STUDY OF SOLUTIONS OF THE ISING PROBLEM IN TWO DIMENSIONS

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ABSTRACT: In this paper we review the classical solution of the Ising problem in two dimensions. Firs we consider and explain the general set up of the Ising problem in Statistical Mathematics related with the result of Onsager in two dimensions. After the formulation, we study the properties of spin representation. Next,the treatment of the value V matrix due to Kaufman is explained, together with the classic eigenvalues of $V+$ and $V-$. Finally Onsager's solution is presented, together with some thermodynamic considerations.

Keywords: Statical Mechanics, Onsager's Problem. Ising Model, Kaufman's Matrix, Two-dimensional Problem, Curie's Temperature

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Foreword

An interesting phenomenon of solid state physics is ferromagnetism in which the spins of the atoms are polarized in the same direction.

This results in a macroscopic magnetic field, even in the absence of an external field.

This phenomenon takes place below a certain temperature known as "Curie's temperature". Above it, the spins orient themselves at random, resulting a null field. At Curie's temperature a transition of phase occurs, thus the remaining magnetism disappears.

The Ising model simulates the structure of a ferromagnetic substance and it presents the peculiarity that in two dimensions it yields an exact solution, given by Onsager in 1943.

This model can also be used to find the partition function of a two dimensional net of gas, where the occupied sites correspond to the upward spins, and the empty sites to the downward. Another problem that this model poses, is that of a mixture of atoms of 2 different kinds in a two dimensional net, where the arrangement is determined by the minimum energy of interaction between atoms. Whereas at low temperatures an order exists, above the critical temperature disorder prevails. In the exact solution of the model a characteristic of the trace is approximated by its largest eigenvalue.

Onsager discomposed the characteristic matrix in a direct product of matrices of second order, in this way resolving **n** quadratic equations.

Another solution is owed to Bruria Kaufman, who uses spin representative matrices of a group of rotations. Thus, the problem of finding the eigenvalues of a matrix of $2^n \times 2^n$ is reduced to finding the eigenvalues of a matrix of $2n \times 2n$. Further symmetry considerations lead to the problem of solving this determinant to **n** quadratic equations, where **n** is the number of atoms of each row and of each column.

The case of infinite crystals is approached by taking the summations to the limit and replacing them by integrals. Singularity in the specific heat is a logarithmic infinite at Curie's temperature.

1 THE FORMULATION OF THE ISING PROBLEM

Our model will be a two dimensional net formed by n rows and n columns. The contour condition is that the last row is equal to the first, thus forming a cylinder, and imposing the last column to be equal to the first, a tore is configured.

$$\mu_\alpha = S_1, S_2, S_3, \ldots, S_n \tag{1–1}$$

where S_1 are the spins of row α

The contour condition is expressed as

$$\mu_{n+1} = \mu_1 \tag{1–2}$$

$$S_{n+1} = S_1 \tag{1–3}$$

The configuration of the net is determined by the totality of μ_α, with α ranging from 1 to n

The interaction among the first neighbors is considered, that is to say that row α interacts with rows $\alpha+1$ and $\alpha-1$

The interaction energy between row α and row α' is given by

$$E(\mu_\alpha, \mu'_\alpha) = -\epsilon \sum_{k=1}^{n} S_k S'_k \tag{1–4a}$$

and the interaction between the spins of a row with its first neighbors, plus the energy of interaction of each of them with the field is given by

$$E(\mu) = \epsilon \sum_{k=1}^{n} S_k S_{k+1} - \beta \sum_{k=1}^{n} S_k \tag{1–4b}$$

where ϵ is the energy of interaction between the two spins and the S_1 can take +1 or 1 values, depending upon whetter they are parallel or not parallel.

The total energy of the net will be

$$E_1 = \sum_{\alpha=1}^{n} \left[E(\mu_\alpha, \mu_{\alpha+1}) + E(\mu_\alpha)\right] \tag{1–5}$$

and the system's partition function will be

$$Q_1(BT) = \sum_{\substack{ALL\\STATES}} e^{-\beta E_1} \tag{1–6}$$

$$Q_1(BT) = \sum_{\mu_1} \sum_{\mu_2} \cdots \sum_{\mu_n} e^{-\beta \sum\limits_{\alpha=1}^{n} \left[E(\mu_\alpha, \mu_{\alpha+1}) + E(\mu_\alpha)\right]}$$

$$Q_1(BT) = \sum_{\mu_1} \sum_{\mu_2} \cdots \sum_{\mu_n} \prod_{\alpha=1}^{n} e^{-\beta \sum\limits_{\alpha=1}^{n} \left[E(\mu_\alpha, \mu_{\alpha+1}) + E(\mu_\alpha)\right]} \tag{1–7}$$

Considering each exponential term as an element of a matrix P, the partition function is given by the trace of P^n, where P is of $2^n \times 2^n$

$$\left\langle \mu_\alpha | P | \mu_{\alpha+1} \right\rangle = e^{-\beta\left[E(\mu_\alpha,\mu_{\alpha+1})+E(\mu_\alpha)\right]} \tag{1–8}$$

$$Q_1(BT) = \sum_{\mu_1}\sum_{\mu_2}\dots\sum_{\mu_n} \left\langle \mu_1|P|\mu_2\right\rangle\left\langle \mu_2|P|\mu_3\right\rangle\dots\left\langle \mu_n|P|\mu_1\right\rangle$$

$$Q_1(BT) = \sum_{\mu_1} \left\langle \mu_1|P^n|\mu_1\right\rangle = TrP^n \tag{1–9}$$

Since the trace is invariant to the similarity transformations, it will be equal to the sum of the eigenvalues.

$$P = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_2^n \end{bmatrix} \qquad Q_1(BT) = \sum_{\alpha=1}^{2^n} (\lambda_\alpha)^n \tag{1–10}$$

If all the eigenvalues are positive, it can be written

$$(\lambda_{\max})^n \leq Q_1(BT) \leq 2^n(\lambda_{\max}^n)$$

$$\frac{1}{n}\ln(\lambda_{\max}) \leq \frac{1}{n^2}\ln(Q_1(BT)) \leq \frac{1}{n}\ln(\lambda_{\max}) + \frac{1}{n}\ln 2 \tag{1–11}$$

When n is large, the trace is approximated through the highest eigenvalue

$$\lim_{n\to\infty}\frac{1}{n^2}\ln(Q_1(BT)) = \lim_{n\to\infty}\frac{1}{n}\ln\lambda_{\max} \tag{1–12}$$

We will search for the highest eigenvalue of P for which we need an appropriate representation. From 1-4 and 1-8 we know that

$$\langle\mu|P|\mu'\rangle = \langle S_1\dots S_n|P|S_1'\dots S_n'\rangle = \mathop{\pi}_{\alpha=1}^{n} e^{\beta\epsilon S_k S_k'}\, e^{\beta\epsilon S_k S_{k+1}}\, e^{\beta B S_k} \tag{1–13}$$

can be considered as a product of three matrices of $2^n \times 2^n$

$$P = V_3 V_2 V_1' \tag{1–14}$$

defined as follow

$$\langle S_1\dots S_n|V_1'|S_1'\dots S_n'\rangle = \mathop{\pi}_{\alpha=1}^{n} e^{\beta\epsilon S_k S_k'} \tag{1–15}$$

$$\langle S_1\dots S_n|V_2|S_1'\dots S_n'\rangle = \mathop{\pi}_{\alpha=1}^{n} e^{\beta\epsilon S_k S_{k-1}}\delta S_k S_k' \tag{1–16}$$

$$\langle S_1\dots S_n|V_2|S_1'\dots S_n'\rangle = \mathop{\pi}_{\alpha=1}^{n} \delta S_k S_k'\, e^{\beta B S_k} \tag{1–17}$$

Since V_2 and V_3 are diagonal matrices, it would only be necessary to diagonalize V_1', but this matrices do not commute and cannot be diagonalized simultaneously.
Let us define the direct product of two matrices A and B of $m \times n$ as the matrix of order $m^2 \times n^2$ whose elements are

$$\langle ii'|A \times B|jj'\rangle = \langle i|A|j\rangle\langle i'|B|j'\rangle \tag{1–18a}$$

This product has the following property

$$(A \times B \times C)\,(D \times E \times F)) = (AD) \times (BE) \times (CF) \tag{1–18b}$$

SPIN MATRICES

Let us see the properties of spin matrices. The following matrices

$$\mathbf{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};\ \mathbf{Y} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix};\ \mathbf{Z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix};\ \mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{1–19}$$

have these properties

$$\begin{gathered} X^2 = Y^2 = Z^2 = I^2 \\ XY + YX = YZ + ZY = XZ + ZX = 0 \\ XY = iZ;\ YZ = iX;\ ZX = iY \end{gathered} \tag{1–20}$$

Let us build matrices of $2^n \times 2^n$ using the direct product of matrices of 2×2, in such a way that

$$\begin{aligned} X_\alpha &= I \times I \times I \times X \times I \times \dots \times I \qquad\qquad n \quad matrices \\ Y_\alpha &= I \times I \times I \times Y \times I \times \dots \times I \\ Z_\alpha &= I \times I \times I \times \underset{\substack{\downarrow \\ \alpha - th}}{Z} \times I \times \dots \times I \end{aligned} \tag{1–21}$$

with the following properties

$$\begin{aligned}
&\left[X_\alpha, X_\beta\right] = \left[Y_\alpha, Y_\beta\right] = \left[Z_\alpha, Z_\beta\right] = \left[X_\alpha, Y_0\right] = 0 \qquad \forall \alpha \neq \beta \\
&X_\alpha^2 = Y_\alpha^2 = Z_\alpha^2 = 1 \\
&X_\alpha Y_\alpha + Y_\alpha X_\alpha = 0 \qquad (1\text{–}22)\\
&X_\alpha Y_\alpha = iZ_\alpha, Y_\alpha Z_\alpha = iX_\alpha, Z_\alpha X_\alpha = iY_\alpha \qquad (1\text{–}23)
\end{aligned}$$

By definition we make

$$e^{\theta x} = \sum_{n=0}^{\infty} \frac{\theta^n}{n!} X^n \qquad (1\text{–}24)$$

if $X^{2n} = I and X^{2n-1} = X$, the summation can be expressed as

$$e^{\theta x} = \sum_{even} \frac{\theta^n}{n!} + x \sum_{odds} \frac{\theta^n}{n!} = \cosh\theta\, I + X \sinh\theta \qquad (1\text{–}25)$$

FORM OF MATRICES V_1', V_2, V_3 (DEFENDING OF THE SPIN MATRICES)

From 1-4

$$\langle S_1, \ldots, S_n | V_1' | S_1', \ldots, S_n' \rangle = \prod_{\alpha=1}^{n} e^{\beta\epsilon S_k S_k'}$$

This matrix can be expressed as a direct product of n matrices of 2×2, so what if $V_1' = \alpha \times \alpha \times \cdots \times \alpha$ then the matrix will be

$$\langle S | \alpha | S' \rangle = e^{\beta\epsilon S S'} \qquad (1\text{–}26)$$

where S and S' can be +1 or -1
It can be seen by direct verification that we can write it as

$$\alpha = \begin{bmatrix} e^{\beta\epsilon} & e^{-\beta\epsilon} \\ e^{-\beta\epsilon} & e^{\beta\epsilon} \end{bmatrix} = e^{\beta\epsilon} I + e^{-\beta\epsilon} X \qquad (1\text{–}27)$$

With 1-24

$$\alpha = \left[2\sinh(2\beta\epsilon)\right]^{\frac{1}{2}} e^{\theta x} \qquad (1\text{–}28)$$

with $\qquad \theta = tgh^{-1} e^{-2\beta\epsilon} \qquad (1\text{–}29)$

$$V_1' = \left(2\sinh 2\beta\epsilon\right)^{\frac{n}{2}} e^{\theta x} \times e^{\theta x} \times \cdots \times e^{\theta x} \qquad (1\text{–}30)$$

by 1-24

$$V_1' = \left(2\sinh 2\beta\epsilon\right)^{\frac{n}{2}} V_1 \qquad \text{where} \quad V_1 = \prod_{\alpha=1}^{n} e^{\theta\alpha} \quad \textit{analogously} \qquad (1\text{–}31)$$

$$V_2 = \prod_{\alpha=1}^{n} e^{\beta\epsilon Z_\alpha Z_{\alpha+1}} \qquad \text{where} \quad Z_{n+2} = Z_1 \qquad (1\text{–}32)$$

$$V_3 = \prod_{\alpha=1}^{n} e^{\beta B Z_\alpha} \qquad (1\text{–}33)$$

V_1 and V_3 are direct products of products of matrices of 2×2 is not, reason by which the problem cannot be formulated through these simple matrices.
For the case where the field is null, V_3 is equal to I and the two dimensional problem is reduced to

$$P = \left(2\sinh 2\beta\epsilon\right)^{\frac{n}{2}} V_2 V_1 \qquad (1\text{–}34)$$

This is the problem that we will solve.

2 STUDY OF THE SPIN REPRESENTATIVE MATRICES AND THEIR PROPERTIES

Let us suppose a group of $2n$ matrices that satisfy the following rule of anticommutation

$$\nabla\mu\nabla\nu + \nabla\nu\nabla\mu = 2\delta\mu\,\nu \qquad (\mu = 1\ldots 2n)\;(\nu = 1\ldots 2n) \tag{2–1}$$

The properties of this group are:

A) The dimension cannot be lesser than $2^n \times 2^n$ (2–2)

B) If $\{\nabla'\mu\}$ and $\{\nabla\mu\}$ are two sets of matrices that satisfy 2-1, then a non-singular matrix S exist such that

$$\nabla\mu = S\,\nabla\mu' S^{-1} \tag{2–3}$$

C) Any matrix of $2^n \times 2^n$ can be expressed as a linear combination of the chosen $\nabla\mu$, of all their independent products and of the matrix I. For the case of $n = 1$ we have the matrices of Pauli,and for $n = 2$, those of Dirac.

An adequate representation for the $\nabla\mu$ of $2^n \times 2^n$ is (2–4)

$$\begin{array}{ll} \nabla_1 = Z_1 & \nabla_2 = Y_1 \\ \nabla_3 = X_1\,Z_2 & \nabla_4 = X_1\,Y_2 \\ \nabla_5 = X_1\,X_2\,Z_3 & \nabla_6 = X_1\,X_2\,Y_3 \\ \nabla_7 = X_1\,X_2\,X_3\,Z_4 & \nabla_8 = X_1\,X_2\,X_3\,Y_4 \\ \nabla_9 = X_1\,X_2\,X_3\,X_4\,Z_5 & \nabla_{10} = X_1\,X_2\,X_3\,X_4\,Y_5 \end{array}$$

This is to say

$$\begin{array}{ll} \nabla_{2\alpha} = X_1\,X_2\,X_3\ldots X_{\alpha-1}Y_\alpha & (\alpha = 1\ldots n) \\ \nabla_{2\alpha-1} = X_1\,X_2\,X_3\ldots X_{\alpha-1}Y_\alpha & (\alpha = 1\ldots n) \end{array} \tag{2–5}$$

If we interchange Z_α for X_α we also have an appropriate expression. Let us consider the set of $\nabla\mu'$ obtained by means of a rotation of $\nabla\mu$ that satisfies 2-1. Therefore $\nabla\mu'$ also satisfies 2-1. Because they are orthogonal rotations it is true that

$$\nabla'\mu = \sum_{\nu=1}^{2n} \omega\;\mu\nu\nabla\nu \tag{2–6}$$

$$\sum_{\mu=1}^{2n} \omega\;\mu\nu\omega\;\mu\lambda = \delta\nu\lambda \tag{2–7}$$

$$\omega\;\omega^T = I$$

Since $\nabla'\mu$ satisfies 2-1 there existes, by c) a matrix S such that

$$S(\omega)\nabla\mu\,S^{-1}(\omega) = \nabla\mu' \tag{2–8}$$

Every group of vectors of a $2^n \times 2^n$ dimension formed by rotations in the $2n$ dimension space can be obtained by transformation in the $2n$ space, associated with the rotation. The $S(\omega)$ form a representation of the rotations. If ω_1 and ω_2 are two rotations, $\omega_1\omega_2$ is a rotation also and

$$S(\omega_1\omega_2) = S(\omega_1)S(\omega_2) \tag{2–9}$$

A rotation on the μ, ν plane of a space of dimension $2n$, will be:

$$\omega(\mu,\nu|\theta) = \begin{array}{c} \begin{array}{ccc} & \mu & \nu \end{array} \\ \begin{bmatrix} 1 & \downarrow & \downarrow & \\ & \cos\theta & \sin\theta & \\ & -\sin\theta & \cos\theta & \\ & & & 1 \end{bmatrix} \end{array} \qquad \begin{array}{ll} \nabla'\lambda = \;\nabla\lambda & \lambda \neq \nu,\mu \\ \nabla'\mu = \;\nabla\mu\cos\theta + \nabla\nu\sin\theta & \\ \nabla'\nu = -\nabla\mu\sin\theta + \nabla\nu\cos\theta & \end{array} \tag{2–10}$$

Properties:

$$\omega(\mu,\nu\,|\theta) = \omega(\nu,\mu\,|-\theta) \qquad \omega(\mu,\nu\,|\theta)\omega^T(\mu,\nu\,|\theta) = I \tag{2–11}$$

a) If ω is the rotation that generates $\lambda'\mu$ originated from $\lambda\mu$, then the matrix S of the similarity transformation will be

$$\omega(\mu\,\nu|\theta) \to S_{\mu\nu}(\omega) = S_{\mu\nu}(\theta) = e^{\frac{1}{2}\theta\nabla\mu\nabla\nu} \tag{2–12}$$

b) If $\omega(\mu\,\nu|\theta)$ is the inverse of ω, the suggests us that

$$S^{-1}\mu\nu(\theta) = e^{\frac{1}{2}\theta\nabla\mu\nabla\nu} \tag{2–13}$$

Proof

$$
\begin{aligned}
&(\nabla\mu\nabla\nu)^2 = \nabla\mu\nabla\nu\nabla\mu\nabla\nu = -\nabla\nu\nabla\mu\nabla\mu\nabla\nu = -1 \\
&e^{\frac{1}{2}\theta\nabla\mu\nabla\nu} = I\sum_{n=0}^{\infty}(-1)^n\frac{(\frac{1}{2}\theta)^{2n}}{2n!} - \nabla\mu\nabla\nu\sum_{n=0}^{\infty}\frac{(\frac{\theta}{2})^{2n}}{2n+1}(-1)^n = \cos\frac{\theta}{2} - \nabla\mu\nabla\nu\sin\frac{\theta}{2} \qquad (2\text{–}14)
\end{aligned}
$$

$$
\begin{aligned}
S\mu\nu(\theta)S^{-1}\mu\nu(\theta) =& \left(\cos\frac{\theta}{2}1 - \nabla\mu\nabla\nu\sin\frac{\theta}{2}\right)\left(\cos\frac{\theta}{2} + \nabla\mu\nabla\nu\sin\frac{\theta}{2}\right) \\
=& \cos^2\frac{\theta}{2}I + \cos\frac{\theta}{2}\sin\frac{\theta}{2}(\nabla\mu\nabla\nu - \nabla\mu\nabla\nu) - (\nabla\mu\nabla\nu)^2\sin^2\frac{\theta}{2} = 1 \\
S\mu\nu(\theta)S^{-1}\mu\nu(\theta) =& I
\end{aligned}
$$

Verification of the transformation

$$
\begin{aligned}
\nabla'\mu =& \left(\cos\frac{\theta}{2}I - \nabla\mu\nabla\nu\sin\frac{\theta}{2}\right)\nabla\mu\left(\cos\frac{\theta}{2}I + \nabla\mu\nabla\nu\sin\frac{\theta}{2}\right) \\
\nabla'\mu =& \nabla\mu\cos^2\frac{\theta}{2} + \cos\frac{\theta}{2}\sin\frac{\theta}{2}(\nabla\mu\nabla\mu\nabla\nu - \nabla\mu\nabla\nu\nabla\mu) - \nabla\mu\nabla\nu\nabla\mu\nabla\mu\nabla\nu\sin^2\frac{\theta}{2} \\
\nabla'\mu =& \nabla\mu\cos\theta + \nabla\nu\sin\theta
\end{aligned}
$$

<u>Lemma 2</u>

$\nabla\mu = Z_1X_2\omega_{(\mu\nu|\theta)}$ are $1, 2n-2$ times degenerate and $e^{\pm i\theta}$. The eigenvalues of $S_{\mu\nu(\theta)}$ are $e^{\pm\frac{i\theta}{2}}$, each 2^{n-1} times degenerate.
The first part is trivial, but for the second we will choose a suitable representation, so the eigenvalues will be independent from the representation.

$$
\begin{aligned}
&\nabla\mu = Z_1X_2 &\qquad (2\text{–}15)\\
&\nabla\nu = Z_1Y_2 \\
&\nabla\mu\nabla\nu = i\bar{Z}_2 &\qquad (2\text{–}16)
\end{aligned}
$$

$$
\begin{aligned}
S\mu\nu(\theta) =& e^{\frac{1}{2}\theta\nabla\mu\nabla\nu} = \cos\frac{\theta}{2}(I\times I\times\cdots\times I) = i\sin\frac{\theta}{2}(I\times Z\times I\times\cdots\times I) \\
=& I\times\begin{bmatrix}\cos\frac{\theta}{2} & 0 \\ 0 & \cos\frac{\theta}{2}\end{bmatrix}\times I\times\cdots\times I - I\times\begin{bmatrix}i\sin\frac{\theta}{2} & 0 \\ 0 & -i\sin\frac{\theta}{2}\end{bmatrix}\times I\times\cdots\times I \\
=& I\times\begin{bmatrix}e^{-i\frac{\theta}{2}} & 0 \\ 0 & e^{i\frac{\theta}{2}}\end{bmatrix}\times I\times I\times\cdots\times I \qquad (2\text{–}17)
\end{aligned}
$$

Therefore the eigenvalues are $e^{\pm i\frac{\theta}{2}}2^{n-2}$ times degenerated.

<u>Lemma 3</u>

If I have a product of n flat rotations that commute, meaning

$$\omega = \omega(\alpha\beta|\theta_1)\,\omega\,(\gamma\delta|\theta_2)\cdots\omega\,(\mu\nu|\theta_n) \qquad (2\text{–}18)$$

a)

$$\omega \to S(\omega) = e^{-\frac{1}{2}\theta_1\nabla\alpha\nabla\beta}e^{-\frac{1}{2}\theta_2\nabla\gamma\nabla\delta}\ldots e^{-\frac{1}{2}\theta_n\nabla\mu\nabla\nu} \qquad (2\text{–}19)$$

b) The eigenvalues of the rotation

$$e^{\pm i\,\theta_1};\ e^{\pm i\,\theta_2};\ldots;e^{\pm i\,\theta_n} \qquad (2\text{–}20)$$

c) The eigenvalues of $S(\omega)$ are the different combinations of possible signs of the following expression

$$S(\omega) = e^{\frac{1}{2}(\pm\theta_1+\theta_2+\cdots+\theta_n)} \qquad (2\text{–}21)$$

We can see that after the eigenvalues of the rotations are known, we can know those of their spin representatives.
This study is made because the matrices V_2 and V_1 are spin representatives of a group of rotations.

FORM OF THE REPRESENTATIVE MATRICES FOR V_2 AND V_1

$$\lim_{N\to\infty}\frac{1}{N}\ln Q_1(BT)=\frac{1}{2}\ln(2\sinh 2\beta\epsilon)+\lim_{n\to\infty}\frac{1}{n}\ln A \tag{2–22}$$

We will use the representation given in 2-5 for the matrices

$$\nabla_{2\alpha}\nabla_{2\alpha-1}=x_1\ldots x_{\alpha-1}Y_\alpha x_1\ldots x_{\alpha-1}Z_\alpha=ix_\alpha$$

$$ix_\alpha=\nabla_{2\alpha}\nabla_{2\alpha-1} \tag{2–23}$$

$$V_1'=\underset{\alpha=1}{\overset{n}{\pi}}e^{\theta\alpha}=\underset{\alpha=1}{\overset{n}{\pi}}e^{-i\theta\nabla_{2\alpha}\nabla_{2\alpha-1}} \tag{2–24}$$

Is a spin representation of the rotation $\underset{\alpha=1}{\overset{n}{\pi}}\omega(2\alpha,2\alpha-1|2i\theta)$ (2–25)

$$\begin{aligned}\nabla_{2\alpha+1}\nabla_{2\alpha}&=x_1\ldots x_\alpha Z_{\alpha+1}x_1\ldots x_{\alpha-1}Y_\alpha=\\&=iZ_\alpha Z_{\alpha-1}\end{aligned} \tag{2–26}$$

$$\begin{aligned}\nabla_1\nabla_{2n}&=Z_1x_1\ldots x_{n-1}Y_n\\&=-Z_1Z_n(x_1\ldots x_{n-1},x_n)\end{aligned} \tag{2–27}$$

Naming $X_1,X_2,X_3,\ldots,X_n=U$and with $U^2=I$ (2–28)

by 1-33

$$V_2=\underset{\alpha=1}{\overset{n-1}{\pi}}e^{\beta\epsilon Z_\alpha Z_{\alpha+1}}\,e^{\beta\epsilon Z_n Z_1}=\underset{\alpha=1}{\overset{n-1}{\pi}}e^{-i\beta\epsilon\nabla_{2\alpha+1}\nabla_{2\alpha}}\,e^{i\beta\epsilon\nabla_1\nabla_{2n}U} \tag{2–29}$$

Consequently the matrix $V=V_1V_2$ except by the contour factor of V_2 is a spin representative. We will transform this factor so we can obtain a spin representative matrix.

$$V=V_2V_1=e^{i\phi\nabla_1\nabla_{2n}U}\left[\underset{\alpha=1}{\overset{n-1}{\pi}}e^{-i\phi\nabla_{2\alpha+1}\nabla_{2\alpha}}\right]\left[\underset{\alpha=1}{\overset{n}{\pi}}e^{-i\phi\nabla_2\gamma\nabla_2\gamma_{-1}}\right] \tag{2–30}$$

The first exponential commutes with the first square bracket.
Properties of U

a) $U^2=I$ (2–31a)

b) $U(I+U)=(I+U)$ (2–31b)

c) $U(I-U)=-(I-U)$ (2–31c)

from 2-5 and 2-1

d) $U=\underset{\alpha=1}{\overset{n}{\pi}}x_\alpha=i^n\nabla_1\nabla_2\cdots\nabla_{2n}$ (2–31d)

e) U commutes with a product of an even number of ∇_μ and anticommutes with a product of an odd number of $\nabla\mu$ (2–31e)

$$\begin{aligned}e^{i\phi\nabla_1\nabla_{2n}U}&=\frac{1}{2}\Big(I+U\Big)+\frac{1}{2}\Big(I-U\Big)\Big[\cos\phi I+\nabla_1\nabla_{2n}\sin\phi\Big] \qquad (2–32)\\&=\frac{1}{2}\Big(I+U\Big)\Big[\cos\phi I+\nabla_1\nabla_{2n}\sin\phi\Big]+\frac{1}{2}\Big(I-U\Big)\Big[\cos\phi I-\nabla_1\nabla_{2n}\sin\phi\Big]\end{aligned}$$

$$e^{i\phi\nabla_1\nabla_{2n}U}=\frac{1}{2}\Big(I+U\Big)e^{i\phi\nabla_1\nabla_{2n}}+\frac{1}{2}\Big(I-U\Big)e^{-i\phi\nabla_1\nabla_{2n}} \tag{2–33}$$

$$V=\frac{1}{2}\Big(I+U\Big)V^++\frac{1}{2}\Big(I-U\Big)V^- \tag{2–34}$$

$$V^+=e^{\pm i\phi\nabla_1\nabla_{2n}}\underset{\alpha=1}{\overset{n-1}{\pi}}e^{-i\phi\nabla_{2\alpha+1}\nabla_{2\alpha}}\underset{\gamma=1}{\overset{n}{\pi}}e^{-i\phi\nabla_2\gamma\nabla_2\gamma_{-1}} \tag{2–35}$$

We see that $V^\pm$ are spin representative matrices of the following rotation

$$\Omega^\pm=\omega\Big(1,2n\big|\mp 2i\phi\Big)\left[\underset{\alpha=1}{\overset{n-1}{\pi}}\Big(2\alpha,2\alpha+1\big|2i\phi\Big)\right]\left[\underset{\gamma=1}{\overset{n}{\pi}}\omega\Big(2\gamma,1,2\gamma\big|-2i\theta\Big)\right] \tag{2–36}$$

We only have to find the eigenvalues of the rotation, and whit them those of $V^\pm$
V^+, V^- and V commute between themselves, therefore they can be diagonalized simultaneously.
A diagonal form of U is $U=ZZZ\ldots Z$ with eigenvalues $+1$ or -1 over the diagonal. We can find a transformation that interchanges the eigenvalues, so that the positives remain in the superior part of the diagonal and the negatives in the inferior part.

$$R\cup R^{-1}=\widetilde{U}=\begin{pmatrix}I&0\\0&-I\end{pmatrix}\qquad RV^\pm R^{-1}=\widetilde{V}^\pm \tag{2–37}$$

Since $\widetilde{V}^{\pm}$ commutes with U it will have the form

$$\widetilde{V}^{\pm} = \begin{pmatrix} U^{\pm} & 0 \\ 0 & B^{\pm} \end{pmatrix} \tag{2–38}$$

$$\frac{1}{2}(I+U)\widetilde{V}^{+} = \begin{pmatrix} U^{+} & 0 \\ 0 & 0 \end{pmatrix} \tag{2–39}$$

$$\frac{1}{2}(I-U)\widetilde{V}^{-} = \begin{pmatrix} 0 & 0 \\ 0 & B^{-} \end{pmatrix} \tag{2–40}$$

Then, diagonalizing both separately we can diagonalize V. For this we search the eigenvalues of $\widetilde{V}^{+}$ that are equal to those of V^{+} and see which fraction of them we must discard; in the matrix that we are interested in, only half of each group contributes.

3 EIGENVALUES OF V^{+} AND V^{-}

Let us transform the rotation associated with V^{+}. Under a transformation the eigenvalues do not change.

$$V^{\pm} = \Omega^{\pm} \tag{3–1}$$

$$\omega^{\pm} = \Delta\Omega^{\pm}\Delta^{1} \tag{3–2}$$

$$\Delta = \left[\underset{\gamma=1}{\overset{n}{\pi}} \omega(2\lambda-1,2\lambda) - 2i\theta\right]^{\frac{1}{2}} \tag{3–3}$$

$$\Delta = \underset{\gamma=1}{\overset{n}{\pi}} \omega(2\lambda-1,2\lambda) - i\theta$$

$$\omega^{\pm} = \Delta X^{\pm}\Delta \tag{3–4}$$

$$\Delta = \omega\Big(1,2\big| -i\theta\Big)\omega\Big(3,4\big| -i\theta\Big)\dots\omega\Big(n-1,n\big| -i\theta\Big) \tag{3–5}$$

$$X^{\pm} = \omega\Big(1,2n\big| \mp 2i\theta\Big)\omega\Big(2,3\big| -2i\theta\Big)\omega\Big(4,5\big| -2i\theta\Big)\dots \tag{3–6}$$

$$\Delta = \begin{bmatrix} j & & \\ & j & \\ & & j \end{bmatrix} \tag{3–7}$$

$$X^{\pm} = \begin{bmatrix} a & & & & \mp b \\ & k & & & \\ & & k & & \\ & & & k & \\ \pm b & & & & a \end{bmatrix} \tag{3–8}$$

$$j = \begin{pmatrix} \cosh\theta & -i\sinh\theta \\ i\sinh\theta & \cosh\theta \end{pmatrix} \tag{3–9}$$

$$k = \begin{pmatrix} \cosh 2\phi & -i\sinh 2\phi \\ i\sinh 2\phi & \cosh 2\phi \end{pmatrix} \tag{3–10}$$

$$a = \cosh 2\phi; \quad b = i\sinh 2\phi \tag{3–11}$$

Performing the matricial product we find certain symmetry; grouping up in matrices, is left

$$\omega^{\pm} = \begin{bmatrix} A & B & 0 & & \mp B^{x} \\ B^{x} & A & B & & \\ 0 & B^{x} & A & & \\ & & B & & \\ & & & & \\ & & & & \\ \mp B & & & & B^{x}A \end{bmatrix} \tag{3–12}$$

$$A = \begin{pmatrix} \cosh 2\phi\cosh 2\theta & i\cosh 2\phi\sinh 2\theta \\ i\cosh 2\phi\sinh 2\theta & \cosh 2\phi\cosh 2\theta \end{pmatrix} \tag{3–13}$$

$$B = \begin{pmatrix} -\frac{1}{2}\sinh 2\theta\sinh 2\theta & i\sinh 2\phi\sinh^2\theta \\ -i\sinh 2\phi\cosh^2\theta & -\frac{1}{2}\sinh 2\phi\sinh 2\theta \end{pmatrix} \tag{3–14}$$

B^{*} is the conjugate transpose of B (3–15)

To find the eigenvalues of $\omega^{\pm}$, due to its symmetry, we form the following eigenvector.

$$\psi = \begin{bmatrix} ZU \\ Z^2U \\ \vdots \\ Z^nU \end{bmatrix} \tag{3–16}$$

Where U is a vector of the components $\mu = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$
The equation of eigenvalues gives the following equations:

$$\omega^{\pm}\psi = \lambda\psi \tag{3–17}$$

$$\Big(ZA + Z^2B \mp Z^nB^x\Big)U = \lambda ZU$$
$$\Big(B^xZ^2 + Z^3A + Z^3B\Big)U = \lambda Z^2U$$
$$\Big(B^xZ^2 + Z^3A + Z^4B\Big)U = \lambda Z^3U \tag{3–18}$$
$$\vdots$$
$$\Big(\mp BZ + Z^{n-1}B^* + Z^nA\Big)U - \lambda Z^nU$$

Diving each equation by the power of z that multiplies the eigenvalue considering $z^n = \pm 1$, all the equations are reduced to

$$\Big(A + Z_kB + Z_k^-1B^*\Big)U = \lambda_kU \tag{3–19}$$

The problem of eigenvalues is reduced to solve a quadratic equation. To different roots os the unit, correspond different eigenvalues

$$= -1 \text{ corresponds to the rotation } \omega^+ \tag{3–20}$$
$$= +1 \text{ corresponds to the rotation } \omega^- \tag{3–21}$$

The different roots are

$$\begin{aligned} Z = e^{i\frac{k\pi}{n}} \qquad & k = 1, 2, 3, \ldots, z_{n-1} \quad \text{for} \quad \omega^+ \to Z^n = -1 \\ & k = 0, 2, \ldots, z_{n-2} \qquad \text{for} \quad \omega^- \to Z^n = +1 \end{aligned} \tag{3–22}$$

The eigenvalues are calculated from the equation:

$$\Big(A + Z_kB + Z_k^{-1}B^*\Big)U = \lambda_kU \tag{3–23}$$

Where for each k there are two possible eigenvalues. This way the $2n$ eigenvalues for each ω are obtained.
As the $\det\Big(A + Z_kB + Z_k^{-1}B^*\Big) = 1$, λ_k might be of the form $e^{\pm\gamma_k}$ where γ_k can be found by equalling the traces.

$$\frac{1}{2}Tr\Big(A + Z_kB + Z_k^{-1}B^*\Big) = \frac{1}{2}\Big(e^{\gamma_k} + e^{-\gamma_k}\Big) = \cosh\gamma k \tag{3–24}$$

Obtaining the trace and $\frac{1}{2}(Z_k + Z_k^{-1}) = \cos\frac{\pi k}{n}$ (3–25)

$$\cosh\gamma\, k = \cosh 2\phi \cosh 2\theta - \cosh\frac{\pi k}{n}\sinh 2\phi \underset{k=0,1,2,\ldots}{\sinh 2\theta} \tag{3–26}$$

We define γ_k as the positive solution.

This way, the eigenvalues of ω^+ are $e^{\pm\gamma_1}, e^{\pm\gamma_3}\, e^{\pm\gamma_5}, \ldots, e^{\pm\gamma_{2n-1}}$ (3–27)

And those of ω^- are $e^{\pm\gamma_0}, e^{\pm\gamma_2}, \ldots, e^{\pm\gamma_{2n-2}}$

$$\gamma_k = \gamma_{2n-k} \tag{3–28}$$
$$0 < \gamma_0 < \gamma_1 < \ldots < \gamma_{2n-1} \tag{3–29}$$

Then given the eigenvalues of $\Omega^{\pm}$, those of $V^{\pm}$ have the form

for $V^+ = e^{\frac{1}{2}(\pm\gamma_1 \pm\gamma_3 \pm\ldots\pm\gamma_{2n-1})}$ (3–30)

for $V^- = e^{\frac{1}{2}(\pm\gamma_0 \pm\gamma_2 \pm\ldots\pm\gamma_{2n-2})}$ (3–31)

We now only need to know which contribute and from those choose the highest.
When the $\nabla\mu$ are rotated, U is multiplied by the determinant of the rotation matrix, which because it is orthogonal is ± 1.
We want to diagonalize $\frac{1}{2}(I + U)V^+$ and $\frac{1}{2}(I - U)V$
T^+ is the associated rotation to the diagonalization of V^+, it is a correct rotation.
T is the associated rotation to the diagonalization of V^-, it is an incorrect rotation.

Then

$$T^+U = U,\ T^{-1}U = -U \to S(T^+) \cup S^{-1}(T^+) = U \text{ and } S(T^+) \cup S(T)^{-1} = -U \tag{3–32}$$

$$S(T^+)\frac{1}{2}(I+U)V^+(T^+) = \frac{1}{2}(I+U)S(T^+)V^+S^{-1}(T^+) \tag{3–33}$$

$$S(T^-)\frac{1}{2}(I-U)V^{-1}S^{-1}(T^-) = \frac{1}{2}(I+U)S(T^-)V^-S^{-1}(T^-) \tag{3–34}$$

Since the eigenvalues of Ω^+ are $e^{+\gamma_{2r-1}}$ and the matrix is diagonal, they can be expressed as

$$S(T^+)V^+S^{-1}(T^+) = \overset{n}{\underset{r=1}{\pi}} e^{\frac{\gamma_{2r-1}}{z} z_{pr}} \tag{3–35}$$

Where p_r is a permutation of n numbers.
Since $1/2(I+U) = 1/2(I + Z_1Z_2Z_3 \dots Z_n)$ and Z has ± 1 elements
$1/2(\mathrm{I{+}U}) = 1$ when there is an even number of products -1
$1/2(\mathrm{I{+}U}) = 0$ when there is an odd number of products -1 (3–36)

Consequently, the eigenvalues that have a even number of minus signs will go in the superior submatrix V^+

$$\overset{n}{\underset{r=1}{\pi}} e^{\frac{\gamma_{2r-1}}{z} zr} = \begin{pmatrix} e^{\gamma_1} & 0 \\ 0 & e^{-\gamma_1} \end{pmatrix} \times \begin{pmatrix} e^{\gamma_2} & 0 \\ 0 & e^{-\gamma_2} \end{pmatrix} \times \cdots \times \begin{pmatrix} e^{\gamma_{2n-1}} & 0 \\ 0 & e^{-\gamma_{2n-1}} \end{pmatrix} \tag{3–37}$$

$$Z_1Z_2Z_3 \dots Z_n = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \times \dots \times \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3–38}$$

Even when the permutation changes the order of the eigenvalues, it will be complied anyway
The eigenvalues of V^+ and V^- that contribute, will be those who have a even number of minus signs.
Bearing in mind the growth of the γ_k, the largest eigenvalue is

$$A = e^{\frac{1}{2}(\gamma_1+\gamma_3+\gamma_5+\cdots+\gamma_{2n+1})} \tag{3–39}$$

4 CALCULATION OF THE LARGEST EIGENVALUE FOR AN INFINITE CRYSTAL

We will explicitly evaluate the largest eigenvalue of V

$$\alpha = \lim_{n\to\infty} \frac{1}{n} \ln \lambda_{max} = \lim_{n\to\infty} \frac{1}{2n}(\gamma_1 + \gamma_3 + \gamma_5 + \cdots + \gamma_{2n-1}) \tag{4–1}$$

$$\alpha = \lim_{n\to\infty} \frac{1}{2n} \sum_{r=1}^{n} \gamma_{2r-1} \tag{4–2}$$

defining

$$\gamma_{2r-1} = \gamma\left(\frac{\pi}{n}|2r-1\right) = \gamma_{(v)} \tag{4–3}$$

$$\Delta v = \frac{2\pi}{n} \to \frac{1}{n} = \frac{\Delta v}{2\pi} \tag{4–4}$$

$$\text{for } n \to \infty \quad \Delta v \to dv \quad \text{and } \sum_{r=1}^{n} \gamma_{2r-1} \to \frac{1}{4\pi}\int_0^{2\pi} \gamma(v)\mathrm{d}v \tag{4–5}$$

as we can see

$\gamma(v) = \gamma(2\pi - v)$ so we can put (4–6)

$\alpha = \dfrac{1}{2n}\displaystyle\int_0^{\pi} \gamma(v)\mathrm{d}v$ and bearing in mind the following relationships.

$$\sinh 2\theta = (\sinh 2\theta)^{-1} \qquad \cosh 2\theta = \coth 2\theta \tag{4–7}$$

the expression for the hyperbolic cosine changes its form to

$$\cosh \gamma(v) = \cosh 2\theta \coth 2\phi - \cos v \tag{4–8}$$

helping ourselves with the following equality, provided by the angle in function of its cosh.

$$|z| = \frac{1}{\pi}\int_0^{\pi} \ln(2\cosh z - 2\cos v')\mathrm{d}v' \tag{4–9}$$

the integral is transformed into a double integral

$$\alpha = \frac{1}{2\pi}\int_0^{\pi} \mathrm{d}v \quad \frac{1}{\pi}\int_0^{\pi} \ln\Big[2\cosh 2\phi \coth 2\phi - 2(\cos v + \cos v')\Big]\mathrm{d}v' \tag{4–10}$$

To solve it let us make a change of variable that consists in a rotation of the axis in 45 degree angle and a change of scale in the axis

$$\frac{v+v'}{2} = \delta_1,\ v - v' = \mathrm{d}z \tag{4–11}$$

Substituting

$$\alpha = \frac{1}{2\pi^2}\int_0^{\pi} \mathrm{d}\,\delta_1 \int_0^{\pi} \mathrm{d}\,\delta_2 \ln\Big[2\cosh 2\phi \coth 2\phi - 4\cos\delta_1 \cos\frac{\delta_2}{2}\Big] \tag{4–12}$$

and taking out cos as common factor, we obtain

$$= \frac{1}{2\pi^2}\int_0^{\pi} \mathrm{d}\,\delta_1 \int_0^{\frac{\pi}{2}} \mathrm{d}\,\delta_2 \left\{\ln(\cos\delta_2) + \ln\frac{2\cosh 2\phi\coth 2\phi - 4\cos\delta_1}{\cos\delta_2}\right\}$$

Since the logarithm of the product is the addition of the logarithms

$$= \frac{1\pi}{2\pi^2}\int_0^{\frac{\pi}{2}} \ln\cos\delta_2 \,\mathrm{d}\delta_2 + \frac{1}{\pi^2}\int_0^{\pi}\int_0^{\frac{\pi}{2}} \ln\left(\frac{2D}{\cos\delta_2} - 4\cos\delta_1\right)\mathrm{d}\delta_1\,\mathrm{d}\delta_2$$

where

$$D = \cosh 2\phi \coth 2\phi \tag{4–13}$$

$$\alpha = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \ln\cos\delta_2 \,\mathrm{d}\delta_2 + \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \cosh^{-1}\left(\frac{2D}{\cos\delta_2}\right)\mathrm{d}\delta_2$$

is an expression for the hyperbolic are cosine is

$$\cosh^{-1} x = \ln\Big[x + \sqrt{x^2-1}\Big] \tag{4–14}$$

and using this relationship in the integral

$$\alpha = \frac{1}{2\pi}\int_0^{\pi} \ln D\Big(1 + (1 - k^2\cos^2\delta_2)^{\frac{1}{2}}\Big)\mathrm{d}\delta_2 \quad \text{with } k = \frac{2}{D} \tag{4–15}$$

separating the logarithm

$$\alpha = \frac{1}{2\pi}\int_0^{\pi} \ln\frac{2\cosh^2\phi}{\sinh 2\phi}\,\mathrm{d}\delta + \frac{1}{2\pi}\int_0^{\pi} \ln\frac{1}{2}\Big(1 + (1 - k^2\cos^2\delta)^{\frac{1}{2}}\Big)\mathrm{d}\delta \tag{4–16}$$

let us change the variable

if $\delta' = \delta + \frac{\pi}{2}$ with which the integral is transformed in (4–17)

$$\alpha = \frac{1}{2}\ln\frac{2\cosh^2\phi}{\sinh 2\phi} + \frac{1}{2\pi}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \ln\frac{1}{2}\Big(1 + (1 - k^2\sin^2\delta')^{\frac{1}{2}}\Big)\mathrm{d}\delta' \tag{4–18}$$

$$\alpha = \frac{1}{2}\ln\frac{2\cosh^2\phi}{\sinh 2\phi} + \frac{1}{2\pi}\int_0^{\pi} \ln\frac{1}{2}\Big(1 + (1 - k^2\sin^2\delta)^{\frac{1}{2}}\Big)\mathrm{d}\delta \tag{4–19}$$

The expression of the partition function for an infinite crystal is

$$\lim_{N\to\infty}\frac{1}{N}\ln Q_1(OT) = \ln 2\cosh 2\beta\epsilon + \frac{1}{2\pi}\int_0^{\pi} \ln\frac{1}{2}\Big(1 + (1 - k^2\sin^2\delta)^{\frac{1}{2}}\Big)\mathrm{d}\delta \tag{4–20}$$

This is the solution provided by Onsager in 1943
Let us proceed to do the thermodynamic calculations.
The free energy of Helmontz as a function of the partition function

$$A_1(BT) = -KT\ln Q_1(BT) \tag{4–21}$$

and the spin free energy

$$B_{a1}(OT) = \ln Q_1(OT) \tag{4–22}$$

The internal energy is the derivative of the free energy with respect to the temperature.

$$\mu_1(OT) = \frac{d(B_{a1}(OT))}{\mathrm{d}B} = -2\epsilon\tanh 2\beta\epsilon + \frac{k}{2\pi} + \frac{\delta k}{\delta\beta}\int_0^{\pi}\frac{\sin^2\phi}{\Delta(1+\Delta)}\mathrm{d}\phi \tag{4–23}$$

with $\Delta = [1 - k^2\sin^2\phi]^{\frac{1}{2}}$ (4–24) let us unfold the equation in two

$$\int_0^{\pi} \frac{\sin^2\phi}{\Delta(1+\Delta)} \frac{(1-\Delta)}{(1-\Delta)} \mathrm{d}\phi = \int_0^{\pi} \frac{\sin^2\phi(1-\Delta)}{\Delta(1-\Delta^2)} \mathrm{d}\phi = -\frac{\pi}{k^2} + \frac{1}{k^2} \int_0^{\pi} \frac{1}{\Delta} \mathrm{d}\phi \tag{4–25}$$

$$\mu_1(OT) = -2\epsilon \tanh 2\beta\epsilon + \frac{1}{2k}\frac{\delta k}{\delta\beta}\left[-1 + \frac{1}{\pi}\int_0^{\pi}[1-k^2-\sin^2\phi]^{\frac{1}{2}}\mathrm{d}\phi\right] \tag{4–26}$$

making the derivative and replacing, we obtain

$$\mu_1(OT) = -\epsilon \coth 2\beta\epsilon \left[1 + \frac{2}{\pi}k' K_1(k)\right] \tag{4–27}$$

where $K_1(k)$ is a first class elliptical integral, defined by

$$K_1(k) = \int_0^{\frac{\pi}{2}} \left[1 - k^2 \sin^2\phi\right]^{-\frac{1}{2}} \mathrm{d}\phi \tag{4–28}$$

and $$k = \frac{2\sinh 2\beta\epsilon}{\cosh^2 2\beta\epsilon} \tag{4–29}$$

$$k' = 2\tanh^2 2\beta\epsilon = 1 \tag{4–30}$$

where it is true that

$$k^2 + k'^2 = 1 \tag{4–31}$$

The calorific capacity in function of the internal energy is

$$C_1(OT) = \frac{\delta U_1(OT)}{\delta T} \tag{4–32}$$

Using 4-27 and 4-32 we get

$$\frac{1}{k}C_1(OT) = \frac{2}{\pi}(\beta\epsilon\coth 2\beta\epsilon)^2 \left\{2K_1(k) - 2E_1(k) - (1-k')\left[\frac{\pi}{2} + k'K_1(k)\right]\right\} \tag{4–33}$$

where we have used the $E_1(k)$ function which is a second class elliptic integral

$$E_1(k) = \int_0^{\frac{\pi}{2}} \mathrm{d}\phi \left[1 - k^2\sin^2\phi\right]^{\frac{1}{2}} \tag{4–34}$$

This is the expression of the calorific capacity for the Ising net. As we see the function 4-28 shows the singularity for $k = 1$.
We will say that the temperature that satisfies this relationship is the critical temperature. Developing $K_1(k)$ and $E_1(k)$ the Tc we obtain the following expression

$$K_1(k) \cong \ln\frac{4}{|k'|} \text{ and } \frac{\mathrm{d}K_1(k)}{\mathrm{d}K} = \frac{\pi}{2} \tag{4–35}$$

$$E_1(k) \approx 1 \tag{4–36}$$

from 4-31 and 4-30, if $k = 1$, we obtain

$$2\tanh^2\left(\frac{2\epsilon}{kTc}\right) = 1 \tag{4–37}$$

$$\frac{\epsilon}{kTc} = 0.4406868 \tag{4–38}$$

$$kTc = (2.269185)\epsilon \tag{4–39}$$

The internal energy is a continuous function, even at critical temperature, at which it takes the following value

$$\mu_1(O_1Tc) = -\epsilon\coth 2\beta_c\epsilon \tag{4–40}$$

Developing the calorific capacity close to the critical point we obtain

$$\frac{1}{k}C_1(O_1Tc) \approx \frac{2}{\pi}\left(\frac{2\epsilon}{kTc}\right)^2\left[-\ln\left|1 - \frac{T}{Tc}\right| + \ln\left(\frac{kTc}{2\epsilon}\right) - \left(1 + \frac{\pi}{4}\right)\right] \tag{4–41}$$

which has, as we can see, a logarithmic infinite at the critical temperature. At this point there is a phase transition because the spontaneous magnetization when the temperature is higher than the critical temperature.

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