $\mathcal{N}=(2,2)$ supersymmetric sigma models [21–24]. In the first part of the thesis, we will explore how to use instanton technique to also calculate the exact NSVZ $\beta-$functions in the heterotic sigma models with $\mathcal{N}=(0,2)$ supersymmetries, and compare it to that of $\mathcal{N}=1$ four-dimensional gauge theories.

It is obvious and fascinating that, if one could further generalize the above argument down to $\mathcal{N} = 0$ four-dimensional gauge theories (including the genuine QCD), we will eventually quantitatively encode, even only in some sectors, the real QCD theory in terms of data from two-dimensional sigma model where we merely encounter simpler problems but have much more theoretical tools, like integrability and etc.. Surly, with no help of supersymmetries, we will return back the origin, and the goal of understanding the full story of real QCD in strong coupling regime seems still quite far from our current situation. However after touring with supersymmetries, we can at least acquire some sense how the non-perturbative physics world looks like and how the exploration of that is challenging but exciting.

1.3 Perspective from mathematical physics

Besides the general $2d/4d$ correspondence, two-dimensional supersymmetric sigma model are also interesting to study on its own right because of their extremely rich mathematical structures. $2d$ sigma models can be defined on generic Riemann manifold M, while scalar fields can be viewed as maps from general Riemann surfaces Σ, not necessarily restricted to flat space time, to target space M. Therefore geometric data of Σ and M can be quite naturally introduced in these mathematical physics models. For examples, the general renormalization group equation, i.e. the β-functions of metric on M, coincides with the Ricci flow up to one-loop order [3]. When one equips supersymmetries on $2d$ sigma models, additional geometric and topological data on Σ and M can be extracted. It is well-known that extended supersymmetries require additional geometrical structures on M: Kähler or complex structure are needed for $\mathcal{N} = (2, 2)$ or $(0, 2)$ supersymmetries [25, 26]; The construction of generalized complex geometries and bi-hermitian geometries were discovered because of the appearance of both chiral and twisted-chiral supermultiplets in $\mathcal{N} = (2, 2)$ sigma models [27, 28]. On the other hand, from string perspective, two-dimensional supersymmetric sigma models concerned as effective string worldsheet theories are living on Calabi-Yau manifolds and therefore manifest themselves with superconformal symmetries. With recent development of super-localization techniques, the partition functions of these sigma models are calculable and identified to the Kähler potentials of the moduli space of Calabi-Yau manifolds [29–31]. Many interesting conjectures on Calabi-Yau manifolds can be thus investigated and proved, at least in physics sense.

On the topological side, Witten long time ago showed that two-dimensional supersymmetric sigma models with $\mathcal{N} = (2, 2)$ or $(0, 2)$ supersymmetries can be performed topological twist and related to Floer homology in mathematics [32, 33]. The topological twist theories also determine chiral ring structures in sigma models and define the quantum cohomology ring of target spaces. Very recently, through compactification of superconformal six-dimensional $\mathcal{N} = (2, 0)$ theories on 4-manifolds [34–36], a correspondence was discovered and explored between the building blocks of 4-manifolds and $\mathcal{N} = (0, 2)$ supersymmetric gauge theories whose renormalization group (RG) flow at IR is to some $\mathcal{N} = (0, 2)$ (superconformal) sigma models.

Although $\mathcal{N} = (0, 2)$ supersymmetric gauge theories and sigma models are of importance in constructing heterotic string models, very little is known about the dynamics of these $(0, 2)$ theories until recent. The main reason is that $2d$ $\mathcal{N} = (0, 2)$ theories often exhibit dynamical supersymmetry breaking and thus hard to be investigated if there is no supersymmetric vacua. Partly motivated by the progresses and difficulties mentioned above, in the thesis we will discuss minimal $\mathcal{N} = (0, 1)$ and $(0, 2)$ supersymmetric sigma models as the fundamental building blocks of the general theories. Here by minimal we mean that there are only, say, left-handed fermions included in models. Unlike $\mathcal{N} = (0, 2)$ heterotic sigma models which are obtained from deformation of $(2, 2)$ theories and ease of anomalies, minimal sigma models may intrinsically have chiral anomalies. After discussing the possibility to remove chiral anomalies, we find that some "curable" models surprisingly have RG flow to superconformal fixed point and thus no supersymmetry breaking.

1.4 Overview and Summary

The thesis is organized as follows: Chapter 2 studies the general $\mathcal{N} = (0, 2)$ heterotic sigma models deformed from $\mathcal{N} = (2, 2)$ sigma models on symmetric Kähler manifolds by adding chiral fermions [38]. Due to the special feature of additional chiral fermions coupled to original $\mathcal{N} = (2, 2)$ theories, we show, in the instanton calculus, that the number [37] and eigenvalues of fermionic zero modes are not changed, rather there is a mixing of the new and old fermions. The holomorphy of coupling constants is thus unbroken and non-zero modes in the instanton background are canceled beyond one-loop. Therefore, analogue to $\mathcal{N} = 1$ four-dimensional Yang-Mills theories, we find the exact NSVZ β-function for heterotically deformed $\mathcal{N} = (0, 2)$ sigma models. We also verify our instanton argument by straightforward two-loop calculation and the "Konishi anomaly" of the hypercurrent (supercurrent multiplet). At last, for the self-consistency of chiral fermions, we investigate isometries on the Kähler spaces, and show that this type of deformation is free of isometry and holonomy anomalies.

In Chapter 3 we turn to analyze the more fundamental minimal construction of chiral sigma models with $\mathcal{N} = (0, 1)$ and $(0, 2)$ supersymmetries. Because only left-handed fermions are contained in theories, there are strong topological constraints on the target spaces to keep theories from chiral anomalies. Our investigation started from the minimal models defined on manifolds S^{N-1}, say the $\mathrm{O}(N)$ model, and CP^{N-1} [39]. We discuss these models in both non-linear and gauged linear formalism and calculate isometry and gauge anomalies respect to the two formalisms. We show how removal of anomalies is implemented in the $\mathrm{O}(N)$ case, whereas same strategy does not work for the $\mathrm{CP}^N (N \geq 2)$. We also observe a correspondence between isometry and gauge anomalies from different formalism. This correspondence is crucial to relate anomalies to topological constraints on target spaces and leads us to more general holonomy anomalies of minimal models on generic homogeneous spaces G/H in next Chapter.

Chapter 4 continues the exploration on anomalies of $\mathcal{N} = (0, 1)$ and $(0, 2)$ minimal models on homogeneous spaces [40]. Thanks to the isomery/gauge anomalies correspondence, we focus on the gauge linear formalism. We explicitly obtain anomalous fermionic effective action of G/H models, and "remedy" it by adding a series of local counterterms. Through the procedure, we derive that the vanishing of anomalies is equivalent to require the target spaces with trivial first Pontryagin class. More interestingly, with the help of local counterterms, we demonstrate two possible scenarios of those cured models in deep IR region: **a.** supersymmetry will be broken for symmetric space; **b.** there exists nontrivial superconformal fixed point for non-symmetric homogeneous spaces with nontrivial third cohomology class. At last, in the spirit of the t'Hooft's anomaly matching condition, we also find such an analog between two-dimensional $\mathcal{N} = (0, 1)$ and $(0, 2)$ gauge theories and minimal sigma model.

Chapter 2

$\mathcal{N} = (0,2)$ Deformation of $(2,2)$ Sigma Models: Geometric Structure, Holomorphic Anomaly and Exact β Functions

2.1 Introduction

Heterotically deformed $\mathcal{N} = (0,2)$ sigma models to be considered below emerged as low-energy world sheet theories on non-Abelian strings supported in some $\mathcal{N} = 1$ four-dimensional Yang-Mills theories [19, 20] (for a recent review see [41]). Particularly, in this case the target space was $\mathrm{CP}(N-1)$ but the heterotic modification could be considered for a wide class of the Kähler manifolds. The heterotic models, although remaining largely unexplored, appear to be important in various problems. Some previous results can be found in [42, 44–47], for a general discussion of (0,2) models see [48–52]. A renewed interest is also due to the recent publication [53]. Here we report further results in the study of geometric structure, holomorphic anomaly and exact β functions in these models.

Heterotic $\mathcal{N} = (0,2)$ models have a rich mathematical structure. In perturbation theory two-dimensional $\mathcal{N} = (0,2)$ models were shown to share some features with $\mathcal{N} = 1$ super-Yang-Mills theories in four dimensions (e.g. [47]). They are asymptotically free in the ultraviolet (UV), have different phases in the infrared (IR), and admit large-N solution [42, 44]. These facts can be interpreted within $2d/4d$ correspondence (e.g. [19, 20]) and the Dijkraaf-Vafa type deformation. The same correspondence was noted in more general $2d/4d$ coupled systems, the study of which requires a thorough knowledge of the two-dimensional side. A number of insights were obtained from string theory, see [51, 52]. However, considerations of (0,2) models in quantum field theory are scarce. In particular, beyond chiral operators, nothing was explored until quite recently.

Two-dimensional sigma models present a natural playground for geometric explorations. They encode the geometry of the target space, that of the worldsheet, and the geometry of various moduli spaces. Essentially everything is known for the undeformed $\mathcal{N}=(2,2)$ models. With $\mathcal{N}=(0,2)$ supersymmetry, one can test the robustness of the target space geometry – whether or not quantization provide us with some kind of geometrical deformation. In particular, the models we will consider have both, isometries and a global symmetry realized in a nontrivial way. The interplay between geometry and quantum effects could be enlightening on both sides.

Finally, implications of current algebra in $\mathcal{N}=(0,2)$ theories were discussed more than once. While the general structure is known [54, 55], explicit examples of how these current-algebraic relations are implemented in particular models and what they imply for quantization were not worked out. We emphasize that the current algebra calculation and the renormalization group (RG) flow of the theory are intertwined [21], and, hence, it is possible to formulate the current algebra as a way to uncover renormalization of a given theory. Moreover, the supercurrent supermultiplet (to be referred to as *hypercurrent*) starts from the $U(1)_R$ current; therefore the overall anomaly is determined by the index theorem for the appropriate Dirac operator.

To explain the nature of heterotic modifications let us start with reminding geometry of unmodified $\mathcal{N}=(2,2)$ sigma models. It was pointed out by Zumino [25] that the target space of these models should have the Kähler geometry. Moreover, to be characterized just by one coupling g, it should be a symmetric space which can be described as a homogeneous space G/H for a Lie group G and the stabilizer H. For the projective $\mathrm{CP}(N-1)$ space $G=\mathrm{SU}(N)$ and $H=\mathrm{S}\left(\mathrm{U}(N-1)\times\mathrm{U}(1)\right)$. It is a particular case of Grassmannian spaces

$$\mathrm{SU}(n+m)/\mathrm{SU}(n)\times\mathrm{SU}(m)\times\mathrm{U}(1)$$

(see, e.g., [24] for a full list of the symmetric Kähler spaces).

In these homogeneous spaces the Ricci tensor $R_{i\bar{j}}$ is proportional to the metric $G_{i\bar{j}}$,

$$R_{i\bar{j}} = b\,\frac{g^2}{2}\,G_{i\bar{j}}\,. \tag{2.1.1}$$

This feature is a definition of the Einstein spaces. The constant b is equal to the dual Coxeter number T_G for the group G.,

$$b_{\,G/H} = T_G\,. \tag{2.1.2}$$

Correspondingly, for the CP$(N-1)$ space

$$b_{\,\mathrm{CP}(N-1)} = T_{\mathrm{SU}(N)} = N\,. \tag{2.1.3}$$

The same constant $b = T_G$ defines the β function

$$\beta\big(\mathcal{N}=(2,2)\big) = \mu\,\frac{dg^2}{d\mu} = -T_G\,\frac{g^4}{4\pi}\,, \tag{2.1.4}$$

which is exhausted by one loop [23, 24] in the $(2,2)$ theories.

To diminish the number of supercharges from 4 in $\mathcal{N}=(2,2)$ to 2 in $\mathcal{N}=(0,2)$ one needs to break partially a partnership between bosonic and fermionic fields. In the $(2,2)$ case each bosonic field ϕ^i has two fermion partners, right- and left-movers, ψ^i_R and ψ^i_L . A simple way to diminish supersymmetry to (0, 2) is just to discard all ψ^i_R. Such chiral models, which can be called minimal, generically suffer from internal diffeomorphism anomaly [56]. In particular, only CP(1) out of the entire CP$(N-1)$ series is anomaly free and presents a consistent minimal model. For the consistency of minimal chiral sigma models, we will present our work in following chapters.

Our study is focused on a different heterotic modification. Namely, instead of deleting right-moving fermions ψ^i_R, one extra right-mover ζ_R is added to the content of the (2,2) theory. A new coupling which mixes ζ_R and ψ^i_R in the background of bosonic field leads to breaking of the $(2,2)$ supersymmetry to $(0,2)$. In contrast to the minimal $(0,2)$ models this heterotic coupling can be switched on perturbatively which is sufficient to show an absense of internal anomaly problem.

Thus, our task is to analyze the heterotically deformed (nonminimal) $\mathcal{N}=(0,2)$ models. We present a more complete geometric formulation of the class of nonminimal models which will be studied in this chapter. Holomorphic properties of such models are revealed. They have two coupling constants, the original g and the heterotic h. Correspondingly, there are two β functions. As in four-dimensional Yang-Mills [21], we have to differentiate between the holomorphic coupling constants which are renormalized at most at one loop, and their nonholomorphic counterparts. The latter appear in conventional perturbation theory and are sometimes referred to as canonic.

We calculate both β functions in more than one way. In particular, we derive exact relations between the β functions and the anomalous dimensions γ, analogous to the NSVZ relations in four-dimensional Yang-Mills [22] using the instanton calculus. For instance, for β_g it will be shown that to all orders in perturbation theory

$$\beta_g = \mu\,\frac{dg^2}{d\mu} = -\frac{g^2}{4\pi}\,\frac{T_G\, g^2\,(1+\gamma_{\psi_R}/2) - h^2\,(\gamma_{\psi_R}+\gamma_\zeta)}{1-(h^2/4\pi)} \tag{2.1.5}$$

where γ_{ψ_R}, γ_ζ are the anomalous dimensions of the ψ_R, ζ_R fields.

We compute the anomalous dimensions up to two loops implying prediction for three loops in β_g. Our two-loop results for anomalous dimensions also confirm the fact that there exists a fixed point for the ratio $\rho \equiv h^2/g^2$. The critical value ρ_c depends on a manifold geometry and equals to $1/2$ for $\mathrm{CP}(N-1)$ [45]. At this point three-loop β_g reduces to

$$\beta_g^{(3)} = -T_G\,\frac{g^4}{4\pi}\,\frac{1}{\big(1-(h^2/4\pi)\big)}\,, \tag{2.1.6}$$

i.e., to the one-loop expression up to the factor $1/(1-(h^2/4\pi))$.

Then we prove that despite the chiral nature of the model no anomaly appears in the isometry currents of $\mathrm{CP}(N-1)$ at any N.

Finally, we consider the $\mathcal{N}=(0,2)$ supercurrent supermultiplet (hypercurrent) and its anomalies, as well as the "Konishi anomaly" [57]. This gives us another method for finding the exact expression for β_g via anomalous dimensions.

2.2 Heterotic $\mathcal{N} = (0, 2)$ models in two-dimensions

2.2.1 Formulation of the model

We start to describe the model to be studied in this chapter by introducing two types of chiral $\mathcal{N} = (0, 2)$ superfields A and B. The first, bosonic superfield A describes a chiral supemultiplet which on mass shell consists of a complex bosonic field and a left-moving Weyl fermion,

$$A(x_R + 2i\theta^\dagger\theta, x_L, \theta) = \phi(x_R + 2i\theta^\dagger\theta,\, x_L) + \sqrt{2}\,\theta\,\psi_L(x_R + 2i\theta^\dagger\theta,\, x_L)\,. \tag{2.2.1}$$

Here $x_{R,L}$ are light-cone coordinates $x_{R,L} = t \pm x$ and θ is the one-component Grassmann variable corresponding to θ_R (see Appendix A for our notation). The second, fermonic superfield B refers to the Fermi supermultiplet which on mass shell contains only the right-moving fermion (F is an auxiliary field),

$$B(x_R + 2i\theta^\dagger\theta, x_L, \theta) = \psi_R(x_R + 2i\theta^\dagger\theta,\, x_L) + \sqrt{2}\,\theta F(x_R + 2i\theta^\dagger\theta,\, x_L)\,. \tag{2.2.2}$$

Note that in the nonlinear formulation here, the fermionic multiplets are taken to be chiral in a strict sense. In the $\mathcal{N} = (0, 2)$ gauged formulation this condition can usually be relaxed. Note also that the $\mathcal{N}=(2, 2)$ chiral field $\Phi(x_R+2i\theta_R^\dagger\theta_R, x_L-2i\theta_L^\dagger\theta_L, \theta_R, \theta_L)$ decomposes in the $\mathcal{N}=(0, 2)$ superfields A and B as

$$\begin{aligned}&\Phi(x_R + 2i\theta_R^\dagger\theta_R, x_L - 2i\theta_L^\dagger\theta_L, \theta_R, \theta_L)\\ = {}&A(x_R + 2i\theta_R^\dagger\theta_R, x_L - 2i\theta_L^\dagger\theta_L, \theta_R) + \sqrt{2}\,\theta_L B(x_R + 2i\theta_R^\dagger\theta_R, x_L, \theta_R).\end{aligned} \tag{2.2.3}$$

The $\mathcal{N} = (0, 2)$ supersymmetry transformations are as follows:

$$\begin{aligned}&\delta x_R = -2i\theta^\dagger\epsilon\,, \quad \delta x_L = 0\,, \quad \delta\theta = \epsilon\,, \quad \delta\theta^\dagger = \epsilon^\dagger\,,\\ &\delta\phi = \sqrt{2}\,\epsilon\,\psi_L\,, \qquad \delta\psi_L = -\sqrt{2}\,i\epsilon^\dagger\partial_L\phi\,,\\ &\delta\psi_R = \sqrt{2}\,\epsilon\,F\,, \qquad \delta F = -\sqrt{2}\,i\epsilon^\dagger\partial_L\psi_R\,,\end{aligned} \tag{2.2.4}$$

where $\partial_L = \partial_t + \partial_x = 2\,\partial_{x_R}$.

The undeformed $\mathcal{N}=(2,2)$ model in terms of the $\mathcal{N}=(0,2)$ superfields (2.2.1) and (2.2.2) contains equal number of bosonic A^i and fermionic B^i superfields. In the particular case of CP$(N-1)$ we have $i=1,2,...,N-1$.

The heterotic deformation to be considered below is induced by adding a singlet fermonic superfield $\mathcal{B}$,

$$\mathcal{B} = \zeta_R(x_R + 2i\theta^\dagger\theta,\, x_L) + \sqrt{2}\,\theta\mathcal{F}(x_R + 2i\theta^\dagger\theta,\, x_L)\,. \tag{2.2.5}$$

The Lagrangian can be written as

$$\begin{aligned}\mathcal{L} &= \frac{1}{4}\int d^2\theta\,\Big[K_i(A,A^\dagger)\left(i\partial_R A^i - 2\kappa\,\mathcal{B}B^i\right) + \text{H.c.}\Big] \\ &+ \frac{1}{2}\int d^2\theta\,\Big[Z\,G_{i\bar{j}}(A,A^\dagger)\,B^{\dagger\bar{j}}B^i + \mathcal{Z}\,\mathcal{B}^\dagger\mathcal{B}\Big]\,.\end{aligned} \tag{2.2.6}$$

Here K is the Kähler potential viewed as a function of the bosonic superfields. By definition

$$K_i(A,A^\dagger) \equiv \partial_{A^i}\,K(A,A^\dagger)\,. \tag{2.2.7}$$

Moreover, $G_{i\bar{j}}$ is the metric on the target space,

$$G_{i\bar{j}} = K_{i\bar{j}}(A,A^\dagger) \equiv \partial_{A^i}\partial_{A^{\dagger\bar{j}}}\,K(A,A^\dagger)\,. \tag{2.2.8}$$

Two Z factors (for the fields B^i and $\mathcal{B}$) are introduced in (2.2.6), in anticipation of their renormalization group (RG) evolution. In the CP$(N-1)$ model

$$K(A,A^\dagger) = \frac{2}{g^2}\,\log\left(1 + \sum_i^{N-1} A^{\dagger\,i}\,A^i\right),$$

see Eq. (2.2.13).

One can check that the above Lagrangian is target-space invariant. However, the target space invariance is implicit in Eq. (2.2.6) because $K_i(A,A^\dagger)$ is not explicitly target-space invariant. It becomes explicit upon passing to the integration over the Grassmann half-space in the first line of Eq. (2.2.6),

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}\int d\theta\, G_{i\bar{j}}(A,A^\dagger)(\bar{D}A^{\dagger\bar{j}})\left(i\partial_R A^i - 2\kappa\,\mathcal{B}B^i\right) + \text{H.c.}\\ &\quad+\frac{1}{2}\int d^2\theta\left[Z\,G_{i\bar{j}}(A,A^\dagger)\,B^{\dagger\bar{j}}B^i + \mathcal{Z}\,\mathcal{B}^\dagger\mathcal{B}\right].\end{aligned}\tag{2.2.9}$$

The F-term structure $G_{i\bar{j}}(\bar{D}A^{\dagger\bar{j}})i\partial_R A^i$ in the first line is an analog of that for the gauge term $W^\alpha W_\alpha$ in 4D gauge theories, while another F-structure, $G_{i\bar{j}}(\bar{D}A^{\dagger\bar{j}})\,\mathcal{B}B^i$, is an analog of superpotential in 4D. The chiral nature of these terms plays a crucial role in their renormalization. Of course, the second line can also be written as an integral over the Grassmann half-space, so that the Lagrangian takes the form

$$\begin{aligned}&\mathcal{L} = \text{Re}\int d\theta\,\mathcal{F} = \frac{1}{2}\int d\theta\,\mathcal{F} + \text{H.c.}\,,\\ &\mathcal{F} = -\frac{1}{2}\,G_{i\bar{j}}(\bar{D}A^{\dagger\bar{j}})\left(i\partial_R A^i - 2\kappa\,\mathcal{B}B^i\right) - \frac{1}{2}\,\bar{D}\left[Z\,G_{i\bar{j}}\,B^{\dagger\bar{j}}B^i + \mathcal{Z}\,\mathcal{B}^\dagger\mathcal{B}\right].\end{aligned}\tag{2.2.10}$$

In the chiral superfield integrand $\mathcal{F}$ the second line of (2.2.9) produces the derivative term with $\bar{D}$ which is not protected under renormalization.

The target space invariance is also transparent if one rewrites the Lagrangian in components,

$$\begin{aligned}\mathcal{L} &= G_{i\bar{j}}\left[\partial_R\phi^{\dagger\bar{j}}\partial_L\phi^i + \psi_L^{\dagger\bar{j}}\,i\nabla_R\,\psi_L^i + Z\,\psi_R^{\dagger\bar{j}}\,i\nabla_L\psi_R^i\right] + Z\,R_{i\bar{j}k\bar{l}}\,\psi_L^{\dagger\bar{j}}\psi_L^i\,\psi_R^{\dagger\bar{l}}\psi_R^k\\ &\quad+\mathcal{Z}\,\zeta_R^\dagger\,i\partial_L\,\zeta_R + \left[\kappa\,\zeta_R\,G_{i\bar{j}}\left(i\,\partial_L\phi^{\dagger\bar{j}}\right)\psi_R^i + \text{H.c.}\right] + \frac{|\kappa|^2}{Z}\zeta_R^\dagger\,\zeta_R\left(G_{i\bar{j}}\,\psi_L^{\dagger\bar{j}}\psi_L^i\right)\\ &\quad-\frac{|\kappa|^2}{\mathcal{Z}}\left(G_{i\bar{j}}\psi_L^{\dagger\bar{j}}\psi_R^i\right)\left(G_{k\bar{l}}\psi_R^{\dagger\bar{l}}\psi_L^k\right).\end{aligned}\tag{2.2.11}$$

Here $\nabla_{L,R}$ are covariant derivatives, $\nabla_{L,R}\,\psi^i_{R,L} = \partial_{L,R}\,\psi^i_{R,L} + \Gamma^i_{kl}\,\partial_{L,R}\,\phi^k\,\psi^l_{R,L}$. The first line in this equation refers to the undeformed $\mathcal{N}=(2,2)$ theory, the subsequent terms bring in the (0,2) deformation.

Actually all the above equations are applicable to the heterotic deformation of any

Kähler manifold. In the particular case of CP($N-1$) the explicit expression for the Fubini-Study metric and related objects are of the form,

$$
\begin{aligned}
K &= \frac{2}{g^2}\,\log\chi\,, & \chi &= 1+\sum_{m}^{N-1}\phi^{\dagger\,m}\phi^{m}\,, \\
G_{i\bar{j}} &= \frac{2}{g^2}\left(\frac{\delta_{i\bar{j}}}{\chi}-\frac{\phi^{\dagger\,i}\phi^{\bar{j}}}{\chi^2}\right), & G^{i\bar{j}} &= \frac{g^2}{2}\,\chi\left(\delta^{i\bar{j}}+\phi^{i}\phi^{\dagger\,\bar{j}}\right), \\
\Gamma^{i}_{kl} &= -\frac{\delta^{i}_{k}\,\phi^{\dagger\,l}+\delta^{i}_{l}\,\phi^{\dagger\,k}}{\chi}\,, & \Gamma^{\bar{i}}_{\bar{k}\bar{l}} &= -\frac{\delta^{\bar{i}}_{\bar{k}}\,\phi^{\bar{l}}+\delta^{\bar{i}}_{\bar{l}}\,\phi^{\bar{l}}}{\chi}\,, \\
R_{i\bar{j}k\bar{l}} &= -\frac{g^2}{2}\left(G_{i\bar{j}}G_{k\bar{l}}+G_{k\bar{j}}G_{i\bar{l}}\right), & R_{i\bar{j}} &= -G^{k\bar{j}}R_{i\bar{j}k\bar{l}}=\frac{g^2N}{2}\,G_{i\bar{j}}\,.
\end{aligned}
\tag{2.2.12}
$$

2.2.2 Geometry of heterotic deformation

What is the geometrical meaning of the heterotic deformation? For the Kähler manifold M of the complex dimension d (for CP($N-1$) the dimension $d = N-1$) we have d right-moving fermions ψ^{i}_{R}, $i = 1,\ldots,d$, plus ζ_R. They can be viewed as defined on the tangent bundle $T(M\times C)$. Let us denote $\zeta_R = \psi^{d+1}_{R}$. Similarly, for superfields $\mathcal{B} = B^{d+1}$. Then, the Lagrangian for the right-moving fermions can be written as

$$
\mathcal{L}_B = \frac{1}{2}\int d^2\theta\left\{G^{(B)}_{i\bar{j}}B^{\dagger\,\bar{j}}B^{i}+\left[T_{ik}B^{i}B^{k}+\text{H.c.}\right]\right\}\,,\quad (i,k,\bar{j}=1,\ldots,d+1). \tag{2.2.13}
$$

Here the metric $G^{(B)}_{i\bar{j}}$ and antisymmetric potential T_{ik} are functions of the bosonic superfields A^{i}, $A^{\dagger\bar{j}}$ with $i,\bar{j} = 1,\ldots,d$. Comparing with the previous definitions we see that nonvanishing components of $G^{(B)}_{i\bar{j}}$ and T_{ik} are

$$
G^{(B)}_{i\bar{j}} = \begin{cases} ZG_{i\bar{j}}\,, & i,\bar{j}=1,\ldots,d, \\ \mathcal{Z}\,, & i=d+1,\quad \bar{j}=d+1\,, \end{cases} \tag{2.2.14}
$$

$$
T_{(d+1)\,i} = -T_{i\,(d+1)} = -\frac{\kappa}{2}\,K_i\,,\quad i=1,\ldots,d\,. \tag{2.2.15}
$$

The potential T_{ik} is not uniquely defined but its curvature

$$\mathcal{H}_{ik\bar{j}} = T_{ik,\bar{j}} = \frac{\partial T_{ik}}{\partial A^{\dagger\,\bar{j}}} \tag{2.2.16}$$

is a good object. This curvature defines the chiral form for the heterotic modification,

$$\mathcal{L}_B = \frac{1}{2}\int d^2\theta\, G^{(B)}_{i\bar{j}} B^{\dagger\,\bar{j}} B^i - \frac{1}{2}\int d\theta \Big[\mathcal{H}_{ik\bar{j}}(\bar{D}A^{\dagger\bar{j}})B^i B^k + \text{H.c.}\Big]\,, \tag{2.2.17}$$

In the model at hand the nonvanishing components of $\mathcal{H}_{ik\bar{j}}$

$$\mathcal{H}_{(d+1)\,i\bar{j}} = -\mathcal{H}_{i\,(d+1)\,\bar{j}} = -\frac{\kappa}{2}\, G_{i\bar{j}} \qquad (i,\bar{j} = 1,\ldots,d\,) \tag{2.2.18}$$

are expressed via the metric tensor $G_{i\bar{j}}$. It looks even simpler for $\mathcal{H}^j_{ik} = \mathcal{H}_{ik\bar{j}} G^{j\bar{j}}$:

$$\mathcal{H}^j_{(d+1)\,i} = -\mathcal{H}^j_{i\,(d+1)} = -\frac{\kappa}{2}\,\delta^j_i \qquad (i,j = 1,\ldots,d)\,. \tag{2.2.19}$$

The chiral field $\mathcal{F}$ in Eq. (2.2.10) which defines the total Lagrangian, $\mathcal{L} = \text{Re}\int d\theta\mathcal{F}$, can be rewritten in the following generic form:

$$\mathcal{F} = -\frac{1}{2}\left[i\, G_{i\bar{j}}(\bar{D}A^{\dagger\bar{j}})\partial_R A^i + \mathcal{H}_{ik\bar{j}}(\bar{D}A^{\dagger\bar{j}})B^i B^k + \bar{D}\big(G^{(B)}_{i\bar{j}} B^{\dagger\bar{j}} B^i\big)\right]\,. \tag{2.2.20}$$

In differential geometry the heterotic construction presented above can be described as an action of the C^* one-dimensional algebra on the odd tangent vector bundle.[1] Note that there is a diagonal U(1) which rotates B^i and $\mathcal{B}$ in the opposite directions.

The consideration above is applicable to modifying any Kähler manifold, we are not limited to CP$(N-1)$. What is less clear, whether or not it is possible to add more right-moving fermions so that the extra fermionic bundle is not just TC. To this end a three-form $\mathcal{H}$ consistent with the target-space invariance should exist. We are not aware of such generalizations.

[1]We are indebted to Alexander Voronov for explanations.

2.2.3 Holomorphy and its breaking

The deformed $(0,2)$ theory contains four bare parameters:

$$\frac{1}{g^2}\,,\ \frac{\kappa}{g^2}\,,\ Z\,,\ \mathcal{Z}\,. \tag{2.2.21}$$

The first two, $1/g^2$ and κ/g^2, enter as coefficients of the F terms in Eq. (2.2.9) and can be taken to be complex, while parameters Z and $\mathcal{Z}$ should be real. The imaginary part of $1/g^2$ defines the vacuum θ angle, $\text{Im}(1/g^2) = \theta/4\pi$, while the phase of (κ/g^2) produces an addition to this θ angle. These angles do not contribute to physical effects due to the presence of massless fermonic fields whose phase can be redefined.

Nonrenormalization of superpotential (i.e. the κ term in (2.2.9)) implies the absence of loop corrections to the holomorphic coupling κ/g^2, and, in particular, the absence of its running,

$$M_{\text{uv}}\frac{d}{dM_{\text{uv}}}\,\frac{\kappa}{g^2} = 0\,. \tag{2.2.22}$$

This means that the curvature $\mathcal{H}_{ik\bar{j}}$ is the renormalization group invariant tensor with no higher-loop corrections. For a detailed derivation of the nonrenormalization theorem see Ref. [46].

The situation is more complicated for the "main" coupling $1/g^2$ appearing in the target space metric. The coupling in the bare Lagrangian (i.e. at M_{uv}) is holomorphic. This means that it can receive only one-loop renormalization, implying one-loop β function. Much in the same way as in four-dimensional $\mathcal{N}=1$ Yang-Mills theory the holomorphic anomaly showing up in loops defies this theorem. The coupling constant the running of which is calculated in conventional perturbation theory is nonholomorphic. For the time being let us denote it by square brackets $1/[g^2]$, as in [21]. In $\mathcal{N} = (2,2)$ sigma models there is no holomorphy violation, and $1/g^2$ and $1/[g^2]$ coincide.

In other words, in the undeformed (2,2) theory, i.e. at $\kappa = 0$, the holomorphicity of $1/g^2$ is maintained. It implies that only one-loop running of $1/g^2$ is allowed, higher loops are absent. In the 4D case this phenomenon is also known in $\mathcal{N}=2$ gauge theories. With less supersymmetry, i.e. in $\mathcal{N}=1$ gauge theories in 4D, holomorphicity is broken. It happens usually at two-loop level but in certain cases appears already at the level of the first loop, see Refs. [58, 59]. Likewise, our κ term leads to breaking of holomorpicity for $1/g^2$. This happens at the level of the first loop. More specifically, the first loop provides a finite $|\kappa|^2$ correction to $1/g^2$ which then leads to the nonholomorphic running of g^2 in the second loop.

Iteration in κ involves integration over the quantum ζ_R and ψ_R fields in the form of a polarization operator, see Fig. 2.1.

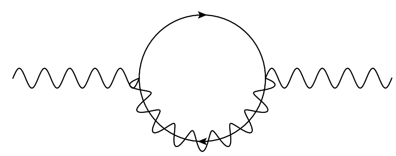

Figure 2.1: One-loop finite correction to the canonical coupling g. The wave line denotes the background field A. The solid line denotes the propagator of B, while the solid line with a wavy line superposed denotes that of $\mathcal{B}$. We shall follow the same notation throughout this chapter.

The polarization operator Π_{RR} is defined as

$$\Pi^{i\bar{j}}_{RR}(x,y) = i|\kappa|^2 \Big\langle T\left\{\zeta_R(x)\psi^i_R(x)\,\psi^{\dagger\bar{j}}_R(y)\zeta^\dagger_R(y)\right\}\Big\rangle_{\rm bck} = i|\kappa|^2 S_\zeta\, S^{i\bar{j}}_{\psi_R}\,, \tag{2.2.23}$$

where S_ζ and S_{ψ_R} are the propagators of ζ and ψ_R fields in the background of the bosonic field ϕ. Referring to the Appendix B for details of calculation we give here the result for Π_{RR},

$$\Pi^{i\bar{j}}_{RR}(x,y) = -\frac{|\kappa|^2}{4\pi Z\mathcal{Z}}\,\langle x|\,G^{i\bar{j}}\,\nabla_R\,\frac{1}{\nabla_L\nabla_R}\,\nabla_R\,|y\rangle\ . \tag{2.2.24}$$

Let us emphasize that there is no ambiguity in the chiral fermion loop for Π_{RR} due to its nonzero Lorentz spin. Generally speaking, the polarization operator $\Pi_{\mu\nu}$ can contain local terms such as $g_{\mu\nu}$ or $\epsilon_{\mu\nu}$. These terms have zero Lorentz spin and do not contribute to Π_{RR}. Note also that in Eq. (2.2.24) the ordering of operators is not important because we neglect by commutator $[\nabla_R,\nabla_L]^i_k = R_k{}^i{}_{m\bar{n}}\big(\partial_R\phi^{\dagger\bar{n}}\partial_L\phi^m - \partial_L\phi^{\dagger\bar{n}}\partial_R\phi^m\big)$. Additional terms with this commutator are infrared ones and do not contribute to the running of couplings we are after.

In Appendix B we also explore an alternative derivation through a relevant UV regularization via modification of the propagator for the ζ fermion. The result for Π_{RR} is the same.

The fermion loop of Fig. 2.1 then results in the following addition to the action,

$$\Delta_\kappa S = \int d^2x d^2y \, G_{i\bar{k}} \partial_L \phi^{\dagger\bar{k}}(x) \, \Pi^{i\bar{j}} i(x,y) \, \partial_L \phi^l(y) G_{l\bar{j}} = -\frac{|\kappa|^2}{4\pi Z \mathcal{Z}} \int d^2x \, G_{i\bar{j}} \partial_L \phi^{\dagger\bar{j}} \partial_R \phi^i \, . \tag{2.2.25}$$

Here we used the relation

$$\nabla_R \, \partial_L \phi = \nabla_L \, \partial_R \phi \, , \tag{2.2.26}$$

which makes the expression for the heterotic correction $\Delta_\kappa \mathcal{L}$ to the original bosonic Lagrangian local,

$$\Delta_\kappa \mathcal{L} = -\frac{|\kappa|^2}{4\pi Z \mathcal{Z}} \, G_{i\bar{j}} \partial_R \phi^{\dagger\bar{j}} \partial_L \phi^i \, . \tag{2.2.27}$$

The resulting correction to the metric $\Delta G_{i\bar{j}}$ can be rewritten in a more geometrical form in terms of the curvature $\mathcal{H}_{ik\bar{j}}$,

$$\Delta_\kappa G_{i\bar{j}} = -\frac{|\kappa|^2}{4\pi Z \mathcal{Z}} \, G_{i\bar{j}} = -\frac{1}{2\pi} \, \mathcal{H}_{lk\bar{j}} \bar{\mathcal{H}}_{\bar{l}\bar{k}i} G^{(B)l\bar{l}} G^{(B)k\bar{k}} = -\frac{1}{2\pi} \, \mathcal{H}_{lk\bar{j}} \bar{\mathcal{H}}_i^{\ lk} \, , \tag{2.2.28}$$

where the indices in $\bar{\mathcal{H}} = \mathcal{H}^*$ are raised by the $G^{(B)i\bar{j}}$ metric tensor (inverse to $G^{(B)}_{i\bar{j}}$ defined in (2.2.14)).

Equation (2.2.27) clearly demonstrates the breaking of holomorphicity by the fer-mion loop depicted in Fig. 2.1. This loop is also related to the axial anomaly in the fermionic current. Indeed, as demonstrated in Appendix B, while classically $\nabla_L \big(\zeta_R \psi_R^i \big) = 0$,[2] for the regularized loop of Π_{RR} we get

$$\nabla_L \Pi_{RR}^{i\bar{j}} = -\frac{|\kappa|^2}{4\pi Z \mathcal{Z}} \, \langle x | \, \nabla_R \, G^{i\bar{j}} \, | y \rangle \ . \tag{2.2.29}$$

Correspondingly we claim the absence of higher-loop corrections to Eq. (2.2.27).

From the above considerations we see that perturbation theory is governed by two real

[2] Strictly speaking this divergence is not vanishing classically but additional terms do not contribute to Π_{RR}.

couplings, $[g^2]$ and a real nonholomorphic combination [3]

$$h^2 = \frac{|\kappa|^2}{Z\mathcal{Z}}\,. \tag{2.2.30}$$

We will also use the ratio ρ of the couplings,

$$\rho \equiv \frac{h^2}{g^2}\,. \tag{2.2.31}$$

Then Eq. (2.2.27) implies

$$\frac{1}{g^2} - \frac{1}{4\pi}\,[\rho] = \frac{1}{[g^2]}\,. \tag{2.2.32}$$

It is convenient to rewrite Eq. (2.2.32) as

$$\frac{1}{g^2} = \frac{1}{[g^2]} + \frac{1}{4\pi}[\rho]\,, \tag{2.2.33}$$

where the holomorphic coupling (i.e. renormalized only at one loop) on the left-hand side is presented as a combination of two nonholomorphic terms.

2.3 Beta functions

2.3.1 Generalities

Considering two, introduced above, couplings, $[g^2]$ and h^2 as functions of the UV cut-off $M_{\rm uv}$ we define two β functions:

$$\beta_g \equiv \frac{d\,[g^2](M_{\rm uv})}{dL}\,, \qquad \beta_h \equiv \frac{d\,h^2(M_{\rm uv})}{dL}\,, \qquad L = \log\, M_{\rm uv}\,. \tag{2.3.1}$$

In what follows we will use also the β function for ρ, see Eq. (2.2.31),

$$\beta_\rho \equiv \frac{d\,[\rho](M_{\rm uv})}{dL}\,. \tag{2.3.2}$$

[3] The definition of h^2 in this chapter corresponds to $\gamma^2 g^4$ in [20]. The reason for this rescaling of the deformation parameter compared to [20] is that g^2 and h^2 as defined here are the genuine loop expansion parameters.

We will omit below the square brackets in $[g^2]$ dealing with the 1PI definition of couplings.

As was discussed above nonrenormalization of the superpotential in Eq. (2.2.9) implies that the ratio κ/g^2 does not depend on $M_{\rm uv}$, see Eq. (2.2.22). This equation can be rewritten as

$$\frac{d}{dL}\,\frac{|\kappa|^2}{g^4} = \frac{d}{dL}\,\frac{h^2 Z\mathcal{Z}}{g^4} = \frac{h^2 Z\mathcal{Z}}{g^4}\left[\frac{\beta_h}{h^2} - 2\,\frac{\beta_g}{g^2} - \gamma\right] = 0\,, \tag{2.3.3}$$

where the anomalous dimension γ is defined as

$$\gamma \equiv -\frac{d\,\log\,(Z\mathcal{Z})}{dL} = \gamma_{\psi_R} + \gamma_\zeta\,, \quad \gamma_{\psi_R} \equiv -\frac{d\,\log\,Z}{dL}\,, \quad \gamma_\zeta \equiv -\frac{d\,\log\,\mathcal{Z}}{dL}\,. \tag{2.3.4}$$

This fixes the β function for h^2 in terms of the β function for g^2 and the sum of the anomalous dimensions for B^i and $\mathcal{B}$ fields,

$$\beta_h = h^2\left[\frac{2}{g^2}\,\beta_g + \gamma\right]\,. \tag{2.3.5}$$

For β_ρ we get

$$\beta_\rho = \rho\left[\frac{1}{g^2}\,\beta_g + \gamma\right]\,. \tag{2.3.6}$$

2.3.2 Beta functions at one loop

The relations (2.3.5) and (2.3.6) are exact to all loops. At the one-loop level all β functions and anomalous dimensions have been calculated earlier [45, 46]:

$$\beta_g^{(1)} = -T_G\,\frac{g^4}{4\pi}\,, \quad \gamma_\zeta^{(1)} = d\,\frac{h^2}{2\pi}\,, \quad \gamma_{\psi_R}^{(1)} = \frac{h^2}{2\pi}\,, \quad \gamma^{(1)} = (d+1)\,\frac{h^2}{2\pi}\,, \tag{2.3.7}$$

$$\beta_h^{(1)} = -\frac{h^2}{2\pi}\left[T_G g^2 - (d+1)h^2\right]\,, \quad \beta_\rho^{(1)} = (d+1)\,\frac{h^2}{2\pi}\left[\rho - \frac{T_G}{2(d+1)}\right]\,. \tag{2.3.8}$$

The results are for the Kähler manifolds of the complex dimension d, which are homogeneous spaces G/H, and T_G is a dual Coxeter number of the group G. It is a straightforward generalization of calculations of Refs. [45, 46], where the CP$(N-1)$ sigma model was considered, for which $d = N-1$ and $T_G = T_{\rm SU(N)} = N$.

An interesting feature of these one-loop results is that they exhibit a fixed point at

$\rho = \rho_c = T_G/2(d+1)$, which becomes $\rho_c = 1/2$ for CP$(N-1)$. At this point

$$\frac{\beta_g^{(1)}}{g^2}\Big|_{\rho=\rho_c} = \frac{\beta_h^{(1)}}{h^2}\Big|_{\rho=\rho_c} = -\gamma^{(1)}\Big|_{\rho=\rho_c} = -\frac{T_G g^2}{4\pi}\,, \qquad \rho_c = \frac{T_G}{2(d+1)}\,. \tag{2.3.9}$$

In terms of the geometrical interpretation we can present all one-loop results as corrections to the bosonic metric $G_{i\bar{j}}$, and to the right-moving fermion extended metric $G^{(B)}_{i\bar{j}}$. These one-loop corrections are

$$\begin{aligned} \Delta G_{i\bar{j}}\big|^{\text{one-loop}} &= -\frac{1}{2\pi}\left\{\mathcal{H}_{\,lk\bar{j}}\bar{\mathcal{H}}_{\bar{l}\bar{k}i}G^{(B)l\bar{l}}G^{(B)k\bar{k}} + R_{i\bar{j}}\log M_{\text{uv}}\right\}, \\ \Delta G^{(B)}_{i\bar{j}}\big|^{\text{one-loop}} &= -\frac{\log M_{\text{uv}}}{2\pi}\left\{4\mathcal{H}_{\,ik\bar{l}}\bar{\mathcal{H}}_{\bar{j}\bar{k}l}G^{(B)k\bar{k}}G^{(B)l\bar{l}} + R^{(B)}_{i\bar{j}}\right\}. \end{aligned} \tag{2.3.10}$$

As mentioned above there are no loop corrections of any order to the heterotic curvature tensor $\mathcal{H}_{ik\bar{j}}$.

Let us parenthetically note that the parameter (κ/g^2) is related to δ introduced in [20, 42, 44], where the large N solution for CP$(N-1)$ was constructed. Namely,

$$\frac{\kappa}{g} = \delta\,, \tag{2.3.11}$$

see Erratum to [44]. The δ parameter appears as the coefficient in a superpotential, see Eq. (C5) in [44], and, as such, is also complexified.

In [44] it is shown that the physical parameter determining (0,2) deformation is

$$u = \frac{16\pi}{N}\,\frac{\delta^2}{g^2} = \frac{16\pi}{Ng^2}\frac{\kappa^2}{g^2}\,, \tag{2.3.12}$$

implying that (a) u is proportional to κ^2/g^4 and, hence, is renormalization group invariant, as was expected; (b) at large N the physical parameter u scales as N^0 while κ^2 and g^2 both scale as $1/N$. The anomalous dimension γ (see Eq. (2.3.4)) then scales as $O(N^0)$, and β_g becomes one-loop exact in the limit $N \to \infty$, see Eq. (2.1.5).

2.4 Beyond one loop from instanton calculus

In this section we will briefly outline the instanton derivation of the β functions along the lines of [21, 22, 46, 47]. In particular, we will use the nonrenormalization theorem for the second and higher loops in the instanton background. Only zero modes and the one-loop contribution have to be considered. All parameters that will appear in the derivation below are those from the bare Lagrangian.

To warm up let us briefly review the instanton calculation in [22]. Consider the instanton measure in four-dimensional $\mathcal{N}=1$ Yang-Mills theory with the SU(N) gauge group. In the quasiclassical approximation the renormalization group invariant (RGI) prefactor in the instanton measure is

$$\mu_{\text{inst}}^{(1)} = M_{\text{uv}}^{3N} \exp\left(-\frac{8\pi^2}{g^2}\right) . \tag{2.4.13}$$

The factor in the exponent is the classical instanton action, while the pre-exponential factor comes from the zero modes. There are $n_B = 4N$ bosonic zero modes and $n_F = 2N$ fermion ones to produce $M^{n_B}/M^{n_F/2} = M^{3N}$. In terms of perturbation theory Eq. (2.4.13) gives us the one-loop running of the holomorphic coupling. What happens in higher loops?

The only change (to all orders in the coupling constant) is the emergence of another pre-exponential factor $g^{n_F}/g^{n_B} = 1/g^{2N}$, due to normalization of the zero modes, namely,

$$\mu_{\text{inst}}^{\text{exact}} = M_{\text{uv}}^{3N} \frac{1}{[g^2]^N} \exp\left(-\frac{8\pi^2}{[g^2]}\right) . \tag{2.4.14}$$

Simultaneously, $1/[g^2]$ in the exponent becomes nonholomorphic. The combination (2.4.14) of M_{uv} and $[g^2](M_{\text{uv}})$ is renormalization group invariant. Differentiating over $\log M_{\text{uv}}$ we arrive at the NSVZ β function for SU(N). Generalization to an arbitrary gauge group G is just a substitution of N in expressions above by the dual Coxeter number T_G.

If matter fields are added, the only further changes in $\mu_{\text{inst}}^{\text{exact}}$ are as follows: (i) the power of M_{uv} is changed appropriately; (ii) $Z^{-1/2}$ factor appears in the pre-exponent for each matter-sector fermion zero mode. The number of such fermion zero modes is given by $2T(R)$ where $T(R)$ is the Dynkin index of representation R. In this way one obtains the full exact NSVZ β function,

$$\beta_{\rm NSVZ}(g^2) = -\frac{g^4}{8\pi^2}\left[3\,T_G - \sum_{\rm matter} T(R_i)(1-\gamma_i)\right]\left(1-\frac{T_G\,g^2}{8\pi^2}\right)^{-1}\,. \tag{2.4.15}$$

Now, let us see how the same strategy can be implemented in the κ deformed CP($N-1$) sigma model under consideration. Let us start first with the non-deformed (2,2) case. At the classical level the instanton-generated exponent is

$$\exp\left(-\frac{4\pi}{g^2}\right)\,, \tag{2.4.16}$$

see e.g. [6]. At the one-loop level (and at higher loops as well) nonzero modes cancel out. In the CP($N-1$) model there are $n_B = 2N$ bosonic zero modes and $n_F = 2N$ fermion zero modes, This produces the $M^{n_B}/M^{n_F/2} = M^N$ pre-exponential factor. As for normalization of the zero modes the corresponding factors cancel out between bosonic and fermion modes, $g^{n_B}/g^{n_F} = 1$. Thus, we come to

$$\mu_{\rm inst}(2,2) = M^N\exp\left(-\frac{4\pi}{g^2}\right) = {\rm RGI}\,, \tag{2.4.17}$$

which leads to the one-loop exact β function and unbroken holomorphicity in the (2,2) theory.

Now let us switch on the κ modification. As we discussed in Sec. 2.2.3 the holomorphicity is broken in perturbation theory already in the order $|\kappa|^2$. Here comes a surprise: such breaking does not occur in the instanton background!

Indeed, the κ terms in the Lagrangian (2.2.11) has the form

$$i\kappa\, G_{i\bar{j}}\,\zeta_R\,\psi_R^i\,\partial_L\phi^{\dagger\bar{j}} - i\kappa^* G_{i\bar{j}}\,\psi_R^{\bar{j}}\,\zeta_R\,\partial_L\phi^i\,; \tag{2.4.18}$$

the product of these two terms enters in the fermion loop calculation. After Euclidean continuation the instanton (or anti-instanton) background leads to vanishing either $\partial_L\phi^{\dagger\bar{j}}$ or $\partial_L\phi^i$. Therefore, the $|\kappa|^2$ iteration is not possible. It means that holomorphicity is not broken in one loop for the instanton, and the instanton action stays $4\pi/g^2$ with the original $1/g^2$. In terms of the running $1/[g^2]$ and $[\rho]$ it means that the combination (2.2.33) enters into the instanton exponent,

$$\exp\left(-\frac{4\pi}{g^2}\right) = \exp\left(-\frac{4\pi}{[g^2]} - [\rho]\right). \tag{2.4.19}$$

Moreover, in the instanton background the only effect of an additional right-mover ζ_R is its admixture to ψ_R. This triangle mixing does not change the eigenvlues so the cancellation of non-zero modes stays the same as in the (2,2) case. Also, the counting of the zero modes does not change. What appears in the pre-exponential factor is an additional $Z^{-1/2}$ factor for each zero mode of ψ_R because of its normalization [37]. There are N such modes so we arrive at the following exact expression for the measure:

$$\mu_{\rm inst}^{\rm exact} = \frac{1}{Z^{N/2}}\, M_{\rm uv}^N \,\exp\left(-\frac{4\pi}{[g^2]} - [\rho]\right)\,. \tag{2.4.20}$$

The Z factor is defined in (2.2.9), see the first term in the second line. All effects due to two loops and higher, associated with nonzero modes, cancel [46,47] much in the same way as in the (2,2) case.

Equation (2.4.20) implies

$$T_G \log M_{\rm uv} - \frac{T_G}{2}\log Z - \frac{4\pi}{[g^2]} - [\rho] = {\rm RGI}\,, \tag{2.4.21}$$

where we substitute N by the Coxeter index to consider a generic Kähler manifold G/H.

Differentiating over $\log M_{\rm uv}$ and using Eqs. (2.3.4) and (2.3.6) for β_ρ we arrive at the full exact β function (2.1.5), relating β_g to the anomalous dimensions, much in the same way as the NSVZ formula.

Combining the full β function in (2.1.5) with Eqs. (2.3.5) and (2.3.6) we derive the "secondary" β functions,

$$\begin{aligned}
\beta_h &= -\frac{h^2}{1-(h^2/4\pi)}\left[T_G\,\frac{g^2}{2\pi}\left(1+\frac{1}{2}\gamma_{\psi_R}\right) - \gamma\left(1+\frac{h^2}{4\pi}\right)\right],\\
\beta_\rho &= -\frac{\rho}{1-(h^2/4\pi)}\left[T_G\,\frac{g^2}{4\pi}\left(1+\frac{1}{2}\gamma_{\psi_R}\right) - \gamma\right].
\end{aligned} \tag{2.4.22}$$

2.5 Explicit two-loop calculations

2.5.1 Two-loop β function for g^2

In this section we will use the superfield method to calculate two-loop beta function for g^2 for the heterotic model at any symmetric Kähler target space. We will use a linear background field method, setting the background field $A_{bk} = fe^{-ix\cdot k}$ (see the review paper [60]). The basic method is roughly the same as that of component field. We expand the action around the chosen background, and calculate all relevant diagrams. To maintain supersymmetry, we use supersymmetric dimensional reduction, which in turn reduces to dimensional regularization in our case. Note that since we are only interested in the renormalization of the canonical coupling, this is compatible. Also we will keep the vector current of the theory conserved. Due to the computational nature, it would not be beneficial to show all steps in detail here. Instead, we will offer some intuitive arguments for the reader to understand our results. For a detailed description of the calculational method and examples, the reader is referred to [60].

To keep our discussion concise, we will only show those Feynman diagrams that are of the leading order with respect to target space curvature (i.e. assuming ϕ and $\phi^\dagger$ to be small).

At two-loop level, the correction of the order g^4 is obviously absent, as predicted by the undeformed model. For the correction of the order g^2h^2 and h^4, the relevant diagrams are those shown in Fig. 2.2 and Table 2.1, at leading order (with respect to the covariant structure) contributed by the superfield A, which renormalizes g^2.

Now it is rather straightforward to show that these diagrams, together with the Hermitian conjugated part, give rise to the following expression:

$$\begin{aligned}\frac{1}{g^2(\mu)} = \quad & \frac{1}{g_0^2}\left[1 - \frac{h_0^2}{4\pi} - \frac{T_G}{2}\, g_0^2 I\right] \\ & + \frac{1}{g_0^2}\left[-\frac{T_G}{4\pi}\, g_0^2 h^2 I + \frac{d+1}{4\pi}\, h_0^4\, I\right],\end{aligned} \tag{2.5.1}$$

where

$$I = \frac{1}{2\pi}\log\left(\frac{M_{\mathrm{uv}}}{\mu}\right). \tag{2.5.2}$$

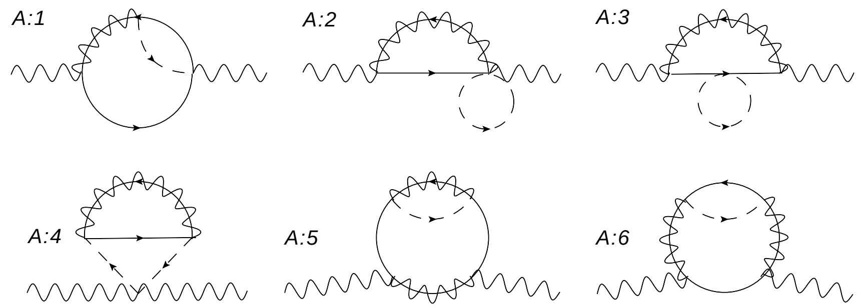

Figure 2.2: Two-loop correction to the canonical coupling g. The dashed line denotes the quantum propagator of A.

For a more accurate definition of I in (2.5.2) in terms of a dimensionally regularized loop integral see Refs. [45,60].

The first line in (2.5.1) contains the one-loop contributions and the second one is the result for two-loop diagrams. From (2.5.1) we get the two-loop β function,

$$\beta_g^{(2)} = -\frac{g^2}{4\pi}\left[T_G g^2\left(1+\frac{h^2}{2\pi}\right) - (d+1)\,\frac{h^4}{2\pi}\right] . \tag{2.5.3}$$

This coincides with the corresponding expansion of the master expression (2.1.5). The one-loop expressions (2.3.7) for the anomalous dimensions are sufficient for this comparison.

2.5.2 Anomalous dimensions at two loops

Now we turn to the calculation of the β function of the deformation coupling h. To this end we will have to understand anomalous dimensions of the fermionic fields ψ_R and ζ_R first. At one-loop level they are given in Eq. (2.3.7), see Fig. 2.3 for the corresponding diagrams.

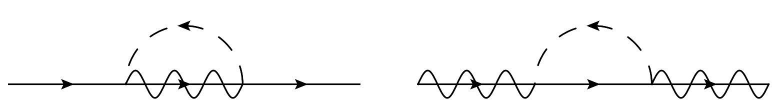

Figure 2.3: One-loop correction to wave function renormalization of ψ_R and ζ_R.

At two-loop level we know that at the order g^4 there is no correction. At the orders

Diagram	Double pole	Single pole
A:1	0	$-T_G\frac{g_0^2 h_0^2}{4\pi}I$
A:2	0	$T_G\frac{g_0^2 h_0^2}{4\pi}I$
A:3	0	$-\frac{T_G}{2}\frac{g_0^2 h_0^2}{4\pi}I$
A:4	0	$-\frac{T_G}{2}\frac{g_0^2 h_0^2}{4\pi}I$
A:5	0	$\frac{h_0^4}{4\pi}I$
A:6	0	$d\,\frac{h_0^4}{4\pi}I$

Table 2.1: Two-loop calculation for the g^2 correction. The labeling of the diagrams follows that in Fig. 2.2.

g^2h^2 and h^4 we have the diagrams (in superfields) shown in Fig. 2.4 and Fig. 2.5 that contribute to γ_ζ and γ_{ψ_R}, respectively.

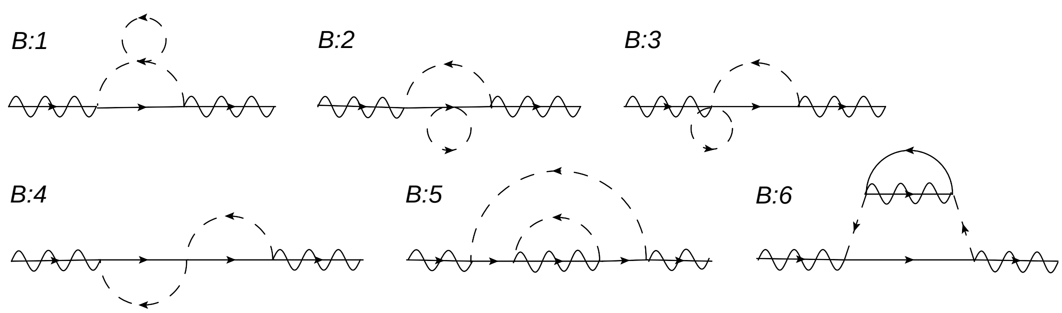

Figure 2.4: Two-loop corrections in the wavefunction renormalization of ζ_R.

The renormalization of ζ_R is easier to understand as $\mathcal{Z}$ is obtained by evaluating all diagrams in Fig. 2.4. Assembling them all we have

$$\mathcal{Z} = 1 + d\,h_0^2 I + d\,\frac{h_0^4}{4\pi}\,I + d\,T_G\,\frac{h_0^2 g_0^2}{2}I^2 - d\,\frac{h_0^4}{2}I^2\,. \tag{2.5.4}$$

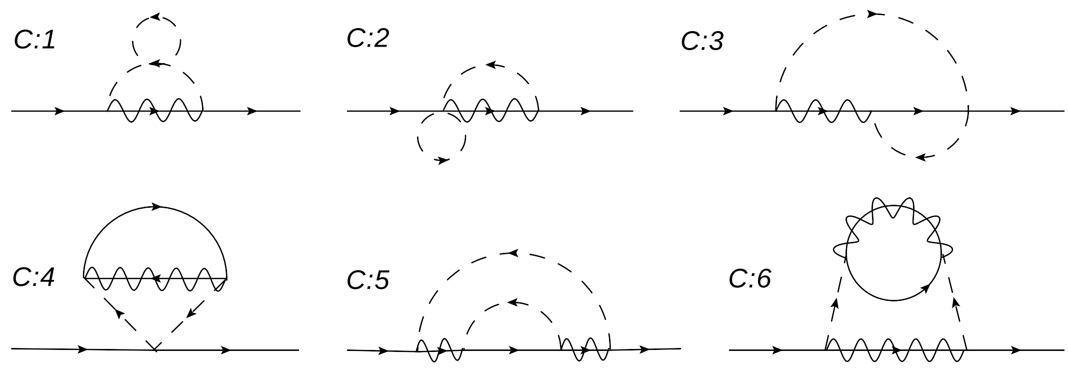

Figure 2.5: Two-loop corrections to the wavefunction renormalization of ψ_R .

The two-loop anomalous dimension γ_ζ then can be written as

$$\gamma_\zeta^{(2)} = -\frac{1}{\mathcal{Z}}\,\mu\frac{d\mathcal{Z}}{d\mu} = d\,\frac{h_0^2}{2\pi}\left(1+\frac{h_0^2}{4\pi}\right)\left[1+I\big(T_G g_0^2-(d+1)h_0^2\big)\right] \tag{2.5.5}$$

The second factor in r.h.s. (the square brackets) just shifts h_0^2 to $h^2(\mu)$ in accord with the one-loop β_h given in Eq. (2.3.8). Thus, we get

$$\gamma_\zeta^{(2)} = d\,\frac{h^2}{2\pi}\left(1+\frac{h^2}{4\pi}\right). \tag{2.5.6}$$

In case of wavefunction renormalization of ψ_R it should be noted that the diagrams shown in Fig. 2.5 in fact do not directly contribute to Z, but, rather, to Z/g^2. Therefore, we have

$$\frac{Z}{g^2} = \frac{1}{g_0^2}\left[1-T_G\,\frac{g_0^2}{2}\,I+h_0^2 I+\frac{h_0^4}{4\pi}\,I-d\,\frac{h_0^4}{2}I^2-T_G\,\frac{h_0^2 g_0^2}{8\pi}I\right]. \tag{2.5.7}$$

Using Eq. (**??**) for $1/g^2$ we get for Z,

$$Z = 1+h_0^2 I+T_G\,\frac{h_0^2 g_0^2}{8\pi}\,I-d\,\frac{h_0^4}{4\pi}\,I+T_G\frac{g_0^2 h_0^2}{2}\,I^2-d\,\frac{h_0^4}{2}I^2\,. \tag{2.5.8}$$

It leads to the following two-loop anomalous dimension of ψ_R ,

$$\gamma_{\psi_R}^{(2)} = -\frac{1}{Z}\,\mu\frac{dZ}{d\mu} = \frac{h_0^2}{2\pi}\left(1+T_G\,\frac{g_0^2}{8\pi}-d\,\frac{h_0^2}{4\pi}\right)\left[1+I\big(T_G g_0^2-(d+1)h_0^2\big)\right] \tag{2.5.9}$$

Again, the factor in the square brackets containing I just shifts h_0^2 to $h^2(\mu)$, so

$$\gamma^{(2)}_{\psi_R} = \frac{h^2}{2\pi}\left(1 + T_G\,\frac{g^2}{8\pi} - d\,\frac{h^2}{4\pi}\right). \tag{2.5.10}$$

2.5.3 Beta functions and fixed point in ρ

The calculated two-loop anomalous dimensions mean that we know β_g at three-loop level. The explicit expression for $\beta_g^{(3)}$ follows from substitution of the anomalous dimensions (2.5.6) and (2.5.10) into the master formula (2.1.5),

$$\beta_g^{(3)} = -\frac{g^2/4\pi}{1-(h^2/4\pi)}\left[T_G\,g^2 + \frac{h^2}{4\pi}\Big(T_G g^2 - 2(d+1)h^2\Big)\left(1 + T_G\,\frac{g^2}{8\pi}\right)\right]. \tag{2.5.11}$$

For β_h and β_ρ we get the two-loop expressions,

$$\beta_h^{(2)} = -\frac{h^2/2\pi}{1-(h^2/4\pi)}\left[T_G g^2 - (d+1)h^2 + \frac{h^2}{8\pi}\Big(T_G g^2 - 2(d+1)h^2\Big)\right], \tag{2.5.12}$$

$$\beta_\rho^{(2)} = (d+1)\,\frac{g^2}{2\pi}\,\frac{\rho}{1-(h^2/4\pi)}\left(\rho - \frac{T_G}{2(d+1)}\right). \tag{2.5.13}$$

The expression for β_ρ differs from the one-loop expression (2.3.7) only by a factor, so the fixed point $\rho_c = T_G/2(d+1)$ stays intact. Certainly, it would be interesting to find a geometrical interpretation of this fixed point but we do not have an answer for this yet.

At this point

$$\beta_g^{(3)}\Big|_{\rho=\rho_c} = -T_G\,\frac{g^4}{4\pi}\,\frac{1}{1-(h^2/4\pi)}\,, \qquad \beta_h^{(2)}\Big|_{\rho=\rho_c} = -T_G\,\frac{h^2 g^2}{4\pi}\,\frac{1}{1-(h^2/4\pi)} \tag{2.5.14}$$

differ only by a factor $1/(1-(h^2/4\pi))$ from the corresponding one-loop results.

2.6 Isometries of the model

In this section we study the isometries of the heterotic models and, in particular, address the question whether they could be broken by loop corrections. For generic sigma model there are the following symmetry transformations of bosonic fields ϕ^i, $\phi^{\dagger\,\bar{j}}$ living on the

Kähler target space:

$$\phi^{\,i} \to \phi^{\,i} + \epsilon^{\,A} V^i_A(\phi)\,, \quad \phi^{\dagger\,\bar{i}} \to \phi^{\dagger\,\bar{i}} + \epsilon^{\,A}\,\overline{V}^{\bar{i}}_A(\phi^\dagger)\,, \tag{2.6.1}$$

where the vector V^i_A is the Killing vector over the target manifold, ϵ^A are real infinitesimal parameters, and the index A labels isometries. Note that in the Kähler cases, the Killing vector V_A has only holomorphic dependence on the bosonic field ϕ.

We are dealing with symmetric homogeneous spaces G/H. Correspondingly, isometries arising from the algebra of H are realized linearly, while the remaining generators in the algebra of the group G are realized nonlinearly, these symmetries are spontaneously broken. For example, in $\mathrm{CP}(N{-}1) = \mathrm{SU}(N)/\mathrm{S}(\mathrm{U}(N{-}1)\times \mathrm{U}(1))$, we have $(N{-}1)^2$ linear symmetries corresponding to $\mathrm{U}(N-1)$ rotations of fields $\{\phi^i, \phi^{\dagger\bar{j}}\}$. The remaining $2N-2$ symmetries are nonlinearly realized. They can be written as

$$\phi^i \to \phi^i + \epsilon^{i\bar{j}}\phi_{\bar{j}} + \beta^i + (\beta^\dagger\phi)\phi^i\,, \quad \phi^{\dagger\bar{j}} \to \phi^{\dagger\bar{j}} - \epsilon^{i\bar{j}}\phi^\dagger_i + \beta^{\dagger\bar{j}} + (\beta\phi^\dagger)\phi^{\dagger\bar{j}}\,, \tag{2.6.2}$$

where the indices of charts $\{\phi^i, \phi^{\dagger\bar{j}}\}$ locally are raised or lowered by $\delta^{i\bar{j}}$ or $\delta_{i\bar{j}}$.

One can supersymmetrize the above model, and write down the general form in the $\mathcal{N}=(2,2)$ case. It can be done in terms of superfields by simply promoting ϕ^i and $\phi^{\dagger\bar{j}}$ to chiral and antichiral superfields. In components, the fermions ψ^i living on tangent space of $\mathrm{CP}(N{-}1)$ transform as tensors corresponding to isometries

$$\psi^i_{R,L} \to \psi^i_{R,L} + \epsilon^\alpha \partial_j V^i_A(\phi)\psi^j_{R,L}\,. \tag{2.6.3}$$

Turning on heterotic deformation does not change the isometries, the additional fermion field ζ_R is a singlet of the group G action. These symmetries can be verified classically in the geometric formulation of the Lagrangian (2.2.17), as long as the curvature $\mathcal{H}_{ik\bar{j}}$ satisfies:

$$\mathcal{L}_A\mathcal{H} = 0\,, \tag{2.6.4}$$

where $\mathcal{L}_A$ is the Lie derivative with respect to the A-th isometry. In the heterotic case, the only nontrivial components of $\mathcal{H}_{ik\bar{j}}$ are proportional to the metric $G_{i\bar{j}}$, see Eq. (2.2.18). It apparently satisfies the above condition. However, the heterotic coupling leads to a change in the expression for the isometry current J_R^A as compared with the (2,2) model,

$$J_R^A = \frac{1}{2}\,\bar{V}_A^{\bar{j}} G_{i\bar{j}} \partial_R \phi^i + \frac{i}{2}\,\nabla_k V_A^i G_{i\bar{j}} \psi_R^{\dagger\bar{j}} \psi_R^k + i\kappa\,\bar{V}_A^{\bar{j}} G_{i\bar{j}} \zeta_R \psi_R^i + \text{H.c.}\,, \tag{2.6.5}$$

while J_L^A does not change.

The question to ask is whether or not the deformation under consideration would deform the classical geometry, since now the chiral fermion ζ_R enters these currents. To answer this, we need to see if these isometric currents have anomalies. It could be verified either by calculating anomalies of these currents or by checking the isometry transformations of the effective action after the one-loop correction. We will proceed along the second route because it is easier and more transparent to demonstrate the isometry invariance in the effective action.

Moreover, even in case when isometry currents happened to be anomalous it does not imply breaking of isometries, anomaly could be a total derivative and does not lead to non-conservation for corresponding generators, at least, in perturbation theory. For this reason examination of the effective action is preferable.

It is worth noting that, if there is any anomaly, it happens due to the fermionic loops. Therefore, we can choose non-zero only bosonic background, and consider the fermion loop corrections to the effective action. Up to one-loop order, keeping terms bilinear in fermionic fields is sufficient. As a result the relevant part of Lagrangian takes the form

$$\begin{aligned} \mathcal{L}_{\text{ferm}} = {} & \mathcal{Z}\,\zeta_R^\dagger\left(1 + \frac{\partial_\mu\partial^\mu}{M^2}\right) i\partial_L\,\zeta_R + G_{i\bar{j}}\left[\psi_L^{\dagger\bar{j}}\,i\nabla_R\,\psi_L^i + Z\psi_R^{\dagger\bar{j}}\,i\nabla_L\psi_R^i\right] \\ & + \left[\kappa\,\zeta_R\,G_{i\bar{j}}\left(i\,\partial_L\phi^{\dagger\bar{j}}\right)\psi_R^i + \text{H.c.}\right], \end{aligned} \tag{2.6.6}$$

where all bosonic fields are background, while fermionic fields are to be integrated out. We also introduced here regularization for the ζ_R field by introducing higher derivatives. This regularization which makes loops with ζ_R convergent is clearly consistent with isometries. It proves then that the heterotic modification of the (2,2) theory does not break any (2,2) isometry.

Our explicit one-loop calculation in Appendix B confirms this. In addition, we also present here geometry of the target space introducing vielbeins $e^a{}_i$ and $\bar{e}_{\bar{j}}{}^{\bar{b}}$ to factorize the metric tensor $G_{i\bar{j}}$ and make sure our effective action preserves explicit geometric structure. Since the fermion fields naturally live on the tangent space, we also redefine the fermions ψ to transform the Lagrangian to the canonical form,

$$\begin{gathered} e^{\,a}_{\;i}\, e^{\,i}_{\;b} = \delta^{\,a}_{\;b}\,, \quad \bar{e}^{\,\bar{j}}_{\bar{a}}\, \bar{e}^{\,\bar{b}}_{\bar{j}} = \delta^{\,\bar{b}}_{\bar{a}}\,, \quad \delta_{a\bar{b}}\, e^{\,a}_{\;i}\, \bar{e}^{\,\bar{b}}_{\bar{j}} = G_{i\bar{j}}\,; \\ \psi^a_{R,L} \equiv e^a_{\;i}\, \psi^i_{R,L}, \quad \bar{\psi}^{\bar{b}}_{R,L} \equiv \bar{\psi}^{\bar{j}}_{R,L}\, \bar{e}^{\,\bar{b}}_{\bar{j}}\,, \end{gathered} \tag{2.6.7}$$

where the tensor indices $\{i, \bar{j}\}$ of vielbein are lowered or raised by the metric $G_{i\bar{j}}$ and $G^{\bar{j}i}$, and the frame indices $\{a, \bar{b}\}$ by the flat metric $\delta_{\bar{b}a}$ and $\delta^{a\bar{b}}$. After these redefinitions the Lagrangian can be rewritten as

$$\mathcal{L}_{\,\mathrm{ferm}} = \mathcal{Z}\zeta^\dagger_R\Big(1+\frac{\partial_\mu\partial^\mu}{M^2}\Big)i\partial_L\zeta_R + \psi^\dagger_{La}\, i\widetilde{\nabla}_R\, \psi^a_L + Z\psi^\dagger_{Ra} i\widetilde{\nabla}_L \psi^a_R + [i\kappa\,\zeta_R\,\bar{e}_{La}\,\psi^a_R + \mathrm{H.c.}]\,, \tag{2.6.8}$$

where

$$\widetilde{\nabla}_{R,L}\,\psi^a_{L,R} = \partial_{R,L}\psi^a_{L,R} + \Omega_{R,L}{}^{a}_{\;c}\,\psi^c_{L,R}$$

and

$$\bar{e}_{La} = \bar{e}_{\bar{j}a}\partial_L \phi^{\dagger\bar{j}}\,.$$

Moreover, $\Omega^a_{R,Lb}$ and $\bar{e}_{La}$ are pull-back spin-connection on the frame bundle and vielbeins, respectively. We express the fermion kinetic term canonically, and the isometries are realized in terms of the frame bundle indices $\{a, \bar{b}\}$ rather than $\{i, \bar{j}\}$.

Next we find the isometry transformations on fermions, vielbeins and spin-connection. It is actually clear how the transformation must look like. Geometrically, once we perform the isometry transformation, there will be effectively an induced rotation on the frame bundle. Now fermions and vielbeins are matter type fields, while spin-connections are gauge fields with respect to U$(N-1)$ gauge symmetries. Therefore, the general form of isometry transformation is

$$\begin{aligned} &\delta\psi^a_{R,L} = v^{\,a}_{A\;\;c}\,\psi^c_{R,L}\,, \quad &&\delta\bar{\psi}_{R,L\,a} = -\bar{\psi}_{R,L\,c}\, v^{\,c}_{A\;\;a}\,, \\ &\delta e^a_L = v^{\,a}_{A\;\;c}\, e^c_L\,, &&\delta\bar{e}_{La} = -\bar{e}_{L\,c} v^{\,c}_{A\;\;a}\,, \\ &\delta\Omega_{R,L}{}^{a}_{\;c} = -\partial_{R,L} v^{\,a}_{A\;\;c} - \big[\Omega_{R,L}\,, v_A\big]^a_{\;\;c}\,. \end{aligned} \tag{2.6.9}$$

The explicit expression of $v_{A\ c}^{a}$ can be found from Eqs. (2.6.2) and (2.6.3), once the vielbeins are given. However, the explicit form of $v^{a}_{\ c}$ is not significant, Eq. (2.6.9) is all we need.

Now, when isometries of the Lagrangian are verified and regularization provides convergence of fermion loop integration we can claim that all original target space isometries are preserved under the heterotic modification. Furthermore the transformation rule of isometries Eq. (2.6.9) is a special case of general holonomy transformations on frame bundle. Therefore the heterotic model is free of holonomy anomaly as well. At last we want to emphasize that we can always introduce appropriate regulators without breaking target space isometries, higher derivative for example, so long as the chiral fermion ζ_R couples to isometry invariant term, see Eq. (2.6.6). It is essentially different from the situation that chiral fermions couple to gauge fields or spin-connections where gauge symmetries or target space symmetries can be only preserved conditionally [39, 40, 56]. We will return to this topic in later chapters.

2.7 Supercurrent multiplet

In this section we analyze the hypercurrent, a superfield which contains supercurrent and energy-momentum tensor among its components. For undeformed $\mathcal{N} = (2,2)$ theories the hypercurrent and its quantum anomalies were studied in Ref. [61]. This study includeds, in particular, the anomaly in the central charge which does not enter the $\mathcal{N}=(0,2)$ algebra. The general formulation in case of $\mathcal{N} = (0,2)$ theories was given in Ref. [55]. We present an explicit superfield form for the hypercurrent and all anomalies in the heterotic models under consideration.

2.7.1 Hypercurrent in the undeformed $\mathcal{N} = (2,2)$ theory

Let us start with the definition of the hypercurrent $\mathcal{T}_\mu$ in the undeformed $\mathcal{N} = (2,2)$ theory. The hypercurrent is the supermultiplet containing a supersymmetry current $s_{\mu\alpha}$ and an energy-momentum tensor $\vartheta_{\mu\nu}$,

$$\mathcal{T}_\mu = v_\mu + \left[\theta\gamma^0 s_\mu + \text{H.c.}\right] - 2\,\bar{\theta}\gamma^\nu\theta\,\vartheta_{\mu\nu} + \dots\,. \tag{2.7.1}$$

Here θ is the spinor $\theta = (\theta^1, \theta^2) = (\theta_L\,, \theta_R)$ and the lowest component $v^\mu = G_{i\bar{j}}\,\bar{\psi}^{\bar{j}}\gamma^\mu\psi^i$ is the fermionic R current.

Introducing spinor indices, $\mathcal{T}_{\alpha\beta} = (\gamma^0\gamma^\mu)_{\alpha\beta}\mathcal{T}_\mu$ we can write the classical hypercurrent in terms of the $\mathcal{N} = (2,2)$ chiral superfields $\Phi^i(x,\theta)$,

$$\mathcal{T}_{\beta\alpha} = G_{i\bar{j}}\bar{D}_\beta\Phi^{\dagger\,\bar{j}}D_\alpha\Phi^i\,, \tag{2.7.2}$$

where D_α, $\bar{D}_\beta$ are conventional spinor derivatives and the metric G is a function of superfields $\Phi^i\,, \Phi^{\dagger\,\bar{j}}$. Actually, only components $\mathcal{T}_{11}$ and $\mathcal{T}_{22}$, presenting nonzero Lorentz spin, are associated with the hypercurrent $\mathcal{T}_\mu$, the scalar $\mathcal{T}_{12} = [\mathcal{T}_{21}]^\dagger$ represents the twisted chiral integrand in the superspace action.

The anomaly equations for the hypercurrent derived in [61] are of the form:[4]

$$\begin{aligned}\bar{D}_1\mathcal{T}_{22} &= \frac{1}{4\pi}\,\bar{D}_2\left[R_{i\bar{j}}\bar{D}_1\Phi^{\dagger\,\bar{j}}D_2\Phi^i\right] = \frac{T_G g^2}{8\pi}\,\bar{D}_2\,\mathcal{T}_{12}\,,\\ \bar{D}_2\mathcal{T}_{11} &= \frac{1}{4\pi}\,\bar{D}_1\left[R_{i\bar{j}}\bar{D}_2\Phi^{\dagger\,\bar{j}}D_1\Phi^i\right] = \frac{T_G g^2}{8\pi}\,\bar{D}_1\,\mathcal{T}_{21}\,.\end{aligned} \tag{2.7.3}$$

In terms of classification of Ref. [55] it is the R_V multiplet, $\partial_{11}\mathcal{T}_{22} + \partial_{22}\mathcal{T}_{11} = 0$.

2.7.2 Hypercurrent in the heterotic $\mathcal{N} = (0,2)$ theory

As shown in Eq. (2.2.3) transition to diminished $\mathcal{N} = (0,2)$ supersymmetry decomposes the $\mathcal{N} = (2,2)$ superfield Φ as

$$\Phi = A + \sqrt{2}\,\theta^1 B\,, \tag{2.7.4}$$

where the $\mathcal{N} = (0,2)$ superfields, introduced in Eq. (2.2.1) and (2.2.2), depend on $\theta^2 = \theta_R$. Correspondingly, the hypercurrent $\mathcal{T}_\mu$ decomposes into two $\mathcal{N} = (0,2)$ supermultiplets,

$$\begin{aligned}\mathcal{J}_L &= \frac{1}{2}\,\mathcal{T}_{22}\Big|_{\theta^1=0} = \frac{1}{2}\,G_{i\bar{j}}\bar{D}A^{\dagger\bar{j}}DA^i\,,\\ \widetilde{\mathcal{T}}_{RR} &= -\frac{1}{2}\left[\bar{D}_1, D_1\right]\mathcal{T}_{11}\Big|_{\theta^1=0} = 2\,G_{i\bar{j}}\Big[\partial_R A^{\dagger\,\bar{j}}\partial_R A^i + i\,B^{\dagger\,\bar{j}}\nabla_R B^i\Big] + \text{H.c.}\,.\end{aligned} \tag{2.7.5}$$

[4] There is a misprint in [61]: the quantum anomalies of the hypercurrent should be multiplied by the overall factor $(-1/2)$.

These supermultiplets introduced in Ref. [55] have the following general structure,[5]

$$\begin{aligned}
\mathcal{J}_L &= v_L + i\theta\, s_{L;L} + i\theta^\dagger s^\dagger_{L;L} - \theta\theta^\dagger \vartheta_{LL}\,, \\
\widetilde{\mathcal{T}}_{RR} &= \vartheta_{RR} + \theta\,\partial_R s_{R;L} - \theta^\dagger \partial_R s^\dagger_{R;L} + \theta\theta^\dagger \partial_R^2 v_L\,.
\end{aligned} \tag{2.7.6}$$

It is clear that the heterotic deformation does not change the expression for $\mathcal{J}_L$ but modifies $\widetilde{\mathcal{T}}_{RR}$ supermultiplet where the lowest superfield component represents the ϑ_{RR} component of energy-momentum tensor. Namely, the expression for $\widetilde{\mathcal{T}}_{RR}$ in Eq. (2.7.5) is modified to

$$\widetilde{\mathcal{T}}_{RR} = \frac{1}{2}\left\{G_{i\bar{j}}\left[\partial_R A^{\dagger\bar{j}}\partial_R A^i + i\,ZB^{\dagger\bar{j}}\nabla_R B^i + i\kappa\,\mathcal{B}B^i\,\partial_R A^{\dagger\bar{j}}\right] + i\,\mathcal{Z}\mathcal{B}^\dagger\partial_R\mathcal{B}\right\} + \text{H.c.}\,. \tag{2.7.7}$$

Quantum anomalies for $\mathcal{J}_L$ and $\widetilde{\mathcal{T}}_{RR}$ according to [55] have the following general form:

$$\begin{aligned}
\partial_R \mathcal{J}_L &= \frac{1}{2}\,D\,\mathcal{W} - \frac{1}{2}\bar{D}\overline{\mathcal{W}}, \\
\bar{D}\widetilde{\mathcal{T}}_{RR} &= \partial_R \mathcal{W}\,,
\end{aligned} \tag{2.7.8}$$

where $\mathcal{W}$ represents the supermultiplets of anomalies,

$$\mathcal{W} = s^\dagger_{R;L} - i\theta\left(\vartheta_{LR} + i\,\partial_R\, v_L\right) - i\theta\theta^\dagger\partial_L s^\dagger_{R;L}\,. \tag{2.7.9}$$

At the one-loop level in the heterotically modified models $\mathcal{W}$ is the $\mathcal{N}=(0,2)$ chiral superfield of the following form,

$$\mathcal{W}^{(1)} = \frac{1}{4\pi}\left\{R_{i\bar{j}}\,\partial_R A^i\,\bar{D}A^{\dagger\bar{j}} - i\bar{D}\left[Z(R_{i\bar{j}} - h^2 G_{i\bar{j}})B^{\dagger\bar{j}}B^i - d\,h^2\mathcal{Z}\mathcal{B}^\dagger\mathcal{B}\right]\right\}. \tag{2.7.10}$$

This expression can be verified through a component calculation of the one-loop graphs in Fig. 2.6 for v_L which is the lowest component of $\mathcal{J}_L$. Rewriting Eq. (2.7.10) in terms of the heterotic curvature $\mathcal{H}$ defined in Eqs. (2.2.16, 2.2.18) we arrive at

$$\mathcal{W}^{(1)} = \frac{1}{4\pi}\left\{R_{i\bar{j}}\,\partial_R A^i\,\bar{D}A^{\dagger\bar{j}} - i\bar{D}\left[R^{(B)}_{i\bar{j}}B^{\dagger\bar{j}}B^i - 4\mathcal{H}_{ik\bar{l}}\,\overline{\mathcal{H}}_{\bar{j}\bar{k}l}G^{(B)k\bar{k}}G^{l\bar{l}}B^{\dagger\bar{j}}B^i\right]\right\}, \tag{2.7.11}$$

[5] Our notations differ: $\mathcal{J}_L$ and $\widetilde{\mathcal{T}}_{RR}$ are the same as S_{++} and $\mathcal{T}_{----}$ in [55].

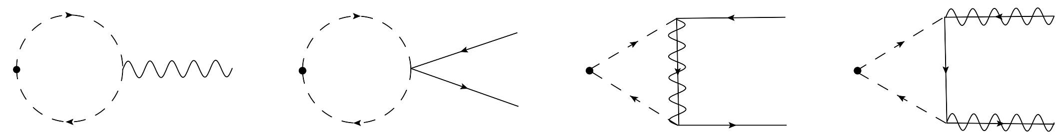

Figure 2.6: One-loop diagrams for v_L current. The dots denote the v_L currents, dashed lines are quantum A fields while wavy lines refer to the background A.

where all right-moving fermions are included in B^i (see Sec. 2.2.2 for details).

Following Eqs. (2.2.10) and (2.3.10) it is simple to verify that the superfield $\mathcal{W}$, which represents the supermultiplets of anomalies, coincides with the one-loop running of the superfield $\mathcal{F}$ associated with the Lagrangian by Eq. (2.2.10),

$$\mathcal{W}^{(1)} = i\, M_{\rm uv} \frac{d}{dM_{\rm uv}}\, \mathcal{F}\,\Big|^{\rm one-loop}\,. \tag{2.7.12}$$

What about higher-loop corrections? They will show up as higher loops in the anomalous dimensions. It means that Eq. (2.7.10) is modified to

$$\mathcal{W} = \frac{1}{4\pi}\Big[R_{i\bar{j}}\partial_R A^i \bar{D} A^{\dagger\bar{j}} - i\bar{D}\big(Z R_{i\bar{j}} B^{\dagger\bar{j}} B^i\big)\Big] + \frac{i}{2}\,\bar{D}\Big[\gamma_{\psi_R} Z G_{i\bar{j}} B^{\dagger\bar{j}} B^i + \gamma_\zeta \mathcal{Z}\mathcal{B}^\dagger\mathcal{B}\Big]. \tag{2.7.13}$$

2.7.3 Analog of the Konishi anomaly and beta function

Here we will discuss a relation between the hypercurrent anomalies and beta functions. It is an example of another analog to 4D gauge theories. We mentioned above that the supermultiplet of anomalies $\mathcal{W}$ is given by differentiation of the effective Lagrangian with respect to $\log M_{\rm uv}$. Let us have a closer look at how it works for β_g .

At the one-loop level the running of the metric $dG_{i\bar{j}}/dL = R_{i\bar{j}}/2\pi$ is given by the Ricci tensor, see Eq. (2.3.10). In $\mathcal{W}$ (see Eq. (2.7.13)) it is represented by the term with the A superfields. The terms with the B fields contribute to the higher loops. We can simplify these terms using equations of motion plus possible anomalies. Using the equation of motion we get

$$\bar{D}\big(ZG_{i\bar{j}}B^{\dagger\bar{j}}B^i\big)\big|_{\text{class}} = -\kappa\, G_{i\bar{j}}\bar{D}A^{\dagger\bar{j}}B^i\mathcal{B}\,,$$
$$\bar{D}\big(\mathcal{Z}\mathcal{B}^\dagger\mathcal{B}\big)\big|_{\text{class}} = -\kappa\, G_{i\bar{j}}\bar{D}A^{\dagger\bar{j}}B^i\mathcal{B}\,. \tag{2.7.14}$$

There is also an anomalous part due to loops in the background of the A field, namely,

$$\bar{D}\big(ZG_{i\bar{j}}B^{\dagger\bar{j}}B^i\big)\big|_{\text{anom}} = -\frac{i}{4\pi}\, R_{i\bar{j}}\partial_R A\bar{D}A^{\dagger\bar{j}}\,. \tag{2.7.15}$$

This part is a clear-cut analog of the Konishi anomaly in 4D [57].

Using both, classical equations (2.7.14) and the anomalous one (2.7.15), as well as $R_{i\bar{j}} = (T_G g^2/2)G_{i\bar{j}}$, we come from $\mathcal{W}$ of Eq. (2.7.13) to

$$\mathcal{W} = \frac{T_G g^2}{8\pi}\, G_{i\bar{j}}\partial_R A^i\,\bar{D}A^{\dagger\bar{j}}\left(1 - \frac{T_G g^2}{8\pi} + \frac{\gamma_{\psi_R}}{2}\right) + \left(\frac{T_G g^2}{8\pi} - \frac{\gamma}{2}\right) i\kappa\, G_{i\bar{j}}\bar{D}A^{\dagger\bar{j}}B^i\mathcal{B}\,. \tag{2.7.16}$$

In the background of the A field the integrating out right-moving fermions in the operator $G_{i\bar{j}}\bar{D}A^{\dagger\bar{j}}B^i\mathcal{B}$ involves the same polarization operator Π_{RR} as in Sec. 2.2.3, see (2.2.23) and Appendix B, and results in

$$\Big\langle i\kappa\, G_{i\bar{j}}\bar{D}A^{\dagger\bar{j}}B^i\mathcal{B}\Big\rangle_A = \frac{h^2}{4\pi}\, G_{i\bar{j}}\partial_R A^i\,\bar{D}A^{\dagger\bar{j}}\,. \tag{2.7.17}$$

Thus, we get

$$\Big\langle\mathcal{W}\Big\rangle_A = \frac{1}{8\pi}\, G_{i\bar{j}}\partial_R A^i\,\bar{D}A^{\dagger\bar{j}}\left[T_G g^2\left(1 - \frac{T_G g^2}{8\pi} + \frac{\gamma_{\psi_R}}{2}\right) + h^2\left(\frac{T_G g^2}{4\pi} - \gamma\right)\right]. \tag{2.7.18}$$

In the above calculations we limit ourselves by one loop (besides higher loops in the anomalous dimensions $\gamma_{\psi_R}\,, \gamma_\zeta$). To see that the higher loops are needed it is sufficient to go to the (2,2) case when $h = 0$. In this limit $\gamma_{\psi_R} = \gamma_\zeta = 0$ but the the factor $1 - (T_G g^2/8\pi)$ remains in (2.7.18). It should cancel out eventually for a pure bosonic field background. Technically it happens in the following way. Accounting for the left-moving fermion anomaly in $\partial_R\big(R_{i\bar{j}}\bar{D}A^{\dagger\bar{j}}DA^i\big)$ which cancels in the (2,2) case the one from right-movers leads to a geometrical progression which turns the factor $1 - (T_G g^2/8\pi)$ into $1/(1+$

$(T_G g^2/8\pi))$. Then, in the bosonic background this factor will be eaten up by integrating out left-moving fermions.

At nonvanishing h one more geometrical progression is generated by the factor $1+(h^2/4\pi)$ which multiplies $T_G g^2$ in (2.7.18). It is simple to understand this as just a summation of a chain of insertions of polarization operator Π_{RR} into a bosonic propagator. Thus, the multi-loop expression for $\langle \mathcal{W} \rangle_A$ becomes

$$\left\langle \mathcal{W} \right\rangle_A^{\text{multi}} = \frac{G_{i\bar{j}} \partial_R A^i \bar{D} A^{\dagger \bar{j}}}{1 + (T_G g^2/8\pi)} \frac{T_G g^2 \left(1 + (\gamma_{\psi_R}/2)\right) - h^2 \gamma}{8\pi(1 - (h^2/4\pi))} \,. \tag{2.7.19}$$

The second factor in this expression is just $(-\beta_g/2g^2)$ which gives the same β_g as in Eq. (2.1.5). The normalzation follows from the one loop. The factor $1/(1+(T_G g^2/8\pi))$ will go away in the bosonic background as it was explained above.

2.8 Conclusion

In this chpater, we analyzed various quantum effects in the $\mathcal{N} = (0,2)$ deformed $(2,2)$ two-dimensional sigma models. The target spaces we consider generalize $\mathrm{CP}(N-1)$ to symmetric Kähler spaces. The $\mathcal{N} = (0,2)$ deformation thoroughly studied in this chapter goes under the name of the *nonminimal* models.

We showed that the isometry currents in the nonminimal models are conserved, which means quantum effects will not deform geometry. Note that this is in drastic contradistinction with the chiral models in the minimal case, see later chapters. The absence of anomaly in the isometry currents in the nonminimal models is a nontrivial fact. Also, since the Lax relation holds classically, one might expect these classes of models to be integrable, as their undeformed cousins.

The models we studied are characterized by two independent coupling constants. We analyzed them in perturbation theory. A crucial role belongs to the graph depicted in Fig. 2.1 which is associated with the anomaly in the current that mixes the right-moving fermions. It is also anomalous in the sense that it produces a nonholomorphic contribution proportional to $|\kappa|^2$ to the renormalization of $1/g^2$ at one loop. This effect then penetrates into higher orders.

Using nonrenormalization theorems [46], analogous to those in [21,22] in four-dimensional Yang-Mills, we derive a number of (perturbatively) exact relations between the β functions and the anomalous dimensions of the fields B and $\mathcal{B}$. Then we calculate the anomalous dimensions up to (and including) two loops thus obtaining explicit β functions up to three loops.

Then we study the relation between the perturbative β functions and the general hypercurrent analysis of Dumitrescu and Seiberg [55]. We find how the general structure of [55] is implemented in the nonminimal models under consideration. We demonstrated that the hypercurrent analysis provides an alternative way for the β-function calculation provided that the two-dimensional analogs of the Konishi anomaly are taken into account.

Recently the $\mathcal{N} = (0,2)$ models attracted attention in connection with developments in the studies of surface operators in four dimensions (see e.g. [62]). In the nonminimal models the protected quantities, i. e. the chiral ring, is preserved under the $(0,2)$ deformation. It is interesting to pursue the calculation beyond the chiral sector exploring nonchiral sector of the world-sheet theory as a part of a $2d/4d$ coupled system. We hope that the results presented here can enlighten the very first step in pursuing such a goal.

Two-dimensional asymptotically free sigma models are long known to be excellent laboratories for modeling four-dimensional Yang-Mills theories. It was forty years ago that A. Polyakov emphasized (in the publication [5]) that asymptotically free two-dimensional sigma models could be the best laboratory for the four-dimensional Yang-Mills theories. His anticipation seems to be materializing. The nonminimal (0,2) sigma model discussed in this chapter presents a close parallel to $\mathcal{N} = 1$ super-Yang-Mills with matter in four dimensions (see also [47]).

Chapter 3

Isometry Anomalies in Minimal $\mathcal{N} = (0,1)$ and $\mathcal{N} = (0,2)$ Sigma Models

3.1 Introduction

As what we have seen in last chapter, nonminimal chiral models are obtained as deformations of $\mathcal{N} = (2,2)$ supersymmetries and contain both *left* and *right*-handed fermions.[1] They are free of anomaly by construction (see Sec.2.6 of chapter 2). In this chapter, we turn to consider *minimal* chiral sigma models with $\mathcal{N} = (0,1)$ and $(0,2)$ supersymmetries where by "minimal" we mean that there are only, say, *left*-handed fermions included in models. Since these minimal chiral theories generically suffer from anomalies, the no-anomaly conditions become criteria for mathematical consistency of these models *per se*. In general the (*left*-handed) chiral fermions can be defined on arbitrary vector bundles over manifolds on which bosonic fields live in various dimensions. There are two types of intrinsic anomalies in the minimal $\mathcal{N} = (0,2)$ sigma models, which can be compared to those in gauge theories in four dimensions.

First, chiral fermions in four dimensions ($4d$) can ruin gauge invariance already at one loop, as it happens, say, in the SU(N) gauge theory with $N > 2$ and a single chiral fermion

[1] Strictly speaking, in two dimensions they are *left*- and *right*-movers.

in the fundamental representation. This anomaly does not appear, however, in the SU(2) gauge theory due to the absence of the d symbols in SU(2). Nevertheless, the SU(2) gauge theory with one chiral fermion in the fundamental representation does not exist since it suffers from a "global" Witten's anomaly [43]. This is the second type of anomalies in four-dimensional Yang-Mills.

The anomalies in the minimal $\mathcal{N} = (0,2)$ sigma models were discussed by many authors in different aspects. A number of authors calculated [63–67] chiral anomalies (i.e. obtained explicit local forms) and discussed the mechanism of the anomaly cancellations. On the other hand, global feature of anomalies were also thoroughly considered in the works [56, 68, 69]. A well-known no-go theorem [56] establishing a global anomaly due to non-zero Pontryagin classes over vector bundles in such minimal $(0,2)$ models such as $\mathrm{CP}(N-1)$ makes them inconsistent (with an exception of CP(1) model).

However, there are two issues that the previous literature failed to cover. Firstly, it is possible that (chiral nonabelian) global and gauge symmetries are involved in the formulation of the geometric model. Even if the global anomaly is absent, it is still desirable that one constructs concrete local effective actions to cure local anomaly. Unfortunately, neither the p_1 vanishing condition nor the arguments like Wess-Zumino consistency condition fixes the two-dimensional effective action and the cure of anomaly uniquely. Unlike the $4d$ sigma model case where the low energy effective action is only defined for the phenomenological consideration, in $2d$, the effective action can be computed directly, and the counter term can be obtained in some concrete cases.

Secondly, the gauge formulation [4] (e.g. variants of the Grassmannian sigma models) of nonlinear sigma models depends on a choice, and for this reason different formulations are allowed at classical level. This might cause problems in the quantum level, since there are different gauge degrees of freedom that chiral fermions might couple to. Note that in previous literatures (which mostly focuses on the 4d models for phenomenological interests), there was no discussion about this point. In Moore-Nelson, their criteria do not quite fixes the gauge formulation precisely because the showing up of gauge formulation is not universal as in their context, and thus needs a more case dependent study. For example, the dual formalism for the $\mathrm{O}(N)$ models we give in the last section of the current chapter has not been written down nor studied carefully elsewhere.

In the present chapter we will discuss the isometries of the target space manifolds O(N) and CP($N-1$) and the corresponding isometry anomalies with multiple formalisms. In the light of chapter 4 (see also [40]), the isometry symmetries of a gauge formulated sigma model is a special manifestation of the holonomy anomaly discussed there. The current chapter provides precise computations and examples. The potential implication of the work, is the following. It has been known that the O(N) models are integrable in the bosonic cases and the $\mathcal{N} = (1,1)$ cases. But the minimal $\mathcal{N} = (0,1)$ case has never been studied. Our result sets a starting point to understand the integrability of the supersymmetric model in terms of the integrability of the bosonic model, which is of interest in its own right.

A few words on terminology. Sigma models are defined on manifolds which typically have to be covered by many local patches. One can specify a local chart of the manifold and then perform the anomaly calculation (typically at one loop). We thus call such anomalies *local*.[2]

In the case of local anomalies offset by counterterms on local patches, one must worry how to patch these counterterms on different charts. It is essentially a cohomology problem [70] which thus is tied up with global features of the manifolds under consideration. When some classes of sigma models admits the gauged formulation, we potentially have to deal with chiral anomalies in the "small" and "large" gauge transformations (analogous to the Witten SU(2) anomaly [43]). Here we shall only be dealing with infinitesimal symmetry transformations. The issue of discrete symmetries also attracts many attentions recently, and we shall briefly comment and the outlooks of that direction in next chapter.

The target spaces in the problems to be considered are homogeneous symmetric spaces of G/H type. In this case it was shown [56, 66] that the criterion of local anomalies is stronger than the global obstruction: the local anomalies imply the global obstruction and *vice versa* for 4d cases. In our examples of the O(N) and CP($N-1$) models we explicitly verify this statement for 2d models. In the first example which is free from the Moore-Nelson global obstruction, we demonstrate that the O(N) model is free from local anomalies. The consistency of our argument is also checked by the dual formalism. In the second example, $\mathcal{N} = (0,2)$ CP($N-1$) models, $N > 2$, in which the first Pontryagin

[2] The isometries in the sigma models under consideration are global symmetries analogous to flavor symmetries in the gauge theories. We will still refer to the isometry anomalies as to local anomalies, to avoid confusion.

class is nontrivial, local anomalies are present, and the gauge formulation of the model is inconsistent.

Note also the local counter term discussion has also been mentioned in previous literatures in the context of gravitational anomalies [71–73]. Similar pattern was shown that the global anomaly manifests itself as a nonlocal term, and the local counter term, when possible, makes the local symmetry well-defined. We emphasize that our discussion is not a simple derivation from those works. The choice of different gauged formulation is a new complication in our problem, which has no counter-part in those works. Also, the fact that the global isometry lifts up to quantum level indicates the existence of conserved charges and their quantum correction, which depends on the metric of the target manifold. It is due to this reason, that the anomaly cancellation can not be simply achieved by assigning spurious transformations to the world sheet fluxes. So both our starting point, computation and conclusion touch more refined data.

As was mentioned, in both cases we examine anomalies in the isometries which decide whether or not geometry of the classical action can be maintained at the quantum level. Only if it can be maintained can the theory be self-consistent. In the minimal O(N) models one can construct anomaly-free isometry currents, while such a construction is impossible in the minimal CP($N-1$) models. The only exception is CP(1) which is equivalent to O(3).

The chapter is organized as follows. In Sec. 3.2 we thoroughly discuss the minimal O(N) models and demonstrate the absence of the isometry anomalies. Section 3.3 is devoted to the minimal CP($N-1$) models. We derive the CP($N-1$) isometry anomalies in this model. Our analysis in this section is somewhat different from the O(N) case. We examine the correspondence between the isometry anomalies in non-linear sigma models ($NL\sigma M$) and gauge anomalies in gauged linear sigma models ($GL\sigma M$), and then derive the isometry anomalies based on the above correspondence. In Sec. 3.4 we consider a dual formalism for the O(N) models and arrive at the same result as in Sec. 3.2 by using the correspondence referred in this section. Section 3.5 summarises our results and outlines questions that will be answered in chapter 4. Appendix C presents details of derivation in Sec. 3.3 through direct calculation as a verification.

3.2 O(N) Sigma Model

Let us first study the "linear" version[3] of the O(N) model [6], investigate the symmetries of the model and then pass to the nonlinear description.

The linear O(N) sigma model contains N real fields n^i, where $i = 1, 2, ..., N$, with the constraint

$$n^i n_i = 1\,. \tag{3.2.1}$$

This means that the target space of the model is the sphere S^{N-1}, which could be viewed as the coset

$$\mathrm{S}^{N-1} = \mathrm{SO}(N)/\mathrm{SO}(N-1)\,. \tag{3.2.2}$$

Thus, the model (3.2.3) can equally be referred to as the S^{N-1} model. In the literature the first name, O(N), is more common, however. It reflects, in particular, that counting of isometries is given by O(N). The bosonic part of Lagrangian is

$$\mathcal{L}_b = \frac{1}{2g_0^2}\,\partial_\mu n^i \partial^\mu n_i + \lambda(n^i n_i - 1) \tag{3.2.3}$$

where λ is a Lagrange multiplier that ensures the constraint above on the n^i fields. As mentioned above there are $N(N-1)/2$ isometry symmetries corresponding to the SO(N) group. For each point in the target space a stationary subgroup $H = \mathrm{SO}(N-1)$ (the denominator in Eq. (3.2.2)) consists of transformations which do not act at this point. We fix a particular choice of H specifying an axis, say n^N, as associated with the stationary under SO($N-1$) point. For other points the transformations from H are realized linearly.

Thus, the first set of isometries is given by linear transformations,

$$\delta_\epsilon n^i = \epsilon^{ij} n_j, \quad \delta_\epsilon n^N = 0, \tag{3.2.4}$$

where $\epsilon^{ij} = -\epsilon^{ji}$, $(i = 1, 2, ..., N-1)$ are infinitesimal parameters. The remaining $N-1$

[3] By linear we mean that the kinetic term of Lagrangian in O(N) model is linear. The model is, for sure, subject to the constraint $n^i n_i = 1$ which makes it nonlinearly realized when we remove the extra redundancy by solving the constraint.

isometries form the second set where the transformations are realized nonlinearly,

$$\delta_\alpha n^i = \alpha^i n^N , \quad \delta_\alpha n^N = -\alpha^i n_i \,, \tag{3.2.5}$$

where α^i, $(i = 1, 2, ..., N-1)$ are infinitesimal parameters. The sub/superscripts are raised or lowered by δ^{ij} or δ_{ij}.

Now, one can rewrite this sigma model through the standard stereographic projection to explicitly solve the constraint $n^i n_i = 1$, by setting

$$\phi^i = \frac{n^i}{1 + n^N} \,, \quad i = 1, 2, ..., N-1 \,. \tag{3.2.6}$$

By recalculating the infinitesimal transformations of ϕ^i with respect to ϵ^{ij} and α^i, one obtains

$$\begin{aligned} \delta_\epsilon \phi^i &= \epsilon^{ij} \phi_j \,, \\ \delta_\alpha \phi^i &= \frac{1-\phi^2}{2}\, \alpha^i + \alpha^j \phi_j \phi^i \,. \end{aligned} \tag{3.2.7}$$

In terms of the field ϕ^i the Lagrangian (3.2.3) takes the form,

$$\mathcal{L}_b = \frac{1}{2}\, g_{ij} \partial_\mu \phi^i \partial^\mu \phi^j \,, \tag{3.2.8}$$

where g_{ij} is the metric tensor of S^{N-1} sphere,

$$g_{ij} = \frac{4}{g_0^2} \frac{\delta_{ij}}{(1+\phi^2)^2} \,. \tag{3.2.9}$$

Supersymmetrizing the O(N) Lagrangian by adding left-handed fermions is straightforward. We couple $N-1$ real left-handed chiral fermions $\psi^i \equiv \psi^i_L$ to the bosonic fields so that the theory has $\mathcal{N} = (0,1)$ supersymmetry,

$$\mathcal{L} = \mathcal{L}_b + \mathcal{L}_f = \frac{1}{2}\, g_{ij} \partial_\mu \phi^i \partial^\mu \phi^j + \frac{i}{2}\, g_{ij} \bar{\psi}^i \gamma^\mu D_\mu \psi^j \tag{3.2.10}$$

where D_μ is the covariant derivative pulled back from the S^{N-1} sphere. The stereographic projection (3.2.6) gives us a local chart $\{\phi^i\}$, for which one can write down the metric (see

(3.2.9)) and connections explicitly,

$$\begin{aligned} D_\mu \psi^i &= \partial_\mu \psi^i + \Gamma^i_{jk} \partial_\mu \phi^j \psi^k \,, \\ \Gamma^i_{jk} &= -\frac{2}{1+\phi^2} \left(\delta^i_j \phi_k + \delta^i_k \phi_j - \delta_{jk} \phi^i \right) . \end{aligned} \tag{3.2.11}$$

Now let us pass to the issue of isometry anomalies. To evaluate the anomalies, one needs to integrate out fermions to find the effective action $\Gamma_{\text{eff}}[\phi]$. Then one performs the isometry transformations (3.2.7). We introduce vielbeins $e^a_{\ i}$ on S^{N-1} to decompose the metric and rewrite fermion fields in the canonic way,

$$e^a_{\ i} = \frac{2}{g_0} \frac{1}{1+\phi^2} \, \delta^a_{\ i} \,, \qquad e^i_{\ b} = \frac{g_0}{2} \, (1+\phi^2) \, \delta^i_{\ b} \,. \tag{3.2.12}$$

Apparently $e^a_{\ i}$ satisfy the conditions

$$e^a_{\ i} e^i_{\ b} = \delta^a_{\ b} \,, \qquad \delta_{ab} e^a_{\ i} e^b_{\ j} = g_{ij} \,. \tag{3.2.13}$$

In Eqs. (3.2.12) and (3.2.13) δ_{ab} and δ^{ab} are for raising and lowering indices $\{a, b, ...\}$, while g_{ij} and g^{ij} for indices $\{i, j, ...\}$. Besides, for local chart $\{\phi^i\}$, one can still use δ_{ij} to write $\phi_i \equiv \delta_{ij} \phi^j$.

As long as conditions (3.2.13) are met, one still has a residual freedom to make different choices for $e^a_{\ i}$. This freedom might lead to the so-called holonomy anomalies which we will discuss in upcoming chapter [40].

Through vielbeins $e^a_{\ i}$ we define $\psi^a \equiv e^a_{\ i} \psi^i$, and thus rewrite the fermion part of the $\mathcal{N} = (0,1)$ Lagrangian,

$$\mathcal{L}_f = \frac{i}{2} \, g_{ij} \bar{\psi}^i \gamma^\mu D_\mu \psi^j = \frac{i}{2} \, \bar{\psi}^a \gamma^\mu (\partial_\mu \delta_{ab} + \omega_{abi} \partial_\mu \phi^i) \psi^b \,, \tag{3.2.14}$$

where

$$\omega^a_{\ bi} = e^a_{\ j} \mathcal{D}_i e^j_{\ b} = e^a_{\ j} \left[\frac{\partial e^j_{\ b}}{\partial \phi^i} + \Gamma^j_{ik} e^k_{\ b} \right] \tag{3.2.15}$$

is the spin-connection on the frame bundle, and $\mathcal{D}_i$ is the covariant derivative on S^{N-1}.

Now, we integrate out fermions and arrive at an effective action Γ_{eff}. This requires calculation of the bi-angle diagram (see Fig. 3.1); higher orders are finite and thus do not contribute into anomalies. Note that there are only chiral fermions ψ_L^a in $\mathcal{L}_f$ coupled to

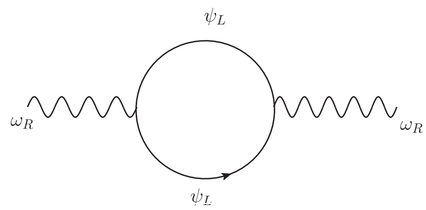

Figure 3.1: The wavy lines denote external spin-connection fields ω_R, and solid lines denote chiral fermion ψ_L.

the spin-connection $\omega_R = \omega_i \partial_R \phi^i$. Therefore, the effective action is a functional of ω_R,

$$\begin{aligned} i\Gamma_{\text{eff}}[\omega_R] &= \frac{i}{16\pi}\int d^2x\, \omega^a{}_{b\mu}\left(g^{\mu\alpha}+\epsilon^{\mu\alpha}\right)\frac{\partial_\alpha\partial_\beta}{\partial^2}\left(g^{\beta\nu}-\epsilon^{\beta\nu}\right)\omega^b{}_{a\nu}+\mathcal{O}(\omega_R^3) \\ &= \frac{i}{16\pi}\int d^2x\, \omega^a{}_{bR}\frac{\partial_L\partial_L}{\partial^2}\omega^b{}_{aR}+\mathcal{O}(\omega_R^3)\,. \end{aligned} \tag{3.2.16}$$

As we mentioned above cubic and higher term in the action are given by well convergent integrals in momentum space. It means that these terms are well defined in UV and anomaly could come only from quadratic in connections terms.

To evaluate the isometry anomalies from Γ_{eff}, we will examine $\delta_\epsilon\Gamma_{\text{eff}}$ and $\delta_\alpha\Gamma_{\text{eff}}$ under isometry transformations (3.2.7). Invariance of Γ_{eff} under linear transformations, $\delta_\epsilon\Gamma_{\text{eff}}=0$ is evident because these symmetries are explicitly maintained. As for nonlinear transformations we will see that for spin-connections they have the gauge form,[4]

$$\delta_v\omega^a{}_{b\mu} = -\partial_\mu v^a{}_b - [\omega_\mu, v]^a{}_b\,, \tag{3.2.17}$$

[4] Since the spin-connections have similar transformation behavior to that of the gauge fields, one should impose the Wess-Zumino consistency condition [63] to obtain correct consistent anomalies. However, in $2d$ gauge theories, consistent anomalies have only two independent candidates [?] :

$$\mathcal{A}_v = \int dx^2\; v_\alpha(c_1\partial^\mu A_\mu^\alpha + c_2\epsilon^{\mu\nu}\partial_\mu A_\nu^\alpha)\,.$$

In our sigma model, the left-handed chiral fermions only couple to ω_R. Therefore, the anomalies are similar to those in the gauge theory, and will *not* reduce to a purely topological term.

where the gauge function v, linear in parameters α^i, depends on fields ϕ^j. Then the anomalies can be obtained by varying Eq. (3.2.16),

$$\begin{aligned}\mathcal{I}_v^{\rm total} &= \delta_v \Gamma_{\rm eff} = \frac{1}{8\pi}\int d^2x \, {\rm Tr}\; v\, \partial_L \omega_R \\ &= \frac{1}{8\pi}\int d^2x \, {\rm Tr}\,(v\, \partial^\mu \omega_\mu - v\, \epsilon^{\mu\nu}\partial_\mu \omega_\nu)\,. \end{aligned} \tag{3.2.18}$$

In Eq. (3.2.18), the first term can be removed by introducing a local counterterm

$$S_{\rm c.t.} = -\frac{1}{16\pi}\int d^2x {\rm Tr}\; \omega_\mu \omega^\mu\,. \tag{3.2.19}$$

This counterterm is in essence equivalent to adding heavy Pauli-Villars (PV) fermions to $\mathcal{L}_f$,

$$\mathcal{L}_{\rm PV} = \frac{i}{2} H_{aL} \nabla_R H_L^a + \frac{i}{2} H_{aR} \nabla_L H_R^a + iM H_{aL} H_R^a\,, \tag{3.2.20}$$

where $H^a_{L,R}$ are real Weyl-Majorana fermions,

$$\nabla_{R,L} H^a_{L,R} \equiv \partial_{R,L} H^a_{L,R} + \omega^a{}_{bR,L} H^b_{L,R}\,,$$

and M is the PV mass. At the very end $M \to \infty$. One can check that, after integrating out the PV fermions $H^a_{L,R}$, one recovers Eq. (3.2.19).

The second term in Eq. (3.2.18) is purely topological. For simplicity, one can write it as a pulled-back form from S^{N-1} where we defined our sigma model by mapping $\phi : \Sigma \to \mathrm{S}^{N-1}$,

$$\begin{aligned}\mathcal{I}_v &= -\frac{1}{8\pi}\int_\Sigma d^2x\, {\rm Tr}\,(v\epsilon^{\mu\nu}\partial_\mu\omega_\nu) = -\frac{1}{8\pi}\int_\Sigma \phi^*(\,{\rm Tr}\,(v d\omega)) \\ &= -\frac{1}{8\pi}\int_{\phi(\Sigma)} {\rm Tr}\,(v d\omega)\,. \end{aligned} \tag{3.2.21}$$

The explicit expression for $\omega^a{}_b = \omega^a{}_{b\mu} dx^\mu$ can be calculated from Eqs. (3.2.12) and (3.2.15),

$$\omega^a{}_b = \frac{2\phi^i d\phi^j}{1+\phi^2}\, E_{ij}{}^a{}_b\,, \qquad E_{ij}{}^a{}_b = \delta_i{}^a \delta_{jb} - \delta_j{}^a \delta_{ib}\,. \tag{3.2.22}$$

Here the E_{ij}'s are the generators of the $\mathfrak{so}(N{-}1)$ Lie algebra in fundamental representation, the holonomy group of S^{N-1} is $\mathrm{SO}(N{-}1)$. Then variation of spin-connection ω with respect to α^i transformations of Eq. (3.2.7) has the form (3.2.17) with v given by

$$v^a{}_b = -\alpha^i \phi^j E_{ij}{}^a{}_b\,. \tag{3.2.23}$$

Therefore the anomaly is given by Eq. (3.2.21) with v from Eq. (3.2.23).

With $v^a{}_b$ being ϕ-dependent the integrand in (3.2.21) does not look as a total derivative. However, it can be rewritten, using integration by parts, as follows:

$$\begin{aligned}
\delta_\alpha \Gamma_{\text{eff}} &= \frac{1}{8\pi}\int_{\phi(S^2)} dv^a{}_b \wedge \omega^b{}_a = \frac{1}{8\pi}\int_{\phi(S^2)} \frac{2\alpha_i\phi_j}{1+\phi^2}\, d\phi^i \wedge d\phi^j \\
&= \frac{1}{8\pi}\int_{\phi(S^2)} d\left[\log(1+\phi^2)\,\alpha_i d\phi^i\right].
\end{aligned}\tag{3.2.24}$$

Then we see that the variation is, in fact, an integral of a total derivative. Therefore, the local anomalies of isometries in the $\mathrm{O}(N)$ models vanish.

3.3 CP$(N{-}1)$ Sigma Model

Our second example is the $\mathrm{CP}(N{-}1) = \mathrm{SU}(N)/\mathrm{S}(\mathrm{U}(N{-}1)\times \mathrm{U}(1))$ sigma model [4]. The model involves N complex fields u^i $(i = 1, 2, ..., N)$ with the constraint

$$\bar{u}_i u^i = 1\,.$$

In addition we need to impose a local U(1) gauge invariance under

$$u^i \to \mathrm{e}^{i\alpha(x)} u^i\,. \tag{3.3.1}$$

To this end one introduces an auxiliary vector field A_μ, and the Lagrangian takes the form

$$\mathcal{L}_b = \frac{2}{g_0^2}(\partial_\mu + iA_\mu)\bar{u}_i(\partial^\mu - iA^\mu)u^i + \lambda(\bar{u}_i u^i - 1)\,. \tag{3.3.2}$$

Similarly to the $\mathrm{O}(N)$ case, to pick up a patch we can chose a “complex”axis, e.g., u^N. The isometries of the model fall into two groups: linear transformations which do not transform u^N,

$$\delta_\epsilon u^i = \epsilon^{i\bar{j}} u_{\bar{j}}\,, \quad \delta_\epsilon u^N = 0\,; \quad i, \bar{j} = 1, 2, ..., N-1\,, \tag{3.3.3}$$

and nonlinear ones which rotate u^N,

$$\delta_\beta u^i = \beta^i u^N, \quad \delta_\beta u^N = -\bar{\beta}_i u^i\,; \quad i = 1, 2, ..., N-1\,. \tag{3.3.4}$$

In the above expressions, $\epsilon^{i\bar{j}}$ is an anti-Hermitian matrix and thus has $(N-1)^2$ real parameters while β^i are $N-1$ complex parameters. The indices can be locally raised or lowered by $\delta^{i\bar{j}}$ or $\delta_{i\bar{j}}$. The total number of isometries is N^2-1 corresponding to SU(N) symmetries of the CP($N-1$) model. Furthermore, since A_μ is nondynamical, we can eliminate it in favor of the u^i fields,

$$A_\mu = -\frac{i}{2}(\bar{u}_i \partial_\mu u^i - \partial_\mu \bar{u}_i u^i)\,. \tag{3.3.5}$$

Now, we can fix the gauge by condition $\mathrm{Im}\, u^N = 0$, and solve the constraint by choosing a set of local coordinates $\{\phi^i,\ \bar{\phi}^{\bar{j}}\}$,

$$\phi^i = \frac{u^i}{u^N}\,, \qquad i = 1, 2, ..., N-1\,. \tag{3.3.6}$$

In terms of the new coordinates the isometry transformations of the model are

$$\begin{aligned} \delta_\epsilon \phi^i &= \epsilon^{i\bar{j}} \phi_{\bar{j}}\,; \\ \delta_\beta \phi^i &= \beta^i\,, \\ \delta_{\bar{\beta}} \phi^i &= (\bar{\beta}\phi)\phi^i\,. \end{aligned} \tag{3.3.7}$$

Parallelizing our discussion of the O(N) model, we can write down the Lagrangian in terms of the fields ϕ^i, $\bar{\phi}^{\bar{j}}$. We then supersymmetrize it to form a $\mathcal{N} = (0,2)$ CP(N$-$1) model by coupling complex *left*-handed Weyl fermions $\psi^i \equiv \psi^i_L$,

$$\mathcal{L} = \mathcal{L}_b + \mathcal{L}_f = g_{i\bar{j}} \partial_\mu \bar{\phi}^{\bar{j}} \partial^\mu \phi^i + g_{i\bar{j}}\, \bar{\psi}^{\bar{j}} i\gamma^\mu D_\mu \psi^i\,, \tag{3.3.8}$$

where

$$g_{i\bar{j}} = \frac{2}{g_0^2}\,\frac{(1+\bar{\phi}_i\phi^i)\delta_{i\bar{j}} - \bar{\phi}_i\phi_{\bar{j}}}{(1+\bar{\phi}_i\phi^i)^2} \tag{3.3.9}$$

is the standard Fubini-Study metric for CP($N-1$).

To explore the isometry anomalies, one can introduce vielbeins as in the O(N) model, but the calculation is lengthy and tedious. We present the calculation details in Appendix. Here, instead, we will find a relation between the gauge anomaly in the gauged linear model and the isometry anomalies in the nonlinear formulation.

The full $\mathcal{N}=(0,2)$ CP($N-1$) gauged model is obtained by adding N complex left-handed fermions ξ_L^i with constraints $u^i\bar{\xi}_{iL}=0$. The corresponding Lagrangian takes the form

$$\mathcal{L} = \mathcal{L}_b + \frac{2}{g_0^2}\,\bar{\xi}_{Li}(i\partial_R + A_R)\xi_L^i + \frac{2}{g_0^2}\left(\kappa_R\bar{\xi}_{iL}u^i + \text{H.c.}\right), \tag{3.3.10}$$

where κ_R is a Lagrange multiplier. The one-loop effective fermionic action following from the bi-angle diagram similar to Fig. 3.1 is

$$i\Gamma_{\text{eff}}[A_R] = -\frac{iN}{8\pi}\int d^2x\; A_R\,\frac{\partial_L\partial_L}{\partial^2}\,A_R\,. \tag{3.3.11}$$

This action obviously suffers from a U(1) anomaly. Similarly to Eq. (3.2.18), this anomaly has longitudinal and topological parts. Since the anomaly in the longitudinal term is always cancelable by a counterterm, as Eq. (3.2.19), we will focus on the topological part.

Keeping in mind that the gauge transformation has the form

$$A_\mu \to A_\mu + \partial_\mu\alpha(x) \tag{3.3.12}$$

we obtain the anomaly

$$\mathcal{A}_\alpha = -\frac{N}{4\pi}\int \alpha dA\,, \tag{3.3.13}$$

where for simplicity we presented the anomaly via a one-form, $A = A_\mu dx^\mu$.

Now let us connect the nonlinear isometry anomalies with the gauge anomaly. To write the nonlinear sigma model, we need fix a gauge and choose a local chart to solve the constraints, see Eq. (3.3.6). Once the gauge and the chart are chosen, the isometries of rotation around u^N are linear, while those rotating the u^N axis have to be nonlinearly realized, see Eqs. (3.3.3) and (3.3.4).

Equation (3.3.5) implies that, since A_μ is isometry invariant, so is the fermion effective action (3.3.11). The only anomaly that exists in the gauged CP$(N-1)$ formulation is the gauge anomaly. Then how can we have isometry anomalies produced? Notice that, within the fixed gauge, the u^N field must be real. However, after rotations of u^N this field becomes complex again. Therefore, to satisfy the reality condition for u^N, the non-linear isometry transformations must be accompanied by a corresponding gauge transformation to offset the imaginary part of u^N. This leads to the gauge anomaly, or equivalently, isometry anomalies in geometric formulation. We also verify that it is indeed the anomalies of the nonlinear isometries by straightforward calculation in Appendix C.

Following the discussion above, we want to find a gauge parameter α, such that $\delta_\alpha u^N + \delta_\beta u^N$ is real. Since

$$\delta_\alpha u^N + \delta_\beta u^N = i\alpha u^N - \bar{\beta}_i u^i \tag{3.3.14}$$

where $i = 1, 2, ..., N-1$, the reality condition is

$$i\alpha u^N - \bar{\beta}_i u^i = -i\alpha u^N - \beta^i \bar{u}_i \,. \tag{3.3.15}$$

Therefore, we can find α in terms of ϕ^i and $\bar{\phi}_i$, namely,

$$\alpha = \frac{i}{2}\,(\beta\bar{\phi} - \bar{\beta}\phi)\,. \tag{3.3.16}$$

Furthermore, we rewrite the gauge field A in terms of ϕ^i and $\bar{\phi}_i$ as well,

$$A = -\frac{i}{2}\,\frac{\bar{\phi}d\phi - d\bar{\phi}\phi}{1 + \bar{\phi}\phi}\,, \tag{3.3.17}$$

what gives for dA

$$dA = \frac{ig_0^2}{2}\, g_{i\bar{j}}\, d\phi^i \wedge d\bar{\phi}^{\bar{j}}\,. \tag{3.3.18}$$

In this way obtain the nonlinear isometry anomalies,

$$\begin{aligned}
\mathcal{I}_\beta = \mathcal{A}_\alpha \;&=\; \frac{i}{8\pi}\int (\bar{\beta}\phi - \beta\bar{\phi})\left[iN\,\frac{(1+\bar{\phi}\phi)\delta_{i\bar{j}} - \bar{\phi}_i\phi_{\bar{j}}}{(1+\bar{\phi}\phi)^2}\,d\phi^i\wedge d\bar{\phi}^{\bar{j}}\right]\\
&=\; \frac{i}{8\pi}\int (\bar{\beta}\phi - \beta\bar{\phi})\,c_1\,,
\end{aligned} \tag{3.3.19}$$

where c_1 is the first Chern class of CP($N-1$).

In contradistinction with the O(N) sigma model (with the exception of CP(1), see below) all other CP($N-1$) sigma models suffer from the isometry anomalies which are neither a total derivative nor cancelable by adding local counter terms. For CP(1), the situation is identical to the O(3) model.

3.3.1 CP(1) is a special case

We find from Eq. (3.3.19) it is indeed total derivative and consistent with previous discussion on O($N-1$) model. The specialty that distinguish CP(1) from other CP($N-1$) models is its low dimension. Since CP(1) is geometrically a two dimensional sphere, locally we only have one ϕ and one $\bar{\phi}$ on one local chart. Equation (3.3.19) can be greatly simplified in this case and written as an integral over total derivative:

$$\mathcal{A}_{\mathrm{CP}(1)} = -\frac{N}{8\pi}\int \frac{\bar{\beta}\phi - \beta\bar{\phi}}{(1+\bar{\phi}\phi)^2}\,d\phi\wedge d\bar{\phi} = -\frac{N}{8\pi}\int d\left(\frac{\bar{\beta}d\phi + \beta d\bar{\phi}}{1+\bar{\phi}\phi}\right)\,. \tag{3.3.20}$$

On the other hand, globally CP(1) sigma model is known to have zero first Pontryagin class p_1, because it at most supports nonzero two-form while p_1 is an element in the fourth de Rham cohomology group. So far the local anomalies calculations are consistent with the global analysis of [56].

In this section we found the relation between isometry anomalies $\mathcal{I}$ and gauge anomaly $\mathcal{A}$. The isometry anomalies in geometric formulation can be understood as gauge anomaly of a special gauge transformation, see Eq. (3.3.16). Following this clue, one can prospect the correspondence of holonomy anomaly versus arbitrary gauge anomaly, and further global anomaly versus “large” gauge anomaly, in geometric and gauge formulations respectively.

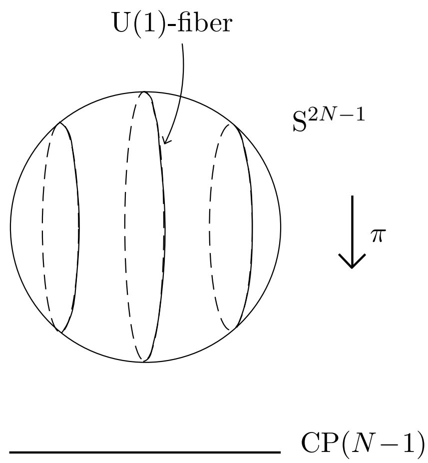

Figure 3.2: The sphere denotes S^{2N-1}, while the solid line below is for $CP(N-1)$. The vertical circles are U(1)-fibers, each of which is projected to a point on $CP(N-1)$.

3.3.2 A closer look at the correspondence between isometry and gauge anomalies

In this subsection, we want to discuss the correspondence between the isometry and gauge anomalies in a more rigorous mathematical way. It will also help us to apply these results in calculating the isometry anomalies in the general coset G/H minimal sigma model in our subsequent work.

First, we want to rephrase the construction of $CP(N-1)$ sigma model in the language of fiber bundle. The Lagrangian, Eq. (3.3.10) or (3.3.8), is constructed through the famous Hopf fibration, see Fig. 3.2 below, by considering CP^{N-1} as the base space of the U(1) principal bundle of S^{2N-1}, i.e.

$$U(1) \xrightarrow{i} S^{2N-1} \xrightarrow{\pi} CP(N-1) \ .$$

Equation (3.3.6) and the gauge condition that u^N is fixed to be real actually assign a map, or, say, a local section, from a local chart $U_s \subset \mathrm{CP}^{N-1}$ to S^{2N-1},

$$\begin{aligned} s:\ U_s &\longrightarrow \mathrm{S}^{2N-1} \\ (\phi^i, \bar{\phi}_j) &\longmapsto (u^i, u^N, \bar{u}_i, \bar{u}_N) \ = \Big(\frac{\phi^i}{\rho}, \frac{1}{\rho}, \frac{\bar{\phi}_j}{\rho}, \frac{1}{\rho}\Big) \text{ for } i, j = 1, 2, ..., N-1\,, \end{aligned}$$

with

$$\rho = (1 + \bar{\phi}_i \phi^i)^{1/2}\,.$$

Therefore it defines a local trivialization Φ of S^{2N-1}, so that S^{2N-1} locally looks like a product space of $U_s \times \mathrm{U}(1)$

$$\begin{aligned} \Phi:\ U_s \times \mathrm{U}(1) &\longrightarrow \mathrm{S}^{2N-1} \\ (\phi^i, \bar{\phi}_j; e^{i\alpha}) &\longmapsto s(\phi, \bar{\phi}) e^{i\alpha} \ \equiv \Big(\frac{\phi^i}{\rho}\, e^{i\alpha}, \frac{1}{\rho}\, e^{i\alpha}, \frac{\bar{\phi}_j}{\rho}\, e^{-i\alpha}, \frac{1}{\rho}\, e^{-i\alpha}\Big). \end{aligned} \quad (3.3.21)$$

It is easy to see that the U(1)-action on fiber $\pi^{-1}(\phi, \bar{\phi})$, is just the gauge transformation (3.3.1). We want to point out that, the U(1)-gauge A_μ in Eq. (3.3.5) is exactly a choice of *connection* 1-form defined on the bundle S^{2N-1}.

To see this, one needs to recall how to define a U(1)-connection on the principal bundle S^{2N-1}. First, the U(1)-action moves any point $p \in \mathrm{S}^{2N-1}$ along the fiber, which defines a one-dimensional subspace of the tangent space $\mathrm{T}_p\mathrm{S}^{2N-1}$, called vertical space V_p (see the tangential direction of the vertical circles in Fig. 3.2). The corresponding tangent vector σ_p spanning V_p is called the fundamental vector, and is given by the trivialization (3.3.21) as

$$\sigma_p = i u^i \frac{\partial}{\partial u^i} - i \bar{u}_i \frac{\partial}{\partial \bar{u}_i}, \text{ for } i = 1, 2, ..., N$$

subject to the constraint $\bar{u}_i u^i = 1$. Now we are about to assign a U(1) connection on S^{2N-1}, or a $2N-2$ dimensional horizontal subspace H, so that the tangent space of bundle S^{2N-1} can be decomposed as direct sum of horizontal and vertical spaces:

$$\mathrm{T}_p S^{2N-1} = H_p \oplus V_p, \text{ for } p \in \mathrm{S}^{2N-1}.$$

Equivalently, using a more familiar language, H is determined by a 1-form

$$\tilde{A} \in \Omega^1(\mathrm{S}^{2N-1})$$

globally defined on S^{2N-1}, so that

$$H_p = \mathrm{Span}\{X_p \in \mathrm{T}_p\mathrm{S}^{2N-1} | \tilde{A}_p(X_p) = 0\} = \ker\tilde{A}_p$$

with $\tilde{A}$ satisfying

$$\tilde{A}_p(\sigma_p) = 1 \quad \text{and} \quad R^*_\alpha \tilde{A}_p = \tilde{A}_{pe^{i\alpha}} \text{ for } p \in \mathrm{S}^{2N-1} \,. \tag{3.3.22}$$

In the second equation R^*_α is the pullback induced by the U(1) action on fibers, which guarantees the equivariance of $\tilde{A}$.

Generically there are various ways to choose the horizontal space H_p corresponding to different connection 1-forms $\tilde{A}$. However, for CP^{N-1} as the quotient space of S^{2N-1} by U(1), the projection map is a Riemann submersion once we assign the standard round metric $\tilde{g}$ on S^{2N-1}. Its tangent space at $\pi(p)$, i.e. $\mathrm{T}_{\pi(p)}\mathrm{CP}^{N-1} = \pi_* H_p$, is an orthogonal complement to V_p.

The metric $\tilde{g}$ defined on the standard sphere S^{2N-1}, with the coordinates $(u^i, \bar{u}_i)$, is given by

$$\tilde{g} = \frac{1}{2}(d\bar{u}_i \otimes du^i + du^i \otimes d\bar{u}_i), \text{ with } \bar{u}_i u^i = 1 \,.$$

Since we choose the horizontal space $H_p = V_p^\perp = (\mathrm{Span}\{\sigma_p\})^\perp$, the connection 1-form $\tilde{A}$ is thereby proportional to $\tilde{g}(\sigma_p)$,

$$\tilde{A} \sim \tilde{g}(\sigma_p) = -i\left(\bar{u}_i du^i - d\bar{u}_i u^i\right) .$$

To meet the condition (3.3.22), we fix the coefficient of $\tilde{A}$ as

$$\tilde{A} = -\frac{i}{2}\left(\bar{u}_i du^i - d\bar{u}_i u^i\right) .$$

One can see that this is just Eq. (3.3.5).

Finally, the connection $\tilde{A}$ is pulled back from S^{2N-1} to the local chart U_s of CP^{N-1} by section s. i.e.

$$s^* : \Omega^1(\mathrm{S}^{2N-1}) \longrightarrow \Omega^1(U_s) \ ,$$

$$\tilde{A} \longmapsto A \ = s^*\tilde{A} = -\frac{i}{2}\frac{\bar{\phi}d\phi - d\bar{\phi}\phi}{1+\bar{\phi}\phi} \,. \tag{3.3.23}$$

Once we assign a new section

$$s' : U_{s'} \to \mathrm{S}^{2N-1} \,,$$

it is clear that, on the intersection $U_s \cap U_{s'}$, any two points on one and the same fiber mapped by s and s' are related by a U(1) action, i.e.

$$s'(\bar{\phi}, \phi) = s(\bar{\phi}, \phi)e^{i\alpha(\bar{\phi},\phi)}, \text{ for } (\bar{\phi}, \phi) \in U_s \cap U_{s'} .$$

Similarly, s' will also pullback the connection $\tilde{A}$ to $A' = s'^*\tilde{A}$. Moreover, A' and A are related by our familiar U(1)-gauge transformation

$$A' = A + d\alpha \,.$$

Based on the discussion above, the Lagrangian, Eq. (3.3.10) and Eq. (3.3.8), could be interpreted as CP^{N-1} model constructed on the bundle S^{2N-1} or a local patch $U_s \subset \mathrm{CP}^{N-1}$. Now, when we consider an isometry transformation f on a local patch, say, U_s, the isometry will induce a change of U_s and thus a gauge transformation of A. Therefore when we calculate the isometry anomalies of CP^{N-1} localized on U_s, they are naturally associated to the gauge anomalies of A locally defined on U_s.

To address the idea in a rigorous manner, we need the concept of the bundle isomorphism. A bundle isomorphism is a $1-1$ bundle map F such that the following diagram commutes:

$$\begin{tikzcd} \mathrm{S}^{2N-1} \arrow[r, "F"] \arrow[d, "\pi"'] & \mathrm{S}^{2N-1} \arrow[d, "\pi"] \\ \mathrm{CP}^{N-1} \arrow[r, "f"] & \mathrm{CP}^{N-1} \end{tikzcd}$$

where f is an induced isomorphism on base space CP^{N-1}, and F satisfies the equivariant condition:

$$F(pe^{i\alpha}) = F(p)e^{i\alpha} \,. \tag{3.3.24}$$

As for the isometry transformations, one needs to consider the corresponding isometric bundle isomorphism, i.e. isomorphisms preserving given metric $\tilde{g}$ on the bundle S^{2N-1} and satisfying equivariance Eq. (3.3.24). For the bundle S^{2N-1}, there are $2N^2 - 2N$ isometries as we discussed in Sec. 3.2, while only the transformations (3.3.3) satisfy the condition (3.3.24) and induce the isometries on CP^{N-1}.

Now we are interested in the transformation of the connection A with respect to isometric bundle morphisms. Generically a bundle isomorphism F will "pushforward" the horizontal space H to

$$H^F \equiv F_* H \,.$$

Therefore the corresponding connection 1-form for H^F is

$$\tilde{A}^F = (F^{-1})^* \tilde{A} \,.$$

We want to calculate the difference of A^F from A pulled back to the base space CP^{N-1}, e.g. at the point $b \in U_s \subset \mathrm{CP}^{N-1}$. Note that the isometric bundle morphism F, i.e. Eq. (3.3.3), also induces an isometric morphism f on CP^{N-1}, see Eq. (3.3.7). Isomorphism f moves the point $b = (\phi, \bar{\phi})$ to $c = f(b)$, located on a different fiber $\pi^{-1}(c)$. One thus needs to further pullback the connection by f^* to compare their difference, see the commuting diagram below,

$$\begin{tikzcd} \mathrm{S}^{2N-1} & \mathrm{S}^{2N-1} \arrow[l, "F^{-1}"'] \\ U_{s'} \arrow[u, "s'"] \arrow[r, "f"] & U_s \arrow[u, "s"'] \end{tikzcd} \qquad \begin{tikzcd} \Omega^1(\mathrm{S}^{2N-1}) \arrow[r, "(F^{-1})^*"] \arrow[d, "s'^*"'] & \Omega^1(\mathrm{S}^{2N-1}) \arrow[d, "s*"] \\ \Omega^1(U_{s'}) & \Omega^1(U_s) \arrow[l, "f^*"'] \end{tikzcd}$$

The variance of connection

$$f^* \circ s^* \tilde{A}^F - s^* \tilde{A} = (F^{-1} \circ s \circ f)^* \tilde{A} - s^* \tilde{A}$$

respect to point $b = (\phi, \bar{\phi})$, will be considered. However, the combination of maps $s' \equiv$

$F^{-1} \circ s \circ f$ defines another section,[5]

$$s' : U_{s'} \to \mathrm{S}^{2N-1}\,.$$

Therefore one has

$$s'(b) = s(b)e^{i\alpha(b)}, \quad \forall\, b \in U_s \cap U_{s'}\,, \tag{3.3.25}$$

and therefore

$$s'^{*}\tilde{A} - s^{*}\tilde{A} = A' - A = d\alpha\,. \tag{3.3.26}$$

Given Eq. (3.3.3) and Eq. (3.3.7) for infinitesimal versions of F and f, we can calculate the infinitesimal transformation of Eq.(3.3.26). We only consider transformations corresponding to the parameters β and $\bar{\beta}$. Since

$$e^{i\alpha(\phi,\bar{\phi})} \sim 1 + i\alpha(\phi,\bar{\phi},\beta,\bar{\beta})\,,$$

the infinitesimal transformation of Eq. (3.3.26) and Eqs. (3.3.3) and (3.3.7) lead us back to Eq. (4.2.43). After a short calculation, we obtain

$$\alpha(\phi,\bar{\phi},\beta,\bar{\beta}) = \frac{i}{2}(\beta\bar{\phi} - \bar{\beta}\phi)\,,$$

which coincides with the previous result (3.3.16).

So far we revisited the anomaly of the CP($N-1$) sigma model. The lesson one can draw is that the nonlinear formalism Lagrangian, see Eq. (3.3.8), is defined on the local patch U_s of CP($N-1$). An isometric transformation f, or F on the bundle will result in a change of the local patch to $U_{s'}$, or equivalently a change of the local section to s'. Therefore the pulled-back connection, or the gauge field (3.3.5), will transform as in Eq.(3.3.26). If there are chiral fermions coupling to the gauge field nontrivially, there must be anomalies produced. In this sense, these anomalies measure the failure of bundle reparametrization from the section s to s' induced by isometric transformation.

[5] It is the the pulled-back section of s by F, and hence depends on the bundle map.

3.4 Dual formalism for the O(N) Model

In Sec. 3.3 we demonstrated that the isometry anomalies in the nonlinear realization of the CP($N-1$) is in one-to-one correspondence with the U(1) anomaly in its gauged linear formulation. This section is motivated by further consistency checks of the gauge versus isometry anomalies. Another motivation is the large-N argument regarding the gauge anomaly in the linear gauged sigma models.

A crucial difference between the O(N) and CP($N-1$) sigma models is that the latter has a U(1) gauge field, and eventually suffers from the U(1) anomaly. At the same time, the O(N) sigma model has no gauge redundancy and therefore is expected to have no isometry anomalies after the passage to its nonlinear formulation.

In Sec. 3.2 we considered the O(N) model (which can also be called the S^{N-1} model) using the realization of the target space in terms of N real fields n^i with the constraint (3.2.1). The real Grassmann model prompts us a dual form of the O(N) model.

The same target space, S^{N-1}, can be implemented as follows. Consider real bosonic matrix fields

$$N^{\alpha}_{\ a}\,,\qquad \alpha=1,2,...,N\,,\qquad a=1,2,...,N-1\,,\tag{3.4.1}$$

and gauge the SO($N-1$) symmetry. The index α in (3.4.1) will play the role of the "color" index of the gauged group SO($N-1$). The index a is the "flavor" index of global SO(N) symmetry. Then we add a constraint

$$(N^T)^a{}_{\alpha} N^{\alpha}_{\ b}=\delta^a_{\ b}\,.\tag{3.4.2}$$

We also add left-handed fermions ψ^{α}_{La} with appropriate constraints to supersymmetrize the model. In this way we arrive at the Lagrangian

$$\begin{aligned}\mathcal{L}&=\frac{1}{2g_0^2}\mathrm{Tr}[(D^{\mu}N)^T D_{\mu}N+i\bar{\psi}_L D_R\psi_L]\,,\\ N^{Ta}{}_{\alpha}N^{\alpha}_{\ b}&=\delta^a_{\ b}\,,\qquad (N^T)^a{}_{\alpha}\psi^{\alpha}_{Lb}=0\,.\end{aligned}\tag{3.4.3}$$

where

$$(D_{\mu}N)^{\alpha}_{\ a}=\partial_{\mu}N^{\alpha}_{\ a}-N^{\alpha}_{\ b}A^{b}_{\mu a}\,,\tag{3.4.4}$$

and the matrix fields $A^b_{\mu a}$ are the SO($N-1$) gauge fields. As previously mentioned, the above gauge fields are nondynamical and can be eliminated in favor of the N fields,

$$A^a_{\mu b} = \frac{1}{2}\left(N^T \partial_\mu N - \partial_\mu N^T \cdot N\right)^a{}_b \,. \tag{3.4.5}$$

Similarly to Eqs. (3.3.5) and (3.3.6) in the CP($N-1$) model, one can fix an SO($N-1$) gauge, and choose local charts to write down a nonlinear sigma model. For example, we treat

$$N^\alpha{}_a = \begin{pmatrix} V^i{}_a \\ \rho_a \end{pmatrix}, \qquad a, i = 1, 2, ..., N-1\,, \tag{3.4.6}$$

where $\rho_a \equiv N^N{}_a$ is an additional row vector.

Now, we fix the gauge in such a way that $V^i{}_a$ becomes symmetric real matrix. Then we define a local chart,

$$\phi_i = \rho_a \left(\frac{1}{1+V}\right)^a{}_i \,. \tag{3.4.7}$$

After solving the constraint, one obtains

$$\begin{aligned} V^i{}_a &= \left(\delta^i{}_j - \frac{2}{1+\phi^2}\phi^i\phi_j\right)\delta^j{}_a\,, \\ \rho_a &= \frac{2\phi_i}{1+\phi^2}\delta^i{}_a\,, \\ A^a_{\mu b} &= \frac{2\phi^i\partial_\mu\phi^j}{1+\phi^2}E_{ij}{}^a{}_b\,. \end{aligned} \tag{3.4.8}$$

The generators $E_{ij}{}^a{}_b$ were defined in Eq. (3.2.22).

One can easily convince oneself that the gauge fields A_μ are just spin connections ω_μ in nonlinear formulation of the S^{N-1} model, see Eq. (3.2.22). Thus, the nonlinear Lagrangian following from (3.4.3) after gauge fixing is in fact identical to that presented in Eq. (3.2.10).

At the perturbative level, the gauge anomalies in the present section and the isometry anomalies in Sec. 3.2 will match each other too. We will discuss only those isometry transformations that involve an interplay between $V^i{}_a$ and ρ_a,

$$\delta_\alpha V^i{}_a = \alpha^i \rho_a\,, \qquad \delta_\alpha \rho_a = -\alpha_i V^i{}_a\,, \tag{3.4.9}$$

since they would induce gauge anomalies for the fixed gauge, see the remark after Eq. (3.4.6). To keep the matrix $V^i_{\ a}$ symmetric, a gauge transformation must accompany (3.4.9), namely,

$$\delta_\lambda V^i_{\ a} = V^i_{\ b}\lambda^b_{\ a}\,, \quad \text{with } \lambda^T = -\lambda\,. \tag{3.4.10}$$

Solving equation

$$\delta_{\alpha+\lambda} V = (\delta_{\alpha+\lambda} V)^T\,, \tag{3.4.11}$$

we arrive at

$$\lambda^a_{\ b} = v^a_{\ b}\,, \tag{3.4.12}$$

where the matrix $v^a_{\ b}$ is given in Eq. (3.2.23). Therefore, the induced gauge anomalies are

$$\mathcal{A}_\lambda = -\frac{1}{8\pi}\int \mathrm{Tr}\,(\lambda dA)\,. \tag{3.4.13}$$

Equations (3.4.8) and (3.4.12) show that $\mathcal{A}_\lambda$ is just the nonlinear isometry anomalies, the same as in Eq. (3.2.21). The theory can be "mended" just in the same way as it was discussed in Sec. 3.2. As a consistency check, we remark that, if one follows the computation in next chapter, see also [40], with unfixed gauge, one could immediately write down a counter term which explicitly contains the gauge field A, and the counter term is not necessarily of the form as given in Eq. (3.2.19). But the special gauge fixing in the dual formalism we high-lighted here, evacuates the possibility, and thus here the counter term is forced to be the same as the one in Sec. 3.2, which can be checked explicitly.

3.5 Conclusions

Two-dimensional chiral sigma models with various degrees of supersymmetry present an excellent theoretical laboratory. While the $(2,2)$ models were thoroughly explored in the 1980s, chiral models received much less attention. Recently non-minimal chiral models reappeared in the focus of theorists' attention because of their special role as world-sheet models on topological vortex solutions supported in certain four-dimensional $\mathcal{N}=1$ Yang-Mills theories. This fact naturally raised interest to the minimal chiral models which serve the fundamental building blocks to compose general $\mathcal{N}=(0,1)$ and $(0,2)$ chiral sigma models.

In this chapter the minimal chiral two-dimensional models are revisited. We demonstrate that the Moore-Nelson consistency condition [56] revealing a global anomaly in CP($N-1$) (with $N>2$) due to a nontrivial first Pontryagin class is in one-to-one correspondence with the local anomalies in isometries. These latter anomalies are generated by fermion loop diagrams which we explicitly calculate.

At the same time the first Pontryagin class in the O(N) models vanishes [56] and, thus, these models are globally self-consistent. We show that the divergence of the isometry currents in these models is anomaly free. Thus, there are no obstructions to quantizing the minimal $\mathcal{N}=(0,1)$ models with the $S^{N-1}=\mathrm{SO}(N)/\mathrm{SO}(N-1)$ target space. CP(1) is self-consistent and presents an exceptional case from the CP($N-1$) series: both the first Pontryagin class vanishes and the local anomalies are absent too. We discuss a relation between the geometric and gauged formulations of the CP($N-1$) models. From the standpoint of the principal fiber bundle, the isometry anomalies on a local patch just reflect the failure of gauge invariance of the theories in passing from one local patch to another. Therefore it relates the local anomalies to global topological criteria [56, 66]. In our next chapter, we will follow this clue to discuss anomalies in general minimal G/H sigma models in both local and global aspects. The obvious distinction between the O(N) and CP($N-1$) target spaces is the fact that in the first case the factor H is a simple group, while in the second case it is a product two factors, SU($N-1$) $\times$ U(1). One can conjecture that the non-simple character of H is behind the emergence of anomalies. For the simple H group the first Pontryagin class of G/H vanishes. We will address this issue and prove it soon [40].

The dual formalism of O(N) models was also discussed carefully. This is not just a crucial statement to enhance our result, but also a hopeful tool to further explore dualities and integrabilities in relevant models. The comparison with the usual formulation shows that the geometric model is essentially defined by the global symmetry together with the topology type while independent of the gauging process. So classically the dual model is equivalent to the normal O(N) formalism by implementing the equation of motion of A_μ, see Eq. (3.4.8), and the ease of anomalies of dual O(N) model at UV regime can be reduced to the discussion in Sec. 3.2. However, quantum mechanically gauge fields are used to acquire kinetic terms and cannot be integrated out as auxiliary fields [4], one should also worry about the genuine gauge anomalies for the dynamical gauge fields, i.e. Eq. (3.4.13), in its own right in IR regime. This has been implied that the gauge dynamics is more involving. In next chapter, we will also take the first step [40] to attack this issue. We will systematically consider this problem with general external gauge fields universally in every reasonable gauge.

Chapter 4

Anomalies in Minimal $\mathcal{N} = (0, 1)$ and $\mathcal{N} = (0, 2)$ Sigma Models on Homogeneous Spaces

4.1 Introduction and summary

In this chapter, we will continue to focus on minimal supersymmetric models with $\mathcal{N} = (0, 1)$ or $\mathcal{N} = (0, 2)$ supersymmetry. As we have seen that such models, generally speaking, exhibit chiral fermion anomaly which imposes severe constraints on the topology of the target manifold [50, 56, 74]. Due to this reason, such minimal supersymmetric models are explored to a lesser extent than non-chiral models. The guiding principle established [56], as well as cases checked in last chapter, for the chiral $\mathcal{N} = (0, 1)$ or $\mathcal{N} = (0, 2)$ sigma models is the first Pontryagin class.

Our present chapter is motivated by the following consideration. Firstly, in supersymmetric theories rather often simplicity of the theory increases with the number of supercharges. By simplicity we mean that the theory under consideration can have special properties allowing one to obtain exact results or uncover elegant mathematical structures. On the other hand, theories with less supersymmetry, presenting more difficulties for theoretical analysis, are sometimes closer to physical phenomena, and as such must be thoroughly studied. Now we can interpret these minimal modes with $\mathcal{N} = (0, 1)$ and

$(0, 2)$ supersymmetries in such a critical region: They are more closed to the real physics, meanwhile highly restricted by anomalies and topological constraints. Therefore study of them is closer to non-supersymmetric world but with relatively more theoretical tools.

Secondly, the global anomaly cancellation condition does not touch the local behavior of the theory. Even when one has a "good" theory, which has no global anomaly [56], it does not automatically mean that one gets the well-defined theory for free. Ease of global anomaly only implies that one is able to introduce "local counterterms" to correctly integrated out chiral fermions and find the anomaly-free fermionic effective action. The "local counterterm" here is not to be confused with the terms added to absorb various divergences in the process of renormalization, since the quantization of fermions in two dimensions is insensitive to RG flow. In fact the roles played by what we call local counterterms are similar to that of the contact term in gauge theories, which is added to keep the transversality of certain polarization operators. Since the latter is sometime also referring to the Schwinger term, we refrain from using it here. Moreover, by explicitly curing such a theory (i.e. adding appropriate local counterterms), one can exhibit many quantum aspects of the theory in a more understandable way, thus enabling one to initiate a discussion of the infrared (IR) behavior of the theory, which was not carried out previously.

Thirdly, many sigma models have more than one equivalent formulations: a nonlinear description based on the Riemannian metric that encodes geometric information, an embedding into a larger linear target space and then imposing extra gauge symmetries, or constraints, or a hybrid way lying between the above two formulations as what we discussed last chapter [39]. Although classically all these formulations are equivalent, at the quantum level one could have different considerations depending on the formulation. For example, in the nonlinear formulation it is easy to understand the global chiral fermion anomaly, while using the gauge formulation, one will be focused on the gauge anomaly. Work has been done on these aspects [56, 63–69, 74], providing us with starting positions. The precise relation between the gauge anomaly and global anomaly for different formulations of the very same model was not thoroughly discussed previously. In this chapter, we will down-to-earth study of the chiral sigma models on homogeneous spaces, for which both nonlinear and gauge formulations are present. We reveal the relation between different anomalies. Our result also provides us with a generalized context for the determinant line bundle consideration of the fermion anomaly. In non-homogeneous spaces one can not compare global and gauge anomalies on the nose, but an analogous structure was revealed in the case of the Kähler manifold. This chapter generalizes and extends the results of the work [39].

Finally, we would like to emphasize possible applications of our results in model building. Models with large supersymmetry may be viewed as being composed of theories with less supersymmetry. In this regard, understanding of the minimal supersymmetric models as the building blocks for all supersymmetric theories is of importance.

In practice, the usual situation is opposite. For example, $\mathcal{N} = (2, 2)$ theories are always better understood than $\mathcal{N} = (0, 2)$ and explored earlier. Softly broken to $\mathcal{N} = (0, 2)$ theories (free from chiral anomalies) they are easier for explorations [19, 20, 35, 36, 38, 41, 42, 44, 45, 47, 53]. We hope that our work on the minimal models can give insights for understanding of more complicated models.

We discuss at length the IR behavior for many models and observe a new connection with superconformal models [36, 51, 52].

The chapter is organized as follows. In Sec. 4.2, we will first construct bosonic and $\mathcal{N} = (0, 1)$ supersymmetric sigma models on homogeneous spaces by virtue of a hidden gauge formulation. The calculation of their isometry anomalies is given in Sec. 4.2.3. As discussed in the previous chapter, we show that the isometry anomalies reflect the failure of bundle re-parameterization from local section s to s' induced by the isometry transformations, where $s,\ s' : U_s \cap U_{s'} \subset M \to G$.

To offset these aforementioned anomalies, we are led to consider (Sec. 4.3) more generic holonomy anomalies, of which isometry anomalies are a special class. We give criteria ensuring holonomy as well as isometry anomalies to be removed by adding well-defined local counterterms (Sec. 4.3.1). With these criteria, and after adding appropriate local counterterms, we discuss the low-energy behavior of the minimal $\mathcal{N} = (0, 1)$ sigma models (Sec. 4.3.2). In Sec. 4.3.3 several concrete examples are given to illustrate the idea. We review appropriate tools that we had developed before. The topological origin of the anomalies and counterterms are discussed in Sec.4.3.4.

In Sec. 4.4 we begin to relate the holonomy and isometry anomalies to topological anomalies in a general context. In Sec. 4.4.1 a discussion of the isometry anomalies in the general Kähler sigma models is given, parallel to the relation between the non-Abelian gauge anomaly and chiral anomaly in gauge theories. In Sec. 4.4.2 we show how the isometry anomaly in pure geometric formulation relates to the topological chiral fermion anomaly in terms of the determinant line bundle discussed by Moore and Nelson, and Freed. In Sec. 4.4.3 we give the determinant line bundle description for the holonomy anomaly for sigma models over homogeneous spaces. This completes a unified picture showing that the holonomy (gauge) anomaly and the topological anomaly are due to the nontriviality of a single determinant line bundle over the space of fields.

4.2 Isometry anomalies

We will formulate this section by following the logic line of our previous work [39] where we construct sigma models on S^{2N-1} and gauge its U(1) factor to deduce the corresponding CP^{N-1} models by the fiberation:

$$\mathrm{U}(1) \xrightarrow{i} \mathrm{S}^{2N-1} \xrightarrow{\pi} \mathrm{CP}^{N-1} .$$

Similarly for homogeneous spaces, we also have a canonical fiberation:

$$H \xrightarrow{i} G \xrightarrow{\pi} M .$$

Therefore we first construct sigma models on group manifold G, and gauge certain subgroup H to obtain sigma model on homogeneous spaces M. Analogue to CP^{N-1} case, to define a sigma model on M, one needs to specify a local patch $U_s \subset M$ and a section $s : U_s \to G$. To discuss isometry anomalies on model M, we will show that an isometric transformation $l_k : M \to M$ will induce a change of section s to s', and thus, a H-gauge transformation. For chiral fermions non-trivially coupled to these H-gauge, there will be isometry anomalies produced. We will calculate them by the end of this section. For simplicity, we only consider G as a connected, compact and semi-simple Lie group and H is its closed Lie subgroup.

4.2.1 Sigma models on M through gauge formulation

For sigma model on M, the construction can be traced back to 70's due to Callan-Coleman-Wess-Zumino (CCWZ) coset construction [1]. In this subsection, we will review this construction but from the so-called "hidden" local gauge formulation, which will be eventually explained in the language of principal bundle.

To have sigma model on M, as mentioned in the beginning, one first construct sigma model on group manifold G, and then "gauge" it down to that of space M. We will see soon that such a construction is just the formulation with a "hidden" local right-H gauge, see [2], in which the Nambu-Goldstone bosons are taking values in the group G instead of M, and the right local H-gauge help eliminate redundant degree of freedom. Each time that one chooses a fixed gauge is equivalent to choose a local section to "pullback" the model defined on bundle G to base space M, and thus the language of principal bundle will be an ideal mathematical framework to interpret the model and further anomalies if there are any.

Since G is semi-simple, one can always use the Killing form $K : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$, which is negative definite, to define the metric $\bar{\gamma}$ of G. We consider the Lie algebra $\mathfrak{g}$ in its fundamental representation,[1] and normalize the anti-hermitian generators F_A as:

$$K(F_A, F_B) = \mathrm{Tr}(F_A F_B) = -\delta_{AB} \ . \tag{4.2.1}$$

In most of the note, we focus on sigma models defined on simple groups G. For bosonic sigma model on such a group G, the action is given by

$$S_G = \frac{1}{2\lambda^2} \int_\Sigma \mathrm{d}^2 x \, \mathrm{Tr}(\partial_\mu g^{-1} \partial^\mu g) = -\frac{1}{2\lambda^2} \int_\Sigma \mathrm{d}^2 x \, \mathrm{Tr}(g^{-1} \partial_\mu g g^{-1} \partial^\mu g), \tag{4.2.2}$$

where Σ is the two-dimensional spacetime manifold, $g = g(x)$ taking value on matrix group G, and λ^2 is a coupling constant.[2] It is seen in Eq. (4.2.2) that $g^{-1}\partial^\mu g$ is the Maurer-Cartan

[1] It is true that Killing form is defined by means of adjoint representation of G, but for semi-simple Lie algebra one is free to rescale a constant for each simple factor and thus we can choose fundamental representation as our bench mark.

[2] For a semi-simple Lie group G, there are as many coupling constants λ_i^2 as the number of its simple factors G_i, and the Killing form K is the direct sum of K_i for each G_i.

form $\theta_g \equiv g^{-1}dg$ pullback to the cotangent space of spacetime Σ. For $g^{-1}dg \in \mathrm{T}^*G$ on G defines map:

$$\theta_g = L_{g^{-1}*} : \mathrm{T}_g G \to \mathrm{T}_e G = \mathfrak{g}\,, \tag{4.2.3}$$

where $L_{g^{-1}*}$ is the pushforward map induced by left translation $L_{g^{-1}}$, and $\mathrm{T}_e G$ is the tangent space of G at group identity e, we thereby have the metric $\bar{\gamma}$ defined as

$$\bar{\gamma}\,(X_g, Y_g) \equiv -K(L_{g^{-1}*}X_g, L_{g^{-1}*}Y_g) = -L^*_{g^{-1}}K(X_g, Y_g)\ , \tag{4.2.4}$$

where X_g and Y_g are two vector fields at point $g \in G$.

On a local chart $\{U, \phi^\alpha\}$ near identity $e \in G$, we can use exponential map to express $g(x)$ as[3]

$$g(x) = \mathrm{Exp}(\delta^A_\alpha \phi^\alpha(x) F_A), \quad \text{for } A, \alpha = 1, 2, ..., \dim\, G\,,$$

where $\phi^\alpha(x)$ are Nambu-Goldstone bosons. Therefore one can express θ_g and $\bar{\gamma}$ in a more familiar way as

$$\begin{aligned} \theta(\phi) &= \theta^A_{\ \alpha}(\phi) d\phi^\alpha F_A\,, \\ \bar{\gamma}_{\alpha\beta}(\phi) &= \delta_{AB}\theta^A_{\ \alpha}\theta^B_{\ \beta}\,, \end{aligned} \tag{4.2.5}$$

where $\theta^A_{\ \alpha}$ is the vielbein to decompose $\bar{\gamma}_{\alpha\beta}$. Notice that the vielbein one-form is left invariant, and right equivariant,

$$\begin{aligned} L^*_{g_0}\theta &= (g_0 g)^{-1} d(g_0 g) = \theta \\ R^*_{g_0}\theta &= (g g_0)^{-1} d(g g_0) = g_0^{-1}\theta g_0, \qquad \text{for } g_0 \in G\,. \end{aligned} \tag{4.2.6}$$

The metric $\bar{\gamma}$ defined above is consequently left and right invariant,

$$L^*_{g_0}\bar{\gamma} = R^*_{g_0}\bar{\gamma} = \bar{\gamma}, \ \text{for any } g_0 \in G\,.$$

Therefore, the action S_G has isometries $G_L \times G_R$.

[3] We use Greek and capital letters to distinguish the indexes of curved coordinates and that of flat vector space $\mathfrak{g}$

Now we consider group G as a principal bundle with fiber H and base space $M \equiv G/H$,

$$H \xrightarrow{i} G \xrightarrow{\pi} M$$

with the projection

$$\begin{aligned} \pi : G &\longrightarrow M \\ g &\longmapsto gH \end{aligned} \tag{4.2.7}$$

and H-group action acting from right on G, satisfies $\pi(gh) = \pi(g)$.

To define a sigma model on M, we notice formula (4.2.7) that H-group action is from right to obtain M coset space. It motivates us to gauge part of right isometries $H \subset G_R$ of sigma model on group G. Consider, $g(x) \to g(x)h(x)$ for a right $h(x) \in H$ transformation, the Maurer-Cartan form changes as:

$$g^{-1}dg \to h^{-1}(g^{-1}dg)h + h^{-1}dh\,. \tag{4.2.8}$$

To make it gauge invariant, we introduce gauge fields

$$A(x) = A^i_\mu(x)dx^\mu H_i\,, \tag{4.2.9}$$

where $H_i \in \mathfrak{h}$ for $i = 1, 2, ..., \dim\mathfrak{h}$, taking values on Lie subalgebra $\mathfrak{h}$. It transforms as

$$A \to h^{-1}Ah + h^{-1}dh$$

to remedy the additional $h^{-1}dh$ part of gauge transformation of $g^{-1}dg$. Therefore

$$g^{-1}dg - A \to h^{-1}(g^{-1}dg - A)h$$

is gauge covariant. The action on M is thus given by

$$S_M = -\frac{1}{2\lambda^2}\int_\Sigma \mathrm{d}^2x\,\mathrm{Tr}\big[(g^{-1}\partial_\mu g - A_\mu)(g^{-1}\partial^\mu g - A^\mu)\big]\ . \tag{4.2.10}$$

After an appropriate gauge fixing, the action above will give usual CCWZ coset construction. To see this, let us work out the action near group identity e, where we will decompose Maurer-Cartan form $\theta_g = g^{-1}dg$ locally, see Eq. (4.2.5), along vertical space $\mathfrak{h}$ and a horizontal space complimentary to $\mathfrak{h}$.

Firstly, for a connected, compact and semi-simple Lie group G with its closed subgroup H, the coset space M is *reductive* homogeneous space, i.e. the Lie algebra $\mathfrak{g}$ of G can be decomposed as

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}\,, \tag{4.2.11}$$

where $\mathfrak{h}$ is the subalgebra corresponding to the subgroup H, and $\mathfrak{m}$ is a transverse subspace that is preserved by the adjoint action of H, i.e.,

$$\mathrm{ad}_H \mathfrak{m} = \mathfrak{m}\,. \tag{4.2.12}$$

In principle, subspace $\mathfrak{m}$ complimentary to $\mathfrak{h}$ is quite arbitrary. However, similar to the discussion of CP^{N-1} embedded into S^{2N-1} [39], we can utilize the Killing form K, see Eq. (4.2.1), to define

$$\mathfrak{m} = \mathfrak{h}^{\perp}\ ,$$

so that homogeneous space M is a Riemann submersion of G, and the tangent space $\mathrm{T}_o M$, with $o \equiv \pi(e)$, is identified with $\mathfrak{m}$. Under this decomposition, for $H_i \in \mathfrak{h}$ and $X_a \in \mathfrak{m}$, we have

$$\mathrm{Tr}(H_i H_j) = -\delta_{ij},\ \ \mathrm{Tr}(X_a X_b) = -\delta_{ab},\ \text{ and } \mathrm{Tr}(H_i X_a) = 0\,. \tag{4.2.13}$$

Now θ_g is decomposed as

$$\theta_g(\phi) = e_g(\phi) + \omega_g(\phi) \equiv e_g^a X_a + \omega_g^i H_i\,, \tag{4.2.14}$$

where e_g are called *basic forms* and ω_g is *canonical connection* for bundle $\pi : G \to M$.

Now we can use gauge fields to eliminate redundant degrees of freedoms. For CCWZ construction, the unitary gauge is chosen to remove all Nambu-Goldstone bosons on $\mathfrak{h}$, i.e.,

$$g(\phi) = \mathrm{Exp}(\delta^a_\alpha \phi^\alpha X_a),$$

on a local chart $\{\phi^\alpha \in U_s \subset M\}$ near $o \in M$. Such a choice, geometrically speaking, is equivalent that we specify a local section $s : U_s \subset M \to G$,

$$s(\phi) \equiv g(\phi) = \mathrm{Exp}(\delta^a_\alpha \phi^\alpha X_a)\,. \tag{4.2.15}$$

Therefore, one can use $s^* : \mathrm{T}^*G \to \mathrm{T}^*M$ pullback basic forms e_g to M ,

$$e_\phi \equiv s^* e_g = e^a{}_\alpha d\phi^\alpha X_a\,,$$

and thus define the vielbein one-form e_ϕ on M. Similarly, canonical connection ω_g is also pullback:

$$\omega_\phi \equiv s^* \omega_g = \omega^i_\alpha d\phi^\alpha H_i$$

as connection one-form locally defined on M.

After fixing the gauge by Eq. (4.2.15), we localized the Lagrangian on a local chart $\{\phi^\alpha \in U_s \subset M\}$:

$$S_M = -\frac{1}{2\lambda^2}\int_\Sigma \mathrm{d}^2x\,\mathrm{Tr}(e^a_\mu X_a + \omega^i_\mu H_i - A^i_\mu H_i)^2\,,$$

where

$$e^a_\mu = e^a{}_\alpha \partial_\mu \phi^\alpha, \quad \omega^i_\mu = \omega^i{}_\alpha \partial_\mu \phi^\alpha$$

are vielbeins and connection one-form further pullback to spacetime Σ by the map $\phi^\alpha : \Sigma \to U_s \subset M$.

Since gauge fields A_μ classically is non-dynamical, one can solve and express them in terms of goldstone fields ϕ^α by equations of motion, and we get

$$A^i_\mu = \omega^i_\alpha \partial_\mu \phi^\alpha\,. \tag{4.2.16}$$

Getting this expression back to action Eq. (4.2.10), we find the action S_M by CCWZ construction,

$$\begin{aligned} S_M &= -\frac{1}{2\lambda^2}\int_\Sigma \mathrm{d}^2x\,\mathrm{Tr}(e^a{}_\alpha X_a e^b{}_\beta X_b)\partial_\mu\phi^\alpha\partial^\mu\phi^\beta \\ &= \frac{1}{2\lambda^2}\int_\Sigma \mathrm{d}^2x\,\delta_{ab} e^a{}_\alpha e^b{}_\beta \partial_\mu\phi^\alpha\partial^\mu\phi^\beta \equiv \frac{1}{2\lambda^2}\int_\Sigma \mathrm{d}^2x\,\gamma_{\alpha\beta}\partial_\mu\phi^\alpha\partial^\mu\phi^\beta\,. \end{aligned} \tag{4.2.17}$$

With that

$$\gamma_{\alpha\beta} = \delta_{ab} e^a{}_\alpha e^b{}_\beta, \text{ for } a, b, \alpha, \beta = 1, 2 \ldots \dim \mathfrak{m}$$

is the metric on M and $e^a{}_\alpha$ is its vielbeins correspondingly.

4.2.2 $\mathcal{N} = (0, 1)$ supersymmetric sigma model on M

In this subsection we will supersymmetrize the action of sigma model on $M = G/H$, see Eq. (4.2.10). In two-dimensional spacetime, we have Weyl-Majorana Grassmannian variable θ_R which helps form the smallest representation of supersymmetry, i.e. (0, 1) supersymmetry. The superderivative in superspace is defined as [4]

$$D_L = -i \frac{\partial}{\partial \theta_R} - \theta_R \partial_{LL}$$

satisfying

$$\{D_L, D_L\} = 2D_L^2 = 2i\partial_{LL}$$

where ∂_{LL} denotes the partial derivative along light-cone coordinate x_L, and ∂_{RR} for that of x_R in what follows. The integration over Grassmannian variable θ_R is equal to differentiation:

$$\int \mathrm{d}\theta_R = \frac{\partial}{\partial \theta_R} = iD_L|_{\theta_R=0} \ .$$

An ordinary bosonic field ϕ will be promoted to its superversion Φ, which is consisted of ϕ and a left-moving fermion ψ_L:

$$\Phi = \phi + i\theta_R \psi_L$$

To supersymmetrize the action Eq. (4.2.10), beside scalar superfield $g(\Phi)$, we also need $(0, 1)$ supergauge multiplets $\{\mathcal{V}_L,\ \mathcal{V}_{RR}\}$ [78]. It is true that one can directly supersymmetrize the local form of Lagrangian in Eq. (4.2.17), which is already localized on certain patch of M, without introducing any auxiliary gauge fields. However, with the help of gauge fields, it is quite easy to track the information of isometric transformations on different local charts,

[4] We here consider $\mathcal{N} = (0, 1)$ supersymmetry, so the notation is different from that in Appendix A. θ_R is real Grassmannian variable parameterizing one-dimensional superspace.

and also facilitate discussion of holonomy anomalies in next section.

The (0, 1) supergauge potential $\{\mathcal{V}_L,\ \mathcal{V}_{RR}\}$ are given as

$$\begin{aligned}\mathcal{V}_L &= \eta_L - \theta_R A_{LL}\,,\\ \mathcal{V}_{RR} &= A_{RR} + i\theta_R \chi_R\,.\end{aligned} \tag{4.2.18}$$

Under supergauge transformation

$$\begin{aligned}\mathcal{V}_L &\to \mathcal{H}^{-1}\mathcal{V}_L\mathcal{H} + \mathcal{H}^{-1}D_L\mathcal{H}\,,\\ \mathcal{V}_{RR} &\to \mathcal{H}^{-1}\mathcal{V}_{RR}\mathcal{H} + \mathcal{H}^{-1}\partial_{RR}\mathcal{H}\,,\end{aligned} \tag{4.2.19}$$

where $\mathcal{H}$ is an arbitrary scalar superfield, one can remove field η_L by choosing Wess-Zumino gauge. After this choice of supergauge the residual is normal gauge transformations on gauge field $A_\mu = (A_{LL}, A_{RR})$ and gaugino field χ_R,

$$\begin{aligned}A_\mu &\to h^{-1}A_\mu h + h^{-1}\partial_\mu h\,,\\ \chi_R &\to h^{-1}\chi_R h\,,\end{aligned} \tag{4.2.20}$$

where field h is the bosonic component of superfield $\mathcal{H}$.

Now we have all ingredients needed to supersymmetrize Lagrangian Eq. (4.2.10). We promote bosonic field $g(x)$ to be scalar superfield $\mathcal{G}(x, \theta_R)$ taking values on group G. The bosonic part of $\mathcal{G}$ is $g(x)$ while fermionic part is defined such as

$$\psi_L = \psi_L^A F_A \equiv \mathcal{G}^{-1} D_L \mathcal{G}|_{\theta_R=0}\,, \tag{4.2.21}$$

and thus,

$$\mathcal{G} = g + i\theta_R g \psi_L^A F_A\,,$$

where F_A are the generators of Lie algebra $\mathfrak{g}$ in fundamental representation as before. Under this definition, the fermionic action of $S_M^{(0,1)}$ becomes canonical. Gauge fields A_μ are also enhanced to $\{\mathcal{V}_L, \mathcal{V}_{RR}\}$ taking values on Lie algebra $\mathfrak{h}$.

The $(0,1)$ supersymmetric action now written in superspace is given as

$$S_M^{(0,1)} = \frac{i}{2}\int_\Sigma \mathrm{d}^2x \int \mathrm{d}\theta_R \mathrm{Tr}\big[(\mathcal{G}^{-1}D_L\mathcal{G} - \mathcal{V}_L)(\mathcal{G}^{-1}\partial_{RR}\mathcal{G} - \mathcal{V}_{RR})\big]. \tag{4.2.22}$$

Superfield $\mathcal{G}$ admits a $\mathcal{H}$ super-gauge transformation as designed,

$$\mathcal{G} \to \mathcal{G}\mathcal{H}\,.$$

To obtain the action in components, we impose Wess-Zumino gauge to remove η_L,

$$\mathcal{V}_L = -\theta_R A_{LL}\,.$$

Integrating θ_R out, we get

$$\begin{aligned} S_M^{(0,1)} =& -\frac{1}{2}\int_\Sigma \mathrm{d}^2x\,\mathrm{Tr}[(g^{-1}\partial_{LL}g - A_{LL})(g^{-1}\partial_{RR}g - A_{RR})] \\ & -\frac{i}{2}\int_\Sigma \mathrm{d}^2x\,\mathrm{Tr}[\psi_L(\partial_{RR} + g^{-1}\partial_{RR}g + A_{RR})\psi_L] \\ & -\frac{i}{2}\int_\Sigma \mathrm{d}^2x\,\mathrm{Tr}(\chi_R\psi_L)\,. \end{aligned} \tag{4.2.23}$$

The action still has ordinary H-gauge invariance,

$$\begin{aligned} & g \to gh\,, \quad \psi_L \to h^{-1}\psi_L h\,; \\ & A_\mu \to h^{-1}A_\mu h + h^{-1}\partial_\mu h\,, \\ & \chi_R \to h^{-1}\chi_R h\,. \end{aligned} \tag{4.2.24}$$

As before we decompose $g^{-1}\partial_\mu g$ and ψ_L along horizontal and vertical directions,

$$\begin{aligned} & g^{-1}\partial_\mu g = e_\mu^a X_a + \omega_\mu^i H_i\,, \\ & \psi_L = \mathcal{G}^{-1}D_L\mathcal{G}|_{\theta_R=0} = \psi_L^a X_a + \psi_L^i H_i\,. \end{aligned} \tag{4.2.25}$$

Since A_μ and χ_R are non-dynamical, we solve these constraints by varying A_μ and χ_R, and have

$$\begin{aligned}
A^i_{RR} &= \omega^i_{RR} \,, \\
A^i_{LL} &= \omega^i_{LL} + \frac{i}{2} C^i_{ab} \psi^a_L \psi^b_L \,, \\
\psi^i_L &= 0 \,,
\end{aligned} \tag{4.2.26}$$

where we have used Eq. (4.2.13), the anti-symmetric property of ψ^a_L, and the commutator relations,

$$[H_i, H_j] = C^k_{ij} H_k \,, \quad [H_i, X_a] = C^c_{ia} X_c \,, \quad [X_a, X_b] = C^k_{ab} H_k + C^c_{ab} X_c \,. \tag{4.2.27}$$

From the first two formulas in Eq. (4.2.27) above, we see that, under this decomposition, Lie subalgebra $\mathfrak{h}$ *reducibly* acts on $\mathfrak{g}$, or say, the adjoint representation of $\mathfrak{g}$ restricted to $\mathfrak{h}$ is decomposed as

$$(\mathrm{ad}\ \mathfrak{g})|_{\mathfrak{h}} = \mathrm{ad}\ \mathfrak{h} \oplus \varrho \,, \tag{4.2.28}$$

where ϱ denotes the representation of $\mathfrak{h}$ acting on subspace $\mathfrak{m}$. We will see soon that this observation is very important to determine if anomalies produced by chiral fermions can be removed, and for us to write the most general action.

Substituting Eq. (4.2.26) back to action (4.2.23), we have

$$\begin{aligned}
S^{(0,1)}_M = \ & \frac{1}{2} \int_\Sigma \mathrm{d}^2 x \ \delta_{ab} e^a_{LL} e^b_{RR} \\
& + \frac{i}{2} \int_\Sigma \mathrm{d}^2 x \ \psi^a_L (\partial_{RR} \delta_{ac} + \omega^i_{RR} C_{aic} + \frac{1}{2} e^b_{RR} C_{abc}) \psi^c_L \,.
\end{aligned} \tag{4.2.29}$$

It is not the final result yet because we should assign coupling constants λ^2, which is related to how vielbein e^a_μ and fermion ψ^a_L transforms under gauge transformation. From Eq. (4.2.24) and (4.2.25), writing the transformations in components:

$$\begin{aligned}
& e^a_\mu \to \rho(h^{-1})^a{}_b e^b_\mu \,, \quad \psi^a_L \to \rho(h^{-1})^a{}_b \psi^b_L \,; \\
& \omega^i_\mu C^a_{ib} \equiv \omega^a_{\mu b} \to \big(\rho(h^{-1}) \omega_\mu \rho(h) + \rho(h^{-1}) \partial_\mu \rho(h)\big)^a{}_b \,,
\end{aligned} \tag{4.2.30}$$

where ρ denotes the H-isotropy representation on $\mathfrak{m}$ corresponding to ϱ, i.e.,[5]

$$h^{-1}X_a h \equiv \rho(h)_a{}^b X_b \;\text{ for } X_{a,b} \in \mathfrak{m}\,. \tag{4.2.31}$$

Equation (4.2.30) implies that the tangent bundle $\mathrm{T}M$ is identified to the associated H-principal fiber bundle with vector space $\mathfrak{m}$,

$$\mathrm{T}M \simeq G \times_{\varrho} \mathfrak{m}\;, \tag{4.2.32}$$

on which vielbeins e^a and fermions ψ_L^a are the basic form, and $\omega^a{}_b$ is the connection in ϱ representation. Now if ρ, or equivalently ϱ, is further reducible on $\mathfrak{m}$,

$$\rho = \bigoplus_{r_a} \rho_{r_a}\;,$$

we can assign different coupling constant λ_a^2 to each independent representation[6] r_a of H on $\mathfrak{m}$. Based on the argument above, we rescale vielbein e_μ^a and fermion ψ_L^a in respect to the representations they belong to,

$$e_\mu^a \to \frac{1}{\lambda_a}\, e_\mu^a, \quad \psi_L^a \to \frac{1}{\lambda_a}\, \psi_L^a\,, \tag{4.2.33}$$

and the action changes to

$$\begin{aligned} S_M^{(0,1)} &= \frac{1}{2\lambda_a^2}\int_\Sigma \mathrm{d}^2x\; \delta_{ab} e_{LL}^a e_{RR}^b \\ &\quad + \frac{i}{2\lambda_a^2}\int_\Sigma \mathrm{d}^2x\; \psi_L^a\Big(\partial_{RR}\delta_{ac} + \omega_{RR}^i C_{aic} + \frac{\lambda_a}{2\lambda_b\lambda_c}\, e_{RR}^b C_{abc}\Big)\psi_L^c\,, \end{aligned} \tag{4.2.34}$$

where we used the fact that connection $\omega_{RR}^i C_{aic}$ is block diagonal and thus indexes a and c are forced in the same representation, say $\lambda_a = \lambda_c$. Further, anticommutativity of fermions

[5] Since we chose normalized and orthogonal bases $\{X_a\}$, ρ is in fact orthogonal real representation of H on $\mathfrak{m}$, i.e. $\rho(h)^a{}_b = \rho(h^{-1})_b{}^a$, by which Eq. (4.2.30) can be verified.

[6] If there exists right isometries after we gauge out $H \subset G_R$, the number of coupling constants will be as many as the independent representation of normalizer of H. For more details, we refer readers to reference [79].

$\psi_L^{a,c}$ requires us to antisymmetrize the indexes a and c of term [7] $\frac{\lambda_a}{2\lambda_b\lambda_c}C_{abc}$,

$$\tau_{abc} \equiv -\frac{1}{2}\frac{\lambda_a}{\lambda_b\lambda_c}C_{abc} \to \kappa_{abc} \equiv \frac{1}{2}\Big(\frac{\lambda_b}{\lambda_a\lambda_c} - \frac{\lambda_c}{\lambda_b\lambda_a} - \frac{\lambda_a}{\lambda_b\lambda_c}\Big)C_{abc}\,.$$

We finally have

$$\begin{aligned} S_M^{(0,1)} &= \frac{1}{2\lambda_a^2}\int_\Sigma \mathrm{d}^2x\ \delta_{ab}e^a_{LL}e^b_{RR} \\ &\quad + \frac{i}{2\lambda_a^2}\int_\Sigma \mathrm{d}^2x\ \psi^a_L\big(\partial_{RR}\delta_{ac} + \omega^i_{RR}C_{aic} - e^b_{RR}\kappa_{abc}\big)\psi^c_L \\ &= \frac{1}{2\lambda_a^2}\int_\Sigma \mathrm{d}^2x\Big[\delta_{ab}e^a_{LL}e^b_{RR} + i\psi^a_L\big(\partial_{RR}\delta_{ac} + \widetilde{\omega}_{RRac}\big)\psi^c_L\Big]\,, \end{aligned} \tag{4.2.35}$$

where κ term is absorbed into connection ω to define:

$$\widetilde{\omega}_{ac} \equiv \omega_{ac} - \kappa_{ac}\ ,$$

as the Levi-Civita connection of homogeneous spaces M. Since field κ_{ab} is tensorial, under H-gauge transformation, we still have

$$\begin{aligned} &e^a_\mu \to \rho(h^{-1})^a{}_b e^b_\mu\,, \quad \psi^a_L \to \rho(h^{-1})^a{}_b\psi^b_L\,, \\ &\widetilde{\omega}^a_{\mu b} \to \big(\rho(h^{-1})\tilde{\omega}_\mu\rho(h) + \rho(h^{-1})\partial_\mu\rho(h)\big)^a{}_b\,. \end{aligned} \tag{4.2.36}$$

4.2.3 Isometry anomalies of sigma model on M

In this subsection, we will disclose the relation between isometric and H-gauge transformations, see Eq. (4.2.30), and then calculate isometry anomalies of the action Eq. (4.2.35). For brevity, in what follows, including also the next section, we will only label one, instead of two, "R" or "L" as the subscription of all quantities when it leads to no confusion.

[7] τ and κ are respectively the torsion and contorsion of homogeneous spaces M, see also in [79].

Now let us consider isometries of the action. We start from the fiberation:

$$H \xrightarrow{i} G \xrightarrow{\pi} M \quad ,$$

that all (left) isometries $l_k : M \to M$ are induced from left translations[8] L_k:

$$L_k : g(x) \mapsto kg(x), \quad \text{for } k \in G\,, \tag{4.2.37}$$

and we have the following commuting diagram:

$$\begin{tikzcd} G \arrow[r, "L_k"] \arrow[d, "\pi"'] & G \arrow[d, "\pi"] \\ M \arrow[r, "l_k"] & M \end{tikzcd}$$

$$\text{with } \pi \circ L_k = l_k \circ \pi \tag{4.2.38}$$

It is easily seen that these left translations keep action Eq. (4.2.23) invariant trivially since $k \in G$ is a constant group element.

When investigating isometric transformation l_k on M, we are required to choose a local trivialization, or say, a local section $s : U_s \subset M \to G$. Physically speaking, we fix a gauge, for example the CCWZ coset construction where unitary gauge is chosen (see Eq. (4.2.15)), and localize the action $S_M^{(0,1)}$ on U_s by the coordinates $\{\phi^\alpha\} \in U_s \subset M$. More explicitly, we have

$$g = s(\phi)\ . \tag{4.2.39}$$

Therefore, vielbeins e^a_μ, connection $\omega^a_{\mu b}$ as well as fermions ψ^a_L are pullback to $U_s \subset M$ and

[8] As mentioned, there may be also right isometries on M induced by right translation on G if the normalizer of H is larger than H itself. There are also corresponding right isometry anomalies, but the discussion of them are similar to that of the left. More detailed can be found in [67].

expressed by Eqs. (4.2.39) and (4.2.25) as

$$
\begin{aligned}
s^*(e^a_\mu) &= e^a{}_\alpha \partial_\mu \phi^\alpha = -\mathrm{Tr}\Big(X^a s^{-1} \frac{\partial s}{\partial \phi^\alpha}\Big) \partial_\mu \phi^\alpha \,, \\
s^*(\omega^a_{\mu b}) &= \omega^a_{\alpha b} \partial_\mu \phi^\alpha = -\mathrm{Tr}\Big(H^i s^{-1} \frac{\partial s}{\partial \phi^\alpha}\Big) \partial_\mu \phi^\alpha C^a_{ib} \,, \\
s^*(\psi^a_L) &= -\mathrm{Tr}\Big(X^a s^{-1} \frac{\partial s}{\partial \Phi^\alpha}\Big) D_L \Phi^\alpha|_{\theta_R=0} = e^a{}_\alpha \psi^\alpha_L \,.
\end{aligned}
\tag{4.2.40}
$$

From now on, we will not label s^* to distinguish these quantities as forms on bundle $G\times_\rho \mathfrak{m}$ or locally pullback to $U_s \subset M$. It should lead no confusion in contexts. Thanks to the gauge fixing, the action localized on U_s is given as

$$
\begin{aligned}
S^{(0,1)}_{U_s}[\phi,\psi_L] = &\frac{1}{2\lambda_a^2}\int_\Sigma \mathrm{d}^2x \; \delta_{ab} e^a{}_\alpha e^b{}_\beta \partial_L \phi^\alpha \partial_R \phi^\beta \\
&+ \frac{i}{2\lambda_a^2}\int_\Sigma \mathrm{d}^2 x \; \psi^a_L \big(\partial_R \delta_{ac} + \partial_R \phi^\alpha \tilde{\omega}_{\alpha a c}\big) \psi^c_L \,.
\end{aligned}
\tag{4.2.41}
$$

This action should be invariant under isometric transformation

$$
l_k : \phi \mapsto l_k(\phi) \,. \tag{4.2.42}
$$

We will show that vielbeins, connections and fermions are transformed under l_k as a special type of H-gauge transformation, see Eq. (4.2.30). Then the invariance of action (4.2.41) is guaranteed.

To see this, one can directly calculate their Lie derivatives respect to isometries l_k (cf. [67] for example). Here instead, we interpret this issue in language of fiber bundle, which we presented and explained in great details for CP^{N-1} case in last chapter [39]. For a given section s, or a fixed gauge, we map the local patch U_s to G by

$$
s(\phi) = g \in G \,.
$$

A left translation L_k acting on $s(\phi)$ not only induces isometric transformation l_k on chart $\{\phi^\alpha\}$ but also changes the fixed gauge. When we consider the isometric transformations of quantities e^a_μ, $\omega^a_{\mu b}$ and ψ^a_L under the original fixed gauge, we are required to accompany them by a H-gauge transformation $h(\phi, k)$ to compensate the change:

$$L_k s(\phi) h(\phi, k) = s(l_k(\phi)), \quad \text{for } k \in G \,. \tag{4.2.43}$$

Or equivalently to say, the composition of $L_k^{-1} \circ s \circ l_k$ define another section s', see the commuting diagram:

$$\begin{tikzcd} G & G \arrow[l, "L_k^{-1}"'] \\ U_{s'} \arrow[u, "s'"] \arrow[r, "l_k"] & U_s \arrow[u, "s"'] \end{tikzcd}$$

Sections s' and s are related by a H-gauge transformation $h(\phi, k)$, i.e. Eq. (4.2.43),

$$s'(\phi) = s(\phi) h(\phi, k) \,.$$

Now, after isometric transformation l_k, vielbeins, connections and fermions are pullback to $U_{s'}$ by s'^* and are related to those pullback by s^* as

$$\begin{aligned} &e_\mu^a \to e'^a_\mu = \rho(h^{-1}_{\phi,k})^a{}_b e^b_\mu, \quad \psi^a_L \to \psi'^a_L = \rho(h^{-1}_{\phi,k})^a{}_b \psi^b_L \,, \\ &\omega^a_{\mu b} \to \omega'^a_{\mu b} = \Big(\rho(h^{-1}_{\phi,k}) \omega_\mu \rho(h_{\phi,k}) + \rho(h^{-1}_{\phi,k}) \partial_\mu \rho(h_{\phi,k})\Big)^a{}_b \,, \end{aligned} \tag{4.2.44}$$

where $h_{\phi,k} \equiv h(\phi, k)$ for short. Infinitesimally one can expand,

$$l_k \simeq 1 + \epsilon^A K_A(\phi) \,, \; L_{k^{-1}} \simeq 1 - \epsilon^A F_A \,, \text{ and } \; h(\phi, k) \simeq 1 + \alpha^i(\phi, \epsilon) H_i \,,$$

and get them back to Eq. (4.2.43) to explicitly solve K_A, the Killing field for isometries l_k, and α^i. However it is unnecessary to know their explicit expression. We only need to know, infinitesimally,

$$\begin{aligned} &\delta_\alpha e^a_\mu = -\varrho(\alpha)^a{}_b e^b_\mu \,, \quad \delta_\alpha \psi^a_L = -\varrho(\alpha)^a{}_b \psi^b_L \,, \\ &\delta_\alpha \omega^a_{\mu b} = \partial_\mu \varrho(\alpha)^a{}_b + [\omega_\mu, \; \varrho(\alpha)]^a{}_b \,, \end{aligned} \tag{4.2.45}$$

where

$$\varrho(\alpha)^a{}_b \equiv \alpha^i \varrho(H_i)^a{}_b = \alpha^i C^a_{ib} \,.$$

One can further show that contortion $\kappa^a{}_b$ transforms tensorially,

$$\delta_\alpha \kappa^a_{\mu b} = [\kappa_\mu,\ \varrho(\alpha)]^a{}_b \ \text{ and thus } \ \delta_\alpha \widetilde{\omega}^a_{\mu b} = \partial_\mu \varrho(\alpha)^a{}_b + [\widetilde{\omega}_\mu,\ \varrho(\alpha)]^a{}_b \ .$$

Now, for isometry anomalies, we use action $S^{(0,1)}_{U_s}[\phi, \psi_L]$ to calculate the effective action. Similarly to the discussion before, anomalies are only produced from fermionic integration effective action. We thereby integrate out the fermionic part of action Eq. (4.2.41) and have

$$i\,\mathcal{W}^s_f[\widetilde{\omega}_R] = \frac{i}{16\pi} \int_\Sigma \mathrm{d}^2x\ \mathrm{Tr}(\widetilde{\omega}_R\, \frac{\partial_L \partial_L}{\partial^2}\, \widetilde{\omega}_R) + \mathcal{O}(\widetilde{\omega}^3_R)\,, \tag{4.2.46}$$

where the superscript s denotes that our perturbative calculation is performed on the local chart U_s. Varying $\mathcal{W}^s_f$, we produce isometry anomalies $\mathcal{I}_\alpha$,

$$\mathcal{I}_\alpha = \delta_\alpha \mathcal{W}^s_f = -\frac{1}{8\pi} \int_\Sigma \mathrm{d}^2x\ \mathrm{Tr}(\alpha \partial_L \widetilde{\omega}_R)\,. \tag{4.2.47}$$

To conclude, in this section we have calculated isometry anomalies of generic $(0,1)$ supersymmetric sigma models defined on manifold $M = G/H$. To perform perturbative calculation, we need to specify a local chart U_s on M to define the model and thus a section s from U_s to G. After integrating out fermions, we find the effective action $\mathcal{W}_f$ which is also defined on the local patch U_s. However in many cases the effective action does not bear isometries l_k , for $k \in G$, it had before and thus produces isometry anomalies. We established a correspondence between the isometry anomalies and some specific H-gauge anomalies when considering to define the effective action $\mathcal{W}_f$ on the intersection of two local patch $U_s \cap U_{s'}$, where local patch $U_{s'} = l^{-1}_k(U_s)$ induced by isometries. This observation actually inevitably leads us to consider anomalies not only localized on a specified coordinates or local chart, but also to evaluate if the effective action $\mathcal{W}_f$ can be consistently defined on different local patches and their intersections. If so, one is able to transit $\mathcal{W}_f$ from patch to patch without producing any H-gauge anomalies, or called holonomy anomalies. Since isometry anomalies is a specific type of H-gauge anomalies, they will for sure vanish in this situation. Otherwise when a model is suffered from holonomy anomalies, it is not even possible to globally define the theory quantum mechanically, and thus makes no sense to consider its isometry anomalies. Therefore in the next chapter, we will focus on holonomy anomalies and the criteria when they vanish or can be canceled by counterterms.

4.3 Holonomy anomalies

In last section, we have argued that, to define sigma models on Homogeneous spaces $M = G/H$, it is required that, prior to consider isometry anomalies, the theory should be independent of the choices of sections $s : U_s \subset M \to G$, or say the ease of holonomy anomalies. The holonomy anomalies will arise when we change from a section s to another s', or physically speaking, from a fixed gauge to another. Therefore they correspond to an arbitrary H-gauge transformation, see Eq. (4.2.24) and Eq. (4.2.30), while isometry transformations are a special type of H-gauge transformation, see Eq. (4.2.44). Therefore, once holonomy anomalies are removed, isometry anomalies will automatically vanish as well. We will thus focus ourselves on holonomy anomalies and their cancellation condition.

4.3.1 Anomaly matching condition

From Eq. (4.2.47), we know that α and $\widetilde{\omega}_R$ are taking values in the ϱ representation of $\mathfrak{h}$ Lie subalgebra. On the other hand, we will show that counterterms that can be introduced is in the $F(\mathfrak{g}|_{\mathfrak{h}})$ representation, say the fundamental representation F of $\mathfrak{g}$ representation restricted on $\mathfrak{h}$. Choosing the fundamental representation F is merely of convention, since we are free to redefine our coupling constants corresponding to other different representation. Roughly speaking, the counterterm we can introduce is an analog of gauged WZW term $\mathcal{W}_{\text{c.t}}[g, A]$, which is well-known to produce gauge anomalies when the gauge fields A taking values in $\mathfrak{h}$ are not in a "safe" representation [80]. When certain matching condition on ϱ and $F(\mathfrak{g}|_{\mathfrak{h}})$ is satisfied, these two anomalies are canceled. From now on to distinguish the difference between representation ϱ and $F(\mathfrak{g}|_{\mathfrak{h}})$, we will use Tr_ϱ and Tr_F to label under which representation we take the trace.

First we will explore more on the structure of effective fermionic action $\mathcal{W}_f$. In what follows, we will not fix ourselves in any specific gauge, and will not solve gauge field A in terms of g as Eq. (4.2.26), because it will help better track the information of gauge transformations on $\mathcal{W}_f$ and, more importantly, give us an explicit expression of fermionic effective action. We thus use Eq. (4.2.23) and rewrite the fermionic part as

$$S_f = -\frac{i}{2}\int_\Sigma \mathrm{d}^2x \left[\mathrm{Tr}_F\ \psi_L(\partial_R\psi_L + [A_R, \psi_L]) + \mathrm{Tr}_F\ \psi_L(g^{-1}\partial_R g - A_R)\psi_L\right] \qquad (4.3.1)$$

with

$$\psi_L = \psi_L^a X_a\,, \quad \text{and} \ \ A_R = A_R^i H_i\,.$$

The two parts of above equation are separately gauge invariant classically. However the second term, $g^{-1}\partial_R g - A_R$ coupling to fermions, transforms tensorially under a H-gauge transformation,

$$g^{-1}\partial_R g - A_R \to h^{-1}(g^{-1}\partial_R g - A_R)h\ ,$$

while in the first term chiral fermions ψ_L couple to gauge fields A_R and will produce genuine anomalies. If we can find counterterms to offset the anomalies from the first term in Eq. (4.3.1), the anomalies from the second one can be removed also by an analog of Bardeen like counterterm in two dimensions. Let us see how it works.

In fact we can ask more for an explicit structure on the anomalous part of $\mathcal{W}_f$ in two-dimensional spacetime due to Polyakov and Wiegmann [81]. In two dimensions, one can parameterize gauge fields as

$$A_R = \tilde{h}^{-1}\partial_R\tilde{h} \ \ \text{and} \ \ A_L = \tilde{h}'^{-1}\partial_L\tilde{h}'\,, \qquad (4.3.2)$$

where fields $\tilde{h}(x)$ and $\tilde{h}'(x)$ are elements in H and under gauge transformation

$$\tilde{h} \to \tilde{h}h\,, \ \text{and} \ \ \tilde{h}' \to \tilde{h}'h\ .$$

Notice that, since $\tilde{h} \neq \tilde{h}'$, A_μ are not flat connection. One can solve $\tilde{h}$ and $\tilde{h}'$ in terms of the Wilson lines of A_R and A_L, although the expressions is surly non-local,

$$\tilde{h}(x) = -P\,\mathrm{e}^{-\int_{C_x}\mathrm{d}\xi_L A_R}, \ \ \text{and} \ \ \tilde{h}'(x) = -P\,\mathrm{e}^{-\int_{C_x}\mathrm{d}\xi_R A_L}\ ,$$

where C_x is a path from certain fixed point to x and P denote path ordered integral. With the help of $\tilde{h}$, one can explicitly write down the anomalous part of $\mathcal{W}_f$. Let us first rewrite term $g^{-1}\partial_R g - A_R$ as

$$g^{-1}\partial_R g - A_R = g^{-1}\partial_R g - \tilde{h}^{-1}\partial_R \tilde{h} = g^{-1}\partial_R (g\tilde{h}^{-1})(g\tilde{h}^{-1})^{-1} g \ . \tag{4.3.3}$$

Clearly $g\tilde{h}^{-1}$ is gauge invariant. Actually, if we redefine fermions ψ_L as

$$\psi_L = \tilde{h}^{-1}\zeta_L \tilde{h} \ \text{ or in components } \ \psi_L^a = \rho(\tilde{h}^{-1})^a{}_b \zeta_L^b \ , \tag{4.3.4}$$

the action S_f changes to

$$S'_f = -\frac{i}{2}\int_\Sigma \mathrm{d}^2x \ \mathrm{Tr}_F \Big[\zeta_L \partial_R \zeta_L + \zeta_L (g\tilde{h}^{-1})^{-1}\partial_R (g\tilde{h}^{-1})\zeta_L\Big]. \tag{4.3.5}$$

Notice now that both ζ_L and $g\tilde{h}^{-1}$ are gauge invariant. After integrating out ζ_L, the effective fermionic action is guaranteed to be gauge invariant as well,

$$\mathcal{W}'_f[g\tilde{h}^{-1}] = -i\log \int \mathcal{D}\zeta_L \ \mathrm{e}^{iS'_f} \ . \tag{4.3.6}$$

Therefore, we can interpret that the anomaly Eq. (4.2.47) is raised in a functional determinant when we change fermionic measure,

$$\int \mathcal{D}\psi_L = \int \mathcal{D}et^{-1}\Big[\frac{\delta\psi_L}{\delta\zeta_L}\Big]\mathcal{D}\zeta_L \, ,$$

and we will calculate the determinant above. The method we will use is mainly based on [81].

The determinant is an integrated version of anomaly Eq. (4.2.47). Now since we keep gauge fields explicitly, the anomaly equation becomes:

$$\begin{aligned} \mathcal{A}_\alpha &= -\frac{1}{8\pi}\int_\Sigma \mathrm{d}^2x \ \mathrm{Tr}_\varrho \alpha \, \partial_L \Big(\frac{1}{2}A_R + \frac{1}{2}\omega_R - \kappa\Big) \\ &= -\frac{1}{8\pi}\int_\Sigma \mathrm{d}^2x \ \mathrm{Tr}_\varrho \alpha \partial_L A_R - \frac{1}{16\pi}\int_\Sigma \mathrm{d}^2x \ \mathrm{Tr}_\varrho \alpha \partial_L (\omega_R - A_R) \end{aligned} \tag{4.3.7}$$

where, in the second equality, we use

$$\mathrm{Tr}_\varrho \alpha\kappa = \frac{1}{2}\Big(\frac{\lambda_b}{\lambda_a\lambda_c} - \frac{\lambda_c}{\lambda_b\lambda_a} - \frac{\lambda_a}{\lambda_b\lambda_c}\Big)\alpha^i e^b C^c_{ia} C^a_{bc} = 0 \ ,$$

because of $\mathrm{Tr}_{\varrho,F} H_i X_b = 0$, see Eq.(4.2.13). The first term in anomaly Eq. (4.3.7) corresponds to the first part of action Eq. (4.3.1):

$$S^1_f = -\frac{i}{2\lambda_a^2}\int_\Sigma \mathrm{d}^2x \ \mathrm{Tr}_F \ \psi_L(\partial_R\psi_L + [A_R,\psi_L]) = \frac{i}{2\lambda_a^2}\int_\Sigma \mathrm{d}^2x \ \psi_{La}(\partial_R\psi^a_L + A^i_R C^a_{ib}\psi^b_L) \ ,$$

where we write the action in components and rescale fermions to make the coupling constants explicit. For A_R parameterized as $\tilde{h}^{-1}\partial_R\tilde{h}$, we now aim to find an effective action $\mathcal{W}^1_f[\tilde{h}]$ which corresponds to S^1_f and satisfies

$$\delta_\alpha \mathcal{W}^1_f[\tilde{h}] = \mathcal{A}^1_\alpha = -\frac{1}{8\pi}\int_\Sigma \mathrm{d}^2x \ \mathrm{Tr}_\varrho(\tilde{h}^{-1}\delta\tilde{h})\partial_L(\tilde{h}^{-1}\partial_R\tilde{h}) \ ,$$

where we also put $\alpha = \tilde{h}^{-1}\delta\tilde{h}$. Due to Polyakov and Weigmann, the effective action can be solved as

$$\mathcal{W}^1_f = \mathcal{W}_{\mathrm{PW}}[\tilde{h}] \equiv \frac{1}{16\pi}\int_\Sigma \mathrm{d}^2x \ \mathrm{Tr}_\varrho(\tilde{h}^{-1}\partial_R\tilde{h})(\tilde{h}^{-1}\partial_L\tilde{h}) - \frac{1}{24\pi}\int_{\tilde{h}(B)} \mathrm{Tr}_\varrho(\tilde{h}^{-1}d\tilde{h})^3 \,, \qquad (4.3.8)$$

where, in the second term, $\tilde{h} = \tilde{h}(x,t)$ has been extended [9] to bulk B bounded by Σ. It is well-known that the second term is multi-valued and can be rewritten as a local form on spacetime Σ, and thus we still have a local theory defined on Σ rather than the bulk B.

Beside this part, there is also the second term left in anomaly Eq. (4.3.7),

$$\mathcal{A}^2_\alpha = -\frac{1}{16\pi}\int_\Sigma \mathrm{d}^2x \ \mathrm{Tr}_\varrho\alpha\partial_L(\omega_R - A_R)\,.$$

We have argued that $g^{-1}\partial_R g - A_R$, as well as $\omega_R - A_R$, transform tensorially and thus do not produce anomalies themselves, unless they are coupled to gauge fields as probes.

[9] The extension of $\tilde{h}(x,t)$, and also that of $g(x,t)$ later, are always assumed to exist. For situations when $\pi_2(H)$ or $\pi_2(G)$ are non-trivial, we will present in our future work on global anomalies.

Therefore we can easily verify that, a Bardeen-like counterterm,

$$\mathcal{W}_f^2[\tilde{h}, \omega_R - A_R] = \frac{1}{16\pi} \int_\Sigma \mathrm{d}^2x \ \mathrm{Tr}_\varrho(\omega_R - A_R)(\tilde{h}^{-1}\partial_L \tilde{h}) \,, \tag{4.3.9}$$

satisfies

$$\delta_\alpha \mathcal{W}_f^2 = \mathcal{A}_\alpha^2 \ ,$$

and, thus, is the second part of the anomalous effective action.

Overall we explicitly solve the anomalous part of effective action $\mathcal{W}_f$, and the whole effective action $\mathcal{W}_f$ is given as

$$\begin{aligned}\mathcal{W}_f &= \mathcal{W}_f^1[\tilde{h}] + \mathcal{W}_f^2[\tilde{h}, \omega_R - A_R] + \mathcal{W}'_f[g\tilde{h}^{-1}] \\ &= -\frac{1}{24\pi}\int_{\tilde{h}(B)} \mathrm{Tr}_\varrho(\tilde{h}^{-1}d\tilde{h})^3 + \frac{1}{16\pi}\int_\Sigma \mathrm{d}^2x \ \mathrm{Tr}_\varrho \omega_R(\tilde{h}^{-1}\partial_L\tilde{h}) + \mathcal{W}'_f[g\tilde{h}^{-1}] \,,\end{aligned} \tag{4.3.10}$$

Now, based on the anomalous effective action above, we are seeking conditions and counterterms $\mathcal{W}_{\text{c.t.}}[g, A_R]$. The key hint from Eq. (4.3.10) is that we need an analog of term $\mathrm{Tr}_\varrho(\tilde{h}^{-1}d\tilde{h})^3$. It should first have same gauge transformation rule as $\tilde{h}^{-1}d\tilde{h}$,

$$\tilde{h}^{-1}d\tilde{h} \to h^{-1}(\tilde{h}^{-1}d\tilde{h})h + h^{-1}dh, \quad \text{for} \ \ \tilde{h} \to \tilde{h}h \ .$$

and secondly be able to pullback to spacetime Σ to define our theory in two dimensions. However the only ingredient we have to satisfy the two conditions is

$$\mathcal{W}_{c.t.} \sim \mathrm{Tr}_F(g^{-1}dg)^3 \ .$$

An infinitesimal H-gauge transformation, c.f. Eq. (4.2.24), is given as:

$$\delta_\alpha(g^{-1}dg) = d\alpha^i H_i + [g^{-1}dg, \alpha^i H_i], \quad \text{for} \ \ \delta_\alpha g = g\alpha^i H_i \,,$$

where we explicitly display α above taking values in $F(\mathfrak{g}|_\mathfrak{h})$. Therefore we have:

$$\delta_\alpha \mathrm{Tr}_F(g^{-1}dg)^3 \sim \mathrm{Tr} \ \alpha d(g^{-1}dg) \sim \alpha^i d\omega^j \mathrm{Tr} H_i H_j \,.$$

Since we have already normalized generators H_i in Eq. (4.2.13), as

$$\mathrm{Tr}_F H_i H_i = -\delta_{ij}, \quad \text{for any} \quad H_{i,j} \in \mathfrak{h}$$

the *anomaly matching condition*, under our conventions, is

$$\mathrm{Tr}_\varrho H_i H_j = c\, \mathrm{Tr}_F H_i H_j = -c\, \delta_{ij}, \quad \text{for any} \quad H_{i,j} \in \mathfrak{h}\,, \tag{4.3.11}$$

with some constant c. So long as the anomaly matching condition is satisfied, we can construct the counterterms $\mathcal{W}_{\text{c.t.}}$ as

$$\mathcal{W}_{\text{c.t.}} = \frac{c}{24\pi}\int_{g(B)} \mathrm{Tr}_F (g^{-1}dg)^3 - \frac{c}{16\pi}\int_\Sigma \mathrm{d}^2x\ \mathrm{Tr}_F A_R (g^{-1}\partial_L g)\,. \tag{4.3.12}$$

One can verify that, when Eq. (4.3.11) is met,

$$\delta_\alpha \mathcal{W}_{\text{c.t.}} + \mathcal{A}_\alpha = 0\,.$$

At last, combining Eq. (4.3.12) and Eq. (4.3.10), we would expect the modified fermionic action $\mathcal{W}_{\text{eff}}$ is gauge invariant,

$$\mathcal{W}_{\text{eff}} = \mathcal{W}_f + \mathcal{W}_{\text{c.t.}} = \frac{c}{24\pi}\int_{g\tilde{h}^{-1}(B)} \mathrm{Tr}_F \Big[(g\tilde{h}^{-1})^{-1} d(g\tilde{h}^{-1})\Big]^3 + \mathcal{W}_f'[g\tilde{h}^{-1}]\,. \tag{4.3.13}$$

4.3.2 Comments on counterterms

So far we derived the anomaly matching condition Eq. (4.3.11), based on which the gauge invariant effective action, Eq. (4.3.13), is constructed above. There are some interesting results and comments we want to put.

i. Anomaly matching condition

The anomaly matching condition is a group theoretical result. In principle, if we understand how a subgroup H is embedded to G, we can determine, by Eq. (4.3.11), whether a minimal $(0,1)$ supersymmetric sigma model can be well-defined. Actually the statement is topological, when we will show in subsection 4.3.4, that Eq. (4.3.11) will be satisfied if and only if the *first Pontryagin form* of M vanishes, i.e. $p_1(M) = 0$.

ii. $\mathcal{W}_{\text{eff}}$ incorporated with $(0,1)$ supersymmetry

Till now, besides requiring $(0,1)$ supersymmetry on model building, we did not fully consider the role supersymmetry may play in the game. The counterterm $\mathcal{W}_{\text{c.t.}}$ we added is apparently non-supersymmetric, but it is required to define our theory. Now we want to proceed one step more, when we find the gauge invariant fermionic action $\mathcal{W}_{\text{eff}}$. For brevity, we use $\varphi \equiv g\tilde{h}^{-1}$ as the gauge invariant field, and $\mathcal{W}_{\text{eff}}$ is rewritten as

$$\mathcal{W}_{\text{eff}}[\varphi] = \frac{c}{24\pi}\int_{\varphi(B)} \text{Tr}_F(\varphi^{-1}d\varphi)^3 + \mathcal{W}'_f[\varphi]\,.$$

The second term is due to a path integral over fermions ζ_L, see Eq.(4.3.5) and Eq. (4.3.6),

$$\mathrm{e}^{i\mathcal{W}'_f[\varphi]} = \int \mathcal{D}\zeta_L \exp \int_\Sigma \mathrm{d}^2x \,\Big(-\frac{i}{2\lambda^2}\Big)\text{Tr}_F\left(\zeta_L\partial_R\zeta_L + \zeta_L\varphi^{-1}\partial_R\varphi\zeta_L\right)\,,$$

which has its supersymmetric counterpart S_b, see Eq. (4.2.10). On the other hand, the first term, as a combination of anomalous and anomaly-counterterms,

$$\mathcal{W}_{\text{WZW}} \equiv \frac{c}{24\pi}\int_{\varphi(B)} \text{Tr}_F(\varphi^{-1}d\varphi)^3\,,$$

has no its supersymmetric pair. Therefore we will supersymmetrize this term. Actually the $\mathcal{N}=(1,1)$ supersymmetrization of $\mathcal{W}_{\mathrm{WZW}}$ is well-known in literatures back to 80's, c.f. [82] and [83] for example. Here we do the similar to equip $\mathcal{W}_{\mathrm{WZW}}$ with a $\mathcal{N}=(0,1)$ supersymmetry. Since field φ is now gauge invariant, its $(0,1)$ super-partner is also gauge invariant, and thus must be ζ_L. The supersymmetrization of $\mathcal{W}_{\mathrm{WZW}}$ can be formally performed in $(0,1)$ superspace as Eq. (4.2.22):

$$\begin{aligned}\mathcal{W}_{\mathrm{sWZW}} &= \frac{c}{16\pi}\int_B \mathrm{d}^2x\mathrm{d}t\int \mathrm{d}\theta_R\ \mathrm{Tr}_F\left(\Psi^{-1}\partial_t\Psi[\Psi^{-1}D_L\Psi,\Psi^{-1}\partial_R\Psi]\right)\\ &= \frac{c}{24\pi}\int_{\varphi(B)}\mathrm{Tr}_F(\varphi^{-1}d\varphi)^3 - \frac{ic}{16\pi}\int_\Sigma \mathrm{d}^2x\ \mathrm{Tr}_F\left(\zeta_L\varphi^{-1}\partial_R\varphi\zeta_L\right),\end{aligned}$$

where we define superfield Ψ, c.f. Eq. (4.2.21):

$$\Psi|_{\theta_R=0}\equiv\varphi\quad\text{and}\quad\Psi^{-1}D_L\Psi|_{\theta_R=0}\equiv\zeta_L\,.$$

As what we mentioned, for now all fields are gauge invariant, one should not worry about anomalies for the fermionic part $\mathcal{W}_{\mathrm{sWZW}}$. Overall we have a supersymmetric effective action:

$$S^{(0,1)} = S_b + S'_f + \mathcal{W}_{\mathrm{sWZW}} \tag{4.3.14}$$

iii. Renormalization flow and superconformal fixed point in IR region

Now we want to investigate some non-perturbative behaviors of the modified theories in deep infrared region. It is interesting to realize that the modified theory contains supersymmetric "WZW" term with gauge invariant variable $\varphi = g\tilde{h}^{-1}$. We are trying to argue that, in an *ad-hoc* gauge:

(a) for M is a symmetric space, the "WZW" action vanishes and the theory is equivalent to a bosonic sigma model with left fermions decoupled. Therefore supersymmetry should be broken in IR region;

(b) for M is a non-symmetric homogeneous spaces with non-trivial third cohomology $H^3(M)\neq 0$, the "WZW" term corresponds to an element in $H^3(M)$. The theory would flow to a (super)conformal fixed point in IR region.

To illustrate part (a), we fix the gauge on variable φ, so that

$$\varphi^{-1}d\varphi \in \Omega^1(M)\otimes\mathfrak{m}, \quad \text{or say} \quad \varphi^{-1}d\varphi = e^a X_a\,, \tag{4.3.15}$$

where e^a will be shown as vielbein 1-forms on M soon. This gauge is alway possible to choose, although φ cannot be expressed in terms of exponential map. It is because that, if we notice $g=\varphi\tilde{h}$, φ is exactly a coset representative for $M=G/H$, and thus $\varphi^{-1}d\varphi$ is a 1-form on T^*M.

Now under this gauge, by the property of symmetric space

$$[\mathfrak{m},\mathfrak{m}]\subset\mathfrak{h},$$

and the orthogonality Eq. (4.2.13), one verifies that:

$$\mathrm{Tr}_F(\varphi^{-1}d\varphi)^3=0, \quad \text{and} \quad \mathrm{Tr}_F\left(\zeta_L\varphi^{-1}\partial_R\varphi\zeta_L\right)=0\,,$$

for $\zeta_L=\zeta_L^a X_a$ as well. Therefore the fermion ζ_L is totally decoupled and free. Now let us turn to bosonic part, see Eq. (4.2.10). We rewrite the action in the light-cone coordinate as

$$S_M=-\frac{1}{2\lambda^2}\int_\Sigma \mathrm{d}^2x\,\mathrm{Tr}_F\Big[(g^{-1}\partial_R g-A_R)(g^{-1}\partial_L g-A_L)\Big].$$

By using Eqs. (4.3.3) and (4.3.2) we further express the action in terms of φ, $\tilde{h}$ and A_L:

$$S_M=-\frac{1}{2\lambda^2}\int_\Sigma \mathrm{d}^2x\,\mathrm{Tr}_F\Big[(\varphi^{-1}\partial_R\varphi)(\varphi^{-1}\partial_L\varphi)+(\varphi^{-1}\partial_R\varphi)(\partial_L\tilde{h}\tilde{h}^{-1})-(\varphi^{-1}\partial_R\varphi)(\tilde{h}A_L\tilde{h}^{-1})\Big].$$

The last two terms vanish because of orthogonality again. Therefore we finally have

$$S_M^{(0,1)}=-\frac{1}{2\lambda^2}\int_\Sigma \mathrm{d}^2x\,\mathrm{Tr}_F\Big[(\varphi^{-1}\partial_R\varphi)(\varphi^{-1}\partial_L\varphi)+i(\zeta_L\partial_R\zeta_L)\Big]. \tag{4.3.16}$$

It is well-known that the bosonic theory is asymptotic free. In the deep infrared region, there is a mass gap generated, while the free fermions ζ_L is chiral, and thus no way to pair mass term. Thereby the supersymmetry will be broken.

For sure when we use Eq.(4.3.2) to parametrize gauge fields, there are functional determinants raised,

$$\int \mathcal{D}A_R \mathcal{D}A_L = \int (\mathcal{D}et\nabla_R)(\mathcal{D}et\nabla_L)\mathcal{D}\tilde{h}\mathcal{D}\tilde{h}' ,$$

where $\nabla_{R,L} \equiv \partial_{R,L} + [A_{R,L},\ ]$. The two determinants combining together is gauge invariant, and gives an additional Polyakov-Wiegmann functional [84, 85],

$$(\mathcal{D}et\nabla_R)(\mathcal{D}et\nabla_L) = \exp(-ic_H \mathcal{W}_{\mathrm{PW}}[\tilde{h}\tilde{h}'^{-1}])\partial_R\partial_L ,$$

where c_H is the eigenvalue of second Casimir operator for $\mathfrak{h}$ in its adjoint representation. Nevertheless, this additional term will not affect our argument above.

Now we are aiming to argue part (b) under the same gauge Eq. (4.3.15). Since for non-symmetric homogeneous spaces, the Lie algebra structure constant C_{abc} is non-zero, we will have non-vanishing WZW term and fermionic interaction, see Eq. (4.3.14). The WZW term

$$\frac{c}{24\pi}\mathrm{Tr}_F(\varphi^{-1}d\varphi)^3 \sim C_{abc}e^a \wedge e^b \wedge e^c$$

is a closed and horizontal basic 3-form, which vanished under action of $\mathfrak{h}$-Lie derivative $\mathcal{L}_{\mathfrak{h}}$. Therefore it is an element in $H^3(M)$, when $H^3(M) \neq 0$, c.f. [86] and [87]. Combining this term with original bosonic action, see Eq. (4.3.16), we have

$$S_{M,b} = -\frac{1}{2\lambda^2}\int_\Sigma \mathrm{d}^2x\ \mathrm{Tr}_F(\varphi^{-1}\partial_R\varphi)(\varphi^{-1}\partial_L\varphi) + \frac{c}{24\pi}\int_{\varphi(B)} \mathrm{Tr}_F(\varphi^{-1}d\varphi)^3 .$$

By standard argument, we know, that for [10]

$$\frac{\lambda^2 c}{8\pi} = 1 \tag{4.3.17}$$

the bosonic theory will be conformal invariant. Now let us temporarily reside at this critical point, and check the fermionic action. Combining Eq. (4.3.5) and the fermionic

[10] For simplicity, we only assume one coupling constant, say λ^2, even though ϱ may be reducible.

part of Eq. (4.3.14), we get

$$
\begin{aligned}
S_{M,f} &= S'_f - \frac{ic}{16\pi}\int_\Sigma \mathrm{d}^2x\ \mathrm{Tr}_F\left(\zeta_L\varphi^{-1}\partial_R\varphi\zeta_L\right) \\
&= -\frac{ic}{16\pi}\int_\Sigma \mathrm{d}^2x\ \mathrm{Tr}_F\left(\zeta_L\partial_R\zeta_L + 2\zeta_L\varphi^{-1}\partial_R\varphi\zeta_L\right).
\end{aligned}
\tag{4.3.18}
$$

Similar to Eq. (4.3.4), we further rotate ζ_L to define a new fermionic variable ξ_L satisfying

$$\zeta_L \equiv \varphi^{-1}\xi_L\varphi\,.$$

We obtain a free fermionic action on ξ_L as:

$$S_{M,f} = -\frac{ic}{16\pi}\int_\Sigma \mathrm{d}^2x\ \mathrm{Tr}_F\xi_L\partial_R\xi_L\,.$$

Certainly such a redefinition on chiral fermions will lead us to the Polyakov-Wiegmann functional as before, although our theory has been gauge invariant as it was modified. Such an additional functional seems only to contribute a shift to the level c of the conformal theory. In sum, because of the existence of WZW term, the theory will flow to a non-trivial infrared conformal fixed point, where fermionic fields are free, while due to conformal symmetries, there is no mass gap for bosonic sector, and thus the $(0,1)$ supersymmetry seems to hold.

4.3.3 Examples

In this subsection, we turn to use anomaly matching condition Eq. (4.3.11) to analyze some examples.

i. Simple Lie group G

Our first example is sigma models defined on simple Lie groups G. Although we construct sigma models on $M = G/H$ by gauging a subgroup H of G, see Eq. (4.2.22), Lie group G itself is a symmetric space as well, i.e.

$$G \simeq G_L \times G_R / G_V \,.$$

The Lie algebra of $G_L \times G_R$ is

$$\mathfrak{g}_L \oplus \mathfrak{g}_R, \quad \text{with that} \quad \mathfrak{g}_L = \mathfrak{g}_R = \mathfrak{g} \;.$$

We label the generators $L_A \in \mathfrak{g}_L$ and $R_A \in \mathfrak{g}_R$, their commutators are

$$[L_A,\ L_B] = C^C_{AB} L_C \,, \quad [R_A,\ R_B] = C^C_{AB} R_C \,, \quad [L_A,\ R_B] = 0 \,.$$

The diagonal group G_V acting on $G_L \times G_R$ gives its Lie subalgebra $H_A \in \mathfrak{g}_V$,

$$H_A = L_A + R_A \,.$$

By using Killing form with normalization

$$\mathrm{Tr}(L_A L_B) = -\delta_{AB}, \quad \mathrm{Tr}(R_A R_B) = -\delta_{AB}, \quad \text{and} \quad \mathrm{Tr}(L_A R_B) = 0 \,,$$

we find other generators belonging to $\mathfrak{m}$, complimentary to $\mathfrak{h} = \mathfrak{g}_V$,

$$X_A = L_A - R_A \,,$$

and their commutator relationship given by

$$[H_A,\ H_B] = C_{AB}^C H_C\,,\quad [H_A,\ X_B] = C_{AB}^C X_C\,,\quad \text{and}\quad [X_A,\ X_B] = C_{AB}^C H_C\,.$$

Therefore $G \simeq G_L \times G_R/G_V$ is a symmetric space with isotropy representation ϱ,

$$(\mathrm{ad}\ \mathfrak{g}_L \oplus \mathfrak{g}_R)|_{\mathfrak{g}_V} = \mathrm{ad}\ \mathfrak{g}_V \oplus \varrho = \mathrm{ad}\ \mathfrak{g}_V \oplus \mathrm{ad}\ \mathfrak{g}_V\,.$$

And we see that

$$\mathrm{Tr}_\varrho(H_A H_B) = -T_G \delta_{AB} = \frac{T_G}{2}\,\mathrm{Tr}_F(H_A H_B)\,,$$

where T_G is the dual Coxeter number of Lie algebra $\mathfrak{g}$. By anomaly matching condition, we know that sigma model is well-defined on simple group manifold G.

Another motivation for us to consider sigma model on G is that: we want to argue that ease of holonomy anomalies, independence of the theory on choices of section s is *prior* to that of isometry anomalies. To illustrate this point, we first look at the action Eq. (4.2.23), without gauge and gaugino fields A_μ and χ_R,

$$S_G^{(0,1)} = -\frac{1}{2\lambda^2}\int_\Sigma \mathrm{d}^2x\ \mathrm{Tr}(g^{-1}\partial_L g)(g^{-1}\partial_R g) + i\mathrm{Tr}\left(\psi_L(\partial_R + g^{-1}\partial_R g)\psi_L\right)\ . \qquad (4.3.19)$$

For this action, in fact, we already fix a gauge, or say a section $s : U_s \subset G \to G_L \times G_R$. Near the identity of $G_L \times G_R$, one can assign coordinates $\{\phi\} \in U_s$ and use exponential map to write s explicitly,

$$g = s(\phi) = \mathrm{e}^{2\phi^A L_A}\ . \qquad (4.3.20)$$

From the above equation, we also know that the gauge fixing is to remove degrees of freedom on G_R. Let us keep it in mind. In the following we will show, under this gauge fixing, there is *no* isometry anomaly.

We consider isometries of the action Eq. (4.3.19). One can either interpret these isometries as left isometries of G_L and right ones of G_R, or as *all left* isometries acting on Eq. (4.3.20). Isometries of G_R, parameterized by $\mathrm{e}^{\epsilon^A R_A}$, acting on $s(\phi)$ from *left*, break the fixed gauge,

$$\mathrm{e}^{\epsilon^A R_A} s(\phi) = \mathrm{e}^{\epsilon^A R_A + 2\phi^A L_A}\,.$$

Therefore we need compensate it by a G_V-gauge transformation $h = \mathrm{e}^{-\epsilon^A H_A}$,

$$\mathrm{e}^{\epsilon^A R_A} s(\phi) h = \mathrm{e}^{2\phi^A L_A} \mathrm{e}^{-\epsilon^A L_A} \ ,$$

where in the two equations above we used the fact that L_A and R_B commutes, and thus it is equivalent to a *right* G_R group action. Since isometries from G_L need no gauge compensation, whereas isometries from G_R need a *constant* gauge compensation, say $h = \mathrm{e}^{-\epsilon^A H_A}$, both G_L and G_R isometries do not produce isometry anomalies in the choice of section s. One can also confirm this statement directly from the fermionic part of action Eq. (4.3.19),

$$\begin{aligned}
&\text{for } g \to kg, \quad g^{-1}\partial_R g \to g^{-1}\partial_R g\,; \\
&\text{for } g \to g\tilde{k}, \quad g^{-1}\partial_R g \to \tilde{k}^{-1}(g^{-1}\partial_R g)\tilde{k}\,,
\end{aligned}$$

where $k, \tilde{k} \in G_L, G_R$ are constant group elements. We get the same result that $g^{-1}\partial_R g$ is invariant under left isometries, and tensorially transformed under right ones. Hence, after integrating out fermions from action Eq. (4.3.19), the fermionic effective action will not produce isometry anomalies.

From the analysis above, it seems that the theory is well-defined even with no need to add counterterms. However in what follows, we will argue that introducing counterterms as Eq.(4.3.12) is a must. First we notice that there is a discrete symmetry *classically* held. On bosonic part of the action Eq. (4.3.19), we realize that

$$g \to g^{-1}, \quad S_{G,b} \to S_{G,b}\,.$$

On the other hand, the fermionic part is changed to

$$S_{G,f} \to -\frac{i}{2\lambda^2} \int_\Sigma \mathrm{d}^2 x \ \mathrm{Tr}\big[\psi_L(\partial_R + g\partial_R g^{-1})\psi_L\big]\,.$$

To get it back to $S_{G,f}$, one need rotate chiral fermions ψ_L simultaneously with g,

$$g \to g^{-1}, \quad \text{and} \quad \psi_L \to g\psi_L g^{-1}\,.$$

Now since the transformation of chiral fermion is x-dependent, such a rotation will produce an integrated anomaly at quantum level, which is a WZW-like term that breaks this symmetry explicitly. Adding a WZW-like counterterm as Eq. (4.3.12) is exactly to offset this anomaly and keep the discrete symmetry above. So far it is still not adequate to require a counterterm, for there is no priori to admit this discrete symmetry in our theory. In fact, on the contrary, a four-dimensional sigma model describing goldstone bosons denies the symmetry $g \to g^{-1}$, but require it accompanied by parity inversion on spacetime, c.f. [88].

Nevertheless, in our case, the anomaly of this discrete symmetry is a signal of non-equivalence of different choices of sections, or gauge fixings. To see this, let us recall CCWZ coset construction on group G manifold, i.e. the unitary gauge Eq. (4.2.15),

$$s'(\phi) = \mathrm{e}^{\phi^A X_A} = \mathrm{e}^{\phi^A L_A - \phi^A R_A} \,. \tag{4.3.21}$$

Under this gauge, we describe our theory by writing its vielbeins and connection. From Eq. (4.2.14) and Eq. (4.2.41), we have the pullback Maurer-Cartan 1-form

$$s'^{-1} ds' = \mathrm{e}^{-\phi^A L_A} d\mathrm{e}^{\phi^A L_A} + \mathrm{e}^{\phi^A R_A} d\mathrm{e}^{-\phi^A R_A}$$

for L_A and R_B commute. Further, because L_A and R_B satisfy the same commutation rules, we will have same functional form, $\theta(\phi)$ for example, for the two terms with arguments up to a minus sign, i.e.,

$$s'^{-1} ds' = \theta^A(\phi) L_A + \theta^A(-\phi) R_A = \frac{1}{2}\big[\theta^A(\phi) + \theta^A(-\phi)\big] H_A + \frac{1}{2}\big[\theta^A(\phi) - \theta^A(-\phi)\big] X_A \,.$$

From it, we can read off the vielbein and connection 1-form under unitary gauge,

$$e'^A(\phi) = \frac{1}{2}\big[\theta^A(\phi) - \theta^A(-\phi)\big], \quad \omega'^A(\phi) = \frac{1}{2}\big[\theta^A(\phi) + \theta^A(-\phi)\big].$$

Apparently, $e'^A(\phi)$ and $\omega'^A(\phi)$ are odd and even 1-forms separately. One can check that, with the help of the parities of e' and ω', the theory indeed has the discrete symmetry mentioned above, which in coordinates ϕ and fermions ψ_L is given as:

$$\phi \to -\phi, \quad \psi_L \to -\psi_L \,. \tag{4.3.22}$$

Since the fermions is intact, at quantum level, this discrete symmetry is still hold. On the other hand, if we choose the section as Eq. (4.3.20), the Maurer-Cartan 1-form is

$$s^{-1}ds = \theta^A(2\phi)L_A = \frac{1}{2}\,\theta^A(2\phi)H_A + \frac{1}{2}\,\theta^A(2\phi)X_A \equiv \omega^A(\phi)H_A + e^A(\phi)X_A\,.$$

In this gauge, vielbeins and connection 1-form coincide [11] with each other, but their parities are sacrificed.

Now we are in the situation that we do not ask for the theory to have or deny the symmetry (4.3.22), but rather require it to be equivalently described in different choices of sections, e.g. s or s'. We know that sections (4.3.20) and (4.3.21) are connected by a H-gauge transformation,

$$s'(\phi) = s(\phi)\mathrm{e}^{-\phi^A H_A}\,.$$

Therefore the theory Eq. (4.3.19) is required to be H-gauge invariant even it has been shown to have vanishing isometry anomalies.

Furthermore, with the counterterm (4.3.12) added, applying the result of Sec.4.3.2, we know that the $\mathcal{N} = (0, 1)$ supersymmetric sigma model defined on simple Lie group G is equivalent to its bosonic principal sigma model plus a free chiral fermions, which is also different from the one predicted by action Eq. (4.2.2).

ii. Oriented real Grassmannian manifolds

Our second example is oriented real Grassmannian manifolds:

$$M = \frac{\mathrm{SO}(p+q)}{\mathrm{SO}(p)\times \mathrm{SO}(q)}\,.$$

We have known from chapter 3 that, for $p = 1$ (or $q = 1$), the manifolds is just sphere S^q with vanishing isometry anomalies [39]. Now we will consider the more general case by anomaly matching condition Eq. (4.3.11).

[11] It should be noticed that, although they have the same form, but they follow different transformation rules, see Eq. (4.2.30). This difference will not be detected by isometric transformation, for they only induce *constant* gauge transformation as what we showed.

In the Grassmannian case, $G = \mathrm{SO}(p+q)$ and $H = \mathrm{SO}(p) \times \mathrm{SO}(q)$ with standard embedding. We choose generators T_{AB} in the fundamental representation of Lie algebra $\mathfrak{g} = \mathfrak{so}(p+q)$ as:

$$(T_{AB})_{CD} = -\delta_{AC}\delta_{BD} + \delta_{AD}\delta_{BC}\,, \quad \text{with } A, B, C, D = 1, 2, ..., p, p+1, ..., p+q\,.$$

Their commutators are

$$[T_{AB},\ T_{CD}] = \delta_{AC}T_{BD} + \delta_{BD}T_{AC} - \delta_{AD}T_{BC} - \delta_{BC}T_{AD}\,,$$

and the normalized by Killing form is

$$\mathrm{Tr}(T_{AB}T_{CD}) = -2(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC})\,.$$

For Lie subalgebra $\mathfrak{h} = \mathfrak{so}(p) \oplus \mathfrak{so}(q) \equiv \mathfrak{h}_p \oplus \mathfrak{h}_q$ we label generators as

$$H_{\mathbf{i}} \equiv T_{ij} \in \mathfrak{so}(p)\,, \quad \text{for } i, j = 1, 2, ..., p\,,$$
$$H_{\mathbf{a}} \equiv T_{ab} \in \mathfrak{so}(p)\,, \quad \text{for } a, b = p+1, p+2, ..., p+q\,,$$

where we use subscripts "**i**" and "**a**" to label two indices for brevity. The rest of generators forms subspace $\mathfrak{m}$ complimentary to $\mathfrak{h}$, where we label them as

$$X_{ia} \equiv T_{ia}\,, \ \text{for } i = 1, 2, ..., p\,; \ a = p+1, p+2, ..., p+q\,.$$

Now we will investigate the isotropy representation of $\mathfrak{h}$ on $\mathfrak{m}$. For

$$[\mathfrak{h}_p,\ \mathfrak{h}_q] = 0\,,$$

we have the decomposition by Eq. (4.2.28),

$$(\mathrm{ad}\ \mathfrak{g})|_{\mathfrak{h}} = (\mathrm{ad}\ \mathfrak{h}) \oplus \varrho = (\mathrm{ad}\ \mathfrak{h}_p \oplus \mathrm{ad}\ \mathfrak{h}_q) \oplus \varrho\,. \tag{4.3.23}$$

Actually we only need to care about $\mathrm{Tr}_\varrho(H_{\mathbf{i}}H_{\mathbf{j}})$, $\mathrm{Tr}_\varrho(H_{\mathbf{a}}H_{\mathbf{b}})$ and $\mathrm{Tr}_\varrho(H_{\mathbf{i}}H_{\mathbf{b}})$. From Eq. (4.3.23), we have the equality

$$\mathrm{Tr}_\varrho = \mathrm{Tr}_{\mathrm{ad}\,\mathfrak{g}} - \mathrm{Tr}_{\mathrm{ad}\,\mathfrak{h}}\,, \tag{4.3.24}$$

while the latter two traces are easy to calculate by the commutation relationship and normalization above. After a short calculation,

$$\mathrm{Tr}_{\mathrm{ad}\,\mathfrak{g}}(H_{\mathbf{i}}H_{\mathbf{j}}) = -2(p+q-2)\delta_{\mathbf{ij}}\,,\ \mathrm{Tr}_{\mathrm{ad}\,\mathfrak{g}}(H_{\mathbf{a}}H_{\mathbf{b}}) = -2(p+q-2)\delta_{\mathbf{ab}}\,,\ \mathrm{Tr}_{\mathrm{ad}\,\mathfrak{g}}(H_{\mathbf{i}}H_{\mathbf{b}}) = 0\,;$$
$$\mathrm{Tr}_{\mathrm{ad}\,\mathfrak{h}}(H_{\mathbf{i}}H_{\mathbf{j}}) = -2(p-2)\delta_{\mathbf{ij}}\,,\ \mathrm{Tr}_{\mathrm{ad}\,\mathfrak{h}}(H_{\mathbf{a}}H_{\mathbf{b}}) = -2(q-2)\delta_{\mathbf{ab}}\,,\ \mathrm{Tr}_{\mathrm{ad}\,\mathfrak{h}}(H_{\mathbf{i}}H_{\mathbf{b}}) = 0\,.$$

Thus, we have

$$\mathrm{Tr}_\varrho(H_{\mathbf{i}}H_{\mathbf{j}}) = -2q\,\delta_{\mathbf{ij}}\,,\ \mathrm{Tr}_\varrho(H_{\mathbf{a}}H_{\mathbf{b}}) = -2p\,\delta_{\mathbf{ab}}\,,\ \mathrm{Tr}_\varrho(H_{\mathbf{i}}H_{\mathbf{b}}) = 0\,.$$

Therefore, to meet anomaly matching condition, we have only two cases when minimal $\mathcal{N}=(0,1)$ supersymmetric sigma models exist.

Case 1: $p=1$, $M=S^q$: $\mathrm{Tr}_\varrho(H_{\mathbf{a}}H_{\mathbf{b}}) = \mathrm{Tr}_F(H_{\mathbf{a}}H_{\mathbf{b}})$.

Case 2: $p=q$, $M=\mathrm{SO}(2p)/(\mathrm{SO}(p)\times\mathrm{SO}(p))$: $\mathrm{Tr}_\varrho(H_{\mathbf{i}}H_{\mathbf{j}}) = p\,\mathrm{Tr}_F(H_{\mathbf{i}}H_{\mathbf{j}})\,.$

The result on case 2 should have no further difficulty to be generalized to the case that H contains more than two identical factors, $H\simeq H_1\times H_2\times\cdots\times H_n$.

For the anomaly on oriented real Grassmannian manifolds M, there is also another interesting observation that helps verify our anomaly matching condition. Instead of constructing sigma models on $\mathrm{SO}(p+q)$ followed by gauging its subgroup $\mathrm{SO}(p)\times\mathrm{SO}(q)$, one can consider another fiberation,

$$\mathrm{SO}(p) \xrightarrow{\ i\ } V_q(\mathbb{R}^{p+q}) \xrightarrow{\ \pi\ } M\ ,$$

where $V_q(\mathbb{R}^{p+q})\simeq\mathrm{SO}(p+q)/\mathrm{SO}(q)$ is the Stiefel manifold which is the set of all orthonormal q-frames in $\mathbb{R}^{p+q}$. Sigma models built on Stiefel manifold is always well-defined which we will show in our next example. Here let us just assume it and consider how a real Grassmannian sigma model can be constructed in the fiberation above.

In previous chapter, we introduced a dual formalism for $\mathrm{O}(N)$ model. With a little modification, we can work out the $\mathcal{N} = (0,1)$ supersymmetric action on Grassmannian manifold,

$$\begin{aligned} S_M &= \frac{1}{2g_0^2}\int \mathrm{d}^2x \,\mathrm{Tr}\left((\nabla_R n)^T \nabla_L n + i\psi_L^T \nabla_R \psi_L\right), \\ (n^T)^a{}_\alpha n^\alpha{}_b &= \delta^a{}_b\,, \qquad (n^T)^a{}_\alpha \psi^\alpha_{Lb} = 0\,, \end{aligned} \tag{4.3.25}$$

where $n^\alpha{}_a$ and ψ^α_{La} $(\alpha = 1,2,...,p+q)$, $(a = 1,2,...,p)$, are real bosonic fields and their chiral fermions partners, and the covariant derivative ∇ is defined as

$$(\nabla_{R,L} n)^\alpha{}_a = \partial_{R,L} n^\alpha{}_a - n^\alpha{}_b A^b_{R,La}\,.$$

The model is obtained by gauging the color symmetries, i.e. those on indexes "a, b...", of the action on Stiefel manifolds. Thus action on Stiefel manifolds is obtained by removing gauge fields, and loosing the constraints on n and ψ_L as

$$n^T \psi_L + \psi_L^T n = 0\,.$$

In standard $(0,1)$ superspace construction, one can introduce a super Lagrange multiplier $\Lambda^a_{R\,b}$,

$$\Lambda^a_{R\,b} = \lambda^a_{R\,b} + \theta_R \sigma^a{}_b\,,$$

with indexes a, b symmetrized. Thus, the super-constraint term is

$$S_c = \int \mathrm{d}^2x \int \mathrm{d}\theta_R \mathrm{Tr}\;\Lambda_R (N^T N - I)\,,$$

where

$$N = n + i\theta_R \psi_L$$

is the superfield version of fields n and ψ_L and I is the $p \times p$ unit matrix. The sigma model on Stiefel manifold is given by

$$S_V = \int \mathrm{d}^2x \int \mathrm{d}\theta_R \Big[-\frac{1}{2g_0^2}\,\mathrm{Tr}\left((D_L N)^T \partial_{RR} N\right) + \mathrm{Tr}\;\Lambda_R (N^T N - I)\Big].$$

Correspondingly, after gauging its "color" symmetry SO(p), we obtain sigma model on

Grassmannian manifold M,

$$S_M = \int \mathrm{d}^2x \int \mathrm{d}\theta_R \Big[-\frac{1}{2g_0^2} \mathrm{Tr}\left((D_L N - N\mathcal{V}_L)^T (\partial_{RR} N - N\mathcal{V}_{RR}) \right) + \mathrm{Tr}\ \Lambda_R (N^T N - I) \Big],$$

where supergauge multiplets $\mathcal{V}_{L,RR}$ were introduced in Eq. (4.2.19).

In fact the above two sigma models can be obtained by considering the (classically) low energy limit of $\mathcal{N} = (0, 1)$ two-dimensional gauge theories and Yukawa theories respectively. We build the Yukawa theories as

$$\begin{aligned} S_Y = &\int \mathrm{d}^2x \int \mathrm{d}\theta_R \Big[-\frac{1}{2g_0^2} \mathrm{Tr}\left((D_L N)^T \partial_{RR} N \right) + \mathrm{Tr}\ \Lambda_R (N^T N - I) \Big] \\ &+ \int \mathrm{d}^2x \int \mathrm{d}\theta_R \Big[-\frac{1}{2\lambda_0^2} \mathrm{Tr}(\Lambda_R^T D_L \Lambda_R) \Big]. \end{aligned} \tag{4.3.26}$$

It is noticed that the coupling constant λ_0 has mass dimension for Λ_R has mass dimension 3/2. In the low energy limit, we put $\lambda_0 \to \infty$, and obtain the action S_V. In the sense we can interpret the UV completion of the sigma model on Stiefel manifold is a Yukawa theory, although the sigma model itself can be considered as a renormalizable theory in two-dimensions.

Similarly, let us find the UV completion of S_M by gauging the Yukawa theory S_Y and adding gauge sectors. Noticing that the Yukawa interaction S_c is gauge invariant, we have

$$\begin{aligned} S_{Y+\mathcal{V}} = &\int \mathrm{d}^2x \int \mathrm{d}\theta_R \Big[-\frac{1}{2g_0^2} \mathrm{Tr}\left((D_L N - N\mathcal{V}_L)^T (\partial_{RR} N - N\mathcal{V}_{RR}) \right) + \mathrm{Tr}\ \Lambda_R (N^T N - I) \Big] \\ &+ \int \mathrm{d}^2x \int \mathrm{d}\theta_R \Big[-\frac{1}{2\lambda_0^2} \mathrm{Tr}\left(\Lambda_R^T (D_L \Lambda_R + [\mathcal{V}_L,\ \Lambda_R]) \right) \Big] \\ &+ \int \mathrm{d}^2x \int \mathrm{d}\theta_R \Big[-\frac{1}{4e_0^2} \mathrm{Tr}\left(W_R (D_L W_R + [\mathcal{V}_L,\ W_R]) \right) \Big], \end{aligned} \tag{4.3.27}$$

where W_R is field strength of gauge potential $\mathcal{V}_{L,RR}$,

$$W_R \equiv [D_L + \mathcal{V}_L\,,\ \partial_{RR} + \mathcal{V}_{RR}\,]\,.$$

Couplings e_0 and λ_0 are of the same nonvanishing dimensionality, so in a low energy limit the last two terms fade away, and we obtain the sigma model S_M on Grassmannian M.

Now due to the observation of t'Hooft's consistency condition, we should expect that $S_{Y+\mathcal{V}}$ and S_M produce same anomalies or be anomaly-free. Therefore we focus ourselves on the gauge fields and bi-fermions interactions of action $S_{Y+\mathcal{V}}$ and calculate their anomalies. The relevant part of the Lagrangian is

$$\begin{aligned}\mathcal{L}_{\text{f.A.f}} &= \frac{i}{2g_0^2}(\psi_L^T)^a{}_\alpha(\partial_R\psi_L - \psi_L A_R)^\alpha{}_a + \frac{i}{2\lambda_0^2}\lambda^a_{R\,b}(\partial_L\lambda^b_{R\,a} + [A_L,\lambda_R]^b{}_a) \\ &+ \frac{i}{2e_0^2}\chi_{R\,\mathbf{i}}(\partial_L\chi^{\mathbf{i}}_R + A^{\mathbf{j}}_L C^{\mathbf{i}}_{\mathbf{jk}}\chi^{\mathbf{k}}_R)\ . \end{aligned} \tag{4.3.28}$$

To see if there are gauge-anomalies produced, we need to consider a vector rotation and compare the gauge anomalies from left and right fermions. For the right, since gauge fields are in the fundamental representation and we also need sum up flavors, say α indexes, we finally have:

$$\mathcal{A}_R \sim (p+q)\mathrm{Tr}_F(H_{\mathbf{i}}H_{\mathbf{j}}) = -2(p+q)\delta_{\mathbf{ij}}\ .$$

On the other hand, gauge fields interacting with gauginos are in adjoint representation of SO(p), and with λ_R are in the fundamental representation. We have:

$$\mathcal{A}_L \sim \mathrm{Tr}_{\text{ad}}(H_{\mathbf{i}}H_{\mathbf{j}}) + (p+2)\mathrm{Tr}_F(H_{\mathbf{i}}H_{\mathbf{j}}) = -4p\,\delta_{\mathbf{ij}}$$

Therefore gauge anomaly vanishes only when $p = q$ consistent with the result we obtained for the sigma model.

This observation on the correspondence of anomalies in two-dimensional gauge theories and sigma models could be useful for considerations theories in deep infrared region. We made some predictions in Sec. 4.3.2. We expect to verify them in anomaly-free gauge theories, by considering Large N-expansion as well. We will present this work somewhere else in the future.

iii. $(G \times \mathrm{U}^r(1))\,/H$ for group G semi-simple and H simple, Toric varieties G/T^r

Our third example is inspired by Eq. (4.3.24), by which we will show that for homogeneous space $M=(G \times \mathrm{U}^r(1))/H$ with G semi-simple and H simple, the anomaly matching condition Eq. (4.3.11) will be satisfied. Therefore, there always exists minimal $N=(0,1)$ supersymmetric sigma model on them.

The proof of the above statement is quite transparent when both G and H are simple groups. Since G and H are simple, their Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ will be simple, and, thus, contain no non-trivial ideals. Therefore, their adjoint representation $\mathrm{ad}\,\mathfrak{g}$ and $\mathrm{ad}\,\mathfrak{h}$ are irreducible respectively. By choosing an appropriate basis, generators $H_i \in \mathfrak{h}$ will satisfy to the following relations,

$$\mathrm{Tr}_{\mathrm{ad}\,\mathfrak{g}}(H_iH_j)=-T_G\delta_{ij}\,,\quad \mathrm{Tr}_{\mathrm{ad}\,\mathfrak{h}}(H_iH_j)=-T_H\delta_{ij}\,;\quad \mathrm{Tr}_{F(\mathfrak{g}|_{\mathfrak{h}})}(H_iH_j)=-\delta_{ij}\,, \tag{4.3.29}$$

where T_G and T_H are the dual Coxeter numbers of $\mathfrak{g}$ and $\mathfrak{h}$ respectively. Combining Eq. (4.2.28) and Eq. (4.3.24), we have

$$\mathrm{Tr}_{\varrho}(H_iH_j) = -(T_G - T_H)\delta_{ij} = (T_G - T_H)\mathrm{Tr}_F(H_iH_j)\ . \tag{4.3.30}$$

Now we can easily improve the above result. For G is semi-simple, the only difference is that we have distinguished normalization for its each simple factor. By assigning independent coupling constants for each simple factors $G_\alpha \subset G$, we can still normalize,

$$\mathrm{Tr}_{F(\mathfrak{g}|_{\mathfrak{h}})}(H_iH_j) = -\delta_{ij}\ ,$$

as our convention. For adjoint representation of $\mathfrak{g}$, we have

$$\mathrm{ad}\,\mathfrak{g} = \oplus_\alpha \mathrm{ad}\,\mathfrak{g}_\alpha\,.$$

Therefore, the first relation in Eq. (4.3.29) turns out to

$$\mathrm{Tr}_{\mathrm{ad}\,\mathfrak{g}}(H_iH_j) = -\Big(\sum_\alpha T_{G_\alpha}\Big)\delta_{ij}\,.$$

On the other hand, since H is simple as before, the second relation of Eq. (4.3.29) holds. Therefore

$$\mathrm{Tr}_\varrho(H_iH_j) = \Big(\Big(\sum_\alpha T_{G_\alpha}\Big) - T_H\Big)\mathrm{Tr}_F(H_iH_j)\,.$$

At last, the subgroup H contains no $U(1)$ factors, so the above result is thus unchanged.

Now let us apply this result to some typical examples. We simply enumerate some classical homogeneous (symmetric) space, satisfying the above condition, on which minimal $\mathcal{N} = (0, 1)$ supersymmetric sigma model can be constructed.

1. $G \times \mathrm{U}^r(1)$ with G semi-simple: For those chiral fermions on U(1) are free;

2. Real, complex and symplectic Stiefel manifolds:

$$\mathrm{SO}(p+q)/\mathrm{SO}(p), \quad \mathrm{SU}(p+q)/\mathrm{SU}(p), \quad \text{and} \quad \mathrm{Sp}(p+q)/\mathrm{Sp}(p)\ ;$$

3. $\mathrm{SU}(n)/\mathrm{SO}(n)$, $\mathrm{SU}(2n)/\mathrm{Sp}(n)$, and $\mathrm{SO}(2n)/\mathrm{Sp}(n)$: All are symmetric spaces.

From the argument above, we see that the condition H is simple is crucial. In fact, as we mentioned earlier, the anomaly matching condition is actually topological. In next subsection, we will show, for example when H is simple, the first Pontryagin class $p_1(M)$ will always vanish.

iv. H containing U(1) factors

Before proceeding to give a topological (characteristic class) explanation on anomaly matching condition Eq. (4.3.11), we want to consider another type of homogeneous space where the subgroup H in turn contains U(1) factors. We are motivated by realizing that, when H contains U(1) factors, the homogeneous spaces will have complex structure, and thus $\mathcal{N} = (0, 1)$ supersymmetry will be enhanced to $\mathcal{N} = (0, 2)$. Unfortunately, however, we will see soon that many of these sigma models are suffered from non-removable anomalies and thus do not exist.

First we want to make some clarification on the method used in Sec. 4.3.1 to derive the anomaly matching condition. Although the Polyakov-Wiegmann functional, see Eq. (4.3.10), was given in the context of non-Abelian gauge theories, it is also true when we have some Abelian U(1) gauge fields. For these Abelian gauge fields, labeled by $B^i_{R,L}T_i$ for example, $T_i \subset \mathfrak{h}$ commute with all other generators in $\mathfrak{h}$, and forms non-trivial center of Lie algebra $\mathfrak{h}$. Therefore, in the fermionic anomalous effective action Eq. (4.3.10), there is no WZW-like terms for them, but only the second one exists, i.e.,

$$\mathcal{W}_{\text{anom.}} = -\frac{1}{24\pi}\int_{\tilde{h}(B)} \mathrm{Tr}_\varrho(\tilde{h}^{-1}d\tilde{h})^3 + \frac{1}{16\pi}\int_\Sigma \mathrm{d}^2x\ \mathrm{Tr}_\varrho \omega_R(\tilde{h}^{-1}\partial_L\tilde{h} + \partial_L u^i T_i)\,,$$

where $\tilde{h}$ as before parameterize those non-Abelian gauge fields A_R, while u^i for B^i_R satisfies

$$B^i_R T_i = \partial_R u^i T_i\ .$$

Meanwhile, the counterterm (4.3.12), which we are able to add, will be also transformed under Abelian gauge rotation. Therefore, the anomaly matching condition will be the same as before.

Nevertheless it is because of the discrepancy above, we will show in the following that the anomaly matching condition can never be fulfilled for $H = H' \times \mathrm{U}^r(1)$ with H' is semi-simple. Therefore lots of minimal $N = (0, 2)$ supersymmetric sigma models, for example complex Grassmannian manifolds $\mathrm{U}(p + q)/(\mathrm{U}(p) \times \mathrm{U}(q))$(except for CP^1), are ruled out.

The proof is quite straightforward. With no loss of generality, let us only consider $H = H' \times \mathrm{U}(1)$ with that both G and H' are simple Lie groups. From a previous result, Eq. (4.3.30), we see that, for $H'_{i,j} \in \mathfrak{h}'$

$$\mathrm{Tr}_\varrho(H'_i H'_j) = (T_G - T_{H'})\mathrm{Tr}_F(H_i H_j)\,,$$

for $[H'_i\,, T] = 0$ and has no contribution to above equation, where T is the generator of U(1). For the same reason,

$$\mathrm{Tr}_\varrho T^2 = \mathrm{Tr}_{\mathrm{ad}\,\mathfrak{g}} T^2 - \mathrm{Tr}_{\mathrm{ad}\,\mathfrak{h}} T^2 = \mathrm{Tr}_{\mathrm{ad}\,\mathfrak{g}} T^2 = T_G \mathrm{Tr}_F T^2\ .$$

The anomaly matching condition is, thus, not satisfied. This finishes our proof.

A corollary we can obtain is that, for homogeneous spaces $M = G/T^r$ with $T = \mathrm{U}(1)$ the torus group, anomaly matching condition Eq. (4.3.11) is satisfied. Therefore minimal $\mathcal{N} = (0,2)$ supersymmetric sigma models on G/T^r can be well-defined.

4.3.4 Topological origin of anomaly cancellation

In this subsection, we will establish a relation between the (local) anomaly matching condition and the global topological property of homogeneous spaces $M = G/H$. More concretely, we will show that the anomaly matching condition Eq. (4.3.11) will be satisfied if and only if the first Pontryagin class on M vanishes, i.e. $p_1(M) = 0$, which thereby agrees with Moore-Nelson's constraint in case of homogeneous spaces [56].

The main argument is based on a proposition in [89], see Prop. (3.2), and a main theorem due to Borel and Hirzebruch, see Theorem 10.7 in [90]. We here only rephrase the result in terms of anomaly matching condition Eq. (4.3.11). The idea can be intuitively interpreted by Eq. (4.3.24),

$$\mathrm{Tr}_{\varrho}(H_i H_j) = \mathrm{Tr}_{\mathrm{ad}\,\mathfrak{g}}(H_i H_j) - \mathrm{Tr}_{\mathrm{ad}\,\mathfrak{h}}(H_i H_j)\,.$$

Since we always can, by rescaling coupling constants, require the equality

$$\mathrm{Tr}_{\mathrm{ad}\,\mathfrak{g}}(H_i H_j) = c'\,\mathrm{Tr}_F(H_i H_j)\,,$$

the anomaly matching condition is thus equivalent to

$$\mathrm{Tr}_{\mathrm{ad}\,\mathfrak{h}}(H_i H_j) = c''\,\mathrm{Tr}_F(H_i H_j) \sim \mathrm{Tr}_{\mathrm{ad}\,\mathfrak{g}}(H_i H_j)\,, \tag{4.3.31}$$

where c' and c'' are some constants. Now that we evaluate the traces in $\mathfrak{h}$ and $\mathfrak{g}$-adjoint representations, one can express them by means of group theoretical invariants, say symmetric functions of roots for $\mathfrak{h}$ and $\mathfrak{g}$ respectively. These symmetric functions are directly related to characteristic classes of M. We will explain that, when Eq. (4.3.31) is satisfied, the first Pontryagin class $p_1(M) = 0$.

In the following, G is assumed to be a compact connected Lie group, H a closed subgroup of G, and $T \subset G$ and $S \subset H$ are maximal tori of G and H respectively, chosen properly so that $S \subset T$. Let $\mathfrak{s} \subset \mathfrak{t}$ be corresponding Lie algebras of S and T, say the Cartan algebra of H and G. We further set $\{\beta_1, ..., \beta_s\} \subset \mathfrak{s}^*$ and $\{\alpha_1, ..., \alpha_t\} \subset \mathfrak{t}^*$ as positive roots in respect to H and G, and arrange them satisfying

$$\beta_1 = \alpha_1|_{\mathfrak{s}}\,,\ \beta_2 = \alpha_2|_{\mathfrak{s}}\,, \ldots,\ \beta_s = \alpha_s|_{\mathfrak{s}}\,,$$

and define

$$\gamma_1 \equiv \alpha_{s+1}|_{\mathfrak{s}}\,,\ \gamma_2 \equiv \alpha_{s+2}|_{\mathfrak{s}}\,, \ldots,\ \gamma_{t-s} \equiv \alpha_t|_{\mathfrak{s}}\,,$$

which are called the roots complimentary to H. With the help of roots, one can rewrite the traces in Eq. (4.3.31) on generators $S_i \in S \subset H$ and $T_i \in T \subset G$ as

$$\mathrm{Tr}_{\mathrm{ad}\,\mathfrak{h}}(S_i S_j) = \sum_{b=1}^{s} \beta_b(S_i)\beta_b(S_j), \quad \text{and} \quad \mathrm{Tr}_{\mathrm{ad}\,\mathfrak{g}}(T_i T_j) = \sum_{a=1}^{t} \alpha_a(T_i)\alpha_a(T_j)\,.$$

Therefore the trace operator can be expressed in terms of quadratic symmetric polynomials on $\sum \alpha_a^2$ or $\sum \beta_b^2$ on Cartan algebra $\mathfrak{s}$ and $\mathfrak{t}$ respectively. Actually it is sufficient to focus ourselves only on the Cartan algebra $\mathfrak{s}$ and $\mathfrak{t}$. It is because that, in our case, Lie algebra $\mathfrak{h}$ and $\mathfrak{g}$ can be always regarded as direct sum of several simple algebras and $\mathfrak{u}(1)$ factors. For each simple factor, with proper basis (Cartan-Weyl basis for example), evaluation of trace on Cartan algebra and other generators can be normalized same, but *not* same among different simple factors.

On the other hand, one can identify $\{\gamma_c\}$, the set of complimentary to H roots, with $H^1(S; Z)$, the first cohomology class of tori S, since they are integral functionals in $\mathrm{Hom}(\pi_1(S), Z) = \mathrm{Hom}(H_1(S; Z), Z)$. The $H^1(S; Z)$ can be further identified with $H^2(BS; Z)$ via transgression, where BS is the classifying space of tori S. Therefore, complimentary roots $\{\gamma_c\}$ will be considered as elements in $H^2(BS; Z)$. In what follows we will only work under real cohomology which will considerably simplifies our argument. First, the inclusion map

$$i : S \to H$$

induces an isomorphism i^* on the cohomology rings of BH and BS,

$$i^* : H^*(BH;R) \simeq H^*(BS;R)^{W(S,H)} \ , \tag{4.3.32}$$

where $H^*(BS;R)^{W(S,H)}$ denotes those elements invariant under Weyl group $W(S,H)$. Secondly, from the universal fiberation:

$$G \xrightarrow{i} EG \xrightarrow{\pi} BG \ ,$$

we have the fiberation by module H,

$$G/H \xrightarrow{j} BH \xrightarrow{q} BG \ .$$

It induces the exact cohomology classes chain

$$H^*(BG,R) \xrightarrow{q^*} H^*(BH;R) \xrightarrow{j^*} H^*(G/H;R) \ .$$

Since $\mathrm{T}(G/H)$ is the vector bundle associated to H-principle bundle, see Eq. (4.2.32), the total Pontryagin classes $p(G/H)$ are pullback elements from some universal elements $a \in H^*(BH;R)$,

$$p(G/H) = j^*(a)\,.$$

Now, with the identification in Eq. (4.3.32), we can express elements $a \in H^*(BH;R)$ in terms of symmetric functions of complimentary roots γ_c in $H^*(BS;R)^{W(S,H)}$,

$$a = \prod_{c=1}^{t-s}(1+\gamma_c^2) \ .$$

Specific to the first Pontryagin class $p_1(G/H)$, we have

$$p_1(G/H) = j^*\Big(\sum_{c=1}^{t-s}\gamma_c^2\Big)\,.$$

From the exact sequence a vanishing p_1 is equivalent to

$$\sum_{c=1}^{t-s} \gamma_c^2 \in \operatorname{Im} q^* \,.$$

At last, it is noticed that

$$\sum_{c=1}^{t-s} \gamma_c^2 = \sum_{a=1}^{t} \alpha_a^2|_{\mathfrak{s}} - \sum_{b=1}^{s} \beta_b^2 \,,$$

while similar to Eq. (4.3.32), we have an isomorphism on BG and BT ,

$$H^*(BG; R) \simeq H^*(BT; R)^{W(T,G)} \ .$$

Since $\sum \alpha_a^2$ is always in $H^*(BG; R)$, the condition $p_1(G/H) = 0$, or say $\sum \gamma_c^2 \in \operatorname{Im} q^*$, is equivalent to requirement $\sum \beta_b^2 \in \operatorname{Im} q^*$. It is just the anomaly matching condition (4.3.31).

4.4 The determinant line bundle of homogeneous space sigma models

The aim of this section is twofold. On the one hand, we would like to see in the nonlinear formulation, how much our understanding of gauge anomalies can benefit us in understanding anomalies in a pure geometric model. Isometries on Riemannian manifolds come in various cases, where some gauge formulation is far from reaching. Still one would like to understand, for example, the relation between chiral anomalies, isometry anomalies and topological anomalies. On the other hand, so far as sigma models on homogeneous spaces are concerned, we would like to see how could the gauge-like holonomy anomalies rise in a view toward determinant line bundle of certain Dirac operators parameterized by the space of bosonic field. The hope is to gain a full picture that touches each of the four corners: local vs global, gauge vs nonlinearity. A context like this can be useful in exploring interesting mathematical structures that closely tied up to each corners.

4.4.1 A digression on Kähler sigma model anomaly in Fujikawa's method

Here we shall look at the issue of local anomaly in geometric formulation. Isometries in our system form a subset of the diffeomorphism group of the target manifold, which is accomplished via field-redefinition alone. We would like to explore, whether such symmetries remain in the quantized system, and what does the anomaly imply. Since we shall not be dealing with unphysical degree of freedom, this is similar to the case of axial anomaly and thus Fujikawa's method can be generalized to our current situation.

We first clarify the types of manipulations we shall use in the discussion. Consider a vector field on X, which locally is given by $V = K^i(x)\partial_i$, where x^is are the local coordinates on X. There are two possible manipulations that can be induced by V — namely the field redefinition and the infinitesimal diffeomorphism. The former is via

$$\phi^i \to \phi^i + \epsilon K^i(\phi)$$

where $\phi \in C^\infty(\Sigma, X)$ is a bosonic field. Since this does not correspond to any symmetry in the action, this shall generally change the interaction. However, the field redefinition is a valid manipulation in field theories which should not cause any observational phenomena. The reason is that one can always get a contribution from the Jacobian of the path integral measure to overcome the change. The diffeomorphism transformation, on the other hand, is the aforementioned field redefinition together with the induced tensorial transformation for all geometric quantities. For example, under such transformation, the metric tensor transforms according to

$$g_{ij}(\phi) \to g_{ij}(\phi)\,\frac{\partial\phi^i}{\partial\phi^k + \epsilon K^k(\phi)}\,\frac{\partial\phi^j}{\partial\phi^l + \epsilon K^l(\phi)}\,,$$

and $\partial\phi^i$ transforms as a tangent vector. This definitely preserves the Lagrangian at the classical level. But in field theory language, when one interprets g_{ij} as a function of the field ϕ, there will also be an accompanying transformation for the "coupling constants" of ϕ in g_{ij}. To make sense of those, one can view those (infinite number of) constants as background fields damped at classical values. The path integral measure would need further justification. This, however, is not the case that we are interested in.

The isometry symmetries are a subset in both classes. Defined solely by field redefinition, it satisfies the property that

$$g_{ij}(\phi)\,\frac{\partial\phi^i}{\partial\phi^k+\epsilon K^k(\phi)}\,\frac{\partial\phi^j}{\partial\phi^l+\epsilon K^l(\phi)}=g_{kl}(\phi+\epsilon K(\phi))+O(\epsilon^2) \tag{4.4.1}$$

and hence preserves the Lagrangian at the classical level. The same is true for the quantum bosonic model, and this is a pure consequence of the property of field redefinition. Indeed, we are forced to have that the Jacobian from path integral measure cancels the anomalous effective action. But a perturbative calculation shows that the effective action respects the isometry, thus forcing the path integral measure to respect the same symmetry up to an overall factor. Indeed, one can see this explicitly by writing down explicitly the measure, where we have used a standard volume form $[D\phi]$ on X associated to the metric g,

$$[D\phi]=\sqrt{\det g_{ij}}\,d\phi^1\wedge\cdots d\phi^n\,. \tag{4.4.2}$$

If our model is coupled with chiral fermions, the path integral measure might not preserve such symmetries, and if this is true, nor shall the effective action after integrating out the fermions. This is the anomaly that we are interested in.

In supersymmetric models with target manifold X, fermions take value in the tangent bundle TX. To build the path integral measure, one has to contract the indexes on TX using a standard volume form. Together with the contribution from the bosonic part, we have that

$$[D\psi]=\frac{1}{\sqrt{\det g_{ij}}}\,d\psi_L^1\cdots d\psi_L^n d\psi_R^1\cdots d\psi_R^n\,. \tag{4.4.3}$$

Note that ψ_R are decoupled from our system, and we write them down to show the comparison between our case, and the nonchiral case. Also to use Fujikawa's method, it is important to have Dirac fermions. Now we perform the isometry transformation induced

	$P_L\psi^i$	$P_R\psi^i$
$\bar{\psi}^{\bar{j}}P_L$	0	$\delta_{i\bar{j}}$
$\bar{\psi}^{\bar{j}}P_R$	$g_{i\bar{j}}$	0

Table 4.1: The metric used in fermion path integral measure in curved indices.

by the Killing vector field K_A, where the index A labels isometries:

$$\begin{aligned}\phi^i(x) &\to \phi^i(y) + \epsilon^A \int \mathrm{d}^2x\, K_A^i[\phi(x)]\delta(x-y)\,, \\ \psi_L^i(x) &\to \psi_L^i(y) + \epsilon^A \int \mathrm{d}^2x\, \partial_j K_A^i[\phi(x)]\delta(x-y)\psi_L^j(y)\,.\end{aligned} \tag{4.4.4}$$

Note that the transformation is linear with respect to the fermionic degrees of freedom. Indeed, we can learn from the case of chiral anomaly that, as far as only the local anomalies are concerned, it is really the phase factor of such transformation that matters.

Let us suppose we have Weyl fermions. Also from here to the end of this section, we shall assume the target manifold to be Kähler, to get the most elegant result. Write explicitly the path integral measure as

$$[D\psi] = \frac{1}{\sqrt{\det G}}\, d\bar{\psi}^{\bar{1}} d\psi^1 \cdots d\bar{\psi}^{\bar{n}} d\psi^n\,, \tag{4.4.5}$$

where each ψ has two components ψ_L and ψ_R. The metric G expanded in basis of $P_L\psi, P_R\psi, \bar{\psi}^{\bar{j}}P_L, \bar{\psi}^{\bar{j}}P_R$ is given by Table. 4.4.1. Hence

$$\det G = \det g_{i\bar{j}}\,. \tag{4.4.6}$$

Now under the transformation

$$\begin{aligned}\psi^i(x) &\to \psi^i(y) + \epsilon^A \mathfrak{Re}(\partial_j K_A^i[\phi(y)])P_L\psi^j(y) + i\epsilon^A \mathfrak{Im}(\partial_j K_A^i[\phi(y)])P_L\psi^j(y)\,, \\ \bar{\psi}^{\bar{i}}(x) &\to \bar{\psi}^{\bar{i}}(y) + \epsilon^A \mathfrak{Re}(\partial_{\bar{j}} \bar{K}_A^{\bar{i}}[\bar{\phi}(y)])\bar{\psi}^{\bar{j}}(y)P_R + i\epsilon^A \mathfrak{Im}(\partial_{\bar{j}} \bar{K}_A^{\bar{i}}[\bar{\phi}(y)])\bar{\psi}^{\bar{j}}(y)P_R \\ &= \bar{\psi}^{i}(y) + \epsilon^A \mathfrak{Re}(\partial_j K_A^i[\phi(y)])\bar{\psi}^{\bar{j}}(y)P_R - i\epsilon^A \mathfrak{Im}(\partial_j K_A^i[\phi(y)])\bar{\psi}^{\bar{j}}(y)P_R\,.\end{aligned} \tag{4.4.7}$$

Recall that the Jacobian, as in the pure bosonic case, has only nontrivial real part, which cancels the change of $d\psi$ and $d\bar{\psi}$ induced by $\mathfrak{Re}(\partial_j K_A^i[\phi(y)])$. But the transformation induced by $\mathfrak{Im}(\partial_j K^i_A[\phi(y)])$ is anomalous. The situation here is precisely the same as in case of chiral anomaly, and for the time being, we take the bosonic degrees of freedom to be external, or classical.

In Fujikawa's method, infinitesimal isometry transformation gives the following extra factor for the fermion integral measure:

$$\delta_{\epsilon^A}\left(\det i\mathcal{D}\right)[\phi,\bar{\phi}] = \exp\Big(-i\epsilon^A\int \mathrm{d}^2x\,\mathrm{Tr}[\mathfrak{Im}(\partial_j K^i_A[\phi(x)])\gamma_5]\Big)\,, \tag{4.4.8}$$

where the trace is taken over the basis from the right eigenstates of the Dirac operator

$$i\mathcal{D}_{i\bar{j}} \equiv i\left(g_{i\bar{j}}\partial\!\!\!/ + g_{j\bar{j}}\Gamma^j_{ik}\partial\!\!\!/\phi^k\right)P_L + i\partial\!\!\!/\delta_{i\bar{j}}P_R = i\mathcal{D}\!\!\!\!/\,P_L + i\partial\!\!\!/ P_R \tag{4.4.9}$$

and its left eigenstates. Evaluation of Eq. (4.4.8) is in general hard, due to the nonflatness of $g_{i\bar{j}}$ and the bosonic degree of freedom. But for the result in 2d, as an analog to the gauge theory case [68], we obtain that, up to the lowest order in external fields,

$$\delta_\epsilon i\Gamma_{\text{eff}}[\phi] = \frac{i\epsilon_A}{4\pi}\int \mathfrak{Im}(\partial_k K^l_A)\mathcal{R}_{i\bar{j}l}{}^k d\phi^i\wedge d\bar{\phi}^{\bar{j}} + \text{higher order terms in}\,\Gamma^{\text{i}}_{\text{jk}}\,. \tag{4.4.10}$$

Indeed, we only need the leading term from $\bar{\partial}_{\bar{j}}\Gamma_{il}{}^k$, which is also, up to a sign, the leading term of the curvature tensor $\mathcal{R}_{i\bar{j}k\bar{l}}$. Note that there is a special feature of nonabelian anomaly (or, correspondingly, the linear isometry anomaly)— if one only cares about the lowest order in the "gauge" field A (or, correspondingly, the Christoffels Γ), then the contribution for nonabelian anomaly is, up to a constant factor, the same as the abelian anomaly [68, 92]. This shows why the anomaly diagram in perturbative calculation looks similar to the one involved in axial anomaly. The constant factor, in $2n$ dimensional spacetime, might depend on the kinematics of the $(n+1)$-gon Feynman diagrams. But in 2d case this is extremely simple. To determine the full structure of such anomaly, one can either do a thorough calculation of Eq. (4.4.8), or use an argument like Wess-Zumino consistency condition as mentioned in [68].

Now we calculate the explicit form of Eq. (4.4.10) using the second method. Recall that the abelian anomaly for a nonlinear sigma model over a Kähler target manifold in 4d is given by

$$\mathrm{Tr}\left[\mathcal{R}^2\right] = (\mathcal{R}_{i\bar{j}}{}^{m}{}_{n}\mathcal{R}_{k\bar{l}}{}^{n}{}_{m} + \mathcal{R}_{i\bar{j}\bar{m}}{}^{\bar{n}}\mathcal{R}_{k\bar{l}\bar{n}}{}^{\bar{m}})d\phi^i \wedge d\bar{\phi}^{\bar{j}} \wedge d\phi^k \wedge d\bar{\phi}^{\bar{l}}\,. \tag{4.4.11}$$

This combination is invariant with respect to isometry transformation, and lifts up to a cohomology class. So locally there is a 3-form ω_3^0 such that $d\omega_3^0 = \mathrm{Tr}\left[\mathcal{R}^2\right]$. Note that making use of Kähler geometry, the bootstrapping procedure is similar to that of non-abelian gauge theories [63], if we consider the following relation:

$$\mathcal{R}_{i\bar{j}}{}^{m}{}_{n}d\phi^i \wedge d\bar{\phi}^{\bar{j}} = -d\Gamma^m{}_n - \Gamma^m{}_b \wedge \Gamma^b{}_n\,, \quad \Gamma^m{}_n \equiv g^{m\bar{a}}\partial_i g_{n\bar{a}} d\phi^i\,. \tag{4.4.12}$$

The isometry transformation is induced by the Killing vector field on the target manifold, which gives,

$$\delta_{\epsilon^A} = dK_A + K_A d\,, \tag{4.4.13}$$

where the vector field is of the form

$$K_A = K_A^i \partial_i + \bar{K}_A^{\bar{i}} \bar{\partial}_{\bar{i}}\,. \tag{4.4.14}$$

Using the fact that the Kähler metric is compatible with the Killing vector fields, we get that

$$\delta_{\epsilon^A}\Gamma^m{}_n = -\partial(\partial_n K_A^m) - \partial_n K_A^a \Gamma^m{}_a + \partial_a K_A^m \Gamma^a{}_n \equiv -d(\partial_m K_A^n) - [\Gamma, \partial K_A]^m{}_n\,. \tag{4.4.15}$$

Finally we have that $\delta_{\epsilon^A}\omega_3^0 = d\omega_2^1[K_A]$, which bears the form

$$\omega_2^1 \propto \left(\partial_m K_A^n d\Gamma^m{}_n - \bar{\partial}_{\bar{n}} \bar{K}_A^{\bar{m}} d\bar{\Gamma}_{\bar{m}}{}^{\bar{n}}\right)\,. \tag{4.4.16}$$

This, at the first order level, coincides with Eq. (4.4.10).

4.4.2 Global vs local anomalies from geometric point of view

In this subsection we shall discuss technical points of the previous calculation, and then deduce the relation between global and local anomalies.

Indeed, we cheated a little in the previous calculation of the anomaly, and what has been hidden is the discussion on geometric condition for the anomaly. Notice that the Dirac operator $\mathcal{D}$, when restricted to the spin bundle on worldsheet, changes the helicity and hence maps ψ_L to ψ_R. However it is not self-composable. This is because in our definition for $\mathcal{D}$, ψ_L lives in the tangent space of a curved target space, while ψ_R lives in a flat space. One consequence of this problem is that we actually do not have a precise definition for the functional determinant of D. This problem can be easily cured by choosing a local diffeomorphism $\mathcal{E}[\phi, 0] : TX \to T\mathbb{C}^m$ and $\mathcal{E}^{-1}[0, \phi] : T\mathbb{C}^m \to TX$.[12] Note the parameter 0 and ϕ in $\mathcal{E}$ merely indicates that ψ_R is decoupled, and ψ_L is coupled to ϕ. Then we can compose $\mathcal{D}$ to obtain an elliptic operator whose image and source are the *same* Hilbert space:

$$\mathcal{D}^2 := \mathcal{E}\not{\mathcal{D}} P_L \mathcal{E}^{-1} \not{\partial} P_R + \mathcal{E}^{-1} \not{\partial} P_R \mathcal{E} \not{\mathcal{D}} P_L \,. \tag{4.4.17}$$

The functional determinant can now can be defined, and a regulator is introduced, by having that

$$\delta_{\epsilon^A} \left(\det i\mathcal{D}\right) [\phi, \bar{\phi}] = \lim_{M\to\infty} \exp\Big(- i\epsilon^A \int \mathrm{d}^2 x \, \mathrm{Tr} \big[\mathfrak{Im}(\nabla_j K^i_A[\phi(x)]) \gamma_5 f\big(\mathcal{D}^2/M^2\big) \big] \Big) \tag{4.4.18}$$

for a smooth function $f(x)$ on $\mathbb{R}$ such that $f(0) = 1$ and $f(\infty) = 0$.

Before proceeding to calculation, we immediately sense a problem, that the maps $\mathcal{E}$ and $\mathcal{E}^{-1}$ are only locally defined. Now if we want to patch the map to make it fibers nicely over the space of bosonic field $C^\infty(\Sigma, X)$ without ambiguity, we would have to view $\delta_{\epsilon^A} \left(\det i\mathcal{D}\right) [\phi, \bar{\phi}]$ firstly as a complex line bundle, and impose the trivialization condition. In fact, one needs no worry here, if the aforementioned model is free of Moore-Nelson anomaly [56]. In their work, the condition to trivialize the line bundle $(\det i\mathcal{D}) [\phi, \bar{\phi}]$ has been given. Suppose our model satisfies their condition, then $(\det i\mathcal{D}) [\phi, \bar{\phi}]$ is a function of $\phi \in C^\infty(\Sigma, X)$, so is $\delta_{\epsilon^A} \left(\det i\mathcal{D}\right) [\phi, \bar{\phi}]$. To conclude here, the vanishing of Moore-Nelson anomaly implies that there is no global obstruction for isometry.

Next, we shall look into the local anomaly. Now we consider the functional determinant $(\det i\mathcal{D}) [\phi, \bar{\phi}]$ to be a function of $\phi \in C^\infty(\Sigma, X)$, then the variation of δ_{ϵ^A} was induced by a vector field on $C^\infty(\Sigma, X)$. Locally we need that there exists a Lie algebroid structure on $TC^\infty(\Sigma, X)$ induced by the infinitesimal isometry on X, i.e., there is a subspace of

[12] The diffeomorphisms $\mathcal{E}$ and $\mathcal{E}^{-1}$ are precisely the isomorphisms $T^{(+)}$ and $T^{(-)}$ used in Section 2 of [56].

$TC^\infty(\Sigma, X)$ over $C^\infty(\Sigma, X)$, which has a Lie bracket coming from the Lie brackets of Killing vector fields on X. This says that, the Lie algebra action can be realized on $(\det i\mathcal{D})\,[\phi, \bar{\phi}]$, i.e., $[\delta_{\epsilon^A}, \delta_{\epsilon^B}] = f^C_{AB}[\phi, \bar{\phi}]\delta_{\epsilon^C}$. Solving the Wess-Zumino consistency condition is equivalent to writing down the explicit form of the effective action (with a counterterm added) as a local functional.

4.4.3 The determinant line bundle analysis

In previous section, we have seen that the relation between isometry anomaly, and the global anomalies for Kählerian manifolds is the following. Once the global anomaly is absent, the functional determinant can be viewed as a function, as opposed to a section of a complex line bundle, over the space of bosonic field. Then, the isometry variation of the theory is via some selected vector fields acting on the determinant (ie, the effective action). The Wess-Zumino condition is then automatically satisfied. In the process of canceling the isometry variation, the counterterm is predicted indeed by the trivialization of a 4-form which represents the first Pontriyagin class.

We want to clarify here when we have Hermitian vector bundles over a Kähler manifold, what do we mean by the first Pontryagin class. Indeed, the argument of [56] gave the anomaly in terms of a second real Chern character, which by definition is defined on real vector bundles by taking the complexification first, and then apply the complex Chern character. In this way, one verify that for real vector bundles, this second real Chern character precisely gives p_1 of the bundle, and in case of a complex vector bundle, this gives $2\,ch_2$ of the complex bundle.

Using Chern-Weil construction, choosing a connection Θ over the bundle E one sees clearly that the 4-form representing the obstruction is

$$\int_{Y\times\Sigma} ev^*\mathrm{tr}\Big[\Big(\frac{i}{2\pi}\,\mathcal{R}^\Theta\Big)^2\Big]\,,$$

where Y is an arbitrary 2-cycle in the space of bosonic field $C^\infty(\Sigma, X)$ and ev is the evaluation map

$$ev : \Sigma \times C^\infty(\Sigma, X) \to X\,.$$

We want to trivialize the expression, and one of the sufficient condition is that ch_2

vanishes before we pulling it back. This Chern-Weil form of ch_2 can always be locally trivialized by the Chern-Simons transgression 3-form $CS(\Theta)$ on X. Moreover, if ch_2 is trivial, the Chern-Simons form is globally defined on X. Then we there is guarantee that the isometry variation of $CS(\Theta)$ is trivialized by a 2-form, which is able to compensate the anomalous transformation of the functional determinant ,

$$\delta_\alpha CS(\Theta) = d(\omega_2)\,, \omega_2 = \mathrm{tr}(\alpha d\Theta) \sim \delta_\alpha \Gamma_{\text{eff}}\,.$$

So the counterterm in this case is given by $CS(\Theta)$. If further more the Chern-Weil form turns out to be trivial, then $CS(\Theta)$ is a closed form, representing a cohomology class in $H^3(X;\mathbb{Q})$. Then the counterterm to be added is genuinely 2-dimensional, which is determined by

$$\begin{aligned} CS(\Theta) &= d\,\Omega_2\,, \\ \delta_\alpha CS(\Theta) &= d(\delta_\alpha \Omega_2) = d\,\mathrm{tr}(\alpha d\Theta)\,. \end{aligned} \tag{4.4.19}$$

Next we explain why holonomy anomaly, as arise genuinely from a gauge description, can be viewed as the nontriviality of certain determinant line bundle, the latter been discussed extensively by [56] and [74].

Starting with the bosonic field $g \in C^\infty(\Sigma, G)$ of the theory, we have that

$$ev : C^\infty(\Sigma, G) \times \Sigma \to G\,, \quad (g, x) \mapsto g(x)\,,$$

and at the level of differential forms, we also have a pushforward map

$$e_* : \Omega^*(C^\infty(\Sigma, G) \times \Sigma) \to \Omega^*(C^\infty(\Sigma, G))$$

induced by integration along Σ. The classical action of the theory should be viewed as a dimΣ-form on $C^\infty(\Sigma, G) \times \Sigma$ pushed down to $\Omega^*(C^\infty(\Sigma, G))$, and hence is a function of the field. Path integral quantization amounts to say that there is also a certain pushforward map by integrating along $C^\infty(\Sigma, G)$. As we do not have applicable mathematical tools to rigidify the process, we shall just consider it as given by the canonical quantization.

To build a coset model using chiral gauge method, we introduce a gauge field A coming from a connection in $Conn(\mathrm{ad}_{\mathfrak{h}}P)$ for an adjoint $\mathfrak{h}$-bundle of $H \to P \to \Sigma$, and now the bosonic field g is promoted to smooth sections $g \in \Gamma(\Sigma, P \times_H G)$. When the bundle P has a global section, g can be viewed as a G-valued smooth map from Σ. In the following analysis, we shall use a local trivialization of P to write g as a smooth map $U \to G$ for $U \subset \Sigma$ while keep in mind the nontrivial gluing of g across open covers of Σ.

The infinite dimensional topological group $C^\infty(\Sigma, H)$ acts on the space of fields:

$$C^\infty(\Sigma, H) \times \Gamma(\Sigma, P \times_H G) \to \Gamma(\Sigma, P \times_H G)) : (h, g) \mapsto gh\,,$$

and

$$C^\infty(\Sigma, H) \times Conn(\mathrm{ad}_{\mathfrak{h}}P) \to Conn(\mathrm{ad}_{\mathfrak{h}}P) : (h, A) \mapsto h^{-1}Ah + h^{-1}dh\,.$$

The action is a functional over the space of field, which is invariant with respect to gauge transformation, and thus is a functional over the orbit space of diagonal action of gauge transformation, which we call the reduced space of field

$$\Gamma(\Sigma, P \times_H G) \times_{C^\infty(\Sigma,H)} Conn(\mathrm{ad}_{\mathfrak{h}}P)\,.$$

Note that the gauge group acts on the bosonic field freely, so the quotient space can be taken as the honest orbit space without invoking ghost degree of freedom.

Now the gauge fixing is a local functional f over the un-reduced space of field whose critical locus intersects $C^\infty(\Sigma, H)$-orbits transversely. By solving out the gauge fixing condition, one picks out a unique element in $\Gamma(\Sigma, P \times_H G)$ for each orbit, and correspondingly the action functional will be restricted to $\mathrm{Crit}(f) \times Conn(\mathrm{ad}_{\mathfrak{h}}P)$, which models the reduced space of bosonic fields. In the gauged formalism of bosonic homogeneous space sigma models, one fixes the gauge by asking the connection to be particular one pulled-back via g from the principal H-bundle $\pi : G \to G/H$. Note that $G \to G/H$ is a Riemannian submersion, and hence this gives a subbundle $\pi^*T(G/H) \subset TG$. Now the Maurer-Cartan form splits into spin connection on G/H and the vielbein 1-form

$$g^{-1}dg = \omega^i H_i + e^a X_a\,. \tag{4.4.20}$$

We now describe the fermions coming from $(0, 1)$ supersymmetry. Those are, from the target side, sections of vector bundles associated to the principal H-bundle via the isotropic representation ϱ where $ad\mathfrak{g} = ad\mathfrak{h} \oplus \varrho$ as before.

Let S_L, S_R denote the bundles associated to $Spin(\Sigma)$ with half spin representation, and we have that

$$\psi \in \Gamma(\Sigma, S_L \otimes g^*G \times_\varrho \mathfrak{m}), \quad g \in \Gamma(\Sigma, P \times_H G);$$

$$D_{RR} : \Gamma(\Sigma, S_L \otimes g^*G \times_\varrho \mathfrak{m}) \to \Gamma(\Sigma, S_R \otimes g^*G \times_\varrho \mathfrak{m}).$$

There is also a linear gauge group action on fermions induced from the isotropic H-actions on ϱ. And due to the pull-backing of $g \in \Gamma(\Sigma, P \times_H G)$, the gauge connection A is coupled to ψ. The Dirac operator we need to consider comes from a Dirac operator on the pulled-back bundle of TG

$$D_{RR}^{\varrho \oplus \mathrm{ad}\,\mathfrak{h}} = \partial_{RR} + g^{-1}\partial_{RR} g + A_{RR},$$

whose component in the isotropy representation is

$$(D_{RR}^{\varrho})^{ab} = \partial_{RR}\delta^{ab} + \frac{1}{2}\, C_i^{ab}(A_{RR}^i + \omega_{RR}^i) + \frac{1}{2}\, C_c^{ab} e_{RR}^c. \tag{4.4.21}$$

The operator is parameterized by $\Gamma(\Sigma, P \times_H G) \times Conn(\mathrm{ad}_\mathfrak{h} P)$ and is gauge covariant. If there is no chiral fermion anomaly, taking the functional determinant of it should result in a gauge invariant expression, and thus descending down to a functional over $\Gamma(\Sigma, P \times_H G) \times_{C^\infty(\Sigma, H)} Conn(\mathrm{ad}_\mathfrak{h} P)$. The presence of fermionic anomaly is because of the fact that the fermionic effective action might be a section of a nontrivial complex line bundle over the space of fields for two reasons. Firstly, it is possible that the effective action is a line bundle already over the un-reduced total space $\Gamma(\Sigma, P \times_H G) \times Conn(\mathrm{ad}_\mathfrak{h} P)$ even before we check the gauge invariance; and secondly, it is possible that the nontriviality of the anomaly comes from the failure of descent condition at quantum level.

Repeating the analysis in [56], one knows that the line bundle is characterized by its first Chern class, which, upon integrating over a two-cycle in the space, gives the Chern number. In this way, one reduce the task of understanding the infinite dimensional space of field $\Gamma(\Sigma, P \times_H G) \times_{C^\infty(\Sigma, H)} Conn(\mathrm{ad}_\mathfrak{h} P)$ to an arbitrary 2-dimensional 2-cycles in it.

We need to be more specific about the choice of 2-cycles. It is hard to lift up a 2-cycle from the base to the larger space precisely because the interaction between the 2-cycle in the base, and the gauge group. But here we have some convenient choice because of the special form of the Dirac operator. Note that the connection-dependence of the Dirac operator decouples into two parts

$$A_1 = \frac{1}{2}\left(\omega_{RR} - A_{RR}\right) , \quad A_2 = A_{RR}|_{\varrho} + \frac{1}{2}\, e_{RR} \, ,$$

the former is covariant with respect to gauge transformation, while the latter is not. In fact, A_1 is the difference of two connections on the very same bundle $P \to \Sigma$. This is based on two facts: 1) ω is a principal H-connection on $G \to G/H$; and 2) a section of the associated bundle $P \times_H G \to \Sigma$ can be used to pull the connection back to $P \to \Sigma$. To understand how the connection can be pulled back, it is enough to see that the sections pullback via $g \in \Gamma(\Sigma, P \times_H G)$, which is obvious. Along this line, one can view an element in $\Gamma(\Sigma, S_L \otimes g^* G \times_{\varrho} \mathfrak{m})$ as one in $\Gamma(\Sigma, S_L \otimes P \times_{\varrho} \mathfrak{m})$. A characteristic computation at rational cohomology level would not depend on A_1. Now the analysis from determinant line bundle says that the anomaly is given by

$$\int_{Y \times \Sigma} \hat{A}(Y \times \Sigma) \cdot ev^* ch(F^{A_2}) \, .$$

The space Y is a 2-cycle in the space of bosonic fields. On the one hand, if we ask Y to be a 2-sphere in $\Gamma(\Sigma, P \times_H G)$ which intersects gauge orbits transversely, then this expression gives rise to the known p_1 anomaly condition. If we take Y to be a 2-sphere suspended from gauge orbit [68], and use A_{RR} as a representative for A_2, this gives the condition on non-abelian chiral gauge anomaly as shown in Sec. 4.3.1.

4.5 Conclusion and outlooks

In this chapter we systematically study the anomalies in minimal $\mathcal{N} = (0,1)$ and $(0,2)$ supersymmetric sigma models on homogeneous spaces. The investigation starts from our previous obersvation [39] on isometry/gauge anomalies correspondence for the sigma models realized in non-linear/linear gauged formalisms respectively. It leads us to consider more general holonomy anomalies and how to remove them.

Following Polyakov and Wiegmann, we systematically explore the anomalous fermion effective action and obtain its explicit form. Later, in the procedure of mending the anomalous action, we derive an anomaly matching condition as criteria to sieve out ill-defined models. This condition is equivalent to the global topological constraint of $p_1(G/H)$ thoroughly discussed by Moore and Nelson [56]. More importantly and surprisingly, we demonstrate that these local counterterms will further modify and constrain the behavior of the "curable" theories in deep IR region. Supersymmetry will be broken in some theories, whereas some others flow to nontrivial infrared superconformal fixed points.

In addition to the general discussion above, we also analyzed various concrete examples, applying the anomaly matching condition to different types of G and H. We find that most survived minimal models are $\mathcal{N} = (0,1)$ supersymmetric, while $\mathcal{N} = (0,2)$ minimal models, due to their nontrivial center in H, are typically topologically obstructed.

We also reveal an interesting correspondence between two-dimensional $\mathcal{N}=(0,1)$ minimal sigma models and gauge theories, analogous to t'Hooft's anomaly matching observation in the four-dimensional case. Finally, we discussed the isometry/holonomy anomalies and the anomaly matching condition from the standpoint of determinant line bundle. We obtained a more general expression on the anomaly equation with the help of a more powerful mathematical tool operative in fields configuration spaces.

Because of the simplicity of the fermion sector in the minimal models we should expect that these models would be either destroyed or strongly constrained by anomalies. This expectation is more or less substantiated in this chapter: our refined treatment of the anomalies and their remedies displays very interesting features of the minimal $\mathcal{N} = (0,2)$ and $(0,1)$ sigma models. Our subsequent work will continue along these lines. It should be interesting to work out some solid examples to further verify our results on the low-energy behavior of the minimal sigma models. Good candidates include models on G/T^r (not necessarily maximal tori), since the complex structures on them will enhance supersymmetry to $\mathcal{N}=(0,2)$, which makes them particularly easy to handle.

On the other hand, it is also noteworthy that the $\mathcal{N}=(0,1)$ minimal sigma model on $\mathrm{SO}(2p)/(\mathrm{SO}(p)\times\mathrm{SO}(p))$ corresponds to a $\mathcal{N}=(0,1)$ two-dimensional gauge theory with the gauge group $\mathrm{SO}(p)$. It is, thus, interesting to ask whether or not every curable minimal model will have its corresponding gauge theory, and how to find them. Investigating these gauge theories may also shed light on the minimal sigma models, and *vice versa*. We expect to answer some of these questions in the subsequent works. If one further considers the dynamics of gauge fields, we'd expect that the different formulations (nonlinear v.s. gauge formalism v.s. dual formalism, etc) leads to different quantum theories, which are possibly connected by phenomena like dualities, which indicates an interesting direction to pursue. Especially, for $\mathrm{O}(N)$ models, it has been known that both the bosonic and the $\mathcal{N}=(1,1)$ theories are integrable at quantum level. It is thus interesting to consider the integrability for $\mathcal{N}=(0,1)$ models. With the help of gauge freedom, the problem is actually easier, the dual formalism deserves more thorough study in this respect.

Besides, the gauge/isometry anomalies correspondence also highlights for us the consistency check of sigma models from its linear gauge formalism. In the case of the fibration:

$$\mathrm{U}(1)\to S^{2N-1}\to \mathrm{CP}^{N-1},$$

as we have shown, the inconsistency of the $\mathrm{CP}(N-1)$ model can be interpreted as a non-removable $\mathrm{U}(1)$ gauge anomaly. Following this line of reasoning, we may consider, for example, the following fibration

$$Z_2\to S^{N-1}\to \mathrm{RP}^{N-1}.$$

It requires one to gauge the discrete Z_2 symmetry to define sigma models on RP^{N-1} distinguished from S^{N-1}. We hence can ask if there are any further esoteric global anomalies, due to chiral fermions, that exists for minimal $\mathcal{N}=(0,1)$ sigma models defined on these manifolds. It is quite an analog of Witten's global $\mathrm{SU}(2)$ anomaly in four-dimensional gauge theories as we mentioned in the beginning of last chapter 3. In literatures similar problems on anomalies of global (gauge) symmetries have been considered both in field theories and condensed matter [43, 75–77]. It is worth asking, both physically and mathematically, to investigate if discrete gauges also produce anomalies and their criteria raised in the context of both chiral gauged linear and nonlinear sigma models.

Last but not least, there is also a seemingly interesting geometric problems arising from RG flow constrained by anomalies. In the beginning of the chapter, we mentioned that the isotropy representation on G/H are usually reducible. Physically speaking, each irreducible summand corresponds to a coupling constant. The isometric invariant metrics on G/H are thus modulied by these constants. The argument of conformal fixed point seems to imply that these parameters are further constrained. On the other hand, RG flow is nothing but Ricci flow, at least up to one-loop order. It hence might be legitimate to ask if geometries on G/H are also modified quantum mechanically through the renormalization flow.

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Appendix A

Notation

We define the left-moving and right-moving derivatives as

$$\partial_L \equiv \partial_{LL} \equiv \partial_t + \partial_z \,, \qquad \partial_R \equiv \partial_{RR} \equiv \partial_t - \partial_z \,. \tag{A.1}$$

Correspondingly, the light-cone coordinates are

$$x_L = t - z \equiv x^0 - x^1 \,, \qquad x_R = t + z \equiv x^0 + x^1 \,. \tag{A.2}$$

We use the following definition for the superderivatives:

$$D_L = \frac{\partial}{\partial \theta_R} - i\theta_R^\dagger \partial_{LL} \,, \qquad \overline{D}_L = -\frac{\partial}{\partial \theta_R^\dagger} + i\theta_R \partial_{LL} \,. \tag{A.3}$$

Their anticommutator gives $\{D_L,\, \overline{D}_L\} = 2i\partial_{LL}\,.$

In the bulk of the paper we do not use θ_L and D_R. Hence we can omit the indices in (A.3),

$$\theta_R \to \theta \,, \qquad D_L \to D = \frac{\partial}{\partial \theta} - i\theta^\dagger \, \partial_L \,, \qquad \overline{D}_L \to \overline{D} = -\frac{\partial}{\partial \theta^\dagger} + i\theta \, \partial_L \,. \tag{A.4}$$

We will consistently use the notation (A.4). Our normalization of the Berezin integral is

$$\int d\theta \, \theta = 1 \,, \tag{A.5}$$

and

$$\int d^2\theta \equiv \int d\theta d\theta^\dagger \,. \tag{A.6}$$

In passing from the ordinary to the light-cone coordinates we must also change the components of Lorenz vectors, tensors, etc. For instance, for the supercurrent we have

$$s_{L;L} = s_{LLL} = (s^0_L + s^1_L)/2\,, \qquad s_{R;L} = s_{RRL} = (s^0_L - s^1_L)/2\,. \tag{A.7}$$

Moreover, for the energy-momentum tensor $T^{\mu\nu}$,

$$\begin{aligned}
T_{LL} = T_{LLLL} \quad &= T_{00} + T_{10} + T_{11} + T_{01}\,,\\
T_{LR} = T_{LLRR} \quad &= T_{00} + T_{10} - T_{11} - T_{01}\,,\\
T_{RL} = T_{RRLL} \quad &= T_{00} - T_{10} - T_{11} + T_{01}\,,\\
T_{RR} = T_{RRRR} \quad &= T_{00} - T_{10} + T_{11} - T_{01}\,.
\end{aligned} \tag{A.8}$$

Appendix B

Calculation of $\Delta_\kappa \mathcal{L}$

In this Appendix a detailed calculation of the crucial diagram presented in Fig. 2.1 is given. In the coordinate space it proceeds as follows (the target space indices which go through are suppressed). We start from the $|\kappa|^2$ correction to the action,

$$\Delta_\kappa S = \int d^2x \, \Delta_\kappa \mathcal{L} = \int d^2x \, d^2y \, \partial_L \phi^{\dagger \bar{k}}(x) G_{i\bar{k}}(x) \Pi_{RR}^{i\bar{j}}(x,y) G_{l\bar{j}} \partial_L \phi^i(y) \,, \tag{B.1}$$

where the polarization operator and its expression via Green function is defined in Eq. (2.2.23). We choose the background field ϕ^i in the form of the plane wave,

$$\phi^i(x) = f^i \, \mathrm{e}^{-ikx} \,, \tag{B.2}$$

where f^i are constants. In such field the fermionic part of the action takes a form,

$$\begin{aligned} S_F = \int d^2x \Big[& \mathcal{Z} \zeta_R^\dagger \left(1 + \frac{\partial_\mu \partial^\mu}{M^2} \right) i\partial_L \zeta_R + Z \psi^{\dagger \bar{j}} G_{i\bar{j}} \left(i\delta_k^i \partial_L + \Gamma_k^i k_L \right) \psi^k \\ & + \left(\kappa \mathrm{e}^{-ikx} \zeta_R \psi_R^i G_{i\bar{j}} f^{\dagger \bar{j}} k_L + \mathrm{H.c.} \right) \Big] . \end{aligned} \tag{B.3}$$

Here $G_{i\bar{j}} = G_{i\bar{j}}(f, f^\dagger)$ and $\Gamma_k^i = \Gamma_{lk}^i(f, f^\dagger) f^k$ are x-independent matrices. Here we also introduced an UV regularization by higher derivatives in the propagator of ζ_R. The Fourier transform of this propagator is

$$S_\zeta(p) = \frac{i}{\mathcal{Z}\, p_L}\, \frac{M^2}{M^2 - p^2}\,. \tag{B.4}$$

The Fourier transform of the ψ_R propagator is

$$S^{i\bar{j}}_{\psi_R} = \frac{i}{Z}\left[\frac{1}{p_L\, I + k_L\, \Gamma}\right]^i_k G^{k\bar{j}} \tag{B.5}$$

Then for the Fourier transform of the polarization operator $\Pi^{i\bar{j}}_{RR}$ we have

$$\Pi^{i\bar{j}}_{RR}(k) = i h^2 \int \frac{d^2p}{(2\pi)^2}\, \frac{M^2}{M^2 - p^2}\, \frac{1}{p_L}\, \left[\frac{1}{(p_L + k_L\, I - k_L\, \Gamma}\right]^i_k G^{k\bar{j}}\,. \tag{B.6}$$

It is simple to do integration which results in

$$\Pi^{i\bar{j}}_{RR}(k) = -\frac{|\kappa|^2}{4\pi Z \mathcal{Z}}\left[\frac{K_R^2}{K_\mu K^\mu}\ \frac{\log(1 - K_\mu K^\mu/M^2)}{(-K_\mu K^\mu/M^2)}\right]^i_k G^{k\bar{j}}\,, \tag{B.7}$$

where we introduced the matrix

$$[K_\mu]^i_k = k_\mu\, [I - \Gamma]^i_k\ , \tag{B.8}$$

representing the covariant derivative $i\nabla_\mu$.

For momenta $k \ll M$ the expression is simple,

$$\Pi^{i\bar{j}}_{RR}(k) = -\frac{|\kappa|^2}{4\pi Z\mathcal{Z}}\, \frac{k_R^2}{k_\mu k^\mu}\, G^{i\bar{j}}\,. \tag{B.9}$$

Substituting this into Eq. (B.1) we come to the result (2.2.27) for $\Delta_\kappa \mathcal{L}$.

The expression for $\Pi^{i\bar{j}}_{RR}(k)$ is related to anomaly in the polarization operator. The way we derived it could be called infrared, the p-integration was contributed dominantly by $p \sim k$. The ultraviolet derivation follows from

$$[K_L]^k_i\, \Pi^{i\bar{j}}_{RR}(k) = i h^2 \int \frac{d^2p}{(2\pi)^2}\, \frac{M^2}{M^2 - p^2}\left[\frac{1}{p_L} - \frac{1}{p_L + K_L}\right]^k_l G^{l\bar{j}}\,. \tag{B.10}$$

Integration here is dominated by $p \sim M$ and gives for $k \ll M$,

$$[K_L]_i^k \, \Pi_{RR}^{i\bar{j}}(k) = -\frac{|\kappa|^2}{4\pi Z \mathcal{Z}} \, [K_R]_l^k \, G^{l\bar{j}} \,, \tag{B.11}$$

what corresponds to Eq. (2.2.29) in the text.

Appendix C

Vielbeins and Anomalies in CP($N-1$)

The Fubini-Study metric $g_{i\bar{j}}$ on CP($N-1$) is

$$g_{i\bar{j}} = \frac{(1+\bar{\phi}_i\phi^i)\delta_{i\bar{j}} - \bar{\phi}_i\phi_{\bar{j}}}{(1+\bar{\phi}_i\phi^i)^2}\,. \tag{C.1}$$

The indices of charts $\{\phi^i, \bar{\phi}^{\bar{j}}\}$ locally are raised or lowered by $\delta^{i\bar{j}}$ or $\delta_{i\bar{j}}$. To explicitly find vielbein of the metric, it is convenient to define

$$\begin{aligned} &r^2 \equiv \bar{\phi}_i\phi^i\,, \quad \rho^2 \equiv 1 + r^2\,, \\ &P_{i\bar{j}} \equiv \delta_{i\bar{j}} - \frac{\bar{\phi}_i\phi_{\bar{j}}}{r^2}\,, \\ &Q_{i\bar{j}} \equiv \frac{\bar{\phi}_i\phi_{\bar{j}}}{r^2}\,, \end{aligned} \tag{C.2}$$

one can easily check the following properties:

$$
\begin{aligned}
&\delta_{i\bar{j}} = P_{i\bar{j}} + Q_{i\bar{j}}\,, \\
&P_{i\bar{j}}\bar{\phi}^{\bar{j}} = P_{i\bar{j}}\phi^i = 0\,, \\
&Q_{i\bar{j}}\bar{\phi}^{\bar{j}} = \bar{\phi}_i, \quad Q_{i\bar{j}}\phi^i = \phi_{\bar{j}}\,, \\
&P^2 = P, \quad Q^2 = Q, \quad PQ = QP = 0\,.
\end{aligned} \tag{C.3}
$$

As a result, the metric and vielbein could be written as

$$
\begin{aligned}
&g_{i\bar{j}} = \frac{1}{\rho^2}(P_{i\bar{j}} + \frac{1}{\rho^2}Q_{i\bar{j}}), \quad g^{i\bar{j}} = \rho^2(P^{i\bar{j}} + \rho^2 Q^{i\bar{j}})\,, \\
&e^a{}_i = \frac{1}{\rho}(P^a_i + \frac{1}{\rho}Q^a{}_i), \quad e^i{}_a = \rho(P^i{}_a + \rho Q^i{}_a)\,, \\
&e_{\bar{j}}^{\bar{b}} = \frac{1}{\rho}(P_{\bar{j}}{}^{\bar{b}} + \frac{1}{\rho}Q_{\bar{j}}{}^{\bar{b}}), \quad e_{\bar{b}}^{\bar{j}} = \rho(P_{\bar{b}}{}^{\bar{j}} + \rho Q_{\bar{b}}{}^{\bar{j}})\,, \\
&e^a{}_i e^i{}_b = \delta^a{}_b, \quad e_{\bar{a}}^{\bar{j}} e_{\bar{j}}^{\bar{b}} = \delta_{\bar{a}}^{\bar{b}}, \quad \delta_{a\bar{b}} e^a{}_i e_{\bar{j}}^{\bar{b}} = g_{i\bar{j}}\,.
\end{aligned} \tag{C.4}
$$

Similarly to the O$(N-1)$ model, the symbols $\delta_{a\bar{b}}$ or $\delta^{a\bar{b}}$ are used to lower or raise frame indices $\{a, \bar{b}, ...\}$. Since CP$(N-1)$ are the Kähler manifolds, there are two sets of vielbein, and correspondingly two sets of spin-connections one-form on the frame bundles Hol$^{(1,0)}$ and Hol$^{(0,1)}$,

$$
\begin{aligned}
&\omega^a{}_b = \omega^a{}_{bi} d\phi^i = e^a{}_j D_i e^j{}_b \; d\phi^i\,, \\
&\bar{\omega}_{\bar{b}}{}^{\bar{a}} = \bar{\omega}_{\bar{b}}{}^{\bar{a}}{}_{\bar{j}} d\bar{\phi}^{\bar{j}} = D_{\bar{j}} e_{\bar{b}}^{\bar{i}} \; e_{\bar{i}}^{\bar{a}} d\bar{\phi}^{\bar{j}}\,, \\
&\omega^\dagger = \bar{\omega}\,.
\end{aligned} \tag{C.5}
$$

Redefining $\psi^a = e^a{}_i \psi^i$, one can present the fermionic part of the CP$(N-1)$ Lagrangian as

$$ig_{i\bar{j}}\bar{\psi}^{\bar{j}}\gamma^{\mu}(\partial_{\mu}\psi^{i}+\Gamma^{i}_{jk}\partial_{\mu}\phi^{j}\psi^{k})=i\bar{\psi}^{\bar{a}}\gamma^{\mu}(\partial_{\mu}\delta_{\bar{a}b}+\Omega_{\bar{a}b\mu})\psi^{b}\,, \tag{C.6}$$

$$\Omega_{\bar{a}b\mu}=\omega_{\bar{a}bi}\partial_{\mu}\phi^{i}-\bar{\omega}_{\bar{a}b\bar{j}}\partial_{\mu}\bar{\phi}^{\bar{j}}\,, \tag{C.7}$$

where Ω is the pulled-back connections from frame bundle $\mathrm{Hol}^{(1,0)}\oplus\mathrm{Hol}^{(0,1)}$ of CP$(N-1)$. Identically to the discussion of O$(N-1)$ models, one can evaluate linear and non-linear isometry transformations on connection Ω, and imposes the Wess-Zumino consistency condition to find consistent anomalies. However the calculation are much more cumbersome than O$(N-1)$ case. The details will be neglected, only main results are listed.

Firstly, the number of isometries of CP$(N-1)$ are $N^2-1=(N-1)^2+2(N-1)$, in which there are $(N-1)^2$ linear symmetries corresponding to U$(N-1)$-rotations of fields $\{\phi^i,\bar{\phi}^{\bar{j}}\}$. It also implies the holonomy group of CP$(N-1)$ is U$(N-1)$. The rest of $2N-2$ symmetries are non-linearly realized,

$$\begin{aligned}
\delta_{\epsilon} &= \epsilon^{i\bar{j}}(\phi_{\bar{j}}\frac{\delta}{\delta\phi^{i}}-\bar{\phi}_{i}\frac{\delta}{\delta\bar{\phi}^{\bar{j}}})\,,\\
\delta_{\beta} &= \beta^{i}\frac{\delta}{\delta\phi^{i}}+(\beta\bar{\phi})\bar{\phi}^{\bar{j}}\frac{\delta}{\delta\bar{\phi}^{\bar{j}}}\,,\\
\delta_{\bar{\beta}} &= \bar{\beta}^{\bar{j}}\frac{\delta}{\delta\bar{\phi}^{\bar{j}}}+(\bar{\beta}\phi)\phi^{i}\frac{\delta}{\delta\phi^{i}}\,.
\end{aligned} \tag{C.8}$$

Further, we calculate the variation of spin-connection Ω. According to the experience from O$(N-1)$, it is not curious that linear symmetries give no anomalies to effective Lagrangian. Therefore only non-linear symmetries are considered as below. Since Ω is anti-Hermitian, one can only evaluate $\delta_{\beta}\Omega$, and take hermitian conjugation to get $\delta_{\bar{\beta}}\Omega$. After explicit calculations following Eq.(C.4) and (C.5) we arrive at

$$\begin{aligned}
\omega^a_{\ b} &= \Big(- \frac{1}{2\rho^2}\bar{\phi}_i \delta^a_{\ b} - \frac{1}{2\rho^2}\bar{\phi}_i Q^a_{\ b} - \frac{\rho-1}{\rho r^2}\bar{\phi}_b P^a_{\ i}\Big) d\phi^i \\
&= \Big(- \frac{1}{2\rho^2}\bar{\phi}_i \delta^a_{\ b} - \frac{\rho-1}{\rho r^2}\bar{\phi}_b \delta^a_{\ i} + \frac{1}{2}\,\frac{(\rho-1)^2}{\rho^2 r^4}\,\bar{\phi}_i \phi^a \bar{\phi}_b\Big) d\phi^i \\
&\equiv \big[- G(r^2)\bar{\phi}_i \delta^a_{\ b} - F(r^2)\bar{\phi}_b \delta^a_{\ i} + \frac{1}{2}\,F^2(r^2)\bar{\phi}_i \phi^a \bar{\phi}_b\big] d\phi^i\,, \\
\bar{\omega}_{\bar{b}}^{\ \bar{a}} &= \big[- G(r^2)\phi_{\bar{j}} \delta_{\bar{b}}^{\ \bar{a}} - F(r^2)\phi_{\bar{b}} \delta_{\bar{j}}^{\ \bar{a}} + \frac{1}{2}F^2(r^2)\phi_{\bar{j}} \phi_{\bar{b}} \bar{\phi}^{\bar{a}}\big] d\bar{\phi}^{\bar{j}}\,,
\end{aligned} \tag{C.9}$$

where real functions G and F are defined as

$$G(r^2) \equiv \frac{1}{2\rho^2}\,, \qquad F(r^2) \equiv \frac{\rho-1}{\rho r^2}\,. \tag{C.10}$$

Varying $\Omega^a_{\ b} = \omega^a_{\ b} - \bar{\omega}^a_{\ b}$, one must have

$$\delta_\beta \Omega^a_{\ b} = -dv_{\beta\ b}^{\ a} - [\Omega,\ v_\beta]^a_{\ b}\,. \tag{C.11}$$

To find $v_{\beta\ b}^{\ a}$ in the easiest way one can consider variation of the torsion equation on CP($N-$1). Since there is no torsion on CP($N-1$), one has

$$de^a + \Omega^a_{\ b} \wedge e^b = 0, \tag{C.12}$$

where $e^a = e^a_{\ i} d\phi^i$ is frame one-form. Acting δ_β on both sides, one can obtain Eq.(C.11) if

$$\delta_\beta e^a = v_{\beta\ b}^{\ a} e^b\,. \tag{C.13}$$

Explicitly calculating $\delta_\beta e^a$, we derive $v_{\beta\ b}^{\ a}$,

$$\begin{aligned}
v_{\beta\ b}^{\ a} &= -\frac{\beta\bar{\phi}}{2}\delta^a_{\ b} - \frac{\beta\bar{\phi}}{2}Q^a_{\ b} - \frac{\rho-1}{r^2}\bar{\phi}_b P^a_{\ i}\beta^i \\
&= -\frac{\beta\bar{\phi}}{2}\delta^a_{\ b} - \rho F \beta^a \bar{\phi}_b - \frac{\beta\bar{\phi}}{2}\rho^2 F^2 \phi^a \bar{\phi}_b\,.
\end{aligned} \tag{C.14}$$

Now the non-linear isometry anomalies of CP($N-1$) can be assembled together by using the Wess-Zumino consistency condition. Similar to the O$(N-1)$ case, anomalies with respect to parameter β are

$$\begin{aligned}\delta_\beta\Gamma_{eff} &= -\frac{1}{4\pi}\int_{\phi(S^2)} v_\beta{}^a{}_b d\Omega^b{}_a \\ &= -\frac{1}{4\pi}\int_{\phi(S^2)}\left\{A\beta_{\bar{i}}\phi_{\bar{j}}d\bar{\phi}^{\bar{j}}\wedge d\bar{\phi}^{\bar{i}} + [B(\beta\bar{\phi})\delta_{i\bar{j}} + C(\beta\bar{\phi})\bar{\phi}_i\phi_{\bar{j}} + D\bar{\phi}_i\beta_{\bar{j}}]d\bar{\phi}^{\bar{j}}\wedge d\phi^i\right\},\end{aligned} \tag{C.15}$$

where the functions $A,\ B,\ C$ and D are

$$\begin{aligned}A(r^2) &= -\frac{1}{4}\rho r^2 F\Big(F^2 + 2\,\frac{dF}{dr^2}\Big)\,, \\ B(r^2) &= NG + F(1 - Fr^2)\,, \\ C(r^2) &= N\,\frac{dG}{dr^2} + \Big(2\,\frac{dF}{dr^2} - F^2\Big)\Big(1 - \frac{1}{2}\,\rho F\Big) + \frac{F}{r^2}\Big(1 - 2\rho F - 2r^4\frac{dF}{dr^2}\Big)\,, \\ D(r^2) &= \frac{1}{2}\,\rho r^2 F\Big(2\frac{dF}{dr^2} - F^2\Big) + 2\rho F^2\,.\end{aligned} \tag{C.16}$$

One can simplify Eq. (C.15) integrating by parts. First note

$$\begin{aligned}&\int_{\phi(S^2)} A(r^2)\beta_{\bar{i}}\phi_{\bar{j}}d\bar{\phi}^{\bar{j}}\wedge d\bar{\phi}^{\bar{i}} \\ &= \int_{\phi(S^2)} A(r^2)\beta_{\bar{i}}dr^2\wedge d\bar{\phi}^{\bar{i}} - A(r^2)\beta_{\bar{i}}\bar{\phi}_j d\phi^j\wedge d\bar{\phi}^{\bar{i}} \\ &= \int_{\phi(S^2)} d\left(\frac{1}{\rho} + \log\rho - 2\log(1+\rho)\beta_{\bar{i}}d\bar{\phi}^{\bar{i}}\right) + A(r^2)\beta_{\bar{j}}\bar{\phi}_i d\bar{\phi}^{\bar{j}}\wedge d\phi^i \\ &= \int_{\phi(S^2)} A(r^2)\beta_{\bar{j}}\bar{\phi}_i d\bar{\phi}^{\bar{j}}\wedge d\phi^i\,.\end{aligned} \tag{C.17}$$

In addition, for the function C term,

$$\begin{aligned}
&\int_{\phi(S^2)} C(r^2)(\beta\bar{\phi})\bar{\phi}_i\phi_{\bar{j}}d\bar{\phi}^{\bar{j}}\wedge d\phi^i \\
&= \int_{\phi(S^2)} C(r^2)(\beta\bar{\phi})\bar{\phi}_i dr^2\wedge d\phi^i \\
&= \int_{\phi(S^2)} d\frac{Nr^2+2(\rho-1)}{2\rho^2r^2}(\beta\bar{\phi})\bar{\phi}_i d\phi^i \\
&= \int_{\phi(S^2)} \frac{-Nr^2-2(\rho-1)}{2\rho^2r^2}[(\beta\bar{\phi})\delta_{i\bar{j}}+\beta_{\bar{j}}\bar{\phi}_i]d\bar{\phi}^{\bar{j}}\wedge d\phi^i \\
&\equiv \int_{\phi(S^2)} \tilde{C}(r^2)[(\beta\bar{\phi})\delta_{i\bar{j}}+\beta_{\bar{j}}\bar{\phi}_i]d\bar{\phi}^{\bar{j}}\wedge d\phi^i\,.
\end{aligned} \tag{C.18}$$

Combining the above two terms into Eq.(C.15), one can find

$$\begin{aligned}
\delta_\beta\Gamma_{eff} &= -\frac{1}{4\pi}\int_{\phi(S^2)}\left\{[B-\tilde{C}](\beta\bar{\phi})\delta_{i\bar{j}}+[A+D-\tilde{C}]\beta_{\bar{j}}\bar{\phi}_i\right\}d\bar{\phi}^{\bar{j}}\wedge d\phi^i \\
&= \frac{N}{4\pi}\int_{\phi(S^2)}\frac{\beta_{\bar{j}}\bar{\phi}_i}{2(1+\bar{\phi}\phi)}d\bar{\phi}^{\bar{j}}\wedge d\phi^i \\
&= -\frac{i}{8\pi}\int_{\phi(S^2)}(\beta\bar{\phi})\,c_1\,.
\end{aligned} \tag{C.19}$$

We also need to add the variation of action with respect to $\bar{\beta}$, which is obtained by Hermitian conjugation of Eq.(C.19). Finally we have the result identical with Eq.(3.3.19),

$$\mathcal{I}_\beta = \frac{i}{8\pi}\int(\bar{\beta}\phi-\beta\bar{\phi})c_1\,. \tag{C.20}$$