

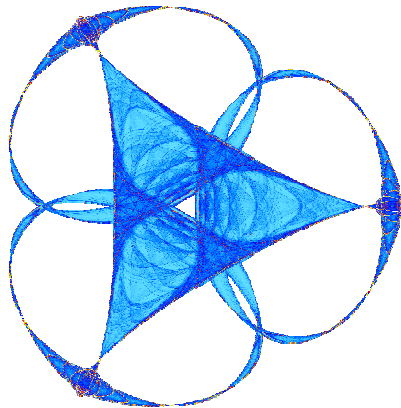
STABILITY OF SOLUTIONS TO GENERALIZED FORCHHEIMER EQUATIONS OF ANY
DEGREE

By

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STABILITY OF SOLUTIONS TO GENERALIZED FORCHHEIMER EQUATIONS OF ANY DEGREE

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ABSTRACT. The non-linear Forchheimer equations are considered as laws of hydrodynamics in porous media in case of high Reynolds numbers, when the fluid flows deviate from the ubiquitous Darcy's law. In this article, the dynamics of generalized Forchheimer equations for slightly compressible fluids are studied by means of the resulting initial boundary value problem for the pressure. We prove that the solutions depend continuously on the boundary data and the Forchheimer polynomials both in finite time and at time infinity. In contrast to related long-time dynamics results which are in the L^2 -context and require a restriction on the degree of the Forchheimer polynomial, the results obtained here are for general L^α -spaces and without this degree restriction. New bounds for the solutions are established in L^α -norm for all $\alpha \geq 1$, and then are used to improve estimates for their spatial and time derivatives. New Poincaré-Sobolev inequalities and non-linear Gronwall-type estimates for non-linear differential inequalities are utilized to achieve better asymptotic bounds.

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1. INTRODUCTION

Fluid flows in porous media are usually described by Darcy's law which is a linear relation between velocity u and pressure gradient ∇p . However this law does not hold in many cases, for instance, when the flows have high velocity or the media have fractures. Non-linear relations are used instead to model the fluid filtration in those situations.

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The common non-linear models are the so-called Forchheimer (or Darcy-Forchheimer) equations; specifically, u and ∇p satisfy the equation $g(|u|)u = -\nabla p$ where $g(s)$ is a polynomial (c.f. [4, 17]). Historically speaking, Darcy in his pioneering book [10] already used such a non-linear equation to match experimental data. While Darcy's law, even with its restriction, is well-established in literature as the basic equation for hydrodynamics in porous media, mathematical studies of the Forchheimer equations only gained attention in the 1990s (see e.g. [9, 19]). Since then there have been a growing number of articles studying them and their variations – the Brinkman-Forchheimer equations – from applied and theoretical points of view. Interested reader can find references in [21]; among recent articles are [7, 8, 15, 18, 22]; for numerical studies, see e.g. [3]. Most of them are devoted to incompressible fluids. In contrast, this article continues our previous studies [2, 13, 14] of slightly compressible fluid flows subject to Forchheimer equations.

Although the original and commonly used Forchheimer equations – the two term, three term and power laws – are only for a polynomial $g(s)$ of up to second degree, there is a need, from practical and theoretical point of view, to allow g to be a generalized polynomial with positive coefficients of higher degrees. In this case corresponding equation is called generalized Forchheimer equation (see [2]) and $g(s)$ is called the Forchheimer polynomial. The generalized Forchheimer equation can be inverted to $u = -K(|\nabla p|)\nabla p$ with conductivity function K degenerating for large ∇p . Namely, $K(\xi) \sim (1 + \xi^a)^{-1}$, where $a = \frac{\deg(g)}{(1+\deg(g))}$ with $\deg(g)$ denoting the degree of the polynomial g . For slightly compressible fluids, the description of fluid dynamics can be deduced, with a slight simplification, to a degenerate parabolic equation for the pressure $p(x, t)$:

$$\frac{\partial p}{\partial t} = \nabla \cdot (K(|\nabla p|)\nabla p) \text{ in } U \times (0, \infty). \quad (1.1)$$

Degenerate equations of this type are extensively studied mostly for their solutions' spatially local behavior and/or finite time properties (see [11] and references therein, and also, for e.g., [1] for a localization result).

Here we study the initial boundary value problem (IBVP) for (1.1) with Dirichlet boundary data on an open bounded domain U in \mathbb{R}^n and focus on its long-time dynamics. In our previous works [2, 13], see also [14], we establish the continuous dependence of the solutions of (1.1) in L^2 - and $W^{1,2-a}$ -norms on the initial, boundary data and on the Forchheimer polynomials. To effectively study the asymptotic dynamics of the solutions, a technical Degree Condition (DC) is imposed, namely, $\deg(g) \leq \frac{n}{n-2}$. This condition also arises naturally in studies of degenerate parabolic equations. (See e.g. [11, 12] for the cruciality of such condition in establishing Harnack inequalities.)

Our goals in this article are to establish a L^α -theory for $\alpha \neq 2$, as a counterpart for the L^2 -results, and to explore the problem when the DC is not met – the case we now refer

to as NDC. Two observations that support our effort: (a) The DC is not needed for time-invariant velocity field (see [2]); and (b) basic numerical simulations hint at the stability at time infinity of the solution in L^α -norm.

In this paper, we establish the continuous dependence of the solution of (1.1) on either the boundary data or coefficients of the Forchheimer polynomial in the space $L^\alpha(U)$ for $\alpha \geq 1$ and in $W^{1,2-a}(U)$. We will mainly focus on the NDC case. In the DC case we obtain sharper estimates which significantly generalize previous L^2 -estimates in [13].

The main features of our results are:

(i) New L^α -estimates of the solutions for all $\alpha \geq 1$. Because of the order of degeneracy, the differential inequality for the L^α -norm has a weak dissipative term, see (4.11). We use a new Poincaré-Sobolev inequality for the mixed term (of derivatives) $|\nabla \bar{p}|^{2-a} |\bar{p}|^{\alpha-2}$ to take advantage of this weak dissipation to gain much sharper estimates. Here \bar{p} is the solution p adjusted by the Dirichlet data.

(ii) The continuous dependence of the solution on the time-dependent boundary data in L^α -norm. Another new weighted Poincaré-Sobolev inequality is used, c.f. Lemma 2.5. Particularly, in the NDC case (part (ii) of the mentioned lemma) the inequality is highly non-linear and involves, in addition to the weight $K(\xi(x))$, a compensating term $|u|^{\theta_2 \alpha}$. We then utilize a new non-linear Gronwall-type estimates, c.f. Lemma A.1, to explore the resulting differential inequality.

(iii) The continuous dependence of the solution on the Forchheimer polynomial in L^α -norm. This requires estimates of the cross-terms $|\nabla p_1|^{2-a} |\bar{p}_2|^{\alpha-2}$ and $|\nabla p_2|^{2-a} |\bar{p}_1|^{\alpha-2}$ of individual solutions p_1 and p_2 . By using the structure of the equation we bound them by the mixed terms which are dealt with in (i).

(iv) The continuous dependence of the pressure gradient in L^{2-a} -norm with respect to the boundary data and the Forchheimer polynomials.

(v) The above continuous dependence results are established in both DC and NDC cases.

The paper is organized as follows. In section 2 we recall the main definitions and relevant results from [2, 13, 14]. We then obtain some inequalities of Poincaré-Sobolev type in Lemmas 2.3 and 2.5. They are suitable for the non-linear degenerate parabolic equation (1.1), and are essential to our analysis of the solutions, particularly their asymptotic behavior. In section 3, using the theory of monotone operators [5, 16, 20, 23], we prove the global existence of weak solutions in $C([0, \infty), L^\alpha(U))$ and $L_{loc}^{2-a}([0, \infty), W^{1,2-a}(U))$. In section 4 we derive more refined estimates of the solutions to (1.1) in $L^\alpha(U)$ for all $\alpha \geq 1$ with explicit dependence on the time-dependent boundary data. (For this, the initial data are required to belong to $L^{\hat{\alpha}}(U)$ with $\hat{\alpha} \geq \alpha$, see Definition 4.2.) These are used to improve estimates of the pressure's spatial and time derivatives, especially for the NDC case. Moreover, asymptotic bounds for the solutions are obtained. As a consequence, we

are able to estimate the mixed terms $\int_U |\nabla p|^{2-a} |p|^\alpha dx$ for arbitrarily large α . This result will be key to proving the structural stability in section 6. In section 5 we establish the continuous dependence of solutions on the boundary data in L^α - and $W^{1,2-a}$ - norms. To investigate this for large time we use a generalization of Gronwall's inequality for certain non-linear differential inequalities at finite time as well as at time infinity. The upper bounds of difference between two solutions for large time and their limits superior when $t \rightarrow \infty$ are given in Theorems 5.2–5.6. It is noteworthy that these bounds are independent of the initial data. In section 6 we prove various theorems on structural stability for equation (1.1). More precisely, we show that the solution depends continuously on the coefficients of the Forchheimer polynomial both for finite time intervals and for time infinity and in both DC and NDC cases. Unlike previous L^2 -estimates in [13] one must control the cross-terms $\int_U |\nabla p_1|^{2-a} |\bar{p}_2|^{\alpha-2} dx$ and $\int_U |\nabla p_2|^{2-a} |\bar{p}_1|^{\alpha-2} dx$, see (6.11) of Lemma 6.1. This cannot be achieved directly from previous sections' estimates. However by exploring the structure of equation (1.1) we can estimate these integrals using the mixed terms $\int_U |\nabla p_1|^{2-a} |\bar{p}_1|^\gamma dx$ and $\int_U |\nabla p_2|^{2-a} |\bar{p}_2|^\gamma dx$ for possibly very large γ ; the latter integrals are already estimated in section 4. The Appendix generalizes our previous results in [13, 14] to give bounds for solutions to a wider class of non-linear differential inequalities.

2. BACKGROUND AND SUPPLEMENTARIES

Consider a fluid in a porous medium occupying a bounded domain U in space \mathbb{R}^n . Throughout this paper, $n \geq 2$ even though for physics problems $n = 2$ or 3 . Let $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$ be the spatial and time variables. The fluid flow has velocity $u(x, t) \in \mathbb{R}^n$, pressure $p(x, t) \in \mathbb{R}$ and density $\rho(x, t) \in \mathbb{R}^+ = [0, \infty)$.

A generalized Forchheimer equation is

$$g(|u|)u = -\nabla p, \quad (2.1)$$

where $g(s) \geq 0$ is a function defined on $[0, \infty)$. It is considered as a momentum equation and is studied in [2, 13, 14]. When $g(s) = \alpha, \alpha + \beta s, \alpha + \beta s + \gamma s^2, \alpha + \gamma_m s^{m-1}$, where $\alpha, \beta, \gamma, m, \gamma_m$ are empirical constants, we have Darcy's law, Forchheimer's two term, three term and power laws, respectively. In this paper, we study the case when the function g in (2.1) is a generalized polynomial with non-negative coefficients. More precisely, the function $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is of the form

$$g(s) = a_0 s^{\alpha_0} + a_1 s^{\alpha_1} + \dots + a_N s^{\alpha_N}, \quad s \geq 0, \quad (2.2)$$

where $N \geq 1, \alpha_0 = 0 < \alpha_1 < \dots < \alpha_N$ are fixed real numbers, the coefficients a_0, \dots, a_N are non-negative numbers with $a_0 > 0$ and $a_N > 0$. From (2.1) one can solve for u in terms of ∇p and obtain a non-linear version of Darcy's equation:

$$u = -K(|\nabla p|)\nabla p, \quad (2.3)$$

where the function $K : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is defined for $\xi \geq 0$ by

$$K(\xi) = \frac{1}{g(s(\xi))}, \text{ with } s = s(\xi) \text{ being the unique non-negative solution of } sg(s) = \xi. \quad (2.4)$$

The number α_N is the degree of g and is denoted by $\deg(g)$. The vector of powers in (2.2) is denoted by $\vec{\alpha} = (\alpha_0, \dots, \alpha_N)$ and the vector $\vec{a} = (a_0, \dots, a_N)$ is referred to as the coefficient vector. When the dependence on \vec{a} needs to be specified, we use notation $g(s, \vec{a})$, $K(\xi, \vec{a})$ to denote the corresponding functions in (2.2) and (2.4).

In addition to (2.1) we have the equation of continuity

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0, \quad (2.5)$$

and the equation of state which, for slightly compressible fluids, is

$$\frac{d\rho}{dp} = \frac{\rho}{\kappa}, \quad \kappa > 0. \quad (2.6)$$

Substituting (2.3) and (2.5) into (2.6) we obtain a scalar equation for the pressure:

$$\frac{\partial p}{\partial t} = \kappa \nabla \cdot (K(|\nabla p|) \nabla p) + K(|\nabla p|) |\nabla p|^2. \quad (2.7)$$

One the right hand side of (2.7) the constant κ is very large for most slightly compressible fluid in porous media, hence we neglect its second term and study the following reduced equation

$$\frac{\partial p}{\partial t} = \kappa \nabla \cdot (K(|\nabla p|) \nabla p). \quad (2.8)$$

(This simplification is used commonly in petroleum engineering.) By scaling the time variable $t \rightarrow \kappa t$, we can assume throughout that $\kappa = 1$.

The class of functions $g(s)$ as in (2.2) is denoted by $FP(N, \vec{\alpha})$ which is the abbreviation of ‘‘Forchheimer polynomials’’. When the function g in (2.1) is one of the $g(s)$ in (2.2), it is referred to as the Forchheimer polynomial.

Let $g = g(s, \vec{a})$ in $FP(N, \vec{\alpha})$. The following exponent is frequently used in our calculations

$$a = \frac{\alpha_N}{\alpha_N + 1} \in (0, 1). \quad (2.9)$$

The function $K(\xi)$ in (2.4) has the following properties: it is decreasing in ξ , maps $[0, \infty)$ onto $(0, \frac{1}{a_0}]$ and

$$\frac{d_1}{(1 + \xi)^a} \leq K(\xi) \leq \frac{d_2}{(1 + \xi)^a}, \quad (2.10)$$

$$d_3(\xi^{2-a} - 1) \leq K(\xi)\xi^2 \leq d_2\xi^{2-a}, \quad (2.11)$$

where d_1, d_2, d_3 are positive constants depending on $\vec{\alpha}$ and \vec{a} .

As in [2, 13, 14], we define

$$H(\xi) = \int_0^{\xi^2} K(\sqrt{s}) ds \quad \text{for } \xi \geq 0.$$

The function $H(\xi)$ can be compared with ξ and $K(\xi)$ by

$$K(\xi)\xi^2 \leq H(\xi) \leq 2K(\xi)\xi^2, \quad (2.12)$$

and hence, as a consequence of (2.11) and (2.12), we have

$$d_3(\xi^{2-a} - 1) \leq H(\xi) \leq 2d_2\xi^{2-a}. \quad (2.13)$$

Since $H'(\xi) = 2\xi K(\xi)$ is non-negative and, by Lemma III.5 of [2], increasing,

$$H(\xi) \text{ is an increasing and convex function on } [0, \infty). \quad (2.14)$$

We recall here Lemma 2.3 in [13] on an important monotonicity property.

Lemma 2.1 (c.f. [13]). (i) For any $y, y' \in \mathbb{R}^n$,

$$(K(|y'|)y' - K(|y|)y) \cdot (y' - y) \geq (1 - a)K(\max\{|y|, |y'|\})|y' - y|^2. \quad (2.15)$$

(ii) For two functions p_1 and p_2 defined on U , one has

$$\begin{aligned} & \int_U (K(|\nabla p_1|)\nabla p_1 - K(|\nabla p_2|)\nabla p_2) \cdot (\nabla p_1 - \nabla p_2) dx \\ & \geq (1 - a) \int_U K(\max\{|\nabla p_1|, |\nabla p_2|\}) |\nabla p_1 - \nabla p_2|^2 dx \\ & \geq d_4 \left[\int_U |\nabla p_1 - \nabla p_2|^{2-a} dx \right]^{\frac{2}{2-a}} (1 + \max\{\|\nabla p_1\|_{L^{2-a}}, \|\nabla p_2\|_{L^{2-a}}\})^{-a}, \end{aligned}$$

where $d_4 = d_4(\vec{\alpha}, \vec{a}, n, U) > 0$.

Definition 2.2. We will refer to the following inequality as the Degree Condition

$$\deg(g) \leq \frac{4}{n-2}. \quad (\text{DC})$$

We label the negation of (DC) by (NDC), that is,

$$\deg(g) > \frac{4}{n-2}. \quad (\text{NDC})$$

We will use the phrase “*the DC case*”, respectively “*the NDC case*”, to refer to the case when condition (DC), respectively condition (NDC), holds.

We establish below a couple of Poincaré-Sobolev inequalities which are suitable to our type of degeneracy in (2.8) and are essential to our estimates in subsequent sections. We first recall the classical Poincaré-Sobolev inequality.

Let $\mathring{W}^{1,r}(U)$ be the space of functions in $W^{1,r}(U)$ with vanishing traces on the boundary. If $1 \leq r < n$, then by Sobolev’s imbedding theorem and Poincaré’s inequality, we have

$$\|f\|_{L^{r^*}(U)} \leq C \|\nabla f\|_{L^r(U)} \text{ for all } f \in \mathring{W}^{1,r}(U), \quad (2.16)$$

where the constant C depends on r, n and the domain U , and

$$r^* = nr/(n-r).$$

Throughout, r_0 is the conjugate of $(2-a)^*$, that is, $\frac{1}{r_0} + \frac{1}{(2-a)^*} = 1$.

We denote

$$\alpha_* = \frac{na}{2-a} = \frac{n\alpha_N}{\alpha_N + 2},$$

a threshold exponent for the validity of our inequalities below.

One can easily verify that

$$\begin{aligned} (DC) &\Leftrightarrow 2 \leq (2-a)^* \Leftrightarrow r_0 \leq 2 \Leftrightarrow \alpha_* \leq 2, \\ (NDC) &\Leftrightarrow 2 > (2-a)^* \Leftrightarrow r_0 > 2 \Leftrightarrow \alpha_* > 2. \end{aligned} \quad (2.17)$$

In the following, C denotes a generic positive constant whose value may vary from one line to another.

Lemma 2.3. *Let U be an open, bounded domain in \mathbb{R}^n and $\alpha \geq \max\{2, \alpha_*\}$. There exists $c_1 = c_1(\alpha, \vec{\alpha}, \vec{a}, n, U) > 0$ such that if u is a function on U vanishing on the boundary ∂U , then*

$$\int_U |u|^\alpha dx \leq c_1 \left\{ \int_U K(|\nabla u|)|\nabla u|^2 |u|^{\alpha-2} dx + \left(\int_U K(|\nabla u|)|\nabla u|^2 |u|^{\alpha-2} dx \right)^{\gamma_0} \right\}, \quad (2.18)$$

where $\gamma_0 = \gamma_0(\alpha) = \frac{\alpha}{\alpha-a}$; consequently,

$$\int_U |\nabla u|^{2-a} |u|^{\alpha-2} dx \geq c_2 \left(\left(\int_U |u|^\alpha dx \right)^{\frac{1}{\gamma_0}} - 1 \right), \quad (2.19)$$

where $c_2 = c_2(\alpha, \vec{\alpha}, \vec{a}, n, U) > 0$.

Proof. Let $\xi = \xi(x)$ be a non-negative function on U . Let $\delta = 1 - \frac{a}{2} \in (0, 1)$, $r = 2 - a \in (1, 2)$ and $m = \frac{\alpha-a}{2-a} \geq 1$. Applying Poincaré-Sobolev's inequality (2.16) to the function $|u|^m$, we have

$$\left(\int_U |u|^{mr^*} dx \right)^{\frac{1}{r^*}} \leq C \left[\int_U (\nabla |u|^m)^r dx \right]^{\frac{1}{r}} \leq C \left[\int_U |\nabla u|^r K^\delta(\xi) \cdot K^{-\delta}(\xi) |u|^{(m-1)r} dx \right]^{\frac{1}{r}}.$$

Applying Hölder's inequality with powers $1/\delta$ and $1/(1-\delta)$ gives

$$\left(\int_U |u|^{mr^*} dx \right)^{\frac{1}{r^*}} \leq C \left[\int_U |\nabla u|^{\frac{r}{\delta}} K(\xi) |u|^{(m-1)r} dx \right]^{\frac{\delta}{r}} \left[\int_U K^{\frac{-\delta}{1-\delta}}(\xi) |u|^{(m-1)r} dx \right]^{\frac{1-\delta}{r}}.$$

Using (2.10) to estimate $K^{\frac{-\delta}{1-\delta}}(\xi)$ yields

$$\begin{aligned} \left(\int_U |u|^{mr^*} dx \right)^{\frac{1}{r^*}} &\leq C \left[\int_U |\nabla u|^{\frac{r}{\delta}} K(\xi) |u|^{(m-1)r} dx \right]^{\frac{\delta}{r}} \left[\int_U (1 + \xi)^{\frac{a\delta}{1-\delta}} |u|^{(m-1)r} dx \right]^{\frac{1-\delta}{r}} \\ &= C \left[\int_U |\nabla u|^2 K(\xi) |u|^{\alpha-2} dx \right]^{\frac{1}{2}} \left[\int_U (1 + \xi)^{2-a} |u|^{\alpha-2} dx \right]^{\frac{a}{2(2-a)}}. \end{aligned}$$

Using relation (2.11) to compare $(1 + \xi)^{2-a}$ with $1 + K(\xi)\xi^2$, we assert

$$\left(\int_U |u|^{mr^*} dx \right)^{\frac{1}{r^*}} \leq C \left[\int_U K(\xi) |\nabla u|^2 |u|^{\alpha-2} dx \right]^{\frac{1}{2}} \left[\int_U (1 + K(\xi)\xi^2) |u|^{\alpha-2} dx \right]^{\frac{a}{2(2-a)}}. \quad (2.20)$$

Since $\alpha \geq \alpha_*$, elementary calculations show $\alpha \leq mr^*$. Then using Holder's inequality and applying (2.20) with $\xi = |\nabla u|$ yield

$$\begin{aligned} \int_U |u|^\alpha dx &\leq C \left(\int_U |u|^{mr^*} dx \right)^{\frac{\alpha}{mr^*}} \\ &\leq C \left[\int_U K(|\nabla u|)|\nabla u|^2 |u|^{\alpha-2} dx \right]^{\frac{\alpha(2-a)}{2(\alpha-a)}} \left[\int_U (1 + K(|\nabla u|)|\nabla u|^2) |u|^{\alpha-2} dx \right]^{\frac{a\alpha}{2(\alpha-a)}} \\ &\leq C \left[\int_U K(|\nabla u|)|\nabla u|^2 |u|^{\alpha-2} dx \right]^{\frac{\alpha(2-a)}{2(\alpha-a)}} \left[\int_U |u|^{\alpha-2} dx + \int_U K(|\nabla u|)|\nabla u|^2 |u|^{\alpha-2} dx \right]^{\frac{a\alpha}{2(\alpha-a)}}. \end{aligned}$$

Hence

$$\begin{aligned} \int_U |u|^\alpha dx &\leq C \left[\int_U K(|\nabla u|)|\nabla u|^2 |u|^{\alpha-2} dx \right]^{\frac{\alpha(2-a)}{2(\alpha-a)}} \left[\int_U |u|^\alpha dx \right]^{\frac{a(\alpha-2)}{2(\alpha-a)}} \\ &\quad + C \left[\int_U K(|\nabla u|)|\nabla u|^2 |u|^{\alpha-2} dx \right]^{\frac{\alpha}{\alpha-a}}. \end{aligned}$$

Applying Young's inequality with powers $\frac{2(\alpha-a)}{\alpha(2-a)}$ and $\frac{2(\alpha-a)}{a(2-a)}$ to the product term on the right-hand side of the previous inequality, we obtain

$$\begin{aligned} \int_U |u|^\alpha dx &\leq C \int_U K(|\nabla u|)|\nabla u|^2 |u|^{\alpha-2} dx + \frac{1}{2} \int_U |u|^\alpha dx \\ &\quad + C \left[\int_U K(|\nabla u|)|\nabla u|^2 |u|^{\alpha-2} dx \right]^{\frac{\alpha}{\alpha-a}}, \end{aligned}$$

and (2.18) follows as a consequence.

We define a family of non-linear functions $\varphi_{c,\gamma}$ by

$$\varphi_{c,\gamma}(z) = c(z + z^\gamma), \quad \text{for } c > 0, \gamma > 0, z \geq 0. \quad (2.21)$$

Since $K(|\nabla u|)|\nabla u|^2 \leq d_2 |\nabla u|^{2-a}$ we have from (2.18) that

$$\int_U |u|^\alpha dx \leq \varphi_{c,\gamma_0} \left(\int_U |\nabla u|^{2-a} |u|^{\alpha-2} dx \right),$$

where $c = c_1 \max\{d_2, d_2^{\gamma_0}\}$, and hence

$$\int_U |\nabla u|^{2-a} |u|^{\alpha-2} dx \geq \varphi_{c,\gamma_0}^{-1} \left(\int_U |u|^\alpha dx \right). \quad (2.22)$$

For $z \geq 0$, let $y = \varphi_{c,\gamma_0}(z) = c(z + z^{\gamma_0})$. Then $y \leq 1 + C_1 z^{\gamma_0}$, where $C_1 = c + c^{\gamma_0}$, hence $z^{\gamma_0} \geq C_1^{-1}(y - 1)$. Noticing that $1/\gamma_0 < 1$, we assert $z \geq C_1^{-1/\gamma_0}(y^{\frac{1}{\gamma_0}} - 1)$, that is, $\varphi_{c,\gamma_0}^{-1}(y) \geq C_1^{-1/\gamma_0}(y^{\frac{1}{\gamma_0}} - 1)$. Therefore we obtain (2.19) from (2.22). \square

Remark 2.4. For $\alpha \geq \max\{2, \alpha_*\}$, a straightforward application of the imbedding from the Sobolev space $\mathring{W}^{1,2-a}(U)$ into $L^{\frac{\alpha(2-a)}{\alpha-a}}(U)$ to function $|u|^\alpha$ gives

$$\int_U |u|^\alpha dx \leq C \left[\int_U |\nabla u|^{2-a} |u|^{\alpha-2} dx \right]^{\frac{\alpha}{2-a}}, \quad (2.23)$$

$$\int_U |\nabla u|^{2-a} |u|^{\alpha-2} dx \geq C \left[\int_U |u|^\alpha dx \right]^{\frac{2-a}{\alpha}}. \quad (2.24)$$

Since $1 < \gamma_0 = \frac{\alpha}{\alpha-a} < \frac{\alpha}{2-a}$, then roughly speaking, (2.18) is better than (2.23), and (2.19) is better than (2.24).

We now prove another Poincaré-Sobolev inequality with a specific weight.

Lemma 2.5. *Let U be an open bounded domain in \mathbb{R}^n and $\xi = \xi(x)$ be a non-negative function defined on U . Assume $\alpha \geq 2$.*

(i) *In the DC case, there is a constant $c_3 = c_3(\alpha, \vec{\alpha}, \vec{a}, n, U) > 0$ such that*

$$\int_U |u|^\alpha dx \leq c_3 \left[\int_U K(\xi) |\nabla u|^2 |u|^{\alpha-2} dx \right] \left[1 + \int_U \xi^{2-a} dx \right]^{\frac{\alpha}{2-a}} \quad (2.25)$$

for any function $u(x)$ vanishing on the boundary ∂U .

(ii) *In the NDC case, given two numbers θ and θ_1 that satisfy*

$$\theta > \frac{2}{(2-a)^*} \text{ and } \max \left\{ 1, \frac{2n}{n\theta + 2} \right\} \leq \theta_1 < 2 - a, \quad (2.26)$$

there is a constant $c_4 = c_4(\alpha, \theta, \theta_1, \vec{\alpha}, \vec{a}, n, U) > 0$ such that

$$\int_U |u|^\alpha dx \leq c_4 \left[\int_U K(\xi) |\nabla u|^2 |u|^{\alpha-2} dx \right]^{\frac{1}{\theta}} \left[1 + \int_U \xi^{2-a} + |u|^{\theta_2 \alpha} dx \right]^{\frac{2-\theta_1}{\theta \theta_1}} \quad (2.27)$$

for any function $u(x)$ vanishing on the boundary ∂U , where

$$\theta_2 = \frac{\theta_1(\theta - 1)(2 - a)}{2(2 - a - \theta_1)} > 0. \quad (2.28)$$

Proof. Suppose θ_1 and m are two numbers that satisfy $1 \leq \theta_1 < n$ and $\frac{\alpha}{m} \leq \theta_1^*$. By the standard Poincaré-Sobolev's inequality corresponding to the imbedding of $\dot{W}^{1,\theta_1}(U)$ into $L^{\frac{\alpha}{m}}(U)$, we have

$$\int_U |u|^\alpha dx = \int_U (|u|^m)^{\frac{\alpha}{m}} dx \leq C \left[\int_U |\nabla |u|^m|^{\theta_1} dx \right]^{\frac{\alpha}{m\theta_1}} \leq C \left[\int_U |\nabla u|^{\theta_1} |u|^{(m-1)\theta_1} dx \right]^{\frac{\alpha}{m\theta_1}}.$$

If $\delta_1 \in [0, 1]$ and $0 < \theta_1 < 2$ then rewriting the above inequality as

$$\int_U |u|^\alpha dx \leq C \left[\int_U \left(|\nabla u|^{\theta_1} K(\xi)^{\frac{\theta_1}{2}} |u|^{\delta_1(m-1)\theta_1} \right) \left(K(\xi)^{-\frac{\theta_1}{2}} |u|^{(1-\delta_1)(m-1)\theta_1} \right) dx \right]^{\frac{\alpha}{m\theta_1}},$$

and then applying Holder's inequality with powers $2/\theta_1$ and $2/(2 - \theta_1)$ give

$$\int_U |u|^\alpha dx \leq C \left[\int_U K(\xi) |\nabla u|^2 |u|^{2\delta_1(m-1)} dx \right]^{\frac{\alpha}{2m}} \left[\int_U K(\xi)^{-\frac{\theta_1}{2-\theta_1}} |u|^{\frac{2\theta_1(1-\delta_1)(m-1)}{2-\theta_1}} dx \right]^{\frac{\alpha(2-\theta_1)}{2\theta_1 m}}. \quad (2.29)$$

(i) Consider the DC case. Set $\theta_1 = 2 - a$, $m = \frac{\alpha}{2}$ and $\delta_1 = 1$, then $1 < \theta_1 < 2$ and $\alpha/m = 2 \leq \theta_1^*$ thanks to (2.17). By (2.29) and (2.10) we have

$$\begin{aligned} \int_U |u|^\alpha dx &\leq C \left[\int_U K(\xi) |\nabla u|^2 |u|^{\alpha-2} dx \right] \left[\int_U K(\xi)^{-\frac{2-a}{a}} dx \right]^{\frac{a}{2-a}} \\ &\leq C \left[\int_U K(\xi) |\nabla u|^2 |u|^{\alpha-2} dx \right] \left[\int_U (1 + \xi^{2-a}) dx \right]^{\frac{a}{2-a}}, \end{aligned}$$

thus obtaining (2.25).

(ii) Next, we consider the NDC case. Note that $\theta > \frac{2}{(2-a)^*}$ is equivalent to $\frac{2n}{n\theta+2} < 2-a$, hence there always exists θ_1 that satisfies the second condition in (2.26). Also, $\theta > 1$ thanks to (2.17).

Set $m = \theta\alpha/2$ and $\delta_1 = (\alpha-2)/(2(m-1))$. Then $m > \alpha/2 \geq 1$ and hence $\delta_1 \in [0, 1)$. Since $\theta_1 \geq \frac{2n}{n\theta+2}$, we have $2m/\alpha = \theta \geq 2/\theta_1^*$ or $\alpha/m \leq \theta_1^*$. Note that $2\delta_1(m-1) = \alpha-2$ and $2(1-\delta_1)(m-1) = 2(m-1) - (\alpha-2) = 2m - \alpha$. Therefore it follows from (2.29) that

$$\begin{aligned} \int_U |u|^\alpha dx &\leq \left[\int_U K(\xi) |\nabla u|^2 |u|^{\alpha-2} dx \right]^{\frac{\alpha}{2m}} \left[\int_U K(\xi)^{-\frac{\theta_1}{2-\theta_1}} |u|^{\frac{\theta_1(2m-\alpha)}{2-\theta_1}} dx \right]^{\frac{\alpha(2-\theta_1)}{2\theta_1 m}} \\ &\leq \left[\int_U K(\xi) |\nabla u|^2 |u|^{\alpha-2} dx \right]^{\frac{\alpha}{2m}} \left[\int_U (1 + \xi)^{\frac{a\theta_1}{2-\theta_1}} |u|^{\frac{\theta_1(2m-\alpha)}{2-\theta_1}} dx \right]^{\frac{\alpha(2-\theta_1)}{2\theta_1 m}}. \end{aligned} \quad (2.30)$$

Since $\theta_1 < 2 - a$ implies $\frac{a\theta_1}{2-\theta_1} < 2 - a$, we apply Young's inequality to obtain

$$\int_U (1 + \xi)^{\frac{a\theta_1}{2-\theta_1}} |u|^{\frac{\theta_1(2m-\alpha)}{2-\theta_1}} dx \leq C \int_U (1 + \xi)^{2-a} + |u|^{\frac{\theta_1(2m-\alpha)(2-a)}{2(2-\theta_1-a)}} dx. \quad (2.31)$$

Note that

$$\frac{\theta_1(2m-\alpha)(2-a)}{2(2-\theta_1-a)} = \theta_2\alpha. \quad (2.32)$$

Therefore we have from (2.30), (2.31) and (2.32) that

$$\int_U |u|^\alpha dx \leq C \left[\int_U K(\xi) |\nabla u|^2 |u|^{\alpha-2} dx \right]^{\frac{\alpha}{2m}} \left[\int_U (1 + |u|^{\theta_2\alpha} + \xi^{2-a}) dx \right]^{\frac{\alpha(2-\theta_1)}{2m\theta_1}},$$

which proves (2.27). \square

We derive below some simple but useful estimates for solutions of certain non-linear ordinary differential inequalities.

Definition 2.6. Given $f(t)$ defined on an interval $I \subset \mathbb{R}$. A function $F(t)$ is called an (upper) envelop of $f(t)$ on I if $F(t) \geq f(t)$ for all $t \in I$. We denote by $Env(f)$ a continuous, increasing envelop function of $f(t)$.

Lemma 2.7. Let $\theta > 0$ and let $y(t) \geq 0, h(t) > 0, f(t) \geq 0$ be continuous functions on $[0, \infty)$ that satisfy

$$y'(t) \leq -h(t)y(t)^\theta + f(t) \quad \text{for all } t > 0.$$

Then

$$y(t) \leq y(0) + [Env(f(t)/h(t))]^{\frac{1}{\theta}} \quad \text{for all } t \geq 0. \quad (2.33)$$

If $\int_0^\infty h(t)dt = \infty$ then

$$\limsup_{t \rightarrow \infty} y(t) \leq \limsup_{t \rightarrow \infty} [f(t)/h(t)]^{\frac{1}{\theta}}. \quad (2.34)$$

In the simple case when $y(t) = \int_0^t e^{-k(t-\tau)} f(\tau) d\tau$, where k is a positive number, we have from (2.34) that

$$\limsup_{t \rightarrow \infty} \int_0^t e^{-k(t-\tau)} f(\tau) d\tau \leq k^{-1} \limsup_{t \rightarrow \infty} f(t). \quad (2.35)$$

Lemma 2.7 is a special case of Lemma A.1 when applied to the function $\phi(s) = s^{1/\theta}$. The reader is referred to Appendix A for its proof.

3. EXISTENCE RESULTS

Our aim is to study the IBVP for equation (2.8) in a bounded domain. Here afterward U is a bounded open connected subset of \mathbb{R}^n , $n = 2, 3, \dots$ with C^2 boundary $\Gamma = \partial U$. In this section the number $N \geq 1$, the vectors $\vec{\alpha}$ and \vec{a} , and the Forchheimer polynomial $g(s, \vec{a}) \in FP(N, \vec{a})$ all are fixed. Denote $g(s) = g(s, \vec{a})$ and let $K(\xi)$ and $H(\xi)$ be defined as in the previous section.

Consider the following IBVP for $p(x, t)$:

$$\begin{cases} \frac{\partial p}{\partial t} = \nabla \cdot (K(|\nabla p|)\nabla p) \text{ in } U \times (0, \infty), \\ p(x, 0) = p_0(x) \text{ in } U, \\ p(x, t) = \psi(x, t) \text{ on } \Gamma \times (0, \infty). \end{cases} \quad (3.1)$$

In order to deal with the non-homogeneous boundary condition, the data $\psi(x, t)$ with $x \in \Gamma$ and $t > 0$ is extended to a function $\Psi(x, t)$ with $x \in \bar{U}$ and $t \geq 0$. Throughout, our results are stated in terms of Ψ instead of ψ . Nonetheless, corresponding results in terms of ψ can be retrieved as performed in [13].

In this section we prove the global (in time) existence and uniqueness of weak solutions to (3.1). The proof is based on the theory of monotone operators (c.f. [5, 16, 20, 23]).

To reduce our problem to the framework of the general theory let us recollect some definitions.

Definition 3.1. Let $\mathbf{V}_1, \mathbf{V}_2$ be Banach spaces, $A : \mathbf{V}_1 \rightarrow \mathbf{V}_2$ be a map. Then A is called:

- **bounded** if it maps bounded sets in \mathbf{V}_1 into bounded sets in \mathbf{V}_2 ,
- **(weakly) continuous** if it maps (weakly) convergent sequences in \mathbf{V}_1 into (weakly) convergent sequences in \mathbf{V}_2 ,
- **completely continuous** if it maps weakly convergent in \mathbf{V}_1 sequences into strongly convergent sequences in \mathbf{V}_2 ,
- **demi-continuous** if it maps strongly convergent sequences in \mathbf{V}_1 into weakly convergent sequences in \mathbf{V}_2 .

In case $\mathbf{V}_1 = \mathbf{V}$ and $\mathbf{V}_2 = \mathbf{V}'$, that is, $A : \mathbf{V} \rightarrow \mathbf{V}'$, the map A is called

- **hemi-continuous** if $t \mapsto \mathbf{V}'\langle A(u + tv), v \rangle_{\mathbf{V}}$ is continuous for all $u, v \in \mathbf{V}$,
- **monotone** if $\mathbf{V}'\langle Au - Av, u - v \rangle_{\mathbf{V}} \geq 0$ for all $u, v \in \mathbf{V}$,
- **of type M** if for every sequence $\{u_n\}_{n=1}^{\infty}$ in \mathbf{V} such that $u_n \rightarrow u$ weakly in \mathbf{V} , $Au_n \rightarrow \xi$ weakly in \mathbf{V}' as $n \rightarrow \infty$, and $\limsup_{n \rightarrow \infty} \mathbf{V}'\langle Au_n, u_n \rangle_{\mathbf{V}} \leq \mathbf{V}'\langle \xi, u \rangle_{\mathbf{V}}$, one has $Au = \xi$.

Above, \mathbf{V}' denotes the dual space of \mathbf{V} and $\mathbf{V}'\langle w, v \rangle_{\mathbf{V}}$ denotes $w(v)$ for $w \in \mathbf{V}'$ and $v \in \mathbf{V}$.

From Lemmas 2.1 and 2.2 in [20] follows:

Proposition 3.2. *Let \mathbf{V} be a Banach space and let $A : \mathbf{V} \rightarrow \mathbf{V}'$. Then:*

- (i) *if A is monotone and hemi-continuous then A is of type M,*
- (ii) *if A is bounded and of type M then it is demi-continuous.*

Next we prove an abstract existence result which will be applied to our particular problem. This follows Propositions 4.1, 4.2 and 5.1 in Chapter III from [20], and Theorem 30.A from [23] with necessary modifications. We provide some details of the proof for the sake of unity and self-containment. Below, for a number $r > 1$ we denote r' its conjugate, that is, $1/r + 1/r' = 1$.

Theorem 3.3. *Let \mathbf{V} be a separable reflexive Banach space, and \mathbf{H} be a separable Hilbert space such that $\mathbf{V} \cap \mathbf{H}$ is dense in \mathbf{H} . Assume that a family of operators*

$$A(t) : \mathbf{V} \rightarrow \mathbf{V}', \quad t \in (0, T),$$

is given such that

- (1) *for every $v \in \mathbf{V}$, the map $t \mapsto A(t)v$ is a measurable map $(0, T) \rightarrow \mathbf{V}'$,*
- (2) *for almost all (a.a.) $t \in (0, T)$, the operator $A(t)$ is monotone, hemi-continuous and bounded with*

$$\|A(t)v\|_{\mathbf{V}'} \leq c(\|v\|_{\mathbf{V}}^{r-1} + k_0^{r-1}(t)), \quad \text{for all } v \in \mathbf{V}, \quad (3.2)$$

with some constant $c > 0$, a number $r \in (1, \infty)$, and a non-negative function $k_0 \in L^r(0, T)$.

- (3) *there exists a semi-norm $[\cdot]$ on \mathbf{V} , constants $\gamma, \lambda > 0$, and a non-negative function $k_1 \in L^r(0, T)$ such that*

$$\lambda\|v\|_{\mathbf{H}} + [v] \geq \gamma\|v\|_{\mathbf{V}} \quad \text{for all } v \in \mathbf{V} \cap \mathbf{H}, \quad (3.3)$$

$$\mathbf{V}'\langle A(t)v, v \rangle_{\mathbf{V}} \geq \gamma[v]^r - k_1^r(t) \quad \text{for all } v \in \mathbf{V}, \text{ a.a. } t \in (0, T). \quad (3.4)$$

Part I. *Then, for every $u_0 \in \mathbf{H}$ and $f \in L^{r'}(0, T; \mathbf{V}')$ there exists a solution u to the abstract Cauchy problem*

$$u' + A(t)u = f(t) \text{ in } L^{r'}(0, T; \mathbf{V}'), \quad u(0) = u_0, \quad (3.5)$$

that satisfies

$$u \in C([0, T]; \mathbf{H}) \cap L^r(0, T; \mathbf{V}), \quad u' \in L^{r'}(0, T; \mathbf{V}').$$

Moreover, this solution is unique in the class

$$\{u \in L^r(0, T; \mathbf{V} \cap \mathbf{H}) : u' \in L^{r'}(0, T; \mathbf{V}' + \mathbf{H})\}.$$

Part II. In addition, assume that there exists a map $\Phi = \Phi(v, t)$ from $\mathbf{V} \times [0, T]$ to \mathbb{R}^+ , convex and locally bounded in the first variable for all $t \in [0, T]$, such that the following chain rule holds

$$\frac{d}{dt}\Phi(v(t), t) = \mathbf{v}' \langle A(t)v(t), v'(t) \rangle_{\mathbf{V}} + \partial_t \Phi(v(t), t) \quad \text{for a.a. } t \in (0, T) \quad (3.6)$$

for every absolutely continuous finite dimensional trajectory $v : (0, T) \rightarrow \mathbf{V} \cap \mathbf{H}$ with $v' \in L^{r'}(0, T; \mathbf{V})$. Assume there is $c > 0$ such that

$$|\partial_t \Phi(v, t)| \leq c(\|v\|_{\mathbf{V}}^r + k_2(t)) \quad \text{for all } v \in \mathbf{V} \cap \mathbf{H}, \text{ a.a. } t \in (0, T), \quad (3.7)$$

where k_2 is a non-negative function. Then the following regularity result holds.

- (i) If $f \in L^2(0, T; \mathbf{H}; t^\mu dt)$, $\Phi(0, \cdot) \in L^1(0, T; t^{\mu-1} dt)$ and $k_2 \in L^1(0, T; t^\mu dt)$ for some $\mu \geq 1$ then

$$u' \in L^2(0, T; H; t^\mu dt) \quad \text{and} \quad \text{ess sup}_{t \in (0, T)} t^\mu \Phi(u(t), t) < \infty.$$

In particular, the identity

$$u'(t) + A(t)u(t) = f(t) \quad \text{holds in } \mathbf{H} \text{ for a.a. } t \in (0, T).$$

Moreover, $u : (0, T] \rightarrow \mathbf{V}$ is weakly continuous.

- (ii) If $f \in L^2(0, T; \mathbf{H})$, $k_2 \in L^1(0, T)$ and $u_0 \in \mathbf{V} \cap \mathbf{H}$ then

$$u' \in L^2(0, T; H) \quad \text{and} \quad \text{ess sup}_{t \in (0, T)} \Phi(u(t), t) < \infty.$$

Moreover, if in addition k_1 is bounded in a neighborhood of zero, then $u : [0, T] \rightarrow \mathbf{V}$ is also weakly continuous at $t = 0$.

Remark 3.4. Let $m > 0$, $t > 0$ and $v \in \mathbf{V} \cap \mathbf{H}$, we define $v(s) = (s/t)^m v$ for $s \in [2^{-\frac{1}{mr}}t, t]$. It follows from (3.6), (3.7), (3.3) and (3.4) that

$$\begin{aligned}
\Phi(v, t) &\geq \Phi((s/t)^m v, s) \Big|_{s=2^{-\frac{1}{m}}t}^{s=t} \\
&= \int_{2^{-\frac{1}{mr}}t}^t \left[\frac{m}{s} \mathbf{V}' \langle A(s)(s/t)^m v, (s/t)^m v \rangle_{\mathbf{V}} + \partial_t \Phi((s/t)^m v, s) \right] ds \\
&\geq \int_{2^{-\frac{1}{mr}}t}^t \gamma \frac{m}{s} [(s/t)^m v]^r ds - \int_{2^{-\frac{1}{mr}}t}^t \frac{m}{s} k_1^r(s) ds \\
&\quad - c \int_{2^{-\frac{1}{mr}}t}^t \|(s/t)^m v\|_{\mathbf{V}}^r ds - c \int_{2^{-\frac{1}{mr}}t}^t k_2(s) ds \\
&= \frac{\gamma}{r} (1 - 2^{-1}) [v]^r - \frac{ct}{mr+1} (1 - 2^{-\frac{mr+1}{mr}}) \|v\|_{\mathbf{V}}^r - \int_{2^{-\frac{1}{mr}}t}^t \left(\frac{m}{s} k_1^r(s) + ck_2(s) \right) ds \\
&\geq \left(\frac{\gamma}{2r} - \frac{2^r ct}{mr\gamma^r} \right) [v]^r - \frac{2^r \lambda^r ct}{mr\gamma^r} \|v\|_{\mathbf{H}}^r - \int_{2^{-\frac{1}{mr}}t}^t \left(\frac{m}{s} k_1^r(s) + ck_2(s) \right) ds.
\end{aligned}$$

Hence, choosing $m = 2^{r+2} ct / \gamma^{r+1}$, we obtain

$$\frac{\gamma}{4r} [v]^r \leq \Phi(v, t) + \frac{\gamma \lambda^r}{4r} \|v\|_{\mathbf{H}}^r + \int_{2^{-\frac{1}{mr}}t}^t \left(\frac{m}{s} k_1^r(s) + ck_2(s) \right) ds.$$

So holds with

$$[v]^r \leq k_3(t) (\Phi(v, t) + \|v\|_{\mathbf{H}}^r + 1) \text{ for all } v \in \mathbf{V} \cap \mathbf{H} \text{ and } t \in (0, T), \quad (3.8)$$

where

$$k_3(t) = \frac{4r}{\gamma} + \lambda^r + \frac{4r}{\gamma} \int_{2^{-\frac{1}{mr}}t}^t \left(\frac{m}{s} k_1^r(s) + ck_2(s) \right) ds.$$

If $k_2 \in L^1_{loc}((0, T])$, then $k_3 \in L^\infty_{loc}((0, T])$. Moreover, since $\int_{2^{-\frac{1}{mr}}t}^t \frac{m}{s} ds = \frac{\ln 2}{r}$, the above function k_3 is bounded in a neighborhood of zero provided k_1 is bounded in a neighborhood of zero and $k_2 \in L^1(0, T)$.

Proof of Theorem 3.3. Note the following.

- By Philips theorem, $L^r(0, T; \mathbf{V})' \simeq L^{r'}(0, T; \mathbf{V}')$.
- By Proposition 3.2, $A(t)$ is demi-continuous for a.a. $t \in (0, T)$. So, given a measurable map $w : (0, T) \rightarrow \mathbf{V}$, and a sequence of simple functions w_n converging to w a.e. in t as $n \rightarrow \infty$, one has $A(t)w_n(t) \rightarrow A(t)w(t)$ weakly in \mathbf{V}' for a.a. $t \in (0, T)$. Hence $t \mapsto A(t)w(t)$ is a measurable map.
- Due to (3.2), the map $\mathcal{A} : L^r(0, T; \mathbf{V}) \rightarrow L^{r'}(0, T; \mathbf{V}')$ defined by $\mathcal{A}w(t) \stackrel{\text{def}}{=} A(t)w(t)$ is a bounded hemi-continuous map. Moreover, since $A(t)$ is monotone for a.a. $t \in (0, T)$, the map \mathcal{A} is monotone as well. Therefore \mathcal{A} is of type M and demi-continuous.

- The uniqueness of the solution u follows from the fact that \mathcal{A} is a monotone operator.

Case A. First we consider the case $\mathbf{V} \hookrightarrow \mathbf{H}$.

Part I. Note that in this case $\mathbf{V} \hookrightarrow \mathbf{H} \simeq \mathbf{H}' \hookrightarrow \mathbf{V}'$, where both embeddings are dense. Since \mathbf{V} is separable, there exists a countable set $F \subset \mathbf{V}$ dense in \mathbf{V} , \mathbf{H} and \mathbf{V}' . The Gram-Schmidt orthonormalization procedure in \mathbf{H} produces a Schauder basis $\{e_k\}_{k \in \mathbb{N}} \subset \mathbf{V}$ which spans \mathbf{V} , \mathbf{H} and \mathbf{V}' , being orthonormal in \mathbf{H} and biorthogonal in $\langle \mathbf{V}', \mathbf{V} \rangle$.

For $n \in \mathbb{N}$, let $\mathbf{V}_n = \text{span}\{e_k : 1 \leq k \leq n\}$ and let P_n denote the projection $\mathbf{V}' \rightarrow \mathbf{V}_n$ defined by $P_n v \stackrel{\text{def}}{=} \sum_{k=1}^n \mathbf{v}' \langle v, e_k \rangle_{\mathbf{V}} e_k$ for any $v \in \mathbf{V}'$.

Let $A_n(t) = P_n A(t) P_n : \mathbf{V}_n \rightarrow \mathbf{V}_n$ and $f_n(t) = P_n f(t)$.

Let $u_{0n} \in \mathbf{V}_n$ be such that $u_{0n} \rightarrow u_0$ in \mathbf{H} as $n \rightarrow \infty$.

Consider the following initial value problem in \mathbf{V}_n :

$$u'_n = A_n(t)u_n + f_n(t), \quad u_n(0) = u_{0n}. \quad (3.9)$$

Since $A(t)$ is demi-continuous on \mathbf{V} for a.a. $t \in (0, T)$, it follows that $A_n(t)$ is continuous on \mathbf{V}_n for a.a. $t \in (0, T)$. Then, by the Cauchy-Peano and Carathéodory theorems, (3.2) implies that (3.9) has a (unique) local solution u_n , called Galerkin approximation, on $[0, T_n]$ with $0 < T_n \leq T$, $u_n \in L^r(0, T_n; \mathbf{V}_n)$ and $u'_n \in L^{r'}(0, T_n; \mathbf{V}_n)$. Note that this implies $u_n \in C([0, T_n], \mathbf{V}_n)$. The solution u_n is extendable to the whole interval $[0, T]$, i.e., $T_n = T$, unless it blows up. We derive an estimate of u_n which is independent of n to show the absence of a blow up and use it in passing to the limit in n .

Multiply (3.9) by u_n and integrate in t to obtain

$$\frac{1}{2} \|u_n(t)\|_{\mathbf{H}}^2 + \int_0^t \mathbf{v}' \langle A(s)u_n(s), u_n(s) \rangle_{\mathbf{V}} ds = \frac{1}{2} \|u_{0n}\|_{\mathbf{H}}^2 + \int_0^t \mathbf{v}' \langle f(s), u_n(s) \rangle_{\mathbf{V}} ds.$$

Let $k(t) = \max\{k_0(t), k_1(t)\}$, $t \in (0, T)$. Note that, by (3.4)

$$\int_0^t \mathbf{v}' \langle A(s)u_n(s), u_n(s) \rangle_{\mathbf{V}} ds \geq \gamma \int_0^t [u_n]^r ds - \|k\|_{L^r(0, T)}^r$$

and, by (3.3) and the Hölder and Young inequalities, there are positive constants $C_{\gamma, r}$ and $C_{\gamma, \lambda, r}$ such that

$$\begin{aligned} \int_0^t \mathbf{v}' \langle f(s), u_n(s) \rangle_{\mathbf{V}} ds &\leq \int_0^t \|f(s)\|_{\mathbf{V}'} \|u_n(s)\|_{\mathbf{V}} ds \leq \frac{1}{\gamma} \int_0^t \|f\|_{\mathbf{V}'} ([u_n] + \lambda \|u_n\|_{\mathbf{H}}) ds \\ &\leq C_{\gamma, r} \|f\|_{r'} + C_{\gamma, \lambda, r} \|f\|_{r'}^2 + \frac{\gamma}{2} \int_0^t [u_n]^r ds + \frac{1}{2} \left(\int_0^t \|u_n\|_{\mathbf{H}}^r ds \right)^{\frac{2}{r}}, \end{aligned}$$

where $\|\cdot\|_{r'}$ denotes the norm in $L^{r'}(0, T; \mathbf{V}')$. So there exists a constant $c_{\gamma, \lambda, r} > 0$ such that

$$\begin{aligned} \|u_n(t)\|_{\mathbf{H}}^2 + \gamma \int_0^t [u_n]^r ds \\ \leq \|u_0\|_{\mathbf{H}}^2 + 2\|k\|_{L^r(0, T)}^r + c_{\gamma, \lambda, r} (\|f\|_{r'} + \|f\|_{r'}^2) + \left(\int_0^t \|u_n\|_{\mathbf{H}}^r ds \right)^{\frac{2}{r}}. \end{aligned} \quad (3.10)$$

In particular, neglecting the integral term on the left-hand side and raising both sides to the power $r/2$ give

$$\|u_n(t)\|_{\mathbf{H}}^r \leq c \left(1 + \int_0^t \|u_n\|_{\mathbf{H}}^r ds\right),$$

with

$$c = 2^{\frac{r}{2}} \left(1 + \|u_0\|_H^2 + 2\|k\|_{L^r(0,T)}^r + c_{\gamma,\lambda,r} (\|f\|_{r'}^{r'} + \|f\|_{r'}^2)\right)^{\frac{r}{2}}$$

independent of n . By Gronwall's inequality, $\|u_n(t)\|_{\mathbf{H}}^r \leq ce^{ct}$ and hence the solution is extendable to the interval $[0, T]$ and, by (3.10) and (3.3),

$$\sup_{t \in [0, T]} \|u_n(t)\|_{\mathbf{H}}^2 + \int_0^T \|u_n\|_{\mathbf{V}}^r ds \text{ is bounded uniformly in } n. \quad (3.11)$$

Since \mathcal{A} is a bounded operator $L^r(0, T; \mathbf{V}) \rightarrow L^{r'}(0, T; \mathbf{V}')$, it follows that $\mathcal{A}u_n$ is bounded in $L^{r'}(0, T; \mathbf{V}')$ uniformly in n . Since a closed ball in $L^r(0, T; \mathbf{V})$ is weakly compact, the sequence $\{u_n\}$ has a weak limit point $u \in L^r(0, T; \mathbf{V})$. Let $\{u_{n_k}\}_{k=1}^\infty$ be a subsequence such that

$$u_{n_k} \rightarrow u \text{ weakly in } L^r(0, T; \mathbf{V}). \quad (3.12)$$

Since closed balls in $L^{r'}(0, T; \mathbf{V}')$ and in \mathbf{H} are weakly compact as well, the sequence $\{\mathcal{A}u_{n_k}\}_{k=1}^\infty$ has a weak limit point $\xi \in L^{r'}(0, T; \mathbf{V}')$ and $\{u_{n_k}(T)\}_{k=1}^\infty$ has a weak limit point $u^* \in \mathbf{H}$. Passing if necessary to a subsequence of $\{u_{n_k}\}_{k=1}^\infty$ (which we denote by the same notation), we may assume that

$$\mathcal{A}u_{n_k} \rightarrow \xi \text{ weakly in } L^{r'}(0, T; \mathbf{V}'), \text{ and} \quad (3.13)$$

$$u_{n_k}(T) \rightarrow u^* \text{ weakly in } \mathbf{H} \text{ as } k \rightarrow \infty. \quad (3.14)$$

Note that

$$-\int_0^T \theta'(t)_{\mathbf{V}'} \langle u_{n_k}(t), v \rangle_{\mathbf{V}} dt = \int_0^T \theta(t)_{\mathbf{V}'} \langle A(t)u_{n_k}(t), v \rangle_{\mathbf{V}} dt + \int_0^T \theta(t)_{\mathbf{V}'} \langle f(t), v \rangle_{\mathbf{V}} dt$$

for all $\theta \in C_c^1(0, T)$ and $v \in \mathbf{V}_{n_k}$. Since $F \stackrel{\text{def}}{=} \cup_n \mathbf{V}_n$ is dense in \mathbf{V} , it follows that

$$-\int_0^T \theta'(t)_{\mathbf{V}'} \langle u(t), v \rangle_{\mathbf{V}} dt = \int_0^T \theta(t)_{\mathbf{V}'} \langle \xi(t), v \rangle_{\mathbf{V}} dt + \int_0^T \theta(t)_{\mathbf{V}'} \langle f(t), v \rangle_{\mathbf{V}} dt$$

for all $\theta \in C_c^1(0, T)$ and $v \in \mathbf{V}$. By Proposition 23.20(b) in [23], this implies u is weakly differentiable and

$$u' = -\xi + f \in L^{r'}(0, T; \mathbf{V}'). \quad (3.15)$$

Hence, by Proposition 23.23(ii) in [23], $u \in C([0, T]; \mathbf{H})$, $u(T) = u^*$ and

$$\int_0^T \mathbf{v}' \langle \xi(t), u(t) \rangle_{\mathbf{V}} dt = \int_0^T \mathbf{v}' \langle f(t), u(t) \rangle_{\mathbf{V}} dt + \frac{1}{2} \|u_0\|_H^2 - \frac{1}{2} \|u(T)\|_H^2. \quad (3.16)$$

We are left to show that $\mathcal{A}u = \xi$. Since \mathcal{A} is of M -type, and one already has (3.12) and (3.13), it suffices to show that

$$\limsup_{k \rightarrow \infty} \int_0^T \mathbf{v}' \langle A(t)u_{n_k}(t), u_{n_k}(t) \rangle_{\mathbf{V}} dt \leq \int_0^T \mathbf{v}' \langle \xi(t), u(t) \rangle_{\mathbf{V}} dt. \quad (3.17)$$

Indeed, it follows from (3.9) that

$$\begin{aligned} & \limsup_{k \rightarrow \infty} \int_0^T \mathbf{v}' \langle A(t)u_{n_k}(t), u_{n_k}(t) \rangle_{\mathbf{V}} dt \\ &= \int_0^T \mathbf{v}' \langle f(t), u(t) \rangle_{\mathbf{V}} dt + \frac{1}{2} \|u_0\|_H^2 - \frac{1}{2} \liminf_{k \rightarrow \infty} \|u_{n_k}(T)\|_H^2 \\ &\leq \int_0^T \mathbf{v}' \langle f(t), u(t) \rangle_{\mathbf{V}} dt + \frac{1}{2} \|u_0\|_H^2 - \frac{1}{2} \|u(T)\|_H^2. \end{aligned}$$

The last inequality is due to (3.14) and $\|\cdot\|_{\mathbf{H}}$ being lower semi-continuous in the weak topology. Combining with (3.16), we obtain (3.17). Thus $\mathcal{A}u = \xi$ and (3.15) implies (3.5). So every weak limit point of $\{u_n\}_{n=1}^{\infty}$ is a solution to (3.5). Since such a solution is unique, we conclude that

$$u_n \rightarrow u \text{ weakly in } L^r(0, T; \mathbf{V}) \text{ as } n \rightarrow \infty. \quad (3.18)$$

Similar argument shows that $\mathcal{A}u$ is the only weak limit point of the sequence $\{\mathcal{A}u_n\}_{n=1}^{\infty}$ in $L^{r'}(0, T; \mathbf{V}')$. So $\mathcal{A}u_n \rightarrow \mathcal{A}u$ weakly in $L^{r'}(0, T; \mathbf{V}')$ as $n \rightarrow \infty$.

The latter implies that, for all $v \in \mathbf{H}$, $t \in [0, T]$,

$$|\mathbf{v}' \langle v, u_n(t) - u(t) \rangle_{\mathbf{V}}| \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (3.19)$$

Since $\sup_{n \geq 1} \max_{t \in [0, T]} \|u_n(t)\|_{\mathbf{H}}$ and $\max_{t \in [0, T]} \|u(t)\|_{\mathbf{H}}$ are bounded, it suffices to prove (3.19) for $v \in F$. Let $m > 1$ and $v \in \mathbf{V}_m$. Let $n > m$. Then

$$\begin{aligned} & |\mathbf{v}' \langle u_n(t) - u(t), v \rangle_{\mathbf{V}}| \leq \|u_{n0} - u_0\|_{\mathbf{H}} \|v\|_{\mathbf{H}} + \left| \int_0^t \mathbf{v}' \langle u'_n(\tau) - u'(\tau), v \rangle_{\mathbf{V}} d\tau \right| \\ &= \|u_{n0} - u_0\|_{\mathbf{H}} \|v\|_{\mathbf{H}} + \left| \int_0^t \mathbf{v}' \langle A(\tau)u_n(\tau) - A(\tau)u(\tau), v \rangle_{\mathbf{V}} d\tau \right| \\ &\quad + \left| \int_0^t \mathbf{v}' \langle f_n(\tau) - f(\tau), v \rangle_{\mathbf{V}} d\tau \right| \\ &= \|u_{n0} - u_0\|_{\mathbf{H}} \|v\|_{\mathbf{H}} + \left| \int_0^t \mathbf{v}' \langle A(\tau)u_n(\tau) - A(\tau)u(\tau), v \rangle_{\mathbf{V}} d\tau \right| \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

(In fact, Theorem 30.A of [23] has much stronger result, namely,

$$\max_{t \in [0, T]} \|u_n(t) - u(t)\|_H \rightarrow 0 \text{ as } n \rightarrow \infty.$$

However, the weak convergence in (3.19) with a short proof adequately serves our purpose at the moment.)

Part II. The proof of this part requires additional estimates of u_n , independent of n .

(i) By (3.6),

$$\frac{d}{dt} \Phi(u_n(t), t) = \mathbf{v}' \langle A(t)u_n(t), u'_n(t) \rangle_{\mathbf{V}} + \partial_t \Phi(u_n(t), t).$$

It follows from (3.9) and (3.7) that

$$\begin{aligned} \|u'_n(t)\|_{\mathbf{H}}^2 + \frac{d}{dt}\Phi(u_n(t), t) &\leq \|u'_n(t)\|_{\mathbf{H}}\|f(t)\|_{\mathbf{H}} + \partial_t\Phi(u_n(t), t) \\ &\leq \frac{1}{2}\|u'_n(t)\|_{\mathbf{H}}^2 + \frac{1}{2}\|f(t)\|_{\mathbf{H}}^2 + C(\|u_n(t)\|_{\mathbf{V}}^r + k_2(t)). \end{aligned}$$

Hence

$$\frac{1}{2}\|u'_n(t)\|_{\mathbf{H}}^2 + \frac{d}{dt}\Phi(u_n(t), t) \leq \frac{1}{2}\|f(t)\|_{\mathbf{H}}^2 + C(\|u_n(t)\|_{\mathbf{V}}^r + k_2(t)). \quad (3.20)$$

Multiply by t^μ and integrate in t to obtain the following:

$$\begin{aligned} &\int_0^T t^\mu \|u'_n(t)\|_{\mathbf{H}}^2 dt + \sup_{t \in [0, T]} t^\mu \Phi(u_n(t), t) \\ &\leq \frac{3}{2} \int_0^T t^\mu \|f(t)\|_{\mathbf{H}}^2 dt + \frac{3}{2} \mu \int_0^T t^{\mu-1} \Phi(u_n(t), t) dt + CT^\mu \| \|u_n\| \|_r^r + C \int_0^T t^\mu k_2(t) dt; \end{aligned}$$

here $\| \|u_n\| \|_r$ stands for the norm of u_n in $L^r(0, T; \mathbf{V})$. Now observe that, since $v \mapsto \Phi(v, t)$ is convex for all $t \in [0, T]$, it follows from (3.2) that

$$\Phi(u_n(t), t) \leq \Phi(0, t) + \mathbf{v}' \langle A(t)u_n, u_n \rangle_{\mathbf{V}} \leq (c+1)\|u_n(t)\|_{\mathbf{V}}^r + k_1^r(t) + \Phi(0, t).$$

Hence there exists a constant $c = c_{\mu, T} > 0$ such that

$$\begin{aligned} &\int_0^T t^\mu \|u'_n(t)\|_{\mathbf{H}}^2 dt + \sup_{t \in [0, T]} t^\mu \Phi(u_n(t), t) \\ &\leq c \left(\int_0^T t^\mu \|f(t)\|_{\mathbf{H}}^2 dt + \| \|u_n\| \|_r^r + \|k_1\|_{L^r(0, T)}^r + \int_0^T t^{\mu-1} \Phi(0, t) dt + \int_0^T t^\mu k_2(t) dt \right). \end{aligned} \quad (3.21)$$

Combining (3.21) with (3.11), we have $\{u'_n\}_{n=1}^\infty$ is uniformly bounded in $L^2(0, T; \mathbf{H}; t^\mu dt)$, and hence it is weakly pre-compact in $L^2(0, T; \mathbf{H}; t^\mu dt)$. Let $w \in L^2(0, T; \mathbf{H}; t^\mu dt)$ be a weak limit point of $\{u'_n\}_{n=1}^\infty$ and $\{u_{n_k}\}$ be a subsequence of $\{u_n\}_{n=1}^\infty$ such that $u'_{n_k} \rightharpoonup w$ weakly in $L^2(0, T; \mathbf{H}; t^\mu dt)$ as $k \rightarrow \infty$. Then, due to (3.18), for every $\theta \in C_c^1(0, T)$ and $v \in \mathbf{V}$,

$$\begin{aligned} \int_0^T \theta(t)_{\mathbf{V}'} \langle w(t), v \rangle_{\mathbf{V}} dt &= \lim_k \int_0^T \theta(t)_{\mathbf{V}'} \langle u'_{n_k}(t), v \rangle_{\mathbf{V}} dt \\ &= - \lim_k \int_0^T \theta'(t)_{\mathbf{V}'} \langle u_{n_k}(t), v \rangle_{\mathbf{V}} dt \\ &= - \int_0^T \theta'(t)_{\mathbf{V}'} \langle u(t), v \rangle_{\mathbf{V}} dt \end{aligned}$$

So $w = u'$, by Proposition 23.20(b) in [23]. Since the weak limit point of $\{u'_n\}_{n=1}^\infty$ is unique, it follows that $u'_n \rightharpoonup u'$ weakly in $L^2(0, T; \mathbf{H}; t^\mu dt)$ as $n \rightarrow \infty$ and $u' \in L^2(0, T; \mathbf{H}; t^\mu dt)$.

Similarly, we have from (3.21) that

$$\sup_n \sup_{t \in [0, T]} t^\mu \Phi(u_n(t), t) < \infty. \quad (3.22)$$

Because of (3.12) and \mathbf{V} being separable, by passing to a subsequence we may assume $u_{n_k}(t) \rightarrow u(t)$ weakly in \mathbf{V} for a.a. $t \in (0, T)$. Since Φ is convex in the first variable, it is lower semi-continuous with respect to the weak topology of \mathbf{V} . Therefore $\text{ess sup}_{t \in (0, T)} t^\mu \Phi(u(t), t) < \infty$. The weak continuity of u in \mathbf{V} will be shown below.

(ii) Note that, since $u_0 \in \mathbf{V}$, we can choose $u_n(0)$ converging to u_0 in \mathbf{V} rather than in \mathbf{H} . Since a locally bounded convex function is continuous, it follows that $\Phi(u_n(0), 0) \rightarrow \Phi(u_0, 0)$ as $n \rightarrow \infty$. Then integration of (3.20) leads to the following:

$$\begin{aligned} & \int_0^T \|u'_n\|_{\mathbf{H}}^2 dt + \sup_{t \in [0, T]} \Phi(u_n(t), t) \\ & \leq c \left(\Phi(u_n(0), 0) + \int_0^T \|f\|_{\mathbf{H}}^2 dt + \|u_n\|_r^r + \int_0^T k_2(t) dt \right). \end{aligned} \quad (3.23)$$

The argument can be completed as in (i).

Now we prove weak continuity of u in \mathbf{V} . It suffices to prove the assertion for $T < \infty$.

Assume (i). By Remark 3.4, we have (3.8) with $k_3 \in L_{loc}^\infty((0, T])$. Then, by (3.3), (3.22) and (3.11) we have that, for every $\varepsilon > 0$,

$$\{u_n(t) : n \in \mathbb{N}, \varepsilon \leq t \leq T\} \text{ is bounded in } \mathbf{V} \text{ and in } \mathbf{H}. \quad (3.24)$$

For each $t \in [\varepsilon, T]$, property (3.19) implies that $u_n(t) \rightarrow u(t)$ weakly in \mathbf{H} , hence by the weak pre-compactness in \mathbf{V} and \mathbf{H} of the set in (3.24), we have $u(t) \in \mathbf{V}$. Since $u \in C([0, T]; \mathbf{H})$, the weak continuity of $u : (0, T] \rightarrow \mathbf{V}$ again follows from (3.24).

Assume (ii) with k_1 bounded in a neighborhood of zero. Then (3.8) holds with $k_3 \in L^\infty(0, T)$. Thanks to (3.23), the boundedness (3.24) holds with $\varepsilon = 0$ and the corresponding weak continuity on $[0, T]$ is proved similarly.

Case B. Now, consider the general case when $\mathbf{V} \cap \mathbf{H}$ is dense in \mathbf{H} . Let $\mathbf{V}_0 = \mathbf{V} \cap \mathbf{H}$. Then

$$\mathbf{V}_0 \hookrightarrow \mathbf{H} \simeq \mathbf{H}' \hookrightarrow \mathbf{V}'_0 \simeq \mathbf{V}' + \mathbf{H} \text{ densely and continuously.}$$

Moreover, assumptions (1)–(3) hold true with \mathbf{V} and \mathbf{V}' replaced with \mathbf{V}_0 and \mathbf{V}'_0 , respectively. Hence problem (3.5) has a unique solution u satisfying

$$u \in C([0, T]; \mathbf{H}) \cap L^r(0, T; \mathbf{V} \cap \mathbf{H}), \quad u' \in L^{r'}(0, T; \mathbf{V}' + \mathbf{H}).$$

However, $C([0, T]; \mathbf{H}) \cap L^r(0, T; \mathbf{V} \cap \mathbf{H}) \simeq C([0, T]; \mathbf{H}) \cap L^r(0, T; \mathbf{V})$, and, since $f \in L^{r'}(0, T; \mathbf{V}')$ and \mathcal{A} is a bounded operator $L^r(0, T; \mathbf{V}) \rightarrow L^{r'}(0, T; \mathbf{V}')$, we have $u' = -\mathcal{A}u + f \in L^{r'}(0, T; \mathbf{V}')$. The other assertions follow from case A above. \square

We apply the above abstract result to our problem to obtain the existence of weak solutions globally in time. Below, for any function $\varphi(x, t)$ of two variables x and t , we

use $\varphi(t)$ to denote the function $t \rightarrow \varphi(\cdot, t)$, hence give meaning to expressions such as $\varphi \in C((0, T), L^2(U))$ and $\|\varphi(t)\|_{L^2(U)}$.

Theorem 3.5. (i) For every $p_0 \in L^2(U)$ and

$$\begin{aligned} \Psi &\in C([0, \infty), L^2(U)) \cap L_{\text{loc}}^\infty([0, \infty), W^{1,2-a}(U)) \\ \text{with } \Psi_t &\in L_{\text{loc}}^{(2-a)'}([0, \infty), (W^{1,2-a}(U))'), \end{aligned}$$

the initial boundary value problem (3.1) has a unique solution p satisfying

$$\begin{aligned} p &\in C([0, \infty), L^2(U)) \cap L_{\text{loc}}^\infty([0, \infty), W^{1,2-a}(U)) \\ \text{and } p_t &\in L_{\text{loc}}^{(2-a)'}([0, \infty), (W^{1,2-a}(U))'). \end{aligned} \quad (3.25)$$

(ii) If, in addition, $\Psi : [0, \infty) \rightarrow W^{1,2-a}(U)$ is weakly continuous,

$$\partial_t \Psi \in L_{\text{loc}}^2([0, \infty), L^2(U)) \cap L_{\text{loc}}^{2-a}([0, \infty), W^{1,2-a}(U)),$$

and $p_0 \in W^{1,2-a}(U)$ with $p_0|_\Gamma = \Psi(0)|_\Gamma$ then

$$p : [0, \infty) \rightarrow W^{1,2-a}(U) \text{ is weakly continuous and } \partial_t p \in L_{\text{loc}}^2([0, \infty), L^2(U)). \quad (3.26)$$

Proof. (i) The uniqueness of a solution $p(x, t)$ is a direct consequence of the monotonicity in Lemma 2.1. For the existence, as usual, we solve for $p(x, t)$ of the form $p(x, t) = \bar{p}(x, t) + \Psi(x, t)$, where $\bar{p}(x, t)$ satisfies

$$\begin{cases} \frac{\partial \bar{p}}{\partial t} = \nabla \cdot (K(|\nabla \bar{p} + \nabla \Psi|)(\nabla \bar{p} + \nabla \Psi)) - \Psi_t \text{ in } U \times (0, \infty), \\ \bar{p}(x, t) = 0 \text{ on } \Gamma \times (0, \infty). \end{cases} \quad (3.27)$$

We will apply Theorem 3.3 to (3.27). Let $\mathbf{V} = \dot{W}^{1,2-a}(U)$, $\mathbf{H} = L^2(U)$ and $f = -\partial_t \Psi$. Let $T > 0$ be arbitrary. Then $f \in L^{(2-a)'}(0, T; \mathbf{V}')$. For the weak formulation, we multiply the equation by $w \in \mathbf{V}$ and integrate over U to obtain

$$\frac{d}{dt} \int_U \bar{p} w dx = - \int_U (K(|\nabla \bar{p} + \nabla \Psi(t)|)(\nabla \bar{p} + \nabla \Psi(t)) \cdot \nabla w dx + \int_U f w dx.$$

Therefore we define $A(t)v$, for $t > 0$ and $v \in \mathbf{V}$, by

$$\mathbf{v}' \langle A(t)v, w \rangle_{\mathbf{V}} = \int_U (K(|\nabla v + \nabla \Psi(t)|)(\nabla v + \nabla \Psi(t)) \cdot \nabla w dx, \quad w \in \mathbf{V}.$$

Then it follows from (2.10) that

$$\begin{aligned} \|A(t)v\|_{\mathbf{V}'} &\leq \|K(|\nabla v + \nabla \Psi(t)|)(\nabla v + \nabla \Psi(t))\|_{L^{(2-a)'}} \\ &\leq C \|\nabla v + \nabla \Psi(t)\|_{L^{2-a}}^{1-a} \leq C (\|v\|_{\mathbf{V}}^{1-a} + \|\nabla \Psi(t)\|_{L^{2-a}}^{1-a}). \end{aligned} \quad (3.28)$$

Hence $A(t)$ is a bounded linear map $\mathbf{V} \rightarrow \mathbf{V}'$ and the estimate (3.2) holds with

$$k_0(t) = \|\nabla \Psi(t)\|_{L^{2-a}} \text{ and the number } r = 2 - a.$$

Obviously, $k_0 \in L^r(0, T)$. Applying (2.10) and the Lebesgue dominated convergence theorem, it is easy to obtain that $A(t)$ is hemi-continuous, (2.15) implies that it is monotone.

We define the seminorm $[v]$, for $v \in \mathbf{V}$, to be $\|\nabla v\|_{L^{2-a}}$. Then (3.3) clearly holds since $2 - a < 2$. It follows from estimates (2.10)–(2.11) that, with some $c > 0$,

$$\begin{aligned} \mathbf{v} \langle A(t)v, v \rangle_{\mathbf{V}} &\geq \int_U K(|\nabla v + \nabla \Psi(t)|) |\nabla v + \nabla \Psi(t)|^2 dx \\ &\quad - \int_U K(|\nabla v + \nabla \Psi(t)|) |\nabla v + \nabla \Psi(t)| |\nabla \Psi(t)| dx \\ &\geq c(\|\nabla v + \nabla \Psi(t)\|_{L^{2-a}}^{2-a} - |U|) - \|\nabla v + \nabla \Psi(t)\|_{L^{2-a}}^{1-a} \|\nabla \Psi(t)\|_{L^{2-a}} \\ &\geq \frac{c}{2} \|\nabla v + \nabla \Psi(t)\|_{L^{2-a}}^{2-a} - c|U| - \left(\frac{c}{2}\right)^{a-1} \|\nabla \Psi(t)\|_{L^{2-a}}^{2-a} \\ &\geq c2^{3-a} \|v\|_{\mathbf{V}}^{2-a} - c|U| - \left(\frac{c}{2} + \left(\frac{c}{2}\right)^{a-1}\right) \|\nabla \Psi(t)\|_{L^{2-a}}^{2-a}. \end{aligned}$$

Thus we obtain the coercivity estimate (3.4) with $k_1 = C(1 + k_0)$ for some $C > 0$. Applying Theorem 3.3 part I, we obtain solution $\bar{p}(x, t)$ and conclude (3.25) for $p(x, t)$ accordingly.

(ii) Here $f \in L^2(0, T; \mathbf{H})$ and $\bar{p}_0 \in \mathbf{V} \cap \mathbf{H}$. For $v \in \mathbf{V}$ and $t \geq 0$, let

$$\Phi(v, t) = \frac{1}{2} \int_U H(|\nabla v + \nabla \Psi(t)|) dx.$$

Then $\Phi(v, t)$ meets the requirements in Theorem 3.3, part II. Indeed, by (2.13),

$$|\Phi(v, t)| \leq C(\|v\|_{\mathbf{V}}^{2-a} + \|\Psi(t)\|_{\mathbf{V}}^{2-a}),$$

hence $\Phi(v, t)$ is bounded in v for each $t \geq 0$.

Let $v, w \in \mathbf{V}$ and $\tau \in (0, 1)$, by (2.14) we have

$$\begin{aligned} \Phi(\tau v + (1 - \tau)w, t) &\leq \frac{1}{2} \int_U H(\tau|\nabla v + \nabla \Psi(t)| + (1 - \tau)|\nabla w + \nabla \Psi(t)|) dx \\ &\leq \frac{1}{2} \int_U \tau H(|\nabla v + \nabla \Psi(t)|) + (1 - \tau)H(|\nabla w + \nabla \Psi(t)|) dx \\ &= \tau\Phi(v, t) + (1 - \tau)\Phi(w, t). \end{aligned}$$

Therefore $\Phi(v, t)$ is convex in v . By direct calculations, one can verify (3.6) and

$$\partial_t \Phi(v, t) = \int_U K(|\nabla v + \nabla \Psi(t)|) (\nabla v + \nabla \Psi(t)) \cdot \nabla \partial_t \Psi(t) dx.$$

Then by (2.10),

$$\begin{aligned} |\partial_t \Phi(v, t)| &\leq C \int_U |\nabla v + \nabla \Psi(t)|^{1-a} |\nabla \partial_t \Psi(t)| dx \\ &\leq C(\|v\|_{\mathbf{V}}^{2-a} + \|\nabla \Psi(t)\|_{L^{2-a}}^{2-a} + \|\nabla \partial_t \Psi(t)\|_{L^{2-a}}^{2-a}). \end{aligned}$$

Hence (3.7) holds with $k_2(t) = \|\nabla \Psi(t)\|_{L^{2-a}}^{2-a} + \|\nabla \partial_t \Psi(t)\|_{L^{2-a}}^{2-a}$. Clearly, $k_2 \in L^1(0, T)$ and $k_1 \in L^\infty(0, T)$. Now the assertion follows from (ii) of part II, Theorem 3.3. \square

In case the initial and boundary data have more regularity, so does our solution. Specifically, we have:

Theorem 3.6. *Let $\alpha \geq 2$. Assume $p_0(x) \in L^\alpha(U)$, $\Psi \in C([0, \infty); L^\alpha(U))$, $\nabla\Psi \in L^{\frac{\alpha(2-a)}{2}}(U \times (0, T))$ and $\Psi_t \in L^\alpha(U \times (0, T))$ for all $T > 0$. Then the corresponding solution $p(x, t)$ to (3.1) satisfies $p \in C([0, \infty); L^\alpha(U))$, and $\bar{p} = p - \Psi$ satisfies*

$$\begin{aligned} \frac{1}{\alpha} \frac{d}{dt} \int_U |\bar{p}(x, t)|^\alpha dx &= -(\alpha - 1) \int_U K(|\nabla p|)(\nabla p \cdot \nabla \bar{p}) |\bar{p}|^{\alpha-2} dx \\ &+ \int_U \Psi_t |\bar{p}|^{\alpha-2} \bar{p} dx \quad \text{for all } t > 0. \end{aligned} \quad (3.29)$$

Proof. Let $T > 0$. For $n \in \mathbb{N}$ and $s \in \mathbb{R}$, let $V_n(s) = \int_0^s (\min\{|\tau|, n\})^{\alpha-2} \tau d\tau$, that is,

$$V_n(s) = \begin{cases} \frac{1}{\alpha} |s|^\alpha, & |s| \leq n; \\ \frac{1}{\alpha} n^\alpha + \frac{1}{2} n^{\alpha-2} (|s| - n)^2, & |s| > n. \end{cases}$$

Clearly, $V_n(s) \uparrow \frac{1}{\alpha} |s|^\alpha$ as $n \uparrow \infty$. Let also $\bar{p}_n = \min\{|\bar{p}|, n\}$. Note that $\bar{p}_n^{\alpha-2} \bar{p} \in C([0, \infty), W^{1,2-a}(U))$ and that

$$\nabla(\bar{p}_n^{\alpha-2} \bar{p}) = [(\alpha - 1)\chi_n |\bar{p}|^{\alpha-2} + n^{\alpha-2} \chi_n^c] \nabla \bar{p},$$

where χ_n is the indicator of the set $\{(x, t) : |\bar{p}(x, t)| \leq n\}$ and $\chi_n^c = 1 - \chi_n$ is the indicator of the complement.

Multiplying the first equation in (3.27) by $\bar{p}_n^{\alpha-2} \bar{p}$ and integrating over domain U give

$$\begin{aligned} \frac{d}{dt} \int_U V_n(\bar{p}) dx &= \int_U \bar{p}_n^{\alpha-2} \bar{p} \frac{\partial \bar{p}}{\partial t} dx \\ &= - \int_U [K(|\nabla p|) |\nabla p|^2 - K(|\nabla p|)(\nabla p) \cdot (\nabla \Psi)] [(\alpha - 1)\chi_n |\bar{p}|^{\alpha-2} + n^{\alpha-2} \chi_n^c] dx \\ &\quad + \int_U \Psi_t \bar{p}_n^{\alpha-2} \bar{p} dx \equiv I_1 + I_2. \end{aligned} \quad (3.30)$$

We will estimate two integrals on the right-hand side of (3.30).

Note from (2.10)–(2.11) that for $\varepsilon > 0$ and $\xi, \eta \geq 0$, we have

$$K(\xi)\xi\eta \leq C\xi^{1-a}\eta \leq C[(K(\xi)\xi^2)^{\frac{1-a}{2}} + 1]\eta \leq \varepsilon K(\xi)\xi^2 + C_\varepsilon(\eta^{2-a} + 1). \quad (3.31)$$

Estimating $K(|\nabla p|)(\nabla p \cdot \nabla \Psi)$ by (3.31) with $\varepsilon = \frac{1}{2}$, there exists $C > 0$ such that

$$\begin{aligned} -[K(|\nabla p|) |\nabla p|^2 - K(|\nabla p|)(\nabla p) \cdot (\nabla \Psi)] &\leq -K(|\nabla p|) |\nabla p|^2 + K(|\nabla p|) |\nabla p| |\nabla \Psi| \\ &\leq -\frac{1}{2} K(|\nabla p|) |\nabla p|^2 + C(|\nabla \Psi|^{2-a} + 1). \end{aligned}$$

Moreover,

$$(\alpha - 1)\chi_n |\bar{p}|^{\alpha-2} + n^{\alpha-2} \chi_n^c \leq \alpha^{\frac{2\alpha-2}{\alpha}} V_n(\bar{p})^{\frac{\alpha-2}{\alpha}},$$

and, since $\alpha \geq 2$, we have $p_n^{\alpha-2} |\bar{p}| \leq C_\alpha V_n(\bar{p})^{\frac{\alpha-1}{\alpha}}$. Then, by the Young inequality,

$$I_1 \leq -\frac{1}{2} \alpha^{\frac{2\alpha-2}{\alpha}} \int_U K(|\nabla p|) |\nabla p|^2 V_n(\bar{p})^{\frac{\alpha-2}{\alpha}} dx + C \left(\int_U V_n(\bar{p}) dx + \int_U |\nabla \Psi|^{\frac{\alpha(2-a)}{2}} dx + 1 \right),$$

$$I_2 \leq \int_U V_n(\bar{p}) dx + \int_U |\Psi_t|^\alpha dx.$$

Hence it follows from (3.30) and the above estimate of I_1 and I_2 that

$$\begin{aligned} \frac{d}{dt} \int_U V_n(\bar{p}) dx + \frac{1}{2} \alpha^{\frac{2\alpha-2}{\alpha}} \int_U K(|\nabla p|) |\nabla p|^2 V_n(\bar{p})^{\frac{\alpha-2}{\alpha}} dx \\ \leq C \left(\int_U V_n(\bar{p}) dx + \int_U |\nabla \Psi|^{\frac{\alpha(2-a)}{2}} dx + \int_U |\Psi_t|^\alpha dx + 1 \right). \end{aligned} \quad (3.32)$$

Drop the second summand on the left-hand side to obtain

$$\frac{d}{dt} \int_U V_n(\bar{p}) dx \leq C \int_U V_n(\bar{p}) dx + C \left(\int_U |\nabla \Psi|^{\frac{\alpha(2-a)}{2}} dx + \int_U |\Psi_t|^\alpha dx + 1 \right).$$

Then it follows from the Gronwall inequality that there exists $C = C(T) > 0$ such that

$$\int_U V_n(\bar{p}) dx \leq C e^{Ct}, \quad 0 < t < T.$$

Passing $n \rightarrow \infty$ and by the Beppo Levi lemma, we have

$$\int_U |\bar{p}|^\alpha dx \leq \alpha C e^{Ct}, \quad 0 < t < T. \quad (3.33)$$

Therefore $\bar{p} \in L^\infty(0, T; L^\alpha(U))$.

Now integrate (3.32) in t to obtain that, for all $T > 0$,

$$\begin{aligned} \int_0^T \int_U K(|\nabla p|) |\nabla p|^2 V_n(\bar{p})^{\frac{\alpha-2}{\alpha}} dx dt \leq C \left(\int_0^T \int_U |\bar{p}|^\alpha dx dt + \int_0^T \int_U |\nabla \Psi|^{\frac{\alpha(2-a)}{2}} dx dt \right. \\ \left. + \int_0^T \int_U |\Psi_t|^\alpha dx dt + \int_U |\bar{p}_0|^\alpha dx + 1 \right). \end{aligned}$$

Again, it follows from the Beppo Levi lemma that

$$K(|\nabla p|) |\nabla p|^2 |\bar{p}|^{\alpha-2} \in L^1(U \times (0, T)) \text{ for all } T > 0. \quad (3.34)$$

Now (3.29) follows from (3.30) as $n \rightarrow \infty$.

To prove that $\bar{p}(t)$ is continuous in $L^\alpha(U)$, observe that it is continuous in measure convergence since it is continuous in $L^2(U)$. Then by the Brezis-Lieb lemma [6], it suffices to prove that $\|\bar{p}(t)\|_{L^\alpha}$ is continuous. However, by the asserted identity (3.29), similar estimates to those of I_1 and I_2 , and properties (3.33) and (3.34), it follows that $\int_0^T \left| \frac{d}{dt} \|\bar{p}(t)\|_{L^\alpha}^\alpha \right| dt < \infty$, hence $\|\bar{p}(t)\|_{L^\alpha}^\alpha$ is absolutely continuous. Consequently the assertion follows. \square

4. ESTIMATES OF SOLUTIONS

In this section, we obtain more refined estimates for solutions of (3.1) than those in the previous section. We emphasize on the asymptotic estimates as $t \rightarrow \infty$ in terms of the asymptotic behavior of the Dirichlet data. They are crucial to the stability analysis of the solutions in the next two sections.

In previous section, we prove that (3.1) possesses a weak solution $p(x, t)$ for all $t > 0$ (see Theorems 3.5 and 3.6). Our solution $p(x, t)$, in fact, has more regularity in spatial and time variables. However, a proof of this fact requires another lengthy treatment which

is beyond the scope of the current paper. Also, since we aim is to study the qualitative properties of the solutions, we take the liberty to assume that $p(x, t)$ has sufficient regularities both in x and t variables such that our calculations hereafterward can be performed legitimately.

Let $p(x, t)$ be a solution to IBVP (3.1) with given data $p_0(x)$ and $\psi(x, t)$. Let $\bar{p} = p - \Psi$, then it satisfies

$$\begin{cases} \frac{\partial \bar{p}}{\partial t} = \nabla \cdot (K(|\nabla p|)\nabla p) - \Psi_t \text{ in } U \times (0, \infty), \\ \bar{p}(x, t) = 0 \text{ on } \Gamma \times (0, \infty). \end{cases} \quad (4.1)$$

We will derive estimates for $\int_U |\bar{p}(x, t)|^\alpha dx$ for all $\alpha > 0$ and $t > 0$, as well as for $\int_U |\nabla p(x, t)|^{2-a} dx$ and $\int_U |\bar{p}_t(x, t)|^2 dx$. The corresponding estimates for $p(x, t)$ can be obtained by using the facts

$$\int_U |p|^\alpha dx \leq 2^\alpha \int_U |\bar{p}|^\alpha + |\Psi|^\alpha dx \text{ and } \int_U |p_t|^2 dx \leq 2 \int_U |\bar{p}_t|^2 + |\Psi_t|^2 dx.$$

Notation for constants. In this section all constants C, C_1, C_2, \dots depend on many parameters, namely, exponent α , vectors $\vec{\alpha}$ and \vec{a} , the spatial dimension n and domain U , but are independent of the initial and boundary data. The constant C is generic and may vary from place to place, even changes values on the same line. Constants C_1, C_2, \dots have values temporarily fixed within one proof. The constants c_1, c_2, \dots , have fixed values for each α . Their dependence on $n, U, \vec{\alpha}$ and \vec{a} is implicitly understood. The constants d_1, d_2, \dots , do not depend on α .

The Lebesgue norms (on U) of $\Psi(x, t)$ or its spatial and time derivatives are always assumed to be continuous in t on $[0, \infty)$ whenever they are used.

For $\alpha > 0$ and $t \geq 0$, we define

$$A(\alpha, t) = A[\Psi](\alpha, t) \stackrel{\text{def}}{=} \left[\int_U |\nabla \Psi(x, t)|^{\frac{\alpha(2-a)}{2}} dx \right]^{\frac{2(\alpha-a)}{\alpha(2-a)}} + \left[\int_U |\Psi_t(x, t)|^\alpha dx \right]^{\frac{\alpha-a}{\alpha(1-a)}}. \quad (4.2)$$

Lemma 4.1. *Suppose $\alpha \geq \max\{2, \alpha_*\}$. Then there exist positive constants $c_5 = c_5(\alpha)$ and C such that*

$$\frac{d}{dt} \int_U |\bar{p}(x, t)|^\alpha dx \leq -c_5 \left(\int_U |\bar{p}(x, t)|^\alpha dx \right)^{\frac{1}{\gamma_0}} + C(1 + A(\alpha, t)), \quad t > 0, \quad (4.3)$$

where $\gamma_0 = \gamma_0(\alpha) = \alpha/(\alpha - a)$.

Proof. Multiplying both sides of the first equation in (4.1) by $|\bar{p}|^{\alpha-1}\text{sign}(\bar{p})$, integrating over domain U , and using Green' formula and the facts $\alpha > 1$, $\bar{p} = 0$ on Γ , we have

$$\begin{aligned}
\frac{1}{\alpha} \frac{d}{dt} \int_U |\bar{p}|^\alpha dx &= \int_U |\bar{p}|^{\alpha-1} \text{sign}(\bar{p}) \frac{\partial \bar{p}}{\partial t} dx \\
&= \int_U [\nabla \cdot (K(|\nabla p|) \nabla p) - \Psi_t] |\bar{p}|^{\alpha-1} \text{sign}(\bar{p}) dx \\
&= -(\alpha - 1) \left[\int_U K(|\nabla p|) |\nabla p|^2 |\bar{p}|^{\alpha-2} dx - \int_U K(|\nabla p|) (\nabla p \cdot \nabla \Psi) |\bar{p}|^{\alpha-2} dx \right] \\
&\quad - \int_U \Psi_t |\bar{p}|^{\alpha-1} \text{sign}(\bar{p}) dx.
\end{aligned} \tag{4.4}$$

We will estimate the last three integrals of (4.4). Let $\varepsilon > 0$. By (2.11) and Hölder's inequality, we have

$$\begin{aligned}
- \int_U K(|\nabla p|) |\nabla p|^2 |\bar{p}|^{\alpha-2} dx &\leq -C \int_U (|\nabla p|^{2-a} - 1) |\bar{p}|^{\alpha-2} dx \\
&\leq -C \int_U |\nabla \bar{p}|^{2-a} |\bar{p}|^{\alpha-2} dx + C \int_U |\nabla \Psi|^{2-a} |\bar{p}|^{\alpha-2} + |\bar{p}|^{\alpha-2} dx \\
&\leq -C \int_U |\nabla \bar{p}|^{2-a} |\bar{p}|^{\alpha-2} dx + C \left(\int_U |\bar{p}|^\alpha dx \right)^{\frac{\alpha-2}{\alpha}} \left(\int_U (1 + |\nabla \Psi|^{2-a})^{\frac{\alpha}{2}} dx \right)^{\frac{2}{\alpha}}.
\end{aligned}$$

Since $\frac{\alpha-2}{\alpha} < \frac{\alpha-a}{\alpha} = \frac{1}{\gamma_0}$, applying Young's inequality to the last product gives

$$\begin{aligned}
- \int_U K(|\nabla p|) |\nabla p|^2 |\bar{p}|^{\alpha-2} dx &\leq -C \int_U |\nabla \bar{p}|^{2-a} |\bar{p}|^{\alpha-2} dx + \varepsilon \left(\int_U |\bar{p}|^\alpha dx \right)^{\frac{1}{\gamma_0}} \\
&\quad + C_\varepsilon \left[1 + \left(\int_U |\nabla \Psi|^{\frac{\alpha(2-a)}{2}} dx \right)^{\frac{2(\alpha-a)}{\alpha(2-a)}} \right].
\end{aligned} \tag{4.5}$$

For the second integral on the right-hand side of (4.4),

$$\begin{aligned}
\left| \int_U K(|\nabla p|) (\nabla p \cdot \nabla \Psi) |\bar{p}|^{\alpha-2} dx \right| &\leq C \int_U |\nabla p|^{1-a} |\nabla \Psi| |\bar{p}|^{\alpha-2} dx \\
&\leq C \int_U |\nabla \bar{p}|^{1-a} |\nabla \Psi| |\bar{p}|^{\alpha-2} dx + \int_U |\nabla \Psi|^{2-a} |\bar{p}|^{\alpha-2} dx.
\end{aligned} \tag{4.6}$$

Applying Hölder's inequality to functions $|\bar{p}|^{\alpha-2}$ and $|\nabla \Psi|^{2-a}$ using powers $\frac{\alpha}{\alpha-2}$ and $\frac{\alpha}{2}$, and then applying Young's inequality we obtain

$$\begin{aligned}
\int_U |\bar{p}|^{\alpha-2} |\nabla \Psi|^{2-a} dx &\leq \left(\int_U |\bar{p}|^\alpha dx \right)^{\frac{\alpha-2}{\alpha}} \left(\int_U |\nabla \Psi|^{\frac{\alpha(2-a)}{2}} dx \right)^{\frac{2}{\alpha}} \\
&\leq \varepsilon \left(\int_U |\bar{p}|^\alpha dx \right)^{\frac{1}{\gamma_0}} + C_\varepsilon \left(\int_U |\nabla \Psi|^{\frac{\alpha(2-a)}{2}} dx \right)^{\frac{2(\alpha-a)}{\alpha(2-a)}}.
\end{aligned} \tag{4.7}$$

Applying Hölder's inequality to three functions $|\nabla \bar{p}|^{1-a} |\bar{p}|^{\frac{(\alpha-2)(1-a)}{2-a}}$, $|\bar{p}|^{\frac{\alpha-2}{2-a}}$, $|\nabla \Psi|$ with powers $\frac{2-a}{1-a}$, $\frac{\alpha(2-a)}{\alpha-2}$, $\frac{\alpha(2-a)}{2}$ respectively, we have

$$\begin{aligned} \int_U |\nabla \bar{p}|^{1-a} |\nabla \Psi| |\bar{p}|^{\alpha-2} dx &= \int_U |\nabla \bar{p}|^{1-a} |\bar{p}|^{\frac{(\alpha-2)(1-a)}{(2-a)}} \cdot |\bar{p}|^{\frac{\alpha-2}{2-a}} \cdot |\nabla \Psi| dx \\ &\leq \left(\int_U |\nabla \bar{p}|^{2-a} |\bar{p}|^{\alpha-2} dx \right)^{\frac{1-a}{2-a}} \left(\int_U |\bar{p}|^\alpha dx \right)^{\frac{\alpha-2}{\alpha(2-a)}} \left(\int_U |\nabla \Psi|^{\frac{\alpha(2-a)}{2}} dx \right)^{\frac{2}{\alpha(2-a)}}. \end{aligned}$$

Since $\frac{1-a}{2-a} < 1$ and $\frac{\alpha-2}{\alpha(2-a)} \leq \frac{\alpha-2}{\alpha} < \frac{1}{\gamma_0}$, applying Young's inequality gives

$$\begin{aligned} \int_U |\nabla \bar{p}|^{1-a} |\nabla \Psi| |\bar{p}|^{\alpha-2} dx \\ \leq \varepsilon \int_U |\nabla \bar{p}|^{2-a} |\bar{p}|^{\alpha-2} dx + \varepsilon \left(\int_U |\bar{p}|^\alpha dx \right)^{\frac{1}{\gamma_0}} + C_\varepsilon \left(\int_U |\nabla \Psi|^{\frac{\alpha(2-a)}{2}} dx \right)^{\frac{2(\alpha-a)}{\alpha(2-a)}}. \end{aligned} \quad (4.8)$$

From (4.6), (4.7) and (4.8), we have

$$\begin{aligned} \left| \int_U K(|\nabla p|) \nabla p \cdot \nabla \Psi |\bar{p}|^{\alpha-2} dx \right| &\leq \varepsilon \int_U |\nabla \bar{p}|^{2-a} |\bar{p}|^{\alpha-2} dx + \varepsilon \left(\int_U |\bar{p}|^\alpha dx \right)^{\frac{1}{\gamma_0}} \\ &\quad + C_\varepsilon \left(\int_U |\nabla \Psi|^{\frac{\alpha(2-a)}{2}} dx \right)^{\frac{2(\alpha-a)}{\alpha(2-a)}}. \end{aligned} \quad (4.9)$$

For the last integral on the right-hand side of (4.4), by Hölder's and Young's inequalities, we similarly obtain

$$\begin{aligned} \left| \int_U \Psi_t |\bar{p}|^{\alpha-1} \text{sign}(\bar{p}) dx \right| &\leq \left(\int_U |\bar{p}|^\alpha dx \right)^{\frac{\alpha-1}{\alpha}} \left(\int_U |\Psi_t|^\alpha dx \right)^{\frac{1}{\alpha}} \\ &\leq \varepsilon \left(\int_U |\bar{p}|^\alpha dx \right)^{\frac{1}{\gamma_0}} + C_\varepsilon \left(\int_U |\Psi_t|^\alpha dx \right)^{\frac{\alpha-a}{\alpha(1-a)}}. \end{aligned} \quad (4.10)$$

Using estimates (4.5), (4.9) and (4.10) in (4.4), we have

$$\begin{aligned} \frac{d}{dt} \int_U |\bar{p}|^\alpha dx &\leq -C \int_U |\nabla \bar{p}|^{2-a} |\bar{p}|^{\alpha-2} dx + \varepsilon \left(\int_U |\bar{p}|^\alpha dx \right)^{\frac{1}{\gamma_0}} + \varepsilon \int_U |\nabla \bar{p}|^{2-a} |\bar{p}|^{\alpha-2} dx \\ &\quad + C_\varepsilon (1 + A(\alpha, t)). \end{aligned}$$

For sufficiently small ε , we obtain

$$\frac{d}{dt} \int_U |\bar{p}|^\alpha dx \leq -C \int_U |\nabla \bar{p}|^{2-a} |\bar{p}|^{\alpha-2} dx + \varepsilon \left(\int_U |\bar{p}|^\alpha dx \right)^{\frac{1}{\gamma_0}} + C_\varepsilon (1 + A(\alpha, t)). \quad (4.11)$$

Applying inequality (2.19) of Lemma 2.3 to $\int_U |\nabla \bar{p}|^{2-a} |\bar{p}|^{\alpha-2} dx$ in (4.11) yields

$$\frac{d}{dt} \int_U |\bar{p}|^\alpha dx \leq -C \left(\left(\int_U |\bar{p}|^\alpha dx \right)^{\frac{1}{\gamma_0}} - 1 \right) + \varepsilon \left(\int_U |\bar{p}|^\alpha dx \right)^{\frac{1}{\gamma_0}} + C_\varepsilon (1 + A(\alpha, t)).$$

Now selecting ε sufficiently small, we obtain (4.3). \square

We can estimate the L^α -norm of \bar{p} for all $\alpha > 0$.

Definition 4.2. For any $\alpha > 0$, we define

$$\hat{\alpha} = \max\{\alpha, 2, \alpha_*\} = \begin{cases} \max\{\alpha, \alpha_*\} & \text{in the NDC case} \\ \max\{\alpha, 2\} & \text{in the DC case.} \end{cases} \quad (4.12)$$

For $\alpha \geq 1$, let

$$A(\alpha) = \limsup_{t \rightarrow \infty} A(\alpha, t) \text{ and } \beta(\alpha) = \limsup_{t \rightarrow \infty} [A'(\alpha, t)]^-.$$

Whenever $\beta(\alpha)$ is in use, it is understood that the function $t \rightarrow A(\alpha, t)$ belongs to $C^1((0, \infty))$.

Theorem 4.3. Let $\alpha > 0$.

(i) For all $t \geq 0$,

$$\int_U |\bar{p}(x, t)|^\alpha dx \leq C \left(1 + \int_U |\bar{p}(x, 0)|^{\hat{\alpha}} dx + [EnvA(\hat{\alpha}, t)]^{\frac{\hat{\alpha}}{\hat{\alpha}-a}} \right). \quad (4.13)$$

(ii) If $A(\hat{\alpha}) < \infty$ then

$$\limsup_{t \rightarrow \infty} \int_U |\bar{p}(x, t)|^\alpha dx \leq C(1 + A(\hat{\alpha})^{\frac{\hat{\alpha}}{\hat{\alpha}-a}}). \quad (4.14)$$

(iii) If $\beta(\hat{\alpha}) < \infty$ then there is $T > 0$ such that

$$\int_U |\bar{p}(x, t)|^\alpha dx \leq C(1 + \beta(\hat{\alpha})^{\frac{\hat{\alpha}}{\hat{\alpha}-2a}} + A(\hat{\alpha}, t)^{\frac{\hat{\alpha}}{\hat{\alpha}-a}}) \quad (4.15)$$

for all $t \geq T$.

Proof. Since $\alpha \leq \hat{\alpha}$, by Young's inequality

$$\int_U |\bar{p}(x, t)|^\alpha dx \leq \int_U 1 + |\bar{p}(x, t)|^{\hat{\alpha}} dx,$$

hence it suffices to prove (4.13), (4.14) and (4.15) for the case $\alpha = \hat{\alpha}$.

Consider $\alpha = \hat{\alpha}$. Let $y(t) = \int_U |\bar{p}(x, t)|^\alpha dx$. Then by (4.3),

$$y'(t) \leq -c_5 y(t)^{\frac{1}{\gamma_0}} + C(1 + A(\alpha, t)), \quad t > 0. \quad (4.16)$$

(i) Applying estimate (2.33) Lemma 2.7 to inequality (4.16), we obtain

$$\int_U |\bar{p}(x, t)|^\alpha dx \leq \int_U |\bar{p}(x, 0)|^\alpha dx + C(1 + [EnvA(\alpha, t)]^{\gamma_0}),$$

hence (4.13) follows.

(ii) Assume $A(\alpha) < \infty$. Applying (2.34) of Lemma 2.7 to (4.16) yields

$$\limsup_{t \rightarrow \infty} \int_U |\bar{p}(x, t)|^\alpha dx \leq C(1 + A(\alpha))^{\gamma_0},$$

hence we obtain (4.14).

(iii) Assume $\beta(\alpha) < \infty$. Let $\phi(z) = c_5^{-\gamma_0} z^{\gamma_0}$ for $z \geq 0$. Then $\phi(z) \leq \varphi_{c, \gamma_0}(z)$, where $c = c_5^{-\gamma_0}$, and $\phi^{-1}(z) \geq \varphi_{c, \gamma_0}^{-1}(z)$, and it follows from (4.16) that

$$y'(t) \leq -\phi^{-1}(y(t)) + C(1 + A(\alpha, t)) \leq -\varphi_{c, \gamma_0}^{-1}(y(t)) + C(1 + A(\alpha, t)), \quad t > 0. \quad (4.17)$$

Note that $1 < \gamma_0 < 2$, applying Proposition 3.7 in [14] to (4.17), there is $T > 0$ such that for all $t > T$,

$$\int_U |\bar{p}(x, t)|^\alpha dx \leq C(1 + \beta(\alpha)^{\frac{\gamma_0}{2-\gamma_0}} + A(\alpha, t)^{\gamma_0}),$$

which yields (4.15). \square

For further estimates of spatial and time derivatives of the pressure, we recall the following quantities from [13, 14]:

$$\begin{aligned} G_1(t) &= G_1[\Psi](t) = \int_U |\nabla \Psi(x, t)|^2 dx + \left[\int_U |\Psi_t(x, t)|^{r_0} dx \right]^{\frac{2-a}{r_0(1-a)}} \\ &\quad + \left[\int_U |\Psi_t(x, t)|^{r_0} dx \right]^{\frac{1}{r_0}}, \\ G_2(t) &= G_2[\Psi](t) = \int_U |\nabla \Psi_t(x, t)|^2 dx + \int_U |\Psi_t(x, t)|^2 dx, \\ G_3(t) &= G_3[\Psi](t) = G_1(t) + G_2(t), \\ G_4(t) &= G_4[\Psi](t) = G_3(t) + \int_U |\Psi_{tt}|^2 dx, \end{aligned}$$

where r_0 is the same as in section 2, or explicitly, $r_0 = \frac{n(2-a)}{(2-a)(n+1)-n}$.

Using the above L^α -estimates, we re-estimate and improve the bounds of the integrals $\int_U H(|\nabla p|) dx$ and $\int_U |\bar{p}_t|^2 dx$ obtained in [13]. Particularly, our new estimates are much sharper in the NDC case.

From estimate (3.25) in [13] we have for all $t \geq 0$ that

$$\int_U H(|\nabla p(x, t)|) dx \leq \int_U H(|\nabla p(x, 0)|) + |\bar{p}(x, 0)|^2 dx + C \int_0^t G_3(\tau) d\tau. \quad (4.18)$$

However, this does not imply the uniform boundedness of $\int_U H(|\nabla p(x, t)|) dx$ for $t \geq 0$ even when $G_3(t)$ are uniformly bounded.

We easily obtain from inequality (3.4) in [13] that

$$\int_0^t \int_U H(|\nabla p(x, \tau)|) dx \leq C \int_U |\bar{p}(x, 0)|^2 dx + C \int_0^t G_1(\tau) d\tau. \quad (4.19)$$

Also, we recall (3.25) of [13]:

$$\int_0^t \int_U |\bar{p}_t(x, \tau)|^2 dx d\tau \leq \int_U H(|\nabla p(x, 0)|) + |\bar{p}(x, 0)|^2 dx + C \int_0^t G_3(\tau) d\tau. \quad (4.20)$$

Because of estimate (4.13), we define for $\alpha > 0$ and $t \geq 0$,

$$B(\alpha, t) = B[\bar{p}(\cdot, 0), \Psi](\alpha, t) \stackrel{\text{def}}{=} \int_U |\bar{p}(x, 0)|^\alpha dx + [EnvA(\alpha, t)]^{\frac{\alpha}{\alpha-a}}. \quad (4.21)$$

Theorem 4.4. (i) If $t \geq 0$ then

$$\begin{aligned} \int_U H(|\nabla p(x, t)|) dx &\leq C \left(1 + \int_U |\bar{p}(x, 0)|^{\hat{2}} dx \right) + e^{-d_5 t} \int_U H(|\nabla p(x, 0)|) dx \\ &+ C \int_0^t e^{-d_5(t-\tau)} \left[[Env(A(\hat{2}), \tau)]^{\frac{\hat{2}}{2-a}} + G_3(\tau) \right] d\tau, \end{aligned} \quad (4.22)$$

where $d_5 > 0$ is introduced in (4.26) below.

(ii) If $A(\hat{2}) < \infty$ then there is $T > 0$ such that for $t > T$,

$$\int_U H(|\nabla p(x, t)|) dx \leq C \left(1 + A(\hat{2})^{\frac{\hat{2}}{2-a}} + \int_0^t e^{-d_5(t-\tau)} G_3(\tau) d\tau \right); \quad (4.23)$$

consequently

$$\limsup_{t \rightarrow \infty} \int_U H(|\nabla p(x, t)|) dx \leq C \left[1 + A(\hat{2})^{\frac{\hat{2}}{2-a}} + \limsup_{t \rightarrow \infty} G_3(t) \right]. \quad (4.24)$$

(iii) If $\beta(\hat{2}) < \infty$ then there is $T > 0$ such that

$$\int_U H(|\nabla p(x, t)|) dx \leq C \left\{ 1 + \beta(\hat{2})^{\frac{\hat{2}}{2-2a}} + \int_0^t e^{-d_5(t-\tau)} \left[A(\hat{2}, \tau)^{\frac{\hat{2}}{2-a}} + G_3(\tau) \right] d\tau \right\} \quad (4.25)$$

for all $t > T$.

Proof. (i) From line 4 into the proof of Corollary 3.7 in [13], we have

$$\frac{d}{dt} \int_U H(|\nabla p|) dx + \frac{1}{2} \int_U |\bar{p}_t|^2 dx \leq -d_5 \int_U H(|\nabla p|) dx + C \int_U \bar{p}^2 dx + CG_3(t), \quad (4.26)$$

hence neglecting the non-negative term on the left-hand side gives

$$\frac{d}{dt} \int_U H(|\nabla p|) dx \leq -d_5 \int_U H(|\nabla p|) dx + C \int_U \bar{p}^2 dx + CG_3(t). \quad (4.27)$$

Applying Gronwall's inequality yields

$$\begin{aligned} \int_U H(|\nabla p(x, t)|) dx &\leq e^{-d_5 t} \int_U H(|\nabla p(x, 0)|) dx \\ &+ C \int_0^t e^{-d_5(t-\tau)} \left[\int_U \bar{p}^2(x, \tau) dx + G_3(\tau) \right] d\tau. \end{aligned} \quad (4.28)$$

Using estimate (4.13) in (4.28) gives

$$\begin{aligned} \int_U H(|\nabla p(x, t)|) dx &\leq e^{-d_5 t} \int_U H(|\nabla p(x, 0)|) dx \\ &+ C \int_0^t e^{-d_5(t-\tau)} \left(1 + \int_U |p(0, x)|^{\hat{2}} dx + [EnvA(\hat{2}, \tau)]^{\frac{\hat{2}}{2-a}} + G_3(\tau) \right) d\tau, \end{aligned}$$

and hence (4.22) follows.

(ii) Assume $A(\hat{2}) < \infty$. We notice that the first term on the right-hand side of (4.28) exponentially decays to 0 as $t \rightarrow \infty$. Next, by (2.35) and (4.14)

$$\limsup_{t \rightarrow \infty} \int_0^t e^{-d_5(t-\tau)} \int_U \bar{p}(x, \tau)^2 dx d\tau \leq C \limsup_{t \rightarrow \infty} \int_U \bar{p}^2(x, t) dx \leq C(1 + A(\hat{2})^{\gamma_0(\hat{2})}).$$

Hence it follows from (4.28) for large t that

$$\int_U H(|\nabla p(x, t)|) dx \leq C(1 + A(\widehat{2})^{\gamma_0(\widehat{2})}) + C \int_U e^{-d_5(t-\tau)} G_3(\tau) d\tau,$$

which is (4.23). Taking limit superior both sides of (4.23) and, again, applying (2.35) to $\int_0^t e^{-d_5(t-\tau)} G_3(\tau) d\tau$ we obtain (4.24).

(iii) Assume $\beta(\widehat{2}) < \infty$. By virtue of Theorem 4.3, there is $T_1 > 0$ such that (4.15) holds for $t > T_1$. Let $t > T_1$. Splitting the time integral in (4.28) gives

$$\begin{aligned} \int_U H(|\nabla p(x, t)|) dx &\leq e^{-d_5 t} \int_U H(|\nabla p(x, 0)|) dx \\ &\quad + C \int_0^{T_1} e^{-d_5(t-\tau)} \left(\int_U \bar{p}(x, \tau)^2 dx + G_3(\tau) \right) d\tau \\ &\quad + C \int_{T_1}^t e^{-d_5(t-\tau)} \left(\int_U \bar{p}(x, \tau)^2 dx + G_3(\tau) \right) d\tau. \end{aligned}$$

To estimate $\int_U \bar{p}(x, \tau)^2 dx$ we use (4.13) for $\tau < T_1$ and use (4.15) for $t > T_1$. Then

$$\begin{aligned} \int_U H(|\nabla p(x, t)|) dx &\leq e^{-d_5 t} \int_U H(|\nabla p(x, 0)|) dx \\ &\quad + C e^{-d_5 t} \int_0^{T_1} e^{d_5 \tau} (1 + B(\widehat{2}, \tau) + G_3(\tau)) d\tau \\ &\quad + C \int_{T_1}^\infty e^{-d_5(t-\tau)} (1 + \beta(\widehat{2})^{\frac{\widehat{2}}{2-2a}} + A(\widehat{2}, \tau)^{\gamma_0(\widehat{2})} + G_3(\tau)) d\tau. \\ &\leq C e^{-d_5 t} \left\{ \int_U H(|\nabla p(x, 0)|) dx + \int_0^{T_1} e^{d_5 \tau} (1 + B(\widehat{2}, \tau) + G_3(\tau)) d\tau \right\} \\ &\quad + C (1 + \beta(\widehat{2})^{\frac{\widehat{2}}{2-2a}}) + C \int_{T_1}^t e^{-d_5(t-\tau)} (A(\widehat{2}, \tau)^{\gamma_0(\widehat{2})} + G_3(\tau)) d\tau. \end{aligned}$$

Due to the exponential decay of the term $e^{-d_5 t} \{ \dots \}$ above, there exists $T > T_1$ such that for $t > T$,

$$\int_U H(|\nabla p(x, t)|) dx \leq C(1 + \beta(\widehat{2})^{\frac{\widehat{2}}{2-2a}}) + C \int_{T_1}^t e^{-d_5(t-\tau)} (A(\widehat{2}, \tau)^{\gamma_0(\widehat{2})} + G_3(\tau)) d\tau,$$

therefore we obtain (4.25). \square

For the time derivative, we have the following.

Theorem 4.5. (i) If $t \geq t_0 > 0$ then

$$\begin{aligned} \int_U H(|\nabla p(x, t)|) + |\bar{p}_t(x, t)|^2 dx &\leq C \left(1 + \int_U |\bar{p}(x, 0)|^2 dx \right) \\ &\quad + C e^{d_6 t_0} t_0^{-1} e^{-d_6 t} \left\{ \int_U |\bar{p}(x, 0)|^2 + H(|\nabla p(x, 0)|) dx + \int_0^{t_0} G_3(\tau) d\tau \right\} \\ &\quad + C \int_0^t e^{-d_6(t-\tau)} \left([EnvA(\widehat{2}, \tau)]^{\frac{\widehat{2}}{2-a}} + G_4(\tau) \right) d\tau, \end{aligned} \quad (4.29)$$

where $d_6 > 0$ is introduced in (4.33) below.

(ii) If $A(\widehat{2}) < \infty$ then there is $T > 0$ such that for $t > T$,

$$\int_U H(|\nabla p(x, t)|) + |\bar{p}_t(x, t)|^2 dx \leq C \left(1 + A(\widehat{2})^{\frac{\widehat{2}}{2-a}} + \int_1^t e^{-d_6(t-\tau)} G_4(\tau) d\tau \right); \quad (4.30)$$

consequently,

$$\limsup_{t \rightarrow \infty} \int_U H(|\nabla p(x, t)|) + |\bar{p}_t(x, t)|^2 dx \leq C \left[1 + A(\widehat{2})^{\frac{\widehat{2}}{2-a}} + \limsup_{t \rightarrow \infty} G_4(t) \right]. \quad (4.31)$$

(iii) If $\beta(\widehat{2}) < \infty$ then there is $T > 0$ such that for $t > T$,

$$\int_U H(|\nabla p(x, t)|) + |\bar{p}_t(x, t)|^2 dx \leq C \left(1 + \beta(\widehat{2})^{\frac{\widehat{2}}{2-2a}} + \int_1^t e^{-d_6(t-\tau)} (A(\widehat{2}, \tau)^{\frac{\widehat{2}}{2-a}} + G_4(\tau)) d\tau \right). \quad (4.32)$$

Proof. (i) The inequality (3.51) of [13] reads

$$\frac{d}{dt} \int_U H(|\nabla p|) + |\bar{p}_t|^2 + |\bar{p}|^2 dx \leq -d_6 \int_U H(|\nabla p|) + |\bar{p}_t|^2 dx + C G_4(t).$$

By Cauchy's inequality (see the derivation of (4.26)), we have

$$\frac{d}{dt} \int_U H(|\nabla p|) + |\bar{p}_t|^2 dx \leq -d_6 \int_U H(|\nabla p|) + |\bar{p}_t|^2 dx + C \int_U \bar{p}^2 dx + C G_4(t). \quad (4.33)$$

Then by Theorem 4.3(i),

$$\frac{d}{dt} \int_U H(|\nabla p|) + |\bar{p}_t|^2 dx \leq -d_6 \int_U H(|\nabla p|) + |\bar{p}_t|^2 dx + C(1 + B(\widehat{2}, t)) + C G_4(t). \quad (4.34)$$

By (4.19) and (4.20)

$$\begin{aligned} \int_0^{t_0} \int_U H(|\nabla p(x, \tau)|) + |\bar{p}_t(x, \tau)|^2 dx d\tau \\ \leq C_1 \left[\int_U H(|\nabla p(x, 0)|) + |\bar{p}(x, 0)|^2 dx + \int_0^{t_0} G_3(\tau) d\tau \right]. \end{aligned}$$

Hence there is $t' \in (0, t_0)$ such that

$$\begin{aligned} \int_U H(|\nabla p(x, t')|) + |\bar{p}_t(x, t')|^2 dx \\ \leq 2C_1 t_0^{-1} \left[\int_U H(|\nabla p(x, 0)|) + |\bar{p}(x, 0)|^2 dx + \int_0^{t_0} G_3(\tau) d\tau \right]. \end{aligned}$$

By (4.34) and Gronwall's inequality, for $t \geq t_0 > t' > 0$,

$$\begin{aligned} \int_U H(|\nabla p(x, t)|) + |\bar{p}_t(x, t)|^2 dx \\ \leq e^{-d_6(t-t')} \int_U H(|\nabla p(x, t')|) + |\bar{p}_t(x, t')|^2 dx + C \int_{t'}^t e^{-d_6(t-\tau)} (1 + B(\widehat{2}, \tau) + G_4(\tau)) d\tau \\ \leq C e^{-d_6(t-t_0)} t_0^{-1} \left\{ \int_U H(|\nabla p(x, 0)|) + |\bar{p}(x, 0)|^2 dx + \int_0^{t_0} G_3(\tau) d\tau \right\} \\ + C \int_0^t e^{-d_6(t-\tau)} \left(1 + \int_U \bar{p}(x, 0)^{\widehat{2}} dx + [EnvA(\widehat{2}, \tau)]^{\gamma_0(\widehat{2})} + G_4(\tau) \right) d\tau. \end{aligned}$$

Therefore (4.29) follows.

(ii) Assume $A(\widehat{2}) < \infty$. By (4.33) and Theorem 4.3(ii), there exists $T_1 \geq 1$ such that

$$\frac{d}{dt} \int_U H(|\nabla p|) + |\bar{p}_t|^2 dx \leq -d_6 \int_U H(|\nabla p|) + |\bar{p}_t|^2 dx + C[1 + A^{\gamma_0(\widehat{2})} + G_4(t)] \quad (4.35)$$

for all $t > T_1$. Let $t > T_1$. Applying Gronwall's inequality to (4.35) yields

$$\begin{aligned} & \int_U H(|\nabla p(x, t)|) + |\bar{p}_t(x, t)|^2 dx \\ & \leq e^{-d_6(t-T_1)} \int_U H(|\nabla p(x, T_1)|) + \bar{p}_t^2(x, T_1) dx + C \int_{T_1}^t e^{-d_6(t-\tau)} (1 + A^{\gamma_0(\widehat{2})} + G_4(\tau)) d\tau \\ & \leq e^{-d_6(t-T_1)} \int_U H(|\nabla p(x, T_1)|) + \bar{p}_t^2(x, T_1) dx + C(1 + A^{\gamma_0(\widehat{2})}) + C \int_1^t e^{-d_6(t-\tau)} G_4(\tau) d\tau. \end{aligned}$$

Since $\int_U H(|\nabla p(x, T_1)|) + \bar{p}_t^2(x, T_1) dx < \infty$, estimate (4.30) easily follows for sufficiently large t . Then (4.31) follows by taking limit superior of (4.30) and using (2.35).

(iii) Assume $\beta(\widehat{2}) < \infty$. By Theorem 4.3 and (4.33) there is $T_2 > 0$ such that for $t > T_2$,

$$\begin{aligned} & \frac{d}{dt} \int_U H(|\nabla p|) + |\bar{p}_t|^2 dx \\ & \leq -d_6 \int_U H(|\nabla p|) + |\bar{p}_t|^2 dx + C \left[1 + \beta(\widehat{2})^{\frac{2}{2-2a}} + A(\widehat{2}, t)^{\gamma_0(\widehat{2})} + G_4(t) \right]. \quad (4.36) \end{aligned}$$

We then obtain (4.34) by using the same argument as in part (ii). \square

Remark 4.6. In the NDC case, despite the initial data being imposed to be in a higher Lebesgue space $L^{\alpha_*}(U)$ the estimates obtained in Theorems 4.4 and 4.5 are much sharper than those in [13]. In particular if the boundary data satisfy $A(\widehat{2}, t) + G_4(t) \leq C$ for all $t \geq 0$, then $\int_U H(|\nabla p(x, t)|) dx$ is uniformly bounded on $[0, \infty)$, and $\int_U |\bar{p}_t(x, t)|^2 dx$ is uniformly bounded on $[1, \infty)$. These cannot be achieved by the results in [13] which only yield the boundedness for finite time intervals.

We now estimate some integrals with mixed products of p and $|\nabla p|$. These will be needed in section 6.

Corollary 4.7. *If $\alpha \geq \max\{2, \alpha_*\}$ and $t > 0$ then*

$$\begin{aligned} & \int_0^t \int_U |\nabla p(x, \tau)|^{2-a} |\bar{p}(x, \tau)|^{\alpha-2} dx d\tau \\ & \leq C \left(\int_U |\bar{p}(x, 0)|^\alpha dx + \int_0^t (1 + A(\alpha, \tau)) d\tau \right), \quad (4.37) \end{aligned}$$

$$\begin{aligned} & \int_U |\nabla p(x, t)|^{2-a} |\bar{p}(x, t)|^{\alpha-2} dx \\ & \leq C \left(1 + A(\alpha, t) + \int_U |\bar{p}(x, t)|^{2(\alpha-1)} dx + \int_U |\bar{p}_t(x, t)|^2 dx \right). \end{aligned} \quad (4.38)$$

Proof. Note in this case that $\hat{\alpha} = \alpha$. Since $|\nabla p| \leq |\nabla \bar{p}| + |\nabla \Psi|$, we have

$$\int_U |\nabla p|^{2-a} |\bar{p}|^{\alpha-2} dx \leq C \int_U |\nabla \bar{p}|^{2-a} |\bar{p}|^{\alpha-2} dx + C \int_U |\nabla \Psi|^{2-a} |\bar{p}|^{\alpha-2} dx. \quad (4.39)$$

Let $t > 0$. Integrating inequality (4.11) from 0 to t with $\varepsilon = 1$ gives

$$\begin{aligned} \int_0^t \int_U |\nabla \bar{p}|^{2-a} |\bar{p}|^{\alpha-2} dx & \leq C \int_U |\bar{p}(x, 0)|^\alpha dx - C \int_U |\bar{p}(x, t)|^\alpha dx \\ & \quad + \int_0^t \left(\int_U |\bar{p}(x, \tau)|^\alpha dx \right)^{\frac{1}{\gamma_0}} d\tau + C \int_0^t (1 + A(\alpha, \tau)) d\tau. \end{aligned} \quad (4.40)$$

Integrating (4.3) from 0 to t yields

$$\int_0^t \left(\int_U |\bar{p}(x, \tau)|^\alpha dx \right)^{\frac{1}{\gamma_0}} d\tau \leq C \int_U |\bar{p}(x, 0)|^\alpha dx + C \int_0^t (1 + A(\alpha, \tau)) d\tau. \quad (4.41)$$

Combining (4.40) with (4.41) we have

$$\int_0^t \int_U |\nabla \bar{p}|^{2-a} |\bar{p}|^{\alpha-2} dx \leq C \int_U |\bar{p}(x, 0)|^\alpha dx + C \int_0^t (1 + A(\alpha, \tau)) d\tau. \quad (4.42)$$

By (4.39) as well as Hölder's and Young's inequalities,

$$\begin{aligned} & \int_0^t \int_U |\nabla p|^{2-a} |\bar{p}|^{\alpha-2} dx d\tau \\ & \leq C \int_0^t \int_U |\nabla \bar{p}|^{2-a} |\bar{p}|^{\alpha-2} dx d\tau + C \int_0^t \int_U |\nabla \Psi|^{2-a} |\bar{p}|^{\alpha-2} dx d\tau \\ & \leq C \int_0^t \int_U |\nabla \bar{p}|^{2-a} |\bar{p}|^{\alpha-2} dx d\tau + C \int_0^t \left(\int_U |\nabla \Psi|^{\frac{\alpha(2-a)}{2}} dx \right)^{\frac{2}{\alpha}} \left(\int_U |\bar{p}|^\alpha dx \right)^{\frac{\alpha-2}{\alpha}} d\tau \\ & \leq C \int_0^t \int_U |\nabla \bar{p}|^{2-a} |\bar{p}|^{\alpha-2} dx d\tau + C \int_0^t \left(\int_U |\nabla \Psi|^{\frac{\alpha(2-a)}{2}} dx \right)^{\frac{2(\alpha-a)}{\alpha(2-a)}} d\tau \\ & \quad + \int_0^t \left(\int_U |\bar{p}|^\alpha dx \right)^{\frac{1}{\gamma_0}} d\tau. \end{aligned}$$

Utilizing estimates (4.41) and (4.42) in the last inequality, and noticing that the quantity $\left(\int_U |\nabla \Psi(x, \tau)|^{\frac{\alpha(2-a)}{2}} dx \right)^{\frac{2(\alpha-a)}{\alpha(2-a)}}$ is a part of $A(\alpha, \tau)$ we obtain (4.37).

Now, we set $\varepsilon = 1$ in (4.11) and rewrite it as

$$\int_U |\nabla \bar{p}|^{2-a} |\bar{p}|^{\alpha-2} dx \leq -C \int_U \frac{\partial |\bar{p}|^\alpha}{\partial t} dx + C \int_U |\bar{p}|^\alpha dx + C(1 + A(\alpha, t)).$$

Hence

$$\begin{aligned} \int_U |\nabla \bar{p}|^{2-a} |\bar{p}|^{\alpha-2} dx & \leq C \int_U |\bar{p}_t| |\bar{p}|^{\alpha-1} dx + C \int_U |\bar{p}|^\alpha dx + C(1 + A(\alpha, t)) \\ & \leq C \left\{ \int_U |\bar{p}_t|^2 dx + \int_U |\bar{p}|^{2(\alpha-1)} dx + \int_U |\bar{p}|^\alpha dx + 1 + A(\alpha, t) \right\}. \end{aligned}$$

Also, by Young's inequality

$$\int_U |\nabla \Psi|^{2-a} |\bar{p}|^{\alpha-2} dx \leq \int_U |\nabla \Psi|^{\frac{\alpha(2-a)}{2}} dx + \int_U |\bar{p}|^\alpha dx \leq 1 + A(\alpha, t) + \int_U |\bar{p}|^\alpha dx.$$

The last inequality is due to the fact that $\frac{2(\alpha-1)}{\alpha(2-a)} \geq 1$. Thus we have from (4.39),

$$\begin{aligned} \int_U |\nabla p|^{2-a} |\bar{p}|^{\alpha-2} dx &\leq C \left\{ \int_U |\bar{p}_t|^2 dx + \int_U |\bar{p}|^{2(\alpha-1)} dx + \int_U |\bar{p}|^\alpha dx + 1 + A(\alpha, t) \right\} \\ &\quad + C \left\{ 1 + A(\alpha, t) + \int_U |\bar{p}|^\alpha dx \right\}. \end{aligned}$$

Since $2(\alpha - 1) \geq \alpha$, estimate (4.38) then follows. \square

Remark 4.8. The estimates and corresponding conditions in Theorems 4.3, 4.4 and 4.5 need be made more flexible in order to compare one to another when the involved parameters are varied. This can be done in the following way. First consider Theorem 4.3. Suppose

$$\alpha' \geq \hat{\alpha} \text{ and } A(\alpha', t)^{\gamma_0(\alpha')} \leq C_{\alpha'}(1 + G(t)), \quad (4.43)$$

where $C_{\alpha'}$ denotes a generic positive constant depending on α' . Then we can replace constant C depending on α by a constant $C_{\alpha'}$ depending on α' ; and

- in part (i), replace $\int_U |\bar{p}(x, 0)|^{\hat{\alpha}} dx$ by $\int_U |\bar{p}(x, 0)|^{\alpha'} dx$, replace $[EnvA(\hat{\alpha}, \cdot)]^{\frac{\hat{\alpha}}{\hat{\alpha}-a}}$ by $EnvG(\cdot)$; and

- in part (ii), replace $A(\hat{\alpha})$ by $[\limsup_{t \rightarrow \infty} G(t)]^{\frac{1}{\gamma_0(\alpha')}}$, replace $A(\hat{\alpha})^{\frac{\hat{\alpha}}{\hat{\alpha}-a}}$ by $\limsup_{t \rightarrow \infty} G(t)$; and

- in part (iii), replace $\beta(\hat{\alpha})$ by $\beta' = \limsup_{t \rightarrow \infty} [(G(t)^{\frac{1}{\gamma_0(\alpha')}})']^-$, replace $\beta(\hat{\alpha})^{\frac{\hat{\alpha}}{\hat{\alpha}-2a}}$ by $\beta'^{\frac{\alpha'}{\alpha'-2a}}$, replace $A(\hat{\alpha}, t)^{\frac{\hat{\alpha}}{\hat{\alpha}-a}}$ by $G(t)$.

Indeed, by Young's inequality

$$\int_U |\bar{p}(x, t)|^\alpha dx \leq \int_U 1 + |\bar{p}(x, t)|^{\alpha'} dx,$$

and by (4.3) and (4.43),

$$\frac{d}{dt} \int_U |\bar{p}(x, t)|^{\alpha'} dx \leq -c_5(\alpha') \left(\int_U |\bar{p}(x, t)|^{\alpha'} dx \right)^{\frac{1}{\gamma_0(\alpha')}} + C_{\alpha'} \left(1 + G(t)^{\frac{1}{\gamma_0(\alpha')}} \right), \quad t > 0.$$

Then we can proceed the proof of Theorem 4.3 with α' replacing α and $G(t)^{\frac{1}{\gamma_0(\alpha')}}$ replacing $A(\alpha, t)$, also noticing that $\hat{\alpha}' = \alpha'$,

$$\begin{aligned} \limsup_{t \rightarrow \infty} [G(t)^{\frac{1}{\gamma_0(\alpha')}}] &= [\limsup_{t \rightarrow \infty} G(t)]^{\frac{1}{\gamma_0(\alpha')}}, \\ \left\{ \limsup_{t \rightarrow \infty} [G(t)^{\frac{1}{\gamma_0(\alpha')}}] \right\}^{\gamma_0(\alpha')} &= \limsup_{t \rightarrow \infty} G(t), \end{aligned}$$

and we can choose $[EnvG(t)]^{\frac{1}{\gamma_0(\alpha')}}$ for $Env[G(t)^{\frac{1}{\gamma_0(\alpha')}}]$, therefore

$$\left\{ Env[G(t)^{\frac{1}{\gamma_0(\alpha')}}] \right\}^{\gamma_0(\alpha')} = EnvG(t).$$

The above replacements can also be applied to Theorems 4.4 and 4.5 with $\hat{\alpha} = \hat{2}$.

5. DEPENDENCE ON THE BOUNDARY DATA

In this section we study the continuous dependence of the solution to IBVP (3.1) on the boundary data $\psi(x, t)$. Let $p_k = p_k(x, t)$, for $k = 1, 2$, be two solutions of (3.1) with boundary values ψ_k respectively. For $k = 1, 2$, let Ψ_k be an extension of ψ_k and denote $\bar{p}_k = p_k - \Psi_k$. Let

$$z = p_1 - p_2, \quad \Phi = \Psi_1 - \Psi_2 \text{ and } \bar{z} = \bar{p}_1 - \bar{p}_2 = z - \Phi.$$

We will estimate $\int_U |\bar{z}(x, t)|^\alpha dx$ and $\int_U |\nabla \bar{z}(x, t)|^{2-a} dx$. The corresponding estimates for z can be retrieved easily by using the relations

$$\begin{aligned} \int_U |z|^\alpha dx &\leq C \left[\int_U |\bar{z}|^\alpha dx + \int_U |\Phi|^\alpha dx \right], \\ \int_U |\nabla z|^{2-a} dx &\leq C \left[\int_U |\nabla \bar{z}|^{2-a} dx + \int_U |\nabla \Phi|^{2-a} dx \right]. \end{aligned}$$

We derive from (5.1) then IBVP for $\bar{z}(x, t)$:

$$\begin{cases} \frac{\partial \bar{z}}{\partial t} = \nabla \cdot (K(|\nabla p_1|)\nabla p_1 - K(|\nabla p_2|)\nabla p_2) - \Phi_t \text{ in } U \times (0, \infty), \\ \bar{z}(x, 0) = \bar{p}_1(x, 0) - \bar{p}_2(x, 0) \text{ in } U, \\ \bar{z}(x, t) = 0 \text{ on } \Gamma \times (0, \infty). \end{cases} \quad (5.1)$$

Throughout this section the numbers θ , θ_1 and θ_2 are as in Lemma 2.5. Denote

$$N(\lambda, t) = \sum_{k=1}^2 \left(\int_U |\bar{p}_k(x, t)|^\lambda dx \right)^{1/\lambda} \text{ for } \lambda > 0, \text{ and } N(0, t) = 1.$$

Lemma 5.1. *Let $\alpha \geq 2$.*

(i) *In the DC case,*

$$\frac{d}{dt} \int_U |\bar{z}(x, t)|^\alpha dx \leq -c_6 \left[\int_U |\bar{z}(x, t)|^\alpha dx \right] M_1(t)^{-\frac{\alpha}{2-a}} + CF(\alpha, t)D(\alpha, t) \quad (5.2)$$

for all $t > 0$, where $c_6 = c_6(\alpha, \vec{\alpha}, \vec{a}, n, U) > 0$,

$$M_1(t) = 1 + \int_U |\nabla p_1(x, t)|^{2-a} + |\nabla p_2(x, t)|^{2-a} dx, \quad (5.3)$$

$$F(\alpha, t) = 1 + N(\alpha, t)^{\alpha-1} + N(\gamma_1, t)^{\alpha-2} M_1(t)^{\frac{1-a}{2-a}}, \quad (5.4)$$

with $\gamma_1 = \gamma_1(\alpha) \stackrel{\text{def}}{=} 2(\alpha - 2)(2 - a)$, and

$$D(\alpha, t) = \begin{cases} \|\nabla \Phi(\cdot, t)\|_{L^{2(2-a)}} + \|\nabla \Phi(\cdot, t)\|_{L^\alpha}^2 + \|\Phi_t(\cdot, t)\|_{L^\alpha} & \text{if } \alpha > 2, \\ \|\nabla \Phi(\cdot, t)\|_{L^{2-a}} + \|\nabla \Phi(\cdot, t)\|_{L^2}^2 + \|\Phi_t(\cdot, t)\|_{L^2} & \text{if } \alpha = 2. \end{cases} \quad (5.5)$$

(ii) *In the NDC case,*

$$\frac{d}{dt} \int_U |\bar{z}(x, t)|^\alpha dx \leq -c_7 \left[\int_U |\bar{z}(x, t)|^\alpha dx \right]^\theta M_2(\alpha, t)^{-\frac{2-\theta_1}{\theta_1}} + CF(\alpha, t)D(\alpha, t) \quad (5.6)$$

for all $t > 0$, where $c_7 = c_7(\alpha, \theta, \theta_1, \vec{\alpha}, \vec{a}, n, U) > 0$,

$$M_2(\alpha, t) = 1 + \int_U |\nabla p_1(x, t)|^{2-a} + |\nabla p_2(x, t)|^{2-a} + |\bar{p}_1(x, t)|^{\theta_2 \alpha} + |\bar{p}_2(x, t)|^{\theta_2 \alpha} dx. \quad (5.7)$$

Proof. Multiplying the first equation of (5.1) by $|\bar{z}|^{\alpha-1}\text{sign}(z)$, integrating over domain U , then applying Green's formula, we have

$$\begin{aligned} & \frac{1}{\alpha} \frac{d}{dt} \int_U |\bar{z}|^\alpha dx \\ &= -(\alpha-1) \int_U (K(|\nabla p_1|)\nabla p_1 - K(|\nabla p_2|)\nabla p_2) \cdot \nabla \bar{z} |\bar{z}|^{\alpha-2} dx - \int_U \Phi_t |\bar{z}|^{\alpha-1} \text{sign}(\bar{z}) dx \\ &= -(\alpha-1) \int_U (K(|\nabla p_1|)\nabla p_1 - K(|\nabla p_2|)\nabla p_2) \cdot (\nabla p_1 - \nabla p_2) |\bar{z}|^{\alpha-2} dx \\ & \quad + (\alpha-1) \int_U (K(|\nabla p_1|)\nabla p_1 - K(|\nabla p_2|)\nabla p_2) \cdot \nabla \Phi |\bar{z}|^{\alpha-2} dx - \int_U \Phi_t |\bar{z}|^{\alpha-1} \text{sign}(\bar{z}) dx. \end{aligned}$$

By the monotonicity, c.f. Lemma 2.1(i),

$$\begin{aligned} \frac{1}{\alpha} \frac{d}{dt} \int_U |\bar{z}|^\alpha dx &\leq -(\alpha-1)(1-a) \int_U K(\max\{|\nabla p_1|, |\nabla p_2|\}) |\nabla p_1 - \nabla p_2|^2 |\bar{z}|^{\alpha-2} dx \\ & \quad + (\alpha-1)J_2 + J_1, \end{aligned}$$

where

$$\begin{aligned} J_1 &= \int_U |\Phi_t| |\bar{z}|^{\alpha-1} dx, \\ J_2 &= \int_U \left(K(|\nabla p_1|)|\nabla p_1| + K(|\nabla p_2|)|\nabla p_2| \right) |\nabla \Phi| |\bar{z}|^{\alpha-2} dx. \end{aligned}$$

Since $|\nabla p_1 - \nabla p_2|^2 = |\nabla \bar{z} - \nabla \Phi|^2 \geq 1/2|\nabla \bar{z}|^2 - |\nabla \Phi|^2$, we have

$$\frac{1}{\alpha} \frac{d}{dt} \int_U |\bar{z}|^\alpha dx \leq -CJ_4 + CJ_3 + (\alpha-1)J_2 + J_1, \quad (5.8)$$

where

$$\begin{aligned} J_3 &= \int_U K(\max\{|\nabla p_1|, |\nabla p_2|\}) |\nabla \Phi|^2 |\bar{z}|^{\alpha-2} dx, \\ J_4 &= \int_U K(\max\{|\nabla p_1|, |\nabla p_2|\}) |\nabla \bar{z}|^2 |\bar{z}|^{\alpha-2} dx. \end{aligned}$$

We now estimate each of J_1, J_2, J_3 .

Consider the case $\alpha > 2$. For J_1 , we have

$$J_1 \leq C \left[\int_U |\Phi_t|^\alpha dx \right]^{\frac{1}{\alpha}} \left[\int_U |\bar{z}|^\alpha dx \right]^{\frac{\alpha-1}{\alpha}} \leq C \|\Phi_t\|_{L^\alpha} N(\alpha, t)^{\alpha-1}. \quad (5.9)$$

For J_2 , using (2.10) and applying Hölder's inequality give

$$\begin{aligned} J_2 &\leq C \int_U \left(\sum_{k=1}^2 |\nabla p_k|^{1-a} \right) |\nabla \Phi| |\bar{z}|^{\alpha-2} dx \\ &\leq C \left[\sum_{k=1}^2 \int_U |\nabla p_k|^{2-a} dx \right]^{\frac{1-a}{2-a}} \left[\int_U |\bar{z}|^{(\alpha-2)(2-a)} |\nabla \Phi|^{2-a} dx \right]^{\frac{1}{2-a}} \\ &\leq CM_1(t)^{\frac{1-a}{2-a}} \left[\int_U |\bar{z}|^{2(\alpha-2)(2-a)} dx \right]^{\frac{1}{2(2-a)}} \left[\int_U |\nabla \Phi|^{2(2-a)} dx \right]^{\frac{1}{2(2-a)}}. \end{aligned}$$

Hence

$$J_2 \leq CM_1(t)^{\frac{1-a}{2-a}} N(2(\alpha-2)(2-a), t)^{\alpha-2} \|\nabla\Phi\|_{L^{2(2-a)}}. \quad (5.10)$$

For J_3 , by the boundedness of $K(\xi)$, Hölder's and Young's inequalities, we get

$$J_3 \leq C \int_U |\nabla\Phi|^2 |\bar{z}|^{\alpha-2} dx \leq C \left[\int_U |\nabla\Phi|^\alpha dx \right]^{\frac{2}{\alpha}} \left[\int_U |\bar{z}|^\alpha dx \right]^{\frac{\alpha-2}{\alpha}} \leq C \|\nabla\Phi\|_{L^\alpha}^2 N(\alpha, t)^{\alpha-2},$$

thus,

$$J_3 \leq C(1 + N(\alpha, t)^{\alpha-1}) \|\nabla\Phi\|_{L^\alpha}^2. \quad (5.11)$$

Combining (5.8) with estimates (5.9), (5.10) and (5.11), we obtain

$$\begin{aligned} \frac{d}{dt} \int_U |\bar{z}|^\alpha dx &\leq -CJ_4 + CN(\alpha, t)^{\alpha-1} \|\Phi_t\|_{L^\alpha} + CM_1(t)^{\frac{1-a}{2-a}} N(\gamma_1, t)^{\alpha-2} \|\nabla\Phi\|_{L^{2(2-a)}} \\ &\quad + C(1 + N(\alpha, t)^{\alpha-1}) \|\nabla\Phi\|_{L^\alpha}^2, \end{aligned}$$

hence

$$\frac{d}{dt} \int_U |\bar{z}|^\alpha dx \leq -CJ_4 + CF(\alpha, t)D(\alpha, t). \quad (5.12)$$

For the case $\alpha = 2$ we use the same estimate for J_1 while quickly estimate

$$J_2 \leq CM_1(t)^{\frac{1-a}{2-a}} \|\nabla\Phi\|_{L^{2-a}} \text{ and } J_3 \leq C \|\nabla\Phi\|_{L^2}^2,$$

hence obtaining (5.12) again with the corresponding $D(2, t)$ defined in (5.5).

(i) Consider the DC case. Applying inequality (2.25) of Lemma 2.5 with $u = \bar{z}$, $\xi = \max\{|\nabla p_1|, |\nabla p_2|\}$ and using $\xi \leq |\nabla p_1| + |\nabla p_2|$ in the last integral of that inequality we obtain

$$\int_U |\bar{z}|^\alpha dx \leq CJ_4 M_1(t)^{\frac{\alpha}{2-a}},$$

which implies

$$-J_4 \leq -C \left[\int_U |\bar{z}|^\alpha dx \right] M_1(t)^{-\frac{\alpha}{2-a}}. \quad (5.13)$$

Then (5.2) follows from (5.12) and (5.13).

(ii) Now consider the NDC case. Similarly, applying inequality (2.27) instead of (2.25) with the same $u(x)$ and $\xi(x)$, and also using $|u| \leq |\bar{p}_1| + |\bar{p}_2|$ in its last integral, we have

$$-J_4 \leq -C \left[\int_U |\bar{z}|^\alpha dx \right]^\theta M_2(\alpha, t)^{-\frac{2-\theta_1}{\theta_1}}. \quad (5.14)$$

Hence (5.6) follows from (5.12) and (5.14). \square

Referring to the notation used in the section 4, we set

$$\mathcal{G}_j(t) = G_j[\Psi_1](t) + G_j[\Psi_2](t) \text{ for } j = 1, 2, 3, 4.$$

To unify may estimates below, we will use the replacements in Remark 4.8. With the condition (4.43) in mind, we observe that

$$\left(\sum_{k=1}^2 A[\Psi_k](\alpha, t) \right)^{\gamma_0(\alpha)} \leq C \sum_{k=1}^2 \left\{ \left(\int_U |\nabla\Psi_k|^{\frac{\alpha(2-a)}{2}} dx \right)^{\frac{2}{2-a}} + \left(\int_U |(\Psi_k)_t|^\alpha dx \right)^{\frac{1}{1-a}} \right\},$$

therefore, by Hölder's inequality,

$$\left(\sum_{k=1}^2 A[\Psi_k](\alpha, t) \right)^{\gamma_0(\alpha)} \leq C\tilde{\mathcal{A}}(\alpha, t), \quad (5.15)$$

where

$$\tilde{\mathcal{A}}(\alpha, t) = \sum_{k=1}^2 \left\{ \int_U |\nabla \Psi_k(x, t)|^\alpha dx + \left(\int_U |(\Psi_k(x, t))_t|^\alpha dx \right)^{\frac{1}{1-a}} \right\}. \quad (5.16)$$

For each $k = 1, 2$, applying (4.22) of Theorem 4.4 for p_k and using replacements in Remark 4.8 with $\alpha' = \alpha \geq \hat{2}$ and, thanks to (5.15), $G(t) = \tilde{\mathcal{A}}(\alpha, t)$, we have

$$M_1(t) \leq C_\alpha \bar{M}_1(\alpha, t) \text{ for all } \alpha \geq \hat{2}, t \geq 0, \quad (5.17)$$

where

$$\begin{aligned} \bar{M}_1(\alpha, t) = 1 + \sum_{k=1}^2 \left[\int_U |\bar{p}_k(x, 0)|^\alpha dx + e^{-d_5 t} \int_U |\nabla p_k(x, 0)|^{2-a} dx \right] \\ + \int_0^t e^{-d_5(t-\tau)} (Env\tilde{\mathcal{A}}(\alpha, \tau) + \mathcal{G}_3(\tau)) d\tau. \end{aligned} \quad (5.18)$$

Particularly, for $\alpha = \hat{2}$

$$M_1(t) \leq C\bar{M}_1(t) \text{ for all } t \geq 0, \text{ where } \bar{M}_1(t) = \bar{M}_1(\hat{2}, t). \quad (5.19)$$

For $\lambda \geq 0$, similar to (5.17) but using (4.13) instead of (4.22), using $\alpha' = \lambda$ and $G(t) = \tilde{\mathcal{A}}(\lambda, t)$ in Remark 4.8, we have

$$\sum_{k=1}^2 \int_U |\bar{p}_k(\cdot, t)|^\lambda dx \leq C_\lambda \bar{E}(\hat{\lambda}, t) \text{ for all } t \geq 0, \quad (5.20)$$

where

$$\bar{E}(\lambda, t) = \begin{cases} 1 + \sum_{k=1}^2 \int_U |\bar{p}_k(\cdot, 0)|^\lambda dx + Env\tilde{\mathcal{A}}(\lambda, t) & \text{if } \lambda > 0, \\ 1 & \text{if } \lambda = 0. \end{cases} \quad (5.21)$$

Let $\lambda \geq \hat{\theta}_2 \alpha \geq \hat{2}$, we observe that

$$\begin{aligned} M_2(\alpha, t) &= M_1(t) + \sum_{k=1}^2 \int_U |\bar{p}_k(x, t)|^{\theta_2 \alpha} dx \\ &\leq C_{\alpha, \lambda} \left(1 + M_1(t) + \sum_{k=1}^2 \int_U |\bar{p}_k(x, t)|^\lambda dx \right). \end{aligned} \quad (5.22)$$

hence by taking α in (5.17) to be λ and using (5.20), we obtain

$$M_2(\alpha, t) \leq C_{\alpha, \lambda} (\bar{M}_1(\lambda, t) + \bar{E}(\lambda, t)). \quad (5.23)$$

Taking into account that $Env\tilde{\mathcal{A}}(\lambda, t)$ is increasing in t , we can bound

$$\int_0^t e^{-d_5(t-\tau)} Env\tilde{\mathcal{A}}(\lambda, \tau) d\tau \leq d_5^{-1} Env\tilde{\mathcal{A}}(\lambda, t). \quad (5.24)$$

Hence

$$M_2(\alpha, t) \leq C_{\alpha, \lambda} \overline{M}_2(\lambda, t) \text{ for all } \lambda \geq \widehat{\theta}_2 \alpha, t \geq 0, \quad (5.25)$$

where

$$\begin{aligned} \overline{M}_2(\lambda, t) = 1 + \sum_{k=1}^2 \left[\int_U |\bar{p}_k(x, 0)|^\lambda dx + e^{-d_5 t} \int_U |\nabla p_k(x, 0)|^{2-a} dx \right] \\ + Env \tilde{\mathcal{A}}(\lambda, t) + \int_0^t e^{-d_5(t-\tau)} \mathcal{G}_3(\tau) d\tau. \end{aligned} \quad (5.26)$$

When $\lambda = \widehat{\theta}_2 \alpha$, estimate (5.25) reads

$$M_2(\alpha, t) \leq C_\alpha \overline{M}_2(\widehat{\theta}_2 \alpha, t) \text{ for all } t \geq 0. \quad (5.27)$$

If $\alpha \geq 0$, then $N(\alpha, t) \leq C_\alpha(1 + N(\widehat{\alpha}, t))$, and by (5.20) we have $N(\alpha, t) \leq C_\alpha \overline{E}(\widehat{\alpha}, t)^{1/\widehat{\alpha}}$, thus

$$N(\alpha, t) \leq C_\alpha \overline{N}(\alpha, t) \text{ for all } \alpha \geq 0, t \geq 0, \quad (5.28)$$

where

$$\overline{N}(\alpha, t) = \begin{cases} 1 + \sum_{k=1}^2 \|\bar{p}_k(\cdot, 0)\|_{L^{\widehat{\alpha}}} + (Env \tilde{\mathcal{A}}(\widehat{\alpha}, t))^{1/\widehat{\alpha}} & \text{if } \alpha > 0, \\ 1 & \text{if } \alpha = 0. \end{cases} \quad (5.29)$$

If $\lambda \geq \alpha \geq 0$, then $N(\alpha, t) \leq C_{\alpha, \lambda}(1 + N(\lambda, t))$, and applying (5.28) with α set to equal λ , we have

$$N(\alpha, t) \leq C_{\alpha, \lambda} \overline{N}(\lambda, t) \text{ for all } \lambda \geq \alpha \geq 0, t \geq 0. \quad (5.30)$$

Consequently, concerning $F(\alpha, t)$ in (5.4), using (5.28) to estimate $N(\alpha, t)$ and $N(\gamma_1, t)$, and using (5.19) to estimate $M_1(t)$, we obtain

$$F(\alpha, t) \leq C_\alpha \overline{F}(\alpha, t) \text{ for all } t \geq 0, \quad (5.31)$$

where

$$\overline{F}(\alpha, t) = \overline{N}(\alpha, t)^{\alpha-1} + \overline{N}(\gamma_1(\alpha), t)^{\alpha-2} \overline{M}_1(t)^{\frac{1-a}{2-a}}. \quad (5.32)$$

Alternatively, for $\lambda \geq \max\{\alpha, \gamma_1(\alpha), \widehat{\theta}_2 \alpha\}$, we use (5.30) to estimate $N(\alpha, t)$ and $N(\gamma_1, t)$, and use (5.17) with α set to equal λ to estimate $M_1(t)$, thus obtaining

$$F(\alpha, t) \leq C_{\alpha, \lambda} \overline{F}(\alpha, \lambda, t) \text{ for all } t \geq 0, \lambda \geq \max\{\alpha, \gamma_1(\alpha), \widehat{\theta}_2 \alpha\}, \quad (5.33)$$

where

$$\overline{F}(\alpha, \lambda, t) = \overline{N}(\lambda, t)^{\alpha-1} + \overline{N}(\lambda, t)^{\alpha-2} \overline{M}_1(\lambda, t)^{\frac{1-a}{2-a}}. \quad (5.34)$$

For large time estimates we define

$$\bar{\alpha}(\alpha) = \limsup_{t \rightarrow \infty} \tilde{\mathcal{A}}(\alpha, t) \text{ and } \bar{\beta}(\alpha) = \limsup_{t \rightarrow \infty} \left[\frac{d}{dt} \left(\tilde{\mathcal{A}}(\alpha, t)^{\frac{1}{\gamma_0(\alpha)}} \right) \right]^-.$$

For $\alpha \geq \widehat{2}$, there is $T_\alpha > 0$ such that

$$M_1(t) \leq C_\alpha \tilde{M}_1(\alpha, t) \text{ for all } t \geq T_\alpha, \quad (5.35)$$

where

$$\tilde{M}_1(\alpha, t) = \begin{cases} 1 + \bar{A}(\alpha) + \int_0^t e^{-d_5(t-\tau)} \mathcal{G}_3(\tau) d\tau & \text{if } \bar{A}(\alpha) < \infty, \\ 1 + \bar{\beta}(\alpha)^{\frac{\alpha}{\alpha-2a}} & \\ + \int_0^t e^{-d_5(t-\tau)} (\tilde{\mathcal{A}}(\alpha, \tau) + \mathcal{G}_3(\tau)) d\tau & \text{if } \bar{A}(\alpha) = \infty, \bar{\beta}(\alpha) < \infty, \\ 1 + Env\tilde{\mathcal{A}}(\alpha, t) & \\ + \int_0^t e^{-d_5(t-\tau)} \mathcal{G}_3(\tau) d\tau & \text{if } \bar{A}(\alpha) = \infty, \bar{\beta}(\alpha) = \infty. \end{cases} \quad (5.36)$$

In particular, when $\alpha = \widehat{2}$ there is $T > 0$ such that

$$M_1(t) \leq C\tilde{M}_1(t) \text{ for all } t \geq T, \text{ where } \tilde{M}_1(t) = \tilde{M}_1(\widehat{2}, t). \quad (5.37)$$

For the proof of (5.35), we combine Theorem 4.4 with the same replacements in Remark 4.8 used for the above derivation of (5.17), and have (4.23) corresponding to the case $\bar{A}(\alpha) < \infty$, have (4.25) corresponding to the case $\bar{A}(\alpha) = \infty, \bar{\beta}(\alpha) < \infty$, and have (4.22) corresponding to the case $\bar{A}(\alpha) = \infty, \bar{\beta}(\alpha) = \infty$.

Similarly, it follows from Theorem 4.3 instead of 4.4 that there is $T_\alpha > 0$, for $\alpha \geq \widehat{2}$, such that

$$\sum_{k=1}^2 \int_U |\bar{p}_k(\cdot, t)|^\alpha dx \leq C_\alpha \tilde{E}(\widehat{\alpha}, t) \text{ for all } t \geq T_\alpha, \quad (5.38)$$

where

$$\tilde{E}(\alpha, t) = \begin{cases} \bar{A}(\alpha) & \text{if } \bar{A}(\alpha) < \infty, \\ \bar{\beta}(\alpha)^{\frac{\alpha}{\alpha-2a}} + \tilde{\mathcal{A}}(\alpha, t) & \text{if } \bar{A}(\alpha) = \infty, \bar{\beta}(\alpha) < \infty, \\ Env\tilde{\mathcal{A}}(\alpha, t) & \text{if } \bar{A}(\alpha) = \infty, \bar{\beta}(\alpha) = \infty, \end{cases} \quad (5.39)$$

For $\alpha \geq 0$ and $\lambda \geq \widehat{\theta_2\alpha}$, by (5.22), (5.35) and (5.38) with $\alpha = \lambda = \widehat{\lambda}$, there is $T_{\alpha,\lambda} > 0$ such that

$$M_2(\alpha, t) \leq C_{\alpha,\lambda} \tilde{M}_2(\lambda, t) \text{ for all } t \geq T_{\alpha,\lambda}, \quad (5.40)$$

where

$$\tilde{M}_2(\alpha, t) = \tilde{M}_1(\alpha, t) + \tilde{E}(\alpha, t). \quad (5.41)$$

Taking $\lambda = \widehat{\theta_2\alpha}$, estimate (5.40) reads

$$M_2(\alpha, t) \leq C_\alpha \tilde{M}_2(\widehat{\theta_2\alpha}, t) \text{ for all } t \geq T_\alpha, \text{ some } T_\alpha > 0. \quad (5.42)$$

The functions $N(\lambda, t)$ and $F(\alpha, t)$ can be treated similarly to obtain counter parts of (5.31) and (5.33). The counter part of $\bar{N}(\lambda, t)$ is

$$\tilde{N}(\lambda, t) = \begin{cases} 1 + \tilde{E}(\widehat{\lambda}, t)^{1/\widehat{\lambda}} & \text{if } \lambda > 0, \\ 1 & \text{if } \lambda = 0, \end{cases} \quad (5.43)$$

the counter part of $\bar{F}(\alpha, t)$ is

$$\tilde{F}(\alpha, t) = \tilde{N}(\alpha, t)^{\alpha-1} + \tilde{N}(\gamma_1(\alpha), t)^{\alpha-2} \tilde{M}_1(t)^{\frac{1-a}{2-a}}, \quad (5.44)$$

and the counter part of $\bar{F}(\alpha, \lambda, t)$ is

$$\tilde{F}(\alpha, \lambda, t) = \tilde{N}(\lambda, t)^{\alpha-1} + \tilde{N}(\lambda, t)^{\alpha-2} \tilde{M}_1(\lambda, t)^{\frac{1-a}{2-a}}. \quad (5.45)$$

For $\alpha \geq 2$, resp. $\alpha \geq 2$ and $\lambda \geq \max\{\alpha, \gamma_1(\alpha), \widehat{2}\}$, using (5.38) we have

$$N(\alpha, t) \leq C_\alpha \tilde{N}(\alpha, t) \text{ and } N(\gamma_1, t) \leq C_\alpha \tilde{N}(\gamma_1, t),$$

$$\text{resp. } N(\alpha, t), N(\gamma_1, t) \leq C_{\alpha, \lambda} \tilde{N}(\lambda, t),$$

for sufficiently large t , hence combining this with (5.37), resp. with (5.35) for α set to equal λ , we obtain

$$F(\alpha, t) \leq C_\alpha \tilde{F}(\alpha, t) \text{ for all } t \geq T_\alpha, \text{ some } T_\alpha > 0, \quad (5.46)$$

$$\text{resp. } F(\alpha, t) \leq C_{\alpha, \lambda} \tilde{F}(\alpha, \lambda, t) \text{ for all } t \geq T_{\alpha, \lambda}, \text{ some } T_{\alpha, \lambda} > 0. \quad (5.47)$$

Note that the quantities $\tilde{M}_1(\alpha, t)$, $\tilde{M}_2(\alpha, t)$, $\tilde{N}(\lambda, t)$, $\tilde{F}(\alpha, t)$ and $\tilde{F}(\alpha, \lambda, t)$ depend on the boundary data $\Psi_1(x, t)$ and $\Psi_2(x, t)$, but are independent of the initial data $p_1(x, 0)$ and $p_2(x, 0)$.

Throughout, we assume all of the quantities $\mathcal{G}_j(t)$, $\tilde{A}(\alpha, t)$, $\overline{M}_k(\alpha, t)$, $\tilde{M}_k(\alpha, t)$, $\overline{N}(\alpha, t)$, $\tilde{N}(\alpha, t)$, $\overline{F}(\alpha, t)$, $\tilde{F}(\alpha, t)$, $\overline{F}(\alpha, \lambda, t)$, $\tilde{F}(\alpha, \lambda, t)$, as functions of t , are continuous on $[0, \infty)$ whenever they are in use.

With the above preparations, we are ready for continuous dependence results.

Theorem 5.2. (i) Assume (DC) and $\alpha \geq 2$. Then

$$\begin{aligned} \int_U |\bar{z}(x, t)|^\alpha dx &\leq e^{-c_8 \int_0^t \overline{M}_1(\tau)^{\frac{-\alpha}{2-\alpha}} d\tau} \int_U |\bar{z}(x, 0)|^\alpha dx \\ &\quad + C \int_0^t e^{-c_8 \int_\tau^t \overline{M}_1(s)^{\frac{-\alpha}{2-\alpha}} ds} \overline{F}(\alpha, \tau) D(\alpha, \tau) d\tau \end{aligned} \quad (5.48)$$

for all $t \geq 0$, where $c_8 = c_8(\alpha) > 0$. Moreover, if $\int_0^\infty \tilde{M}_1(t)^{-\frac{\alpha}{2-\alpha}} dt = \infty$ then

$$\limsup_{t \rightarrow \infty} \int_U |\bar{z}(x, t)|^\alpha dx \leq C \limsup_{t \rightarrow \infty} \left[\tilde{F}(\alpha, t) \tilde{M}_1(t)^{\frac{\alpha}{2-\alpha}} D(\alpha, t) \right]. \quad (5.49)$$

(ii) Assume (NDC) and $\alpha \geq \alpha_*$. Then

$$\int_U |\bar{z}(x, t)|^\alpha dx \leq \int_U |\bar{z}(x, 0)|^\alpha dx + C \left[\text{Env} \left(\overline{F}(\alpha, t) \overline{M}_2(\widehat{\theta}_2 \alpha, t)^{\frac{2-\theta_1}{\theta_1}} D(\alpha, t) \right) \right]^{\frac{1}{\theta}} \quad (5.50)$$

for all $t \geq 0$. Moreover, if $\int_0^\infty \tilde{M}_2(t)^{-\frac{2-\theta_1}{\theta_1}} dt = \infty$ then

$$\limsup_{t \rightarrow \infty} \int_U |\bar{z}(x, t)|^\alpha dx \leq C \limsup_{t \rightarrow \infty} \left[\tilde{F}(\alpha, t) \tilde{M}_2(\widehat{\theta}_2 \alpha, t)^{\frac{2-\theta_1}{\theta_1}} D(\alpha, t) \right]^{\frac{1}{\theta}}. \quad (5.51)$$

Proof. (i) By (5.2), estimates (5.19) and (5.31):

$$\frac{d}{dt} \int_U |\bar{z}|^\alpha dx \leq -c_8 \left[\int_U |\bar{z}|^\alpha dx \right] \overline{M}_1(t)^{-\frac{\alpha}{2-\alpha}} + C \overline{F}(\alpha, t) D(\alpha, t).$$

Applying Gronwall's inequality we obtain (5.48).

By (5.2), (5.37) and (5.46), we have for $t > T$ that

$$\frac{d}{dt} \int_U |\bar{z}|^\alpha dx \leq -C_1 \left[\int_U |\bar{z}|^\alpha dx \right] \tilde{M}_1(t)^{-\frac{\alpha}{2-\alpha}} + C_2 \tilde{F}(\alpha, t) D(\alpha, t).$$

Then applying estimate (2.34) in Lemma 2.7 with $y(t) = \int_U |\bar{z}(x, t)|^\alpha dx$, $h(t) = C_1 \tilde{M}_1(t)^{-\frac{a}{2-a}}$, $f(t) = C_2 \tilde{F}(\alpha, t) D(\alpha, t)$ and $\theta = 1$ yields (5.49).

(ii) By (5.6), (5.27) and (5.31) we have

$$\frac{d}{dt} \int_U |\bar{z}|^\alpha dx \leq -C_3 \left[\int_U |\bar{z}|^\alpha dx \right]^\theta \overline{M}_2(\widehat{\theta_2 \alpha}, t)^{-\frac{2-\theta_1}{\theta_1}} + C_4 \overline{F}(\alpha, t) D(\alpha, t). \quad (5.52)$$

Applying estimate (2.33) in Lemma 2.7 to (5.52) with $y(t) = \int_U |\bar{z}(x, t)|^\alpha dx$, $h(t) = C_3 \overline{M}_2(\widehat{\theta_2 \alpha}, t)^{-\frac{2-\theta_1}{\theta_1}}$, $f(t) = C_4 \overline{F}(\alpha, t) D(\alpha, t)$, we obtain (5.50).

Thanks to (5.42) and (5.46), we have for $t > T$ that

$$\frac{d}{dt} \int_U |\bar{z}|^\alpha dx \leq -C_5 \left[\int_U |\bar{z}|^\alpha dx \right]^\theta \tilde{M}_2(\widehat{\theta_2 \alpha}, t)^{-\frac{2-\theta_1}{\theta_1}} + C_6 \tilde{F}(\alpha, t) D(\alpha, t), \quad (5.53)$$

Applying (2.34) in Lemma 2.7 to (5.53) on (T, ∞) , we obtain (5.51). \square

According to Theorem 5.2, even when the individual boundary data Ψ_1 and Ψ_2 , characterized here by \overline{M}_k and \overline{F} , are asymptotically large as $t \rightarrow \infty$, if their difference Φ , characterized by $D(t)$, is asymptotically sufficiently small to diminish their growth, then $\int_U |\bar{z}(x, t)|^\alpha dx$ is asymptotically small. For all time estimates, $\int_U |\bar{z}(x, t)|^\alpha dx$ can be controlled by $\int_U |\bar{z}(x, 0)|^\alpha dx$ and small $D(t)$. Below, we present a simple scenario.

Corollary 5.3. *Let $\alpha \geq \max\{2, \alpha_*\}$. Assume the functions Ψ_k ($k = 1, 2$) satisfy*

$$\sup_{[0, \infty)} \{ \|\nabla \Psi_k(\cdot, t)\|_{L^\infty}, \|(\Psi_k)_t(\cdot, t)\|_{L^\infty}, \|\nabla(\Psi_k)_t(\cdot, t)\|_{L^\infty} \} < \infty, \quad (5.54)$$

$$\lim_{t \rightarrow \infty} \|\nabla \Phi(\cdot, t)\|_{L^\gamma} = \lim_{t \rightarrow \infty} \|\Phi_t(\cdot, t)\|_{L^\alpha} = 0, \quad (5.55)$$

where $\gamma = \max\{\alpha, 2(2-a)\}$. Then

$$\lim_{t \rightarrow \infty} \int_U |\bar{z}(x, t)|^\alpha dx = 0. \quad (5.56)$$

Proof. On the one hand, we have, thanks to (5.54), that $A[\Psi_k](\widehat{2}, t)$, $A[\Psi_k](\widehat{\alpha}, t)$, $A[\Psi_k](\widehat{\theta_2 \alpha}, t)$, $A[\Psi_k](\widehat{\gamma}_1, t)$ and $G_3[\Psi_k](t)$, for $k = 1, 2$, are uniformly bounded on $[0, \infty)$, hence so are $\tilde{M}_1(t)$, $\tilde{M}_2(\widehat{\theta_2 \alpha}, t)$ and $\tilde{F}(\alpha, t)$. On the other hand, by (5.55) we have $\lim_{t \rightarrow \infty} D(\alpha, t) = 0$. Therefore inequality (5.49), resp. (5.51), implies (5.56) in the DC, resp. NDC, case. \square

We turn to the continuous dependence for the pressure gradient. Because the DC case was treated in [14], we now focus on the NDC case.

Lemma 5.4. *In the NDC case, we have for all $t > 0$ that*

$$\left[\int_U |\nabla z|^{2-a} dx \right]^{\frac{2}{2-a}} \leq C M_1(t)^{\frac{1}{2-a}} \|\nabla \Phi(\cdot, t)\|_{L^{2-a}} + C M_1(t)^{\frac{a}{2-a}} M_3(t)^{\frac{1}{2}} \left[\int_U |\bar{z}|^{\alpha_*} dx \right]^{\frac{1}{\alpha_*}}, \quad (5.57)$$

where

$$M_3(t) = \int_U |(\bar{p}_1)_t(x, t)|^2 + |(\bar{p}_2)_t(x, t)|^2 dx + \int_U |\Phi_t(x, t)|^2 dx. \quad (5.58)$$

Proof. Multiplying the first equation of (5.1) by \bar{z} and integrating over U give

$$\int_U \bar{z} \bar{z}_t dx = - \int_U (K(|\nabla p_1|) \nabla p_1 - K(|\nabla p_2|) \nabla p_2) \cdot (\nabla z - \nabla \Phi) dx - \int_U \Phi_t \bar{z} dx.$$

Hence

$$\begin{aligned} & \int_U (K(|\nabla p_1|) \nabla p_1 - K(|\nabla p_2|) \nabla p_2) \cdot (\nabla z) dx \\ & \leq \int_U (K(|\nabla p_1|) |\nabla p_1| + K(|\nabla p_2|) |\nabla p_2|) |\nabla \Phi| dx + \int_U (|\bar{z}_t| + |\Phi_t|) |\bar{z}| dx. \end{aligned}$$

By Lemma 2.1(ii),

$$\int_U (K(|\nabla p_1|) \nabla p_1 - K(|\nabla p_2|) \nabla p_2) \cdot (\nabla z) dx \geq CM_1(t)^{\frac{-a}{2-a}} \left(\int_U |\nabla z|^{2-a} dx \right)^{\frac{2}{2-a}}.$$

By relation (2.10),

$$\begin{aligned} & \int_U (K(|\nabla p_1|) |\nabla p_1| + K(|\nabla p_2|) |\nabla p_2|) |\nabla \Phi| dx \leq C \int_U (|\nabla p_1|^{1-a} + |\nabla p_2|^{1-a}) |\nabla \Phi| dx \\ & \leq C \left(\int_U |\nabla p_1|^{2-a} + |\nabla p_2|^{2-a} dx \right)^{\frac{1-a}{2-a}} \left(\int_U |\nabla \Phi|^{2-a} dx \right)^{\frac{1}{2-a}} \leq CM_1(t)^{\frac{1-a}{2-a}} \|\nabla \Phi\|_{L^{2-a}}. \end{aligned}$$

Recall from (2.17) that $\alpha_* > 2$ in this NDC case. Applying Hölder's inequality and using $|\bar{z}_t| \leq |(\bar{p}_1)_t| + |(\bar{p}_2)_t|$, we have

$$\begin{aligned} \int_U (|\bar{z}_t| + |\Phi_t|) |\bar{z}| dx & \leq C \left(\int_U |(\bar{p}_1)_t|^2 + |(\bar{p}_2)_t|^2 + |\Phi_t|^2 dx \right)^{\frac{1}{2}} \left(\int_U |\bar{z}|^{\alpha_*} dx \right)^{\frac{1}{\alpha_*}} \\ & \leq CM_3(t)^{\frac{1}{2}} \left(\int_U |\bar{z}|^{\alpha_*} dx \right)^{\frac{1}{\alpha_*}}. \end{aligned}$$

Combining all the above, we obtain

$$M_1(t)^{-\frac{a}{2-a}} \left(\int_U |\nabla z|^{2-a} dx \right)^{\frac{2}{2-a}} \leq CM_1(t)^{\frac{1-a}{2-a}} \|\nabla \Phi\|_{L^{2-a}} + CM_3(t)^{\frac{1}{2}} \left(\int_U |\bar{z}|^{\alpha_*} dx \right)^{\frac{1}{\alpha_*}}.$$

Then multiplying by $M_1(t)^{\frac{a}{2-a}}$ yields (5.57) \square

By virtue of Lemma 5.4 we need to estimate $M_3(t)$. For each $k = 1, 2$, applying inequality (4.29) in Theorem 4.4 for p_k with $t_0 = 1$ and using Remark 4.8 with $\alpha' = \alpha \geq \widehat{2}$ and $G(t) = \tilde{\mathcal{A}}(\alpha, t)$, we have

$$M_3(t) \leq C_\alpha \overline{M}_3(\alpha, t) \text{ for all } t \geq 1, \quad (5.59)$$

where

$$\begin{aligned} \overline{M}_3(\alpha, t) & = 1 + \sum_{k=1}^2 \int_U |\bar{p}_k(x, 0)|^\alpha + |\nabla p_k(x, 0)|^{2-a} dx \\ & + \int_0^1 \mathcal{G}_3(\tau) d\tau + \int_0^t e^{-d_6(t-\tau)} (Env\tilde{\mathcal{A}}(\alpha, \tau) + \mathcal{G}_4(\tau)) d\tau + \int_U |\Phi_t(x, t)|^2 dx. \end{aligned} \quad (5.60)$$

In particular, when $\alpha = \widehat{2}$

$$M_3(t) \leq C \overline{M}_3(t) \text{ for all } t \geq 1, \text{ where } \overline{M}_3(t) = \overline{M}_3(\widehat{2}, t). \quad (5.61)$$

For $\alpha \geq \widehat{2}$, same as estimate (5.35) but using Theorem 4.5 (setting $t_0 = 1$) instead of Theorem 4.4 and using the same replacements for Remark 4.8, there exists $T_\alpha \geq 1$ such that

$$M_3(t) \leq C_\alpha \widetilde{M}_3(\alpha, t) \text{ for all } t \geq T_\alpha, \quad (5.62)$$

where

$$\begin{aligned} \widetilde{M}_3(\alpha, t) &= 1 + \int_U |\Phi_t(x, t)|^2 dx \\ &+ \begin{cases} \bar{A}(\alpha) + \int_1^t e^{-d_6(t-\tau)} \mathcal{G}_4(\tau) d\tau & \text{if } \bar{A}(\alpha) < \infty, \\ \bar{\beta}(\alpha)^{\frac{\alpha}{\alpha-2a}} + \int_1^t e^{-d_6(t-\tau)} (\widetilde{\mathcal{A}}(\alpha, \tau) + \mathcal{G}_4(\tau)) d\tau & \text{if } \bar{A}(\alpha) = \infty, \bar{\beta}(\alpha) < \infty, \\ Env \widetilde{\mathcal{A}}(\alpha, t) + \int_1^t e^{-d_6(t-\tau)} \mathcal{G}_4(\tau) d\tau & \text{if } \bar{A}(\alpha) = \infty, \bar{\beta}(\alpha) = \infty. \end{cases} \end{aligned} \quad (5.63)$$

Particularly, when $\alpha = \widehat{2}$ there exists $T \geq 1$ such that

$$M_3(t) \leq C \widetilde{M}_3(t) \text{ for all } t \geq T, \text{ where } \widetilde{M}_3(t) = \widetilde{M}_3(\widehat{2}, t). \quad (5.64)$$

Combining Lemma 5.4 and previous estimates for the quantities $M_1(t)$, $M_2(t)$, $M_3(t)$ and $\int_U |\bar{z}(x, t)|^{\alpha_*} dx$ will give continuous dependence results for the pressure gradient. We will explicate in details below. Let

$$\begin{aligned} \mu_1 &= \max\{2, \alpha_*, \theta_2 \alpha_*, \gamma_1(\alpha_*)\}, \\ \mu_2 &= \max\left\{\frac{\alpha_* - 1}{\mu_1}, \frac{\alpha_* - 2}{\mu_1} + \frac{1-a}{2-a}\right\}, \\ \mu_3 &= \frac{a}{2-a} + \frac{1}{2} + \frac{1}{\theta \alpha_*} \left(\mu_2 + \frac{2-\theta_1}{\theta_1}\right). \end{aligned}$$

Taking $\alpha = \mu_1$ in (5.35), taking $\alpha = \alpha_*$ and $\lambda = \mu_1$ in (5.40), taking $\alpha = \mu_1$ in (5.62), and taking $\alpha = \alpha_*$ and $\lambda = \mu_1$ in (5.47), we assert that there is $T > 0$ such that for $t \geq T$,

$$M_1(t) \leq C \widetilde{M}_1(\mu_1, t), \quad M_2(\alpha_*, t) \leq C \widetilde{M}_2(\mu_1, t), \quad M_3(t) \leq C \widetilde{M}_3(\mu_1, t), \quad (5.65)$$

$$F(\alpha_*, t) \leq C \widetilde{F}(\alpha_*, \mu_1, t). \quad (5.66)$$

Theorem 5.5. *Assume (NDC).*

(i) *Then*

$$\left(\int_U |\nabla z(x, t)|^{2-a} dx\right)^{\frac{2}{2-a}} \leq C \overline{M}_1(t)^{\frac{1}{2-a}} \|\nabla \Phi(\cdot, t)\|_{L^{2-a}} + C \overline{M}_1(t)^{\frac{a}{2-a}} \overline{M}_3(t)^{\frac{1}{2}} \overline{D}(t) \quad (5.67)$$

for all $t \geq 1$, where

$$\overline{D}(t) = \|\bar{z}(\cdot, 0)\|_{L^{\alpha_*}} + \left[Env\left(\overline{F}(\alpha_*, t) \overline{M}_2(\widehat{\theta_2 \alpha_*}, t)^{\frac{2-\theta_1}{\theta_1}} D(\alpha_*, t)\right)\right]^{\frac{1}{\theta \alpha_*}}, \quad (5.68)$$

with $\overline{F}(\alpha_*, t)$, $\overline{M}_2(\widehat{\theta_2 \alpha_*}, t)$ and $D(\alpha_*, t)$ defined by (5.32), (5.26) and (5.5), respectively.

(ii) Let $\eta_1 = 1 + \bar{A}(\mu_1) + \limsup_{t \rightarrow \infty} \mathcal{G}_4(t)$. If $\eta_1 < \infty$ then

$$\begin{aligned} \limsup_{t \rightarrow \infty} \left(\int_U |\nabla z(x, t)|^{2-a} dx \right)^{\frac{2}{2-a}} &\leq C \eta_1^{\frac{1}{2-a}} \limsup_{t \rightarrow \infty} \|\nabla \Phi(\cdot, t)\|_{L^{2-a}} \\ &+ C \eta_1^{\mu_3} \limsup_{t \rightarrow \infty} \left\{ \|\nabla \Phi(\cdot, t)\|_{L^{2(2-a)}} + \|\nabla \Phi(\cdot, t)\|_{L^{\alpha_*}}^2 + \|\Phi_t(\cdot, t)\|_{L^{\alpha_*}} \right\}^{\frac{1}{\theta \alpha_*}}. \end{aligned} \quad (5.69)$$

Proof. (i) For the last integral of (5.57), we use (5.50) to bound

$$\left(\int_U |\bar{z}|^{\alpha_*} dx \right)^{\frac{1}{\alpha_*}} \leq C \bar{D}(t). \quad (5.70)$$

Thus (5.67) follows from inequality (5.57) and estimates (5.19), (5.59) and (5.70).

(ii) Note in this NDC case that $\hat{2} = \alpha_*$. By (5.65) and (2.35), we obtain

$$\begin{cases} \limsup_{t \rightarrow \infty} M_1(t) &\leq C \limsup_{t \rightarrow \infty} \tilde{M}_1(\mu_1, t) \leq C \eta_1, \\ \limsup_{t \rightarrow \infty} M_2(\alpha_*, t) &\leq C \limsup_{t \rightarrow \infty} \tilde{M}_2(\mu_1, t) \leq C \eta_1, \\ \limsup_{t \rightarrow \infty} M_3(t) &\leq C \limsup_{t \rightarrow \infty} \tilde{M}_3(\mu_1, t) \leq C \eta_1. \end{cases} \quad (5.71)$$

Above, for $M_3(t)$ we first bound $\int_U |\Phi_t(x, t)|^2 dx$ by $\sum_{k=1,2} \int_U |(\Psi_k)_t(x, t)|^2 dx$ and then absorb it into $\mathcal{G}_4(t)$. Note $\gamma_1(\alpha_*) > 0$ in this case, then

$$\tilde{N}(\mu_1, t) \equiv 1 + \bar{A}(\mu_1)^{\frac{1}{\mu_1}} \leq C \eta_1^{\frac{1}{\mu_1}}$$

and hence by (5.66),

$$\limsup_{t \rightarrow \infty} F(\alpha_*, t) \leq C \limsup_{t \rightarrow \infty} \tilde{F}(\alpha_*, \mu_1, t) \leq C \left(\eta_1^{\frac{\alpha_*-1}{\mu_1}} + \eta_1^{\frac{\alpha_*-2}{\mu_1}} \eta_1^{\frac{1-a}{2-a}} \right) \leq C \eta_1^{\mu_2}. \quad (5.72)$$

Applying (2.34) of Lemma 2.7 to (5.6) yields

$$\begin{aligned} \limsup_{t \rightarrow \infty} \int_U |\bar{z}(x, t)|^{\alpha_*} dx &\leq C \left(\limsup_{t \rightarrow \infty} F(\alpha_*, t) \limsup_{t \rightarrow \infty} M_2(\alpha_*, t)^{\frac{2-\theta_1}{\theta_1}} \limsup_{t \rightarrow \infty} D(\alpha_*, t) \right)^{\frac{1}{\theta}} \\ &\leq C \left(\eta_1^{\mu_2 + \frac{2-\theta_1}{\theta_1}} \limsup_{t \rightarrow \infty} D(\alpha_*, t) \right)^{\frac{1}{\theta}}. \end{aligned} \quad (5.73)$$

Combining (5.71), (5.73) with (5.57) gives

$$\begin{aligned} \limsup_{t \rightarrow \infty} \left(\int_U |\nabla z(x, t)|^{2-a} dx \right)^{\frac{2}{2-a}} &\leq C \limsup_{t \rightarrow \infty} M_1(t)^{\frac{1}{2-a}} \limsup_{t \rightarrow \infty} \|\nabla \Phi(\cdot, t)\|_{L^{2-a}} \\ &+ C \limsup_{t \rightarrow \infty} M_1(t)^{\frac{a}{2-a}} \limsup_{t \rightarrow \infty} M_3(t)^{\frac{1}{2}} \left(\eta_1^{\mu_2 + \frac{2-\theta_1}{\theta_1}} \limsup_{t \rightarrow \infty} D(\alpha_*, t) \right)^{\frac{1}{\theta \alpha_*}} \\ &\leq C \eta_1^{\frac{1}{2-a}} \limsup_{t \rightarrow \infty} \|\nabla \Phi(\cdot, t)\|_{L^{2-a}} + C \eta_1^{\frac{a}{2-a}} \eta_1^{\frac{1}{2}} \left(\eta_1^{\mu_2 + \frac{2-\theta_1}{\theta_1}} \limsup_{t \rightarrow \infty} D(\alpha_*, t) \right)^{\frac{1}{\theta \alpha_*}}, \end{aligned}$$

hence (5.69) follows, noting that $D(\alpha_*, t)$ is defined by (5.5) with $\alpha = \alpha_* > 2$. \square

Next we deal with the case when $\eta_1 = \infty$. Let

$$\mu_4 = \alpha_*(\theta - 1) \left(\frac{a}{2-a} + \frac{1}{2} \right) + \frac{2-\theta_1}{\theta_1} \text{ and } \mu_5 = \mu_4 + \alpha_* \left(\frac{a}{2-a} + \frac{1}{2} \right).$$

Theorem 5.6. Assume (NDC) and $\eta_1 = \infty$. Define for $t \geq 0$,

$$\omega(t) = \begin{cases} 1 + \bar{A}(\mu_1) + \int_0^t e^{-d_7(t-\tau)} \mathcal{G}_4(\tau) d\tau & \text{if } \bar{A}(\mu_1) < \infty, \\ 1 + \bar{\beta}(\mu_1)^{\frac{\mu_1}{\mu_1-2a}} + \tilde{\mathcal{A}}(\mu_1, t) \\ \quad + \int_0^t e^{-d_7(t-\tau)} (\tilde{\mathcal{A}}(\mu_1, \tau) + \mathcal{G}_4(\tau)) d\tau & \text{if } \bar{A}(\mu_1) = \infty, \bar{\beta}(\mu_1) < \infty, \\ 1 + \text{Env} \tilde{\mathcal{A}}(\mu_1, t) + \int_0^t e^{-d_7(t-\tau)} \mathcal{G}_4(\tau) d\tau & \text{if } \bar{A}(\mu_1) = \infty, \bar{\beta}(\mu_1) = \infty, \end{cases}$$

where $d_7 = \min\{d_5, d_6\}$. If $\int_0^\infty \omega(t)^{-\mu_4} dt = \infty$ and

$$\eta_2 \stackrel{\text{def}}{=} \limsup_{t \rightarrow \infty} [(\omega(t)^{\mu_5})']^+ < \infty \quad (5.74)$$

then

$$\begin{aligned} \limsup_{t \rightarrow \infty} \left(\int_U |\nabla z(x, t)|^{2-a} dx \right)^{\frac{2}{2-a}} &\leq C \limsup_{t \rightarrow \infty} \left(\omega(t)^{\frac{1}{2-a}} \|\nabla \Phi(\cdot, t)\|_{L^{2-a}} \right) \\ &+ C \left[\limsup_{t \rightarrow \infty} \omega(t)^{\mu_6} D(t) \right]^{\frac{1}{\theta \alpha_*}} + C \eta_2^{\frac{\theta}{\alpha_*}} \left[\limsup_{t \rightarrow \infty} \omega(t)^{\mu_7} D(t) \right]^{\frac{1}{\theta^2 \alpha_*}}, \end{aligned} \quad (5.75)$$

where μ_6 and μ_7 are positive numbers defined below by (5.79) and (5.82), respectively.

Proof. Same as for (5.71) and (5.72), one can verify that

$$\tilde{M}_1(\mu_1, t), \tilde{M}_2(\mu_1, t), \tilde{M}_3(\mu_1, t) \leq C\omega(t) \text{ and } \tilde{F}(\alpha_*, \mu_1, t) \leq C\omega(t)^{\mu_2} \text{ for all } t \geq 0.$$

Then by (5.65) and (5.66), there is $T > 1$ such that

$$M_1(t), M_2(\alpha_*, t), M_3(t) \leq C\omega(t) \text{ and } F(\alpha_*, t) \leq C\omega(t)^{\mu_2} \text{ for all } t \geq T. \quad (5.76)$$

Combining (5.57) with estimates in (5.76) we have for $t \geq T$ that

$$\left(\int_U |\nabla z(x, t)|^{2-a} dx \right)^{\frac{2}{2-a}} \leq C\omega(t)^{\frac{1}{2-a}} \|\nabla \Phi(\cdot, t)\|_{L^{2-a}} + C\omega(t)^{\frac{a}{2-a} + \frac{1}{2}} Z(t)^{\frac{1}{\alpha_*}},$$

where $Z(t) = \int_U |\bar{z}(x, t)|^{\alpha_*} dx$. Hence by setting $W(t) = \omega(t)^{\alpha_* (\frac{a}{2-a} + \frac{1}{2})}$ we obtain

$$\left(\int_U |\nabla z(x, t)|^{2-a} dx \right)^{\frac{2}{2-a}} \leq C\omega(t)^{\frac{1}{2-a}} \|\nabla \Phi(\cdot, t)\|_{L^{2-a}} + C \left(W(t) Z(t) \right)^{\frac{1}{\alpha_*}}, \quad t \geq T. \quad (5.77)$$

We now estimate the limit superior of the product $W(t)Z(t)$ as $t \rightarrow \infty$. As treated in [13], we derive a differential inequality for WZ first. Denote $\tilde{M}_2(t) = \tilde{M}_2(\widehat{\theta_2 \alpha_*}, t)$, $\tilde{F}(t) = \tilde{F}(\alpha_*, t)$ and $D(t) = D(\alpha_*, t)$. For $t > T$, setting $\alpha = \alpha_*$ in (5.6) and in estimates (5.42), (5.46), we have

$$\begin{aligned} (WZ)' &= Z'W + ZW' \leq -CZ^\theta \tilde{M}_2^{-\frac{2-\theta_1}{\theta_1}} W + C\tilde{F}DW + ZW' \\ &\leq -C(W^{1-\theta} \tilde{M}_2^{-\frac{2-\theta_1}{\theta_1}}) (WZ)^\theta + CW\tilde{F}D + W'Z \\ &\leq -C\omega^{-\alpha_*(\theta-1)(\frac{a}{2-a} + \frac{1}{2}) - \frac{2-\theta_1}{\theta_1}} (WZ)^\theta + C\omega^{\mu_2 + \alpha_*(\frac{a}{2-a} + \frac{1}{2})} D + C\omega' \omega^{\alpha_*(\frac{a}{2-a} + \frac{1}{2}) - 1} Z. \end{aligned}$$

Thus

$$(WZ)' \leq -C\omega^{-\mu_4} (WZ)^\theta + C\omega^{\mu_2 + \alpha_*(\frac{a}{2-a} + \frac{1}{2})} D + C[\omega']^+ \omega^{\alpha_*(\frac{a}{2-a} + \frac{1}{2}) - 1} Z.$$

Since $\int_0^\infty \omega(t)^{-\mu_4} dt = \infty$, applying (2.34) in Lemma 2.7 for $y(t) = W(t+T)Z(t+T)$ gives

$$\begin{aligned} & \limsup_{t \rightarrow \infty} W(t)Z(t) \\ & \leq C \limsup_{t \rightarrow \infty} \left(\omega(t)^{\mu_4 + \mu_2 + \alpha_* \left(\frac{a}{2-a} + \frac{1}{2} \right)} D(t) + [\omega'(t)]^+ \omega(t)^{\mu_4 + \alpha_* \left(\frac{a}{2-a} + \frac{1}{2} \right) - 1} Z(t) \right)^{\frac{1}{\theta}}, \end{aligned}$$

or

$$\limsup_{t \rightarrow \infty} W(t)Z(t) \leq C \limsup_{t \rightarrow \infty} \left(\omega(t)^{\mu_6} D(t) + [(\omega(t)^{\mu_5})']^+ Z(t) \right)^{\frac{1}{\theta}}, \quad (5.78)$$

where

$$\mu_6 = \mu_4 + \mu_2 + \alpha_* \left(\frac{a}{2-a} + \frac{1}{2} \right) = \mu_5 + \mu_2. \quad (5.79)$$

Then by (5.78) and (5.74)

$$\limsup_{t \rightarrow \infty} W(t)Z(t) \leq C \left(\limsup_{t \rightarrow \infty} \omega(t)^{\mu_6} D(t) \right)^{\frac{1}{\theta}} + C \left(\eta_2 \limsup_{t \rightarrow \infty} Z(t) \right)^{\frac{1}{\theta}}.$$

Therefore (5.77) now gives

$$\begin{aligned} \limsup_{t \rightarrow \infty} \left(\int_U |\nabla z(x, t)|^{2-a} dx \right)^{\frac{2}{2-a}} & \leq C \limsup_{t \rightarrow \infty} \left(\omega(t)^{\frac{1}{2-a}} \left(\int_U |\nabla \Phi(x, t)|^{2-a} dx \right)^{\frac{1}{2-a}} \right) \\ & + C \limsup_{t \rightarrow \infty} \left[\omega^{\mu_6}(t) D(t) \right]^{\frac{1}{\alpha_* \theta}} + C \eta_2^{\frac{1}{\alpha_* \theta}} \left[\limsup_{t \rightarrow \infty} Z(t) \right]^{\frac{1}{\alpha_* \theta}}. \end{aligned} \quad (5.80)$$

Also, by (5.51) and (5.76):

$$\limsup_{t \rightarrow \infty} Z(t) \leq \limsup_{t \rightarrow \infty} \left[\omega(t)^{\mu_7} D(t) \right]^{\frac{1}{\theta}}, \quad (5.81)$$

where

$$\mu_7 = \mu_2 + \frac{2 - \theta_1}{\theta_1}. \quad (5.82)$$

(Since $\theta_1 < 2 - a$, the numbers μ_4, μ_5, μ_6 and μ_7 are positive.) We then obtain (5.75) from (5.80) and (5.81). The proof is complete. \square

6. DEPENDENCE ON THE FORCHHEIMER POLYNOMIAL

In this section we study the dependence of solutions of IBVP (4.1) on the coefficients of the Forchheimer polynomial $g(s)$ in (2.2). Let $N \geq 1$, the exponent vector $\vec{\alpha} = (0, \alpha_1, \dots, \alpha_N)$ and the boundary data $\psi(x, t)$ be fixed. For each Forchheimer polynomial $g(s, \vec{\alpha})$ in class $FP(N, \vec{\alpha})$ we denote $p(x, t; \vec{\alpha})$ the solution of (4.1) with $K = K(\xi, \vec{\alpha})$ and initial data $p(x, 0, \vec{\alpha})$.

Let \mathbf{D} be a compact subset of $\{\vec{\alpha} = (a_0, a_1, \dots, a_N) : a_0, a_N > 0, a_1, \dots, a_{N-1} \geq 0\}$.

Let $g_1(s) = g(s, \vec{\alpha}^{(1)})$ and $g_2(s) = g(s, \vec{\alpha}^{(2)})$ be two functions of class $FP(N, \vec{\alpha})$, where $\vec{\alpha}^{(1)}$ and $\vec{\alpha}^{(2)}$ belong to \mathbf{D} . Let $p_k = p_k(x, t; \vec{\alpha}^{(k)})$ and $\bar{p}_k = p_k - \Psi$ for $k = 1, 2$, where Ψ

is an extension of ψ . Then

$$\begin{cases} \frac{\partial \bar{p}_k}{\partial t} = \nabla \cdot (K(|\nabla p_k|, \vec{a}^{(k)}) \nabla p_k) - \Psi_t \text{ in } U \times (0, \infty), \\ \bar{p}_k(x, t) = 0 \text{ on } \Gamma \times (0, \infty). \end{cases} \quad (6.1)$$

We denote the difference between p_1 and p_2 by $z = p_1 - p_2 = \bar{p}_1 - \bar{p}_2$. Then

$$\begin{cases} \frac{\partial z}{\partial t} = \nabla \cdot (K(|\nabla p_1|, \vec{a}^{(1)}) \nabla p_1) - \nabla \cdot (K(|\nabla p_2|, \vec{a}^{(2)}) \nabla p_2) \text{ in } U \times (0, \infty), \\ z(x, t) = 0 \text{ on } \Gamma \times (0, \infty). \end{cases} \quad (6.2)$$

We use the following notation for convenience in further discussions: let $\vec{x} = (x_1, x_2, \dots)$ and $\vec{x}' = (x'_1, x'_2, \dots)$ be two arbitrary vectors of the same length, including possible length 1. We denote by $\vec{x} \vee \vec{x}'$ and $\vec{x} \wedge \vec{x}'$ their maximum and minimum vectors, respectively, with components

$$(\vec{x} \vee \vec{x}')_j = \max\{x_j, x'_j\} \text{ and } (\vec{x} \wedge \vec{x}')_j = \min\{x_j, x'_j\}.$$

We will derive estimates for $\int_U |z|^\alpha dx$ and $\int_U |\nabla z|^{2-a} dx$. Those estimates contain different constants that depend on $\vec{a}^{(1)}$, $\vec{a}^{(2)}$, $\vec{a}^{(1)} \vee \vec{a}^{(2)}$ or $\vec{a}^{(1)} \wedge \vec{a}^{(2)}$. To simplify those dependences, we define for $\vec{a} \in \mathbf{D}$

$$\chi(\vec{a}) = \max\left\{a_0, a_1, \dots, a_N, \frac{1}{a_0}, \frac{1}{a_N}\right\} \in [1, \infty) \text{ and set } \hat{\chi}(\mathbf{D}) = \max\{\chi(\vec{a}) : \vec{a} \in \mathbf{D}\}.$$

Then $\hat{\chi}(\mathbf{D})$ is a number in $[1, \infty)$. As shown in [13], all constants d_j , c_j , C_j and C appearing in estimates in the previous sections when \vec{a} varies among the vectors $\vec{a}^{(1)}$, $\vec{a}^{(2)}$, $\vec{a}^{(1)} \vee \vec{a}^{(2)}$ and $\vec{a}^{(1)} \wedge \vec{a}^{(2)}$, can be made independent of \vec{a} ; they depend only on n , U , $\hat{\chi}(\mathbf{D})$, $\vec{\alpha}$, θ , θ_1 , and possibly α . We still denote them by d_j , c_j , C_j and C , respectively, in this section.

Let $M_1(t)$ and $M_2(\alpha, t)$ be defined by (5.3) and (5.7), respectively.

Lemma 6.1. (i) *In the DC case, if $\alpha \geq 3$ then for $t > 0$,*

$$\frac{d}{dt} \int_U |z(x, t)|^\alpha dx \leq -c_9 \left[\int_U |z(x, t)|^\alpha dx \right] M_1(t)^{-\frac{\alpha}{2-a}} + C |\vec{a}^{(1)} - \vec{a}^{(2)}| R(t), \quad (6.3)$$

where $c_9 = c_9(\alpha) > 0$ and $R(t) = R(\alpha, t) = 1 + R_1(t) + R_2(t) + R_3(t) + R_4(t)$ with

$$R_1(t) = R_1(\alpha, t) = \int_U |\nabla \Psi(x, t)|^{4-2\alpha} + |\Psi_t(x, t)|^\alpha dx,$$

$$R_2(t) = \sum_{k=1}^2 \int_U |\nabla p_k(x, t)|^{2-a} + |(\bar{p}_k)_t(x, t)|^2 dx,$$

$$R_3(t) = R_3(\alpha, t) = \sum_{k=1}^2 \int_U |\bar{p}_k(x, t)|^{\gamma_2} dx,$$

$$R_4(t) = R_4(\alpha, t) = \sum_{k=1}^2 \int_U |\nabla p_k(x, t)|^{2-a} |\bar{p}_k(x, t)|^{\gamma_3} dx,$$

$$\gamma_2 = \gamma_2(\alpha) \stackrel{\text{def}}{=} \max\{2\alpha, (4-2\alpha)(\alpha-2)\}, \quad (6.4)$$

$$\gamma_3 = \gamma_3(\alpha) \stackrel{\text{def}}{=} \max \left\{ \alpha - 2, \frac{2-a}{1-a}, (2-a)(\alpha-3) \right\}. \quad (6.5)$$

(ii) In the NDC case, if $\alpha \geq \max\{3, \alpha_*\}$ then for $t > 0$,

$$\frac{d}{dt} \int_U |z(x, t)|^\alpha dx \leq -c_{10} \left[\int_U |z(x, t)|^\alpha dx \right]^\theta M_2(\alpha, t)^{-\frac{2-\theta_1}{\theta_1}} + C |\vec{a}^{(1)} - \vec{a}^{(2)}| R(t), \quad (6.6)$$

where $c_{10} = c_{10}(\alpha, \theta, \theta_1) > 0$.

Proof. Multiplying both sides of the first equation of (6.2) by $\alpha|z|^{\alpha-1}\text{sign}(z)$, integrating over domain U , then applying Green's formula, we have

$$\begin{aligned} & \frac{1}{\alpha} \frac{d}{dt} \int_U |z|^\alpha dx \\ &= -(\alpha-1) \int_U (K(|\nabla p_1|, \vec{a}^{(1)}) \nabla p_1 - K(|\nabla p_2|, \vec{a}^{(2)}) \nabla p_2) \cdot (\nabla p_1 - \nabla p_2) |z|^{\alpha-2} dx. \end{aligned} \quad (6.7)$$

According to the perturbed monotonicity, c.f. Lemma 5.2 in [13], we have

$$\begin{aligned} & (K(|\nabla p_1|, \vec{a}^{(1)}) \nabla p_1 - K(|\nabla p_2|, \vec{a}^{(2)}) \nabla p_2) \cdot (\nabla p_1 - \nabla p_2) \\ & \geq (1-a) K(|\nabla p_1| \vee |\nabla p_2|, \vec{a}^{(1)} \vee \vec{a}^{(2)}) |\nabla p_1 - \nabla p_2|^2 \\ & \quad - C |\vec{a}^{(1)} - \vec{a}^{(2)}| K(|\nabla p_1| \vee |\nabla p_2|, \vec{a}^{(1)} \wedge \vec{a}^{(2)}) (|\nabla p_1| \vee |\nabla p_2|) |\nabla p_1 - \nabla p_2|. \end{aligned}$$

Thus

$$\frac{d}{dt} \int_U |z|^\alpha dx \leq -C J_1 + C |\vec{a}^{(1)} - \vec{a}^{(2)}| J_2, \quad (6.8)$$

where

$$\begin{aligned} J_1 &= \int_U K(|\nabla p_1| \vee |\nabla p_2|, \vec{a}^{(1)} \vee \vec{a}^{(2)}) |\nabla z|^2 |z|^{\alpha-2} dx, \\ J_2 &= \int_U K(|\nabla p_1| \vee |\nabla p_2|, \vec{a}^{(1)} \wedge \vec{a}^{(2)}) (|\nabla p_1| \vee |\nabla p_2|) |\nabla p_1 - \nabla p_2| |z|^{\alpha-2} dx. \end{aligned}$$

Regarding J_2 , we have

$$J_2 \leq C \int_U K(|\nabla p_1| \vee |\nabla p_2|, \vec{a}^{(1)} \wedge \vec{a}^{(2)}) (|\nabla p_1| \vee |\nabla p_2|)^2 |z|^{\alpha-2} dx,$$

then by (2.11),

$$J_2 \leq C \int_U (|\nabla p_1| \vee |\nabla p_2|)^{2-a} |\bar{z}|^{\alpha-2} dx \leq C \int_U (|\nabla p_1| + |\nabla p_2|)^{2-a} (|\bar{p}_1|^{\alpha-2} + |\bar{p}_2|^{\alpha-2}) dx.$$

Hence

$$J_2 \leq C(L_1 + L_2), \quad (6.9)$$

where

$$L_1 = \int_U |\nabla p_1|^{2-a} |\bar{p}_1|^{\alpha-2} dx + \int_U |\nabla p_2|^{2-a} |\bar{p}_2|^{\alpha-2} dx, \quad (6.10)$$

$$L_2 = \int_U |\nabla p_1|^{2-a} |\bar{p}_2|^{\alpha-2} dx + \int_U |\nabla p_2|^{2-a} |\bar{p}_1|^{\alpha-2} dx. \quad (6.11)$$

We estimate the crossed term L_2 . Multiplying the first equation of (6.1) for \bar{p}_1 by $\bar{p}_1|\bar{p}_2|^{\alpha-2}$, multiplying the first equation of (6.1) for \bar{p}_2 by $\bar{p}_2|\bar{p}_1|^{\alpha-2}$, summing up and integrating over domain U , we have

$$\begin{aligned}
& \int_U (\bar{p}_1)_t \bar{p}_1 |\bar{p}_2|^{\alpha-2} dx + \int_U (\bar{p}_2)_t \bar{p}_2 |\bar{p}_1|^{\alpha-2} dx \\
&= - \int_U K(|\nabla p_1|, \bar{a}^{(1)}) \nabla p_1 \cdot (\nabla p_1 - \nabla \Psi) |\bar{p}_2|^{\alpha-2} dx \\
&\quad - \int_U K(|\nabla p_2|, \bar{a}^{(2)}) \nabla p_2 \cdot (\nabla p_2 - \nabla \Psi) |\bar{p}_1|^{\alpha-2} dx \\
&\quad - (\alpha - 2) \int_U K(|\nabla p_1|, \bar{a}^{(1)}) (\nabla p_1 \cdot \nabla \bar{p}_2) \bar{p}_1 |\bar{p}_2|^{\alpha-3} \text{sign}(\bar{p}_2) dx \\
&\quad - (\alpha - 2) \int_U K(|\nabla p_2|, \bar{a}^{(2)}) (\nabla p_2 \cdot \nabla \bar{p}_1) \bar{p}_2 |\bar{p}_1|^{\alpha-3} \text{sign}(\bar{p}_1) dx \\
&\quad - \int_U \Psi_t (\bar{p}_1 |\bar{p}_2|^{\alpha-2} + \bar{p}_2 |\bar{p}_1|^{\alpha-2}) dx \\
&\leq -I + I_1 + (\alpha - 2)I_2 + I_3,
\end{aligned}$$

where

$$\begin{aligned}
I &= \int_U K(|\nabla p_1|, \bar{a}^{(1)}) |\nabla p_1|^2 |\bar{p}_2|^{\alpha-2} + K(|\nabla p_2|, \bar{a}^{(2)}) |\nabla p_2|^2 |\bar{p}_1|^{\alpha-2} dx, \\
I_1 &= \int_U K(|\nabla p_1|, \bar{a}^{(1)}) |\nabla p_1| |\nabla \Psi| |\bar{p}_2|^{\alpha-2} + K(|\nabla p_2|, \bar{a}^{(2)}) |\nabla p_2| |\nabla \Psi| |\bar{p}_1|^{\alpha-2} dx, \\
I_2 &= \int_U K(|\nabla p_1|, \bar{a}^{(1)}) |\nabla p_1| |\nabla \bar{p}_2| |\bar{p}_1| |\bar{p}_2|^{\alpha-3} + K(|\nabla p_2|, \bar{a}^{(2)}) |\nabla p_2| |\nabla \bar{p}_1| |\bar{p}_2| |\bar{p}_1|^{\alpha-3} dx, \\
I_3 &= \int_U |\Psi_t| |\bar{p}_1| |\bar{p}_2|^{\alpha-2} + |\Psi_t| |\bar{p}_2| |\bar{p}_1|^{\alpha-2} dx.
\end{aligned}$$

Hence

$$I \leq I_1 + (\alpha - 2)I_2 + I_3 + I_4, \quad (6.12)$$

where

$$I_4 = \int_U |(\bar{p}_1)_t| |\bar{p}_1| |\bar{p}_2|^{\alpha-2} + |(\bar{p}_2)_t| |\bar{p}_2| |\bar{p}_1|^{\alpha-2} dx.$$

By (2.11) we have

$$\begin{aligned}
I &\geq C \left[\int_U |\nabla p_1|^{2-a} - 1 |\bar{p}_2|^{\alpha-2} dx + \int_U (|\nabla p_2|^{2-a} - 1) |\bar{p}_1|^{\alpha-2} dx \right] \\
&= C \left[L_2 - \int_U |\bar{p}_1|^{\alpha-2} + |\bar{p}_2|^{\alpha-2} dx \right].
\end{aligned}$$

Together with (6.12) we have

$$\begin{aligned}
L_2 &\leq CI + C \left[\int_U |\bar{p}_1|^{\alpha-2} + |\bar{p}_2|^{\alpha-2} dx \right] \\
&\leq C \left[\int_U |\bar{p}_1|^{\alpha-2} + |\bar{p}_2|^{\alpha-2} dx + I_1 + I_2 + I_3 + I_4 \right].
\end{aligned} \quad (6.13)$$

• *Estimating I_1* : Applying Young's inequality to three functions $|\nabla p_k|^{1-a}$, $|\nabla \Psi|$, $|\bar{p}_{3-k}|^{\alpha-2}$, for $k = 1, 2$, with powers $\frac{2-a}{1-a}$, $2(2-a)$, $2(2-a)$ respectively, we have

$$\begin{aligned} I_1 &\leq C \int_U |\nabla p_1|^{1-a} |\nabla \Psi| |\bar{p}_2|^{\alpha-2} dx + \int_U |\nabla p_2|^{1-a} |\nabla \Psi| |\bar{p}_1|^{\alpha-2} dx \\ &\leq C \sum_{k=1}^2 \left[\int_U |\nabla p_k|^{2-a} + |\nabla \Psi|^{4-2a} + |\bar{p}_{3-k}|^{(4-2a)(\alpha-2)} dx \right]. \end{aligned} \quad (6.14)$$

• *Estimating I_2* : Applying Young's inequality to two functions $|\nabla p_k|^{1-a} |\bar{p}_k|$ and $|\nabla \bar{p}_{3-k}| |\bar{p}_{3-k}|^{\alpha-3}$, for $k = 1, 2$, with powers $\frac{2-a}{1-a}$ and $2-a$, respectively, we have

$$\begin{aligned} I_2 &\leq C \left[\int_U |\nabla p_1|^{1-a} |\bar{p}_1| |\nabla \bar{p}_2| |\bar{p}_2|^{\alpha-3} dx + \int_U |\nabla p_2|^{1-a} |\bar{p}_2| |\nabla \bar{p}_1| |\bar{p}_1|^{\alpha-3} dx \right] \\ &\leq C \sum_{k=1}^2 \left[\int_U |\nabla p_k|^{2-a} |\bar{p}_k|^{\frac{2-a}{1-a}} + |\nabla \bar{p}_{3-k}|^{2-a} |\bar{p}_{3-k}|^{(2-a)(\alpha-3)} dx \right] \\ &\leq C \sum_{k=1}^2 \left[\int_U |\nabla p_k|^{2-a} |\bar{p}_k|^{\frac{2-a}{1-a}} + |\nabla p_{3-k}|^{2-a} |\bar{p}_{3-k}|^{(2-a)(\alpha-3)} + |\nabla \Psi|^{2-a} |\bar{p}_{3-k}|^{(2-a)(\alpha-3)} dx \right]. \end{aligned}$$

Applying Cauchy's inequality to the last product gives

$$\begin{aligned} I_2 &\leq C \sum_{k=1}^2 \left[\int_U |\nabla p_k|^{2-a} \left(|\bar{p}_k|^{\frac{2-a}{1-a}} + |\bar{p}_k|^{(2-a)(\alpha-3)} \right) + |\bar{p}_k|^{(4-2a)(\alpha-3)} dx \right] \\ &\quad + C \int_U |\nabla \Psi|^{4-2a} dx. \end{aligned} \quad (6.15)$$

• *Estimating I_3* : Similarly, applying Young's inequality to three functions $|\Psi_t|$, $|\bar{p}_k|$, $|\bar{p}_{3-k}|^{\alpha-2}$, for $k = 1, 2$, with powers α , α , $\frac{\alpha}{\alpha-2}$ respectively, we have

$$I_3 \leq C \sum_{k=1}^2 \left[\int_U |\Psi_t|^\alpha + |\bar{p}_k|^\alpha + |\bar{p}_{3-k}|^\alpha dx \right] = C \int_U |\Psi_t|^\alpha dx + C \sum_{k=1}^2 \int_U |\bar{p}_k|^\alpha dx. \quad (6.16)$$

• *Estimating I_4* : Applying Young's inequality to three functions $|(\bar{p}_k)_t|$, $|\bar{p}_k|$, $|\bar{p}_{3-k}|^{\alpha-2}$, for $k = 1, 2$, with powers 2 , α , $\frac{2\alpha}{\alpha-2}$ respectively, we have

$$I_4 \leq C \sum_{k=1}^2 \left[\int_U |(\bar{p}_k)_t|^2 + |\bar{p}_k|^\alpha + |\bar{p}_{3-k}|^{2\alpha} dx \right]. \quad (6.17)$$

From (6.10), (6.13) and (6.14)–(6.17) we have

$$\begin{aligned} L_1 + L_2 &\leq C \sum_{k=1}^2 \int_U |\bar{p}_k|^{\alpha-2} + |\bar{p}_k|^\alpha + |\bar{p}_k|^{2\alpha} + |\bar{p}_k|^{(4-2a)(\alpha-3)} + |\bar{p}_k|^{(4-2a)(\alpha-2)} dx \\ &\quad + C \sum_{k=1}^2 \int_U |\nabla p_k|^{2-a} \left(1 + |\bar{p}_k|^{\alpha-2} + |\bar{p}_k|^{\frac{2-a}{1-a}} + |\bar{p}_k|^{(2-a)(\alpha-3)} \right) + |(\bar{p}_k)_t|^2 dx \\ &\quad + C \int_U |\nabla \Psi|^{4-2a} + |\Psi_t|^\alpha dx. \end{aligned}$$

By Young's inequality, we then obtain

$$\begin{aligned} L_1 + L_2 &\leq C \sum_{k=1}^2 \int_U 1 + |\bar{p}_k|^{\gamma_2} dx + C \sum_{k=1}^2 \int_U |\nabla p_k|^{2-a} (1 + |\bar{p}_k|^{\gamma_3}) + |(\bar{p}_k)_t|^2 dx \\ &\quad + C \int_U |\nabla \Psi|^{4-2a} + |\Psi_t|^\alpha dx, \end{aligned}$$

thus

$$L_1 + L_2 \leq CR(t). \quad (6.18)$$

Combining (6.8), (6.9) and (6.18) gives

$$\frac{d}{dt} \int_U |z|^\alpha dx \leq -CJ_1 + C|\bar{a}^{(1)} - \bar{a}^{(2)}|R(t). \quad (6.19)$$

We now use Lemma 2.5 to estimate J_1 . Note that $z = \bar{z}$ in this case.

(i) In the DC case, for $\alpha \geq 3$, we have similar to (5.13) that

$$-J_1 \leq -C_1 \left[\int_U |z|^\alpha dx \right] M_1(t)^{-\frac{\alpha}{2-a}}. \quad (6.20)$$

Hence combining (6.19) and (6.20) yields (6.3).

(ii) In the NDC case, for $\alpha \geq \max\{3, \alpha_*\}$, similar to (5.14) we have

$$-J_1 \leq -C_1 \left[\int_U |z|^\alpha dx \right]^\theta M_2(\alpha, t)^{-\frac{2-\theta_1}{\theta_1}}. \quad (6.21)$$

Then (6.6) follows (6.19) and (6.21). The proof is complete. \square

By Hölder's and Young's inequalities, we easily see that

$$A(\alpha, t) \leq C(1 + A(\alpha', t)) \text{ for } \alpha < \alpha'. \quad (6.22)$$

We obtain a dependence result for finite time intervals.

Theorem 6.2. *If $\alpha \geq \max\{3, \alpha_*\}$ and $T > 0$ then*

$$\int_U |z(x, t)|^\alpha dx \leq \int_U |z(x, 0)|^\alpha dx + CM_{0,T} |\bar{a}^{(1)} - \bar{a}^{(2)}| \text{ for all } t \in [0, T], \quad (6.23)$$

where

$$M_{0,T} = 1 + \sum_{k=1}^2 \int_U |\bar{p}_k(x, 0)|^{\gamma_4} + |\nabla p_k(x, 0)|^{2-a} dx + \int_0^T 1 + R_1(\tau) + A(\gamma_4, \tau) + G_3(\tau) d\tau, \quad (6.24)$$

with $\gamma_4 = \gamma_4(\alpha) \stackrel{\text{def}}{=} \max\{\gamma_2 + a, \gamma_3 + 2\}$.

Proof. Dropping the negative term on the right-hand side of inequality (6.3) in the DC case, or of inequality (6.6) in the NDC case, then integrating from 0 to t , we have

$$\int_U |z(x, t)|^\alpha dx \leq \int_U |z(x, 0)|^\alpha dx + C|\bar{a}^{(1)} - \bar{a}^{(2)}| \int_0^t R(\tau) d\tau. \quad (6.25)$$

We estimate each integral $\int_0^t R_j(\tau) d\tau$, for $j = 2, 3, 4$, in $\int_0^t R(\tau) d\tau$.

- Using relation (2.13), estimates (4.19) and (4.20), we have for $t \geq 0$ that

$$\begin{aligned} \int_0^t R_2(\tau) d\tau &= \sum_{k=1}^2 \int_0^t \int_U |(\bar{p}_k)_t|^2 + |\nabla p_k|^{2-a} dx d\tau \\ &\leq C + C \sum_{k=1}^2 \int_U |\nabla p_k(x, 0)|^{2-a} + |\bar{p}_k(x, 0)|^2 dx + C \int_0^t 1 + G_3(\tau) + G_1(\tau) d\tau. \end{aligned}$$

Since $G_1(t) \leq G_3(t)$, above inequality becomes

$$\int_0^t R_2(\tau) d\tau \leq C + C \sum_{k=1}^2 \int_U |\nabla p_k(x, 0)|^{2-a} + |\bar{p}_k(x, 0)|^2 dx + C \int_0^t 1 + G_3(\tau) d\tau. \quad (6.26)$$

- Observe that

$$\int_U |\bar{p}_k|^{\gamma_2} dx \leq C \left(\int_U |\bar{p}_k|^{\gamma_2+a} dx \right)^{\frac{\gamma_2}{\gamma_2+a}} = C \left(\int_U |\bar{p}_k|^{\gamma_2+a} dx \right)^{\frac{1}{\gamma_0(\gamma_2+a)}}.$$

Applying (4.41) with $\alpha = \gamma_2 + a$ yields

$$\begin{aligned} \int_0^t R_3(\tau) d\tau &\leq C \sum_{k=1}^2 \int_0^t \left(\int_U |\bar{p}_k(x, \tau)|^{\gamma_2+a} dx \right)^{\frac{1}{\gamma_0(\gamma_2+a)}} d\tau \\ &\leq C \sum_{k=1}^2 \int_U |\bar{p}_k(x, 0)|^{\gamma_2+a} dx + C \int_0^t 1 + A(\gamma_2 + a, \tau) d\tau. \end{aligned} \quad (6.27)$$

- Note that $\gamma_3 + 2 \geq \max\{2, \alpha_*\}$. Applying inequality (4.37) in Corollary 4.7 with $\alpha = \gamma_3 + 2$ gives

$$\int_0^t R_4(\tau) d\tau \leq C \sum_{k=1}^2 \int_U |\bar{p}_k(x, 0)|^{\gamma_3+2} dx + C \int_0^t 1 + A(\gamma_3 + 2, \tau) d\tau. \quad (6.28)$$

Summing up (6.26), (6.27) and (6.28), and using Young's inequality as well as relation (6.22), we obtain for $t \in [0, T]$ that

$$\begin{aligned} \int_0^t R(\tau) d\tau &\leq \int_0^T R(\tau) d\tau \leq C + C \sum_{k=1}^2 \left\{ \int_U |\nabla p_k(x, 0)|^{2-a} + |\bar{p}_k(x, 0)|^{\gamma_4} dx \right\} \\ &\quad + C \int_0^T 1 + R_1(\tau) + G_3(\tau) + A(\gamma_4, \tau) d\tau, \end{aligned}$$

hence $\int_0^t R(\tau) d\tau \leq CM_{0,T}$. This and (6.25) prove inequality (6.23). \square

We now estimate $\int_U |z(x, t)|^\alpha dx$ for large t , the case that is not covered well by Theorem 6.2. As indicated by differential inequalities in Lemma 6.1, we need to bound $M_1(t)$, $M_2(\alpha, t)$ and $R(t)$. The first two quantities were estimated in section 5 with the use of the function $\tilde{A}(\alpha, t)$, see (5.16). Taking $\Psi_1 = \Psi_2 = \Psi$, we recast $\tilde{A}(\alpha, t)$ as

$$\tilde{A}(\alpha, t) = \int_U |\nabla \Psi|^\alpha dx + \left(\int_U |\Psi_t|^\alpha dx \right)^{\frac{1}{1-\alpha}}, \quad (6.29)$$

and have, the same as (5.15),

$$A(\alpha, t) \leq 1 + A(\alpha, t)^{\gamma_0(\alpha)} \leq C(1 + \tilde{A}(\alpha, t)). \quad (6.30)$$

By Young's inequality,

$$\tilde{A}(\alpha, t) \leq C_{\alpha, \alpha'}(1 + \tilde{A}(\alpha', t)) \text{ for } \alpha < \alpha'. \quad (6.31)$$

The estimates (5.19) and (5.27) can be rewritten as

$$M_1(t) \leq C\overline{M}_1(t) \text{ and } M_2(t) \leq C\overline{M}_2(t) \text{ for all } t \geq 0, \quad (6.32)$$

where

$$\begin{aligned} \overline{M}_1(t) = 1 + \sum_{k=1}^2 \left[\int_U |\bar{p}_k(x, 0)|^{\widehat{2}} dx + e^{-d_5 t} \int_U |\nabla p_k(x, 0)|^{2-a} dx \right] \\ + \int_0^t e^{-d_5(t-\tau)} (Env\tilde{A}(\widehat{2}, \tau) + G_3(\tau)) d\tau, \end{aligned} \quad (6.33)$$

$$\begin{aligned} \overline{M}_2(t) \stackrel{\text{def}}{=} \overline{M}_2(\widehat{\theta_2\alpha}, t) = 1 + \sum_{k=1}^2 \left[\int_U |\bar{p}_k(x, 0)|^{\widehat{\theta_2\alpha}} dx + e^{-d_5 t} \int_U |\nabla p_k(x, 0)|^{2-a} dx \right] \\ + Env\tilde{A}(\widehat{\theta_2\alpha}, t) + \int_0^t e^{-d_5(t-\tau)} G_3(\tau) d\tau. \end{aligned} \quad (6.34)$$

We now estimate $R(t)$. Let $\alpha \geq \max\{3, \alpha_*\}$. Applying (4.38) in Corollary 4.7 with $\alpha - 2 = \gamma_3$, we have

$$\int_U |\nabla p_k|^{2-a} |\bar{p}_k|^{\gamma_3} \leq C \int_U |(\bar{p}_k)_t|^2 dx + C \left(1 + A(\gamma_3 + 2, t) + \int_U |\bar{p}_k|^{2(\gamma_3+1)} dx \right), \quad k = 1, 2.$$

Hence

$$\begin{aligned} R(t) \leq R_1(t) + C(1 + A(\gamma_3 + 2, t)) \\ + C \sum_{k=1}^2 \left\{ \int_U |(\bar{p}_k)_t|^2 + |\nabla p_k|^{2-a} dx + \int_U |\bar{p}_k|^{2(\gamma_3+1)} dx + |\bar{p}_k|^{\gamma_2} dx \right\}. \end{aligned}$$

By Young's inequality, we obtain for $t > 0$ that

$$R(t) \leq R_1(t) + C \left\{ 1 + A(\gamma_3 + 2, t) + \sum_{k=1}^2 \left(\int_U |\bar{p}_k|^{\gamma_5} dx + \int_U |(\bar{p}_k)_t|^2 + |\nabla p_k|^{2-a} dx \right) \right\}, \quad (6.35)$$

where $\gamma_5 = \gamma_5(\alpha) = \max\{\gamma_2, 2\gamma_3 + 2\}$.

Using relation (2.13) and using (4.29) with $t_0 = 1$, we have for $t \geq 1$ that

$$\begin{aligned} \int_U |\nabla p_k|^{2-a} + |(\bar{p}_k)_t|^2 dx \leq C \left(1 + \int_U |\bar{p}_k(x, 0)|^{\widehat{2}} + |\nabla p_k(x, 0)|^{2-a} dx + \int_0^1 G_3(\tau) d\tau \right) \\ + C \int_0^t e^{-d_6(t-\tau)} (Env\tilde{A}(\widehat{2}, \tau) + G_4(\tau)) d\tau. \end{aligned}$$

Note that $\gamma_5 \geq \gamma_2 \geq 2\alpha \geq \widehat{2}$. By Theorem 4.3(i),

$$\int_U |\bar{p}_k|^{\gamma_5} dx \leq C \left\{ 1 + \int_U |\bar{p}_k(x, 0)|^{\gamma_5} dx + Env\tilde{A}(\gamma_5, t) \right\}.$$

Summing up, using (6.35) and Young's inequality with $\gamma_5 \geq \widehat{2}$, we obtain

$$\begin{aligned} R(t) &\leq C \left\{ 1 + \sum_{k=1}^2 \int_U |\bar{p}_k(x, 0)|^{\gamma_5} + |\nabla p_k(x, 0)|^{2-a} dx + \int_0^1 G_3(\tau) d\tau \right\} \\ &\quad + R_1(t) + C(A(\gamma_3 + 2, t) + Env\tilde{A}(\widehat{2}, t) + Env\tilde{A}(\gamma_5, t)) + C \int_0^t e^{-d_6(t-\tau)} G_4(\tau) d\tau. \end{aligned}$$

Therefore

$$R(t) \leq C\bar{R}(t), \quad t \geq 1, \quad (6.36)$$

where

$$\bar{R}(t) = \bar{R}(\alpha, t) \stackrel{\text{def}}{=} M_0 + \int_0^1 G_3(\tau) d\tau + R_1(t) + Env\tilde{A}(\gamma_5, t) + \int_0^t e^{-d_6(t-\tau)} G_4(\tau) d\tau, \quad (6.37)$$

with

$$M_0 = 1 + \sum_{k=1}^2 \left\{ \int_U |\bar{p}_k(x, 0)|^{\gamma_5} + |\nabla p_k(x, 0)|^{2-a} dx \right\}. \quad (6.38)$$

To avoid complicated expressions, we continue to estimate $\bar{R}(t)$ more explicitly in terms of $\Psi(x, t)$. Let $t \geq 1$. Note that $\int_0^1 G_3(t) dt \leq (EnvG_4)(1) \leq (EnvG_4)(t)$ and, similar to (5.24), the last integral of (6.37) is bounded by $d_6^{-1} EnvG_4(t)$. Therefore

$$\bar{R}(t) \leq C(M_0 + R_1(t) + Env\tilde{A}(\gamma_5, t) + EnvG_4(t)). \quad (6.39)$$

To unify different dependences on $\Psi(x, t)$ of the terms on the right-hand side of (6.39), we introduce, for $\gamma > 0$, the function

$$S(\gamma, t) = \int_U |\nabla \Psi(x, t)|^\gamma + \left(\int_U |\Psi_t(x, t)|^\gamma dx \right)^{\frac{1}{1-a}} + \int_U |\nabla \Psi_t(x, t)|^2 + |\Psi_{tt}(x, t)|^2 dx, \quad t \geq 0.$$

We have $\tilde{A}(\gamma, t) \leq S(\gamma, t)$. Regarding $G_1(t)$ as part of $G_4(t)$, we note that if $\gamma \geq \max\{2 - a, r_0\}$ then

$$\begin{aligned} \left[\int_U |\Psi_t|^{r_0} dx \right]^{\frac{1}{r_0}} &\leq 1 + \left[\int_U |\Psi_t|^{r_0} dx \right]^{\frac{2-a}{r_0(1-a)}} \leq 1 + C \left[\int_U |\Psi_t|^\gamma dx \right]^{\frac{2-a}{\gamma(1-a)}} \\ &\leq C + C \left[\int_U |\Psi_t|^\gamma dx \right]^{\frac{1}{1-a}}, \end{aligned}$$

hence we easily find $G_4(t) \leq C(1 + S(\gamma, t))$. Therefore, if $\alpha \geq 2$, $\lambda > 0$ and $t > 0$ then

$$R_1(\alpha, t) + \tilde{A}(\lambda, t) + G_4(t) \leq C(1 + S(\gamma, t)), \quad \text{where } \gamma = \max\{\alpha, 4 - 2a, \lambda, r_0\}. \quad (6.40)$$

By (6.39) and (6.40), we obtain

$$\bar{R}(t) \leq C\bar{\bar{R}}(t), \quad t > 0, \quad (6.41)$$

where

$$\overline{\overline{R}}(t) = \overline{\overline{R}}(\alpha, t) \stackrel{\text{def}}{=} M_0 + \text{Env}S(\tilde{\gamma}_5, t), \quad (6.42)$$

with $\tilde{\gamma}_5 = \max\{\alpha, 4 - 2a, \gamma_5, r_0\}$. Since $\gamma_5 \geq \gamma_2 \geq 2\alpha \geq 6 > 4 - 2a$, we have

$$\tilde{\gamma}_5 = \max\{\gamma_5, r_0\}. \quad (6.43)$$

Thanks to (6.36) and (6.41),

$$R(t) \leq C\overline{\overline{R}}(t), \quad t \geq 1. \quad (6.44)$$

Theorem 6.3. *Assume (DC) and $\alpha \geq 3$.*

(i) *If $t \geq 1$ then*

$$\begin{aligned} \int_U |z(x, t)|^\alpha dx &\leq \int_U |z(x, 0)|^\alpha dx \\ &\quad + C|\vec{a}^{(1)} - \vec{a}^{(2)}| \left\{ M_{0,1} + \int_1^t e^{-c_{11} \int_\tau^t \overline{M}_1(s)^{-\frac{2}{2-a}} ds} \overline{\overline{R}}(\tau) d\tau \right\}, \end{aligned} \quad (6.45)$$

where $c_{11} = c_{11}(\alpha) > 0$, and $M_{0,1}$ is the constant $M_{0,T}$ defined by (6.24) with $T = 1$.

(ii) *Let $\gamma_6 = \max\{\gamma_4, \gamma_5, r_0\}$. If*

$$\Upsilon_0 \stackrel{\text{def}}{=} 1 + \sum_{k=1}^2 \int_U |\bar{p}_k(x, 0)|^{\gamma_6} + |\nabla p_k(x, 0)|^{2-a} dx + \sup_{[0, \infty)} S(\gamma_6, t)$$

is finite then

$$\sup_{[0, \infty)} \int_U |z(x, t)|^\alpha dx \leq \int_U |z(x, 0)|^\alpha dx + C\Upsilon_0^{\frac{2}{2-a}} |\vec{a}^{(1)} - \vec{a}^{(2)}|. \quad (6.46)$$

Proof. (i) By virtue of (6.32) and (6.44), we can replace $M_1(t)$ and $R(t)$ by $\overline{M}_1(t)$ and $\overline{\overline{R}}(t)$, respectively, in (6.3) for $t \geq 1$, and then apply Gronwall's inequality to obtain

$$\int_U |z(x, t)|^\alpha dx \leq \int_U |z(x, 1)|^\alpha dx + C|\vec{a}^{(1)} - \vec{a}^{(2)}| \int_1^t e^{-c_{11} \int_\tau^t \overline{M}_1(s)^{-\frac{2}{2-a}} ds} \overline{\overline{R}}(\tau) d\tau. \quad (6.47)$$

Theorem 6.2 with $T = 1$ yields

$$\sup_{[0, 1]} \int_U |z(x, t)|^\alpha dx \leq \int_U |z(x, 0)|^\alpha dx + CM_{0,1} |\vec{a}^{(1)} - \vec{a}^{(2)}|. \quad (6.48)$$

Then using (6.48) to estimate $\int_U |z(x, 1)|^\alpha dx$ in (6.47) results in (6.45).

(ii) Note that $M_{0,1}, M_0, \overline{M}_1(t) \leq C\Upsilon_0$. Also, $\tilde{\gamma}_5 \leq \gamma_6$ implies

$$S(\tilde{\gamma}_5, t) \leq C(1 + S(\gamma_6, t)) \leq C\Upsilon_0,$$

hence we can select $\text{Env}S(\tilde{\gamma}_5, t) \equiv C\Upsilon_0$. Therefore it follows from (6.45) for $t \geq 1$ that

$$\int_U |z(x, t)|^\alpha dx \leq \int_U |z(x, 0)|^\alpha dx + C|\vec{a}^{(1)} - \vec{a}^{(2)}| \left[\Upsilon_0 + \int_1^t e^{-C\Upsilon_0^{-\frac{2}{2-a}}(t-\tau)} \Upsilon_0 d\tau \right],$$

thus

$$\int_U |z(x, t)|^\alpha dx \leq \int_U |z(x, 0)|^\alpha dx + C\Upsilon_0^{\frac{2}{2-a}} |\vec{a}^{(1)} - \vec{a}^{(2)}|. \quad (6.49)$$

Combining estimate (6.49) for $t \geq 1$ with (6.48), we obtain (6.46). \square

Theorem 6.4. *Assume (NDC) and $\alpha \geq \max\{3, \alpha_*\}$.*

(i) *If $t \geq 1$ then*

$$\begin{aligned} \int_U |z(x, t)|^\alpha dx &\leq \int_U |z(x, 0)|^\alpha dx + CM_{0,1} |\bar{a}^{(1)} - \bar{a}^{(2)}| \\ &\quad + C |\bar{a}^{(1)} - \bar{a}^{(2)}|^{\frac{1}{\theta}} \left[\text{Env} \left(\bar{R}(t) \bar{M}_2(t)^{\frac{2-\theta_1}{\theta_1}} \right) \right]^{\frac{1}{\theta}}. \end{aligned} \quad (6.50)$$

(ii) *Let $\gamma_7 = \gamma_7(\alpha) = \max\{\theta_2\alpha, \gamma_4, \gamma_5, r_0\}$. If*

$$\Upsilon_0 \stackrel{\text{def}}{=} 1 + \sum_{k=1}^2 \int_U |\bar{p}_k(x, 0)|^{\gamma_7} + |\nabla p_k(x, 0)|^{2-a} dx + \sup_{[0, \infty)} S(\gamma_7, t)$$

is finite then

$$\sup_{[0, \infty)} \int_U |z(x, t)|^\alpha dx \leq \int_U |z(x, 0)|^\alpha dx + C\Upsilon_0 |\bar{a}^{(1)} - \bar{a}^{(2)}| + C\Upsilon_0^{\frac{2}{\theta_1\theta}} |\bar{a}^{(1)} - \bar{a}^{(2)}|^{\frac{1}{\theta}}. \quad (6.51)$$

Proof. (i) By virtue of (6.32) and (6.36), we replace $M_2(t)$, $R(t)$ by $\bar{M}_2(t)$, $\bar{R}(t)$, respectively, in inequality (6.6) for $t \geq 1$ to obtain

$$\frac{d}{dt} \int_U |z|^\alpha dx \leq -C_1 \left[\int_U |z|^\alpha dx \right]^\theta \bar{M}_2(t)^{-\frac{2-\theta_1}{\theta_1}} + C_2 |\bar{a}^{(1)} - \bar{a}^{(2)}| \bar{R}(t).$$

Applying Lemma 2.7 to the preceding inequality on $[1, \infty)$ yields

$$\int_U |z(x, t)|^\alpha dx \leq \int_U |z(x, 1)|^\alpha dx + C |\bar{a}^{(1)} - \bar{a}^{(2)}|^{\frac{1}{\theta}} \left[\text{Env}(\bar{R}(t) \bar{M}_2(t)^{\frac{2-\theta_1}{\theta_1}}) \right]^{\frac{1}{\theta}}. \quad (6.52)$$

Again, using (6.48) to estimate $\int_U |z(x, 1)|^\alpha dx$ in (6.52) yields (6.50).

(ii) We have $M_{0,1}, M_0 \leq C\Upsilon_0$, and $\bar{M}_2(t) \leq C\Upsilon_0$ for all $t \geq 0$. Also, thanks to (6.41), $\bar{R}(t) \leq C\bar{R}(t) \leq C\Upsilon_0$ for all $t \geq 1$. Employing these in (6.50), we have for $t \geq 1$ that

$$\begin{aligned} \int_U |z(x, t)|^\alpha dx &\leq \int_U |z(x, 0)|^\alpha dx + C\Upsilon_0 |\bar{a}^{(1)} - \bar{a}^{(2)}| + C |\bar{a}^{(1)} - \bar{a}^{(2)}|^{\frac{1}{\theta}} (\Upsilon_0 \Upsilon_0^{\frac{2-\theta_1}{\theta_1}})^{\frac{1}{\theta}} \\ &= \int_U |z(x, 0)|^\alpha dx + C\Upsilon_0 |\bar{a}^{(1)} - \bar{a}^{(2)}| + C\Upsilon_0^{\frac{2}{\theta_1\theta}} |\bar{a}^{(1)} - \bar{a}^{(2)}|^{\frac{1}{\theta}}. \end{aligned}$$

Combining this with (6.48), we obtain (6.51). \square

We now study the limit superior of $\int_U |z(x, t)|^\alpha dx$ as $t \rightarrow \infty$.

In the DC case, if $\int_0^\infty M_1(t)^{-a/(2-a)} dt = \infty$ then by (6.3) and Lemma 2.7 we have

$$\limsup_{t \rightarrow \infty} \int_U |z(x, t)|^\alpha dx \leq C |\bar{a}^{(1)} - \bar{a}^{(2)}| \limsup_{t \rightarrow \infty} (R(t) \cdot M_1(t)^{\frac{a}{2-a}}). \quad (6.53)$$

Similarly, in the NDC case, if $\int_0^\infty M_2(t)^{-\frac{2-\theta_1}{\theta_1}} dt = \infty$ then by (6.6) and Lemma 2.7 we have

$$\limsup_{t \rightarrow \infty} \int_U |z(x, t)|^\alpha dx \leq C |\bar{a}^{(1)} - \bar{a}^{(2)}|^{\frac{1}{\theta}} \limsup_{t \rightarrow \infty} (R(t) \cdot M_2(t)^{\frac{\theta_1}{2-\theta_1}})^{\frac{1}{\theta}}. \quad (6.54)$$

Hence we need to estimate the limits superior of $M_1(t)$, $M_2(t)$ and $R(t)$.

Let $\alpha \geq \max\{3, \alpha_*\}$. In (6.35), by applying Theorem 4.3(ii), with α set to be γ_5 , and Theorem 4.5(ii) combined with relation (6.30) we have

$$\limsup_{t \rightarrow \infty} R(t) \leq C \limsup_{t \rightarrow \infty} (1 + R_1(t) + A(\gamma_3 + 2, t) + \tilde{A}(\widehat{\gamma}_5, t) + \tilde{A}(\widehat{2}, t) + G_4(t)).$$

Since $\gamma_5 \geq \max\{\widehat{2}, 2\alpha, \gamma_3 + 2\}$, then $\widehat{\gamma}_5 = \gamma_5$ and by taking into account (6.22), (6.30), (6.31) and (6.40) we assert

$$\limsup_{t \rightarrow \infty} R(t) \leq C \limsup_{t \rightarrow \infty} (1 + R_1(t) + \tilde{A}(\gamma_5, t) + G_4(t)) \leq C \limsup_{t \rightarrow \infty} (1 + S(\tilde{\gamma}_5, t)). \quad (6.55)$$

Similarly, from part (ii) of Theorems 4.4 and 4.3 follow

$$\limsup_{t \rightarrow \infty} M_1(t) \leq C \limsup_{t \rightarrow \infty} (1 + \tilde{A}(\widehat{2}, t) + G_3(t)) \leq C \limsup_{t \rightarrow \infty} (1 + S(\tilde{\gamma}_5, t)), \quad (6.56)$$

$$\limsup_{t \rightarrow \infty} M_2(t) \leq C \limsup_{t \rightarrow \infty} (1 + \tilde{A}(\widehat{2}, t) + G_3(t) + \tilde{A}(\widehat{\theta_2 \alpha}, t)) \leq C \limsup_{t \rightarrow \infty} (1 + S(\gamma_8, t)), \quad (6.57)$$

where $\gamma_8 = \gamma_8(\alpha) = \max\{\gamma_5, \theta_2 \alpha, r_0\}$.

Theorem 6.5. (i) In the DC case, if $\alpha \geq 3$ and $\Upsilon_1 \stackrel{\text{def}}{=} \limsup_{t \rightarrow \infty} S(\gamma_5, t)$ is finite, then

$$\limsup_{t \rightarrow \infty} \int_U |z(x, t)|^\alpha dx \leq C(1 + \Upsilon_1)^{\frac{2}{2-a}} |\bar{a}^{(1)} - \bar{a}^{(2)}|. \quad (6.58)$$

(ii) In the NDC case, if $\alpha \geq \max\{3, \alpha_*\}$ and

$$\Upsilon_2 = \Upsilon_2(\alpha) \stackrel{\text{def}}{=} \limsup_{t \rightarrow \infty} S(\gamma_8, t) \quad (6.59)$$

is finite, then

$$\limsup_{t \rightarrow \infty} \int_U |z(x, t)|^\alpha dx \leq C(1 + \Upsilon_2)^{\frac{2}{\theta\theta_1}} |\bar{a}^{(1)} - \bar{a}^{(2)}|^{\frac{1}{\theta}}. \quad (6.60)$$

Proof. (i) Assume (DC), $\alpha \geq 3$ and $\Upsilon_1 < \infty$. By (2.17) We have $\widehat{2} = 2$, $\widehat{3} = 3$ and $r_0 \leq 2$. Hence $\alpha \geq \max\{3, \alpha_*\}$ and $\tilde{\gamma}_5 = \gamma_5$. We have $1 + S(\gamma_5, t) \leq C$ for all $t \geq 0$. Thanks to this uniform boundedness, we have from (6.53), (6.55) and (6.56) that

$$\limsup_{t \rightarrow \infty} \int_U |z(x, t)|^\alpha dx \leq C |\bar{a}^{(1)} - \bar{a}^{(2)}| \limsup_{t \rightarrow \infty} (1 + S(\gamma_5, t)) \limsup_{t \rightarrow \infty} (1 + S(\gamma_5, t))^{\frac{\alpha}{2-a}},$$

which proves (6.58).

(ii) Assume (NDC), $\alpha \geq \max\{3, \alpha_*\}$ and $\Upsilon_2 < \infty$. Note that $\gamma_5 \geq \gamma_2 \geq 2\alpha > \alpha_*$ and $r_0 > 2$. Similar to part (i), we obtain from (6.54), (6.55) and (6.57) that

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \int_U |z(x, t)|^\alpha dx \\ & \leq C |\bar{a}^{(1)} - \bar{a}^{(2)}|^{\frac{1}{\theta}} \limsup_{t \rightarrow \infty} (1 + S(\tilde{\gamma}_5, t))^{\frac{1}{\theta}} \limsup_{t \rightarrow \infty} (1 + S(\gamma_8, t))^{\frac{2-\theta_1}{\theta\theta_1}}. \end{aligned} \quad (6.61)$$

Since $\tilde{\gamma}_5 \leq \gamma_8$, we have $S(\tilde{\gamma}_5, t) \leq C(1 + S(\gamma_8, t))$. Thus (6.61) gives

$$\limsup_{t \rightarrow \infty} \int_U |z(x, t)|^\alpha dx \leq C |\bar{a}^{(1)} - \bar{a}^{(2)}|^{\frac{1}{\theta}} \left(1 + \limsup_{t \rightarrow \infty} S(\gamma_8, t)\right)^{\frac{1}{\theta} + \frac{2-\theta_1}{\theta\theta_1}}$$

and (6.59) follows. \square

We now find estimates for $\int_U |\nabla z(x, t)|^{2-a} dx$. Since the DC case was already studied in [14], we focus on the NDC case here. Recall that $\hat{3} = \max\{3, \alpha_*\}$.

Theorem 6.6. *Assume (NDC).*

(i) For all $t \geq 1$,

$$\begin{aligned} & \left(\int_U |\nabla z(x, t)|^{2-a} dx \right)^{\frac{2}{2-a}} \leq C |\bar{a}^{(1)} - \bar{a}^{(2)}| \bar{M}_1(t)^{\frac{2}{2-a}} \\ & + C \bar{M}_1(t)^{\frac{a}{2-a}} \bar{M}_3(t)^{\frac{1}{2}} \left\{ \left[\int_U |z(x, 0)|^{\hat{3}} dx \right]^{1/\hat{3}} \right. \\ & \left. + \left[M_{0,1} |\bar{a}^{(1)} - \bar{a}^{(2)}| + \left(\text{Env} \{ \bar{R}(\hat{3}, t) \bar{M}_{2,\hat{3}}(t)^{\frac{2-\theta_1}{\theta_1}} \} \right)^{\frac{1}{\theta}} |\bar{a}^{(1)} - \bar{a}^{(2)}|^{\frac{1}{\theta}} \right]^{1/\hat{3}} \right\}, \end{aligned} \quad (6.62)$$

where $\bar{M}_3(t)$ is defined in (5.61) and (5.60) with the boundary data $\Psi_1 = \Psi_2 = \Psi$ and the difference $\Phi = 0$, while $M_{0,1}$ is defined by (6.24) and $\bar{M}_{2,\hat{3}}(t)$ is defined by (6.34) with $\alpha = \hat{3}$.

(ii) Let $\Upsilon_2 = \Upsilon_2(\hat{3})$, the limit defined by (6.59) with $\alpha = \hat{3}$. If $\Upsilon_2 < \infty$ then

$$\begin{aligned} \limsup_{t \rightarrow \infty} \left(\int_U |\nabla z(x, t)|^{2-a} dx \right)^{\frac{2}{2-a}} & \leq C(1 + \Upsilon_2)^{\frac{2}{2-a}} |\bar{a}^{(1)} - \bar{a}^{(2)}| \\ & + C(1 + \Upsilon_2)^{1/2+a/(2-a)+2/(\hat{3}\theta\theta_1)} |\bar{a}^{(1)} - \bar{a}^{(2)}|^{1/(\hat{3}\theta)}. \end{aligned} \quad (6.63)$$

Proof. (i) Inequality (5.20) of Lemma 5.3 in [13] reads

$$\begin{aligned} \int_U z z_t dx & \leq -C \left(\int_U |\nabla z|^{2-a} dx \right)^{\frac{2}{2-a}} M_1(t)^{\frac{-a}{2-a}} \\ & + C |\bar{a}^{(1)} - \bar{a}^{(2)}| \int_U (|\nabla p_1|^{2-a} + |\nabla p_2|^{2-a}) dx, \end{aligned}$$

hence

$$\left(\int_U |\nabla z|^{2-a} dx \right)^{\frac{2}{2-a}} \leq C |\bar{a}^{(1)} - \bar{a}^{(2)}| M_1(t)^{\frac{2}{2-a}} + C M_1(t)^{\frac{a}{2-a}} \int_U |z| |z_t| dx.$$

Same note as above, the constant C here can be made dependent on $\hat{\chi}(\mathbf{D})$ instead of on $\bar{a}^{(1)}$ and $\bar{a}^{(2)}$. Since $z_t = \bar{z}_t$ and $\hat{3} > 2$, applying Hölder's inequality to the last integral we have

$$\left[\int_U |\nabla z|^{2-a} dx \right]^{\frac{2}{2-a}} \leq C |\bar{a}^{(1)} - \bar{a}^{(2)}| M_1(t)^{\frac{2}{2-a}} + C M_1(t)^{\frac{a}{2-a}} \left[\int_U |\bar{z}_t|^2 dx \right]^{\frac{1}{2}} \left[\int_U |z|^{\hat{3}} dx \right]^{1/\hat{3}}. \quad (6.64)$$

We use (6.32) to bound $M_1(t)$, use (5.61) to bound

$$\int_U |\bar{z}_t|^2 dx \leq 2 \int_U |(\bar{p}_1)_t|^2 + |(\bar{p}_2)_t|^2 dx \leq C \bar{M}_3(t)$$

and use (6.50) with $\alpha = \widehat{3}$ to estimate $\int_U |z|^{\widehat{3}} dx$, hence obtain

$$\begin{aligned} \left(\int_U |\nabla z|^{2-a} dx \right)^{\frac{2}{2-a}} &\leq C |\bar{a}^{(1)} - \bar{a}^{(2)}| \overline{M}_1(t)^{\frac{2}{2-a}} + C \overline{M}_1(t)^{\frac{a}{2-a}} \overline{M}_3(t)^{\frac{1}{2}} \left\{ \int_U |z(x, 0)|^{\widehat{3}} dx \right. \\ &\quad \left. + M_{0,1} |\bar{a}^{(1)} - \bar{a}^{(2)}| + \left[Env(\overline{R}(\widehat{3}, t) \overline{M}_{2,\widehat{3}}(t)^{\frac{2-\theta_1}{\theta_1}}) \right]^{\frac{1}{\theta}} |\bar{a}^{(1)} - \bar{a}^{(2)}|^{\frac{1}{\theta}} \right\}^{1/\widehat{3}}. \end{aligned}$$

This proves inequality (6.62).

(ii) Assume $\Upsilon_2 < \infty$. Let $R(t) = R(\widehat{3}, t)$ and $S(t) = S(\gamma_8(\widehat{3}), t)$ defined as in Lemma 6.1 and Theorem 6.5, respectively, with $\alpha = \widehat{3}$. Similar to the previous theorem, one can easily verify that

$$\limsup_{t \rightarrow \infty} M_1(t), \limsup_{t \rightarrow \infty} R(t) \leq C(1 + \limsup_{t \rightarrow \infty} S(t)) \leq C(1 + \Upsilon_2). \quad (6.65)$$

Note from the definition of $R(t)$ in Lemma 6.1 that

$$\int_U |\bar{z}_t|^2 dx \leq 2 \int_U |(\bar{p}_1)_t|^2 + |(\bar{p}_2)_t|^2 dx \leq 2R_2(t) \leq 2R(t).$$

We then have

$$\limsup_{t \rightarrow \infty} \int_U |\bar{z}_t(x, t)|^2 dx \leq 2 \limsup_{t \rightarrow \infty} R(t) \leq C(1 + \Upsilon_2). \quad (6.66)$$

By (6.65), (6.66) and (6.60) with $\alpha = \widehat{3}$, each time-dependent term on the right-hand side of (6.64) has finite limit superior. Hence taking the limit superior of inequality (6.64) and using the mentioned estimates we obtain

$$\begin{aligned} \limsup_{t \rightarrow \infty} \left(\int_U |\nabla z(x, t)|^{2-a} dx \right)^{\frac{2}{2-a}} &\leq C |\bar{a}^{(1)} - \bar{a}^{(2)}| (1 + \Upsilon_2)^{\frac{2}{2-a}} \\ &\quad + C(1 + \Upsilon_2)^{\frac{a}{2-a}} (1 + \Upsilon_2)^{\frac{1}{2}} \left\{ (1 + \Upsilon_2)^{\frac{2}{\theta\theta_1}} |\bar{a}^{(1)} - \bar{a}^{(2)}|^{\frac{1}{\theta}} \right\}^{1/\widehat{3}}, \end{aligned}$$

therefore (6.63) follows. \square

APPENDIX A

We generalize the results in section 3 of [14]. This covers the special case in Lemma 2.7.

Lemma A.1. *Let ϕ be a continuous, strictly increasing function from $[0, \infty)$ onto $[0, \infty)$. Suppose $y(t) \geq 0$ is a continuous function on $[0, \infty)$ such that*

$$y' \leq -h(t)\phi^{-1}(y(t)) + f(t), \quad t > 0, \quad (A.1)$$

where $h(t) > 0$, $f(t) \geq 0$ for $t \geq 0$ are continuous functions on $[0, \infty)$.

(i) *If $M(t) = Env\left(\frac{f(t)}{h(t)}\right)$ on $[0, \infty)$ then*

$$y(t) \leq y(0) + \phi(M(t)) \text{ for all } t \geq 0. \quad (A.2)$$

(ii) *If $\int_0^\infty h(\tau) d\tau = \infty$ then*

$$\limsup_{t \rightarrow \infty} y(t) \leq \phi \left(\limsup_{t \rightarrow \infty} \frac{f(t)}{h(t)} \right). \quad (A.3)$$

Here, we use the notation $\phi(\infty) = \infty$.

Proof. (i) **Claim 1.** Given any $\varepsilon > 0$,

$$y(t) \leq y(0) + \varepsilon + \phi(M(t)) \text{ for all } t \geq 0. \quad (\text{A.4})$$

Suppose (A.4) fails. Then there exist $t_2 > t_1 > 0$ such that

$$\begin{aligned} y(t_1) &= y(0) + \phi(M(t_1)), \\ y(t_2) &= y(0) + \varepsilon + \phi(M(t_2)), \\ y(t) &\geq y(0) + \phi(M(t)) \geq \phi\left(\frac{f(t)}{h(t)}\right), \text{ for all } t \in [t_1, t_2]. \end{aligned}$$

The last inequality and (A.1) give

$$y'(t) \leq -h(t)\phi^{-1}(y(t)) + f(t) \leq -h(t)\frac{f(t)}{h(t)} + f(t) = 0$$

for all $t \in (t_1, t_2)$, thus $y(t)$ is decreasing on $[t_1, t_2]$. This and the fact that $M(t)$ is increasing give

$$y(t_2) \leq y(t_1) = y(0) + \phi(M(t_1)) < y(0) + \varepsilon + \phi(M(t_2)) = y(t_2).$$

It is a contradiction and hence Claim 1 holds true. Now letting $\varepsilon \rightarrow 0$ in (A.4) yields (A.2).

(ii) Assume $\int_0^\infty h(\tau)d\tau = \infty$. Let $A = \limsup_{t \rightarrow \infty} \phi\left(\frac{f(t)}{h(t)}\right)$. It suffices to prove (A.3) for finite A . Given $\delta > 0$, there is $T_0 > 0$ such that for all $t > T_0$

$$\phi\left(\frac{f(t)}{h(t)}\right) \leq A + \delta \Rightarrow f(t) \leq \phi^{-1}(A + \delta)h(t).$$

Thus

$$y'(t) \leq -h(t)\phi^{-1}(y(t)) + \phi^{-1}(A + \delta)h(t) \text{ for all } t > T_0. \quad (\text{A.5})$$

Claim 2. Given any $\varepsilon > 0$, there is $T > 0$ such that

$$y(t) \leq \phi\left(\phi^{-1}(A + \delta) + \varepsilon\right) \text{ for all } t > T. \quad (\text{A.6})$$

We infer from (A.6) that

$$\limsup_{t \rightarrow \infty} y(t) \leq \phi\left(\phi^{-1}(A + \delta) + \varepsilon\right).$$

Now letting $\varepsilon \rightarrow 0$ and then $\delta \rightarrow 0$, and using the continuity of ϕ and ϕ^{-1} we obtain (A.3).

Proof of Claim 2. Given $\varepsilon > 0$. If (A.6) holds for $T = T_0$ then we are done. Suppose (A.6) fails for $T = T_0$, we will prove that (A.6) holds for some $T > T_0$. Because (A.6) fails for $T = T_0$, then there is $T_* \geq T_0$ such that

$$y(T_*) > \phi\left(\phi^{-1}(A + \delta) + \varepsilon\right). \quad (\text{A.7})$$

We claim that there is $t_* > T_*$ such that

$$y(t_*) \leq \phi \left(\phi^{-1}(A + \delta) + \frac{\varepsilon}{2} \right). \quad (\text{A.8})$$

Indeed, if this is not the case then

$$y(t) > \phi \left(\phi^{-1}(A + \delta) + \frac{\varepsilon}{2} \right) \text{ for all } t > T_*.$$

It implies for all $t > T_*$ that

$$y'(t) \leq -h(t) \left[\phi^{-1}(A + \delta) + \frac{\varepsilon}{2} \right] + \phi^{-1}(A + \delta)h(t) = -\frac{\varepsilon}{2}h(t).$$

Thus for $t > T_*$,

$$y(t) = y(T_*) + \int_{T_*}^t y'(\tau) d\tau \leq y(T_*) - \frac{\varepsilon}{2} \int_{T_*}^t h(\tau) d\tau \rightarrow -\infty \text{ as } t \rightarrow \infty,$$

since $\int_0^\infty h(\tau) d\tau = \infty$ and $h(t)$ is continuous on $[0, T_*]$. This contradicts the fact that $y(t) \geq 0$ for all $t \geq 0$ and hence (A.8) must be true.

By (A.7) and (A.8) there exists $T \in (T_*, t_*]$ such that

$$\phi \left(\phi^{-1}(A + \delta) + \frac{\varepsilon}{2} \right) = y(T) < \phi \left(\phi^{-1}(A + \delta) + \varepsilon \right).$$

Now we prove that (A.6) holds for such T . Suppose otherwise, then there is $T' > T$ such that $y(T') > \phi \left(\phi^{-1}(A + \delta) + \varepsilon \right)$. Hence there are t_1 and t_2 with $T \leq t_1 < t_2 \leq T'$ such that

$$\phi \left(\phi^{-1}(A + \delta) + \frac{\varepsilon}{2} \right) \leq y(t) < y(t_2) = \phi \left(\phi^{-1}(A + \delta) + \varepsilon \right), \text{ for all } t_1 \leq t < t_2. \quad (\text{A.9})$$

Then we have from (A.5) and (A.9) that

$$\begin{aligned} y(t_2) - y(t_1) &= \int_{t_1}^{t_2} y'(t) dt \leq \int_{t_1}^{t_2} -h(t) \phi^{-1}(y(t)) + \phi^{-1}(A + \delta)h(t) dt \\ &\leq \int_{t_1}^{t_2} -h(t) \left(\phi^{-1}(A + \delta) + \frac{\varepsilon}{2} \right) + \phi^{-1}(A + \delta)h(t) dt \\ &= \int_{t_1}^{t_2} -\frac{\varepsilon}{2}h(t) dt < 0, \end{aligned}$$

since $h(t) > 0$ and $h(t)$ is continuous on $[t_1, t_2]$. Thus $y(t_2) < y(t_1)$ which contradicts (A.9) with $t = t_1$. Therefore (A.6) holds for T . The proof is complete. \square

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