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# Interior Estimates for Generalized Forchheimer Flows of Slightly Compressible Fluids

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**Abstract**

The generalized Forchheimer flows are studied for slightly compressible fluids in porous media with time-dependent Dirichlet boundary data for the pressure. No restrictions on the degree of the Forchheimer polynomial are imposed. We derive, for all time, the interior $L^\infty$-estimates for the pressure and its partial derivatives, and the interior $L^2$-estimates for its Hessian. The De Giorgi and Ladyzhenskaya-Uraltseva iteration techniques are used taking into account the special structures of the equations for both pressure and its gradient. These are combined with the uniform Gronwall-type bounds in establishing the asymptotic estimates when time tends to infinity.

## 1 Introduction

In this paper, we study the generalized Forchheimer flows for slightly compressible fluids in porous media. Forchheimer equations are used to describe the fluids' dynamics when the ubiquitous Darcy's law is not applicable. They are nonlinear relations between the fluid's velocity and gradient of pressure which were realized by Darcy [3] and Dupuit [7] in early works, formulated by Forchheimer [9, 10], and were studied more extensively afterward in physics and engineering, see [2, 16, 17, 20, 23] and references therein. Mathematics of Darcy flows has been studied thoroughly with a vast literature for a long time dated back to the 1960s, see the treaty [22] for references. In contrast, Forchheimer flows were investigated mathematically much later in the 1990s. Even less are the mathematical works on compressible Forchheimer flows. The reader is referred to [1, 11, 12] for more information about this topic.

Generalized Forchheimer equations were proposed and studied in our previous works [1, 11–14] in order to cover a large class of fluid flows formulated from experiments. For compressible fluids, they form a new class of degenerate parabolic equations with their own characteristics compared to other models of porous medium equations. Among a small number of papers on these flows, recent work [14] is focused on studying the pressure and its time derivative in space $L^\infty$, the pressure gradient in $L^s$ for $s \in [1, \infty)$ and the pressure's Hessian in $L^{2-\delta}$ for $\delta \in (0, 1)$. However, it requires the so-called Strict Degree Condition (SDC), that is, the degree of the Forchheimer polynomial is less than $4/(n-2)$, where $n$ is the spatial dimension. Another related paper [13], in contrast, does not require (SDC), but the analysis is mainly for pressure in space $L^s$, pressure gradient in space $L^{2-a}$ for a specific number $a \in (0,1)$, and pressure's time derivative in $L^2$. Our current work is to unite the two approaches and develop them further without any restrictions on the Forchheimer polynomials. Specifically, we consider the initial boundary value problem (IBVP) for the pressure in a bounded domain with time-dependent Dirichlet boundary data. The interior $W^{1,\infty}$ and $W^{2,2}$ norms of the solutions are estimated, particularly for large time. Such estimates for degenerate parabolic equations usually require much work, see, for e.g., [6,18,19,21]. Improving on [14], we adapt and refine techniques by De Giorgi [4], Ladyzhenskaya-Uraltseva [15], and also DiBenedetto [5] for parabolic equations, and combine them with those used for Navier-Stokes equations [8]. These techniques are utilized successfully here thanks to the special structure of our equation.

The paper is organized as follows. We recall the model, equations and basic facts in section 2. In section 3, we derive the uniform Gronwall-type estimates that sharpen the previous results in [11,13] especially for large time. In section 4, the interior $L^\infty$-estimates for pressure are established by using $L^\alpha$-based De Giorgi iteration with sufficiently large $\alpha$. This is different from [14] which is based on the $L^2$-estimate and, hence, requires (SDC). As a result, no restrictions on the Forchheimer polynomials are needed for our estimates. In section 5, we first use Ladyzhenskaya-Uraltseva-type imbedding and iteration to estimate the pressure gradient in $L^s$ for any $s \geq 1$. Our estimates only require the initial data to be in the spaces $W^{1,2-a}$ and $L^\alpha$ for a fixed $\alpha$. When $s$ is large, this requirement is much less than the $W^{1,s}$ condition in [14]. In subsection 5.2, we derive the $W^{1,\infty}$ estimates for the pressure, which were not previously studied in [13,14]. We make use of the equation (5.20) for the pressure gradient, which it is highly nonlinear but has a special structure. Thanks to this and previous $W^{1,s}$ ($s < \infty$) bounds, we utilize the De Giorgi technique to obtain the $L^\infty$-estimates for the pressure gradient. Section 6 contains the $L^\infty$-estimates for the time derivative of the pressure, while section 7 contains new $L^2$-bounds for the pressure's Hessian. It is noteworthy that, thanks to the uniform Gronwall-type inequalities in section 3, the asymptotic bounds obtained as time goes to infinity depend only on the asymptotic behavior of the boundary data. This paper is focused primarily on the interior estimates. The issues of the solutions' estimates for on the entire domain as well as their continuous dependence on the data will be addressed in our future works.

## 2 Background

Consider a fluid in a porous medium occupying a bounded domain $U$ with boundary $\Gamma = \partial U$ in space $\mathbb{R}^n$. For physics problem $n = 3$, but here we consider any $n \geq 2$. Let $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$ be the spatial and time variables. The fluid flow has velocity $v(x,t) \in \mathbb{R}^n$, pressure $p(x,t) \in \mathbb{R}$ and density $\rho(x,t) \in [0,\infty)$.

The generalized Forchheimer equations studied in [1,11–14] are of the the form:

$$g(|v|)v = -\nabla p, \tag{2.1}$$

where $g(s) \geq 0$ is a function defined on $[0,\infty)$. When $g(s) = \alpha, \alpha + \beta s, \alpha + \beta s + \gamma s^2, \alpha + \gamma_m s^{m-1}$, where $\alpha, \beta, \gamma, m, \gamma_m$ are empirical constants, we have Darcy's law, Forchheimer's two-term, three-term and power laws, respectively.

In this paper, we study the case when the function $g$ in (2.1) is a generalized polynomial with non-negative coefficients, that is,

$$g(s) = a_0 s^{\alpha_0} + a_1 s^{\alpha_1} + \ldots + a_N s^{\alpha_N} \quad \text{for} \quad s \geq 0, \tag{2.2}$$

where $N \geq 1$, the powers $\alpha_0 = 0 < \alpha_1 < \ldots < \alpha_N$ are fixed real numbers (not necessarily integers), the coefficients $a_0, a_1, \ldots, a_N$ are non-negative with $a_0 > 0$ and $a_N > 0$. This function $g(s)$ is referred to as the Forchheimer polynomial in equation (2.1), and $\alpha_N$ is the degree of $g$.

From (2.1) one can solve $v$ implicitly in terms of $\nabla p$ and derives a nonlinear version of Darcy's equation:

$$v = -K(|\nabla p|)\nabla p, \tag{2.3}$$

where the function $K : [0, \infty) \to [0, \infty)$ is defined by

$$K(\xi) = \frac{1}{g(s(\xi))}, \text{ with } s = s(\xi) \geq 0 \text{ satisfying } sg(s) = \xi, \quad \text{for } \xi \geq 0. \tag{2.4}$$

In addition to (2.1), we have the continuity equation

$$\phi\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0, \tag{2.5}$$

where number $\phi \in (0, 1)$ is the constant porosity. Also, for slightly compressible fluids, the equation of state is

$$\frac{d\rho}{dp} = \frac{\rho}{\kappa}, \quad \text{with } \kappa = const. > 0. \tag{2.6}$$

From (2.3), (2.5) and (2.6) one derives a scalar equation for the pressure:

$$\phi\frac{\partial p}{\partial t} = \kappa\nabla \cdot (K(|\nabla p|)\nabla p) + K(|\nabla p|)|\nabla p|^2. \tag{2.7}$$

On the right-hand side of (2.7), the constant $\kappa$ is very large for most slightly compressible fluids in porous media [16], hence we neglect its second term and by scaling the time variable, we study the reduced equation

$$\frac{\partial p}{\partial t} = \nabla \cdot (K(|\nabla p|)\nabla p). \tag{2.8}$$

Note that this reduction is commonly used in engineering.

Our aim is to study the IBVP for equation (2.8) in a bounded domain. Here afterward $U$ is a bounded open connected subset of $\mathbb{R}^n$, $n = 2, 3, \ldots$ with $C^2$ boundary $\Gamma = \partial U$. Throughout, the Forchheimer polynomial $g(s)$ is fixed. The following number is frequently used in our calculations:

$$a = \frac{\alpha_N}{1 + \alpha_N} \in (0, 1). \tag{2.9}$$

The function $K(\xi)$ in (2.4) has the following properties (c.f. [1,11]): it is decreasing in $\xi$ mapping $\xi \in [0, \infty)$ onto $(0, 1/a_0]$, and

$$\frac{C_1}{(1+\xi)^a} \leq K(\xi) \leq \frac{C_2}{(1+\xi)^a}, \tag{2.10}$$

$$C_3(\xi^{2-a} - 1) \leq K(\xi)\xi^2 \leq C_2\xi^{2-a}, \tag{2.11}$$

$$-aK(\xi) \leq K'(\xi)\xi \leq 0, \tag{2.12}$$

where $C_1$, $C_2$, $C_3$ are positive constants depending on $g$. As in previous works, we use the function $H(\xi)$ defined by

$$H(\xi) = \int_0^{\xi^2} K(\sqrt{s})ds \quad \text{for } \xi \geq 0.$$

It satisfies $K(\xi)\xi^2 \le H(\xi) \le 2K(\xi)\xi^2$, thus, by (2.11),

$$C_3(\xi^{2-a}-1) \le H(\xi) \le 2C_2\xi^{2-a}. \tag{2.13}$$

The following parabolic Poincaré-Sobolev inequalities are needed for our study. For each $T>0$, denote $Q_T = U\times(0,T)$. We define a threshold exponent

$$\alpha_* = \frac{an}{2-a}. \tag{2.14}$$

**Lemma 2.1.** *Assume*

$$\alpha \ge 2 \quad \textit{and} \quad \alpha > \alpha_*. \tag{2.15}$$

*Let*

$$p = \alpha\Big(1+\frac{2-a}{n}\Big) - a. \tag{2.16}$$

*Then*

$$\|u\|_{L^p(Q_T)} \le C(1+\delta T)^{1/p}[[u]], \tag{2.17}$$

*where $\delta=1$ in general, $\delta=0$ in case $u$ vanishes on the boundary $\partial U$, and*

$$[[u]] = \operatorname*{ess\,sup}_{[0,T]} \|u(t)\|_{L^\alpha(U)} + \Big(\int_0^T\int_U |u|^{\alpha-2}|\nabla u|^{2-a}dxdt\Big)^{\frac{1}{\alpha-a}}. \tag{2.18}$$

*In case $U=B_R$ – a ball of radius $R$ – the inequality (2.17) holds with*

$$[[u]] = R^{\frac{n}{p}-\frac{n}{\alpha}} \operatorname*{ess\,sup}_{[0,T]} \|u(t)\|_{L^\alpha(B_R)} + R^{\frac{n}{p}-\frac{n-(2-a)}{\alpha-a}}\Big(\int_0^T\int_{B_R} |u|^{\alpha-2}|\nabla u|^{2-a}dxdt\Big)^{\frac{1}{\alpha-a}} \tag{2.19}$$

*and the constant $C$ independent of $R$.*

The proof of Lemma 2.1 is given in Appendix A. The next is a particular embedding with spatial weights from Lemma 2.4 of [14] (see inequality (2.28) with $m=2$ there).

**Lemma 2.2** (cf. [14], Lemma 2.4)**.** *Given $W(x,t)>0$ on $Q_T$. Let $r$ be a number that satisfies*

$$\frac{2n}{n+2} < r < 2. \tag{2.20}$$

*Set*

$$\varrho = \varrho(r) \stackrel{\text{def}}{=\!=} 4(1-1/r^*). \tag{2.21}$$

*Then*

$$\|u\|_{L^\varrho(Q_T)} \le C[[u]]_{2,W;T}\Big\{\delta T^{\frac{1}{\varrho}} + \operatorname*{ess\,sup}_{t\in[0,T]}\Big(\int_U W(x,t)^{-\frac{r}{2-r}}\chi_{\operatorname{supp} u}(x,t)dx\Big)^{\frac{2-r}{\varrho r}}\Big\}, \tag{2.22}$$

*where $\delta=1$ in general, $\delta=0$ in case $u$ vanishes on the boundary $\partial U$, and*

$$[[u]]_{2,W;T} = \operatorname*{ess\,sup}_{[0,T]} \|u(t)\|_{L^2(U)} + \Big(\int_0^T\int_U W(x,t)|\nabla u|^2 dxdt\Big)^{\frac{1}{2}}. \tag{2.23}$$

The following is a generalization of the convergence of fast decay geometry sequences in Lemma 5.6, Chapter II of [15]. It will be used in the De Giorgi iterations.

**Lemma 2.3** (cf. [14], Lemma A.2)**.** *Let $\{Y_i\}_{i=0}^\infty$ be a sequence of non-negative numbers satisfying*

$$Y_{i+1} \le \sum_{k=1}^m A_k B_k^i Y_i^{1+\mu_k}, \quad i=0,1,2,\ldots,$$

*where $A_k>0$, $B_k>1$ and $\mu_k>0$ for $k=1,2,\ldots,m$. Let $B=\max\{B_k : 1\le k\le m\}$ and $\mu=\min\{\mu_k : 1\le k\le m\}$. If $Y_0 \le \min\{(m^{-1}A_k^{-1}B^{-\frac{1}{\mu}})^{1/\mu_k} : 1\le k\le m\}$ then $\lim_{i\to\infty} Y_i = 0$.*

## 3 Uniform Gronwall-type estimates

We study the following IBVP for $p(x,t)$:

$$\begin{cases} \dfrac{\partial p}{\partial t} = \nabla \cdot (K(|\nabla p|)\nabla p) & \text{in } U \times (0,\infty), \\ p(x,0) = p_0(x) & \text{in } U, \\ p(x,t) = \psi(x,t) & \text{on } \Gamma \times (0,\infty). \end{cases} \tag{3.1}$$

In order to deal with the non-homogeneous boundary condition, the data $\psi(x,t)$ with $x \in \Gamma$ and $t > 0$ is extended to a function $\Psi(x,t)$ with $x \in \bar{U}$ and $t \geq 0$. Throughout, our results are stated in terms of $\Psi$ instead of $\psi$. Nonetheless, corresponding results in terms of $\psi$ can be retrieved as performed in [11]. The function $\Psi$ is always assumed to have adequate regularities for all calculations in this paper.

It is proved in Section 3 of [13] that (3.1) possesses a weak solution $p(x,t)$ for all $t > 0$. It, in fact, has more regularity in spatial and time variables, see [5]. For the current study, we assume that solution $p(x,t)$ has sufficient regularities both in $x$ and $t$ variables such that our calculations hereafterward can be performed legitimately.

In this section, we obtain improved estimates for solutions of (3.1). We emphasize the asymptotic estimates as $t \to \infty$ in terms of the asymptotic behavior of $\Psi(x,t)$.

**Notation.** (a) Hereafter, symbol $C$ is used to denote a positive number independent of the initial and boundary data, and the time variables $t$, $T_0$, $T$; it may depend on many parameters, namely, exponents and coefficients of polynomial $g$, the spatial dimension $n$ and domain $U$, other involved exponents $\alpha$, $s$, etc. in calculations. The value of $C$ may vary from place to place, even on the same line. (b) For partial derivative notation, we will alternatively use $\partial p/\partial t = \partial_t p = p_t$, and $\partial p/\partial x_m = \partial_m p = p_{x_m}$. (c) The Lebesgue norm $\|\cdot\|_{L^s}$ means $\|\cdot\|_{L^s(U)}$. (d) For a function $f(x,t)$, we denote by $f(t)$ the function $x \to f(x,t)$.

For $\alpha \geq 1$, we define

$$A(\alpha,t) = \Big[\int_U |\nabla\Psi(x,t)|^{\frac{\alpha(2-a)}{2}} dx\Big]^{\frac{2(\alpha-a)}{\alpha(2-a)}} + \Big[\int_U |\Psi_t(x,t)|^{\alpha} dx\Big]^{\frac{\alpha-a}{\alpha(1-a)}} \tag{3.2}$$

for $t \geq 0$, and

$$A(\alpha) = \limsup_{t\to\infty} A(\alpha,t) \quad \text{and} \quad \beta(\alpha) = \limsup_{t\to\infty} [A'(\alpha,t)]^-. \tag{3.3}$$

Also, define for $\alpha > 0$ the number

$$\widehat{\alpha} = \max\big\{\alpha, 2, \alpha_*\big\}. \tag{3.4}$$

Whenever $\beta(\alpha)$ is in use, it is understood that the function $t \to A(\alpha,t)$ belongs to $C^1((0,\infty))$.

For a function $f : [0,\infty) \to \mathbb{R}$, we denote by $Env f$ a continuous and increasing function $F : [0,\infty) \to \mathbb{R}$ such that $F(t) \geq f(t)$ for all $t \geq 0$.

Let $p(x,t)$ be a solution to IBVP (3.1). Denote $\overline{p} = p - \Psi$ and $\bar{p}_0 = p_0 - \Psi(\cdot,0)$. We recall relevant results from [13] below.

**Theorem 3.1** (cf. [13], Theorem 4.3)**.** *Let $\alpha > 0$.*

(i) *For all $t \geq 0$,*

$$\int_U |\bar{p}(x,t)|^{\alpha} dx \leq C\Big(1 + \int_U |\bar{p}_0(x)|^{\widehat{\alpha}} dx + [Env A(\widehat{\alpha},t)]^{\frac{\widehat{\alpha}}{\widehat{\alpha}-a}}\Big). \tag{3.5}$$

(ii) *If* $A(\widehat{\alpha})<\infty$ *then*

$$\limsup_{t\to\infty}\int_U |\bar p(x,t)|^\alpha dx \le C\big(1+A(\widehat{\alpha})^{\frac{\widehat{\alpha}}{\widehat{\alpha}-a}}\big). \tag{3.6}$$

(iii) *If* $\beta(\widehat{\alpha})<\infty$ *then there is* $T>0$ *such that*

$$\int_U |\bar p(x,t)|^\alpha dx \le C\big(1+\beta(\widehat{\alpha})^{\frac{\widehat{\alpha}}{\widehat{\alpha}-2a}}+A(\widehat{\alpha},t)^{\frac{\widehat{\alpha}}{\widehat{\alpha}-a}}\big) \quad \textit{for all } t\ge T. \tag{3.7}$$

For gradient and time derivative estimates, we denote

$$\begin{aligned} G_1(t) &= \int_U |\nabla\Psi(x,t)|^2dx + \Big[\int_U |\Psi_t(x,t)|^{r_0}dx\Big]^{\frac{2-a}{r_0(1-a)}} + \Big[\int_U |\Psi_t(x,t)|^{r_0}dx\Big]^{\frac{1}{r_0}},\\ G_2(t) &= \int_U |\nabla\Psi_t(x,t)|^2dx + \int_U |\Psi_t(x,t)|^2dx,\\ G_3(t) &= G_1(t)+G_2(t), \quad G_4(t)=G_3(t)+\int_U |\Psi_{tt}|^2dx. \end{aligned}$$

with $r_0=\frac{n(2-a)}{(2-a)(n+1)-n}$. For $t\ge 0$, recall from (4.20) in [13] and from (3.25) in [11] that

$$\int_0^t\int_U H(|\nabla p(x,\tau)|)dxd\tau \le C\int_U \bar p_0^2(x)dx + C\int_0^t G_1(\tau)d\tau, \tag{3.8}$$

$$\int_U H(|\nabla p(x,t)|)dx + \int_0^t\int_U |\bar p_t(x,\tau)|^2dxd\tau \le \int_U \big[H(|\nabla p_0(x)|)+\bar p_0^2(x)\big]dx + C\int_0^t G_3(\tau)d\tau. \tag{3.9}$$

Let $\alpha\ge\widehat{2}$. For $t>0$, applying Theorem 4.5 in [13] with $t_0=t$ and following Remark 4.8 there to replace $\widehat{2}$ by $\alpha$, we have

$$\begin{aligned} \int_U |\bar p_t(x,t)|^2dx \le C\Big\{1+\int_U |\bar p_0(x)|^\alpha dx + t^{-1}\Big(\int_U |\bar p_0(x)|^2 + H(|\nabla p_0(x)|)dx + \int_0^t G_3(\tau)d\tau\Big)\Big\}\\ + C\int_0^t e^{-d_0(t-\tau)}\big([EnvA(\alpha,\tau)]^{\frac{\alpha}{\alpha-a}}+G_4(\tau)\big)d\tau, \end{aligned} \tag{3.10}$$

where $d_0>0$ is independent of $\alpha$. Since $EnvA(\alpha,t)$ is increasing in $t$, and $G_3\le G_4$ we simplify (3.10) as

$$\begin{aligned} \int_U |\bar p_t(x,t)|^2dx \le C(1+t^{-1})\Big(1+\int_U |\bar p_0(x)|^\alpha dx + \int_U H(|\nabla p_0(x)|)dx\\ + [EnvA(\alpha,t)]^{\frac{\alpha}{\alpha-a}} + \int_0^t G_4(\tau)d\tau\Big). \end{aligned} \tag{3.11}$$

Below are improved estimates for time $t$ large by using uniform Gronwall-type inequalities.

**Lemma 3.2.** *For* $t\ge 1$,

$$\int_{t-1}^t\int_U H(|\nabla p(x,\tau)|)dxd\tau \le C\int_U \bar p^2(x,t-1)dx + C\int_{t-1}^t G_1(\tau)d\tau, \tag{3.12}$$

$$\int_U H(|\nabla p(x,t)|)dx + \frac12\int_{t-1/2}^t\int_U \bar p_t^2(x,\tau)dxd\tau \le C\int_U \bar p^2(x,t-1)dx + C\int_{t-1}^t G_3(\tau)d\tau, \tag{3.13}$$

$$\int_U \bar p_t^2(x,t)dx \le C\int_U \bar p^2(x,t-1)dx + C\int_{t-1}^t G_4(\tau)d\tau. \tag{3.14}$$

*Proof.* The proof follows Theorems 4.4 and 4.5 in [12].

**Proof of (3.12).** Using inequality (3.4) of Lemma 3.1 in [11], one has

$$\frac{d}{dt}\int_U \bar{p}^2(x,t)dx \le -C\int_U H(|\nabla p|)dx + CG_1(t). \tag{3.15}$$

Dropping the negative term on the right-hand side of (3.15), and integrating it from $t-1$ to $t$, we obtain (3.12).

**Proof of (3.13).** Using (3.17) of Lemma 3.3 in [11] with $\varepsilon = 1$, one has

$$\frac{d}{dt}\int_U H(|\nabla p|)dx \le -\int_U \bar{p}_t^2(x,t)dx + \int_U H(|\nabla p|)dx + CG_2(t). \tag{3.16}$$

Let $s \in [t-1,t]$. Integrating (3.16) from $s$ to $t$ we have

$$\int_U H(|\nabla p(x,t)|)dx + \int_s^t\int_U \bar{p}_t^2(x,\tau)dxd\tau \le \int_U H(|\nabla p|)(x,s)dx + \int_s^t\int_U H(|\nabla p|)dxd\tau + C\int_s^t G_2(\tau)d\tau.$$

Thus,

$$\int_U H(|\nabla p(x,t)|)dx + \int_s^t\int_U \bar{p}_t^2(x,\tau)dxd\tau \le \int_U H(|\nabla p|)(x,s)dx + \int_{t-1}^t\int_U H(|\nabla p|)dxd\tau + C\int_{t-1}^t G_2(\tau)d\tau. \tag{3.17}$$

Integrating (3.17) in $s$ from $t-1$ to $t$, and using (3.12), we obtain

$$\begin{aligned}\int_U H(|\nabla p(x,t)|)dx + \int_{t-1}^t\int_s^t\int_U \bar{p}_t^2(x,\tau)dxd\tau ds &\le 2\int_{t-1}^t\int_U H(|\nabla p|)dxd\tau + C\int_{t-1}^t G_2(\tau)d\tau\\ &\le C\int_U \bar{p}^2(x,t-1)dx + C\int_{t-1}^t G_1(\tau)d\tau + C\int_{t-1}^t G_2(\tau)d\tau. \end{aligned}\tag{3.18}$$

Observe that

$$\begin{aligned}&\int_{t-1}^t\int_s^t\int_U \bar{p}_t^2(x,\tau)dxd\tau ds = \int_{t-1}^t\int_{t-1}^\tau\int_U \bar{p}_t^2(x,\tau)dxdsd\tau\\ &\ge \int_{t-1/2}^t\int_{t-1}^\tau\int_U \bar{p}_t^2(x,\tau)dxdsd\tau \ge \frac{1}{2}\int_{t-1/2}^t\int_U \bar{p}_t^2(x,\tau)dxd\tau.\end{aligned}$$

Therefore (3.13) follows from (3.18).

**Proof of (3.14).** From (3.37) of Proposition 3.11 in [11] we have

$$\frac{d}{dt}\int_U \bar{p}_t^2 dx \le -C\int_U K(|\nabla p|)|\nabla p_t|^2dx + C\int_U |\nabla\Psi_t|^2dx - \int_U \bar{p}_t\Psi_{tt}dx.$$

Dropping the first term on the right-hand side and using Cauchy's inequality give

$$\frac{d}{dt}\int_U \bar{p}_t^2 dx \le C\int_U |\nabla\Psi_t|^2dx + \frac{1}{2}\int_U |\bar{p}_t|^2dx + \frac{1}{2}\int_U |\Psi_{tt}|^2dx. \tag{3.19}$$

For $s \in [t-1/2, t]$, integrating (3.19) from $s$ to $t$ gives

$$\int_U \bar{p}_t^2(x,t)dx \le \int_U \bar{p}_t^2(x,s)dx + C\int_s^t\int_U |\nabla\Psi_t|^2 + |\Psi_{tt}|^2 dxd\tau + \frac{1}{2}\int_s^t\int_U |\bar{p}_t(x,\tau)|^2 dxd\tau,$$
$$\int_U \bar{p}_t^2(x,t)dx \le \int_U \bar{p}_t^2(x,s)dx + C\int_{t-1}^t\int_U |\nabla\Psi_t|^2 + |\Psi_{tt}|^2 dxd\tau + \frac{1}{2}\int_{t-1/2}^t\int_U |\bar{p}_t(x,\tau)|^2 dxd\tau.$$

Integrating the last inequality in $s$ from $t-1/2$ to $t$ yields

$$\frac{1}{2}\int_U \bar{p}_t^2(x,t)dx \le \frac{5}{4}\int_{t-1/2}^t\int_U \bar{p}_t^2(x,s)dxds + C\int_{t-1}^t\int_U |\nabla\Psi_t)|^2 + |\Psi_{tt}|^2 dxd\tau.$$

Using (3.13) for the first integral on the right-hand side, we have

$$\int_U \bar{p}_t^2(x,t)dx \le C\int_U \bar{p}^2(x,t-1)dx + C\int_{t-1}^t G_3(\tau)d\tau + C\int_{t-1}^t\int_U |\nabla\Psi_t)|^2 + |\Psi_{tt}|^2 dxd\tau. \quad (3.20)$$

Using the fact $\int_U |\nabla\Psi_t|^2 dx \le CG_2(t) \le CG_3(t)$, we obtain (3.14) from (3.20). The proof is complete. □

Combining the above, we have the following specific estimates which will be used conveniently in subsequent sections.

**Corollary 3.3.** *Let $\alpha \ge \widehat{2}$.*

(i) *For $t \ge 1$,*

$$\int_{t-1}^t\int_U H(|\nabla p(x,\tau)|)dxd\tau \le C\Big(1 + \int_U |\bar{p}_0(x)|^\alpha dx + [EnvA(\alpha,t)]^{\frac{\alpha}{\alpha-a}} + \int_{t-1}^t G_1(\tau)d\tau\Big), \quad (3.21)$$

$$\int_{t-1/2}^t\int_U \bar{p}_t^2(x,\tau)dxd\tau \le C\Big(1 + \int_U |\bar{p}_0(x)|^\alpha dx + [EnvA(\alpha,t)]^{\frac{\alpha}{\alpha-a}} + \int_{t-1}^t G_3(\tau)d\tau\Big). \quad (3.22)$$

(ii) *If $A(\alpha) < \infty$ then*

$$\limsup_{t\to\infty}\int_U H(|\nabla p(x,t)|)dx \le C\Big(1 + A(\alpha)^{\frac{\alpha}{\alpha-a}} + \limsup_{t\to\infty}\int_{t-1}^t G_3(\tau)d\tau\Big), \quad (3.23)$$

$$\limsup_{t\to\infty}\int_U \bar{p}_t^2(x,t)dx \le C\Big(1 + A(\alpha)^{\frac{\alpha}{\alpha-a}} + \limsup_{t\to\infty}\int_{t-1}^t G_4(\tau)d\tau\Big), \quad (3.24)$$

$$\limsup_{t\to\infty}\int_{t-1}^t\int_U H(|\nabla p(x,\tau)|)dxd\tau \le C\Big(1 + A(\alpha)^{\frac{\alpha}{\alpha-a}} + \limsup_{t\to\infty}\int_{t-1}^t G_1(\tau)d\tau\Big), \quad (3.25)$$

$$\limsup_{t\to\infty}\int_{t-1/2}^t\int_U \bar{p}_t^2(x,\tau)dxd\tau \le C\Big(1 + A(\alpha)^{\frac{\alpha}{\alpha-a}} + \limsup_{t\to\infty}\int_{t-1}^t G_3(\tau)d\tau\Big). \quad (3.26)$$

(iii) *If $\beta(\alpha) < \infty$ then there is $T > 0$ such that one has for all $t \ge T$ that*

$$\int_U H(|\nabla p(x,t)|)dx \le C\Big(1 + \beta(\alpha)^{\frac{\alpha}{\alpha-2a}} + A(\alpha,t-1)^{\frac{\alpha}{\alpha-a}} + \int_{t-1}^t G_3(\tau)d\tau\Big), \quad (3.27)$$

$$\int_U \bar{p}_t^2(x,t)dx \le C\Big(1 + \beta(\alpha)^{\frac{\alpha}{\alpha-2a}} + A(\alpha,t-1)^{\frac{\alpha}{\alpha-a}} + \int_{t-1}^t G_4(\tau)d\tau\Big), \quad (3.28)$$

$$\int_{t-1}^{t}\int_U H(|\nabla p(x,\tau)|)dxd\tau \le C\Big(1+\beta(\alpha)^{\frac{\alpha}{\alpha-2a}} + A(\alpha,t-1)^{\frac{\alpha}{\alpha-a}} + \int_{t-1}^{t} G_1(\tau)d\tau\Big), \tag{3.29}$$

$$\int_{t-1/2}^{t}\int_U \bar{p}_t^2(x,\tau)dxd\tau \le C\Big(1+\beta(\alpha)^{\frac{\alpha}{\alpha-2a}} + A(\alpha,t-1)^{\frac{\alpha}{\alpha-a}} + \int_{t-1}^{t} G_3(\tau)d\tau\Big). \tag{3.30}$$

*Proof.* Note that $\widehat{\alpha} = \alpha$.

(i) For (3.21), resp. (3.22), we combine (3.12), resp. (3.13), with inequality

$$\int_U \bar{p}^2(x,t-1)dx \le \int_U (1+|\bar{p}(x,t-1)|^\alpha)dx, \tag{3.31}$$

and the estimate (3.5) for $t-1$.

(ii) We combine estimates in Lemma 3.2 with (3.31) and the limit (3.6).

(iii) This part is similar to part (ii) with the use of (3.7) in place of (3.6). □

# 4 Interior $L^\infty$-estimates for pressure

In this section we estimate the $L^\infty$-norm of the pressure. *Hereafter, $\alpha$ is a number that satisfies (2.15).* We use $\kappa_j$ to denote a number of exponents that depend on $\alpha$. Let

$$\kappa_0 = \alpha(1+(2-a)/n) - a, \quad \kappa_1 = 1/\delta_1 \quad \text{and} \quad \kappa_2 = 1/\delta_2,$$

where $\delta_1 = 1-\alpha/\kappa_0$ and $\delta_2 = (1-a/\alpha)(1-\alpha_*/\alpha)$. Note that $\delta_1,\delta_2 \in (0,1)$, hence, $\kappa_1,\kappa_2 > 1$.

We start with estimating the $L^\infty$-norm in terms of the $L^\alpha$-norm.

**Theorem 4.1.** *Let $U' \Subset U$. If $T_0 \ge 0$, $T>0$ and $\theta \in (0,1)$ then*

$$\sup_{[T_0+\theta T, T_0+T]} \|p(t)\|_{L^\infty(U')} \le C(1+T)^{\frac{\kappa_1}{\kappa_0}}\Big(1+(\theta T)^{-1}\Big)^{\frac{\kappa_1}{\alpha-a}}\Big(1+\|p\|_{L^\alpha(U\times(T_0,T_0+T))}\Big)^{\kappa_2}. \tag{4.1}$$

*Proof.* Without loss of generality, we assume $T_0 = 0$. For $k \ge 0$, define $p^{(k)} = \max\{p-k,0\}$ and denote by $\chi_k(x,t)$ the characteristic function of the set $\operatorname{supp} p^{(k)}$. Let $\phi_1(x)$ and $\phi_2(t)$ be cut-off functions with $\phi_1 = 1$ on $U'$, $\phi_1 = 0$ on a neighborhood of $\partial U$, and $\phi_2(0) = 0$. Let $\zeta(x,t) = \phi_1(x)\phi_2(t)$. Multiplying the first equation in (3.1) by $|p^{(k)}|^{\alpha-1}\zeta^2$ and integrating over $U$, we have

$$\begin{aligned}
&\frac{1}{\alpha}\frac{d}{dt}\int_U |p^{(k)}|^\alpha \zeta^2 dx + (\alpha-1)\int_U K(|\nabla p^{(k)}|)|\nabla p^{(k)}|^2 |p^{(k)}|^{\alpha-2}\zeta^2 dx \\
&= \frac{2}{\alpha}\int_U |p^{(k)}|^\alpha \zeta\zeta_t dx + 2\int_U K(|\nabla p^{(k)}|)(\nabla p^{(k)}\cdot\nabla\zeta)|p^{(k)}|^{\alpha-1}\zeta dx.
\end{aligned}$$

By Cauchy's inequality, we have for the last integral that

$$|K(|\nabla p^{(k)}|)(\nabla p^{(k)}\cdot\nabla\zeta)|p^{(k)}|^{\alpha-1}\zeta| \le \frac{\alpha-1}{2}K(|\nabla p^{(k)}|)|\nabla p^{(k)}|^2|p^{(k)}|^{\alpha-2}\zeta^2 + CK(|\nabla p^{(k)}|)|p^{(k)}|^\alpha|\nabla\zeta|^2.$$

Combining the above, we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_U |p^{(k)}|^\alpha\zeta^2 dx + \frac{\alpha-1}{2}\int_U K(|\nabla p^{(k)}|)|\nabla p^{(k)}|^2|p^{(k)}|^{\alpha-2}\zeta^2 dx \\
&\le C\int_U |p^{(k)}|^\alpha \zeta|\zeta_t| dx + C\int_U K(|\nabla p^{(k)}|)|p^{(k)}|^\alpha|\nabla\zeta|^2 dx.
\end{aligned}$$

Using (2.11), we then have

$$\begin{aligned}
&\frac{d}{dt}\int_U |p^{(k)}|^\alpha \zeta^2 dx + \int_U |\nabla p^{(k)}|^{2-a}|p^{(k)}|^{\alpha-2}\zeta^2 dx\\
&\quad \le C\int_U |p^{(k)}|^\alpha \zeta|\zeta_t| dx + C\int_U K(|\nabla p^{(k)}|)|p^{(k)}|^\alpha |\nabla\zeta|^2 dx + C\int_U |p^{(k)}|^{\alpha-2}\zeta^2 dx.
\end{aligned}$$

Using the boundedness of $K(\cdot)$ in the second to last integral, and applying Young's inequality to the last terms yield

$$\begin{aligned}
&\frac{d}{dt}\int_U |p^{(k)}|^\alpha \zeta^2 dx + \int_U |\nabla p^{(k)}|^{2-a}|p^{(k)}|^{\alpha-2}\zeta^2 dx\\
&\quad \le C\int_U |p^{(k)}|^\alpha (\zeta^2 + \zeta|\zeta_t| + |\nabla\zeta|^2) dx + C\int_U \chi_k \zeta^2 dx.
\end{aligned}$$

For $t \in [0,T]$, integrating from 0 to $T$, and taking supremum in $t$, we obtain

$$\begin{aligned}
&\sup_{[0,T]}\int_U |p^{(k)}|^\alpha \zeta^2 dx + \int_0^T\int_U |\nabla p^{(k)}|^{2-a}|p^{(k)}|^{\alpha-2}\zeta^2 dxdt\\
&\le C\int_0^T\int_U |p^{(k)}|^\alpha (\zeta^2 + \zeta|\zeta_t| + |\nabla\zeta|^2) dx + C\int_0^T\int_U \chi_k \zeta^2 dx.
\end{aligned} \tag{4.2}$$

Let $x_0$ be any given point in $\bar U'$. Denote $\rho = \mathrm{dist}(\bar U', \partial U) > 0$. Let $M_0 > 0$ be fixed which will be determined later. For $i \ge 0$, define

$$k_i = M_0(1-2^{-i}), \quad t_i = \theta T(1-2^{-i}), \quad \rho_i = \frac{1}{4}\rho(1+2^{-i}). \tag{4.3}$$

Then $t_0 = 0 < t_1 < \ldots < \theta T$ and $\rho_0 = \rho/2 > \rho_1 > \ldots > \rho/4 > 0$. Note that

$$\lim_{i\to\infty} t_i = \theta T \quad \text{and} \quad \lim_{i\to\infty} \rho_i = \rho/4. \tag{4.4}$$

Let $U_i = \{x : \|x - x_0\| < \rho_i\}$ then $U_{i+1} \Subset U_i$ for $i = 0, 1, 2, \ldots$. For $i, j \ge 0$, we denote

$$\mathcal{Q}_i = U_i \times (t_i, T) \quad \text{and} \quad A_{i,j} = \{(x,t) \in \mathcal{Q}_j : p(x,t) > k_i\}. \tag{4.5}$$

For each $\mathcal{Q}_i$, we use a cut-off function $\zeta_i(x,t)$ with $\zeta_i \equiv 1$ in $\mathcal{Q}_{i+1}$ and $\zeta_i \equiv 0$ on $Q_T \setminus \mathcal{Q}_i$, and

$$|(\zeta_i)_t| \le \frac{C}{t_{i+1}-t_i} = \frac{C2^{i+1}}{\theta T}, \quad \text{and} \quad |\nabla\zeta_i| \le \frac{C}{\rho_i - \rho_{i+1}} = \frac{C2^{i+1}}{4\rho}. \tag{4.6}$$

for some $C > 0$. Applying (4.2) with $k = k_{i+1}$ and $\zeta = \zeta_i$ gives

$$\begin{aligned}
&\sup_{[0,T]}\int_U |p^{(k_{i+1})}|^\alpha \zeta_i^2 dx + \int_0^T\int_U |\nabla p^{(k_{i+1})}|^{2-a}|p^{(k_{i+1})}|^{\alpha-2}\zeta_i^2 dxdt\\
&\le \int_0^T\int_U |p^{(k_{i+1})}|^\alpha (\zeta_i^2 + \zeta_i|\zeta_{it}| + |\nabla\zeta_i|^2) dxdt + C\int_0^T\int_U \chi_{k_{i+1}}\zeta_i^2 dxdt.
\end{aligned} \tag{4.7}$$

Define

$$F_i \overset{\text{def}}{=\!=} \sup_{[t_{i+1},T]}\int_{U_{i+1}} |p^{(k_{i+1})}|^\alpha dx + \int_{t_{i+1}}^T\int_{U_{i+1}} |\nabla p^{(k_{i+1})}|^{2-a}|p^{(k_{i+1})}|^{\alpha-2} dxdt.$$

Then (4.7) yields

$$\begin{aligned} F_i &\le \int_{t_i}^T \int_U |p^{(k_{i+1})}|^\alpha (\zeta_i^2 + \zeta_i|\zeta_{it}| + |\nabla \zeta_i|^2) dxdt + C\Big(\int_{t_i}^T \int_{U_i} \chi_{k_{i+1}} dxdt\Big) \\ &\le C4^i((\theta T)^{-1} + 1)\|p^{(k_{i+1})}\|^\alpha_{L^\alpha(A_{i+1,i})} + C|A_{i+1,i}|. \end{aligned} \tag{4.8}$$

Since $\|p^{(k_i)}\|_{L^\alpha(A_{i,i})} \ge \|p^{(k_i)}\|_{L^\alpha(A_{i+1,i})} \ge (k_{i+1} - k_i)|A_{i+1,i}|^{1/\alpha}$, we have

$$|A_{i+1,i}| \le (k_{i+1} - k_i)^{-1}\|p^{(k_i)}\|^\alpha_{L^\alpha(A_{i,i})} \le C2^{\alpha i} M_0^{-\alpha}\|p^{(k_i)}\|^\alpha_{L^\alpha(A_{i,i})}. \tag{4.9}$$

This and (4.8) imply

$$\begin{aligned} F_i &\le C4^i(1 + (\theta T)^{-1})\|p^{(k_{i+1})}\|^\alpha_{L^\alpha(A_{i+1,i})} + C2^{\alpha i} M_0^{-\alpha}\|p^{(k_i)}\|^\alpha_{L^\alpha(A_{i,i})} \\ &\le C2^{\alpha i}\Big(1 + (\theta T)^{-1} + M_0^{-\alpha}\Big)\|p^{(k_i)}\|^\alpha_{L^\alpha(A_{i,i})}. \end{aligned} \tag{4.10}$$

Note that $\kappa_0$ is the exponent defined in (2.16). Applying Lemma 2.1 to domain $U_i$, interval $[t_{i+1}, T]$ and function $p^{(k_{i+1})}$ with the use of (2.19), we have

$$\|p^{(k_{i+1})}\|_{L^{\kappa_0}(A_{i+1,i+1})} = \|p^{(k_{i+1})}\|_{L^{\kappa_0}(\mathcal{Q}_{i+1})} \le C(1+T)^{\frac{1}{\kappa_0}}(F_i^{1/\alpha} + F_i^{1/(\alpha-a)}). \tag{4.11}$$

Note that the constant $C$ depends on $\rho$. Holder's inequality gives

$$\|p^{(k_{i+1})}\|_{L^\alpha(A_{i+1,i+1})} \le \|p^{(k_{i+1})}\|_{L^{\kappa_0}(A_{i+1,i+1})}|A_{i+1,i+1}|^{1/\alpha - 1/\kappa_0} \le \|p^{(k_{i+1})}\|_{L^{\kappa_0}(A_{i+1,i+1})}|A_{i+1,i}|^{1/\alpha - 1/\kappa_0}.$$

It follows from this, (4.11), (4.10), and (4.9) that

$$\begin{aligned} \|p^{(k_{i+1})}\|_{L^\alpha(A_{i+1,i+1})} &\le C(1+T)^{\frac{1}{\kappa_0}}(F_i^{1/\alpha} + F_i^{1/(\alpha-a)})|A_{i+1,i}|^{1/\alpha - 1/\kappa_0} \\ &\le C(1+T)^{\frac{1}{\kappa_0}} B^i\Big\{\Big(1 + (\theta T)^{-1} + M_0^{-\alpha}\Big)^{1/\alpha}\|p^{(k_i)}\|_{L^\alpha(A_{i,i})} \\ &\quad + \Big(1 + (\theta T)^{-1} + M_0^{-\alpha}\Big)^{1/(\alpha-a)}\|p^{(k_i)}\|^{\alpha/(\alpha-a)}_{L^\alpha(A_i)}\Big\} M_0^{-1+\alpha/\kappa_0}\|p^{(k_i)}\|^{1-\alpha/\kappa_0}_{L^2(A_{i,i})}, \end{aligned}$$

where $B = 2^\alpha$. Now selecting $M_0 \ge 1$ we have

$$\begin{aligned} \|p^{(k_i)}\|_{L^\alpha(A_{i+1,i+1})} &\le C(1+T)^{\frac{1}{\kappa_0}} B^i\Big\{\Big(1 + (\theta T)^{-1}\Big)^{1/\alpha} M_0^{-1+\alpha/\kappa_0}\|p^{(k_i)}\|^{2-\alpha/\kappa_0}_{L^\alpha(A_{i,i})} \\ &\quad + \Big(1 + (\theta T)^{-1}\Big)^{1/(\alpha-a)} M_0^{-1+\alpha/\kappa_0}\|p^{(k_i)}\|^{\alpha/(\alpha-a)+1-\alpha/\kappa_0}_{L^\alpha(A_{i,i})}\Big\}. \end{aligned}$$

Let $\nu_1 = \alpha/(\alpha - a) - \alpha/\kappa_0 > 0$. Let $Y_i = \|p^{(k_i)}\|_{L^\alpha(A_{i,i})}$. Then

$$Y_{i+1} \le CB^i\Big(D_1 Y_i^{1+\delta_1} + D_2 Y_i^{1+\nu_1}\Big),$$

where $D_1 = (1+T)^{\frac{1}{\kappa_0}}\big(1 + (\theta T)^{-1}\big)^{1/\alpha} M_0^{-\delta_1}$, $D_2 = (1+T)^{\frac{1}{\kappa_0}}\big(1 + (\theta T)^{-1}\big)^{1/(\alpha-a)} M_0^{-\delta_1}$. Take $M_0$ sufficiently large such that

$$Y_0 \le C\min\{D_1^{-1/\delta_1}, D_2^{-1/\nu_1}\}. \tag{4.12}$$

Then by Lemma 2.3, $\lim_{i\to\infty} Y_i = 0$, consequently, $\int_{\theta T}^T \int_{B_{\rho/4}(x_0)} |p^{(M_0)}|^\alpha dxdt = 0$, that is

$$p(x,t) \le M_0 \quad \text{a.e. in } B_{\rho/4}(x_0) \times (\theta T, T).$$

Since $Y_0 \le \|p\|_{L^\alpha(Q_T)}$, condition (4.12) is met if $\|p\|_{L^\alpha(Q_T)} \le C\min\{D_1^{-1/\delta_1}, D_2^{-1/\nu_1}\}$. This, in turn, is satisfied if

$$M_0 \ge C(1+T)^{\frac{1}{\kappa_0\delta_1}}\Big\{(1+(\theta T)^{-1})^{1/(\alpha\delta_1)}\|p\|_{L^\alpha(Q_T)} + (1+(\theta T)^{-1})^{1/((\alpha-a)\delta_1)}\|p\|_{L^\alpha(Q_T)}^{\nu_1/\delta_1}\Big\}.$$

Note that $\nu_1/\delta_1 = 1/\delta_2 = \kappa_2 > 1$. Combining this and condition $M_0 \ge 1$ we choose

$$M_0 = C(1+T)^{\frac{\kappa_1}{\kappa_0}}(1+\theta T)^{-1})^{\frac{\kappa_1}{\alpha-a}}(1+\|p\|_{L^\alpha(Q_T)})^{\kappa_2},$$

with an appropriate positive constant $C$. By using a finite covering of $\bar U'$, we obtain

$$p(x,t) \le M_0 \quad \text{a.e. in } U' \times (\theta T, T).$$

Repeating the argument for $-p$ instead of $p$, we obtain $|p(x,t)| \le M_0$ a.e. in $U' \times [\theta T, T]$, and hence (4.1) follows. The proof is complete. □

Combining Theorem 4.1 with estimates in Section 3, we obtain the following specific estimates for the $L^\infty$-norm.

**Theorem 4.2.** *Let $U' \Subset U$.*

(i) *If $t \in (0,1)$ then*

$$\|p(t)\|_{L^\infty(U')} \le Ct^{-\frac{\kappa_1}{\alpha-a}}\Big(1 + \|\bar p_0\|_{L^\alpha} + [EnvA(\alpha,t)]^{\frac{1}{\alpha-a}} + \|\Psi\|_{L^\alpha(U\times(0,t))}\Big)^{\kappa_2}. \tag{4.13}$$

*If $t \ge 1$ then*

$$\|p(t)\|_{L^\infty(U')} \le C\Big(1 + \|\bar p_0\|_{L^\alpha} + [EnvA(\alpha,t)]^{\frac{1}{\alpha-a}} + \|\Psi\|_{L^\alpha(U\times(t-1,t))}\Big)^{\kappa_2}. \tag{4.14}$$

(ii) *If $A(\alpha) < \infty$ then*

$$\limsup_{t\to\infty}\|p(t)\|_{L^\infty(U')} \le C\big(1 + A(\alpha)^{\frac{1}{\alpha-a}} + \limsup_{t\to\infty}\|\Psi\|_{L^\alpha(U\times(t-1,t))}\big)^{\kappa_2}. \tag{4.15}$$

(iii) *If $\beta(\alpha) < \infty$ then there is $T > 0$ such that*

$$\|p(t)\|_{L^\infty(U')} \le C\big(1 + \beta(\alpha)^{\frac{1}{\alpha-2a}} + A(\alpha,t)^{\frac{1}{\alpha-a}} + \|\Psi\|_{L^\alpha(U\times(t-1,t))}\big)^{\kappa_2} \tag{4.16}$$

*for all $t \ge T$.*

*Proof.* (i) Note that $\alpha = \widehat\alpha$. Let $t \in (0,1)$. Applying (4.1) for $T = t$ and $\theta = 1/2$, we have

$$\begin{aligned}\|p(t)\|_{L^\infty(U')} &\le C(1 + t^{-\frac{\kappa_1}{\alpha-a}})(1 + \|p\|_{L^\alpha(U\times(0,t))}^{\kappa_2})\\ &\le Ct^{-\frac{\kappa_1}{\alpha-a}}(1 + \|\bar p\|_{L^\alpha(U\times(0,t))} + \|\Psi\|_{L^\alpha(U\times(0,t))})^{\kappa_2}.\end{aligned}$$

Using (3.5), we obtain (4.13).

For $t \ge 1$, applying (4.1) with $T_0 = t-1$, $T = 1$ and $\theta = 1/2$ we obtain

$$\|p(t)\|_{L^\infty(U')} \le C\Big(1 + \|p\|_{L^\alpha(U\times(t-1,t))}^{\kappa_2}\Big) \le C\Big(1 + \|\bar p\|_{L^\alpha(U\times(t-1,t))} + \|\Psi\|_{L^\alpha(U\times(t-1,t))}\Big)^{\kappa_2}. \tag{4.17}$$

Again, using inequality (3.5), we obtain (4.14).

(ii) From (4.17) we have

$$\limsup_{t\to\infty}\|p(t)\|_{L^\infty(U')} \le C\Big(1 + \limsup_{t\to\infty}\|\bar p\|_{L^\alpha(U\times(t-1,t))} + \limsup_{t\to\infty}\|\Psi\|_{L^\alpha(U\times(t-1,t))}\Big)^{\kappa_2}.$$

By (3.6),

$$\limsup_{t\to\infty}\|\bar p\|_{L^\alpha(U\times(t-1,t))}^\alpha \le \limsup_{t\to\infty}\int_U |\bar p(x,t)|^\alpha dx \le C(1 + A(\alpha))^{\alpha/(\alpha-a)}.$$

Thus (4.15) follows.

(iii) Using (3.7) in (4.17) we obtain (4.16). □

# 5 Interior estimates for pressure gradient

We will first estimate the pressure gradient in $L^s$-norm (for $s < \infty$) in subsection 5.1, and then in $L^\infty$-norm in subsection 5.2.

## 5.1 $L^s$-estimates

The following imbedding lemma from [14] is a suitable extension of Lemma 5.4 on page 93 in [15].

**Lemma 5.1** (cf. [14], Lemma 3.7)**.** *For each $s \geq 1$, there exists a constant $C > 0$ depending on $s$ such that for each smooth cut-off function $\zeta(x) \in C_c^\infty(U)$, the following inequality holds*

$$\int_U K(|\nabla p|)|\nabla p|^{2s+2}\zeta^2 dx \leq C \max_{\mathrm{supp}\zeta} |p|^2 \Big[ \int_U K(|\nabla p|)|\nabla p|^{2s-2}|\nabla^2 p|^2 \zeta^2 dx + \int_U K(|\nabla p|)|\nabla p|^{2s}|\nabla \zeta|^2 dx \Big],$$

*for every sufficiently regular function $p(x)$ such that the right hand side is well-defined.*

We establish the basic step for the Ladyzhenskaya-Uraltseva iteration.

**Lemma 5.2.** *For each $s \geq 0$, if $T_0 \geq 0$, $T > 0$, and $\zeta(x,t)$ is a smooth cut-off function then*

$$\sup_{[T_0,T_0+T]} \int_U |\nabla p(x,t)|^{2s+2}\zeta^2 dx + \int_{T_0}^{T_0+T}\int_U K(|\nabla p|)|\nabla^2 p|^2 |\nabla p|^{2s}\zeta^2 dxdt \leq C\int_{T_0}^{T_0+T}\int_U K(|\nabla p|)|\nabla p|^{2s+2}|\nabla\zeta|^2 dxdt + C\int_{T_0}^{T_0+T}\int_U |\nabla p|^{2s+2}\zeta|\zeta_t| dx. \tag{5.1}$$

*Proof.* Without loss of generality, assume $T_0 = 0$. Let $\zeta(x,t)$ be the cut off function on $U \times [0,T]$ with $\zeta(\cdot,0) = 0$ and $\mathrm{supp}\,\zeta(\cdot,t) \subset \bar{U}' \Subset U$. Multiplying the equation (3.1) by $-\nabla\cdot(|\nabla p|^{2s}\zeta^2\nabla p)$, integrating the resultant over $U$ and using integration by parts, we obtain

$$\frac{1}{2s+2}\frac{d}{dt}\int_U |\nabla p|^{2s+2}\zeta^2 dx = -\int_U \partial_j(K(|\nabla p|)\partial_i p)\,\partial_i(|\nabla p|^{2s}\partial_j p\zeta^2)dx + \frac{1}{s+1}\int_U |\nabla p|^{2s+2}\zeta\zeta_t dx.$$

Calculated as in Lemma 3.6 of [14] above equation is rewritten as

$$\begin{aligned}\frac{1}{2s+2}\frac{d}{dt}\int_U |\nabla p|^{2s+2}\zeta^2 dx &= -\int_U \Big[\partial_{y_l}(K(|y|)y_i)\Big|_{y=\nabla p}\partial_j\partial_l p\Big]\partial_j\partial_i p|\nabla p|^{2s}\zeta^2 dx \\ &- 2\int_U \Big[\partial_{y_l}(K(|y|)y_i)\Big|_{y=\nabla p}\partial_j\partial_l p\Big]\partial_j p\,|\nabla p|^{2s}\,\zeta\partial_i\zeta dx \\ &- 2s\int_U \Big[\partial_{y_l}(K(|y|)y_i)\Big|_{y=\nabla p}\partial_j\partial_l p\Big]\partial_j p\,(|\nabla p|^{2s-2}\partial_i\partial_m p\partial_m p)\,\zeta^2 dx + \frac{1}{s+1}\int_U |\nabla p|^{2s+2}\zeta\zeta_t dx.\end{aligned}$$

We denote the four terms on the right-hand side by $I_1$, $I_2, I_3$ and $I_4$. It follows from the calculations in Lemma 3.6 of [14] that

$$I_1 \leq -(1-a)\sum_j \int_U K(|\nabla p|)|\nabla(\partial_j p)|^2|\nabla p|^{2s}\zeta^2 dx,$$

$$|I_2| \leq 2(1+a)\int_U K(|\nabla p|)|\nabla^2 p||\nabla p|^{2s+1}\zeta|\nabla\zeta| dx,$$

and

$$I_3 \le -2(1-a)s\int_U K(|\nabla p|)\Big|\nabla\Big(\frac{1}{2}|\nabla p|^2\Big)\Big|^2|\nabla p|^{2s-2}\zeta^2 dx \le 0.$$

Combining these estimates with Young's inequality, we find that

$$\begin{aligned}\frac{1}{2s+2}\frac{d}{dt}\int_U |\nabla p|^{2s+2}\zeta^2 dx + \frac{1-a}{2}\int_U K(|\nabla p|)|\nabla^2 p|^2|\nabla p|^{2s}\zeta^2 dx\\ \le C\int_U K(|\nabla p|)|\nabla p|^{2s+2}|\nabla\zeta|^2 dx + \frac{1}{s+1}\int_U |\nabla p|^{2s+2}\zeta|\zeta_t| dx.\end{aligned}\tag{5.2}$$

Inequality (5.1) follows directly by integrating (5.2) from 0 to $T$. □

Next, we reduce estimates for $W^{1,s}$-norm, with large $s$, down to $W^{1,2-a}$ and $L^\infty$ norms.

**Proposition 5.3.** *Let $U' \Subset V \Subset U$, $T_0 \ge 0$, $T > 0$ and $\theta \in (0,1)$. If $s \ge 2$ then*

$$\begin{aligned}\int_{T_0+\theta T}^{T_0+T}\int_{U'} K(|\nabla p|)|\nabla p|^s dxdt \le C\Big(1+(\theta T)^{-1}\Big)^{s-2}\big(1+\sup_{[T_0+\theta T/2,T_0+T]}\|p\|^2_{L^\infty(V)}\big)^{s-2}\\ \cdot\int_{T_0}^{T_0+T}\int_U (1+K(|\nabla p|)|\nabla p|^2) dxdt,\end{aligned}\tag{5.3}$$

*and*

$$\begin{aligned}\sup_{t\in[T_0+\theta T,T_0+T]}\int_{U'} |\nabla p(x,t)|^s dxdt \le C\Big(1+(\theta T)^{-1}\Big)^{s+a-1}\big(1+\sup_{[T_0+\theta T/2,T_0+T]}\|p\|^2_{L^\infty(V)}\big)^{s-2+a}\\ \cdot\int_{T_0}^{T_0+T}\int_U (1+K(|\nabla p|)|\nabla p|^2) dxdt,\end{aligned}\tag{5.4}$$

*where constant $C > 0$ is independent of $T_0$, $T$, and $\theta$.*

*Proof.* Without loss of generality, assume $T_0 = 0$. First, we prove a more general version of (5.3).

*Claim.* For $0 < \theta' < \theta < 1$ we have

$$\begin{aligned}\int_{\theta T}^{T}\int_{U'} K(|\nabla p|)|\nabla p|^s dxdt \le C\Big(1+[(\theta-\theta')T]^{-1}\Big)^{s-2}\big(1+\sup_{[\theta' T,T]}\|p\|^2_{L^\infty(V)}\big)^{s-2}\\ \cdot\int_0^T\int_U (1+K(|\nabla p|)|\nabla p|^2) dxdt,\end{aligned}\tag{5.5}$$

where constant $C > 0$ is independent of $T$, $\theta$, and $\theta'$.

The proof of (5.5) consists in a number of steps.

**Step 1.** Let $\zeta(x,t)$ be the cut-off function with $\zeta = 0$ for $t \le \theta' T$ and $\zeta = 1$ for $t \ge \theta T$. By applying Lemma 5.1 with $s+1$ in place of $s$, we have

$$\begin{aligned}\int_0^T\int_U K(|\nabla p|)|\nabla p|^{2s+4}\zeta^2 dxdt \le C\max_{\mathrm{supp}\zeta}|p|^2\Big[\int_0^T\int_U K(|\nabla p|)|\nabla p|^{2s}|\nabla^2 p|^2\zeta^2 dxdt\\ + \int_0^T\int_U K(|\nabla p|)|\nabla p|^{2s+2}|\nabla\zeta|^2 dxdt\Big].\end{aligned}$$

Let $N_0 = \sup_{[\theta' T,T]} \|p\|^2_{L^\infty(V)}$. Using (5.1) to estimate the first integral on the right-hand side, we find that

$$\begin{aligned}&\int_0^T\int_U K(|\nabla p|)|\nabla p|^{2s+4}\zeta^2 dxdt\\&\quad\le CN_0\Big[\int_0^T\int_U K(|\nabla p|)|\nabla p|^{2s+2}|\nabla\zeta|^2 dxdt + \int_0^T\int_U |\nabla p|^{2s+2}\zeta|\zeta_t|dx\Big].\end{aligned}\tag{5.6}$$

Since

$$|\nabla p|^{2s+2}\le 1+K(|\nabla p|)|\nabla p|^{2s+2+a} \text{ and } K(|\nabla p|)|\nabla p|^{2s+2}\le C(1+K(|\nabla p|)|\nabla p|^{2s+2+a}),$$

inequality (5.6) leads to

$$\begin{aligned}&\int_0^T\int_U K(|\nabla p|)|\nabla p|^{2s+4}\zeta^2 dxdt\\&\qquad\le CN_0\int_0^T\int_U (1+K(|\nabla p|)|\nabla p|^{2s+2+a})(|\nabla\zeta|^2+\zeta|\zeta_t|)dxdt.\end{aligned}\tag{5.7}$$

**Step 2.** Let $s=0$ in (5.6), for $\varepsilon>0$,

$$\begin{aligned}\int_0^T\int_U K(|\nabla p|)|\nabla p|^4\zeta^2 dx &\le \varepsilon\int_0^T\int_U K(|\nabla p|)|\nabla p|^4\zeta^2 + C\varepsilon^{-1}N_0\int_0^T\int_U K(|\nabla p|)^{-1}\zeta_t^2 dxdt\\&\quad + CN_0\int_0^T\int_U K(|\nabla p|)|\nabla p|^2|\nabla\zeta|^2 dxdt\Big].\end{aligned}\tag{5.8}$$

Taking $\varepsilon = (2(N_0+1))^{-1}$ and using Young's inequality yield

$$\begin{aligned}\int_{\theta T}^T\int_{U'} K(|\nabla p|)|\nabla p|^4 dxdt &\le C(N_0+1)N_0(\theta T)^{-2}\int_{\theta T/2}^T\int_U (1+|\nabla p|)^a dxdt\\&\quad + CN_0\int_{\theta T/2}^T\int_U K(|\nabla p|)|\nabla p|^2 dxdt\\&\le C(1+N_0)^2\Big(1+\frac{1}{\theta T}\Big)^2\int_{\theta T/2}^T\int_U (1+K(|\nabla p|)|\nabla p|^2)dxdt.\end{aligned}$$

This implies (5.5) when $s=4$.

**Step 3.** When $s\in(2,4)$, let $\gamma$ and $\beta$ be two positive numbers such that $\frac{1}{s}=\frac{\gamma}{2}+\frac{\beta}{4}$ and $\gamma+\beta=1$. Then, by interpolation inequality:

$$\Big(\int_{\theta T}^T\int_{U'} K(|\nabla p|)|\nabla p|^s dxdt\Big)^{\frac1s}\le\Big(\int_{\theta T}^T\int_U K(|\nabla p|)|\nabla p|^2 dxdt\Big)^{\frac{\gamma}{2}}\Big(\int_{\theta T}^T\int_{U'} K(|\nabla p|)|\nabla p|^4 dxdt\Big)^{\frac{\beta}{4}}.$$

Note that $\beta s/4 = s/2-1$. Using (5.8) to estimate the last double integral, we obtain

$$\int_{\theta T}^T\int_{U'} K(|\nabla p|)|\nabla p|^s dxdt\le C(1+\frac{1}{\theta T})^{s-2}(1+N_0)^{s-2}\int_{\theta T/2}^T\int_U 1+K(|\nabla p|)|\nabla p|^2 dx.\tag{5.9}$$

This implies (5.3) for $s\in(2,4)$. Therefore, we have proved (5.5) with $s\in(2,4]$.

**Step 4.** When $s>4$, let $m=[\frac{s-4}{2-a}]+1\in\mathbb{N}$, then $s-m(2-a)\in(2,4]$. Then, let $\{U_k\}_{k=0}^m$ be a family of open, smooth domains in $U$ such that $U'\Subset U_0\Subset U_1\Subset U_2\Subset\ldots\Subset U_{m-1}\Subset U_m\Subset V\subset U$.

For each $k = 1, 2, \ldots, m$, let $\theta_k = \theta - (\theta - \theta')k/m$ and $\tilde{Q}_k = U_k \times (\theta_k T, T)$. Also, let $\zeta_k(x,t)$ be a smooth cut-off function which is equal to one on $\tilde{Q}_k$ and zero on $Q_T \setminus \tilde{Q}_{k-1}$. There is a positive constant $c > 0$ depending on all $U_k$, $k = 1, 2, \ldots, m$, such that $|\nabla \zeta_k| \le c$ and $|\zeta_{k,t}| \le 2^{k+1} c[(\theta - \theta')T]^{-1}$ , for all $k = 1, 2, \ldots, m$. From (5.7) we have

$$\begin{aligned}\int_{\theta T}^{T}\int_{U'} K(|\nabla p|)|\nabla p|^s dxdt &\le C_m(1 + [(\theta - \theta')T]^{-1})N_0 \int_{\theta_1 T}^{T}\int_{U_1} (1 + K(|\nabla p|)|\nabla p|^{s-(2-a)})dxdt\\ &= C_m(1 + [(\theta - \theta')T]^{-1})N_0\Big[T + \int_{\theta_1 T}^{T}\int_{U_1} K(|\nabla p|)|\nabla p|^{s-(2-a)}dxdt\Big].\end{aligned}$$

Recursively, we obtain

$$\begin{aligned}\int_{\theta T}^{T}\int_{U'} K(|\nabla p|)|\nabla p|^s dxdt &\le CT\sum_{i=1}^{m} d^i + Cd^m \int_{\theta_m T}^{T}\int_{U_m} K(|\nabla p|)|\nabla p|^{s-m(2-a)}dxdt\\ &\le C_m d^m\Big[T + \int_{\theta_m T}^{T}\int_{U_m} K(|\nabla p|)|\nabla p|^{s-m(2-a)}dxdt\Big],\end{aligned}$$

where $d = (1 + [(\theta - \theta')T]^{-1})(1 + N_0)$. Since $s - m(2 - a) \in (2, 4]$, estimating the last integral by (5.9) gives

$$\int_{\theta T}^{T}\int_{U'} K(|\nabla p|)|\nabla p|^s dxdt \le Cd^m T + Cd^{s-m(2-a)-2+m}\int_{\theta_m T}^{T}\int_{U} K(|\nabla p|)|\nabla p|^2 dx. \tag{5.10}$$

Since $d \ge 1$ and $m, s - m(2 - a) - 2 + m \le s - 2$, the desired estimate (5.5) follows (5.10). This completes the proof of (5.5) for all $s$.

The inequality (5.3) then is an obvious consequence of (5.5) with $\theta' = \theta/2$.

Now, we turn to the proof of (5.4). For $s > 2 - a$, it follows (5.1) that

$$\begin{aligned}\sup_{[0,T]}\int_U |\nabla p|^s \zeta^2 dx &\le C\int_0^T\int_U K(|\nabla p|)|\nabla p|^s|\nabla\zeta|^2 dxdt + C\int_0^T\int_U |\nabla p|^s\zeta|\zeta_t|dx\\ &\le C\int_0^T\int_U |\nabla p|^s(|\nabla\zeta|^2 + \zeta|\zeta_t|)dxdt.\end{aligned}$$

Since $|\nabla p|^s \le CK(|\nabla p|)(1 + |\nabla p|^{s+a}) \le C(1 + K(|\nabla p|)|\nabla p|^{s+a})$, we have

$$\sup_{[0,T]}\int_U |\nabla p|^s\zeta^2 dx \le C\int_0^T\int_U (1 + K(|\nabla p|)|\nabla p|^{s+a})(|\nabla\zeta|^2 + \zeta|\zeta_t|)dx. \tag{5.11}$$

Let $V_1$ be a set such that $V \Subset V_1 \Subset U$. Choose $\zeta(x,t)$ such that $\zeta = 0$ for $t \le 3\theta T/4$ or $x \notin V_1$, and $\zeta = 1$ for $(x,t) \in U' \times [\theta T, T]$. Then we have from (5.11) that

$$\sup_{[\theta T,T]}\int_U |\nabla p|^s\zeta^2 dx \le C(1 + (\theta T)^{-1})\int_{3\theta T/4}^{T}\int_{V_1} (1 + K(|\nabla p|)|\nabla p|^{s+a})dxdt. \tag{5.12}$$

To estimate the last integral, we apply (5.5) with parameters $\theta'$, $\theta$, $U'$ being replaced by $\theta/2$, $3\theta/4$, $V_1$. Therefore, we obtain

$$\sup_{[\theta T,T]}\int_{U'} |\nabla p|^s dx \le C\Big\{((1 + (\theta T)^{-1})^{s+a-1}(1 + N_0)^{s+a-2}\int_0^T\int_U 1 + K(|\nabla p|)|\nabla p|^2 dxdt\Big\},$$

hence proving (5.4). The proof is complete. □

Now, we combine Proposition 5.3 with estimates in section 3 to express the bounds in terms of the initial and boundary data.

**Theorem 5.4.** *Let $U' \Subset U$ and $s \geq 2$. If $t \in (0,2)$ then*

$$\int_{U'} |\nabla p(x,t)|^s dx \leq Ct^{-\mu_1}(1+\|\bar{p}_0\|_{L^\alpha})^{\mu_2+2}\Big(1+[EnvA(\alpha,t)]^{\frac{1}{\alpha-a}}+\|\Psi\|_{L^\alpha(U\times(0,t))}\Big)^{\mu_2} \cdot \Big(1+\int_0^t G_1(\tau)d\tau\Big), \quad (5.13)$$

*where*

$$\mu_1 = \mu_1(\alpha,s) \overset{\text{def}}{=\!=} \big[1+\frac{2\kappa_1}{\alpha-a}\big](s+a-2)+1 \quad \textit{and} \quad \mu_2=\mu_2(\alpha,s) \overset{\text{def}}{=\!=} 2\kappa_2(s-2+a). \quad (5.14)$$

*If $t \geq 2$ then*

$$\int_{U'} |\nabla p(x,t)|^s dx \leq C(1+\|\bar{p}_0\|_{L^\alpha})^{\mu_2+\alpha}\Big(1+[EnvA(\alpha,t)]^{\frac{1}{\alpha-a}}+\|\Psi\|_{L^\alpha(U\times(t-2,t))}\Big)^{\mu_2+\alpha} \cdot \Big(1+\int_{t-1}^t G_1(\tau)d\tau\Big). \quad (5.15)$$

*Proof.* Let $t \in (0,2)$. Applying (5.4) to $T_0 = 0$, $T = t$ and $\theta = 1/2$, and using (4.13), (3.8) and relation (2.13), we obtain

$$\int_{U'} |\nabla p(x,t)|^s dx \leq Ct^{-\mu_1} \cdot \Big(1+\|\bar{p}_0\|_{L^\alpha}+[EnvA(\alpha,t)]^{\frac{1}{\alpha-a}}+\|\Psi\|_{L^\alpha(U\times(0,t))}\Big)^{2\kappa_2(s-2+a)} \cdot \Big(1+\|\bar{p}_0\|_{L^2}^2+\int_0^t G_1(\tau)d\tau\Big).$$

Then (5.13) follows. Let $t \geq 2$. Applying (5.4) with $T_0 = t-1$, $T = 1$, $\theta = 1/2$, then combining it with (4.14) and (3.21), we obtain

$$\int_{U'} |\nabla p(x,t)|^s dx \leq C\Big(1+\|\bar{p}_0\|_{L^\alpha}+[EnvA(\alpha,t)]^{\frac{1}{\alpha-a}}+\|\Psi\|_{L^\alpha(U\times(t-2,t))}\Big)^{\mu_2} \cdot \Big(1+\|\bar{p}_0\|_{L^\alpha}^\alpha+EnvA(\alpha,t)^{\frac{\alpha}{\alpha-a}}+\int_{t-1}^t G_1(\tau)d\tau\Big).$$

Then (5.15) follows. □

For large time estimates, we have:

**Theorem 5.5.** *Let $U' \Subset U$, $s \geq 2$, and let $\mu_2$ be defined as in Theorem 5.4.*

(i) *If $A(\alpha) < \infty$ then*

$$\limsup_{t\to\infty}\int_{U'} |\nabla p(x,t)|^s dx \leq C\Big(1+A(\alpha)^{\frac{1}{\alpha-a}}+\limsup_{t\to\infty}\|\Psi\|_{L^\alpha(U\times(t-1,t))}\Big)^{\mu_2+\alpha}\Big(1+\limsup_{t\to\infty}\int_{t-1}^t G_1(\tau)d\tau\Big). \quad (5.16)$$

(ii) *If $\beta(\alpha) < \infty$ then there is $T > 0$ such that for all $t > T$,*

$$\int_{U'} |\nabla p(x,t)|^s dx \leq C\Big(1+\beta(\alpha)^{\frac{1}{\alpha-2a}}+\sup_{[t-1,t]} A(\alpha,\cdot)^{\frac{1}{\alpha-a}}+\|\Psi\|_{L^\alpha(U\times(t-2,t))}\Big)^{\mu_2+\alpha}\Big(1+\int_{t-1}^t G_1(\tau)d\tau\Big). \quad (5.17)$$

*Proof.* For $t \ge 1$ Applying (5.4) with $T_0 = t-1$, $T = 1$, $\theta = 1/2$,

$$\int_{U'} |\nabla p(x,t)|^s dxdt \le C\big(1 + \sup_{[t-1,t]} \|p\|^2_{L^\infty(V)}\big)^{s-2+a} \cdot \int_{t-1}^t \int_U (1 + K(|\nabla p|)|\nabla p|^2) dxdt. \tag{5.18}$$

(i) Taking limit superior as $t \to \infty$ and using (4.15), (3.25) give

$$\limsup_{t\to\infty} \int_{U'} |\nabla p(x,t)|^s dx \le C\big(1 + A(\alpha)^{\frac{1}{\alpha-a}} + \limsup_{t\to\infty} \|\Psi\|_{L^\alpha(U\times(t-1,t))}\big)^{\mu_2}$$
$$\cdot \Big(1 + A(\alpha)^{\frac{\alpha}{\alpha-a}} + \limsup_{t\to\infty} \int_{t-1}^t G_1(\tau) d\tau\Big).$$

Then estimate (5.16) follows.

(ii) By combining (5.18) with (4.16) and (3.29), we have

$$\int_{U'} |\nabla p(x,t)|^s dx \le C\Big\{1 + \beta(\alpha)^{\frac{1}{\alpha-2a}} + \sup_{[t-1,t]} A(\alpha,\cdot)^{\frac{1}{\alpha-a}} + \|\Psi\|_{L^\alpha(U\times(t-2,t))}\Big\}^{\mu_2}$$
$$\cdot \Big(1 + \beta(\alpha)^{\frac{\alpha}{\alpha-2a}} + A(\alpha, t-1)^{\frac{\alpha}{\alpha-a}} + \int_{t-1}^t G_1(\tau) d\tau\Big).$$

Then (5.17) follows. $\square$

## 5.2 $L^\infty$-estimates

In this subsection, we obtain interior $L^\infty$-estimates for the gradient of pressure. For each $m = 1, 2, \dots, n$, denote $u_m = p_{x_m}$ and $u = (u_1, u_2, \dots, u_n) = \nabla p$. We have

$$\frac{\partial u_m}{\partial t} = \partial_m(\nabla\cdot(K(|u|)u)) = \nabla\cdot(K(|u|)\partial_m u) + \nabla\cdot\Big[K'(|u|)\frac{\sum_i u_i \partial_m u_i}{|u|} u\Big]. \tag{5.19}$$

Since $\partial_i u_m = \partial_m u_i$, we have $\partial_m u = (\partial_m u_1, \dots, \partial_m u_n) = (\partial_1 u_m, \dots, \partial_n u_m) = \nabla u_m$, and $\sum_i u_i \partial_m u_i = \sum_i u_i \partial_i u_m = u\cdot\nabla u_m$. Therefore, we rewrite (5.19) as

$$\frac{\partial u_m}{\partial t} = \nabla\cdot(K(|u|)\nabla u_m) + \nabla\cdot\Big[K'(|u|)\frac{u\cdot\nabla u_m}{|u|} u\Big]. \tag{5.20}$$

We will apply De Giorgi's technique to equation (5.20). In the following, we fix a number $s_0$ such that $r = s_0$ satisfies (2.20). Note that $s_0^* > 2$. We will also use $s_j$ for $j \ge 1$ to denote some exponents that depend on $s_0$ but are independent of $\alpha$. Let

$$s_1 = (1 - 2/s_0^*)^{-1} > 1.$$

**Theorem 5.6.** *Let $U' \Subset V \Subset U$. For any $T_0 \ge 0$, $T > 0$, and $\theta \in (0,1)$, if $t \in [T_0 + \theta T, T_0 + T]$ then*

$$\|\nabla p(t)\|_{L^\infty(U')} \le C(1 + (\theta T)^{-1})^{\frac{s_1+1}{2}} \lambda^{\frac{s_1}{2}} \|\nabla p\|_{L^2(V\times(T_0+\theta T/2, T_0+T))}, \tag{5.21}$$

*where*

$$\lambda = \lambda(T_0, T, \theta; V) = \Big(\int_{T_0+\theta T/2}^{T_0+T} \int_V (1 + |\nabla p|)^{\frac{a s_0}{2-s_0}} dxdt\Big)^{\frac{2-s_0}{s_0}}. \tag{5.22}$$

*and constant $C > 0$ is independent of $T_0$, $T$, and $\theta$.*

*Proof.* Without loss of generality, assume $T_0 = 0$. Fix $m \in \{1, 2, \ldots, n\}$. We will show for $t \in [\theta T, T]$ that

$$\|p_{x_m}(t)\|_{L^\infty(U')} \le C(1 + (\theta T)^{-1})^{\frac{s_1+1}{2}} \lambda^{\frac{s_1}{2}} \|p_{x_m}\|_{L^2(V\times(T_0+\theta T/2, T_0+T))}. \tag{5.23}$$

Let $\zeta(x,t) = \phi(x)\varphi(t)$ be a cut-off function with $\varphi(t) = 0$ for $t \le \theta T/2$, and $\operatorname{supp}\phi \subset U$. We define for $k \ge 0$

$$u_m^{(k)} = \max\{u_m - k, 0\}, \quad S_k(t) = \{x \in U : u_m^{(k)}(x,t) \ge 0\},$$

and denote by $\chi_k(x,t)$ the characteristic function of $S_k(t)$.

Multiplying (5.20) by $u_m^{(k)}\zeta^2$ and integrating over $U$, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_U |u_m^{(k)}|^2\zeta^2 dx = \int_U |u_m^{(k)}|^2 \zeta|\zeta_t| dx - \int_U K(|u|)|\nabla u_m^{(k)}|^2\zeta^2 dx - 2\int_U K(|u|)(\nabla u_m^{(k)}\cdot\nabla\zeta)u_m^{(k)}\zeta dx \\ - \int_U \frac{1}{|u|}K'(|u|)(u\cdot\nabla u_m^{(k)})(u\cdot\nabla(u_m^{(k)}\zeta^2))dx. \end{aligned} \tag{5.24}$$

By the product rule,

$$-\frac{1}{|u|}K'(|u|)(u\cdot\nabla u_m^{(k)})(u\cdot\nabla(u_m^{(k)}\zeta^2)) = -\frac{1}{|u|}K'(|u|)(u\cdot\nabla u_m^{(k)})\Big(u\cdot\nabla u_m^{(k)}\zeta^2 + 2u_m^{(k)}\zeta u\cdot\nabla\zeta\Big).$$

It follows property (2.12) that

$$-\frac{1}{|u|}K'(|u|)(u\cdot\nabla(u_m^{(k)})^2\zeta^2 \le \frac{aK(|u|)}{|u|^2}|u|^2|\nabla u_m^{(k)}|^2\zeta^2 = aK(|u|)|\nabla u_m^{(k)}|^2\zeta^2$$

and

$$-\frac{2}{|u|}K'(|u|)u^2 u_m^{(k)}\zeta\nabla u_m^{(k)}\cdot\nabla\zeta \le \frac{2}{|u|}|K'(|u|)|u|^2|\nabla u_m^{(k)}||u_m^{(k)}|\zeta|\nabla\zeta| \le CK(|u|)|\nabla u_m^{(k)}||u_m^{(k)}|\zeta|\nabla\zeta|.$$

Then we obtain from (5.24) that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_U |u_m^{(k)}|^2\zeta^2 \le \int_U |u_m^{(k)}|^2\zeta|\zeta_t|dx - (1-a)\int_U K(|u|)|\nabla u_m^{(k)}|^2\zeta^2 dx \\ + C\int_U K(|u|)|\nabla u_m^{(k)}||u_m^{(k)}|\zeta|\nabla\zeta|dx. \end{aligned}$$

Applying Cauchy's inequality to the last term in previous inequality yields

$$\frac{1}{2}\frac{d}{dt}\int_U |u_m^{(k)}|^2\zeta^2 \le \int_U |u_m^{(k)}|^2\zeta|\zeta_t|dx - \frac{1-a}{2}\int_U K(|u|)|\nabla u_m^{(k)}|^2\zeta^2 dx + C\int_U K(|u|)|u_m^{(k)}|^2|\nabla\zeta|^2 dx.$$

Since $|\zeta\nabla u_m^{(k)}|^2 = |\nabla(u_m^{(k)}\zeta) - u_m^{(k)}\nabla\zeta|^2 \ge \frac{1}{2}|\nabla(u_m^{(k)}\zeta)|^2 - |u_m^{(k)}\nabla\zeta|^2$, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_U |u_m^{(k)}\zeta|^2 dx + \frac{1-a}{4}\int_U K(|u|)|\nabla(u_m^{(k)}\zeta)|^2 dx \\ \le \int_U |u_m^{(k)}|^2\zeta|\zeta_t|dx + C\int_U K(|u|)|u_m^{(k)}|^2|\nabla\zeta|^2 dx. \end{aligned} \tag{5.25}$$

Integrating (5.25) from 0 to $t$ for $t \in [0,T]$, and then taking supremum in $t$ give

$$\begin{aligned}\max_{[0,T]} \int_U |u_m^{(k)}\zeta|^2 dx + C\int_0^T \int_U K(|u|)|\nabla(u_m^{(k)}\zeta)|^2 dxdt \\ \le \int_0^T \int_U |u_m^{(k)}|^2 \zeta|\zeta_t| dx + C\int_0^T \int_U K(|u|)|u_m^{(k)}|^2 |\nabla\zeta|^2 dxdt.\end{aligned} \tag{5.26}$$

Let

$$\nu_2 = 4(1 - 1/s_0^*) > 2. \tag{5.27}$$

Applying Lemma 2.2 to function $u_m^{(k)}\zeta$ which vanishes on the boundary, weight $W = K(|u|)$ and exponents $r = s_0$, $\varrho = \varrho(s_0) = \nu_2$, we have

$$\begin{aligned}\|u_m^{(k)}\zeta\|_{L^{\nu_2}(Q_T)} \le C\Big[\operatorname*{ess\,sup}_{t\in[0,T]} \|u_m^{(k)}\zeta\|_{L^2(U)} + \Big(\int_0^T \int_U K(|u|)|\nabla(u_m^{(k)}\zeta)|^2 dxdt\Big)^{\frac12}\Big] \\ \cdot \Big[\int_0^T \int_{\mathrm{supp}\zeta} K(|u|)^{-\frac{s_0}{2-s_0}} dxdt\Big]^{\frac{2-s_0}{\nu_2 s_0}}.\end{aligned}$$

Using (2.11), we have $K(|u|)^{-\frac{s_0}{2-s_0}} \le C(1+|u|)^{\frac{as_0}{2-s_0}}$, hence

$$\|u_m^{(k)}\zeta\|_{L^{\nu_2}(Q_T)} \le C\lambda^{1/\nu_2}\Big[\max_{t\in[0,T]} \int_U |u_m^{(k)}\zeta|^2 dx + \int_0^T \int_U K(|u|)|\nabla(u_m^{(k)}\zeta)|^2 dxdt\Big]^{\frac12}. \tag{5.28}$$

By (5.28), (5.26) and the boundedness of function $K(\cdot)$ we find that

$$\begin{aligned}\|u_m^{(k)}\zeta\|_{L^{\nu_2}(Q_T)} \le C\lambda^{1/\nu_2}\Big(\max_{[0,T]} \int_U |u_m^{(k)}\zeta|^2 dx + C\int_0^T \int_U K(|u|)|\nabla(u_m^{(k)}\zeta)|^2 dxdt\Big)^{\frac12} \\ \le C\lambda^{1/\nu_2}\Big(\int_0^T \int_U |u_m^{(k)}|^2\zeta|\zeta_t| dx + \int_0^T \int_U |u_m^{(k)}|^2 |\nabla\zeta|^2 dxdt\Big)^{\frac12}.\end{aligned} \tag{5.29}$$

Here, we use the same notation $x_0$, $\rho$, $M_0$, $k_i$, $\rho_i$, $U_i$, $\mathcal{Q}_i$ and $\zeta_i$ as introduced in the proof of Theorem 4.1 from (4.3) to (4.6). Also, the sets $A_{i,j}$ are defined by (4.5) with $p$ being replaced by $u_m$.

Define $F_i = \|u_m^{(k_{i+1})}\zeta_i\|_{L^{\nu_2}(A_{i+1,i})}$. Applying (5.29) with $k = k_{i+1}$ and $\zeta = \zeta_i$ gives

$$F_i \le C\lambda^{1/\nu_2}\Big\{\int_0^T \int_U |u_m^{(k_{i+1})}|^2 \zeta_i |(\zeta_i)_t| dx + \int_0^T \int_U |u_m^{(k_{i+1})}|^2 |\nabla\zeta_i|^2\Big) dxdt\Big\}^{1/2}. \tag{5.30}$$

Using (4.6), we obtain

$$F_i \le C2^i\lambda^{1/\nu_2}(1 + \frac{1}{\theta T})^{1/2}\|u_m^{(k_{i+1})}\|_{L^2(A_{i+1,i})} \le C2^i(1+\frac{1}{\theta T})^{1/2}\lambda\|u_m^{(k_i)}\|_{L^2(A_{i,i})}.$$

Since $\nu_2 > 2$, it follows from Hölder's inequality that

$$\begin{aligned}\|u_m^{(k_{i+1})}\zeta_i\|_{L^2(A_{i+1,i+1})} \le \|u_m^{(k_{i+1})}\zeta_i\|_{L^{\nu_2}(A_{i+1,i+1})}|A_{i+1,i+1}|^{1/2-1/\nu_2} \\ \le \|u_m^{(k_{i+1})}\zeta_i\|_{L^{\nu_2}(A_{i+1,i+1})}|A_{i+1,i}|^{1/2-1/\nu_2} \le CF_i|A_{i+1,i}|^{1/2-1/\nu_2}.\end{aligned} \tag{5.31}$$

Note that $\|u_m^{(k_i)}\|_{L^2(A_{i,i})} \ge \|u_m^{(k_i)}\|_{L^2(A_{i+1,i})} \ge (k_{i+1}-k_i)|A_{i+1,i}|^{1/2}$. Thus,

$$|A_{i+1,i}| \le (k_{i+1}-k_i)^{-2}\|u_m^{(k_i)}\|^2_{L^2(A_i)} \le C4^i M_0^{-2}\|u_m^{(k_i)}\|^2_{L^2(A_{i,i})}. \tag{5.32}$$

Then it follows (5.31), (5.30) and (5.32) that

$$\begin{aligned}\|u_m^{(k_{i+1})}\|_{L^2(A_{i+1,i+1})} &\le C2^i(1+\frac{1}{\theta T})^{1/2}\lambda^{1/\nu_2}\|u_m^{(k_i)}\|_{L^2(A_i)}2^{i-\frac{2i}{\nu_2}}M_0^{-1+2/\nu_2}\|u_m^{(k_i)}\|_{L^2(A_{i,i})}^{1-2/\nu_2}\\ &\le C4^i(1+\frac{1}{\theta T})^{1/2}\lambda^{1/\nu_2}M_0^{-1+2/\nu_2}\|u_m^{(k_i)}\|_{L^2(A_{i,i})}^{2-2/\nu_2}.\end{aligned}$$

Denote $\nu_3 = 1 - 2/\nu_2$. Let $Y_i = \|u_m^{(k_i)}\|_{L^2(A_{i,i})}$, $B = 4$ and $D = C(1 + \frac{1}{\theta T})^{1/2}\lambda^{1/\nu_2}M_0^{-\nu_3}$. We obtain

$$Y_{i+1} \le DB^iY_i^{1+\nu_3} \quad \text{for all } i \ge 0.$$

We now determine $M_0$ so that $Y_0 \le D^{-1/\nu_3}B^{-1/\nu_3^2}$. This condition is met if

$$M_0 \ge C\Big[\lambda^{1/\nu_2}(1+\frac{1}{\theta T})^{1/2}\Big]^{1/\nu_3}Y_0 = C\lambda^{s_1/2}(1+\frac{1}{\theta T})^{\frac{1+s_1}{2}}Y_0.$$

Since $Y_0 = \|u_m^{(k_0)}\|_{L^2(A_{0,0})} \le \|u_m\|_{L^2(V\times(\theta T/2,T))}$, it suffices to choose $M_0$ as

$$M_0 = C\lambda^{s_1/2}(1+\frac{1}{\theta T})^{\frac{s_1+1}{2}}\|u_m\|_{L^2(V\times(\theta T/2,T))}. \tag{5.33}$$

Then Lemma 2.3 gives $\lim\limits_{i\to\infty} Y_i = 0$. Hence,

$$\int_{\theta T}^T\int_{B(x_0,\rho/4)}|u_m^{(M_0)}|^2dxdt = 0.$$

Thus, $u_m(x,t) \le M_0$ a.e. in $B(x_0,\rho/4)\times(0,T)$. Replace $u_m, u$ by $-u_m, -u$ and use the same argument we obtain $|u_m(x,t)| \le M_0$ a.e. in $B(x_0,\rho/4)\times(0,T)$. Now by covering $U'$ by finitely many such balls $B(x_0,\rho/4)$, we come to conclusion

$$|u_m(x,t)| \le M_0 \quad \text{a.e. in } U'\times(\theta T,T). \tag{5.34}$$

By the choice of $M_0$ we obtain from (5.34) that

$$|u_m(x,t)| \le C(1+\frac{1}{\theta T})^{\frac{s_1+1}{2}}\lambda^{s_1/2}\|u_m\|_{L^2(V\times(\theta T/2,T))}$$

for all $m = 1,\dots,n$. Then (5.23) follows. □

We will combine Theorem 5.6 with the high integrability of $\nabla p$ in subsection 5.1 to obtain the $L^\infty$-estimates. Let

$$\begin{aligned}&s_2 = \max\Big\{2, \frac{as_0}{2-s_0}\Big\}, \quad s_3 = s_1(2-s_0)/s_0 + 1,\\ &\kappa_3 = (s_2+a-2)\big(1+\frac{2\kappa_1}{\alpha-a}\big), \quad \kappa_5 = \kappa_2(s_2-2+a), \quad \kappa_4 = 1+s_1+\kappa_3 s_3.\end{aligned}$$

**Theorem 5.7.** *If $t \in (0,2)$ then*

$$\|\nabla p(t)\|_{L^\infty(U')} \le Ct^{-\kappa_4/2}(1+\|\bar p_0\|_{L^\alpha})^{s_3(\kappa_5+1)}\Big(1+EnvA(\alpha,t)^{\frac{1}{\alpha-a}}+\|\Psi\|_{L^\alpha(U\times(0,t))}\Big)^{s_3\kappa_5} \cdot\Big(1+\int_0^t G_1(\tau)d\tau\Big)^{s_3/2}. \tag{5.35}$$

*If $t \ge 2$ then*

$$\|\nabla p(t)\|_{L^\infty(U')} \le C(1+\|\bar p_0\|_{L^\alpha})^{s_3(\kappa_5+\alpha/2)}\Big(1+[EnvA(\alpha,t)]^{\frac{1}{\alpha-a}}+\|\Psi\|_{L^\alpha(U\times(t-2,t))}\Big)^{s_3(\kappa_5+\alpha/2)} \cdot\Big(1+\int_{t-1}^t G_1(\tau)d\tau\Big)^{s_3/2}. \quad (5.36)$$

*Proof.* First, we have from (5.21) that

$$\sup_{[T_0+\theta T,T_0+T]}\|\nabla p(t)\|_{L^\infty(U')} \le C(1+(\theta T)^{-1})^{\frac{1+s_1}{2}}\Big(\int_{T_0+\theta T/2}^{T_0+T}\int_V (1+|\nabla p|)^{s_2}dxdt\Big)^{s_3/2}. \quad (5.37)$$

Let $t\in(0,2)$, applying (5.37) with $T_0=0$, $T=t$, and $\theta=1/2$, we obtain

$$\|\nabla p(t)\|_{L^\infty(U')} \le Ct^{-\frac{1+s_1}{2}}\Big(\int_{t/4}^t\int_V (1+|\nabla p|)^{s_2}dxdt\Big)^{s_3/2}. \quad (5.38)$$

We apply (5.13) with $s=s_2$ and $U'=V$. Note from formulas in (5.14) that

$$\mu_1(\alpha,s_2)=\kappa_3+1 \quad\text{and}\quad \mu_2(\alpha,s_2)=2\kappa_5. \quad (5.39)$$

We obtain

$$\begin{aligned}\|\nabla p(t)\|_{L^\infty(U')} &\le Ct^{-\frac{1+s_1}{2}}\Big\{t\cdot t^{-(\kappa_3+1)}(1+\|\bar p_0\|_{L^\alpha})^{2(\kappa_5+1)}\\ &\quad\cdot\Big(1+EnvA(\alpha,t)^{\frac{1}{\alpha-a}}+\|\Psi\|_{L^\alpha(U\times(0,t))}\Big)^{2\kappa_5}\Big(1+\int_0^t G_1(\tau)d\tau\Big)\Big\}^{s_3/2}\\ &\le Ct^{-\frac{1+s_1+\kappa_3 s_3}{2}}(1+\|\bar p_0\|_{L^\alpha})^{s_3(\kappa_5+1)}\\ &\quad\cdot\Big(1+EnvA(\alpha,t)^{\frac{1}{\alpha-a}}+\|\Psi\|_{L^\alpha(U\times(0,t))}\Big)^{s_3\kappa_5}\Big(1+\int_0^t G_1(\tau)d\tau\Big)^{s_3/2}.\end{aligned}$$

Then (5.35) follows. Let $t\ge 2$. Applying (5.37) with $T_0=t-3/4$, $T=3/4$, $\theta=2/3$, then

$$\|\nabla p(t)\|_{L^\infty(U')} \le C\Big(1+\int_{t-1/2}^t\int_V |\nabla p|^{s_2}dxdt\Big)^{s_3/2}. \quad (5.40)$$

Thanks to (5.15) with $s=s_2$, we obtain (5.36). □

Combining (5.40) with Theorem 5.5, we have the following asymptotic estimates.

**Theorem 5.8.** (i) *If $A(\alpha)<\infty$ then*

$$\limsup_{t\to\infty}\|\nabla p(t)\|_{L^\infty(U')} \le C\Big(1+A(\alpha)^{\frac{1}{\alpha-a}}+\limsup_{t\to\infty}\|\Psi\|_{L^\alpha(U\times(t-1,t))}\Big)^{s_3(\kappa_5+\alpha/2)} \cdot\Big(1+\limsup_{t\to\infty}\int_{t-1}^t G_1(\tau)d\tau\Big)^{s_3/2}. \quad (5.41)$$

(ii) *If $\beta(\alpha)<\infty$ then there is $T>0$ such that when $t>T$ we have*

$$\|\nabla p(t)\|_{L^\infty(U')} \le C\Big(1+\beta(\alpha)^{\frac{1}{\alpha-2a}}+\sup_{[t-1,t]}A(\alpha,\cdot)^{\frac{1}{\alpha-a}}+\|\Psi\|_{L^\alpha(U\times(t-2,t))}\Big)^{s_3(\kappa_5+\alpha/2)} \cdot\Big(1+\int_{t-1}^t G_1(\tau)d\tau\Big)^{s_3/2}. \quad (5.42)$$

## 6 Interior estimates for time derivative of pressure

In this section, we estimate the $L^\infty$-norm of $p_t(x,t)$ for $t>0$. Let $q=p_t$. Then

$$\frac{\partial q}{\partial t} = \nabla\cdot\big(K(|\nabla p|)\nabla p\big)_t. \tag{6.1}$$

Using (6.1), we first derive a local-in-time estimate for $L^\infty$-norm of $p_t$.

**Proposition 6.1.** *Let $U'\Subset V\Subset U$. If $T_0\ge 0$, $T>0$ and $\theta\in(0,1)$, then*

$$\sup_{[T_0+\theta T, T_0+T]}\|p_t\|_{L^\infty(U')}\le C\lambda^{\frac{s_1}{2}}\big(1+(\theta T)^{-1}\big)^{\frac{s_1+1}{2}}\|p_t\|_{L^2(U\times(T_0,T_0+T))}, \tag{6.2}$$

*where $s_1$ and $\lambda=\lambda(T_0,T,\theta;V)$ are defined in Theorem 5.6, and constant $C>0$ is independent of $T_0$, $T$, and $\theta$.*

*Proof.* Without loss of generality, assume $T_0=0$. For $k\ge 0$, let $q^{(k)}=\max\{q-k,0\}$ and $S_k(t)=\{x\in U: q(x,t)>k\}$, and $\chi_k(x,t)$ be the characteristic function of set $\{(x,t)\in U\times(0,T): q(x,t)>k\}$. On $S_k(t)$, we have $(\nabla p)_t=\nabla q=\nabla q^{(k)}$.

Let $\zeta=\zeta(x,t)$ be the cut-off function on $U\times[0,T]$ satisfying $\zeta(\cdot,0)=0$ and $\zeta(\cdot,t)$ having compact support in $U$. We will use test function $q^{(k)}\zeta^2$, noting that $\nabla(q^{(k)}\zeta^2)=\zeta[\nabla(q^{(k)}\zeta)+q^{(k)}\nabla\zeta]$. Multiplying (6.1) by $q^{(k)}\zeta^2$ and integrating the resultant on $U$, we get

$$\begin{aligned}\frac12\frac{d}{dt}\int_U|q^{(k)}\zeta|^2dx &= \int_U|q^{(k)}|^2\zeta\zeta_t dx-\int_U(K(|\nabla p|))_t\nabla p\cdot[\nabla(q^{(k)}\zeta)+q^{(k)}\nabla\zeta]\zeta dx\\ &\quad-\int_U K(|\nabla p|)(\nabla p)_t\cdot[\nabla(q^{(k)}\zeta)+q^{(k)}\nabla\zeta]\zeta dx.\end{aligned}$$

Put $z=\zeta[\nabla(q^{(k)}\zeta)+q^{(k)}\nabla\zeta]$. We have

$$(\nabla p)_t\cdot z=\zeta\nabla q^{(k)}\cdot[\nabla(q^{(k)}\zeta)+q^{(k)}\nabla\zeta]=|\nabla(q^{(k)}\zeta)|^2-|q^{(k)}\nabla\zeta|^2.$$

Taking into account (2.12),

$$|(K(|\nabla p|))_t\nabla p\cdot z| \;=|K'(|\nabla p|)|\frac{|\nabla p\cdot\nabla p_t|}{|\nabla p|}|\nabla p\cdot z|\le aK(|\nabla p|)|\nabla q||z|.$$

Note that

$$\begin{aligned}|\nabla q||z|&=|\zeta\nabla\bar q^{(k)}||\nabla(q^{(k)}\zeta)+q^{(k)}\nabla\zeta|\le\big\{|\nabla(q^{(k)}\zeta)|+|q^{(k)}||\nabla\zeta|\big\}^2\\ &=|\nabla(q^{(k)}\zeta)|^2+2|q^{(k)}||\nabla\zeta||\nabla(q^{(k)}\zeta)|+|q^{(k)}\nabla\zeta|^2.\end{aligned}$$

and by Cauchy's inequality:

$$2|q^{(k)}\nabla\zeta||\nabla(q^{(k)}\zeta)|\le\frac{1-a}{2a}K(|\nabla p|)|\nabla(q^{(k)}\zeta)|^2+\frac{2a}{1-a}K(|\nabla p|)|\bar q^{(k)}\nabla\zeta|^2.$$

It follows that

$$\begin{aligned}&\frac{d}{dt}\int_U|q^{(k)}\zeta|^2dx+(1-a)\int_U K(|\nabla p|)|\nabla(q^{(k)}\zeta)|^2dx\\ &\qquad\le 2\int_U|q^{(k)}|^2\zeta|\zeta_t|dx+C\int_U K(|\nabla p|)|q^{(k)}\nabla\zeta|^2dx.\end{aligned}$$

Integrating this inequality from 0 to $T$, we obtain

$$\begin{aligned}
\max_{[0,T]}\int_U |q^{(k)}\zeta|^2 dx + \int_0^T\int_U K(|\nabla p|)|\nabla(q^{(k)}\zeta)|^2 dxdt\\
&\le C\Big[\int_0^T\int_U |q^{(k)}|^2\zeta|\zeta_t| dxdt + \int_0^T\int_U K(|\nabla p|)|q^{(k)}\nabla\zeta|^2 dxdt\Big]\\
&\le C\int_0^T\int_U |q^{(k)}|^2(\zeta|\zeta_t| + |\nabla\zeta|^2) dxdt.
\end{aligned}$$

The last inequality uses the fact that function $K(\cdot)$ is bounded above. Applying Lemma 2.2 to $q^{(k)}\zeta$ with $W = K(|\nabla p|)$ and $\varrho = \nu_2$ defined by (5.27), we have

$$\|q^{(k)}\zeta\|_{L^{\nu_2}(Q_T)} \le C\lambda^{1/\nu_2}\cdot\Big\{\max_{[0,T]}\int_U |q^{(k)}\zeta|^2 dx + \int_0^T\int_U K(|\nabla p|)|\nabla(q^{(k)}\zeta)|^2 dxdt\Big\}^{1/2}.$$

and hence,

$$\|q^{(k)}\zeta\|_{L^{\nu_2}(Q_T)} \le C\lambda^{1/\nu_2}\Big\{\int_0^T\int_U |q^{(k)}|^2(|\zeta_t|\zeta + |\nabla\zeta|^2) dxdt\Big\}^{1/2}. \tag{6.3}$$

This is similar to inequality (5.29). Then by following arguments of Theorem 5.6 applied for $q^{(k)}$ instead of $u_m^{(k)}$, we obtain (6.2). The proof is complete. □

The next theorem contains the estimates in terms of the initial and boundary data for all $t > 0$. Let

$$\kappa_6 = 1 + s_1 + \kappa_3(s_3 - 1),\quad \kappa_7 = (s_3 - 1)(\kappa_5 + 1) + 1,$$
$$\kappa_8 = (s_3 - 1)\kappa_5,\quad \kappa_9 = (s_3 - 1)(\kappa_5 + \alpha/2) + \alpha/2.$$

**Theorem 6.2.** *Let $U' \Subset U$. For $t \in (0, 2)$,*

$$\begin{aligned}
\|p_t(t)\|_{L^\infty(U')} \le Ct^{-\kappa_6/2}\Big(1 + \|\bar p_0\|_{L^\alpha}\Big)^{\kappa_7}\Big(1 + \int_U H(|\nabla p_0(x)|)dx\Big)^{1/2}\cdot\\
\cdot\Big(1 + [EnvA(\alpha, t)]^{\frac{1}{\alpha - a}} + \|\Psi\|_{L^\alpha(U\times(0,t))}\Big)^{\kappa_8}\Big(1 + \int_0^t G_3(\tau)d\tau\Big)^{s_3/2}.
\end{aligned}\tag{6.4}$$

*For $t \ge 2$,*

$$\begin{aligned}
\|p_t(t)\|_{L^\infty(U')} \le C(1 + \|\bar p_0\|_{L^\alpha})^{\kappa_9}\Big(1 + [EnvA(\alpha, t)]^{\frac{1}{\alpha - a}} + \|\Psi\|_{L^\alpha(U\times(t-2,t))}\Big)^{\kappa_9}\\
\cdot\Big(1 + \int_{t-1}^t G_3(\tau)d\tau\Big)^{s_3/2}.
\end{aligned}\tag{6.5}$$

*Proof.* Let $t \in (0, 2)$, apply (6.2) with $T_0 = 0$, $T = t$ and $\theta = 1/2$. By (5.22) and (5.3), we have

$$\lambda^{s_1/2} = \Big(\int_{t/2}^t\int_V (1 + |\nabla p|)^{\frac{as_0}{2 - s_0}} dxd\tau\Big)^{\frac{(2 - s_0)s_1}{2s_0}} \le \Big(\int_{t/2}^t\int_V (1 + |\nabla p|)^{s_2} dxd\tau\Big)^{\frac{\nu_4}{2}},$$

where

$$\nu_4 = (2 - s_0)s_1/s_0 = s_3 - 1. \tag{6.6}$$

By (5.13) and relation (5.39),

$$\lambda^{s_1/2} \le C\Big\{t\cdot t^{-(\kappa_3+1)}(1+\|\bar p_0\|_{L^\alpha})^{2(\kappa_5+1)}\Big(1+[EnvA(\alpha,t)]^{\frac{1}{\alpha-a}}+\|\Psi\|_{L^\alpha(U\times(0,t))}\Big)^{2\kappa_5}$$
$$\cdot\Big(1+\int_0^t G_1(\tau)d\tau\Big)\Big\}^{\nu_4/2} \le Ct^{\frac{-\kappa_3\nu_4}{2}}(1+\|\bar p_0\|_{L^\alpha})^{(\kappa_5+1)\nu_4}$$
$$\cdot\Big(1+[EnvA(\alpha,t)]^{\frac{1}{\alpha-a}}+\|\Psi\|_{L^\alpha(U\times(0,t))}\Big)^{\kappa_5\nu_4}\Big(1+\int_0^t G_1(\tau)d\tau\Big)^{\nu_4/2}.$$

Combining this with (6.2) and (3.9) gives

$$\|p_t(t)\|_{L^\infty(U')} \le C\lambda^{\frac{s_1}{2}}t^{-\frac{1+s_1}{2}}\|p_t\|_{L^2(U\times(0,t))}$$
$$\le Ct^{\frac{-\kappa_3\nu_4-(1+s_1)}{2}}(1+\|\bar p_0\|_{L^\alpha})^{(\kappa_5+1)\nu_4}\Big(1+[EnvA(\alpha,t)]^{\frac{1}{\alpha-a}}+\|\Psi\|_{L^\alpha(U\times(0,t))}\Big)^{\kappa_5\nu_4}$$
$$\cdot\Big(1+\int_0^t G_1(\tau)d\tau\Big)^{\nu_4/2}$$
$$\cdot\Big(\int_U[H(|\nabla p_0(x)|)+\bar p_0^2(x)]dx+\int_0^t G_3(\tau)d\tau+\int_0^t\int_U|\Psi_t(x,\tau)|^2dxd\tau\Big)^{1/2}.$$

Therefore,

$$\|p_t(t)\|_{L^\infty(U')} \le Ct^{-\kappa_6/2}\Big(1+\|\bar p_0\|_{L^\alpha}\Big)^{(\kappa_5+1)\nu_4+1}\Big(1+\int_U H(|\nabla p_0(x)|)dx\Big)^{1/2}.$$
$$\cdot\Big(1+[EnvA(\alpha,t)]^{\frac{1}{\alpha-a}}+\|\Psi\|_{L^\alpha(U\times(0,t))}\Big)^{\kappa_5\nu_4}\Big(1+\int_0^t G_3(\tau)d\tau\Big)^{(\nu_4+1)/2}.$$

We obtain (6.4).

Now consider $t\ge 2$. We apply (6.2) with $T_0=t-1$, $T=1$ and $\theta=1/2$. Then by (5.22) and (5.15),

$$\lambda^{s_1/2} \le C\Big(\int_{t-1}^t\int_V(1+|\nabla p|)^{s_2}dxd\tau\Big)^{\nu_4/2}$$
$$\le C(1+\|\bar p_0\|_{L^\alpha})^{\nu_4(\kappa_5+\alpha/2)}\Big(1+[EnvA(\alpha,t)]^{\frac{1}{\alpha-a}}+\|\Psi\|_{L^\alpha(U\times(t-2,t))}\Big)^{\nu_4(\kappa_5+\alpha/2)}$$
$$\cdot\Big(1+\int_{t-1}^t G_1(\tau)d\tau\Big)^{\nu_4/2}.$$

Combining this with (6.2) and (3.22) gives

$$\|p_t(t)\|_{L^\infty(U')} \le C\lambda^{\frac{s_1}{2}}\|p_t\|_{L^2(U\times(t-1/2,t))}$$
$$\le C(1+\|\bar p_0\|_{L^\alpha})^{\nu_4(\kappa_5+\alpha/2)}\Big(1+[EnvA(\alpha,t)]^{\frac{1}{\alpha-a}}+\|\Psi\|_{L^\alpha(U\times(t-2,t))}\Big)^{\nu_4(\kappa_5+\alpha/2)}$$
$$\cdot\Big(1+\int_{t-1}^t G_1(\tau)d\tau\Big)^{\nu_4/2}\cdot\Big(1+\int_U|\bar p_0(x)|^\alpha dx+[EnvA(\alpha,t)]^{\frac{\alpha}{\alpha-a}}+\int_{t-1}^t G_3(\tau)d\tau\Big)^{1/2}.$$

Thus,

$$\|p_t(t)\|_{L^\infty(U')} \le C(1+\|\bar p_0\|_{L^\alpha})^{\nu_4(\kappa_5+\alpha/2)+\alpha/2}$$
$$\cdot\Big(1+[EnvA(\alpha,t)]^{\frac{1}{\alpha-a}}+\|\Psi\|_{L^\alpha(U\times(t-2,t))}\Big)^{\nu_4(\kappa_5+\alpha/2)+\alpha/2}\Big(1+\int_{t-1}^t G_3(\tau)d\tau\Big)^{\nu_4/2+1/2},$$

and we obtain (6.5). The proof is complete. □

For large time or asymptotic estimates, we have:

**Theorem 6.3.** *Let $U' \Subset U$.*

(i) *If $A(\alpha) < \infty$ then*

$$\limsup_{t\to\infty} \|p_t(t)\|_{L^\infty(U')} \le C\Big(1 + A(\alpha)^{\frac{1}{\alpha-a}} + \limsup_{t\to\infty} \|\Psi\|_{L^\alpha(U\times(t-1,t))}\Big)^{\kappa_9} \cdot \Big(1 + \limsup_{t\to\infty} \int_{t-1}^t G_3(\tau)d\tau\Big)^{s_3/2}. \tag{6.7}$$

(ii) *If $\beta(\alpha) < \infty$ then there is $T > 0$ such that for all $t > T$,*

$$\|p_t(t)\|_{L^\infty(U')} \le C\Big(1 + \beta(\alpha)^{\frac{1}{\alpha-2a}} + \sup_{[t-1,t]} A(\alpha,\cdot)^{\frac{1}{\alpha-a}} + \|\Psi\|_{L^\alpha(U\times(t-2,t))}\Big)^{\kappa_9} \cdot \Big(1 + \int_{t-1}^t G_3(\tau)d\tau\Big)^{s_3/2}. \tag{6.8}$$

*Proof.* (i) For large $t$, we apply (6.2) with $T_0 = t-1$, $T = 1$ and $\theta = 1/2$. We have

$$\|p_t(t)\|_{L^\infty(U')} \le C\lambda^{s_1/2}\|p_t\|_{L^2(U\times(t-1/2,t))} \le C\Big(\int_{t-1}^t \int_V (1+|\nabla p|)^{s_2} dx d\tau\Big)^{\nu_4/2} \|p_t\|_{L^2(U\times(t-1/2,t))}, \tag{6.9}$$

where $\nu_4$ is defined by (6.6). Take the limit superior and using (5.16) with $s = s_2$, and (3.26), we obtain

$$\begin{aligned}
&\limsup_{t\to\infty} \|p_t(t)\|_{L^\infty(U')}\\
&\le C\Big(1 + A(\alpha)^{\frac{1}{\alpha-a}} + \limsup_{t\to\infty} \|\Psi\|_{L^\alpha(U\times(t-1,t))}\Big)^{\nu_4(\kappa_5+\alpha/2)} \Big(1 + \limsup_{t\to\infty} \int_{t-1}^t G_1(\tau)d\tau\Big)^{\nu_4/2}\\
&\quad \cdot \Big(1 + A(\alpha)^{\frac{\alpha}{\alpha-a}} + \limsup_{t\to\infty} \int_{t-1}^t \int_U |\Psi_t(x,\tau)|^2 dx d\tau + \limsup_{t\to\infty} \int_{t-1}^t G_3(\tau)d\tau\Big)^{1/2}\\
&\le C\Big(1 + A(\alpha)^{\frac{1}{\alpha-a}} + \limsup_{t\to\infty} \|\Psi\|_{L^\alpha(U\times(t-1,t))}\Big)^{\nu_4(\kappa_5+\alpha/2)+\alpha/2} \Big(1 + \limsup_{t\to\infty} \int_{t-1}^t G_3(\tau)d\tau\Big)^{\nu_4/2+1/2}.
\end{aligned}$$

We obtain (6.7).

(ii) Using (6.9), (5.17), and (3.30), we obtain

$$\begin{aligned}
\|p_t(t)\|_{L^\infty(U')} &\le C\Big(1 + \beta(\alpha)^{\frac{1}{\alpha-2a}} + \sup_{[t-1,t]} A(\alpha,\cdot)^{\frac{1}{\alpha-a}} + \|\Psi\|_{L^\alpha(U\times(t-2,t))}\Big)^{\nu_4(\kappa_5+\alpha/2)} \Big(1 + \int_{t-1}^t G_1(\tau)d\tau\Big)^{\nu_4/2}\\
&\quad \cdot \Big(1 + \beta(\alpha)^{\frac{\alpha}{\alpha-2a}} + A(\alpha,t-1)^{\frac{\alpha}{\alpha-a}} + \int_{t-1}^t \int_U |\Psi_t(x,\tau)|^2 dx d\tau + \int_{t-1}^t G_3(\tau)d\tau\Big)^{1/2}.
\end{aligned}$$

Therefore (6.8) follows. □

## 7 Interior estimates for pressure's Hessian

In this section, we estimate the $L^2$-norm of the Hessian $\nabla^2 p = (p_{x_i x_j})_{i,j=1,2,\ldots,n}$.

**Lemma 7.1.** *Let $U' \Subset V \Subset U$. For $t > 0$,*

$$\|\nabla^2 p(t)\|_{L^2(U')} \le C\Big(1 + \|\nabla p(t)\|_{L^\infty(V)}\Big)^a \Big(\int_U [|\nabla p|^{2-a} + |p_t|^2] dx\Big)^{1/2}. \tag{7.1}$$

*Proof.* From (5.2) of Lemma 5.2 with $\zeta = \zeta(x)$ being a cut-off function in space, we have

$$\begin{aligned}\frac{1}{2s+2}\frac{d}{dt}\int_U |\nabla p|^{2s+2}\zeta^2 dx + \frac{1-a}{2}\int_U K(|\nabla p|)|\nabla^2 p|^2|\nabla p|^{2s}\zeta^2 dx \\ \le C\int_U K(|\nabla p|)|\nabla p|^{2s+2}|\nabla\zeta|^2 dx.\end{aligned} \tag{7.2}$$

Clearly,

$$\frac{1}{2s+2}\frac{d}{dt}\int_U |\nabla p|^{2s+2}\zeta^2 dx = -\int_U p_t\nabla\cdot(|\nabla p|^{2s}\nabla p\zeta^2)dx. \tag{7.3}$$

Combining (7.2) and (7.3) with $s = 0$, we have

$$\int_U K(|\nabla p|)|\nabla^2 p|^2\zeta^2 dx \le C\int_U K(|\nabla p|)|\nabla p|^2|\nabla\zeta|^2 dx + C\int_U |p_t\nabla\cdot(\nabla p\zeta^2)|dx.$$

Since

$$\begin{aligned}|p_t\nabla\cdot(\nabla p\zeta^2)| &\le |p_t||\nabla^2 p|\zeta^2 + 2|p_t||\nabla p|\zeta|\nabla\zeta| \\ &\le 1/2K(|\nabla p|)|\nabla^2 p|^2\zeta^2 + CK(|\nabla p|)|\nabla p|^2|\nabla\zeta|^2 + C|p_t|^2K^{-1}(|\nabla p|)\zeta^2,\end{aligned}$$

and, by (2.11), $K^{-1}(\xi) \le C(1+\xi)^a$ we find that

$$\int_U K(|\nabla p|)|\nabla^2 p|^2\zeta^2 dx \le C\int_U K(|\nabla p|)|\nabla p|^2|\nabla\zeta|^2 dx + C\int_U |p_t|^2(1+|\nabla p|)^a\zeta^2 dx. \tag{7.4}$$

Constructing appropriate $\zeta$ in (7.4) with $\zeta \equiv 1$ on $U'$ and supp $\zeta \subset V$ we obtain

$$\int_{U'} K(|\nabla p|)|\nabla^2 p|^2 dx \le C(1+\|\nabla p\|_{L^\infty(V)})^a\int_V [|\nabla p|^{2-a} + |p_t|^2]dx. \tag{7.5}$$

For $x \in U'$,

$$K(|\nabla p|) \ge C(1+|\nabla p|)^{-a} \ge C((1+\|\nabla p\|_{L^\infty(V)})^{-a} \tag{7.6}$$

From (7.5), (7.6) and Young's inequality we obtain (7.1). □

Now, we combine Lemma 7.1 with estimates in section 3 and subsection 5.2 to obtain particular bounds for the Hessian. Let

$$\begin{aligned}&s_4 = as_3 + 1,\\ &\kappa_{10} = a\kappa_4 + 1, \quad \kappa_{11} = as_3(\kappa_5 + \alpha/2) + \alpha/2, \quad \kappa_{12} = as_3\kappa_5 + \alpha/2.\end{aligned}$$

**Theorem 7.2.** *Let $U' \Subset U$.*

(i) *If $t \in (0,2)$ then*

$$\begin{aligned}\|\nabla^2 p(t)\|_{L^2(U')} \le Ct^{-\kappa_{10}/2}(1+\|\bar p_0\|_{L^\alpha})^{\kappa_{11}}\Big(1+\int_U H(|\nabla p_0(x)|)dx\Big)^{1/2} \\ \cdot\Big(1+EnvA(\alpha,t)^{\frac{1}{\alpha-a}} + \|\Psi\|_{L^\alpha(U\times(0,t))}\Big)^{\kappa_{12}}\Big(1+\int_0^t G_4(\tau)d\tau\Big)^{s_4/2}.\end{aligned} \tag{7.7}$$

(ii) *If $t \ge 2$ then*

$$\begin{aligned}\|\nabla^2 p(t)\|_{L^2(U')} \le C(1+\|\bar p_0\|_{L^\alpha})^{\kappa_{11}}\Big(1+[EnvA(\alpha,t)]^{\frac{1}{\alpha-a}} + \|\Psi\|_{L^\alpha(U\times(t-2,t))}\Big)^{\kappa_{11}} \\ \cdot\Big(1+\int_{t-1}^t G_4(\tau)d\tau\Big)^{s_4/2}.\end{aligned} \tag{7.8}$$

(iii) *If $A(\alpha)<\infty$ then*

$$\limsup_{t\to\infty}\|\nabla^2 p\|_{L^2(U')}\le C\Big(1+A(\alpha)^{\frac{1}{\alpha-a}}+\limsup_{t\to\infty}\|\Psi\|_{L^\alpha(U\times(t-1,t))}\Big)^{\kappa_{11}} \cdot\Big(1+\limsup_{t\to\infty}\int_{t-1}^t G_4(\tau)d\tau\Big)^{s_4/2}. \tag{7.9}$$

(iv) *If $\beta(\alpha)<\infty$ then there is $T>0$ such that for $t>T$,*

$$\|\nabla^2 p(t)\|_{L^2(U')}\le C\Big(1+\beta(\alpha)^{\frac{1}{\alpha-2a}}+\sup_{[t-1,t]}A(\alpha,\cdot)^{\frac{1}{\alpha-a}}+\|\Psi\|_{L^\alpha(U\times(t-2,t))}\Big)^{\kappa_{11}} \cdot\Big(1+\int_{t-1}^t G_4(\tau)d\tau\Big)^{s_4/2}. \tag{7.10}$$

*Proof.* (i) Using (7.1), (5.35), (3.9) and (3.11):

$$\begin{aligned}\|\nabla^2 p(t)\|_{L^2(U')}&\le Ct^{-a\kappa_4/2}(1+\|\bar p_0\|_{L^\alpha})^{as_3(\kappa_5+1)}\Big(1+EnvA(\alpha,t)^{\frac{1}{\alpha-a}}+\|\Psi\|_{L^\alpha(U\times(0,t))}\Big)^{as_3\kappa_5}\\&\cdot\Big(1+\int_0^t G_1(\tau)d\tau\Big)^{as_3/2}\Big\{t^{-1}\Big(1+\int_U|\bar p_0(x)|^\alpha dx+\int_U H(|\nabla p_0(x)|)dx\\&+[EnvA(\alpha,t)]^{\frac{\alpha}{\alpha-a}}+\int_U|\Psi_t(x,t)|^2dx+\int_0^t G_4(\tau)d\tau\Big)\Big\}^{1/2}.\end{aligned}$$

Then (7.7) follows (noting that $\alpha\ge 2$).

(ii) Using (7.1), (5.36), (3.13), (3.14) and (3.5):

$$\begin{aligned}\|\nabla^2 p(t)\|_{L^2(U')}&\le C(1+\|\bar p_0\|_{L^\alpha})^{as_3(\kappa_5+\alpha/2)}\Big(1+[EnvA(\alpha,t)]^{\frac{1}{\alpha-a}}+\|\Psi\|_{L^\alpha(U\times(t-2,t))}\Big)^{as_3(\kappa_5+\alpha/2)}\\&\cdot\Big(1+\int_{t-1}^t G_1(\tau)d\tau\Big)^{as_3/2}\\&\cdot\Big(1+\int_U|\bar p_0(x)|^\alpha dx+[EnvA(\alpha,t-1)]^{\frac{\alpha}{\alpha-a}}+\int_U|\Psi_t(x,t-1)|^2dx+\int_{t-1}^t G_4(\tau)d\tau\Big)^{1/2}.\end{aligned}$$

Then (7.8) follows.

(iii) Using (7.1), (5.41), (3.23) and (3.24):

$$\begin{aligned}\limsup_{t\to\infty}\|\nabla^2 p\|_{L^2(U')}&\le C\Big(1+A(\alpha)^{\frac{1}{\alpha-a}}+\limsup_{t\to\infty}\|\Psi\|_{L^\alpha(U\times(t-1,t))}\Big)^{as_3(\kappa_5+\alpha/2)}\\&\cdot\Big(1+\limsup_{t\to\infty}\int_{t-1}^t G_1(\tau)d\tau\Big)^{as_3/2}\Big(1+A(\alpha)^{\frac{\alpha}{\alpha-a}}+\limsup_{t\to\infty}\int_{t-1}^t G_4(\tau)d\tau\Big)^{1/2}.\end{aligned}$$

Then (7.9) follows.

(iv) Using (7.1), (5.42), (3.27) and (3.28):

$$\begin{aligned}\|\nabla^2 p(t)\|_{L^2(U')}&\le C\Big(1+\beta(\alpha)^{\frac{1}{\alpha-2a}}+\sup_{[t-1,t]}A(\alpha,\cdot)^{\frac{1}{\alpha-a}}+\|\Psi\|_{L^\alpha(U\times(t-2,t))}\Big)^{as_3(\kappa_5+\alpha/2)}\\&\cdot\Big(1+\int_{t-1}^t G_1(\tau)d\tau\Big)^{as_3/2}\Big(1+\beta(\alpha)^{\frac{\alpha}{\alpha-2a}}+A(\alpha,t-1)^{\frac{\alpha}{\alpha-a}}+\int_{t-1}^t G_4(\tau)d\tau\Big)^{1/2}.\end{aligned}$$

Then (7.10) follows. □

# A Appendix

*Proof of Lemma 2.1.* First, we have the following Poincaré-Sobolev inequality

$$\|\phi\|_{L^{r^*}} \le C\|\nabla\phi\|_{L^r} + C\delta\|\phi\|_{L^1}. \tag{A.1}$$

Let $r = 2 - a$ and $\phi = |u|^m$, where $m = (\alpha - a)/(2 - a) \ge 1$. Applying (A.1) to $\phi$, we obtain

$$\|u\|_{L^{mr^*}} \le C\Big(\int_U |u|^{\alpha-2}|\nabla u|^{2-a}dx\Big)^{\frac{1}{\alpha-a}} + C\delta\|u\|_{L^m} \le C\Big(\int_U |u|^{\alpha-2}|\nabla u|^{2-a}dx\Big)^{\frac{1}{\alpha-a}} + C\delta\|u\|_{L^\alpha}. \tag{A.2}$$

The last inequality comes from Hölder's inequality and the fact $\alpha > m$. Consider a number $p \in (\alpha, mr^*)$. Let $\theta \in (0,1)$ such that

$$\frac{1}{p} = \frac{\theta}{\alpha} + \frac{1-\theta}{mr^*}. \tag{A.3}$$

Combining interpolation inequality and (A.2) yields

$$\begin{aligned}\|u\|^p_{L^p_x} &\le \|u\|^{\theta p}_{L^\alpha_x}\|u\|^{(1-\theta)p}_{L^{mr^*}_x} \le C\|u\|^{\theta p}_{L^\alpha_x}\Big\{\Big(\int_U |u|^{\alpha-2}|\nabla u|^{2-a}dx\Big)^{\frac{1}{\alpha-a}} + C\delta\|u\|_{L^\alpha_x}\Big\}^{(1-\theta)p}\\ &\le C\delta\|u\|^p_{L^\alpha_x} + \|u\|^{\theta p}_{L^\alpha_x}\Big(\int_U |u|^{\alpha-2}|\nabla u|^{2-a}dx\Big)^{\frac{(1-\theta)p}{\alpha-a}}.\end{aligned}$$

Selecting $(1-\theta)p = \alpha - a$ and integrating the preceding inequality from 0 to $T$ give

$$\begin{aligned}\int_0^T \|u(t)\|^p_{L^p_x}dt &\le C\delta T \operatorname*{ess\,sup}_{[0,T]} \|u\|^p_{L^\alpha_x} + C\operatorname*{ess\,sup}_{[0,T]} \|u\|^{\theta p}_{L^\alpha_x}\int_0^T\int_U |u|^{\alpha-2}|\nabla u|^{2-a}dxdt\\ &\le C\delta T[[u]]^p + C[[u]]^{\theta p}[[u]]^{(1-\theta)p} = C(1+\delta T)[[u]]^p.\end{aligned}$$

Therefore (2.17) follows. It remains to calculate $p$ and check sufficient conditions. Note that $\theta p = p - \alpha + a$, and from (A.3), we derive

$$1 = \frac{\theta p}{\alpha} + \frac{(1-\theta)p}{mr^*} = \frac{p-\alpha+a}{\alpha} + \frac{\alpha-a}{mr^*}.$$

Solving for $p$ from this gives $p = \alpha + (\alpha - a)(1 - \frac{\alpha}{mr^*})$. Simple calculations yield $p$ in the form of (2.16). It is elementary to verify that the second condition in (2.15) guarantees $\alpha < p < mr^*$.

In case $U$ is a ball $B_R(x_0)$, the inequality with (2.19) follows from the scaling $y = (x - x_0)/R$ for $x \in B_R(x_0)$ and (2.17) for $B_1(0)$. □

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