REGULARITY AND MULTIGRID ANALYSIS FOR LAPLACE-TYPE AXISYMMETRIC EQUATIONS

By

Hengguang Li
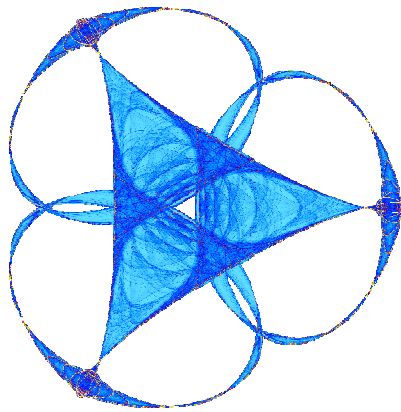
REGULARITY AND MULTIGRID ANALYSIS FOR LAPLACE-TYPE AXISYMMETRIC EQUATIONS

HENGGUANG LI

ABSTRACT. Consider axisymmetric equations associated with Laplace-type operators. We establish *full regularity* estimates in high-order Kondrat′ve-type spaces for possible singular solutions due to the non-smoothness of the domain and to the singular coefficients in the operator. Then, we show suitable graded meshes can be used in high-order finite element methods to achieve the optimal convergence rate, even when the solution is singular. Using these results, we further propose multigrid V-cycle algorithms solving the system from linear finite element discretizations on optimal graded meshes. We prove the multigrid algorithm is a contraction, with the contraction number independent of the mesh level. Numerical tests are provided to verify the theorem.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}_+^2 := \{(r, z), \ r > 0\}$ be a bounded polygonal domain in the rz-plane and the integer $\beta = 0$ or 1. Let Γ_0 be the interior part of $\partial\Omega \cap \{r = 0\}$ and $\Gamma_1 := \partial\Omega \backslash \Gamma_0$. See Figure 1 for example. We consider equations with parameter β,

$$(1) \qquad \mathcal{L}_\beta u_\beta := -(\partial_r^2 + r^{-1}\partial_r + \partial_z^2 - \beta r^{-2\beta})u_\beta = f_\beta \quad \text{in} \quad \Omega,$$

with the β-dependent boundary conditions

$$(2) \qquad\qquad u_0 = 0 \quad \text{on} \quad \Gamma_1, \qquad \partial_r u_0 = 0 \quad \text{on} \quad \Gamma_0;$$

$$(3) \qquad\qquad\qquad u_1 = 0 \quad \text{on} \quad \partial\Omega.$$

Equation (1) plays an important role in the study of various equations in 3D axisymmetric domains. For instance, denote by $\mathbb{R}^3 \ni \tilde{\Omega} := (\Omega \cup \Gamma_0) \times [0, 2\pi)$ the domain obtained by the rotation of Ω (meridian domain) about the z-axis (Figure 1). Recall the Cartesian coordinates (x, y, z) and the cylindrical coordinates (r, θ, z) on $\tilde{\Omega}$. For any $\tilde{v} \in L^2(\tilde{\Omega})$, its Fourier coefficients with respect to θ are defined by

$$(4) \qquad v_k(r, z) = \mathcal{F}_k(\tilde{v}) := \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \tilde{v}(r, \theta, z)e^{-ik\theta}d\theta, \qquad k \in \mathbb{Z}.$$

Let $\tilde{w}$ be the solution of the 3D Poisson equation

$$(5) \qquad\qquad -\Delta\tilde{w} = \tilde{f} \quad \text{in} \quad \tilde{\Omega}, \qquad \tilde{w} = 0 \quad \text{on} \quad \partial\tilde{\Omega}.$$

Then, the first three Fourier coefficients w_k, $|k| \leq 1$, of $\tilde{w}$ are given by equation (1) as follows: $w_0 = u_0$, given $f_0 = \mathcal{F}_0(\tilde{f})$ ($\beta = 0$); $w_1 = u_1$, given $f_1 = \mathcal{F}_1(\tilde{f})$ ($\beta = 1$); and $w_{-1} = u_1$, given $f_1 = \mathcal{F}_{-1}(\tilde{f})$ ($\beta = 1$). In particular, when the 3D data are axisymmetric, u_0 is the meridian trace of $\tilde{w}$ and $\mathcal{L}_1 u_1 = f_1$ is regarded as

FIGURE 1. The 3D axisymmetric domain $\tilde{\Omega}$ (left); the meridian polygonal domain Ω (right).

the azimuthal Stokes equation [21, 22]. We also mention that $\mathcal{L}_1$ coincides with the principal part of the azimuthal Maxwell operator [6, 15].

This dimension reduction from 3D to 2D has the potential for substantial computational savings in numerical approximations of equations on 3D axisymmetric domains. However, the coordinate transformation introduces new differential operators with singular coefficients as in (1) and new function spaces with singular or degenerate weights. The development of robust numerical methods for these equations calls for rigorous numerical analysis, which has recently drawn much attention from the scientific community. For example, [6] provided a comprehensive introduction on spectral methods for various axisymmetric equations. Under the assumption that the solution has sufficient regularity, finite element analysis for axisymmetric problems was discussed in [1, 5, 12, 21, 26]. In the case that the solution possesses singularities, the singular expansion of the solution for equation (1) was studied in [17, 27]. As the most relevant results, we mention the papers [23] and [15], both for equation (1) with $\beta = 0$. In [23], a *second-order* regularity estimate was established in a class of weighted (Kondrat′ev-type) spaces for singular solutions. Consequently, the author proposed new *linear* finite element methods approximating singular solutions in the optimal rate. In [15], the multigrid V-cycle algorithm was studied for finite element spaces augmented with non-polynomial functions. The result was derived only for solutions with required regularity.

This paper contains our systematic study for equation (1) on its regularity, finite element approximation, and multigrid analysis. Motivated by [18, 24, 22], we first introduce *high-order* Kondrat′ev-type weighted spaces (Definition 2.2) to handle possible singular solutions from the non-smoothness of the domain and from the singular coefficients. Using appropriate isomorphic mappings in weighted spaces, we prove the *full-regularity* estimates (Theorem 3.5) for singular solutions of equation (1). Based on this regularity result and properties of interpolation operators in weighted spaces, we give an explicit construction of nested graded meshes, such that the associated *high-order* finite element methods achieve the optimal convergence rate for singular solutions (Theorem 3.16). Then, we analyze the system from the linear finite element discretization on optimal graded meshes. With the growth rate of the condition numbers for nested subspaces (Lemma 4.5), we provide smoothing properties of the Richardson smoother and approximation properties of the numerical solution in various weighted spaces. This leads to our convergence result (Theorem 4.12) on the multigrid V-cycle algorithm for singular solutions of equation (1).

For equation (1), our high-order regularity result and optimal high-order finite element methods approximating singular solutions are new. These results generalize the results in [23] (low-order regularity and linear finite element results only for the case $\beta = 0$). The uniform convergence of the proposed multigrid method on graded meshes is also new for the axisymmetric equation (1), which recovers the classical multigrid result for elliptic equations with full-regularity solutions [7, 11]. We mention that high-order Fourier coefficients ($|k| \geq 2$) of $\tilde{w}$ in (5) are determined by equations similar to equation (1) with $\beta = 1$ [6]. Therefore, our analysis shall provide building blocks for approximating general 2D Fourier coefficients of the 3D problem (5). The multigrid method studied here shall be a crucial ingredient in developing efficient multigrid solves for more complex axisymmetric problems.

The rest of the paper is organized as follows. In Section 2, we introduce weighted Sobolev spaces for our problem. We also summarize estimates for functions in these spaces. In Section 3, we establish high-order regularity results in Kondrat′ev-type weighted spaces for the possible singular solution of equation (1). Then, we give high-order finite element methods approximating the singular solutions in the optimal rate. In Section 4, we first describe the multigrid V-cycle algorithm for equation (1). Then, we show the proposed multigrid algorithm converges uniformly, independent of the mesh level, even when the solution is singuar. In Section 5, we include multiple numerical tests for the multigrid method to verify the theory.

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2. Preliminaries and notation

We introduce appropriate weighted spaces to study equation (1). Useful estimates regarding these spaces will be collected in the second part of this section.

2.1. **Function spaces.** We first recall a class of weighted Sobolev spaces from [6].

Definition 2.1. (Type I weighted spaces). For an integer $m \geq 0$, define

$$L^2_1(\Omega) := \{v, \int_\Omega v^2 r dr dz < \infty\}, \quad H^m_1(\Omega) := \{v, \ \partial^\alpha_c v \in L^2_1(\Omega), \ |\alpha| \leq m\},$$

where the muti-index $\alpha = (\alpha_1, \alpha_2)$ is a pair of nonnegative integers, $|\alpha| = \alpha_1 + \alpha_2$, and $\partial^\alpha_c = \partial^{\alpha_1}_r \partial^{\alpha_2}_z$. The norms and the semi-norms for any $v \in H^m_1(\Omega)$ are

$$\|v\|^2_{H^m_1(\Omega)} := \sum_{|\alpha| \leq m} \int_\Omega (\partial^\alpha_c v)^2 r dr dz, \quad |v|^2_{H^m_1(\Omega)} := \sum_{|\alpha| = m} \int_\Omega (\partial^\alpha_c v)^2 r dr dz.$$

Furthermore, we define two spaces $H^m_+(\Omega)$ and $H^m_-(\Omega)$.

For $H^m_+(\Omega)$, if m is not even,

(6) $$H^m_+(\Omega) := \{v \in H^m_1(\Omega), \ \partial^{2i-1}_r v|_{\{r=0\}} = 0, \ 1 \leq i < \tfrac{m}{2}\},$$
$$\|v\|_{H^m_+(\Omega)} = \|v\|_{H^m_1(\Omega)};$$

if m is even, besides the condition in (6), we require $\int_\Omega (\partial^{m-1}_r v)^2 r^{-1} dr dz < \infty$ for any $v \in H^m_+(\Omega)$, and the corresponding norm is

$$\|v\|_{H^m_+(\Omega)} = \left(\|v\|^2_{H^m_1(\Omega)} + \int_\Omega (\partial^{m-1}_r v)^2 r^{-1} dr dz\right)^{1/2}.$$

For $H_-^m(\Omega)$, if m is not odd,

(7)
$$H_-^m(\Omega) := \{v \in H_1^m(\Omega), \ \partial_r^{2i}v|_{\{r=0\}} = 0, \ 0 \le i < \tfrac{m-1}{2}\}$$
$$\|v\|_{H_-^m(\Omega)} = \|v\|_{H_1^m(\Omega)};$$

if m is odd, besides the condition in (7), we require $\int_\Omega (\partial_r^{m-1}v)^2 r^{-1} dr dz < \infty$, for any $v \in H_-^m(\Omega)$, and the corresponding norm is

$$\|v\|_{H_-^m(\Omega)} = \left(\|v\|_{H_1^m(\Omega)}^2 + \int_\Omega (\partial_r^{m-1}v)^2 r^{-1} dr dz\right)^{1/2}.$$

Thus, we denote different subspaces:

$$H_{+,0}^1(\Omega) := H_+^1(\Omega) \cap \{v|_{\Gamma_1} = 0\}, \quad H_{-,0}^1(\Omega) := H_-^1(\Omega) \cap \{v|_{\partial\Omega} = 0\}.$$

We now introduce another type of weighted spaces for our analysis on singular solutions of equation (1).

Definition 2.2. (Type II weighted spaces). Let Q_i be the ith vertex of Ω and define the vertex set $\mathcal{Q} := \{Q_i\}_{i=1}^I$. Denote by $2L$ the smallest distance from a vertex to any disjoint edge of $\partial\Omega$. Let $B(x, r_0)$ be the ball centered at x with radius r_0. Let $\vartheta \in \mathcal{C}^\infty(\bar{\Omega}\backslash\mathcal{Q})$ be a function, such that $\vartheta = |x - Q_i|$ in $\mathcal{V}_i := \Omega \cap B(Q_i, L/2)$ and $\vartheta \ge L/2$ in $\Omega\backslash \cup_{i=1}^I \mathcal{V}_i$. Note that $\mathcal{V}_i$ and $\mathcal{V}_j$ are disjoint if $i \ne j$. Thus, we define for $\mu \in \mathbb{R}$ and for any open set $G \subset \Omega$,

$$\mathcal{K}_{\mu,1}^m(G) := \{v, \ \vartheta^{-\mu+|\alpha|}\partial_c^\alpha v \in L_1^2(G), \ |\alpha| \le m\}.$$

with the semi-norm and norm

$$|v|_{\mathcal{K}_{\mu,1}^m(G)}^2 := \sum_{|\alpha|=m} \|\vartheta^{m-\mu}\partial_c^\alpha v\|_{L_1^2(G)}^2, \quad \|v\|_{\mathcal{K}_{\mu,1}^m(G)}^2 := \sum_{l=0}^m |v|_{\mathcal{K}_{\mu,1}^l(G)}^2.$$

In addition, for $0 \le l \le m$, we define the subspaces of $\mathcal{K}_{\mu,1}^m(\Omega)$:

(8) $\mathcal{K}_{\mu,+}^m(\Omega) := \{v \in \mathcal{K}_{\mu,1}^m(\Omega), \ \int_\Omega \left(\partial_r^{2i-1}(\vartheta^{-\mu+l}v)\right)^2 r^{-1} dr dz < \infty, \ 1 \le i \le \tfrac{l}{2}\}$,

(9) $\mathcal{K}_{\mu,-}^m(\Omega) := \{v \in \mathcal{K}_{\mu,1}^m(\Omega), \ \int_\Omega \left(\partial_r^{2i}(\vartheta^{-\mu+l}v)\right)^2 r^{-1} dr dz < \infty, \ 0 \le i \le \tfrac{l-1}{2}\}$.

The corresponding norms are

$$\|v\|_{\mathcal{K}_{\mu,+}^m(\Omega)} = \left(\|v\|_{\mathcal{K}_{\mu,1}^m(\Omega)}^2 + \sum_{l \le m}\sum_{1 \le i \le \frac{l}{2}} \int_\Omega \left(\partial_r^{2i-1}(\vartheta^{-\mu+l}v)\right)^2 r^{-1} dr dz\right)^{1/2},$$

$$\|v\|_{\mathcal{K}_{\mu,-}^m(\Omega)} = \left(\|v\|_{\mathcal{K}_{\mu,1}^m(\Omega)}^2 + \sum_{l \le m}\sum_{0 \le i \le \frac{l-1}{2}} \int_\Omega \left(\partial_r^{2i}(\vartheta^{-\mu+l}v)\right)^2 r^{-1} dr dz\right)^{1/2}.$$

Recall the Fourier coefficients v_k of $\tilde{v}$ in (4). We then have the following isomorphism (Theorem II.3.1) [6].

Proposition 2.3. *For $m \ge 0$, $\tilde{v} \to (v_k)_{k\in\mathbb{Z}}$ defines an isomorphic mapping*

(10)
$$H^m(\tilde{\Omega}) \to \Pi_{k\in\mathbb{Z}}H_{(k)}^m(\Omega),$$

where the space $H_{(k)}^m(\Omega)$ is defined as follows. For $|k| > m - 1$,

$$\|w\|_{H_{(k)}^m(\Omega)}^2 = \|w\|_{H_1^m(\Omega)}^2 + |k|^{2m}\|r^{-m}w\|_{L_1^2(\Omega)}^2.$$

If $|k| \le m-1$, for $0 \le j \le |k|-1$,

$$H^m_{(k)}(\Omega) = H^m_+(\Omega) \cap \{\partial^j_r w|_{\Gamma_0} = 0\} \text{ and } \|w\|_{H^m_{(k)}(\Omega)} = \|w\|_{H^m_+(\Omega)} \quad \text{(even } k\text{)};$$

$$H^m_{(k)}(\Omega) = H^m_-(\Omega) \cap \{\partial^j_r w|_{\Gamma_0} = 0\} \text{ and } \|w\|_{H^m_{(k)}(\Omega)} = \|w\|_{H^m_-(\Omega)} \quad \text{(odd } k\text{)}.$$

Remark 2.4. For $\tilde{v} \in H^m(\tilde{\Omega})$, $H^m_{(k)}(\Omega)$ are natural spaces to study the traces of its Fourier coefficients on Ω. We shall show that the Kondrat'ev-type spaces $\mathcal{K}^m_{\mu,1}(\Omega)$ are the appropriate spaces to study the trace when $\tilde{v}$ only has reduced regularity. Note that for $m \ge 0$,

$$H^m_{(0)}(\Omega) = H^m_+(\Omega), \qquad H^m_{(\pm 1)}(\Omega) = H^m_-(\Omega).$$

Thus, we shall consider equation (1) in $H^m_+(\Omega)$ if $\beta = 0$ and in $H^m_-(\Omega)$ if $\beta = 1$. Inheriting from the 3D Poisson equation (5), we have the boundary condition $u_\beta|_{\Gamma_1} = 0$. The restrictions on u_β in type I weighted spaces lead to different boundary conditions on $\Gamma_0 := \partial\Omega\backslash\Gamma_1$ in (2) and (3). Since the Fourier coefficients may be complex even though the original function is real, we shall denote the complex extensions of all the spaces above without change of notation.

The Fourier transform (4) leads to the variational formulation for equation (1) [6]: Find $u_0 \in H^1_{+,0}(\Omega)$ for any $v_0 \in H^1_{+,0}(\Omega)$ (resp. $u_1 \in H^1_{-,0}(\Omega)$ for $v_1 \in H^1_{-,0}(\Omega)$), such that

$$(11) \qquad a_\beta(u_\beta, v_\beta) := \int_\Omega (\partial_r u_\beta \partial_r v_\beta + \partial_z u_\beta \partial_z v_\beta + \frac{\beta}{r^2} u_\beta v_\beta) r\,dr\,dz = \langle f_\beta, v_\beta \rangle,$$

where $f_0 \in H^1_{+,0}(\Omega)'$ (resp. $f_1 \in H^1_{-,0}(\Omega)'$), the dual space of $H^1_{+,0}(\Omega)$ (resp. $H^1_{-,0}(\Omega)$) with the pivot space $L^2_1(\Omega)$. For $f_\beta \in L^2_1(\Omega)$, the right hand side is the L^2_1-inner product. Then, we have the well-posedness result.

Corollary 2.5. *The variational formulation (11) defines a unique solution $u_0 \in H^1_{+,0}(\Omega)$ (resp. $u_1 \in H^1_{-,0}(\Omega)$) and there is a constant $C > 0$, independent of u_β and f_β, such that*

$$(12) \qquad \|u_0\|_{H^1_+(\Omega)} \le C\|f_0\|_{H^1_{+,0}(\Omega)'} \quad \text{and} \quad \|u_1\|_{H^1_-(\Omega)} \le C\|f_1\|_{H^1_{-,0}(\Omega)'}.$$

Proof. For $\beta = 0$, the well-posedness follows from the isomorphism (10) and the 3D Poincaré inequality; for $\beta = 1$, it follows from the fact that $a_\beta(\cdot,\cdot)$ is coercive and continuous in $H^1_{-,0}(\Omega)$. The estimates in (12) are immediate consequences of the well-posedness. $\square$

2.2. **Weighted estimates.** We now recall several estimates from [22]. These estimates give relations between various spaces and will be used to formulate our high-order regularity results in the next section.

We distinguish the vertices on and off the z-axis as follows. A vertex on the z-axis will be denoted by Q^z; A vertex away from the z-axis will be denoted by Q^r. Recall the neighborhood $\mathcal{V} := B(Q, L/2) \cap \Omega$ of the vertex Q. For a vertex Q^z, let $\Gamma_\mathcal{V} = \Gamma_0 \cap \bar{\mathcal{V}}$ and we denote by $\tilde{\mathcal{V}} := (\mathcal{V} \cup \Gamma_\mathcal{V}) \times [0, 2\pi) \subset \tilde{\Omega}$ the neighborhood of Q^z in $\tilde{\Omega}$. For an integer $l \ge 1$, $\mathcal{V}/l := B(Q, L/(2l)) \cap \Omega$ and $\tilde{\mathcal{V}}/l := (\mathcal{V}/l \cup \Gamma_{\mathcal{V}/l}) \times [0, 2\pi)$.

Throughout the paper, by H', we mean the dual space of H. As in Definition 2.1, we also use the multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ for a 3D domain, such that $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$ and $\partial^\alpha = \partial^{\alpha_1}_x \partial^{\alpha_2}_y \partial^{\alpha_3}_z$. The generic constant $C > 0$ in our analysis below may be different at different occurrences. It will depend on the computational

domain, but not on the functions involved in the estimates or the mesh level in the finite element algorithms.

The following lemma, derived as Lemmas 2.8 and 2.9 in [22], contains useful weighted estimates in usual Sobolev spaces in the 3D neighborhood $\tilde{\mathcal{V}}$ of a vertex Q^z. We refer to [22] for the detailed proof.

Lemma 2.6. *Let $\tilde{\mathcal{V}} \subset \tilde{\Omega}$ be the neighborhood of a vertex Q^z and ρ be the distance to Q^z. Suppose $\sum_{|\alpha| \leq m} \|\rho^{-a+|\alpha|} \partial^\alpha v\|_{L^2(\tilde{\mathcal{V}})}^2 < \infty$, for $m \geq 0$ and $a \in \mathbb{R}$. Then, for any $0 \leq l \leq m$,*

$$C\|\rho^{-a+l} v\|_{H^l(\tilde{\mathcal{V}})}^2 \leq \sum_{|\alpha| \leq m} \|\rho^{-a+|\alpha|} \partial^\alpha v\|_{L^2(\tilde{\mathcal{V}})}^2 \leq C \sum_{l \leq m} |\rho^{-a+l} v|_{H^l(\tilde{\mathcal{V}})}^2.$$

Recall the multi-index $\alpha = (\alpha_1, \alpha_2)$ and the notation ∂_c^α from Definition 2.1. Then, the following lemma concerns the connection between the two types of weighted spaces in the 2D neighborhood $\mathcal{V}$ of a vertex Q^z in the rz-plane (Lemmas 2.10 and 2.11 in [22]).

Lemma 2.7. *Let $\mathcal{V} \subset \Omega$ be the neighborhood of a vertex Q^z and ρ be the distance to Q^z. Suppose $v \in \mathcal{K}_{a,1}^m(\mathcal{V})$. Then, for $0 \leq l \leq m$ and $a \in \mathbb{R}$,*

$$C\|\rho^{-a+l} v\|_{H_1^l(\mathcal{V})}^2 \leq \|v\|_{\mathcal{K}_{a,1}^m(\mathcal{V})}^2 \leq C \sum_{l \leq m} |\rho^{-a+l} v|_{H_1^l(\mathcal{V})}^2.$$

The following two lemmas concern the local behavior of functions in weighted spaces near the vertices.

Lemma 2.8. *Recall ϑ in Definition 2.2. In the neighhood $\mathcal{V}$ of a vertex Q^r away from the z-axis, for any $v \in H_{+,0}^1(\Omega)$ (resp. $v \in H_{-,0}^1(\Omega)$), we have*

$$\|\vartheta^{-1} v\|_{L_1^2(\mathcal{V})} \leq C\|v\|_{H_+^1(\mathcal{V})}, \qquad (\text{resp. } \|\vartheta^{-1} v\|_{L_1^2(\mathcal{V})} \leq C\|v\|_{H_-^1(\mathcal{V})}).$$

Proof. On $\mathcal{V}$, both H_+^1 and H_-^1 (resp. L_1^2) are equivalent to the usual Sobolev space H^1 (resp. L^2), since r is bounded away from 0. Therefore, it suffices to show for any $v \in H^1(\mathcal{V}) \cap \{v|_{\Gamma_1} = 0\}$,

$$(13) \qquad\qquad \|\vartheta^{-1} v\|_{L^2(\mathcal{V})} \leq C\|v\|_{H^1(\mathcal{V})}.$$

However, the estimate in (13) is well known based on a local Poincaré inequality. See [20, 19, 25, 4]. □

Lemma 2.9. *Recall ϑ in Definition 2.2. Let $v \in H_0^1(\tilde{\Omega})$ and let $\tilde{\mathcal{V}} = (\mathcal{V} \cup \Gamma_{\mathcal{V}}) \times [0, 2\pi)$ be the 3D neighborhood of a vertex Q^z on the z-axis. Then,*

$$\|\vartheta^{-1} v\|_{L^2(\tilde{\mathcal{V}})} \leq C\|v\|_{H^1(\tilde{\mathcal{V}})}.$$

Proof. $\tilde{\mathcal{V}}$ can be characterized in the spherical coordinates (ρ, θ, ϕ) centered at Q^z:

$$\tilde{\mathcal{V}} = \{(\rho, \omega), \ 0 < \rho < L/2, \ \omega \in \omega_{Q^z}\},$$

where $\omega_{Q^z} \subset S^2$ is the polygonal domain on the unit sphere S^2. Then, for any $v \in H^1(\tilde{\mathcal{V}}) \cap \{v|_{\partial\tilde{\Omega}} = 0\}$,

$$|\nabla v|^2 = v_x^2 + v_y^2 + v_z^2 = v_\rho^2 + \frac{v_\phi^2}{\rho^2} + \frac{v_\theta^2}{\rho^2 \sin^2 \phi},$$

and

$$\int_{\omega_{Q^z}} v^2 dS \le C \int_{\omega_{Q^z}} (v_\phi^2 + \frac{v_\theta^2}{\sin^2 \phi}) \sin \phi d\theta d\phi,$$

which is just the Poincaré inequality on ω_{Q^z} and $dS = \sin \phi d\theta d\phi$ is the volume element on ω_{Q^z}. Thus, we obtain

$$\int_{\tilde{\mathcal{V}}} \frac{v^2}{\rho^2} dxdydz = \int_0^{L/2} \int_{\omega_{Q^z}} v^2 dS d\rho \le C \int_0^{L/2} \int_{\omega_{Q^z}} (v_\rho^2 + \frac{v_\phi^2}{\rho^2} + \frac{v_\theta^2}{\rho^2 \sin^2 \phi}) \rho^2 dS d\rho$$

$$= C \int_{\tilde{\mathcal{V}}} |\nabla v|^2 dxdydz,$$

which completes the proof. $\qquad\qquad\square$

3. Regularity and finite element analysis

The solution u_β for equation (1) is well defined in weighted spaces (Corollary 2.5). When the 3D domain $\tilde{\Omega}$ is smooth, using the isomorphism in (10), we will have full regularity estimates in type I weighted spaces. These estimates will not hold when Ω is a general polygonal domain, since the regularity of u_β is determined by the smoothness of Ω and by the operator $\mathcal{L}_\beta$. In this section, we show type II weighted spaces are suitable to study singular solutions. We derive high-order regularity estimates in these spaces for singular solutions and propose finite element algorithms approximating singular solutions in the optimal rate.

3.1. **Regularity analysis.** Before formulating our main regularity result (Theorem 3.5), we first need several local regality estimates. All these results are written for different values of the parameter β, namely, 0 or 1.

Lemma 3.1. *Near a vertex Q^r away from the z-axis, there exists $\eta_\beta > 0$, such that for any $0 \le a_\beta < \eta_\beta$, if $f_\beta \in \mathcal{K}^m_{a_\beta-1,1}(\mathcal{V})$, $m \ge 0$, the solution u_β of equation (11) satisfies*

$$(14) \qquad \|u_0\|_{\mathcal{K}^{m+2}_{a_0+1,1}(\mathcal{V}/2)} \le C(\|f_0\|_{\mathcal{K}^m_{a_0-1,1}(\mathcal{V})} + \|f_0\|_{H^1_{+,0}(\Omega)'})$$

$$(15) \qquad \|u_1\|_{\mathcal{K}^{m+2}_{a_1+1,1}(\mathcal{V}/2)} \le C(\|f_1\|_{\mathcal{K}^m_{a_1-1,1}(\mathcal{V})} + \|f_1\|_{H^1_{-,0}(\Omega)'}).$$

Proof. We only show (14), since the proof of (15) follows similarly. We apply a localization argument. Let ζ be a smooth cutoff function, such that $\zeta = 1$ on $\mathcal{V}/2$ and $\zeta = 0$ outside $\mathcal{V}$. Thus, ζu_0 has the Dirichlet boundary condition on $\mathcal{V}$. Then,

$$(16) \qquad -(\partial_r^2 + r^{-1}\partial_r + \partial_z^2)\zeta u_0 = \zeta f_0 + g_0 \quad \text{in } \mathcal{V},$$

where

$$g_0 = u_0(\partial_r^2 + \partial_z^2)\zeta + 2(\partial_r \zeta \partial_r u_0 + \partial_z \zeta \partial_z u_0) + r^{-1} u_0 \partial_r \zeta.$$

Note that the support of g_0 is within $\mathcal{V} \backslash \mathcal{V}/2$ and $\mathcal{K}^m_{\mu,1} = H^m$ on $\mathcal{V} \backslash \mathcal{V}/2$. Therefore, using the interior regularity estimate in H^m [14] and (12), we have

$$\|g_0\|_{\mathcal{K}^m_{a_0-1,1}(\mathcal{V})} = \|g_0\|_{\mathcal{K}^m_{a_0-1,1}(\mathcal{V} \backslash \mathcal{V}/2)} \le C\|g_0\|_{H^m(\mathcal{V} \backslash \mathcal{V}/2)}$$

$$\le C\|u_0\|_{H^{m+1}(\mathcal{V} \backslash \mathcal{V}/2)} \le C(\|f_0\|_{H^m(\mathcal{V} \backslash \mathcal{V}/4)} + \|f_0\|_{H^1_{+,0}(\Omega)'})$$

$$\le C(\|f_0\|_{\mathcal{K}^m_{a_0-1,1}(\mathcal{V} \backslash \mathcal{V}/4)} + \|f_0\|_{H^1_{+,0}(\Omega)'})$$

By Corollary 2.5, equation (16) has a unique solution $\zeta u_0 \in H^1_+(\mathcal{V})$. Since r is bounded away from 0 on $\mathcal{V}$, $\zeta u_0 \in H^1(\mathcal{V})$ and the space $\mathcal{K}^m_{\mu,1}(\mathcal{V})$ is the same as the

Kondrat′ev space $\mathcal{K}_\mu^m(\mathcal{V})$ (Definition 1.1 in [3]). In addition, the regularity of ζu_0 is determined by the principle part $-\partial_r^2 - \partial_z^2$ of the operator in (16). Then, based on the 2D regularity estimates in the Kondrat′ev spaces for the Laplace operator [18, 24, 16], for the solution $\zeta u_0 \in H^1(\mathcal{V})$, there exists $\eta_0 > 0$, such that for any $0 \le a_0 < \eta_0$,

$$\|\zeta u_0\|_{\mathcal{K}_{a_0+1,1}^{m+2}(\mathcal{V})} \le C\|\zeta f_0 + g_0\|_{\mathcal{K}_{a_0-1,1}^m(\mathcal{V})} \le C(\|\zeta f_0\|_{\mathcal{K}_{a_0-1,1}^m(\mathcal{V})} + \|g_0\|_{\mathcal{K}_{a_0-1,1}^m(\mathcal{V})})$$
$$\le C(\|f_0\|_{\mathcal{K}_{a_0-1,1}^m(\mathcal{V})} + \|f_0\|_{H_{+,0}^1(\Omega)'}).$$

The lemma is thus proved due to the definition of the function ζ.   □

Remark 3.2. Near the vertex Q^r, since r is away from 0, the principle part of the operator $\mathcal{L}_\beta$ coincides with the Laplace operator $-\partial_r^2 - \partial_z^2$. Therefore, the regularity index η_β is the same for $\beta = 0$ and $\beta = 1$. Let ω be the interior angle at Q^r. Then,

(17) $$\eta_0 = \eta_1 = \pi/\omega,$$

which is the first eigenvalue of the operator pencil associated with the Laplace operator [16, 18, 19, 24].

Define $\tilde{u}_\beta(x,y,z) := \frac{1}{\sqrt{2\pi}} u_\beta(r,z) e^{i\beta\theta}$ and $\tilde{f}_\beta(x,y,z) := \frac{1}{\sqrt{2\pi}} f_\beta(r,z) e^{i\beta\theta}$. Then, we see that, by the change of variables, $\tilde{u}_\beta$ satisfies the 3D Poisson equation,

(18) $$-\Delta\tilde{u}_\beta = \tilde{f}_\beta \quad \text{in} \quad \tilde{\Omega} \qquad \tilde{u}_\beta = 0 \quad \text{on} \quad \partial\tilde{\Omega}, \qquad \beta = 0,1.$$

In addition, its Fourier coefficients

$$\mathcal{F}_k(\tilde{u}_\beta) = \begin{cases} u_\beta & \text{if} \quad k = \beta, \\ 0 & \text{if} \quad k \ne \beta. \end{cases}$$

We now give a regularity estimate for $\tilde{u}_\beta$ near a vertex on the z-axis.

Lemma 3.3. *For a vertex Q^z on the z-axis, there is $\eta_\beta > 0$ for all $m \ge 0$, such that for any $0 \le a_\beta < \eta_\beta$, the solution $\tilde{u}_\beta$ of equation (18) satisfies*

$$\Big(\sum_{|\alpha| \le m+2} \|\vartheta^{-a_\beta-1+|\alpha|}\partial^\alpha\tilde{u}_\beta\|_{L^2(\tilde{\mathcal{V}}/2)}^2 \Big)^{1/2}$$
$$\le C\Big(\Big(\sum_{|\alpha| \le m} \|\vartheta^{-a_\beta+1+|\alpha|}\partial^\alpha\tilde{f}_\beta\|_{L^2(\tilde{\mathcal{V}})}^2 \Big)^{1/2} + \|\tilde{f}_\beta\|_{H^{-1}(\tilde{\Omega})}\Big).$$

Proof. We use a localization augment similar to the one in Lemma 3.1. Let ζ be a smooth cutoff function, such that $\zeta = 1$ on $\tilde{\mathcal{V}}/2$ and $\zeta = 0$ outside $\tilde{\mathcal{V}}$. Then,

(19) $$-\Delta\zeta\tilde{u}_\beta = \zeta\tilde{f}_\beta + \tilde{g}_\beta,$$

where

$$\tilde{g}_\beta = \tilde{u}_\beta\Delta\zeta + 2\partial_x\tilde{u}_\beta\partial_x\zeta + 2\partial_y\tilde{u}_\beta\partial_y\zeta + 2\partial_z\tilde{u}_\beta\partial_z\zeta.$$

Since $\tilde{g}_\beta$ vanishes near Q^z, using the well-posedness of the Poisson problem (18), the usual interior regularity estimate [14], and the expression of $\tilde{g}_\beta$ above, we have

$$\Big(\sum_{|\alpha| \le m} \|\vartheta^{-a_\beta+1+|\alpha|}\partial^\alpha\tilde{g}_\beta\|_{L^2(\tilde{\mathcal{V}})}^2 \Big)^{1/2} \le C(\|\tilde{u}_\beta\|_{H^{m+1}(\tilde{\mathcal{V}}\backslash\tilde{\mathcal{V}}/2)})$$
$$\le C(\|\tilde{f}_\beta\|_{H^{m-1}(\tilde{\mathcal{V}}\backslash\tilde{\mathcal{V}}/4)} + \|\tilde{f}_\beta\|_{H^{-1}(\tilde{\Omega})})$$
(20) $$\le C\Big(\Big(\sum_{|\alpha| \le m} \|\vartheta^{-a_\beta+1+|\alpha|}\partial^\alpha\tilde{f}_\beta\|_{L^2(\tilde{\mathcal{V}})}^2 \Big)^{1/2} + \|\tilde{f}_\beta\|_{H^{-1}(\tilde{\Omega})}\Big).$$

Based on the 3D regularity estimates in the Kondrat$'$ev spaces for equation (19) (Theorem 1.2 in [2]) and (20), there exists $\eta_\beta > 0$, such that for any $0 \leq a_\beta < \eta_\beta$, the solution $\zeta \tilde{u}_\beta$ satisfies

$$
(\sum_{|\alpha| \leq m+2} \|\vartheta^{-a_\beta - 1 + |\alpha|} \partial^\alpha (\zeta \tilde{u}_\beta)\|^2_{L^2(\tilde{\mathcal{V}})})^{1/2}
$$

$$
\leq C (\sum_{|\alpha| \leq m} \|\vartheta^{-a_\beta + 1 + |\alpha|} \partial^\alpha (\zeta \tilde{f}_\beta + \tilde{g}_\beta)\|^2_{L^2(\tilde{\mathcal{V}})})^{1/2}
$$

$$
\leq C ((\sum_{|\alpha| \leq m} \|\vartheta^{-a_\beta + 1 + |\alpha|} \partial^\alpha (\zeta \tilde{f}_\beta)\|^2_{L^2(\tilde{\mathcal{V}})})^{1/2} + (\sum_{|\alpha| \leq m} \|\vartheta^{-a_\beta + 1 + |\alpha|} \partial^\alpha \tilde{g}_\beta\|^2_{L^2(\tilde{\mathcal{V}})})^{1/2})
$$

$$
\leq C ((\sum_{|\alpha| \leq m} \|\vartheta^{-a_\beta + 1 + |\alpha|} \partial^\alpha \tilde{f}_\beta\|^2_{L^2(\tilde{\mathcal{V}})})^{1/2} + \|\tilde{f}_\beta\|_{H^{-1}(\bar{\Omega})}).
$$

The lemma is thus proved due to the definition of ζ. $\qquad\square$

Remark 3.4. Different from the situation in Lemma 3.1 and Remak 3.2, near the vertex Q^z, the index η_β is determined by the operator pencil of the 3D Laplace operator in a bounded region on the unit sphere [16, 18]. To be more precise, in the neighborhood $\tilde{\mathcal{V}}$ of Q^z, let ω be the projection of $\tilde{\mathcal{V}}$ on the unit sphere. We write equation (18) in the polar coordinations, $-\Delta \tilde{u}_\beta = -\rho^{-2}((\rho \partial_\rho)^2 + \rho \partial_\rho + \Delta')\tilde{u}_\beta$, where Δ' is the Laplace-Beltrami operator on ω,

$$
\Delta' = \cot \phi \partial_\phi + \partial_\phi^2 + \sin^{-2} \phi \partial_\theta^2.
$$

For any $v(r,z) \in L_1^2(\mathcal{V})$, define $\tilde{v}_\beta := \frac{1}{\sqrt{2\pi}} v e^{i\beta\theta}$. The smallest eigenvalue λ_β of Δ' with respect to $\tilde{v}_\beta$ is given by

$$
\lambda_0 \tilde{v}_0 = -\Delta' \tilde{v}_0 = -(\cot \phi \partial_\phi + \partial_\phi^2)\tilde{v}_0,
$$

$$
\lambda_1 \tilde{v}_1 = -\Delta' \tilde{v}_1 = -(\cot \phi \partial_\phi + \partial_\phi^2 - \sin^{-2} \phi)\tilde{v}_1,
$$

which is determined by the associated Legendre's differential equation [13, 27]. Then, the explicit formula for the index η_β near Q^z is

(21) $$\eta_\beta = \sqrt{\lambda_\beta + 1/4}.$$

In contrast to the case near Q^r in (17), η_0 and η_1 may be different near Q^z, since λ_β are eigenvalues associated with different differential operators.

Combining the local estimates in the lemmas above, we derive the global regularity estimate for equation (1).

Theorem 3.5. *Let $u_0 \in H^1_{+,0}(\Omega)$ and $u_1 \in H^1_{-,0}(\Omega)$ be the solutions of equation (1) with $\beta = 0$ and 1, respectively. For $m \geq 0$, there exist $\Upsilon_0 > 0$ and $\Upsilon_1 > 0$, such that for any $0 \leq a_0 < \Upsilon_0$ and $0 \leq a_1 < \Upsilon_1$, we have*

(22) $$\|u_0\|_{\mathcal{K}^{m+2}_{a_0+1,1}(\Omega)} \leq C\|f_0\|_{\mathcal{K}^m_{a_0-1,+}(\Omega)},$$

(23) $$\|\vartheta^{-a_1} r^{-1} u_1\|_{L_1^2(\Omega)} + \|u_1\|_{\mathcal{K}^{m+2}_{a_1+1,1}(\Omega)} \leq C\|f_1\|_{\mathcal{K}^m_{a_1-1,-}(\Omega)},$$

given that $f_0 \in \mathcal{K}^m_{a-1,+}(\Omega)$ and $f_1 \in \mathcal{K}^m_{a-1,-}(\Omega)$.

Proof. Let η_β^i be the upper bound of the parameter a_β for the ith vertex Q_i in Lemmas 3.1 and 3.3. Define $\Upsilon_0 = \min_i(\eta_0^i)$ and $\Upsilon_1 = \min_i(\eta_1^i)$. We only show (23), since the proof for (22) follows similarly.

Recall that the weighted space $\mathcal{K}_{a,1}^m$ (resp., $\mathcal{K}_{a,+}^m$ and $\mathcal{K}_{a,-}^m$) is equivalent to the weighted space H_1^m (resp., H_+^m and H_-^m) in a subdomain $\Omega_{sub} \subset \Omega$ that is away from the vertex set. Based on the isomorphism in (10), the well-posedness and the usual interior regularity estimate for the 3D Poisson equation (18), we have

$$\|\vartheta^{-a_1}r^{-1}u_1\|_{L_1^2(\Omega_{sub})} + \|u_1\|_{\mathcal{K}_{a_1+1,1}^{m+2}(\Omega_{sub})} \leq C\|\tilde{f}_1\|_{H^m(\tilde{\Omega}')} + \|\tilde{f}_1\|_{H^{-1}(\tilde{\Omega})}$$

$$(24) \qquad\qquad\qquad \leq C(\|f_1\|_{\mathcal{K}_{a_1-1,-}^m(\Omega')} + \|f_1\|_{\mathcal{K}_{a_1-1,-}^0(\Omega)}),$$

where $\Omega_{sub} \subset \Omega' \subset \Omega$ such that Ω' is away from the vertex set, and $\tilde{\Omega}' = \big(\Omega' \cup (\bar{\Omega}' \cap \{r = 0\})\big) \times [0, 2\pi)$ is from the rotation of Ω' about the z-axis.

Let $\mathcal{V}$ be the neighborhood of a vertex Q^r away from the z-axis. By Lemma 3.1, the fact that r is bounded away from 0 on $\mathcal{V}$, Lemma 2.8, and $a_1 \geq 0$, we have

$$\|\vartheta^{-a_1}r^{-1}u_1\|_{L_1^2(\mathcal{V}/2)} + \|u_1\|_{\mathcal{K}_{a_1+1,1}^{m+2}(\mathcal{V}/2)} \leq C(\|f_1\|_{\mathcal{K}_{a_1-1,1}^m(\mathcal{V})} + \|f_1\|_{H_{-,0}^1(\Omega)'})$$

$$= C(\|f_1\|_{\mathcal{K}_{a_1-1,1}^m(\mathcal{V})} + \sup_{v \in H_{-,0}^1(\Omega), v \neq 0} (f_1, v)_{L_1^2}/\|v\|_{H_{-,0}^1(\Omega)})$$

$$(25) \qquad\qquad\qquad \leq C(\|f_1\|_{\mathcal{K}_{a_1-1,-}^m(\mathcal{V})} + \|f_1\|_{\mathcal{K}_{a_1-1,-}^0(\Omega)}).$$

Now, let $\mathcal{V}$ be the small neighborhood of a vertex Q^z on the z-axis. By Lemma 2.7, we first have for any $0 \leq l \leq m$,

$$(26) \qquad\qquad \|\vartheta^{1-a_1+l}f_1\|_{H_1^l(\mathcal{V})} \leq C\|f_1\|_{\mathcal{K}_{a_1-1,1}^m(\mathcal{V})}.$$

Then, for $f_1 \in \mathcal{K}_{a_1-1,-}^m(\mathcal{V})$, (26) and the condition in (9)

$$\int_\Omega \big(\partial_r^{2i}(\vartheta^{1-a_1+l}f_1)\big)^2 r^{-1}drdz < \infty, \quad 0 \leq i \leq (l-1)/2$$

lead to $\vartheta^{1-a_1+l}f_1 \in H_-^l(\mathcal{V})$. Then, by Lemma 2.6, the isomorphism in (10), and the definitions of the weighted spaces in (7) and (9), we have

$$\sum_{|\alpha| \leq m} \|\vartheta^{1-a_1+|\alpha|}\partial^\alpha \tilde{f}_1\|_{L^2(\tilde{\mathcal{V}})}^2 \leq C \sum_{0 \leq l \leq m} \|\vartheta^{1-a_1+l}\tilde{f}_1\|_{H^l(\tilde{\mathcal{V}})}^2$$

$$(27) \qquad \leq C \sum_{0 \leq l \leq m} \|\vartheta^{1-a_1+l}f_1\|_{H_-^l(\mathcal{V})}^2 \leq C\|f_1\|_{\mathcal{K}_{a_1-1,-}^m(\mathcal{V})}^2.$$

Then, by Lemma 2.7, the isomorphism in (10), Lemma 2.6, Lemma 3.3, Lemma 2.9, (27), and $a_1 \geq 0$, we have

$$\|\vartheta^{-a_1}r^{-1}u_1\|_{L_1^2(\mathcal{V}/2)} + \|u_1\|_{\mathcal{K}_{a_1+1,1}^{m+2}(\mathcal{V}/2)}$$

$$\leq C\big(\|\vartheta^{-a_1}r^{-1}u_1\|_{L_1^2(\mathcal{V}/2)} + \big(\sum_{l \leq m+2} \|\vartheta^{-a_1-1+l}u_1\|_{H_1^l(\mathcal{V}/2)}^2\big)^{1/2}\big)$$

$$\leq C\big(\sum_{l \leq m+2} \|\vartheta^{-a_1-1+l}\tilde{u}_1\|_{H^l(\tilde{\mathcal{V}}/2)}^2\big)^{1/2}$$

$$\leq C\big(\big(\sum_{|\alpha| \leq m} \|\vartheta^{-a_1+1+|\alpha|}\partial^\alpha \tilde{f}_1\|_{L^2(\tilde{\mathcal{V}})}^2\big)^{1/2} + \sup_{v \in H_0^1(\tilde{\Omega}), v \neq 0} (\tilde{f}_1, v)_{L^2(\tilde{\Omega})}/\|v\|_{H^1(\tilde{\Omega})}\big)$$

$$\leq C\big(\big(\sum_{|\alpha| \leq m} \|\vartheta^{-a_1+1+|\alpha|}\partial^\alpha \tilde{f}_1\|_{L^2(\tilde{\mathcal{V}})}^2\big)^{1/2} + \|\vartheta^{-a_1+1}\tilde{f}_1\|_{L^2(\tilde{\Omega})}\big)$$

$$(28) \quad \leq C(\|f_1\|_{\mathcal{K}_{a_1-1,-}^m(\mathcal{V})} + \|f_1\|_{\mathcal{K}_{a_1-1,-}^0(\Omega)}).$$

Then, the proof is completed by combining (24), (25), and (28). $\qquad\qquad \square$

Remark 3.6. We used the global parameter Υ_β in Theorem 3.5 in order to simplify the exposition. The proof in fact shows that Theorem 3.5 holds as long as for the ith vertex Q_i, we choose the parameter $0 \le a_\beta < \eta_\beta^i$, where η_β^i is determined as in Remarks 3.2 and 3.4. This will result in the analogue of Theorem 3.5 in Kondrat′ev-type spaces with vector subindices, in which the function ϑ is allowed to have different exponents near different vertices to capture the local behavior of the solution. See [24] for such a presentation for elliptic equations.

3.2. **Finite element analysis.** Let $\mathcal{T}_n = \{T_i\}$ be a triangulation of Ω with shape-regular triangles T_i. Denote by $\mathbb{V}_n^\beta$ the continuous Lagrange finite element space of order m with the corresponding Dirichlet boundary conditions described in (2) and (3), respectively. Then, we have [21]

$$\text{(29)} \qquad \mathbb{V}_n^0 \subset H_{+,0}^1(\Omega) \quad \text{and} \quad \mathbb{V}_n^1 \subset H_{-,0}^1(\Omega).$$

In turn, the finite element solution $u_n^\beta \in \mathbb{V}_n^\beta$ for equation (11) is defined by

$$\text{(30)} \qquad a_\beta(u_n^\beta, v_n^\beta) = \int_\Omega f_\beta v_n^\beta r\,dr\,dz, \qquad \forall\, v_n^\beta \in \mathbb{V}_n^\beta.$$

We denote a point in the rz-plane by $\mathbf{x} = (r, z)$. Let $\Pi : \mathcal{C}^0(\Omega) \to \mathbb{V}_k^\beta$ be the usual nodal interpolation operator, such that $\Pi v(\mathbf{x}_i) = v(\mathbf{x}_i)$, where $\mathbf{x}_i$ is the ith node in the triangulation. Note that we use the same notation for the interpolation operators into different spaces $\mathbb{V}_n^\beta$, $\beta = 0$ and 1. In the text below, by a triangle T, we mean the closed set including both the interior and boundary of the triangle.

We first recall two approximation results from Lemmas 6.1, 6.2 and 6.3 in [26], which are based on embedding theorems in weighted spaces.

Lemma 3.7. *Let $\mathbf{x}_i$ be a node in the triangulation, which is on the z-axis. Let K_i be the union of triangles that contain $\mathbf{x}_i$. Then, for $v \in H_1^{m+1}(K_i)$, we have*

$$\text{(31)} \qquad |v - \Pi v|_{H_1^1(K_i)} \le Ch_{K_i}^m |v|_{H_1^{m+1}(K_i)};$$

for $v \in H_1^{m+1}(K_i) \cap H_-^1(\Omega)$,

$$\text{(32)} \qquad \|r^{-1}(v - \Pi v)\|_{L_1^2(K_i)} \le Ch_{K_i}^m |v|_{H_1^{m+1}(K_i)},$$

where h_{K_i} denotes the diameter of K_i and $m \ge 1$.

Lemma 3.8. *Let $T \in \mathcal{T}_n$ be a triangle that does not intersect the z-axis. Suppose $\min_{\mathbf{x} \in T} r(\mathbf{x}) \ge Ch_T$. Then, for $v \in H_1^{m+1}(T)$, we have*

$$\text{(33)} \qquad |v - \Pi v|_{H_1^1(T)} \le Ch_T^m |v|_{H_1^{m+1}(T)},$$

$$\text{(34)} \qquad \|r^{-1}(v - \Pi v)\|_{L_1^2(T)} \le Ch_T^m |v|_{H_1^{m+1}(T)},$$

where h_T denotes the diameter of T and $m \ge 1$.

Remark 3.9. By Lemma 3.7 and Lemma 3.8, one can derive the convergence result for the finite element solution in (30) on quasi-uniform meshes

$$\|u_0 - u_n^0\|_{H_+^1(\Omega)} \le Ch^m \|u_0\|_{H_+^{m+1}(\Omega)} \quad \text{and} \quad \|u_1 - u_n^1\|_{H_-^1(\Omega)} \le Ch^m \|u_1\|_{H_-^{m+1}(\Omega)},$$

given that u_β is sufficiently regular and h is the mesh size. As discussed above, the regularity of the solution, however, depends on the differential operator $\mathcal{L}_\beta$ and the domain Ω. When the solution u_β is singular (e.g., $u_\beta \notin H_1^2(\Omega)$), the linear finite element method on the quasi-uniform mesh only gives a suboptimal convergence rate [6].

Based on our regularity estimates in Theorem 3.5 and Remark 3.6, we use the following process to generate a sequence of graded triangulations approximating the singular solutions of equation (1).

Definition 3.10. (The κ-refinement). Let $\kappa \in (0, 1/2]$ be the grading parameter and $\mathcal{T}$ be a triangulation of Ω such that no two vertices of Ω belong to the same triangle of $\mathcal{T}$. Then the κ-refinement of $\mathcal{T}$, denoted by $\kappa(\mathcal{T})$, is obtained by dividing each edge AB of $\mathcal{T}$ into two parts as follows. If neither A nor B is in the vertex set $\mathcal{Q}$, then we divide AB into two equal parts. Otherwise, if $A \in \mathcal{Q}$, we divide AB into AC and CB such that $|AC| = \kappa|AB|$. This will divide each triangle of $\mathcal{T}$ into four triangles and leads to a finer mesh. Suppose the initial mesh $\mathcal{T}_0$ satisfies the above conditions. Then the nth level mesh with κ-refinement is obtained recursively by $\mathcal{T}_n = \kappa(\mathcal{T}_{n-1})$, $n = 1, 2, \ldots$.

This graded process was proposed in [23] for the linear finite element method approximating the axisymmetric Poisson equation. We here generalize it to high-order finite element methods for the new equation (1).

We need the following notation to carry out the analysis on graded meshes. Let n be the number of κ-refinements of the domain. Thus, the final triangulation is $\mathcal{T}_n$. Let $\mathbb{T}_{i,j} \subset \mathcal{T}_j$, $j \leq n$, be the union of triangles in $\mathcal{T}_j$ that contain the vertex $Q_i \in \mathcal{Q}$. Note that $\mathbb{T}_{i,j} \subset \mathbb{T}_{i,l}$ for $j \geq l$ and $\cup_i \mathbb{T}_{i,j}$ occupies the neighborhood of the vertex set $\mathcal{Q}$ in the triangulation $\mathcal{T}_j$. Recall the regularity estimate for the solution and the parameter Υ_β in Theorem 3.5. We choose the grading parameter in the κ-refinement, for different values of β,

$$(35) \qquad \kappa_\beta = \min(1/2, 2^{-m/a_\beta}), \quad \text{for any } 0 < a_\beta < \Upsilon_\beta,$$

where $m \geq 1$ is the degree of piecewise polynomials in the finite element space $\mathbb{V}_n^\beta$. Then, based on analysis on $\mathcal{T}_n \setminus \cup_i \mathbb{T}_{i,0}$, on $\cup_i \mathbb{T}_{i,j-1} \setminus \cup_i \mathbb{T}_{i,j}$, and on $\mathbb{T}_{i,n}$, our error estimates are summarized in the following lemmas.

Lemma 3.11. For κ_β defined in (35), let $U \subset \mathcal{T}_n$ be the union of triangles that intersect $\mathcal{T}_n \setminus \cup_i \mathbb{T}_{i,0}$. Then,

$$\|u_0 - \Pi u_n^0\|_{H^1_+(\mathcal{T}_n \setminus \cup_i \mathbb{T}_{i,0})} \leq C 2^{-nm} \|u_0\|_{H_1^{m+1}(U)}$$

$$\|u_1 - \Pi u_n^1\|_{H^1_-(\mathcal{T}_n \setminus \cup_i \mathbb{T}_{i,0})} \leq C 2^{-nm} \|u_1\|_{H_1^{m+1}(U)}.$$

Proof. Assume U is away from the vertices of the domain (this is true when $n > 2$). Then, based on Definition 3.10, the mesh size on U is $\mathcal{O}(2^{-n})$. Summing up the estimates in (31), (32), (33), and (34) completes the proof. $\qquad\square$

For the estimates on $\mathbb{T}_{i,0}$, the union of initial triangles containing the vertex Q_i, we consider the new coordinate system that is a simple translation of the old rz-coordinate system, now with Q_i at the origin. Then, for a subset $G \subset \mathbb{T}_{i,0}$ and $0 < \lambda < 1$, we define the dilations of G and of a function as follows, for $(r, z) \in G$,

$$(36) \qquad G_\lambda := G/\lambda = \{(r_\lambda, z_\lambda) = (\lambda^{-1}r, \lambda^{-1}z)\}, \quad v_\lambda(r_\lambda, z_\lambda) := v(r, z).$$

Then, we have the following scaling estimates from Lemma 4.5, [22].

Lemma 3.12. Suppose G_λ is in the neighborhood $\mathcal{V}$ of Q_i. If Q_i is on the z-axis,

$$\|v_\lambda\|_{\mathcal{K}_{a,1}^m(G_\lambda)} = \lambda^{a-3/2}\|v\|_{\mathcal{K}_{a,1}^m(G)},$$

$$\|r_\lambda^{-1}v_\lambda\|_{L_1^2(G_\lambda)} = \lambda^{-1/2}\|r^{-1}v\|_{L_1^2(G)};$$

if Q_i is not on the z-axis,

$$C\lambda^{a-1}\|v\|_{\mathcal{K}^m_{a,1}(G)} \leq \|v_\lambda\|_{\mathcal{K}^m_{a,1}(G_\lambda)} \leq C\lambda^{a-1}\|v\|_{\mathcal{K}^m_{a,1}(G)}.$$

Now, we are ready to give estimates on the region $\mathbb{T}_{i,j-1}\backslash\mathbb{T}_{i,j}$.

Lemma 3.13. *Let $U \subset \mathcal{T}_n$ be the union of triangles that intersect $G := \mathbb{T}_{i,j-1}\backslash\mathbb{T}_{i,j}$. Suppose $G \subset \mathcal{V}$. Let h be the mesh size on U and $\xi = \sup_{\mathbf{x} \in G} \vartheta(\mathbf{x})$. Let $a \geq 0$. Then, for $v \in \mathcal{K}^{m+1}_{a+1,1}(U)$,*

$$(37) \qquad \|v - \Pi v\|_{H^1_+(G)} \leq C\xi^a (h/\xi)^m \|v\|_{\mathcal{K}^{m+1}_{a+1,1}(U)},$$

and for $v \in H^1_-(U) \cap \mathcal{K}^{m+1}_{a+1,1}(U)$,

$$(38) \qquad \|v - \Pi v\|_{H^1_-(G)} \leq C\xi^a (h/\xi)^m \|v\|_{\mathcal{K}^{m+1}_{a+1,1}(U)}.$$

Proof. Recall the new coordinate system with Q_i as the origin. Recall the dilations in (36) and the parameter L in Definition 2.2. Note that $(\Pi v)_\lambda = \Pi(v_\lambda)$. Then, we choose $\lambda = 2\xi/L$, such that $G_\lambda \subset \mathcal{V}$.

If Q_i is on the z-axis, by Lemma 3.12, the definitions of the weighted spaces, (31), and (32), we have

$$\|r^{-1}(v - \Pi v)\|_{L^2_1(G)} + \|v - \Pi v\|_{H^1_1(G)}$$
$$\leq C\|r^{-1}(v - \Pi v)\|_{L^2_1(G)} + \|v - \Pi v\|_{\mathcal{K}^1_{1,1}(G)}$$
$$= \lambda^{1/2}(\|r_\lambda^{-1}(v_\lambda - \Pi(v_\lambda))\|_{L^2_1(G_\lambda)} + \|v_\lambda - \Pi(v_\lambda)\|_{\mathcal{K}^1_{1,1}(G_\lambda)})$$
$$\leq C\lambda^{1/2}(\|r_\lambda^{-1}(v_\lambda - \Pi(v_\lambda))\|_{L^2_1(G_\lambda)} + \|v_\lambda - \Pi(v_\lambda)\|_{H^1_1(G_\lambda)})$$
$$\leq C\lambda^{1/2}(h/\lambda)^m\|v_\lambda\|_{H^{m+1}_1(U_\lambda)} \leq C\lambda^{1/2}(h/\lambda)^m\|v_\lambda\|_{\mathcal{K}^{m+1}_{1,1}(U_\lambda)}$$
$$\leq C(h/\xi)^m\|v\|_{\mathcal{K}^{m+1}_{1,1}(U)} \leq C\xi^a(h/\xi)^m\|v\|_{\mathcal{K}^{m+1}_{a+1,1}(U)}.$$

If Q_i is not on the z-axis, the proof is similar. With the corresponding estimate in Lemma 3.12, the definitions of the weighted spaces, (33), and (34), we have

$$\|r^{-1}(v - \Pi v)\|_{L^2_1(G)} + \|v - \Pi v\|_{H^1_1(G)}$$
$$\leq C\|r^{-1}(v - \Pi v)\|_{L^2_1(G)} + \|v - \Pi v\|_{\mathcal{K}^1_{1,1}(G)} \leq C\|v - \Pi v\|_{\mathcal{K}^1_{1,1}(G)}$$
$$\leq C\|v_\lambda - \Pi(v_\lambda)\|_{\mathcal{K}^1_{1,1}(G_\lambda)} \leq C\|v_\lambda - \Pi(v_\lambda)\|_{H^1_1(G_\lambda)} \leq C(h/\lambda)^m\|v_\lambda\|_{H^{m+1}_1(U_\lambda)}$$
$$\leq C(h/\lambda)^m\|v_\lambda\|_{\mathcal{K}^{m+1}_{1,1}(U_\lambda)} \leq C(h/\xi)^m\|v\|_{\mathcal{K}^{m+1}_{1,1}(U)} \leq C\xi^a(h/\xi)^m\|v\|_{\mathcal{K}^{m+1}_{a+1,1}(U)}.$$

Thus, the estimate in (38) is proved.

The estimate in (37) follows similarly using Lemma 3.12, the definitions of the weighted spaces, (31), and (33). □

Lemma 3.14. *For κ_β defined in (35), let $U \subset \mathcal{T}_n$ be the union of triangles that intersect $G := \mathbb{T}_{i,j-1}\backslash\mathbb{T}_{i,j}$. Then, for $0 < a_\beta < \Upsilon_\beta$,*

$$\|u_0 - \Pi u_0\|_{H^1_+(G)} \leq C2^{-nm}\|u_0\|_{\mathcal{K}^{m+1}_{a_0+1,1}(U)},$$
$$\|u_1 - \Pi u_1\|_{H^1_-(G)} \leq C2^{-nm}\|u_1\|_{\mathcal{K}^{m+1}_{a_1+1,1}(U)}.$$

Proof. Definition 3.10 shows that the mesh size on U is $\mathcal{O}(\kappa_\beta^{j-1}2^{j-1-n})$. Using the notation of Lemma 3.13, we have $\xi = \mathcal{O}(\kappa_\beta^{j-1})$ on $\mathbb{T}_{i,j-1}\backslash\mathbb{T}_{i,j}$. Therefore, using

Lemma 3.13, we have

$$\|u_1 - \Pi u_1)\|_{H^1_-(G)} \leq C\kappa_1^{(j-1)a_1} 2^{(j-1-n)m} \|u_1\|_{\mathcal{K}^{m+1}_{a_1+1,1}(U)}$$
$$\leq C2^{-(j-1)m} 2^{(j-1-n)m} \|u_1\|_{\mathcal{K}^{m+1}_{a_1+1,1}(U)} = C2^{-nm} \|u_1\|_{\mathcal{K}^{m+1}_{a_1+1,1}(U)}.$$

Then, we have proved the second estimate in this lemma. The first estimate can be proved similarly by Lemma 3.13 and the observation on the mesh size for the regions G and U. □

The following lemma gives the error bounds on the last patch $\mathbb{T}_{i,n}$ of triangles that have the vertex Q_i as the common node.

Lemma 3.15. *For κ_β defined in (35), let $U \subset \mathcal{T}_n$ be the union of triangles that intersect $\mathbb{T}_{i,n}$. Suppose $\mathbb{T}_{i,0} \subset \mathcal{V}$ near Q_i. Then,*

$$\|u_0 - \Pi u_0\|_{H^1_+(\mathbb{T}_{i,n})} \leq C2^{-nm} \|u_0\|_{\mathcal{K}^{m+1}_{a_0+1,1}(U)}$$
$$\|u_1 - \Pi u_1)\|_{H^1_-(\mathbb{T}_{i,n})} \leq C2^{-nm} (\|u_1\|_{\mathcal{K}^{m+1}_{a_1+1,1}(U)} + \|\vartheta^{-a_1} r^{-1} u_1\|_{L^2_1(U)}).$$

Proof. Definition 3.10 shows that the mesh on U has the size $\mathcal{O}(\kappa_\beta^n)$. Let ζ be a smooth function, such that $\zeta = 0$ in the small neighborhood of the vertex Q_i and $\zeta = 1$ on a region containing all the other nodes in $\mathbb{T}_{i,n}$. Let $v = u_1 - \zeta u_1$. Note that $\Pi u_1 = \Pi(\zeta u_1)$ and $\mathbb{T}_{i,0} = \mathbb{T}_{i,n}/\lambda$, where $\lambda = \kappa_\beta^n$. If Q_i is on the z-axis, by Lemma 3.12, we have

$$\|r^{-1}\zeta u_1\|_{L^2_1(\mathbb{T}_{i,n})} = \lambda^{1/2} \|r_\lambda^{-1} \zeta_\lambda (u_1)_\lambda\|_{L^2_1(\mathbb{T}_{i,0})}$$
$$\leq C\lambda^{1/2} \|r_\lambda^{-1}(u_1)_\lambda\|_{L^2_1(\mathbb{T}_{i,0})} = C\|r^{-1}u_1\|_{L^2_1(\mathbb{T}_{i,n})},$$

and

$$\|\zeta u_1\|_{\mathcal{K}^m_{1,1}(\mathbb{T}_{i,n})} = \lambda^{1/2} \|\zeta_\lambda (u_1)_\lambda\|_{\mathcal{K}^m_{1,1}(\mathbb{T}_{i,0})} \leq C\lambda^{1/2} \|(u_1)_\lambda\|_{\mathcal{K}^m_{1,1}(\mathbb{T}_{i,0})} = C\|u_1\|_{\mathcal{K}^m_{1,1}(\mathbb{T}_{i,n})}.$$

Therefore, by Lemma 2.9, (10), Lemma 3.7, Lemma 3.12, and the estimates above,

$$\|r^{-1}(u_1 - \Pi u_1)\|_{L^2_1(\mathbb{T}_{i,n})} + \|u_1 - \Pi u_1\|_{H^1_-(\mathbb{T}_{i,n})}$$
$$\leq \|r^{-1}v\|_{L^2_1(\mathbb{T}_{i,n})} + \|r^{-1}(\zeta u_1 - \Pi u_1)\|_{L^2_1(\mathbb{T}_{i,n})} + \|v\|_{H^1_-(\mathbb{T}_{i,n})} + \|\zeta u_1 - \Pi u_1\|_{\mathcal{K}^1_{1,1}(\mathbb{T}_{i,n})}$$
$$\leq C(\|r^{-1}u_1\|_{L^2_1(\mathbb{T}_{i,n})} + \|u_1\|_{\mathcal{K}^1_{1,1}(\mathbb{T}_{i,n})}$$
$$+ \lambda^{1/2}(\|r_\lambda^{-1}(\zeta_\lambda (u_1)_\lambda - \Pi(u_1)_\lambda)\|_{L^2_1(\mathbb{T}_{i,0})} + \|\zeta_\lambda (u_1)_\lambda - \Pi(u_1)_\lambda\|_{\mathcal{K}^1_{1,1}(\mathbb{T}_{i,0})})$$
$$\leq C(\|r^{-1}u_1\|_{L^2_1(\mathbb{T}_{i,n})} + \|u_1\|_{\mathcal{K}^1_{1,1}(\mathbb{T}_{i,n})} + \lambda^{1/2}|\zeta_\lambda (u_1)_\lambda|_{H^{m+1}_1(U_\lambda)})$$
$$\leq C(\|r^{-1}u_1\|_{L^2_1(\mathbb{T}_{i,n})} + \|u_1\|_{\mathcal{K}^1_{1,1}(\mathbb{T}_{i,n})} + \lambda^{1/2}\|(u_1)_\lambda\|_{\mathcal{K}^{m+1}_{1,1}(U_\lambda)})$$
$$\text{(39)} \qquad = C(\|r^{-1}u_1\|_{L^2_1(\mathbb{T}_{i,n})} + \|u_1\|_{\mathcal{K}^{m+1}_{1,1}(U)}).$$

If Q_i is not on the z-axis, r is bounded away from 0, by Lemma 3.12,

$$\|r^{-1}\zeta u_1\|_{L^2_1(\mathbb{T}_{i,n})} = \|(r^{-1}\vartheta)\vartheta^{-1}\zeta u_1\|_{L^2_1(\mathbb{T}_{i,n})} \leq C\|\zeta u_1\|_{\mathcal{K}^0_{1,1}(\mathbb{T}_{i,n})}$$
$$\leq C\|\zeta_\lambda (u_1)_\lambda\|_{\mathcal{K}^0_{1,1}(\mathbb{T}_{i,0})} \leq C\|(u_1)_\lambda\|_{\mathcal{K}^0_{1,1}(\mathbb{T}_{i,0})} \leq C\|u_1\|_{\mathcal{K}^0_{1,1}(\mathbb{T}_{i,n})};$$
$$\|\zeta u_1\|_{\mathcal{K}^m_{1,1}(\mathbb{T}_{i,n})} \leq C\|\zeta_\lambda (u_1)_\lambda\|_{\mathcal{K}^m_{1,1}(\mathbb{T}_{i,0})} \leq C\|(u_1)_\lambda\|_{\mathcal{K}^m_{1,1}(\mathbb{T}_{i,0})} \leq C\|u_1\|_{\mathcal{K}^m_{1,1}(\mathbb{T}_{i,n})}.$$

Thus, by Lemma 2.8, Lemma 3.8, Lemma 3.12, and the estimates above,

$$\|r^{-1}(u_1 - \Pi u_1)\|_{L_1^2(\mathbb{T}_{i,n})} + \|u_1 - \Pi u_1\|_{H_1^1(\mathbb{T}_{i,n})}$$

$$\leq \|r^{-1}v\|_{L_1^2(\mathbb{T}_{i,n})} + \|r^{-1}(\zeta u_1 - \Pi u_1)\|_{L_1^2(\mathbb{T}_{i,n})} + \|v\|_{H_1^1(\mathbb{T}_{i,n})} + \|\zeta u_1 - \Pi u_1\|_{H_1^1(\mathbb{T}_{i,n})}$$

$$\leq C(\|u_1\|_{\mathcal{K}_{1,1}^1(\mathbb{T}_{i,n})} + \|\zeta u_1 - \Pi u_1\|_{\mathcal{K}_{1,1}^1(\mathbb{T}_{i,n})})$$

$$\leq C(\|u_1\|_{\mathcal{K}_{1,1}^1(\mathbb{T}_{i,n})} + \|\zeta_\lambda(u_1)_\lambda - \Pi(u_1)_\lambda\|_{\mathcal{K}_{1,1}^1(\mathbb{T}_{i,0})})$$

$$(40) \qquad \leq C(\|u_1\|_{\mathcal{K}_{1,1}^1(\mathbb{T}_{i,n})} + \|\zeta_\lambda(u_1)_\lambda\|_{\mathcal{K}_{1,1}^{m+1}(\mathbb{T}_{i,0})}) \leq C\|u_1\|_{\mathcal{K}_{1,1}^{m+1}(\mathbb{T}_{i,n})}.$$

Note that $\max_{\mathbf{x} \in \mathbb{T}_{i,n}} \vartheta(\mathbf{x}) = \mathcal{O}(\kappa_\beta^n)$. Combining (39) and (40), we have

$$\|r^{-1}(u_1 - \Pi u_1)\|_{L_1^2(\mathbb{T}_{i,n})} + \|u_1 - \Pi u_1\|_{H_1^1(\mathbb{T}_{i,n})} \leq C(\|r^{-1}u_1\|_{L_1^2(\mathbb{T}_{i,n})} + \|u_1\|_{\mathcal{K}_{1,1}^{m+1}(U)})$$

$$\leq C\kappa_1^{na_1}(\|\vartheta^{-a_1}r^{-1}u_1\|_{L_1^2(\mathbb{T}_{i,n})} + \|u_1\|_{\mathcal{K}_{a_1+1,1}^{m+1}(U)})$$

$$\leq C2^{-nm}(\|\vartheta^{-a_1}r^{-1}u_1\|_{L_1^2(\mathbb{T}_{i,n})} + \|u_1\|_{\mathcal{K}_{a_1+1,1}^{m+1}(U)}).$$

This completes the proof of the second estimate of the lemma. The first estimate follows from a similar process using Lemmas 3.12, 2.8, 2.9, 3.7, and 3.8. □

The local estimates above lead to the main result on the convergence rate of our finite element method.

Theorem 3.16. *Let u_β be the solution of equation (1) and $u_n^\beta \in \mathbb{V}_n^\beta$ be the finite element solution defined in (30). Recall κ_β from (35) for $0 < a_\beta < \Upsilon_\beta$ as defined in Theorem 3.5. Then, for $f_0 \in \mathcal{K}_{a_0-1,+}^{m-1}(\Omega)$ and $f_1 \in \mathcal{K}_{a_1-1,-}^{m-1}(\Omega)$,*

$$\|u_0 - u_n^0\|_{H_+^1(\Omega)} \leq C\mathrm{dim}(\mathbb{V}_n^0)^{-m/2}\|f_0\|_{\mathcal{K}_{a_0-1,+}^{m-1}(\Omega)},$$

$$\|u_1 - u_n^1\|_{H_-^1(\Omega)} \leq C\mathrm{dim}(\mathbb{V}_n^1)^{-m/2}\|f_1\|_{\mathcal{K}_{a_1-1,-}^{m-1}(\Omega)}.$$

where $\mathrm{dim}(\mathbb{V}_n^\beta) = \mathcal{O}(4^n)$ is the dimension of $\mathbb{V}_n^\beta$ and $m \geq 1$.

Proof. The proof follows by summing up the estimates in Lemmas 3.11, 3.14, and 3.15, together with the regularity results in Theorem 3.5. □

Remark 3.17. The graded meshes from the κ-refinements are nested and consist of shape-regular triangles. Various numerical tests for the linear finite element methods ($m = 1$) approximating singular solutions in the case $\beta = 0$ can be found in [23], which convincingly verify Theorem 3.16. As mentioned in Remark 3.6, instead of a uniform α_β for all vertices, we can choose different regularity indices a_β near different vertices Q_i. This will result in different grading parameters κ_β close to the vertices. Nevertheless, as long as $0 < a_\beta < \eta_\beta^i$ for each Q_i, we shall have the optimal convergence rate indicated in Theorem 3.16. This flexibility can help to improve the shape regularity of the triangulation.

4. THE MULTIGRID ALGORITHM

From now on, we concentrate on the multigrid analysis for the *linear* finite element approximation of equation (1) on graded meshes given in the last section.

4.1. **The multigrid V-cycle algorithm.** We denote the β-dependent *linear* finite element spaces

$$\mathbb{V}_0^\beta \subset \mathbb{V}_1^\beta \subset \cdots \subset \mathbb{V}_K^\beta$$

that are defined on graded meshes $\mathcal{T}_k$ in the last section, $0 \le k \le K$, where the parameter κ_β is chosen as in (35) for $m = 1$. For simplicity, we assume that there is only one singular vertex Q for the solution. Namely, the solution is in H_1^2 (or $\eta_\beta^i > 1$) except in the neighborhood of Q. Therefore, the graded mesh ($\kappa_\beta < 0.5$) is implemented only for triangles touching Q, while the usual quasi-uniform ($\kappa = 0.5$) decomposition is performed for other triangles. If we modify the weight function in the definition of the space $\mathcal{K}_{\mu,1}^m(\Omega)$, letting ϑ be the distance to Q in its neighborhood $\mathcal{V}$ and be 1 in the neighborhoods of other vertices, then Theorem 3.16 in this case reads

(41) $$\|u_0 - u_k^0\|_{H_+^1(\Omega)} \le C\dim(\mathbb{V}_k^0)^{-1/2}\|f_0\|_{\mathcal{K}_{a_0-1,+}^0(\Omega)},$$

(42) $$\|u_1 - u_k^1\|_{H_-^1(\Omega)} \le C\dim(\mathbb{V}_k^1)^{-1/2}\|f_1\|_{\mathcal{K}_{a_1-1,-}^0(\Omega)}.$$

In this section, we use this modified function ϑ and also define $\nabla := (\partial_r, \partial_z)^t$.

Let $\mathbb{T}_k \subset \mathcal{T}_k$ be the union of all the triangles in $\mathcal{T}_k$ touching Q. Define the jth layer of $\mathcal{T}_k$,

(43) $$L_j := \mathbb{T}_{j-1}\backslash\mathbb{T}_j, \quad 0 \le j \le k,$$

where $\mathbb{T}_{-1} := \mathcal{T}_0$. Then, we define the piecewise-constant function on $\mathcal{T}_k$

(44) $$\omega_k|_{L_j} := (2\kappa_\beta)^{-j},$$

where $\kappa_\beta = 2^{-1/a_\beta} < 0.5$ is the grading parameter for Q.

Then, we define the following mesh-dependent weighted inner product

(45) $$(v_k, w_k)_k := \int_\Omega \omega_k^2 v_k w_k r dr dz, \quad \forall\, v_k, w_k \in \mathbb{V}_k^\beta,$$

and the norm induced by the inner product,

$$\|v_k\|_k^2 := \int_\Omega \omega_k^2 v_k^2 r dr dz, \quad \forall\, v_k \in \mathbb{V}_k^\beta.$$

Let $\mathcal{I}_{k-1}^k : \mathbb{V}_{k-1}^\beta \to \mathbb{V}_k^\beta$ be the coarse-to-fine operator, which is the natural injection. The fine-to-coarse operator $\mathcal{I}_k^{k-1} : \mathbb{V}_k^\beta \to \mathbb{V}_{k-1}^\beta$ is the transpose with respect to the inner product in (45),

$$(\mathcal{I}_k^{k-1} v_k, w_{k-1})_{k-1} := (v_k, \mathcal{I}_{k-1}^k w_{k-1})_k, \quad \forall\, v_k \in \mathbb{V}_k^\beta, \forall\, w_{k-1} \in \mathbb{V}_{k-1}^\beta.$$

Let $A_k^\beta : \mathbb{V}_k^\beta \to \mathbb{V}_k^\beta$ be the operator associated with equation (30),

(46) $$(A_k^\beta v_k, w_k)_k := a_\beta(v_k, w_k), \quad \forall\, v_k, w_k \in \mathbb{V}_k^\beta.$$

In addition, let $f_k^\beta \in \mathbb{V}_k^\beta$ be the function, such that

$$(f_k^\beta, v_k)_k = (f_\beta, v_k)_{L_1^2(\Omega)}, \quad \forall\, v_k \in \mathbb{V}_k^\beta.$$

Thus, we are ready to formulate the abstract multigrid V-cycle algorithm for equation (1) with the Richardson smoother.

Algorithm 4.1. (The V-cycle algorithm). The kth level V-cycle algorithm produces $MG(k, f_k^\beta, z_0, l)$ as an approximate solution for

$$A_k^\beta u_k^\beta = f_k^\beta$$

with initial guess $z_0 \in \mathbb{V}_k^\beta$, where l denotes the number of pre-smoothing and post-smoothing steps.

For $k = 0$, we define

$$MG(0, f_0^\beta, z_0, l) = (A_0^\beta)^{-1} f_0^\beta.$$

For $k \geq 1$, the approximate solution $MG(k, f_k^\beta, z_0, l)$ is computed recursively in three steps:

Pre-smoothing. For $1 \leq j \leq l$, compute z_j by

$$z_j = z_{j-1} + \gamma_k (f_k^\beta - A_k^\beta z_{j-1}),$$

where the constant γ_k is the damping factor that we will specify later.

Coarse grid correction. Let $g_{k-1} = \mathcal{I}_k^{k-1}(f_k^\beta - A_k^\beta z_l)$ be the restriction of the residue on the $k-1$st level. Define $q \in \mathbb{V}_{k-1}^\beta$ by

$$q := MG(k - 1, g_{k-1}, 0, l).$$

Then, we compute z_{l+1} by

$$z_{l+1} = z_l + \mathcal{I}_{k-1}^k q.$$

Post-smoothing. For $l + 2 \leq j \leq 2l + 1$, compute z_j by

$$z_j = z_{j-1} + \gamma_k (f_k^\beta - A_k^\beta z_{j-1}).$$

We then define $MG(k, f_k^\beta, z_0, l) = z_{2l+1}$.

Remark 4.2. Let $E_k : \mathbb{V}_k^\beta \to \mathbb{V}_k^\beta$ be the error-propagation operator for the kth level V-cycle algorithm defined above. Namely, $E_k(u_k^\beta - z_0) = u_k^\beta - MG(k, f_k^\beta, z_0, l)$. The relation below follows from a straightforward calculation:

$$(47) \qquad E_k = R_k^l (Id_k - \mathcal{I}_{k-1}^k P_k^{k-1} + \mathcal{I}_{k-1}^k E_{k-1} P_k^{k-1}) R_k^l,$$

where Id_k is the identity operator on $\mathbb{V}_k^\beta$, $R_k : \mathbb{V}_k^\beta \to \mathbb{V}_k^\beta$ measuring the effect of the smoothing step is defined by

$$(48) \qquad\qquad\qquad R_k := Id_k - \gamma_k A_k^\beta,$$

and $P_k^{k-1} : \mathbb{V}_k^\beta \to \mathbb{V}_{k-1}^\beta$ is the transpose of $\mathcal{I}_{k-1}^k$ with respect to the bilinear forms, i.e., for any $v_k \in \mathbb{V}_k^\beta$,

$$(49) \qquad a_\beta(P_k^{k-1} v_k, w_{k-1}) = a_\beta(v_k, \mathcal{I}_{k-1}^k w_{k-1}), \qquad \forall \, w_k \in \mathbb{V}_{k-1}^\beta.$$

Note that P_k^{k-1} is in fact the projection on $\mathbb{V}_{k-1}^\beta$ with respect to $a_\beta(\cdot, \cdot)$. Then, for any $v_{k-1}, w_{k-1} \in \mathbb{V}_{k-1}^\beta$, we have

$$a_\beta(P_k^{k-1} \mathcal{I}_{k-1}^k v_{k-1}, w_{k-1}) = a_\beta(\mathcal{I}_{k-1}^k v_{k-1}, \mathcal{I}_{k-1}^k w_{k-1}) = a_\beta(v_{k-1}, w_{k-1}).$$

Then, for any $v_k \in \mathbb{V}_{k-1}^\beta$, $w_k \in \mathbb{V}_k^\beta$,

$$a_\beta\left(\mathcal{I}_{k-1}^k v_{k-1}, (Id_k - \mathcal{I}_{k-1}^k P_k^{k-1}) w_k\right)$$
$$= a_\beta(\mathcal{I}_{k-1}^k v_{k-1}, w_k) - a_\beta(\mathcal{I}_{k-1}^k v_{k-1}, \mathcal{I}_{k-1}^k P_k^{k-1} w_k)$$
$$(50) \qquad = a_\beta(v_{k-1}, P_k^{k-1} w_k) - a\left(v_{k-1}, P_k^{k-1} \mathcal{I}_{k-1}^k P_k^{k-1} w_k\right) = 0.$$
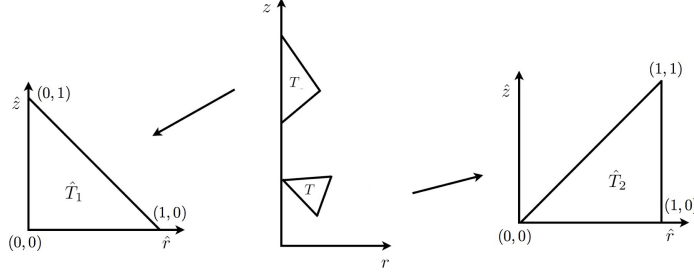
FIGURE 2. Reference triangles $\hat{T}_1$ and $\hat{T}_2$ corresponding to different triangles touching the z-axis.

For the sake of economy of notation, we use the same notation for the operators $Id_k, \mathcal{I}_{k-1}^k, \mathcal{I}_k^{k-1}, E_k, R_k$, and P_k^{k-1}, even though $\mathbb{V}_k^0$ and $\mathbb{V}_k^1$ are different subspaces of $H_1^1(\Omega)$ (see (29)).

Let $\{\mathbf{x}_i\}$ be the set of nodes in the triangulation $\mathcal{T}_k$ and $d_{i,k}$ be the diameter of the support of the basis function associated with the node $\mathbf{x}_i$. Throughout the text, by $A \simeq B$, we mean there are generic constants $C_1, C_2 > 0$, such that $C_1 B \leq A \leq C_2 B$. For a triangle T, $r_{max}(T) := \max_{\mathbf{x} \in T} r(\mathbf{x})$ and $r_{min}(T) := \min_{\mathbf{x} \in T} r(\mathbf{x})$. For a triangle $T \in \mathcal{T}_k$ away from the z-axis, we denote by $\hat{T}$ its standard reference triangle of diameter one. If T touches the z-axis, its reference triangle $\hat{T}$ will be either $\hat{T}_1$ or $\hat{T}_2$ (Figure 2), to preserve the intersection set between T and the z-axis. For $v_k \in \mathbb{V}_k^\beta$ and $\mathbf{x} \in T$, we define $\hat{v}_k(\hat{\mathbf{x}}) := v_k(\mathbf{x})$, where $\hat{\mathbf{x}} \in \hat{T}$ is the image of $\mathbf{x}$ after the affine mapping.

We now investigate necessary properties of the inner products and of the operators in Algorithm 4.1.

Lemma 4.3. *Let $T_i \in \mathcal{T}_k$ be a triangle and $h_{i,k}$ be its diameter. Then,*

$$\int_{T_i} v_k^2 r\, dr\, dz \simeq h_{i,k}^2 \sum_{\mathbf{x}_l \in T_i} v_k^2(\mathbf{x}_l) \max\left(d_{l,k}, r(\mathbf{x}_l)\right), \quad \forall\, v_k \in \mathbb{V}_k^\beta,$$

where $\mathbf{x}_l$, $1 \leq l \leq 3$, is the vertex of T_i and $d_{l,k}$ is defined as above.

Proof. **Case I:** $T_i \cap \{r = 0\} = \emptyset$. Thus, $r_{max}(T_i)/r_{min}(T_i) < M$ with a uniform constant $M > 0$ for any T_i. We map T_i to the reference triangle $\hat{T}$. Based on the norm equivalence on finite dimensional spaces, we have

$$
\begin{aligned}
\int_{T_i} v_k^2 r\, dr\, dz &\simeq & r_{max}(T_i) \int_{T_i} v_k^2\, dr\, dz = r_{max}(T_i)|T_i| \int_{\hat{T}} \hat{v}_k^2\, d\hat{r}\, d\hat{z} \\
&\simeq & r_{max}(T_i) h_{i,k}^2 \sum_{\hat{\mathbf{x}}_l \in \hat{T}} \hat{v}_k^2(\hat{\mathbf{x}}_l) \simeq h_{i,k}^2 \sum_{\mathbf{x}_l \in T_i} v_k^2(x_l) \max\left(d_{l,k}, r(\mathbf{x}_l)\right),
\end{aligned}
$$

where we used the fact that $r_{max}(T_i) \simeq \max\left(d_{l,k}, r(\mathbf{x}_l)\right)$ for any triangle away from the z-axis.

Case II: $T_i \cap \{r = 0\} \neq \emptyset$. Let $\{\lambda_m\}_{m=1,2,3}$ be the barycentric coordinates associated with the three vertices of T_i.

1. ($T_i \cap \{r = 0\}$ is a line segment.) We map T_i to the the first reference $\hat{T}_1$. Without loss of generality, we assume the first node of T_i is away from the z-axis.

Let r_1 be the distance from its first node to the z-axis. Note that $r_1 \simeq h_{i,k}$. Based on the norm equivalence on finite dimensional spaces, we have

$$\int_{T_i} v_k^2 r dr dz \;=\; r_1 \int_{T_i} v_k^2 \lambda_1 dr dz = r_1 |T_i| \int_{\hat{T}_1} \hat{v}_k^2 \hat{\lambda}_1 d\hat{r} d\hat{z}$$

$$\simeq\; h_{i,k}^3 \sum_{\hat{\mathbf{x}}_l \in \hat{T}_1} \hat{v}_k^2(\hat{\mathbf{x}}_l) \simeq h_{i,k}^2 \sum_{\mathbf{x}_l \in T_i} v_k^2(\mathbf{x}_l) \max\big(d_{l,k}, r(\mathbf{x}_l)\big).$$

2. ($T_i \cap \{r = 0\}$ is a point.) We map T_i to the second reference $\hat{T}_2$. Suppose that its third vertex is on the z-axis. Denote by r_1 and r_2 the distances form the first and second vertices of T_i to the z-axis, respectively. Then, for any $\mathbf{x} \in T_i$, $r(\mathbf{x}) = r_1 \lambda_1(\mathbf{x}) + r_2 \lambda_2(\mathbf{x})$ with $r_1 \simeq r_2 \simeq h_{i,k}$. Using the norm equivalence on finite element spaces, we have

$$\int_{T_i} v_k^2 r dr dz \;=\; \int_{T_i} v_k^2 (r_1 \lambda_1 + r_2 \lambda_2) dr dz \simeq h_{i,k} |T_i| \int_{\hat{T}_2} \hat{v}_k^2 (\hat{\lambda}_1 + \hat{\lambda}_2) d\hat{r} d\hat{z}$$

$$\simeq\; h_{i,k}^3 \sum_{\hat{\mathbf{x}}_l \in \hat{T}_2} \hat{v}_k^2(\hat{\mathbf{x}}_l) \simeq h_{i,k}^2 \sum_{\mathbf{x}_l \in T_i} v_k^2(\mathbf{x}_l) \max\big(d_{l,k}, r(\mathbf{x}_l)\big).$$

Hence, the lemma is proved for any $T_i \in \mathcal{T}_k$. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 4.4. *The weighted inner product in* (45) *satisfies*

$$(v_k, v_k)_k \simeq \dim(\mathbb{V}_k^\beta)^{-1} \sum_i v_k^2(\mathbf{x}_i) \max\big(d_{i,k}, r(\mathbf{x}_i)\big), \quad \forall\, v_k \in \mathbb{V}_k^\beta.$$

Proof. Recall $\dim(\mathbb{V}_k^\beta) \simeq 4^k$. Thus, for any $v_k \in \mathbb{V}_k^\beta$, we have

$$\dim(\mathbb{V}_k^\beta)^{-1} \sum_{i=1}^{\dim(\mathbb{V}_k^\beta)} v_k^2(\mathbf{x}_i) \max\big(d_{i,k}, r(\mathbf{x}_i)\big)$$

$$\simeq \sum_{i=1}^{\dim(\mathbb{V}_k^\beta)} (2\kappa)^{-2\Im(i)} \big(2^{2(\Im(i)-k)} \kappa^{2\Im(i)}\big) v_k^2(\mathbf{x}_i) \max\big(d_{i,k}, r(\mathbf{x}_i)\big),$$

where the function $\Im : \mathbb{N} \to \mathbb{N} \cup \{0\}$ is such that $\Im(i) = j$ if the node $\mathbf{x}_i$ is in the jth layer L_j of $\mathcal{T}_k$. By Definition 3.10, for a node $\mathbf{x}_i \in L_j$, the diameter of a triangle T_l touching $\mathbf{x}_i$ satisfies $h_{l,k} \simeq 2^{j-k} \kappa^j$. Therefore, using Lemma 4.3, we have

$$\dim(\mathbb{V}_k^\beta)^{-1} \sum_{i=1}^{\dim(\mathbb{V}_k^\beta)} v_k^2(\mathbf{x}_i) \max\big(d_{i,k}, r(\mathbf{x}_i)\big)$$

$$\simeq \sum_{T_l \in \mathcal{T}_k} \sum_{\mathbf{x}_i \in T_l} (2\kappa)^{-2\Im(i)} \big(2^{2(\Im(i)-k)} \kappa^{2\Im(i)}\big) v_k^2(\mathbf{x}_i) \max\big(d_{i,k}, r(\mathbf{x}_i)\big)$$

$$\simeq \sum_{T_l \in \mathcal{T}_k} \omega_k^2|_{T_l} h_{l,k}^2 \sum_{\mathbf{x}_i \in T_l} v_k^2(\mathbf{x}_i) \max\big(d_{i,k}, r(\mathbf{x}_i)\big)$$

$$\simeq \sum_{T_l \in \mathcal{T}_k} \omega_k^2|_{T_l} \int_{T_l} v_k^2 r dr dz = \int_\Omega \omega_k^2 v_k^2 r dr dz = (v_k, v_k)_k,$$

which completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 4.5. *The spectral radus of A_k^β defined in* (46) *satisfies*

$$0 < \rho(A_k^\beta) \le C\dim(\mathbb{V}_k^\beta).$$

Proof. According to the definition, A_k^β is symmetric positive definite (SPD) with respect to the inner product $(\cdot, \cdot)_k$. Therefore, it has a discrete spectrum and all its eigenvalues are real and positive. Thus, we only need to show for any $\mathbb{V}_k^\beta \ni v_k \ne 0$,

$$(51) \qquad \frac{(A_k^\beta v_k, v_k)_k}{(v_k, v_k)_k} \le C\dim(\mathbb{V}_k^\beta).$$

Recall that near the singular vertex Q, the grading parameter $\kappa_\beta = 2^{-1/a_\beta} < 0.5$. If a triangle T_i is in the jth layer L_j of $\mathcal{T}_k$, $0 \le j \le k$, its diameter $h_{i,k} \simeq \kappa^j 2^{j-k}$, and so is the diameter $d_{l,k}$ of the support of the basis function associated with the vertex $\mathbf{x}_l$ of T_i.

Case I: $T_i \cap \{r = 0\} = \emptyset$. Recall that $r_{max}(T_i)/r_{min}(T_i) \le M$ is bounded with $M > 0$ for any T_i. Then,

$$\int_{T_i} (\nabla v_k \cdot \nabla v_k) r \, dr dz \quad \simeq \quad r_{max}(T_i) \int_{T_i} \nabla v_k \cdot \nabla v_k dr dz \le C h_{i,k}^{-2} r_{max}(T_i) \int_{T_i} v_k^2 dr dz$$

$$(52) \qquad\qquad \simeq \quad r_{max}(T_i) \sum_{\mathbf{x}_l \in T_i} v_k^2(\mathbf{x}_l) \simeq \sum_{\mathbf{x}_l \in T_i} v_k^2(\mathbf{x}_l) r(\mathbf{x}_l),$$

where we used the usual H^1-L^2 inverse inequality in the finite element space and the norm equivalence on finite dimensional spaces.

Note $r_{max}(T_i)^{-1} \le C h_{i,k}^{-1}$. By the norm equivalence on finite dimensional spaces,

$$\int_{T_i} r^{-1} v_k^2 dr dz \quad \simeq \quad r_{max}(T_i)^{-1} \int_{T_i} v_k^2 dr dz = r_{max}(T_i)^{-1} |T_i| \int_{\hat{T}} \hat{v}_k^2 d\hat{r} d\hat{z}$$

$$(53) \qquad\qquad \le \quad C h_{i,k} \sum_{\hat{\mathbf{x}}_l \in \hat{T}} \hat{v}_k^2(\hat{\mathbf{x}}_l) \le C \sum_{\mathbf{x}_l \in T_i} v_k^2(\mathbf{x}_l) \max\left(d_{l,k}, r(\mathbf{x}_l)\right).$$

Case II: $T_i \cap \{r = 0\} \ne \emptyset$. Let $\{\lambda_m\}_{m=1,2,3}$ be the barycentric coordinates associated with the three vertices of T_i.

1. ($T_i \cap \{r = 0\}$ is a line segment.) We map T_i to $\hat{T}_1$. Suppose the first node of T_i is away from the z-axis. Let r_1 be the distance from the first node to the z-axis. Then, $r_1 \simeq h_{i,k}$. Based on the norm equivalence on finite dimensional spaces,

$$(54) \qquad \int_{\hat{T}_1} (\nabla \hat{v}_k \cdot \nabla \hat{v}_k) \hat{\lambda}_1 d\hat{r} d\hat{z} + \int_{\hat{T}_1} \hat{v}_k^2 \hat{\lambda}_1 d\hat{r} d\hat{z} \le C \int_{\hat{T}_1} \hat{v}_k^2 d\hat{r} d\hat{z},$$

since each side above defines the square of a norm for $\hat{v}_k$. Then,

$$\int_{T_i} (\nabla v_k \cdot \nabla v_k) r \, dr dz \quad = \quad r_1 \int_{T_i} (\nabla v_k \cdot \nabla v_k) \lambda_1 dr dz$$

$$\simeq \quad r_1 |T_i| \int_{\hat{T}_1} h_{i,k}^{-2} (\nabla \hat{v}_k \cdot \nabla \hat{v}_k) \hat{\lambda}_1 d\hat{r} d\hat{z} \le C r_1 \int_{\hat{T}_1} \hat{v}_k^2 d\hat{r} d\hat{z}$$

$$(55) \qquad\qquad \simeq \quad r_1 \sum_{\hat{\mathbf{x}}_l \in \hat{T}_1} \hat{v}_k^2(\hat{\mathbf{x}}_l) \simeq \sum_{\mathbf{x}_l \in T_i} v_k^2(\mathbf{x}_l) \max\left(d_{l,k}, r(\mathbf{x}_l)\right),$$

where we used the scaling argument, the norm equivalence on finite dimensional spaces, and (54).

For $v_k \in \mathbb{V}_k^1$, $v_k|_{T_i}$ is a linear function that vanishes on the z-axis. Therefore, $v_k|_{T_i} = \nu_k r$ for a constant ν_k. Then, we have

$$
\begin{aligned}
\int_{T_i} r^{-1} v_k^2 dr dz &= \int_{T_i} \nu_k^2 r dr dz = r_1 |T_i| \int_{\hat{T}_1} \nu_k^2 \hat{\lambda}_1 d\hat{r} d\hat{z} \simeq h_{i,k}^3 \nu_k^2 \\
(56) \qquad &\simeq h_{i,k} \sum_{\mathbf{x}_l \in T_i} v_k^2(\mathbf{x}_l) \simeq \sum_{\mathbf{x}_l \in T_i} v_k^2(\mathbf{x}_l) \max\left(d_{l,k}, r(\mathbf{x}_l)\right),
\end{aligned}
$$

where we used the norm equivalence on finite dimensional spaces.

2. ($T_i \cap \{r = 0\}$ is a point.) We map T_i to $\hat{T}_2$. Suppose that its first two vertices are not on the z-axis. Denote by r_1 and r_2 the distances from the first vertex and the second vertex of T_i to the z-axis, respectively. For any point $\mathbf{x} \in T_i$, $r(\mathbf{x}) = r_1 \lambda_1(\mathbf{x}) + r_2 \lambda_2(\mathbf{x})$ and $r_1 \simeq r_2 \simeq h_{i,k}$. Using the norm equivalence on finite dimensional spaces, we have

$$
(57) \quad \int_{\hat{T}_2} (\nabla \hat{v}_k \cdot \nabla \hat{v}_k)(\hat{\lambda}_1 + \hat{\lambda}_2) d\hat{r} d\hat{z} + \int_{\hat{T}_2} \hat{v}_k^2 (\hat{\lambda}_1 + \hat{\lambda}_2) d\hat{r} d\hat{z} \le C \int_{\hat{T}_2} \hat{v}_k^2 d\hat{r} d\hat{z},
$$

since each side of (57) defines the square of a norm for $\hat{v}_k$. Similarly, we have

$$
\begin{aligned}
\int_{T_i} (\nabla v_k \cdot \nabla v_k) r dr dz &\simeq h_{i,k} |T_i| \int_{\hat{T}_2} h_{i,k}^{-2} (\nabla \hat{v}_k \cdot \nabla \hat{v}_k)(\hat{\lambda}_1 + \hat{\lambda}_2) d\hat{r} d\hat{z} \\
(58) \qquad &\le C h_{i,k} \int_{\hat{T}_2} \hat{v}_k^2 d\hat{r} d\hat{z} \simeq h_{i,k} \sum_{\hat{\mathbf{x}}_l \in \hat{T}_2} \hat{v}_k^2(\hat{\mathbf{x}}_l) \simeq \sum_{x_l \in T_i} v_k^2(\mathbf{x}_l) \max\left(d_{l,k}, r(\mathbf{x}_l)\right).
\end{aligned}
$$

For $v_k \in \mathbb{V}_k^1$, we can write the linear function $v_k|_{T_i} = \sum_l \nu_l \phi_l$, where for $l = 1, 2$, ν_l is a constant and ϕ_l is the linear basis function associated with the lth vertex of T_i. It was shown that $\int_{T_i} r^{-1} \phi_l^2 dr dz < \infty$ in [21]. Therefore, $\int_{T_i} r^{-1} v_k^2 dr dz$ is bounded. We then have

$$
\begin{aligned}
\int_{T_i} r^{-1} v_k^2 dr dz &= \int_{T_i} (r_1 \lambda_1 + r_2 \lambda_2)^{-1} v_k^2 dr dz \simeq h_{i,k}^{-1} |T_i| \int_{\hat{T}_2} (\hat{\lambda}_1 + \hat{\lambda}_2)^{-1} \hat{v}_k^2 d\hat{r} d\hat{z} \\
(59) \qquad &\simeq h_{i,k} \sum_{\hat{\mathbf{x}}_l \in \hat{T}_2} \hat{v}_k^2(\hat{\mathbf{x}}_l) = h_{i,k} \sum_{\mathbf{x}_l \in T_i} v_k^2(\mathbf{x}_l) \simeq \sum_{\mathbf{x}_l \in T_i} v_k^2(\mathbf{x}_l) \max\left(d_{l,k}, r(\mathbf{x}_l)\right),
\end{aligned}
$$

where the key step is to use the norm equivalence between $\left(\int_{\hat{T}_2} (\hat{\lambda}_1 + \hat{\lambda}_2)^{-1} \hat{v}_k^2 d\hat{r} d\hat{z}\right)^{1/2}$ and $\left(\sum_{\hat{\mathbf{x}}_l \in \hat{T}_2} \hat{v}_k^2(\hat{\mathbf{x}}_l)\right)^{1/2}$. This can be done because the first term is finite for all linear functions that vanish on the z-axis.

Hence, based on (52), (55), and (58), we have for any $T_i \in \mathcal{T}_k$

$$
(60) \quad \int_{T_i} (\nabla v_k \cdot \nabla v_k) r dr dz \le C \sum_{\mathbf{x}_l \in T_i} v_k^2(\mathbf{x}_l) \max\left(d_{l,k}, r(\mathbf{x}_l)\right), \quad \forall \, v_k \in \mathbb{V}_k^\beta.
$$

Summing up (53), (56), and (59), we obtain that for any $v_k \in \mathbb{V}_k^1$ and $T_i \in \mathcal{T}_k$,

$$
(61) \quad \int_{T_i} r^{-1} v_k^2 dr dz \le C \sum_{\mathbf{x}_l \in T_i} v_k^2(\mathbf{x}_l) \max\left(d_{l,k}, r(\mathbf{x}_l)\right).
$$

Consequently, using (60), (61), and Lemma 4.4, for any $v_k \in \mathbb{V}_k^\beta$, we have

$$
\begin{aligned}
(A_k^\beta v_k, v_k)_k &= a_\beta(v_k, v_k) = \sum_{T_i \in \mathcal{T}_k} \int_{T_i} (\nabla v_k \cdot \nabla v_k + \beta r^{-2} v_k^2) r\,dr\,dz \\
&\leq C \sum_{T_i \in \mathcal{T}_k} \Big(\sum_{\mathbf{x}_l \in T_i} v_k^2(\mathbf{x}_l) \max\big(d_{l,k}, r(\mathbf{x}_l)\big) \Big) \\
&\simeq \sum_{1 \leq i \leq \dim(\mathbb{V}_k^\beta)} v_k^2(\mathbf{x}_i) \max\big(d_{i,k}, r(\mathbf{x}_i)\big) \simeq \dim(\mathbb{V}_k)(v_k, v_k)_k.
\end{aligned}
$$

$\square$

Remark 4.6. Based on Lemma 4.5, we can choose the damping factor

$$(62) \qquad\qquad \gamma_k = \mathcal{O}\big(\dim(\mathbb{V}_k^\beta)^{-1}\big)$$

for the multigrid V-cycle, such that the spectral radius

$$(63) \qquad\qquad 0 < \rho(\gamma_k A_k^\beta) \leq 1.$$

4.2. **The multigrid analysis.** We now concentrate on the convergence analysis of the multigrid V-cycle defined in Algorithm 4.1 for the axisymmetric equation (1).

We first have the smoothing property for the Richardson smoother.

Lemma 4.7. *Recall the operator R_k from (48). Then,*

$$
a_\beta\big((Id_k - R_k)R_k^{2l} v_k, v_k\big) \leq \frac{1}{2l} a_\beta\big((Id_k - R_k^{2l}) v_k, v_k\big), \quad \forall\, v_k \in \mathbb{V}_k^\beta.
$$

Proof. Since $R_k = Id_k - \gamma_k A_k^\beta$, by (63), for any $0 \leq j \leq i$, we have

$$
\begin{aligned}
a_\beta\big((Id_k - R_k)R_k^i v_k, v_k\big) &= \gamma_k(R_k^i A_k^\beta v_k, A_k^\beta v_k)_k \\
&\leq \gamma_k(R_k^j A_k^\beta v_k, A_k^\beta v_k)_k = a_\beta\big((Id_k - R_k)R_k^j v_k, v_k\big).
\end{aligned}
$$

Then, the lemma is proved by the following telescopic cancellation

$$
\begin{aligned}
(2l)a_\beta\big((Id_k - R_k)R_k^{2l} v_k, v_k\big) &\leq \sum_{j=0}^{2l-1} a_\beta\big((Id_k - R_k)R_k^j v_k, v_k\big) \\
&\leq a_\beta\big((Id_k - R_k^{2l}) v_k, v_k\big).
\end{aligned}
$$

$\square$

Lemma 4.8. *The error-propagation operator E_k in (47) is symmetric positive semi-definite with respect to $a_\beta(\cdot, \cdot)$ for $k \geq 0$.*

Proof. We prove it by induction. For $k = 0$, the lemma holds. Assume the statement holds for $k - 1$, $k \geq 1$. We first show the symmetry of E_k. By (47), (48), (49), and the assumption for E_{k-1}, we have

$$
\begin{aligned}
a_\beta(E_k v_k, w_k) &= a_\beta\big(R_k^l(Id_k - P_k^{k-1} + E_{k-1}P_k^{k-1})R_k^l v_k, w_k\big) \\
&= a_\beta\big(R_k^l v_k, (Id_k - P_k^{k-1})R_k^l w_k\big) + a_\beta\big(E_{k-1}P_k^{k-1}R_k^l v_k, R_k^l w_k\big) \\
&= a_\beta\big(R_k^l v_k, (Id_k - P_k^{k-1})R_k^l w_k\big) + a_\beta\big(R_k^l v_k, E_{k-1}P_{k-1}R_k^l w_k\big) \\
&= a_\beta\big(v_k, R_k^l(Id_k - P_k^{k-1} + E_{k-1}P_k^{k-1})R_k^l w_k\big) = a_\beta(v_k, E_k w_k).
\end{aligned}
$$

We now show that E_k is positive and semi-definite. By (47), (49), and the assumption on E_{k-1},

$$
\begin{aligned}
a_\beta(E_k v_k, v_k) &= a_\beta(R_k^l v_k, R_k^l v_k) - a_\beta\big((Id_k - E_{k-1})P_k^{k-1}R_k^l v_k, P_k^{k-1}R_k^l v_k\big) \\
&= a_\beta\big((Id_k - P_k^{k-1})R_k^l v_k, (Id_k - P_k^{k-1})R_k^l v_k\big) \\
&\quad + a_\beta(E_{k-1}P_k^{k-1}R_k^l v_k, P_k^{k-1}R_k^l v_k) \geq 0.
\end{aligned}
$$

$\square$

In the next two lemmas, we give approximation results needed in the analysis.

Lemma 4.9. *Let $\mathcal{E}$ be $H^1_{+,0}(\Omega)$ for $\beta = 0$ and be $H^1_{-,0}(\Omega)$ for $\beta = 1$. For any $v_k \in \mathbb{V}_k^\beta$ and $k \geq 1$, we have*

$$
\|(Id_k - \mathcal{I}_{k-1}^k P_k^{k-1})v_k\|_k \leq C\mathrm{dim}(\mathbb{V}_k^\beta)^{-1/2}\|(Id_k - \mathcal{I}_{k-1}^k P_k^{k-1})v_k\|_\mathcal{E}.
$$

Proof. We now give the proof for the case $\beta = 1$. Consider the auxiliary variational problem: Find $\zeta \in H^1_{-,0}(\Omega)$, such that

$$
\text{(64)} \qquad a_1(\zeta, \xi) = \int_\Omega \omega_k^2 (Id_k - \mathcal{I}_{k-1}^k P_k^{k-1})v_k \xi \, r dr dz, \quad \forall \, \xi \in H^1_{-,0}(\Omega),
$$

where $a_1(\zeta, \xi) = (\mathcal{L}_1 \zeta, \xi)_{L^2_1(\Omega)}$ is the bilinear form in (11) and ω_k is the function defined in (44). Let $\zeta_{k-1} \in \mathbb{V}_{k-1}^\beta$ be the finite element solution of (64) on the $k - 1$st level mesh. Recall the layer L_j of the triangulation $\mathcal{T}_k$ in (43). Then, since the regularity index $0 < a_1 < 1$, by Theorem 3.5, (42), (44), and (64), we have

$$
\|\zeta - \mathcal{I}_{k-1}^k \zeta_{k-1}\|_{H^1_-(\Omega)}^2 = \|\zeta - \zeta_{k-1}\|_{H^1_-(\Omega)}^2 \leq C\mathrm{dim}(\mathbb{V}_{k-1}^1)^{-1}\|\mathcal{L}_1\zeta\|_{\mathcal{K}^0_{a_1-1,-}(\Omega)}^2
$$

$$
\simeq \mathrm{dim}(\mathbb{V}_k^1)^{-1}\|\mathcal{L}_1\zeta\|_{\mathcal{K}^0_{a_1-1,-}(\Omega)}^2 = \mathrm{dim}(\mathbb{V}_k^1)^{-1}\sum_{j=0}^k \|\vartheta^{1-a_1}\mathcal{L}_1\zeta\|_{L^2_1(L_j)}^2
$$

$$
\leq C\mathrm{dim}(\mathbb{V}_k^1)^{-1}\sum_{j=0}^k \|\kappa_1^{(1-a_1)j}\mathcal{L}_1\zeta\|_{L^2_1(L_j)}^2 = C\mathrm{dim}(\mathbb{V}_k^1)^{-1}\sum_{j=0}^k \|\kappa_1^j 2^j \mathcal{L}_1\zeta\|_{L^2_1(L_j)}^2
$$

$$
\text{(65)} \quad = C\mathrm{dim}(\mathbb{V}_k^1)^{-1}\|(Id_k - \mathcal{I}_{k-1}^k P_k^{k-1})v_k\|_k^2.
$$

Then, by (50), (65), and Lemma 4.4, we have

$$
\begin{aligned}
\|(Id_k - \mathcal{I}_{k-1}^k P_k^{k-1})v_k\|_k^2 &= a_1\big(\zeta, (Id_k - \mathcal{I}_{k-1}^k P_k^{k-1})v_k\big) \\
&= a_1\big(\zeta - \mathcal{I}_{k-1}^k \zeta_{k-1}, (Id_k - \mathcal{I}_{k-1}^k P_k^{k-1})v_k\big) \\
&\leq \|\zeta - \mathcal{I}_{k-1}^k \zeta_{k-1}\|_{H^1_-(\Omega)}\|(Id_k - \mathcal{I}_{k-1}^k P_k^{k-1})v_k\|_{H^1_-(\Omega)} \\
&\leq C\mathrm{dim}(\mathbb{V}_k^1)^{-1/2}\|(Id_k - \mathcal{I}_{k-1}^k P_k^{k-1})v_k\|_k\|(Id_k - \mathcal{I}_{k-1}^k P_k^{k-1})v_k\|_{H^1_-(\Omega)}.
\end{aligned}
$$

Thus, we obtain for $\beta = 1$,

$$
\|(Id_k - \mathcal{I}_{k-1}^k P_k^{k-1})v_k\|_k \leq C\mathrm{dim}(\mathbb{V}_k^1)^{-1/2}\|(Id_k - \mathcal{I}_{k-1}^k P_k^{k-1})v_k\|_{H^1_-(\Omega)}.
$$

The proof for $\beta = 0$ follows similarly.

$\square$

Lemma 4.10. *Recall the space $\mathcal{E}$ from Lemma 4.9. For any $v_k \in \mathbb{V}_k^\beta$ and $k \geq 1$,*

$$
\|(Id_k - \mathcal{I}_{k-1}^k P_k^{k-1})v_k\|_\mathcal{E} \leq C\mathrm{dim}(\mathbb{V}_k^\beta)^{-1/2}\|A_k^\beta v_k\|_k.
$$

Proof. Using the Cauchy-Schwarz inequality, (50), and Lemma 4.9, we have

$$\|(Id_k - \mathcal{I}_{k-1}^k P_k^{k-1})v_k\|_{\mathcal{E}}^2 = a_\beta\big((Id_k - \mathcal{I}_{k-1}^k P_k^{k-1})v_k, (Id_k - \mathcal{I}_{k-1}^k P_k^{k-1})v_k\big)$$

$$= a_\beta\big((Id_k - \mathcal{I}_{k-1}^k P_k^{k-1})v_k, v_k\big) = \big((Id_k - \mathcal{I}_{k-1}^k P_k^{k-1})v_k, A_k^\beta v_k\big)_k$$

$$\leq \|(Id_k - \mathcal{I}_{k-1}^k P_k^{k-1})v_k\|_k \|A_k^\beta v_k\|_k$$

$$\leq C \dim(\mathbb{V}_k^\beta)^{-1/2} \|(Id_k - \mathcal{I}_{k-1}^k P_k^{k-1})v_k\|_{\mathcal{E}} \|A_k^\beta v_k\|_k,$$

which completes the proof. $\qquad\square$

Then, we have the estimate for the error-propagation operator E_k in (47).

Theorem 4.11. *Recall the number of smoothing steps l in Algorithm 4.1. Then,*

$$(66) \qquad a_\beta(E_k v_k, v_k) \leq \frac{C}{l+C} a_\beta(v_k.v_k), \quad \forall\, v_k \in \mathbb{V}_k^\beta,$$

where $C > 0$ is independent of k.

Proof. Recall the space $\mathcal{E}$ defined in Lemma 4.9. By Lemma 4.10, (62), and Lemma 4.7, we first have

$$a_\beta\big((Id_k - P_k^{k-1})R_k^l v_k, (Id_k - P_k^{k-1})R_k^l v_k\big) = \|((Id_k - P_k^{k-1})R_k^l v_k)\|_{\mathcal{E}}^2$$

$$\leq C\dim(\mathbb{V}_k^\beta)^{-1} a_\beta(A_k^\beta R_k^l v_k, R_k^l v_k) = C\dim(\mathbb{V}_k^\beta)^{-1}\gamma_k^{-1} a_\beta\big((Id_k - R_k)R_k^l v_k, R_k^l v_k\big)$$

$$(67) \qquad\qquad \leq C a_\beta\big((Id_k - R_k)R_k^{2l} v_k, v_k\big) \leq \frac{C}{l} a_\beta\big((Id_k - R_k^{2l})v_k, v_k\big).$$

We now give the proof by induction. For $k = 0$, (66) holds since $E_0 = 0$. Assume (66) holds for $k - 1$, $k \geq 1$. Let $c = C(l+C)^{-1}$ where C is the constant in (67). Note that $c = (1-c)Cl^{-1}$. Then, by (47), (49), the assumption on E_{k-1}, and (67),

$$a_\beta(E_k v_k, v_k) = a_\beta\big((Id_k - P_k^{k-1})R_k^l v_k, (Id_k - P_k^{k-1})R_k^l v_k\big)$$

$$+ a_\beta(E_{k-1}P_k^{k-1}R_k^l v_k, P_k^{k-1}R_k^l v_k)$$

$$\leq a_\beta\big((Id_k - P_k^{k-1})R_k^l v_k, (Id_k - P_k^{k-1})R_k^l v_k\big) + c a_\beta(P_k^{k-1}R^l v_k, P_k^{k-1}R^l v_k)$$

$$= (1-c)a_\beta\big((Id_k - P_k^{k-1})R_k^l v_k, (I_k - P_k^{k-1})R_k^l v_k\big) + c a_\beta(R^l v_k, R^l v_k)$$

$$\leq (1-c)Cl^{-1}a_\beta\big((Id_k - R_k^{2l})v_k, v_k\big) + c a_\beta(R^l v_k, R^l v_k)$$

$$= c a_\beta\big((Id_k - R_k^{2l})v_k, v_k\big) + c a_\beta(R^l v_k, R^l v_k) = c a_\beta(v_k, v_k),$$

which completes the proof. $\qquad\square$

Hence, the multigrid V-cycle algorithm is a contraction with contraction number strictly less than 1, independent of the mesh level k.

Theorem 4.12. *With the choice of the damping factor γ_k in (63), the multigrid V-cycle scheme defined in Algorithm 4.1 solving the axisymmetric equation (30) converges uniformly,*

$$\|E_k v_k\|_{\mathcal{E}} \leq \frac{C}{l+C}\|v_k\|_{\mathcal{E}}, \quad \forall\, v_k \in \mathbb{V}_k^\beta,$$

where C is independent of the mesh level k and $\mathcal{E}$ is the space in Lemma 4.9.

Proof. By Lemma 4.8, let $0 \leq \alpha_1 \leq \cdots \leq \alpha_{n_k}$ be the eigenvalues of E_k and $\nu_1, \cdots, \nu_{n_k}$ be the corresponding orthonormal eigenvectors with respect to $a_\beta(\cdot, \cdot)$. Then, for any $v \in \mathbb{V}_k^\beta$, we write $v_k = \sum_{1 \leq i \leq n_k} \omega_i \nu_i$. Therefore, $a_\beta(E_k v_k, v_k) =$

$\sum_{1 \leq i \leq n_k} \alpha_i \omega_i^2$. The estimate in (66) indicates that all the eigenvalues $\alpha_i \leq C(l + C)^{-1}$. Then, the proof is completed by

$$\|E_k v_k\|_{\mathcal{E}}^2 = a_\beta(E_k v_k, E_k v_k) = \sum_{1 \leq i \leq n_k} \alpha_i^2 \omega_i^2$$

$$\leq (\frac{C}{l+C})^2 \sum_{1 \leq i \leq n_k} \omega_i^2 \leq (\frac{C}{l+C})^2 \|v_k\|_{\mathcal{E}}^2.$$

$\square$

Remark 4.13. The convergence result (Theorem 4.12) is the generalization of the Braess-Hackbusch estimate for self-adjoint second order elliptic equations with the full regularity assumption [7], for axisymmetric equations with singular solutions. It clearly holds when the solution is in H_1^2 and quasi-uniform meshes are used. In particular, the V-cycle algorithm converges with one pre-smoothing and post-smoothing step. The smoothing property (Lemma 4.7) and approximation properties (Lemmas 4.9 and 4.10) are essential. When the solution is singular and only quasi-uniform meshes are used, with the partial-regularity result [6], it is possible to obtain multigrid analysis following the methods in [9, 10, 8, 28]. Multigrid algorithms with smoothers other than the Richardson smoother will be investigated in the future.

5. Numerical Illustrations

We implemented two sets of numerical tests for the multigrid V-cycle algorithm (Algorithm 4.1) with different values of β on different domains. In all these tests, based on the regularity result in Theorem 3.5, suitable graded or quasi-uniform meshes are chosen to achieve the optimal convergence rate in the finite element solution, as indicated in Theorem 3.16. It is shown in all the cases, that the proposed multigrid V-cycle algorithms are contractions with contraction numbers decreasing as the number of smoothing steps l increases, but independent of mesh level k, which convincingly verifies Theorem 4.12.

5.1. **Numerical tests.** The first set of tests is for equation (30) with $\beta = 0$ on three domains (Figure 3): the square Ω_1, the L-shape domain Ω_2, and the polygon with a 170° interior angle on the z-axis Ω_3. By the calculation of regularity indices (Remarks 3.2 and 3.4), given $f \in L_1^2$, the corresponding solutions satisfy $u_0 \in H_1^2(\Omega_1)$, $u_0 \in \mathcal{K}_{a_0+1}^2(\Omega_2)$ for $a_0 < \eta_0 = \frac{2}{3}$, and $u_0 \in \mathcal{K}_{a_0+1}^2(\Omega_3)$ for $a_0 < \eta_0 \approx 0.7$. Consequently, the linear finite element approximation obtains the optimal convergence rate on quasi-uniform meshes for Ω_1 and on graded meshes toward the vertex Q with $\kappa_0 < 0.354$ for Ω_2 and $\kappa_0 < 0.372$ for Ω_3. Some examples of optimal meshes are given in Figure 3.

The contraction numbers of the proposed multigrid V-cycle algorithms are listed in Tables 1-3, where k denotes the mesh level and l is the number of pre-smoothing (post-smoothing) steps. It is clear that for fixed l, the contraction numbers are independent of the mesh level k for all domains with optimal meshes. Note that the contraction number becomes smaller when we increase the number of smoothing steps. This is also predicted by Theorem 4.12.

The second set of tests is for equation (30) with $\beta = 1$ on two domains (Figure 3): the square Ω_1 and the L-shape domain Ω_2. For $f_1 \in L_1^2$, our regularity result implies that the solutions $u_1 \in H_1^2(\Omega_1)$ and $u_1 \in \mathcal{K}_{a_1+1}^2(\Omega_2)$ for $a_1 < \eta_1 = \frac{2}{3}$.
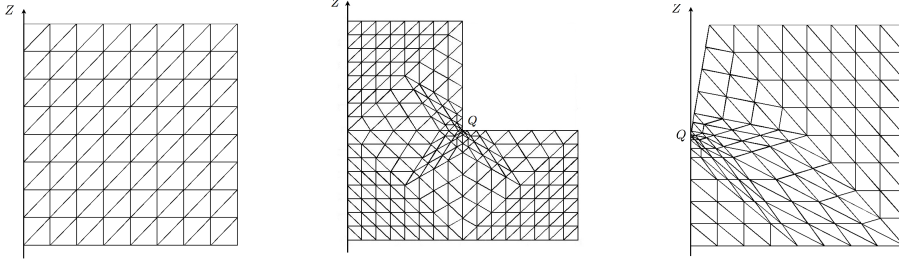
FIGURE 3. The quasi-uniform mesh for the square domain Ω_1 (left); the graded mesh for the L-shape domain Ω_2, $\kappa = 0.2$ (center); the graded mesh for the domain with the $170°$ interior angle Ω_3, $\kappa = 0.2$ (right).

$(l,l)\backslash k$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
1 - 1	0.72	0.72	0.73	0.73	0.73	0.73	0.73
2 - 2	0.52	0.53	0.53	0.53	0.53	0.53	0.53
4 - 4	0.28	0.30	0.30	0.30	0.30	0.30	0.30
8 - 8	0.11	0.12	0.12	0.12	0.12	0.12	0.12

TABLE 1. Contraction numbers of the multigrid V-cycle algorithm for equation (30) ($\beta = 0$) on Ω_1 with uniform meshes (Figure 3).

$(l,l)\backslash k$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
1 - 1	0.51	0.51	0.56	0.60	0.62	0.64	0.64
2 - 2	0.26	0.37	0.44	0.47	0.49	0.51	0.52
4 - 4	0.13	0.25	0.31	0.34	0.36	0.37	0.38
8 - 8	0.05	0.14	0.19	0.22	0.23	0.24	0.25

TABLE 2. Contraction numbers of the multigrid V-cycle algorithm for equation (30) ($\beta = 0$) on Ω_2, $\kappa_0 = 0.2$ (Figure 3).

$(l,l)\backslash k$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
1 - 1	0.85	0.87	0.90	0.91	0.91	0.92	0.92
2 - 2	0.73	0.78	0.81	0.83	0.84	0.85	0.85
4 - 4	0.56	0.64	0.69	0.72	0.74	0.75	0.75
8 - 8	0.36	0.47	0.54	0.58	0.60	0.61	0.62

TABLE 3. Contraction numbers of the multigrid V-cycle algorithm for equation (30) ($\beta = 0$) on Ω_3, $\kappa_0 = 0.2$ (Figure 3).

In particular, for a polygonal domain, the possible H_1^2-singularity in the solution only appears near vertices away from the z-axis. Namely, u_1 is always in H_1^2 near vertices on the z-axis. This is the reason that we do not consider the solution on Ω_3 in this set, since it will have the same character as the solution on the square. Then, the optimal meshes for the linear finite element approximations are quasi-uniform on Ω_1 and graded with $\kappa_1 < 0.354$ toward the vertex Q on Ω_2.

$(l,l) \backslash k$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
1 - 1	0.32	0.39	0.43	0.44	0.44	0.44	0.44
2 - 2	0.14	0.22	0.26	0.27	0.28	0.28	0.28
4 - 4	0.06	0.11	0.14	0.15	0.16	0.16	0.17
8 - 8	0.03	0.05	0.07	0.08	0.09	0.09	0.09

TABLE 4. Contraction numbers of the multigrid V-cycle algorithm for equation (30) ($\beta = 1$) on Ω_1 with uniform meshes (Figure 3).

$(l,l) \backslash k$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
1 - 1	0.34	0.49	0.56	0.60	0.62	0.63	0.64
2 - 2	0.22	0.36	0.43	0.47	0.49	0.51	0.51
4 - 4	0.11	0.23	0.30	0.33	0.35	0.37	0.38
8 - 8	0.03	0.12	0.18	0.21	0.23	0.24	0.25

TABLE 5. Contraction numbers of the multigrid V-cycle algorithm for equation (30) ($\beta = 1$) on Ω_2, $\kappa_1 = 0.2$ (Figure 3).

The contraction numbers of the V-cycle algorithms for $\beta = 1$ are listed in Tables 4-5. Similar to the case of $\beta = 0$, the algorithms converge uniformly independent of the mesh level k. These contraction numbers also well represent the asymptotic rate of decay with respect to the number of smoothing steps, given in Theorem 4.12.

5.2. **Summary.** We tested the multigrid algorithm for different values of β, with both regular and singular solutions. All the numerical tests clearly show the uniform convergence of the multigrid algorithm. The estimates of the regularity in weighted spaces and finite element approximations on graded meshes are the ingredients of our multigrid analysis.

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Hengguang Li, Department of Mathematics, Wayne State University, Detroit, MI 48202, USA

E-mail address: hli@math.wayne.edu