

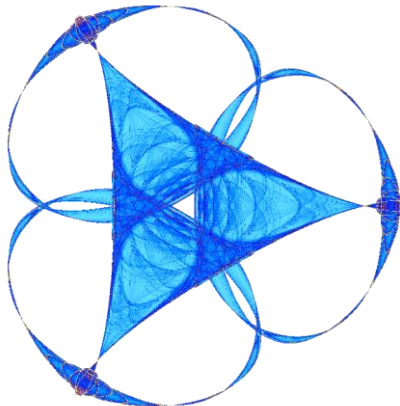
A DIFFERENCE EQUATION INVOLVING FIBONACCI NUMBERS

By

**Ezio Marchi**

**IMA Preprint Series #2471**

(March 2016)



INSTITUTE FOR MATHEMATICS AND ITS APPLICATIONS  
UNIVERSITY OF MINNESOTA

400 Lind Hall

207 Church Street S.E.

Minneapolis, Minnesota 55455-0436

Phone: 612-624-6066 Fax: 612-626-7370

URL: <http://www.ima.umn.edu>

A DIFFERENCE EQUATION INVOLVING FIBONACCI NUMBERS

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Ezio Marchi \*)

**Abstract:** In this short note we solve a non linear difference equation which becomes related to Fibonacci numbers.

\*) Director del Instituto de Matemática Aplicada San Luis. CONICET.  
Universidad Nacional de San Luis. Ejercito de los Andes 950  
(5700) San Luis, Argentina.

## 1- A NON LINEAR DIFFERENCE EQUATION

Consider the difference equation given by:

$$x_{n-1} = \bar{\alpha}_n x_n x_{n-1} + \bar{\beta}_n x_{n-2} \quad n = 1, 2, \dots \quad (1)$$

where  $x_n$  is obtained from the previous values in a non linear way in the form expressed in it. The problem to be solved is to find  $x_n$  in terms of the firsts values of it. We will propose a general form for the values of  $x_n$  called quadrature and generating function procedure. The values of  $\bar{\beta}_n$  will not be arbitrary but in the solution they will depend upon other parameters. Simply the proposal is as follows

$$x_n = \alpha_n c(n) + \frac{\beta_n}{c(n)} \quad (2)$$

then

$$x_n x_{n-1} = \alpha_n \alpha_{n-1} c(n) c(n-1) + \frac{\beta_n}{c(n)} \alpha_{n-1} c(n-1) + \alpha_n \beta_{n-1} \frac{c(n)}{c(n-1)} + \frac{\beta_n \beta_{n-1}}{c(n) c(n-1)} \quad (3)$$

Introducing this expression in (1) and  $x_{n-2}$ , then after identifying terms we obtain the following recursive of difference equations

$$\alpha_{n+1} c(n+1) = \bar{\alpha}_n \alpha_n \alpha_{n-1} c(n) c(n-1) \quad (4)$$

$$\frac{\beta_{n+1}}{c(n+1)} = \bar{\alpha}_n \frac{\beta_n \beta_{n-1}}{c(n) c(n-1)} \quad (5)$$

On the other hand,

$$\alpha_n \beta_{n-1} \frac{c(n)}{c(n-1)} = \beta_{n-1} \bar{\alpha}_{n-1} \alpha_{n-1} \alpha_{n-2} c(n-2) \quad (6)$$

then

$$\bar{\alpha}_n \beta_{n-1} \bar{\alpha}_{n-1} \alpha_{n-1} + \bar{\beta}_n = 0 \quad (7)$$

In a similar way

$$\beta_{n-1} \alpha_n \frac{1}{c(n-2)} \frac{\beta_n}{\beta_{n-1}} \frac{\alpha_{n-1}}{\bar{\alpha}_{n-1}} \alpha_{n-1} \alpha_{n-2} = \beta_n \alpha_{n-1} \frac{c(n-1)}{c(n)} \quad (8)$$

from which it follows

$$\bar{\alpha}_n \alpha_n \beta_n + \bar{\beta}_n \beta_{n-1} \alpha_{n-1} \alpha_{n-2} = 0 \quad (9)$$

or

$$\bar{\beta}_n = -\frac{\bar{\alpha}_n \alpha_n \beta_n}{\beta_{n-1} \alpha_{n-1} \alpha_{n-2}} \quad (10)$$

which is a condition to be fulfilled by the coefficient  $\bar{\beta}_n$  as we already mentioned.

From the equality

$$\bar{\beta}_n = \bar{V} \bar{\alpha}_n \beta_{n-1} \bar{\alpha}_{n-1} \alpha_{n-1} = -\frac{\bar{\alpha}_n \alpha_n \beta_n}{\beta_{n-2} \bar{\alpha}_{n-1} \alpha_{n-2}} \quad (11)$$

one derives

$$\alpha_n \beta_n = \bar{\alpha}_{n-1}^2 \beta_{n-1} \alpha_{n-1} \beta_{n-2} \alpha_{n-2} \quad (12)$$

This last difference equation is the same as that equation obtained by multiplying **(4)** and **(5)**.

Now the equation **(12)** has the general form of

$$\gamma_n = \varepsilon_{n-1} \gamma_{n-1} \gamma_{n-2} \quad (13)$$

Developing the first terms of **(13)** we have

$$\begin{aligned} \gamma_n &= \varepsilon_{n-1} \varepsilon_{n-2} \gamma_{n-2}^2 \gamma_{n-3} = \varepsilon_{n-1} \varepsilon_{n-2} \varepsilon_{n-3}^2 \gamma_{n-3}^3 \gamma_{n-4} \\ &= \varepsilon_{n-1} \varepsilon_{n-2} \varepsilon_{n-3}^2 \varepsilon_{n-4}^3 \gamma_{n-4}^5 \gamma_{n-5}^3 = \varepsilon_{n-1} \varepsilon_{n-2} \varepsilon_{n-3}^2 \varepsilon_{n-4}^3 \varepsilon_{n-5}^5 \gamma_{n-5}^8 \gamma_{n-6}^5 \\ &= \varepsilon_{n-1} \varepsilon_{n-2} \varepsilon_{n-3}^2 \varepsilon_{n-4}^3 \varepsilon_{n-5}^5 \varepsilon_{n-6}^8 \gamma_{n-6}^{13} \gamma_{n-7}^8 \end{aligned} \quad (14)$$

Where you immediately see that Fibonacci numbers appear in the developing of  $\gamma_n$ .

We define

0	1	1	2	3	5	8	13	21	<b>(15)</b>
p(0)	p(1)	p(2)	p(3)	p(4)	p(5)	p(6)	p(7)	p(8)	

as the Fibonacci numbers which are recursively given by

$$p(n+1) = p(n) + p(n-1)$$

with  $p(0) = 0$  and  $p(1) = 1$ .

In order to find the solution of **(13)** let us consider some equalities. First we will prove that

$$\prod_{s=1}^{n-1} \varepsilon_{n-s}^{p(s)} \prod_{s=1}^{n-2} \varepsilon_{n-1-s}^{p(s)} = \prod_{s=2}^n \varepsilon_{n-1-s}^{p(s)} \quad (16)$$

In order to see that this equality holds true consider the equality

$$\prod_{s=1}^{n-2} \varepsilon_{n-1-s}^{p(s)} = \prod_{s=2}^{n-1} \varepsilon_{n-s}^{p(s-1)} \quad (17)$$

which is obtained just by a change of variables  $s + 1 = \bar{s}$ . Using this last equality then since  $p(0) = 0$  and  $p(1) + p(0) = p(2)$  then

$$\prod_{s=1}^{n-1} \varepsilon_{n-s}^{p(s)} \prod_{s=2}^{n-1} \varepsilon_{n-s}^{p(s-1)} = \prod_{s=2}^{n-1} \varepsilon_{n-s}^{p(s+1)} \varepsilon_{n-1}^{p(1)} = \prod_{s=2}^{n-1} \varepsilon_{n-s}^{p(s+1)} \varepsilon_{n-1}^{p(2)} = \prod_{s=1}^{n-1} \varepsilon_{n-s}^{p(s+1)} \quad (18)$$

Now with the change of variable  $s - 1 = \bar{s}$  we have

$$\prod_{s=2}^n \varepsilon_{n+1-s}^{p(s)} = \prod_{s=1}^{n-1} \varepsilon_{n-s}^{p(s+1)} \quad (19)$$

and therefore the equality **(16)** is valid.

Now we will claim that

$$\gamma_n = \prod_{s=1}^{n-1} \varepsilon_{n-s}^{p(s)} \gamma_1^{p(n)} \gamma_0^{p(n-1)} \quad (20)$$

is the solution of the difference equation **(13)**.

We will prove it by induction.

For  $n=2$  we have

$$\gamma_2 = \prod_{s=1}^1 \varepsilon_{2-s}^{p(s)} \gamma_1^{p(2)} \gamma_0^{p(1)} = \varepsilon_1 \gamma_1 \gamma_0 \quad (21)$$

Now assume that the formula is true for  $j \leq n - 1$  then we have to prove that it is valid for  $n$ . This is equivalent to prove that the next equality is true

$$\gamma_n = \prod_{s=1}^{n-1} \varepsilon_{n-s}^{p(s)} \gamma_1^{p(n)} \gamma_0^{p(n-1)} \quad (22)$$

We remember that

$$\gamma_n = \varepsilon_{n-1} \gamma_{n-1} \gamma_{n-2} \quad (13)$$

$$\prod_{s=1}^{n-1} \varepsilon_{n-s}^{p(s)} \gamma_1^{p(n)} \gamma_0^{p(n-1)} = \varepsilon_{n-1} \prod_{s=1}^{n-2} \varepsilon_{n-1-s}^{p(s)} \gamma_1^{p(n-1)} \gamma_0^{p(n-2)} \prod_{s=1}^{n-3} \varepsilon_{n-2-s}^{p(s)} \gamma_1^{p(n-2)} \gamma_0^{p(n-3)} \quad (23)$$

or equivalently

$$\prod_{s=1}^{n-1} \varepsilon_{n-s}^{p(s)} = \varepsilon_{n-1} \prod_{s=1}^{n-2} \varepsilon_{n-1-s}^{p(s)} \prod_{s=1}^{n-3} \varepsilon_{n-2-s}^{p(s)} \quad (24)$$

Make the change of variable  $s + 2 = \bar{s}$  in the first product of the right hand of (24) and  $s + 1 = \bar{s}$  in the second product of the same hand, we have

$$\prod_{s=1}^{n-1} \varepsilon_{n-s}^{p(s)} = \varepsilon_{n-1} \prod_{s=2}^{n-1} \varepsilon_{n-s}^{p(s)} \prod_{s=3}^{n-1} \varepsilon_{n-s}^{p(s-2)} = \varepsilon_{n-1} \varepsilon_{n-2}^{p(1)} \prod_{s=3}^{n-1} \varepsilon_{n-s}^{p(s-2)+p(s-1)} = \varepsilon_{n-1} \varepsilon_{n-2}^{p(1)} \prod_{s=3}^{n-1} \varepsilon_{n-s}^{p(s)} \quad (25)$$

and since  $p(1) = p(2)$

$$= \varepsilon_{n-1} \varepsilon_{n-2}^{p(2)} \prod_{s=3}^{n-1} \varepsilon_{n-s}^{p(s)}$$

and this the equality given by (24) or (25) is true.

Then applying the formula (22) to the recurrence equation (12) one gets the solution

$$\alpha_n \beta_n = \prod_{s=1}^{n-1} \alpha_{n-s}^{-2p(s)} \alpha_1^{p(n)} \beta_1^{p(n)} \alpha_0^{p(n-1)} \beta_0^{p(n-1)} \quad (26)$$

On the other hand the equation (4) has the solution

$$\alpha_n c(n) = \prod_{s=1}^{n-1} \alpha_{n-s}^{-p(s)} \alpha_1^{p(n)} c_1^{p(n)} \alpha_0^{p(n-1)} c_0^{p(n-1)} \quad (27)$$

and equation (5)

$$\frac{\beta_n}{c(n)} = \prod_{s=1}^{n-1} \alpha_{n-s}^{-p(s)} \frac{\beta_1^{p(n)} \beta_0^{p(n-1)}}{c_1^{p(n)} c_0^{p(n-1)}} \quad (28)$$

Thus the general solution **(2)** is given by

$$x_n = \prod_{s=1}^{n-1} \alpha_{n-s}^{-p(s)} \left[ \alpha_1^{p(n)} c_1^{p(n)} \alpha_0^{p(n-1)} c_0^{p(n-1)} + \frac{\beta_1^{p(n)} \beta_0^{p(n-1)}}{c_1^{p(n)} c_0^{p(n-1)}} \right] \quad (29)$$

The equality **(11)** which is a restriction on the coefficients  $\bar{\beta}_n$  can be expressed as

$$\bar{\beta}_n = -\bar{\alpha}_n \bar{\alpha}_{n-1} \prod_{s=1}^{n-2} \alpha_{n-s}^{-2p(s)} \alpha_1^{p(n-1)} \beta_1^{p(n-1)} \alpha_0^{p(n-2)} \beta_0^{p(n-2)} \quad (30)$$

This giving the values  $\alpha_1, \alpha_0, \beta_0, \beta_1$  and  $c_1, c_0$  we have solved in a suitable way the problem of finding a general solution for equation **(1)** by quadrature.