A MATHEMATICAL MODEL FOR COMPETITION AMONG POPULATIONS OF ORGANISMS

By

Cristina di Pasquale and Ezio Marchi

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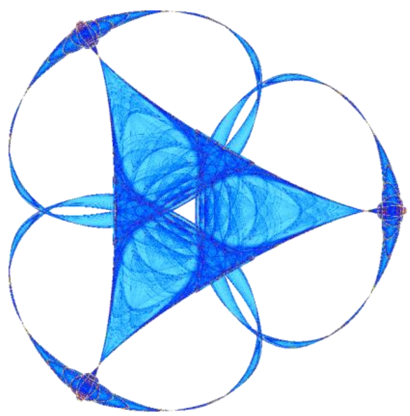

INSTITUTE FOR MATHEMATICS AND ITS APPLICATIONS
UNIVERSITY OF MINNESOTA
400 Lind Hall
207 Church Street S.E.
Minneapolis, Minnesota 55455-0436
Phone: 612-624-6066 Fax: 612-626-7370
URL: http://www.ima.umn.edu

A Mathematical Model for Competition among Populations of Organisms

*CRISTINA DI PASQUALE AND EZIO MARCHI *)*

*) Emeritus Professor UNSL, San Luis, Argentina.
Founder and First Director IMASL, UNSL- CONICET
(ex) Superior Researcherr CONICET

ABSTRACT

A theory of spacial niche for different species under a situation of competition among them is considered.

The descriptive situation is given analytically by means of a strategy game in normal form, where the players involved in it are the respective species as separate entities. The total competition is given by means of the payoff functions and the strategy sets.

A concept of solution modified for competitive normal games is used: the nucleolar solution. Its existence is proved.

We also find sets that include the set of nucleolar solutions as a subset. These results allow us to calculate the nucleolar solutions in a simplified way.

1- INTRODUCTION

Among other important authors devoted to the study of competitive situations, using Game Theory in Biology and Ecology, we can mention for example Lewontin [16], and more recently Maynard Smith [18, 19], also Hammerstein [11], with different aspect of the competitive questions in this field. However, much more has to be done in this important and intricate area.

In this paper we are concerned with a theory of spacial niche for different species in a situation of competition among them. Here, the term competition has to be understood as interaction with possibility of conflict or fight among some of the species. For example the species can compete for space, food, etc. In other words, we consider a general situation where all possibilities and degrees of interaction might be presented among subjects belonging to different species.

Although the concept of competition is one of the most important points in the modern ecology, it is quite hard and complex to handle in the experimental aspect.

Frequently, competition in nature is hard to understand, study and quantify. Anyway, important experiments involving competition have been made, see Park [22], Vanderment [26], Wilbur [28], Neil [20].

However, much more has to be done related to theoretical and experimental aspects involving interactions of competition.

The aim of this paper is to study competition among species in a new way, using the concept of ecological niche or more precisely spacial niche.

The concept of ecological niche has been studied for a long time by ecologists, in different aspects. For example Grinnell [8, 9] has been one of the first who used the term niche. He considered the niche in the functional aspect and the position of organisms in the community.

Other authors, as Elton [7], Dice [4], Clarke [1] used the concept niche in different ways.

A modern treatment of niche, due to Hutchinson [12], considers the term in a more formal way.

More recently, Odum [21] defined the ecological niche as the position of organism in the community.

Weatherley [27] suggested that the definition of niche be restricted to the relation of the organisms with the populations they consume.

Some ecologists prefer to define the term niche in a very general way and to consider the “spacial niche ”, that is to say the place where the organism lives; or the “trophic niche ”, in other words its trophic position in the ecosystem. We are interested now in the spacial niche of the populations of organisms.

This descriptive situation is given analytically by means of a strategy game in normal form where the players involved in it are the respective species as separate entities. The game is assumed to be played by all the populations living in the field considered.

The total competition is given by means of the payoff functions and the strategy sets. The first ones take into account the general interaction situation among the species. The strategy sets are given by the different positions of the individuals in the space where they usually live.

The different movements in the field of all the species of the community replace the process of choice by the rational players of classical game theory.

In the study of biological populations it is important to find equilibrium points, which will be useful among other things, to avoid superabundance and extintion cases. We are interested in new concepts to find these equilibrium points. For example it would be interesting to find the best positions among the worst possible situations that can occur in the field referring to populations space. This intuitive concept is formally given by the nucleolar solutions, which allow us to work wich all the species of the ecosystem at the same time.

So, as a general tool of the theory we now wish to present, we use a modern concept of solution for cooperative games due to Schmeidler [25] but modified to handle the competitive normal games needed in order to present the new theory for the spacial niche. The concept of solution used is that of nucleolar solution, which is important from many points of view. In particular for its conceptual solution, which in a sense involves the minimax concept.

All these concepts will be explained in detail in this paper.

2- MODEL DESCRIPTION

In this paragraph we are going to describe the model of the interactive situation among an arbitrary number of species.

Thus, consider the set of species described by a set $I = \{1,...,n\}$ which is technically called the set of players. An $i\epsilon I$ describes a unique species.

For the spacial distributions of the agents of each species in the geographical configuration we assume that the field where the study is done is described as a grid in a two dimensional space. Consider K_j with $j = 1,2$ as a finite interval of natural numbers; then the grid is given by

$$K = K_1 x K_2$$

that is to say the set of all positions $k = (k_1, k_2)$ with $k_1 \in K_1$ and $k_2 \in K_2$. This indeed described in a discrete way the ecological positions of the geographical space. Only for simplicity we assume it is rectangular and in two dimensions. However more general shapes of the ecological space in higher dimensions can be described without any problem.

From an ecological point of view $k = (k_1, k_2)$ describes a unit cell where individuals of the different species may live together.

The concept of spacial niche is therefore intruduced in the following way by means of those points $k \in K$ with a distribution given by the number $x_{ik} \geq 0$ of individuals of species $i \in I$ living in the geographical unit cell $k \in K$. Thus, the spacial region where the species $i \in I$ live is

$$N_i(x) = \{k: x_{ik} > 0\}$$

In each cell of $N_i(x)$ we find individuals of the species $i \in I$. Thus, in static term the $N_i(x)$ gives us the geographical part of the niche.

Heuristically speaking, we have that the more overlapping among the $N_i(x)$ the more interaction among the species appears.

Moreover, since here we wish to study only a static theory of the spacial niche, that is to say, we do not take into consideration the variation in time of the individuals of the species, we have that the total number of individuals of each species

$$\sum_k x_{ik} = m_i > 0 \qquad (1)$$

is already specified and constant.

Let an element x satisfying:

$$x \in X = \underset{i \in I}{X} X_i = \underset{i \in I}{X} \qquad \{x_i: \textstyle\sum_k x_{ik} = m_i > 0\},$$

that is to say a joint strategy for all the species, where x_i is a vector obtained with the numbers x_{ik} varying k.

Now the competitive situation among the species is therefore given by payoff functions

$$A_i(x)$$

which take into consideration all the interactions among the species in the context of game theory in the geographical unit cells $k = (k_1, k_2)$.

Thus , x_{ik} plays the role of a strategy of species $i \in I$ in the geographic cell k.

We note that the food webs are not considered explicitly as in Cohen [2], but are implicitly assumed in the payoff functions of all the species. The payoff functions biologically describe the utility or benefit for each species of having a determined number of individuals in the geographic cells k under a strategic distribution of x_{ik} by all the species.

Since the real number of individuals of the species $i \in I$ in general might be considered to be large, x_{ik} is assumed to be real non negative number.

Thus, the ecological competition among the species is described by strategy game in normal from with strategies x_{ik}, varyng k, for the species $i \in I$. This involves all the competitive situations in all the k geographical cells, contrained with the condition (1).

3- CONCEPT OF SOLUTION

We have been considering all species as if they were rational players "choosing" their strategies, and we are interested now in finding the best option for the worst possibilities presented.

Therefore it is important to introduce a somewhat strong concept of solution for our set of games. We are going to define a type of nucleolos already studied by Schmeidler [25], and later used in other publications, see Kohlberg [14], and Justman [13]. Here the result are in the spirit of the nucleolar concepts, but presented for our situation in a different way.

First, given an element $x \in X$, we order the numbers

$$A_i(x)$$

varying i in a non-decreasing way. Thus, with these numbers we obtain a vector

$$\theta(x).$$

We comparate now the first component of $\theta(x)$ with the first component of $\theta(y)$. The second of $\theta(x)$ with the second of $\theta(y)$ and so on.

Then we say that x dominates y for $x, y \in X$, if the first component of $\theta(x)$ different from the corresponding component of $\theta(y)$ is greater. We write it as $x > y$ or equivalently $\theta(x) > \theta(y)$.

We can also define $x \sim y$, meaning that all the corresponding components of both vectors equal.

Similarly we define $x \geq y$ or $\theta(x) \geq \theta(y)$.

We say that a distribution $x \in X$ is a nucleolar solution if $x \geq y$ for each $y \in X$.

Such a point x is one of the best in the intuitive sence that the utility of the species in the worst situation for the geographical cells is maximized. We recall that this is related with the concept of nucleolus, as already mentioned, but is different in the context and in concept technically speaking.

Our next task is the determination of the existence of such a nucleolar solution. We call the set of nucleolar solution $N(x)$.

Assuming that all the payoff functions A_i for the different geographical cells are continuous, we have the result that the set $N(x)$ is non-empty.

Indeed, the set of all joint strategies X for all the species is convex, bounded, closed and non-empty.

Define now the following number, which corresponds to the first component of $\theta(x)$:

$$\theta_1(x) = min\{max[A_1(x): i \in J]: J \subset I, |J| = 1\}$$

where $|J|$ is the cardinality of the set J. Analogously the 1-th component of $\theta(x)$ may be seen to be given by:

$$\theta_1(x) = min\{max[A_1(x): i \in J]: J \subset I, |J| = 1\}$$

With these functions let us define the sets:

$$X^0 = X$$

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$$A_1(x) < A_1(\lambda\, x + (1-\lambda)y) < A_1(y)\; for\; all\; 0 < \lambda < 1$$

This means that if for two distributions in the space, x, y the utility of the species 1 is greater in y than in x, then the utility of the distribution obtained as combination of x and y, is smaller than $A_1(y)$ and greater than $A_1(x)$.

Usually it is a bit too hard to find the set $N(x)$. We simplify this problem finding sets that include $N(x)$ as a subset. These new sets are not arbitrarlly obtained.

They are strongly related to the nucleolar solutions, and it is possible to give them a right meaninful interpretation. This important result is given in the following theorem. In order to present it, let us consider some notations and results.

First, let S be the following set:

$$S = \{x \in X\; such\; that\; for\; each\; i \in I, there\; is\; only\; one\; k\; such\; that\; x_{ik} = m_i\}$$

Consider an element $x \in S$, then for a given i let be k such that $x_{ik} = m_i$, therefore for each $\bar{k} \neq k;\; x_{i\bar{k}} = 0$.

If $x \in S$, it is clear from the convexity of X that it is an extremal point.

Another result that can be proved is that if the payoff function A_1 has:

<u>Property A)</u>

Then the maximum point or maximum points of A_1 are extremal points and the segments joining them are maximum too. This is useful in the applicability of the theorem. This result can be seen by writing:

$$A_1(x^1) \leq \cdots \leq A_i(x^{k'}) \qquad (3)$$

Where $x^1 \ldots x^{k'}$ are all the extremal points.

With these points and all combinations $\lambda\, x\, j + (1-\lambda)xj'$ where xj and xj' are extremal points or combination of extremal points and $0 \leq \lambda \leq 1$; we obtain the set of geographical positions X. Using also the fact that A_1 has Property A) and inequality (3) it can be easlly seen that the maximum points are extremal points. The segments joining them are maximum too, which is obtained using this result:

If A_1 has Proprty A) then for each $x, x' \in X$ such that

$A_1(x) = A_1(\lambda\, x + (1-\lambda)\, y) \qquad 0 \leq \lambda \leq 1$ for proof see Di Pasquale [5].

With these result let us introduce now the mentioned Theorem, that is proved in the Appendix.

Theorem 1:

Let A_1 be payoff functions that have Property A).

There exists i such that $A_i \neq cte.$

$$I' = \{i \in I \; such \; that \; A_i \; \neq cte.\}, \qquad |I'| = q$$

Let $S_{i''_0} \dots i''_r : \left\{ \begin{matrix} A_{i''_0}(x) = \cdots = A_{i''_r}(x) \\ x \in X \end{matrix} \right\}$

Proof:

Let $\theta(x) = \left(\theta_1(x), \dots, \theta_q(x)\right)$

Where $\theta_1(x) = \min[\max[A_i\,(x) : i \in J\,] : J \subset I';\ |\,J\,| = 1]$

Let $\theta_1(x) = \; A_{i_1}(x) \;\; if \;\; x \; \in N(x)$

$$Z_p^1 = \left\{ \begin{matrix} x' \in X \;\; such \; that \; there \; exist \; i'_1 \dots \; i'_p \in I' - [i_1] \\ such \; that \; x' is \; solution \; of \; S_{i_1} i_1 \dots i_p \end{matrix} \right\}$$

$$1 = i, \dots, p \qquad\qquad p = 1, \dots, q - 1$$

Let

$Z = \{x \in X \; such \; that \; there \; exists \; i \in I' such \; that \; x \; is \; maximum \; of \; A_i\} = \; \bigcup_{i \in I}[x \in X \; such \; that \; x \; is \; maximum \; of \; A_i]$

I) If $Z_1^1 = \emptyset \; then \; N(X) \subset \{x \in X \; such \; that \; x \; is \; maximum \; of \; A_{i_1}\}$

II) If $Z_1^1 \neq \emptyset \; then \; there \; exist \; 1, r \; such \; that \; N(x) \subset \mathrm{Z}_\mathrm{r}^1 \; \cup \mathrm{Z}$

Note that we have assumed in the theorems the condition that there exists $i \in I$ such that A_i is not the constant function; otherwise, it can be proved that $N(x) = X$, result that is intuitively obvious.

Anyway there is no problem in proving it rigorously.

Finally, we remark that the study of $N(x)$ is reduced to the sets of solutions of.

$$S_{i_1} \quad i'_1, \dots, i'_p \quad , wich \quad i'_1, \dots, i'_p \;\; \in I' - \{i_1\};\;\; 1 = 1, \dots, q\,, p = 1, \dots, q - 1$$

and the set of extremal points. That is, it is reduced to the study of $a11 \;\; Z_p^1$ such that $Z_p^1 \; \neq \emptyset$, and the set S.

From an intuitive point of view the theorem gives us a simplified way of finding nucleolar solutions. The problem is reduced to find the intersections of the graphs of the payoff functions, or the points where they meet. The same sets can be interpreted as the sets of points where two or more payoff functions take the same value or obtain the same utility.

On the other hand, we must study the set of extremal points, which can be seen as the set where each species distributes its organism in only one cell.

In one or both of these cases that we have just interpreted, we will find the nucleolar solution.

If we are interested in different aspects of the niche, others than the space, this theorem may also be applied, or modified for the new reformulation. This new approach can be performed only taking into account that the set of options X must be in each case bounded, closed and convex.

4- DISCUSSION

This analysis of the ecological niche by means of game theory suggests to us some comments about the applicability.

This model allows us to deal with all the species living in the place consideted at the same time, and to obtain result and decisions for the whole ecosystem.

Once the populations of organisms living in a field are know, the number of individuals belonging to each one can be estimated using statiscal methods, see Cormack, Patil and Robson [3] and Ravinovich [24].

The payoff functions can be determined now with the condition

$$\sum_k x_i k = m_i$$

They can be given considering for example the populations that cause damage to the field, the number of organisms belonging to each species, the preservation of natural resources, the environmental conditions, etc.

We can be obtain now the nucleolar solutions with the Theorem 1. Once the nucleolar solutions are found, control of men can be helpful and maybe necessary to allow the subjects of the species to remain in the “best position” or in the “adquate cell”, avoiding movements to inadequate places. These result turn into very interesting cases when the nucleolar solution is unique. Some cases are studied in Di Pasquale [5].

Much more can be studied in this area. For example it would be interensting to study other aspect of the ecological niche, others than the space. We could also consider dispute over other environmental resources, or cooperation in the sense the might help each other.

Another important concept that can be studied with some of the definitions we have explained is that of predation, see Di Pasquale [5], where a real example is also developed.

A dynamit theory could also be studied beginning with the material we have presented and considering time dependence.

It would also be interesting that these kind of solutions could be related in the future with models applied to different situations, as many important authors have studied, for example:

Hallam [10], May [17], Levin [15].

We expect to present more result related with these models in a next future.

APENDIX A

Before proving Theorem 1 announced in Section 4, we will give some Lemmas which will be needed in order to prove the theorem.

Lemma A.1:

a) If for each $x' \in Z_1^1 \cap X^{1-1}, A_{i_1}(x) > A_{i_1}(x')$ for each $x \in N(x)$.

If $1 = 1$ then there exists $x'' \in X$ maximum point of A_{i_1}: $\theta_1(x'') = A_{i_1}(x'')$

Proof:

Since A_{i_1} is continuous and X is closed and bounded, using a known result there exists $x'' \in X$ maximum point of A_{i_1}.

We will show now that: there exists $x'' \in X$ maximum point of A_{i_1}:

$$A_{i_1}(x'') = \theta_1(x''),\ \ 1 = 1$$

we prove it by contradiction assuming that:

$$\theta_1(x'') = A_{1'_1}(x''), i'_1 \neq i_1$$

Three parts are considered:

1) $A_{i'_1}(x) > A_{i_1}(x)$

As also $A_{i'_1}(x'') < A_{i_1}(x'')$, (by definition of $\theta_1(x'')$, $1 = i$)

We could find $x' \in Z_1^1 \cap X^{1-1}$: $A_{i_1}(x) < A_{i_1}(x')$

In other words, as $A_{i_1}(x)$ has Property A) and $A_{i_1}(x) < A_{i_1}(x'')$ then

$A_{i_1}(x) < A_{i_1}(\lambda\, x + (1-\lambda)\, x'') < A_{i_1}(x'')$ with $0 < \lambda < 1$ hence $A_{i_1}(x) < A_{i_1}(x')$, but this contradicts.

a) For $x' = \lambda\, x + (1-\lambda)\, x'' \in Z_1^1 \cap X^{1-1}$, with $A_{i_1}(x') = A_{i'_1}(x')$

2) $A_{i'_1}(x') = A_{i_1}(x)$ then $x \in Z_1^1 \cap X^{1-1}$. By a) $A_{i_1}(x) > A_{i_1}(x)$, a contradiction.

3) $A_{i_1}(x) < A_{i_1}(x)$

This is impossible by definition of $\theta_1(x), 1 = 1$.

Lemma A.2:

If there exists $x \in N(X)$ such that $x \in Z_p^1$ then for each $y \in N(X), y \in Z_p^1$.

Proof:

We show now this result assuming that there exists $y \in N(X)$ but $y \notin Z_p^1$. Since $x, y \in N(X), x \in Z_p^1,\ and\ y \notin Z_p^1$ we can write:

$$A_{i_1}(y) = \theta_1(y) = \theta_{1_1}(y) = \cdots = \theta_{1_{p'}}(y) = A_{1_1}(x) = A_{1'_1}(x) = \cdots = A_{1'_p}(x) = \theta_1(x) = \theta_{1_1}(x) = \cdots$$
$$= \theta_{1_p}(x).$$

$0 \le p' < p$ because $y \notin Z_p^1$ and $p' = 0$ means $y \notin Z_p^1$.

From this we can also conclude that:

(A1) $\theta_{1_{p'+1}}(y) \neq \theta_{1_{p'+1}}(x) \ldots \theta_{1_p}(y) \neq \theta_{1_p}(x)$

and the contradiction is obtained because $x, y \in N(X)$.

Lemma A.3:

b) If there exists $x' \in Z_1^1 \cap X^{1-1}$: $A_{i_1}(x') > A_{i_1}(x)$ for each $x \in N(X)$. If it is not possible to find $x' \in Z_1^1 \cap X^{1-1}$:

$A_{i_1}(x') > A_{i_1}(x)$ and $\theta_1(x') = A_{i_1}(x')$ for each $x \in N(X)$ then (A.2). There exists $i'_1 \in I' - \{1_1\}$: $A_{1'_1}(x) = A_{1_1}(x)$ for each $x \in N(X)$ or $A_{1_1}(x) = A_{1_1}(x')$ for some $x' \in Z_1^1 \cap X^{1-1}$, for each $x \in N(X)$.

Proof:

This result is proved now by contradiction using Lemma A.2.

We begin assuming that there exists $x \in N(x)$:

(A.3) For each $i'_1 \in I' - \{i_1\}$ $A_{i'_1}(x) \neq A_{i_1}(x)$ and $A_{i_1}(x) \neq A_{i_1}(x')$ for each $x' \in Z_1^1 \cap X^{1-1}$.

As we also assume (by b) that there exists $x' \in Z_1^1 \cap X^{1-1}$:

$A_{i_1}(x') \geq A_{i_1}(x)$, we find $x'_1 \in Z_1^1 \cap X^{1-1}$;

$A_{i_1}(x) \leq A_{i_1}(x'_1) \leq A_{i_1}(x')$, for each $x' \in Z_1^1 \cap X^{1-1}$: $A_{i_1}(x) \leq A_{i_1}(x')$.

- If $A_{i_1}(x) < A_{i_1}(x'_1)$

By construction it is $\theta_1(x'_1) = A_{i_1}(x'_1)$ but we asumed that it was impossible to find x' like this.

- If $A_{i_1}(x) = A_{i_1}(x'_1)$

This is impossible because we assumed $A_{i_1}(x) \neq A_{i_1}(x')$ for each $x' \in Z_1^1 \cap X^{1-1}$ and also by (A.3) $x \notin Z_1^1 \cap X^{1-1}$.

Lemma A.4:

If (A.2) is not true and if 1=1 then

$$N(X) \subset \{x \in X : x \text{ } is \text{ } maximum \text{ } of \text{ } A_{i_1}\}$$

Proof:

We assume that $x \in N(X)$ but $x \notin \{x \in X : x \text{ } is \text{ } maximum \text{ } of \text{ } A_{i_1}\}$

If we prove that there exists $x'' \in X$ maximum point of A_{i_1}: $A_{i_1}(x'') = \theta_1(x'')$, and using that $A_{i_1}(x'') > A_{i_1}(x)$ (because x is not maximum point of A_{i_1}), we obtain:

$$\theta_1(x'') = A_{i_1}(x'') > A_{i_1}(x) = \theta_1(x)$$

but this is a contradiction because x is nucleolar solution.

So we prove now that:

$$A_{i_1}(x'') = \theta_1(x'')$$

We assume that

$$A_{i'_1}(x'') = \theta_1(x''), \qquad i'_1 \neq i_1$$

As before we consider 3 cases:

1) (A.4) $\quad A_{i'_1}(x) > A_{i_1}(x)$

By definition of $\theta_1(x'')$ with $1 = 1$

(A.5) $\quad A_{i_1}(x'') < A_{i_1}(x'')$

The following inequality

(A.6) $A_{i_1}(x) < A_{i_1}(x'')$

is obtained because x is not maximum point and x'' it is.

It follows by Propety A) that:

$$A_{i_1}(x) < A_{i_1}(\lambda\, x + (1-\lambda)\, x'') \; < A_{i_1}(x'')$$

By (A.4), (A.5) and (A.6) we could find $x' \in Z_1^1 \cap X^{1-1}$ such that $A_{i_1}(x) < (x'_1)$. We find x'_1 such that $A_{i_1}(x) < A_{i_1}(x'_1) \le A_{i_1}(x') \le A_{i_1}(x'')$

for each $x' \in Z_1^1 \cap X^{1-1}$. From these results and using that (A.2) is not true, we conclude that $A_{i_1}(x'_1) = \theta_1(x'_1)$. But $\theta_1(x) = A_{i_1}(x)$, so if $x \in N(X)$ and

$\theta_1(x) = A_{i_1}(x) < A_{i_1}(x'_1) = \theta_1(x'_1)$ this would be impossible because $x \in N(X)$.

2) $A_{i'_1}(x) = A_{i_1}(x)$

is impossible, because we assumed that (A.2) was not true.

3) $A_{i'_1}(x) < A_{i_1}(x)$

is impossible, by definition of $\theta_1(x)$ $1 = 1$.

We prove now <u>Theorem 1</u> which was announced in Section 4.

First, we have by definition of Z_p^1 that:

$$Z_1^1 \supset \ \dots \ \supset Z_{q-1}^1 \quad , \quad 1 = i, \dots, q$$

Let us prove now part I).

I) We consider first: $Z_1^1 = \emptyset$

$Z_1^1 = \emptyset$ means that there are no solutions for any

$S_{1_1 i'_1 \dots i'_{p''}}$ $p = 1, \dots, q-1$. In this case, for each $x \in X$

$\theta_1(x) = A_{i_1}(x)$

We will prove now that $N(x) \subset \{x \in X: x \; is \; maximum \; of \; A_{i_1}\}$.

We assume that x is a nucleolar solution but it is not maximum of A_{i_1}.

Since X is a compact set and functions are continuous, there exists x''a maximum point of A_{i_1} and $x'' \ne x$, because x is not maximum.

As x'' is maximum point of A_{i_1} and x is not $A_{i_1}(x'') > A_{i_1}(x)$.

Therefore, we obtain that:

$\theta_1(x'') = A_{i_1}(x'') > A_{i_1}(x) = \theta_1(x)$

As a second step we prove II).

II) We consider now $Z_1^1 \neq \emptyset$

Let $X^0, X^1, \dots, X^{q-1}$ be defined as in (2) Section 3.

Let 1=1

a) If for each $x' \in Z_1^1 \cap X^{1-1}, A_{\ i_1}(x) > A_{i_1}(x')$ for each $x \in N(x)$.

If 1=1 we will show by contradiction that:

$N(X) \subset \{x \in X: x\ is\ maximum\ point\ of\ A_{i_1}\}$.

We assume that $x \in N(x)$ but $x \notin \{x \in X: x\ is\ maximum\ point\ of\ A_{i_1}\}$.

Using Lemma A.1 there exists $x'' \in X$ maximum point of A_{i_1}:

$$A_{i_1}(x'') = \theta_1(x'')$$

$A_{i_1}(x'') > A_{i_1}(x)$ because x is not maximum point of A_{i_1}.

Using these result: $\theta_1(x'') = A_{i_1}(x'') > A_{i_1}(x) = \theta_1(x)$ but this is a contradiction because x is nucleolar solution.

Therefore: $N(X) \subset \{x \in X: x\ is\ maximum\ point\ of\ A_{i_1}\} \subset \mathrm{Z}$, and the proof is continued in (A.10).

If $1 > 1$ and a) is not true the case b) is considered.

If $1 > 1$ and a) is not true nothing else can be obtained, the proof continue in (A.10).

b) If a) is not true, this means that:

There exists $x' \in Z_1^1 \cap X^{1-1} : \ A_{i_1}(x') \geq \ A_{i_1}(x)$ for each $x \in N(X)$.

- If there exists $x' \in Z_1^1 \cap X^{1-1} : \ A_{i_1}(x') > \ A_{i_1}(x)$

and $\theta_1(x') = A_{i_1}(x')$ for each $x \in N(X)$, then

$\theta_1(x') = A_{i_1}(x') > \ A_{i_1}(x) = \theta_1(x)$, but x is nucleolar solution and the contradiction is obtained.

- If it is not possible to find $x' \in Z_1^1 \cap X^{1-1} : A_{i_1}(x') > \ A_{i_1}(x)$ and

$\theta_1(x') = A_{i_1}(x')$ for each $x \in N(X)$, then by Lemma A.3:

(A.7) there exists $1'_1 \in I' - \{i_1\} : \ A_{i'_1}(x) = \ A_{i_1}(x)$ for each $x \in N(X)$ or (A.8) $A_{i_1}(x) = A_{i_1}(x')$ for some $x' \in Z_1^1 \cap X^{1-1}$, for each $x \in N(X)$.

The case (A.7) means that $x \in Z_1^1$, and we obtain:

(A.9) there exist $1, r : N(X) \subset Z_r^1 \cup Z,\ r = max\ \ \{p : N(X) \subset \mathrm{Z}_\mathrm{p}^1\}$.

We can continue proof in (A.10).

If (A.7) is impossible and (A.8) is true, in Lemma A.4 we showed that

If 1=1, $N(X) \subset \{x \in X: x\ is\ maximum\ point\ of\ A_{i_1}\} \subset Z$. in this case we continue the proof in (A.10).

If $1 > 1$ nothing else can be obtained, in this case:

(A.10) if $1 = q$ the proof is finished. If this is not the case, let 1=1+1

If $Z_1^1 \cap x^{1-1} \neq \emptyset$ we continue the proof in part a).

If $Z_1^1 \cap x^{1-1} = \emptyset$ nothing else can be obtained, we continue in (A.10).

And the proof of Theorem 1 has been completed.

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