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APPROXIMATE SOLUTIONS TO SECOND ORDER PARABOLIC EQUATIONS *II*: TIME-DEPENDENT COEFFICIENTS

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Abstract. We consider second order parabolic equations with coefficients that vary both in space and in time (non-autonomous). We derive closed-form approximations to the associated fundamental solution by extending the Dyson-Taylor commutator method that we recently established for autonomous equations. We establish error bounds in Sobolev spaces and show that by including enough terms, our approximation can be proven to be accurate to arbitrary high order in the short-time limit. We show how our method extends to give an approximation of the solution for any fixed time and within any given tolerance. Some applications to option pricing are presented. In particular, we perform several numerical tests, and specifically include results on Stochastic Volatility models.

Contents

Introduction

The aim of the paper is to provide accurate, efficiently computable approximations for the Green's function or fundamental solution of parabolic equations in $\mathbb{R}^N$, with variable coefficients depending *both on space and time*.

More precisely, we consider operators of the form $\partial_t - L$, where

$$L = \sum_{i,j}^{N} a_{ij}(t,x)\partial_i\partial_j + \sum_{i}^{N} b_i(t,x)\partial_i + c(t,x), \tag{0.1}$$

with $x = (x_1, ..., x_N) \in \mathbb{R}^N$ and $\partial_k := \frac{\partial}{\partial x_k}$. The coefficients a_{ij}, b_i, and c are assumed smooth and all their derivatives are assumed to be uniformly bounded.

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(We write $a_{ij}, b_j, c \in \mathcal{C}_b^\infty\left(\mathbb{R}^+ \times \mathbb{R}^N\right)$.) Without loss of generality we can assume that $a_{ij} = a_{ji}$ as well. These type of equations are sometimes referred to as Fokker-Planck equations and they arise naturally in many contexts, in particular in statistical mechanics and probability. An important example is given by heat equations on manifolds with geometry evolving in time. In this case, $L(t) = \Delta_{g(t)}$ is the Laplace-Beltrami operator associated to a time-dependent metric g. In a non-relativistic context, such metris appear for instance in covariant formulations of continuum mechanics [24].

We impose a uniform strong ellipticity condition on the operators $L(t)$, *i.e.,* there exists a constant $\gamma > 0$, such that

$$\sum a_{ij}(t,x)\xi_i\xi_j \geq \gamma\|\xi\|^2, \quad \forall t \geq 0,\ x, \xi \in \mathbb{R}^N, \tag{0.2}$$

and we consider the initial value problem (IVP)

$$\begin{cases} \partial_t u(t,x) - L(t)u(t,x) = g(t,x) & \text{in } (0,\infty) \times \mathbb{R}^N, \\ u(0,x) = f(x), & \text{on } \{0\} \times \mathbb{R}^N, \end{cases} \tag{0.3}$$

where u, f, and g are in suitable spaces. Even when the initial value problem (0.3) is well posed, it is difficult to obtain exact solution formulas. Only in special cases, for example when $L(t)$ is a constant-coefficient operator, the solution can be obtained explicitly in terms of the initial data. Such explicit representation of the solution is not available in general for variable-coefficient operators. The dependence of the coefficients on time poses additional difficulties as discussed later in this Introduction.

Several methods are available to approximate the exact solution: numerically, asymptotically, and even analytically. However, the majority of these methods are slow. Our goal is to derive approximate closed-form solutions for the IVP (0.3) that are accurate, yet efficiently computable for initial data that usually arises in practice. A fast solution method is crucial when calibrating unknown parameters, especially in the Baeysian inference framework.

We first develop a general approach to approximate the Green's function or fundamental solution for the operator $\partial_t - L$, which then leads to an approximate formula for the solution of (0.3) by convolution with the initial data. The conditions given above on the coefficients of L together with the strong ellipticity condition on the operator ensures the well-posedness of the IVP in Sobolev and other function spaces (see for example [1, 18, 23] and references therein). In addition, the solution operator forms a so-called *evolution system*, informally a generalization of the semi-group e^{tL}, which gives the solution operator in the case L is independent of time. (See Definition 1.10.) We will denote the evolution system by $U(t,r)$, $0 \leq r \leq t$, throughout the paper.

The approximation scheme for the Green's function is an extension to the case of time-dependent coefficients of a method recently introduced by the authors and their collaborators in [5]. This method, which will be referred to in the paper as the *Dyson-Taylor commutator method*, combines known techniques in a novel way and is based on a suitable *local* parabolic rescaling, Taylor expansion of the coefficients, and Duhamel's formula, together with exact commutator formulas. It is more elementary than those found in the literature, yet accurate to arbitrary order in time in the short-time limit. When L is independent of t, global error estimates as $t \to 0^+$ on the operator norm of the approximate solution operator in Sobolev spaces $W^{r,p}$ were derived in [5]. The Sobolev spaces can be exponentially weighted, a setting of interest for certain applications to probability (see the discussion in [5] and Definition 1.19). Deriving such estimates is more challenging here, since it is more difficult to obtain the needed mapping properties for an evolution system than for a semigroup, in particular because the equation generally is not invariant under time rescaling. We use these more general result to obtain new applications to stochastic volatility models.

The main result of this paper is the Theorem 0.1 below. We will collectively denote by $\mathbb{L}_\gamma$ the class of operators L of the form (0.1) satisfying the ellipticity condition (0.2) (this class is formally introduced in Definition 1.1). We also need to introduce the concept of an *admissible function*, that is a function $z = z(x, y)$ such that $z(x, x) = x$ and all its derivatives $\partial^\alpha z$ are bounded for $\alpha \neq 0$. The pint z is the center for the dilations used in approximating the Green's function $\mathcal{G}_t^L(x, y)$. One can choose $z(x, y) = z$, but our results are more general. This is a key novelty of our method. In current work, we are investigating how different choices of z influence the accuracy of the approximation in examples of practical interest [3]. Lastly, we denote by $W^{r,p}_{a,z}$ the Sobolev space with weight $exp(a(1 + |x - z|^2)^{1/2})$, formally introduced in Definition 1.19. The reason for introducing the weighted Sobolev spaces $W^{r,p}_{a,z}$ is that we want the typical initial conditions that arise in applications (such as pay-offs of continent claims) to be covered by our results.

Theorem 0.1. *Let m be a positive integer, L an operator in $\mathbb{L}_\gamma$, and $z = z(x, y)$ an admissible function. Then L generates an evolution system $U(t, t')$ in the Sobolev space $W^{r,p}_{a,z}(\mathbb{R}^N)$, $r \in \mathbb{R}_+$, $1 < p < \infty$, $a \in \mathbb{R}$. Furthermore, the Green's function for $\partial_t - L$ has the asymptotic expansion*

$$\mathcal{G}_t^L(x, y) := \mathcal{G}_t^{[m,z]}(x, y) + s^{m+1} E_m^{t,z}(x, y)$$

where $\mathcal{G}_t^{[m,z]}(x, y)$ is the mth order approximate *Green's function given by*

$$\mathcal{G}_t^L(x, y) = s^{-N} \Big(e^{L_0}(z + s^{-1}(x - z), z + s^{-1}(y - z)) \\ + \sum_{\ell=1}^{m} s^\ell \Lambda_z^\ell (z + s^{-1}(x - z), z + s^{-1}(y - z)) \Big)$$

with Λ_z^ℓ defined in Equations (3.19) *and* (3.18) *and Lemma 3.10 and the error term $s^{m+1} E_m^{t,z}(x, y)$ is small in the following sense. For any $f \in W^{k,p}_{a,z}$, $a, k, r \in \mathbb{R}$, $1 < p < \infty$, define the error operator $\mathcal{E}^{[m,z]_t}$ by*

$$\mathcal{E}_t^{[m,z]} f(x) = \int E_m^{t,z}(x, y) f(y) dy,$$

then

$$\|\mathcal{E}_t^{[m,z]} f\|_{W^{r+k,p}_{a,z}} \leq C_{L,m,a,z,p} t^{-r/2} \|f\|_{W^{k,p}_{a,z}},$$

for any $t \in [0, T]$, $s > 0$, where the constant $C_{L,m,a,z,p}$ does not depend on $t \in [0, T]$.

There is a well-established literature on deriving asymptotic formulas for the fundamental solution of parabolic equations in the short-time regime, but mostly for the autonomous case. A fundamental approach consists in viewing L as a Laplace-Beltrami operator on a manifold with metric tensor g given by $[a_{ij}]^{-1}$, plus lower-order terms. Asymptotic formulas for heat kernels on manifolds are discussed, for example, in [2, 14, 15, 19, 21, 29, 29, 30], (see also [9, 20, 27] for a pseudo-differential operator perspective). Several of these results are for compact manifolds, and they translate to local estimates on noncompact, complete manifolds. We remark that the error bounds in our main theorem are *global* in space.

Minakshisundaram-Pleijel [21] in particular obtained the following asymptotic expansion

$$(0.4)\quad \mathcal{G}_t(x,y) = (4\pi t)^{-\frac{N}{2}} e^{-\frac{d(x,y)^2}{4t}} \left(\mathcal{G}^{(0)}(x,y) + \mathcal{G}^{(1)}(x,y)t + \mathcal{G}^{(2)}(x,y)t^n + \dots\right),$$

as $t \to 0_+$, where $d(x,y)$ is the geodesic distance between x and y in the metric g, and $\mathcal{G}^{(j)}(x,y)$ are smooth functions in x and y. Greiner [9] constructed the same type of expansion as a *parametrix* for $\partial_t - L$ via a Volterra series. These types of asymptotic formulas have proven very successful from a theoretical point of view, for example in estimating eigenvalues for the operator L and obtaining trace formulas (see the articles cited above). In applications to probability, Henry-Labordére [12] used the asymptotic expansion (0.4) to compute the implied volatility in stochastic volatility models. This geometric approach has proven less successful in practical implementations, given that in general there are no exact formulas for the geodesic distance $d(x,y)$, which therefore needs to be computed numerically or otherwise estimated, usually asymptotically for $x \sim y$.

Pseudo-differential calculus gives a related parametrix expansion for $\partial_t - L$, using transport equations for amplitudes:

$$(0.5)\qquad \mathcal{G}^L(t,x,y) \sim \sum_{j\geq 0} t^{\frac{j-n}{2}} p_j\big(x, t^{-\frac{1}{2}}(x-y)\big) e^{-\frac{(x-y)^T A(x)^{-1}(x-y)}{4t}},$$

as $t \to 0_+$, where $p_j(x,w)$ is a polynomial of degree j in w, and $A(x) := [a_{ij}(x)]$. (See Taylor [27, Chapter 7, Section 13] for a derivation and for error estimates on compact manifolds.) This expansion using pseudodifferential operators is akin to semiclassical asymptotics for Schrödinger operators, which are sometimes approached using the Wentzel-Kramers-Brillouin (WKB) method. In this context, the WKB method gives

$$(0.6)\qquad \mathcal{G}_t(x,y) = \frac{1}{(2\pi t)^{N/2}} \exp\Big(-\frac{a(x,y)^2}{2t} + \sum_{k\geq 0} c_k(x,y)t^k\Big),$$

with $a(x,y)$, $c_k(x,y)$ are to be determined. For details about this method, see for example Kampen [15].

Finally, we mention the Lie group approach based on the groups on invariant transformations for the differential equations. In some cases, it is possible to find the exact fundamental solutions by reducing the equation to an equivalent one that can be solved exactly (see [16, 17] for applications). For a further discussion, we refer the reader to Olver's monograph [22]. A main difficulty of this approach is that usually a large system of ODEs for the generators of the group needs to be solved.

In this paper, we shall use the so called Dyson-Taylor commutator method to derive an asymptotic series of the Green's function of (0.3) similar to (0.4) and (0.5), but based on elementary methods, algorithmically computable, and of arbitrary high accuracy (if enough terms are included in the approximation). Furthermore, the numerical tests we perform show that our methods works even for certain classes of PDEs not satisfying our assumptions of regularity and uniform ellipticity. These equations are usually degenerate and their coefficients unbounded. Examples arises in probability and financial mathematics, for instance the well-known Black-Scholes equation

$$\partial_t u(t,x) = \frac{\sigma^2 x^2}{2} u(t,x) + rx\partial_x u(t,x) - ru(t,x).$$

(See [3] for numerical results for the Black-Scholes equation and other models in 1D.) In a forthcoming paper, we plan to extend extend our method to degenerate parabolic equations (typically with polynomial coefficients) and will justify the use of our method for the examples in this paper using a partition of unity and geometric properties of our equation, as in [3].

The paper is organized as follows. In Section 1, we present some preliminary results on non-autonomous, second order strongly elliptic operators $L \in \mathbb{L}_\gamma$ time-dependent coefficients. For such operators, we study the existence and mapping properties of the solution operator (or Green function) of the parabolic equation $(\partial_t - L(t))u(t,x) = 0$ and establish estimates needed in our later error analysis. In Section 2, we consider a time-independent strongly elliptic operator $L_0 \in \mathbb{L}_\gamma$ perturbed by another possibly time-dependent operator V, and study the mapping properties of the evolution system (solution operator) generated by $L := L_0 + V$. As a particular case, we apply interpolation theory to extend and refine the mapping properties obtained in Section 1. Section 3.1 continues the discussion of Section 2, in this section, using a parabolic scaling argument, we transform our problem into another equivalent problem. Taylor expansion allows us to split the original operator into the sum of a time-independent operator and another time-dependent operator. After an iterative procedure, we obtain a formal expansion of the evolution system of our equation $(\partial_t - L(t))u(t,x) = 0$. In section 4, we justify the convergence of the series of the evolution system obtained in Subsection 3.1. In Section 5, we show that our asymptotic expansion is invariant under affine transformations. In the last section, Section 6, we address some issues that arise in then numerical implementations of our approach. In particular, we extend our results from small time to any time by a bootstrap scheme. We also illustrate the *Dyson-Taylor commutator method* by a concrete example, and present important applications of our approach to stochastic volatility models that appear in option pricing theory.

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1. Preliminaries on evolution systems

We begin by describing in more details the class of operators to which our main results applies and some of their main properties.

We first introduce some notations and recall several definitions. In what follows, we denote the inner product on $\mathbb{R}^N$ by

$$(u,v) = \int_{\mathbb{R}^N} u(x)v(x)dx. \tag{1.1}$$

Let us denote

$$\langle \xi \rangle := (1 + |\xi|^2)^{1/2} \tag{1.2}$$

and let

$$\hat{u}(\xi) = \int_{\mathbb{R}^N} e^{-\imath \xi \cdot x} u(x) dx \tag{1.3}$$

be the Fourier transform of u. We also recall the definition of some basic facts about L^p-based Sobolev spaces $W^{r,p}(\mathbb{R}^N)$: for any $1 < p < \infty$, $r \in \mathbb{R}$, we define

$$\begin{aligned} W^{r,p} = W^{r,p}(\mathbb{R}^N) &:= \{u : \mathbb{R}^N \to \mathbb{C}\,,\ \langle \xi \rangle^r \hat{u} \in L^p(\mathbb{R}^N)\} \\ &= \{u : \mathbb{R}^N \to \mathbb{C}\,,\ (1-\Delta)^{r/2} u \in L^p(\mathbb{R}^N)\}, \end{aligned} \tag{1.4}$$

If $r \in \mathbb{Z}_+$, then $W^{r,p}(\mathbb{R}^N) = \{u : \mathbb{R}^N \to \mathbb{C},\ \partial^\alpha u \in L^p(\mathbb{R}^N),\ |\alpha| \leq r\}$, so we recover the usual definition.

Since the dimension N is fixed throughout the paper, we will usually write $W^{r,p}$ for $W^{r,p}(\mathbb{R}^N)$. Similarly, we shall often write L^p instead of $L^p(\mathbb{R}^N)$. When $1 < p < \infty$, the dual of $W^{r,p}$ is the Sobolev space $W^{-r,p'}$ with $1/p + 1/p' = 1$.

Let

$$\mathcal{C}_b^\infty(\mathbb{R}^+ \times \mathbb{R}^N) := \{f : \mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{C},\, \partial^\alpha f \text{ bounded for all } \alpha\}. \tag{1.5}$$

We remark that, if $f \in \mathcal{C}_b^\infty(\mathbb{R}^+ \times \mathbb{R}^N)$, then f is Hölder continuous with respect to the time variable t uniformly in the space variable $x \in \mathbb{R}^N$.

Definition 1.1. *We shall denote by $\mathbb{L}$ the set of second-order differential operators $L(t)$ of the form*

$$L(t) := \sum_{i,j=1}^N a_{ij}(t,x)\partial_i\partial_j + \sum_{k=1}^N b_k(t,x)\partial_k + c(t,x), \tag{1.6}$$

where the matrix (a_{ij}) is symmetric and $a_{ij}, b_k, c \in \mathcal{C}_b^\infty(\mathbb{R}^+ \times \mathbb{R}^N)$ are real valued. *We shall denote by $\mathbb{L}_\gamma$ the subset of operators $L(t) \in \mathbb{L}$ satisfying the uniform strong ellipticity estimate* (0.2) *with the ellipticity constant γ*

Let $A : D(A) \to X$ be a closed linear operator defined on a subspace $D(A) \subset X$ of the Banach space X. We shall denote by $\rho(A)$ the *resolvent* set of A, namely the set

$$\rho(A) := \{\lambda \in \mathbb{C},\, \lambda - A : D(A) \to X \text{ is bijective}\}. \tag{1.7}$$

We next recall the definition of a sectorial operator [18].

Definition 1.2. *A closed operator $A : D(A) \to X$, where $D(A)$ is a linear subspace of the Banach space X is called* sectorial *if there are constants $\omega \in R$, $\theta \in (\pi/2, \pi)$, and $M > 0$ such that*

$$\begin{cases} \rho(A) \supset S_{\theta,\omega} := \{\lambda \in \mathcal{C} : \lambda \neq \omega, |arg(\lambda - \omega)| < \theta\} \\ \|R(\lambda, A)\| \leq \frac{M}{|\lambda - \omega|}, \forall \lambda \in S_{\theta,\omega}, \end{cases}$$

where $\rho(A)$ is the resolvent set of A.

Later on, we will give a sufficient condition to guarantee that $L(t)$ is sectorial and use sectoriality to obtain mapping properties for the evolution system $U(t,r)$ that $L(t)$ generates.

1.1. **Properties of the class $\mathbb{L}_\gamma$.** Elliptic pseudo-differential operators, in particular elements of the class $\mathbb{L}_\gamma$, generate equivalent norms in Sobolev spaces. (See [26, 27, 31] for definition and basic properties of pseudodifferential operators.)

Proposition 1.3. *Let $m \geq 0$ and $Q \in \Psi^m_{1,0}(\mathbb{R}^N)$ be a uniformly elliptic operator and $1 < p < \infty$. Then the following two norms are equivalent*

$$\|u\|_{W^{m,p}} \sim \|u\|_{L^p} + \|Qu\|_{L^p}. \tag{1.8}$$

We sketch the proof for the reader's convenience. Recall that $W^{m,p} = W^{m,p}(\mathbb{R}^N)$ and $L^p = L^p(\mathbb{R}^N)$.

Proof. Since $Q \in \Psi^m_{1,0}(\mathbb{R}^N)$, Q is a bounded operator from $W^{m,p}$ to L^p by [26], and hence there exists C_1 such that

$$\|u\|_{L^p} + \|Qu\|_{L^p} \leq C_1 \|u\|_{W^{m,p}}$$

On the other hand, since Q is uniformly elliptic, there exists a pseudodifferential operator $R \in \Psi^{-m}_{1,0}(\mathbb{R}^N)$ such that $I = RQ - S$, where I is the identity operator, and $S \in \Psi^{-\infty}(\mathbb{R}^N)$. Thus by mapping properties of pseudodifferential operators, we have

$$\|u\|_{W^{m,p}} \leq \|RQu\|_{W^{m,p}} + \|Su\|_{W^{m,p}} \leq C(\|Qu\|_{L^p} + \|u\|_{L^p})$$

The proof is complete. $\square$

We then obtain the following.

Corollary 1.4. *Suppose $L \in \mathbb{L}_\gamma$, $1 < p < \infty$, and m is a nonnegative integer. Then for each t the following two norms are equivalent*

$$\|u\|_{W^{2m,p}} \sim \|u\|_{L^p} + \|L^m(t)u\|_{L^p}. \tag{1.9}$$

Proof. One can easily check that if $L \in \mathbb{L}_\gamma$, then $L^m(t)$ is uniformly elliptic. An application of the above proposition then completes the proof. $\square$

The constants implicit in Equation (1.8) and Equation (1.9) above are uniform in the class $\mathbb{L}_\gamma$.

Next we show that if $L \in \mathbb{L}_\gamma$, then $L(t)$ is Hölder continuous in t and sectorial for each $t \in [0, T]$ between suitable Sobolev spaces. Their properties in turn give the needed mapping bounds for the evolution system. (See [1, 18, 23] for instance.) We split the proof into several propositions.

Proposition 1.5. *Suppose $L \in \mathbb{L}_\gamma$, then for any $k \in \mathbb{N}$.*

$$L(t) : W^{k+2,p} \to W^{k,p}$$

is Hölder continuous in t of exponent $\alpha = 1$.

Proof. For any $0 \leq t_1 \leq t_2 \leq T$, we have

$$\begin{aligned} L(t_2) - L(t_1) \quad &= \quad \sum (a_{ij}(t_2, x) - a_{i,j}(t_1, x))\, \partial_i \partial_j \\ &\quad + \sum (b_i(t_2, x) - b_i(t_1, x))\, \partial_i + c(t_2, x) - c(t_1, x) \end{aligned}$$

We also notice that for any multi-index β with $|\beta| \leq k$, by our assumption on the coefficients of the operator L,

$$\|a_{ij}(t_2, x) - a_{ij}(t_1, x)\|_{W^{|\beta|,\infty}} \leq C(t_2 - t_1)$$

similarly,

$$\begin{aligned}\|b_i(t_2,x)-b_i(t_1,x)\|_{W^{|\beta|,\infty}} &\leq C(t_2-t_1)\\ \|c(t_2,x)-c(t_1,x)\|_{W^{|\beta|,\infty}} &\leq C(t_2-t_1)\end{aligned}$$

Therefore, for any $u \in W^{k+2,p}$,

$$\|(L(t_2)-L(t_1))u\|_{W^{k,p}} \leq \|\sum (a_{ij}(t_2,x)-a_{ij}(t_1,x))\frac{\partial^2}{\partial x_i \partial x_j}u\|_{W^{k,p}}$$

$$+\|\sum (b_i(t_2,x)-b_i(t_1,x))\frac{\partial}{\partial x_i}u\|_{W^{k,p}} + \|(c(t_2,x)-c(t_1,x))u\|_{W^{k,p}}$$

$$\leq \sum_{|\beta|\leq k}\sum_{i,j}\|a_{ij}(t_2,x)-a_{ij}(t_1,x)\|_{W^{|\beta|,\infty}}\|u\|_{W^{k+2,p}}$$

$$+\sum_{|\beta|\leq k}\sum_{i}\|b_i(t_2,x)-b_i(t_1,x)\|_{W^{|\beta|,\infty}}\|u\|_{W^{k+2,p}}$$

$$+\sum_{|\beta|\leq k}\|c(t_2,x)-c(t_1,x)\|_{W^{|\beta|,\infty}}\|u\|_{W^{k+2,p}} \leq C(t_2-t_1)\|u\|_{W^{k+2,p}}$$

i.e., $\|L(t_2)-L(t_1)\|_{W^{k+2,p}\to W^{k,p}} \leq C(t_2-t_1)$. The proof is complete. □

The following well known proposition gives a sufficient condition to guarantee that an operator is sectorial.

Proposition 1.6. *Let $A : D(A) \subset X \to X$ be a linear operator such that $\rho(A)$ contains a half plane $\{\lambda \in \mathbb{C} : Re\lambda \geq \omega\}$, and*

$$\|\lambda R(\lambda, A)\|_{L(X)} \leq M, Re\lambda \geq \omega$$

with $\omega \in R, M > 0$. Then A is sectorial.

Proof. See [18], page 43. □

Our goal is to prove that $L(t) : W^{2k+2,p} = W^{2k+2,p}(\mathbb{R}^N) \to W^{2k,p}$ is sectorial for any non-negative integer k. The following special case is well known in the literature (see Lunardi [18], page 73 for example).

Proposition 1.7. *Let $L \in \mathbb{L}_\gamma$. For each t, $L(t)$ defines a continuous map $W^{2,p} \to L^p$ and its resolvent $(\lambda - L(t))^{-1}$, $\lambda \in \rho(L(t))$, satisfy the conditions of Proposition (1.6), thus $L(t)$ is sectorial from $W^{2,p}$ to L^p.*

Proof. Let us fix t_0 and write $\tilde{L} = L(t_0)$. By uniform, strong ellipticity, if λ is in the resolvent set of $L(t)$, then Re$\lambda > 0$. Hence we can write $\lambda = r^2 e^{i\theta}$ with $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

We now define an augmented, auxiliary operator in $N+1$ variables:

$$L_\theta = \tilde{L} + e^{i\theta}\partial_s^2.$$

One can easily check that L_θ is still a secon-order, elliptic, pseudo-differential operator on $\mathbb{R}^{N+1}$. Therefore, by Equation (1.8), we have the following norm equivalence

$$\|v\|_{W^{2,p}} \sim \|v\|_{L^p} + \|L_\theta v\|_{L^p} \tag{1.10}$$

for any $v \in W^{2,p}(\mathbb{R}^{N+1})$.

We next introduce a smooth cut-off function $\varphi \in C_c^\infty(\mathbb{R})$ such that $\varphi(s) = 1$ for $|s| \leq \frac{1}{2}$ and $\varphi(s) = 0$ for $|s| \geq 1$. For any $u \in W^{2,p}(\mathbb{R}^N)$ and $r > 0$, we set

$$v(x,s) = \varphi(s)e^{irs}u(x).$$

A direct computation yields

$$L_\theta v = \varphi(s)e^{irs}(\tilde{L} - r^2e^{i\theta})u + e^{i(\theta+rs)}(\varphi^{''}(s) + 2ir\varphi^{'}(s))u.$$

Therefore, the norm equivalence (1.10) implies that

$$\begin{aligned}
&\|v\|_{W^{2,p}} \le C\left(\|v\|_{L^p} + \|L_\theta v\|_{L^p}\right) \\
&= C\left(\|v\|_{L^p} + \|\varphi e^{irs}(\tilde{L} - r^2e^{i\theta})u + e^{i(\theta+rs)}(\varphi^{''} + 2ir\varphi^{'})u\|_{L^p}\right) \\
(1.11)\qquad &\le C\left(\|u\|_{L^p}\left(\|\varphi\|_{L^p} + 2r\|\varphi'\|_{L^p)} + \|\varphi^{''}\|_{L^p}\right) + \|\tilde{L} - r^2e^{i\theta}u\|_{L^p}\right) \\
&\le C\left((1+r)\|u\|_{L^p} + \|\tilde{L} - r^2e^{i\theta}u\|_{L^p}\right)
\end{aligned}$$

On the other hand, by our construction of φ, we have

$$(1.12)\qquad \|v\|^p_{W^{2,p}} \ge \int_{\mathbb{R}^N\times(-1/2,1/2)} |\partial_s^2(ue^{irs})|^p dxds = r^{2p}\|u\|^p_{L^p}.$$

Combining the inequalities (1.11) and (1.12), we obtain

$$r^2\|u\|_{L^p} \le \|v\|_{W^{2,p}} \le C\left((1+r)\|u\|_{L^p)} + \|(\tilde{L} - r^2e^{i\theta})u\|_{L^p}\right).$$

Then set $\lambda = r^2e^{i\theta}$ and choose r such that $r^2/2 \ge C(1+r)$ to obtain

$$\|\lambda u\|_{L^p} \le C\|(\tilde{L} - \lambda)u\|_{L^p}.$$

This gives

$$\|\lambda R(\lambda, \tilde{L})\|_{L^p\to L^p} \le C.$$

On the other hand, it is well known that the resolvent set $\rho(\tilde{L})$ contains a half plane (see [18] page 73 for example). Therefore, $\tilde{L}$ satisfies all the conditions in Proposition (1.6) and thus is sectorial. The proof is complete. $\square$

We have next the following generalization.

Lemma 1.8. *If $L \in \mathbb{L}_\gamma$, then for each t and k, $L(t)$ defines a continous map $W^{2k+2,p} \to W^{2k,p}$ with the property that the the resolvent set of $L(t)$ contains a half plane $\{\lambda \in \mathbb{C} : Re\lambda \ge \omega\}$.*

Proof. When $k = 0$, the result is obvious by Proposition (1.7). Choose any λ in $\rho_{L^p}(L(t))$, the resolvent of $L(t) : W^{2,p} \to L^p$ and denote by $R_\lambda = (\lambda - L(t))^{-1} : L^p \to W^{2,p}$. Then R_λ is bounded by the definition of the resolvent set. We claim that $R_\lambda W^{2k,p} \subset W^{2k+2,p}$ and that the resulting linear operator $W^{2k,p} \to W^{2k+2,p}$ is bounded.

Indeed, this is proved as follows. First, let us notice that $R_\lambda L(t)f = f = R_\lambda L(t)$ for $f \in D(L(t)) = W^{2,p}$. Let now $f \in W^{2k,p}$. Then $L^{k+1}(t)R_\lambda f = L^k(t)f \in L^p$. By the norm equivalence (1.8), we conclude that $R_\lambda f \in W^{2k+2,p}$ and hence our claim holds.

On the other hand, it is obvious that $(\lambda - L(t))W^{2k+2,p} \subset W^{2k,p}$. So actually we have $R_\lambda W^{2k,p} = W^{2k+2,p}$ and hence $R_\lambda : W^{2k,p} \to W^{2k+2,p}$ and $(\lambda - L(t)) : W^{2k+2,p} \to W^{2k,p}$ are inverses to each other. This fact tells us that $\rho_{L^p}(L(t))$ is contained in the resolvent set of the operator $L(t) : W^{2k+2,p} \to W^{2k,p}$ for $k > 0$. $\square$

Finally, we prove that for any $L \in \mathbb{L}_\gamma$ and any t, the operator $L(t)$ is sectorial between suitable spaces.

Lemma 1.9. *If $L \in \mathbb{L}_\gamma$, then for each t and k, the operator $L(t) : W^{2k+2,p} \to W^{2k,p}$ is sectorial.*

Proof. For each fixed $t = t_0$, since the sectorial property does not rely on t, we drop the time dependence and simply write $L_0 = L(t_0)$. We shall apply proposition (1.6) to prove that L_0 is sectorial between $W^{2k+2,p}$ and $W^{2k,p}$. For any $u \in W^{2k,p}$ and $\lambda \in \rho(L_0)$, then by Lemma (1.8), $R(\lambda, L_0)u \in W^{2k,p}$. Therefore, using the norm equivalence (1.8) twice, we obtain

$$\begin{aligned}\|\lambda R(\lambda, L_0)u\|_{W^{2k,p}} &\leq C(\|\lambda R(\lambda, L_0)u\|_{L^p} + \|\lambda L_0^k R(\lambda, L_0)u\|_{L^p}) \\ &= C(\|\lambda R(\lambda, L_0)u\|_{L^p} + \|\lambda R(\lambda, L_0)L_0^k u\|_{L^p}) \\ &\leq C(\|u\|_{L^p} + \|L_0^k u\|_{L^p}) \leq C\|u\|_{W^{2k,p}},\end{aligned}$$

that is,

$$\|\lambda R(\lambda, L_0)\|_{W^{2k,p} \to W^{2k,p}} \leq C.$$

In the second equality, as in Lemma (1.8), we used the same fact that if $u \in D(L_0^k)$, then $R(\lambda, L_0)L_0^k = L_0^k R(\lambda, L_0)$. (See [6],chapter 7). We also applied Proposition (1.7) that $\lambda R(\lambda, L_0)$ is bounded from L^p to L^p.

Then by Lemma (1.8) and Proposition (1.6), $L_0 : W^{2k+2,p} \to W^{2k,p}$ is sectorial. The proof is complete. □

1.2. **Existence and properties of the evolution system.** If $L(t) = L_0$, that is, if all the coefficients are time independent, then under some hypothesis L will generate a semigroup e^{tL}, it can be considered as the solution operator. However, in the nonautonomous case, the solution operator is not a semigroup, instead it is an evolution system, if it exists.

Definition 1.10. *A two parameter family of bounded linear operators $U(t, r)$ on X, $0 \leq r \leq t \leq T$, is called* an evolution system *if the following two conditions are satisfied*

(1) $U(r, r) = I$, $U(t, r)U(r, s) = U(t, s)$ for $0 \leq s \leq r \leq t \leq T$,
(2) $(t, r) \to U(t, r)$ is strongly continuous for $0 \leq r \leq t \leq T$.

We now introduce the standard assumption made in the literature that gives the existence of the evolution system. See for instance [1, 23], or [18], page 212.

Definition 1.11. *A family of strongly elliptic operators $L(t) : \mathcal{D}(L(t)) \subset X \to X$, $t \in [0, T]$, will be called* uniformly sectorial *if the following conditions are satisfied:*

(1) The domains $\mathcal{D}(t)$ are independent of t. Denote the common domain by $\mathcal{D}$. Then $\mathcal{D}$ is dense in X.
(2) We can endowe $\mathcal{D}$ with a Banach space norm such that $\mathcal{D} \to X$ is continuous and $L(t)$ is Hölder continues, i.e., *there exists $\alpha \in (0, 1]$, such that*

$$\|L(t) - L(r)\|_{D \to X} \leq C|t - r|^\alpha,$$

(3) There are constants $\omega \in \mathbb{R}, \theta \in (\pi/2, \pi), M > 0$ such that, for any $t \in [0, T]$

$$\begin{cases} \rho(L(t)) \supset S_{\theta,\omega} = \{\lambda \in \mathbb{C} : \lambda \neq \omega, |arg(\lambda - \omega)| < \theta\} \\ \|R(\lambda, L(t))\| \leq \frac{M}{|\lambda - \omega|}, \forall \lambda \in S_{\theta,\omega} \end{cases}$$

For later use, we recall the following useful result [1, 18, 23].

Theorem 1.12. *Suppose L is uniformly sectorial, then L generates an evolution system $U(t,r)$, and for any $0 \le r \le t \le T$*

$$\|U(t,r)\|_X \le C\,, \quad \|U(t,r)\|_{X\to D} \le \frac{C}{t-r}\,,$$
$$\|\partial_t U(t,r)\| = \|L(t)U(t,r)\|_X \le \frac{C}{t-r}\,,$$
$$\|L(t)U(t,r)\|_{D\to X} \le C\,, \quad \partial_s U(t,s) = U(t,s)L(s).$$

Proof. See, for example, Lunardi[18], Corollary 6.1.8, page 219. □

Proposition (1.5) and Lemma (1.9) show that $\mathbb{L}_\gamma$ consists of uniformly sectorial operators.

Corollary 1.13. *Suppose $L \in \mathbb{L}_\gamma$. Then L generates an evolution system $U(t,r)$, $0 \le r \le t \le T$, and for any real $k, l \ge 0$, and $1 < p < \infty$, we have*

$$\|U(t,r)\|_{W^{k,p}\to W^{k,p}} \le C\,, \quad \|U(t,r)\|_{W^{k,p}\to W^{k+2,p}} \le \frac{C}{(t-r)},$$
$$\|L(t)U(t,r)\|_{W^{k+2,p}\to W^{k,p}} \le C\,, \quad \textit{and} \quad \|L(t)U(t,r)\|_{W^{k,p}\to W^{k,p}} \le C.$$

Proof. The proof is mainly an application of a duality argument and space interpolation. First, by Theorem (1.12), for any nonnegative even integer k and $1 < p < \infty$, the above inequalities are true. Next we apply the duality method to pass the mapping properties to negative Sobolev spaces. Note that $U(t,s)$ satisfies the equation

$$\begin{cases} U(t,t) = 1 \\ \partial_t U(t,r) = L(t)U(t,r). \end{cases} \tag{1.13}$$

We define the adjoint operator $V(t,r) = U(T-r, T-t)^*$, then

$$V(t,t) = U(T-t, T-t)^* = 1$$

and

$$\begin{aligned} V(t,s)V(s,r) &= U(T-s, T-t)^* U(T-r, T-s)^* \\ &= [U(T-r, T-t)]^* = V(t,r). \end{aligned}$$

Therefore, $V(t,r)$ is also an evolution system, and moreover,

$$\begin{aligned} &\partial_t V(t,r) = \partial_t U(T-r, T-t)^* = \partial_t U(T-r, T-t)]^* \\ &= [U(T-r, T-t)L(T-t)]^* = L^*(T-t)U(T-r, T-t)^* \\ &= L^*(T-t)V(t,r). \end{aligned} \tag{1.14}$$

i.e., $\partial_t V(t,r) = L^*(T-t)V(t,r)$. According to our assumptions, the family $L^*(T-t)$ is of the same type as $L(t)$. Thus the evolution system should satisfy the same mapping properties with $U(t,r)$. Then for any positive k, we have

$$\begin{aligned} &\|U(t,r)\|_{W^{-k,p}\to W^{-k,p}} = \|U(t,r)^*\|_{W^{k,q}\to W^{k,q}} \\ &= \|V(T-r, T-t)\|_{W^{k,q}\to W^{k,q}} \le C \end{aligned} \tag{1.15}$$

and

$$\begin{aligned} &\|U(t,r)\|_{W^{-k-2,p}\to W^{-k,p}} = \|U(t,r)^*\|_{W^{k,q}\to W^{k+2,q}} \\ &= \|V(T-r, T-t)\|_{W^{k,q}\to W^{k+2,q}} \le \frac{C}{(T-r)-(T-t)} = \frac{C}{t-r}, \end{aligned} \tag{1.16}$$

where q is the conjugate of p, *i.e.*, $\frac{1}{p} + \frac{1}{q} = 1$.

At last, we apply the spaces interpolation technique to obtain mapping properties between non-integer Sobolev spaces. For any $\ell \in \mathbb{R}$, assume $k \le \ell < k+2$ where k is an even integer. Then by the complex interpolation,

$$W^{\ell,p} = \left(W^{2k,p'}, W^{2k+2,p''}\right)_{[\theta]}, W^{\ell+2,p} = \left(W^{2k+2,p'}, W^{2k+4,p''}\right)_{[\theta]},$$

where

$$\begin{aligned} \ell &= (1-\theta)\cdot 2k + \theta \cdot (2k+2) \\ \frac{1}{p} &= \frac{1-\theta}{p'} + \frac{1}{p''}. \end{aligned}$$

Therefore,

$$\begin{aligned} \|U(t,r)\|_{W^{\ell,p}\to W^{\ell,p}} &\le C_1^{1-\theta} C_2^{\theta} \le C, \\ \|U(t,r)\|_{W^{\ell,p}\to W^{\ell+2,p}} &\le \left(\frac{C_1}{t-r}\right)^{1-\theta} \left(\frac{C_2}{t-r}\right)^{\theta} \le \frac{C}{t-r}. \end{aligned}$$

□

Since our approximation is asymptotic near zero, without loss of generality, henceforth we will assume $T = 1$. We also mention that throughout this paper C is a generic constant, it may be different at different appearance.

Let us now return to the study of the initial value problem (0.3), in the literature there are several types of solutions (mild, classical, strong) for (0.3). Thus we need to clarify what we mean by a solution of (0.3).

We shall use the following notion of solution, see [18], page 123-124.

Definition 1.14. *Let $g \in \mathcal{C}([0,\infty), X)$. By a* strong solution in X *of* (0.3) *we mean a function*

$$u \in \mathcal{C}([0,\infty), X) \cap \mathcal{C}^1((0,\infty), X) \cap \mathcal{C}((0,\infty), \mathcal{D}(L(t))), \tag{1.17}$$

such that $\partial_t u(t) = L(t)u(t) + g(t)$ in X, for $t > 0$, and $u(0) = f \in X$.

We are also interested in the case that f is in a larger space, because, in concrete applications, the initial data f may be unbounded, even not L^p-integrable. An example is provided by the payoff function of a European call option. To include such cases, we therefore introduce *exponentially weighted Sobolev spaces*. Given a fixed point $z \in \mathbb{R}^N$, we set

$$\langle x\rangle_z := \langle x - z\rangle = (1 + |x-z|^2)^{1/2} \tag{1.18}$$

and define $W^{k,p}_{a,z}(\mathbb{R}^N)$ for $k \in \mathbb{Z}_+$, $a \in \mathbb{R}$, $1 < p < \infty$, by

$$\begin{aligned} W^{k,p}_{a,z} = W^{k,p}_{a,z}(\mathbb{R}^N) &:= e^{-a\langle x\rangle_z} W^{k,p}(\mathbb{R}^N) \\ &= \{u : \mathbb{R}^N \to \mathbb{C},\ \partial_x^\alpha\big(e^{a\langle x\rangle_z} u(\cdot)\big), \in L^p(\mathbb{R}^N),\ |\alpha| \le k\}, \quad \text{if } k \in \mathbb{Z}_+, \end{aligned} \tag{1.19}$$

with norm

$$\|u\|^p_{W^{k,p}_{a,z}} := \|e^{a\langle x\rangle_z} u\|^p_{W^{k,p}} = \sum_{|\alpha|\le k} \|\partial_\xi^\alpha\big(e^{a\langle x\rangle_z} u(x)\big)\|^p_{L^p}.$$

The parameter z will be called *the weight center*. The space $W^{k,p}_{a,z}$ is independent of z as a vector space, but the norm does depend on z. The reason for introducing z is to obtain uniform constants indepenent of the norms used.

Let us consider the operator $L_a(t) = e^{a<x>_z} L(t) e^{-a<x>_z}$. We notice that proving a result for $L(t)$ acting between the weighted Sobolev spaces $W^{k,p}_{a,z}$ is the same thing as proving the corresponding result for $L_a(t)$ acting between the weighted Sobolev spaces $W^{k,p} = W^{k,p}_{0,z}$. But in order to pass from the conjugated operator $L_a(t)$ to the ordinary operator $L(t)$, we require that $L(t)$ and $L_a(t)$ have the same properties.

Lemma 1.15. *If $L(t) \in \mathbb{L}_\omega$ and $a \in \mathbb{R}$, then $L_a(t) \in \mathbb{L}_\omega$.*

Proof. Suppose $u \in W^{k,p}$ and $L(t) \in \mathbb{L}_\omega$. Denote $\gamma(x) = e^{-a<x>_z}$. Then

$$
\begin{aligned}
&L_a(t)u = e^{a<x>_z} L(t) e^{-a<x>_z} u \\
&= \gamma^{-1}(x) \left[\sum a_{ij}(t,x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum b_i(t,x) \frac{\partial}{\partial x_i} + c(t,x)\right] \gamma(x) u(x) \\
&= \gamma^{-1}(x) \Big[\sum a_{ij}(t,x) \left(\gamma(x)\partial_{ij} u(x) + \partial_i \gamma(x) \partial_j u(x) + \partial_j \gamma(x) \partial_i u(x)\right. \\
&\left. + \partial_{ij}\gamma(x) u(x)\right) + \sum b_i(t,x) \left(\partial_i \gamma(x) u(x) + \gamma(x) \partial_i u(x)\right) + c(t,x)\gamma(x) u(x)\Big] \\
&= L(t)u + \gamma^{-1}(x) \left[\sum 2a_{ij}(t,x) \partial_i \gamma(x) \partial_j + \left(\sum_{i,j} \partial_i \partial_j \gamma(x)\right.\right. \\
&\left.\left. + \sum b_i(t,x) \partial_i \gamma(x)\right)\right] u(x).
\end{aligned}
$$

Notice that $L_a(t) - L(t)$ is a first order differential operator whose coefficients are smooth with all their derivatives uniformly bounded. Therefore, $L_a(t)$ satisfies all the assumptions that we make for $L(t)$. So $L_a(t) \in \mathbb{L}_\omega$. □

Therefore, by the above lemma (1.15) and our foregoing discussion, we may reduce our arguments to the case $a = 0$ and z is arbitrary. In particular, $L(t) : W^{k+2,p}_{a,z} \to W^{k,p}_{a,z}$ is well defined and continuous for any a, since this is true for $a = 0$.

Lemma 1.16. *If $L \in \mathbb{L}_\gamma$, $\gamma > 0$, and $U(t,r)$ is the resulting evolution system. Then*

$$\|U(t,r) - I\|_{W^{k+2,p}_{a,z} \to W^{k,p}_{a,z}} \le C|t-r|,$$

for a constant C independent of t, r, and z. In particular, if we fix r, then

$$[r,\infty) \ni t \to U(t,r) \in \mathcal{B}(W^{k+2,p}_{a,z}, W^{k,p}_{a,z})$$

defines a continuous operator.

Proof. Notice that if $f \in W^{k+2,p}_{a,z}$, then $\frac{\partial}{\partial_t} U(t,r)f = L(t)U(t,r)f$, and hence

$$
\begin{aligned}
\|(U(t,r) - I)f\|_{W^{k,p}_{a,z}} &\le \int_r^t \|L(\tau)U(\tau,r)f\|_{W^{k,p}_{a,z}} d\tau \\
&\le \int_r^t \|U(\tau,r)f\|_{W^{k+2,p}_{a,z}} d\tau \le C|t-r| \|f\|_{W^{k+2,p}_{a,z}},
\end{aligned}
$$

by Theorem (1.12). Then, assuming $t_1 \ge t_2 \ge r$, we obtain

$$
\begin{aligned}
\|U(t_1,r)f - U(t_2,r)f\|_{W^{k,p}_{a,z}} &\le \|(U(t_1,t_2) - I)U(t_2,r)f\|_{W^{k,p}_{a,z}} \\
&\le C|t_1 - t_2| \|U(t_2,r)f\|_{W^{k+2,p}_{a,z}} \le C|t_1 - t_2| \|f\|_{W^{k+2,p}_{a,z}}.
\end{aligned}
$$

This completes the proof of the second part. □

2. Perturbative expansions

Let us assume now that $L, L_0 \in \mathbb{L}_\gamma$, with $\gamma > 0$ fixed, and with L_0 *time independent.* We shall write

$$L(t) = L_0 + V(t). \tag{2.1}$$

In this section we study the effect of above splitting of the operator $L(t)$. More precisely, we shall investigate the classical question of relating the evolution system $U(t,s)$ generated by L to the semigroup e^{tL_0} generated by L_0 [18, 1].

2.1. **Analytic semigroups.** Let A be sectorial, more precisely, to fix notation, let us assume that the resolvent set $\rho(A)$ contains a sector $S_{\theta,\omega} = \{\lambda \in \mathbb{C} : \lambda \neq \omega, |arg(\lambda - \omega)| < \theta\}$ and that $\|R(\lambda, L(t))\| \leq \frac{M}{|\lambda-\omega|}, \forall \lambda \in S_{\theta,\omega}$. For such an operator, we can define the integral

$$e^{tA} = \frac{1}{2\pi i}\int_{\omega+\gamma_{r,\eta}} e^{t\lambda} R(\lambda, A) d\lambda, t > 0,$$

where $r > 0, \eta \in (\pi/2, \theta)$ and the integral curve

$$\gamma_{r,\eta} = \{\lambda \in \mathbb{C} : |arg(\lambda)| = \eta, |\lambda| \geq r\} \cup \{\lambda \in \mathbb{C} : |arg(\lambda)| \leq \eta, |\lambda| = r\}$$

is oriented counterclockwise. We also set $e^{0A}x = x, \forall x \in X$. Then the family of operators $\{e^{tA}\}$ is said to be the analytic semigroup generated by A.

The following proposition concerning mapping properties of the semigroup generated by L_0 is taken from [5].

Proposition 2.1. *Assume $L_0 \in \mathbb{L}_\gamma$ is time independent, as above, then e^{tL_0} is bounded on $[0,1]$ and*

$$\|e^{tL_0} f\|_{W^{r,p}_{a,z}(\mathbb{R}^N)} \leq C(r,s) t^{(s-r)/2} \|f\|_{W^{s,p}_{a,z}(\mathbb{R}^N)}, \quad r \geq s,$$

with $C(r,s)$ independent of t.

An immediate consequence of the above result is

Corollary 2.2. *Let $s, r \in \mathbb{R}$ be arbitrary and $L_0 \in \mathbb{L}_\gamma$ be time independent, as above. We then have that the map*

$$(0,\infty) \ni t \to e^{tL_0} \in \mathcal{B}(W^{s,p}_{a,z}, W^{r,p}_{a,z})$$

is infinitely many times differentiable.

Proof. Notice that $\partial_t^k e^{tL_0} = e^{tL_0} L_0^k$, so it suffices to show that the map $(0,\infty) \ni t \to e^{tL_0} \in \mathcal{B}(W^{s-2k,p}_{a,z}, W^{r,p}_{a,z})$ is continuous. Let $t \geq \delta > 0$. Then $e^{\delta L_0}$ maps $W^{s-2k,p}_{a,z}$ to $W^{r+2,p}_{a,z}$ continuously, by Proposition 2.1. Writing $e^{tL_0} = e^{(t-\delta)L_0} e^{\delta L_0}$ and using the continuity of $[\delta,\infty) \ni t \to e^{(t-\delta)L_0} \in \mathcal{B}(W^{r+2,p}_{a,z}, W^{r,p}_{a,z})$, again by proposition (2.1), we obtain the result. $\square$

2.2. **Mapping properties of $U(t,s)$.** We now proceed to develop for the evolution system $U(t,s)$, generated by $L \in \mathbb{L}_\gamma$, similar mapping properties to the mapping properties of the semigroup e^{tL_0} obtained in Proposition (2.1). We start by studying the perturbation of evolution systems. If we fix $r = 0$, our evolution system $U(t,s)$ generated by L becomes a one parameter evolution system. We donote $U(t) =$

$U(t,0)$. Recall that L_0 generates an analytic semigroup. Now consider the following equation

$$\begin{cases} \partial_t u(t,x) - L_0 u(t,x) = g(t,x) & \text{in } (0,\infty)\times\mathbb{R}^N \\ u(0,x) = f(x), & \text{on } \{0\}\times\mathbb{R}^N\,, \end{cases} \tag{2.2}$$

Then we have

Lemma 2.3. *If $g \in L^1([0,1],L^p)\bigcap\mathcal{C}((0,1],L^p)$, and $u(t,x)\in L^p$ is a classical solution to* (2.2)*, then*

$$u(t,x) = e^{tL_0}f + \int_0^t e^{(t-\tau)L_0}g(\tau)d\tau, \quad 0\le t\le 1$$

for any initial data $f\in L^p$. Moreover, assume $L(t)$ generates an evolution system $U(t,r)$ and $U(t) = U(t,0)$ is the one parameter system, then the classical L^p-solution to the equation (0.3) *is given by*

$$u(t) = U(t)f = e^{tL_0}f + \int_0^t e^{(t-\tau)L_0}V(\tau)u(\tau)d\tau \tag{2.3}$$

for any $f\in L^p$.

Proof. Define $h(\tau) = e^{(t-\tau)L_0}u(\tau)$, $0\le\tau\le t$. Since $u(t,x)$ is a classical solution, $h(\tau)$ is continuously differentiable when $\tau>0$,

$$h(0) = e^{tL_0}f, \quad h(t) = u(t),$$

and

$$h'(\tau) = -L_0e^{(t-\tau)L_0}u(\tau) + e^{(t-\tau)L_0}u'(\tau) = e^{(t-\tau)L_0}g(\tau),\ 0<\tau<t.$$

Integrating it from ϵ to $t-\epsilon$, we have

$$e^{\epsilon L_0}u(t-\epsilon) = e^{(t-\epsilon)L_0}u(\epsilon) + \int_\epsilon^{t-\epsilon} e^{(t-\tau)L_0}g(\tau)d\tau.$$

Sending ϵ to zero completes the proof of the first part. For the second part, we first assume that $f\in W^{2,p}$, since $L(t)$ generates an evolution system, suppose $u(t,x)$ is the classical solution of (0.3). Define

$$g(t) = Vu(t,x) = (L(t)-L_0)u(t,x) = u_t(t,x) - L_0U(t)f.$$

On the one hand, $U(t): W^{2,p}\to W^{2,p}$ is continuous and bounded, and $L_0: W^{2,p}\to L^p$ is bounded, so $L_0U(t)f$ is continuous and L^p-integrable. On the other hand, obviously $u_t(x)$ is continuous by definition of the solution and is also integrable. Therefore, By the result of the first part of the lemma, $u(t,x)$ has the form (2.3). For the general case when $f\in L^p$, since $W^{2,p}$ is dense in L^p, We only need to show that the right hand side of (2.3) is bounded in the L^p norm. (We make a note here that in this case the right hand side is not necessarily in $L^1((0,T),L^p)$, we shall only apply the density argument, not the first part of the Lemma.) This is true by applying the mapping properties of the semigroup e^{tL_0} (Proposition 2.1) and the evolution system $U(t)$ (Corollary 1.13), that is,

$$\begin{aligned}
& \| \int_0^t e^{(t-\tau)L_0} V(\tau) U(\tau) d\tau \|_{L^p \to L^p} \\
\leq & \int_0^t \|e^{(t-\tau)L_0}\|_{W^{-1,p},L^p} \|V(\tau)\|_{W^{1,p}\to W^{-1,p}} \|U(\tau)\|_{L^p \to W^{1,p}} d\tau \\
\leq & \int_0^t \frac{1}{\sqrt{t-\tau}} \frac{1}{\sqrt{\tau}} d\tau < \infty
\end{aligned}$$

The proof is complete. □

Lemma 2.4. *(Mapping properties of $U(t,r)$) Suppose $U(t,r)$ is the two parameter evolution system introduced before, and $k \leq r, t > 0, a \in \mathbb{R}$. Then*

$$\|U(t_1,t_2)\|_{W^{k,p}_{a,z} \to W^{r,p}_{a,z}} \leq C(t_1 - t_2)^{(k-r)/2}.$$

Proof. As discussed before, we only need to prove the case $a = 0$. We also omit p from the notation, because it is the same, hence we write $W^{k,p} = W^k$. We first assume that $k \leq r < k+2$, then by Corollary 1.13 and Proposition 2.1, starting from equation (2.3) we have

$$\begin{aligned}
& \|U(t_1,t_2)\|_{W^k \to W^r} \leq \|e^{(t_1-t_2)L_0}\|_{W^k \to W^r} \\
& + \int_0^{\frac{t_1-t_2}{2}} \|e^{(t_1-t_2-\tau)L_0}\|_{W^{k-2}\to W^r} \|V\|_{W^k \to W^{k-2}} \|U(\tau+t_2,t_2)\|_{W^k \to W^k} d\tau \\
& + \int_{\frac{t_1-t_2}{2}}^{t_1-t_2} \|e^{(t_1-t_2-\tau)L_0}\|_{W^k \to W^r} \|V\|_{W^{k+2}\to W^k} \|U(\tau+t_2,t_2)\|_{W^k \to W^{k+2}} d\tau \\
& \leq C\left((t_1-t_2)^{\frac{k-r}{2}} + \int_0^{\frac{t_1-t_2}{2}} (t_1-t_2-\tau)^{\frac{k-2-r}{2}} d\tau + \int_{\frac{t_1-t_2}{2}}^{t_1-t_2} (t_1-t_2-\tau)^{\frac{k-r}{2}} \frac{d\tau}{\tau} \right) \\
& \leq C(t_1-t_2)^{(k-r)/2}
\end{aligned}$$

that is

$$\|U(t_1,t_2)\|_{W^k \to W^r} \leq C(t_1-t_2)^{(k-r)/2}, t_1 \geq t_2 \geq 0.$$

For the general case, let $\delta = \frac{r-k}{m}$, where m is an integer and $m > \frac{r-k}{2}$. Then by our above argument, for $j = 1, 2, \cdots, m$

$$\|U(t_1 - (j-1)\frac{t_1-t_2}{m}, t_1 - j\frac{t_1-t_2}{m})\|_{W^{k+(j-1)\delta}\to W^{k+j\delta}} \leq C\left(\frac{t_1-t_2}{m}\right)^{\frac{k-r}{2m}}$$

Therefore,

$$\|U(t_1,t_2)\|_{W^k \to W^r} \leq C\left(\frac{t_1-t_2}{m}\right)^{m\frac{k-r}{2m}} = C(t_1-t_2)^{(k-r)/2},$$

where C depends on k, r, p, and a. □

In particular, consider the one parameter evolution system $U(t)$, then the meaning of Proposition 2.1 and Lemma 2.4 is that both e^{tL_0} and $U(t)$ are smoothing operators as long as $t \geq \delta > 0$, *i.e.,* they map a function from any Sobolev space to another Sobolev space continuously. Moreover, they do not decrease the regularity of this function for any $t \geq 0$. The worst case is that $t = 0$, and e^{tL_0} and $U(t)$ become the identity operator. This fact will be useful in later applications. Another consequence of Lemma (2.4) is that the Green function for any $L \in \mathbb{L}_\gamma$ exists by Schwartz Kernel Theorem, *i.e.,* there exists $\mathcal{G}_t^L(x,y) \in \mathcal{C}^\infty((0,\infty)\times\mathbb{R}^N\times\mathbb{R}^N)$ such that

$$U(t)f(x) = \int_{\mathbb{R}^N} \mathcal{G}_t^L(x,y)f(y)dy \tag{2.4}$$

and explicitly, we have

$$\mathcal{G}_t^L(x,y) =< \delta_x, U(t)\delta_y >$$

where $< \cdot,\cdot >$ is the duality pairing between $\mathcal{C}^\infty(\mathbb{R}^N)$ and compactly supported distributions $\mathcal{E}'(\mathbb{R}^N)$, δ_z is the Dirac delta distribution. As we mentioned before, one purpose of this paper is to approximate $\mathcal{G}_t^L(x,y)$.

As a consequence of Lemma (2.4), we also have the following corollary similar to Corollary (2.2)

Corollary 2.5. *If $L(t) \in \mathbb{L}_\gamma$, and $U(t,r), t \geq r \geq 0$ is the resulting two parameter evolution system, then*

$$(r,+\infty) \ni t \to U(t,r) \in \mathcal{B}(W^{s,p}_{a,z}, W^{m,p}_{a,z})$$

is infinitely many times differentiable for any s and m.

Proof. For any positive integer k, it is easy to show that formally $\partial_t^k U(t,r) = h(L(t),\partial_t L(t))U(t,r)$ where $h(L(t),\partial_t L(t))$ is a $2k$ order differential operators with smooth and bounded coefficients. For any fixed δ with $t \geq \delta > r$,$U(\delta,r)$ is a continuous map from $W^{s,p}_{a,z}$ to $W^{m+2k+2,p}_{a,z}$ by lemma (2.4). Moreover, $U(t,\delta)$ is continuous from $W^{m+2k+2,p}_{a,z}$ to $W^{m+2k,p}_{a,z}$ by lemma (1.16). Lastly, clearly $h(L(t),\partial_t L(t)) \in \mathcal{B}(W^{r+2k,p}_{a,z}, W^{r,p}_{a,z})$ is also continuous. Therefore, combining the three operators and using the definition of evolution system we conclude that $\partial_t^k U(t,r)$ is continuous from $W^{s,p}_{a,z}$ to $W^{r,p}_{a,z}$ ☐

Next we proceed to expand the operator $U(t)$ at $t = 1$. For each $k \in \mathbb{Z}_+$, we denote

$$\begin{aligned}\Sigma_k &:= \{\tau = (\tau_0,\tau_1,\ldots,\tau_k) \in \mathbb{R}^{k+1},\ \tau_j \geq 0, \sum \tau_j = 1\} \\ &\simeq \{\sigma = (\sigma_1,\ldots,\sigma_k) \in \mathbb{R}^k,\ 1 \geq \sigma_1 \geq \sigma_2 \geq \ldots \sigma_{k-1} \geq \sigma_k \geq 0\}\end{aligned}$$

the *standard unit simplex* of dimension k. The bijection above is given by $\sigma_j = \tau_j + \tau_{j+1} + \ldots + \tau_k$. Using this bijection, for any operator-valued function f of $\mathbb{R}^N$ we can write

$$\int_{\Sigma_k} f(\tau)d\tau = \int_0^1\int_0^{\sigma_1}\ldots\int_0^{\sigma_{k-1}} f(1-\sigma_1,\sigma_1-\sigma_2,\ldots,\sigma_{k-1}-\sigma_k,\sigma_k)d\sigma_k\ldots d\sigma_1$$

Throughout, operator-valued integrals are taken in the sense of Bochner.

Lemma 2.6. *Let $L_j \in \mathbb{L}_\gamma$ and let V_j be such that $e^{-b_j\langle x\rangle}V_j \in \mathbb{L}$, $j = 1,\ldots,k$, for some $b = (b_1,\ldots,b_k) \in \mathbb{R}^k_+$, $k \in \mathbb{Z}_+$. Then*

$$\Phi(\tau) = e^{\tau_0 L_0}V_1 e^{\tau_1 L_1}\ldots e^{\tau_{k-1}L_{k-1}}V_k E(\tau_k), \qquad \tau \in \Sigma_k$$

defines a continuous function $\Phi : \Sigma_k \to \mathcal{B}(W^{s,p}_{a,z}(\mathbb{R}^N), W^{r,p}_{a-|b|}(\mathbb{R}^N))$ for any $a \in \mathbb{R}$ $r \geq s$, and $1 < p < \infty$, where either $E(\tau_k) = e^{\tau_k L_k}$ or $E(\tau_k) = U(\tau_k) = U(\tau_k, 0)$.

In this lemma, we used the standard multi-index notation $|b| = \sum_{j=1}^k b_j$.

Proof. Our proof is based on the fact that both $e^{\tau L}$ and $U(\tau)$ are smoothing operators when $\tau > 0$, and they have the same type mapping properties (see Proposition 2.1 and Lemma 2.4). It suffices to prove that Φ is continuous on each of the sets $\mathcal{V}_j := \{\tau_j > 1/(k+2)\}$, $j = 0, \dots, k$, since they cover Σ_k. Let us assume that $j = 0$, for the simplicity of notation.(If $j = k$, it is slightly different. We will indicate later.)

By assumption and by Proposition 2.1 and Lemma 1.16, each of the functions

$$[0, \infty) \ni \tau_j \to V_j e^{\tau_j L_j} \in \mathcal{B}(W^{r_j+4,p}_{c_j}, W^{r_j,p}_{c_j - b_j}), \quad 1 \le j < k,$$

$$[0, \infty) \ni \tau_k \to V_k E(\tau_k) \in \mathcal{B}(W^{r_k+4,p}_{c_k}, W^{r_k,p}_{c_k - b_k})$$

is continuous. For a suitable choice of c_j and r_j (more precisely, $c_j = c_{j+1} - b_{j+1}$, $c_k = a$, $r_j = r_{j+1} - 4$, $r_k = s$), we obtain that the map

$$[0, \infty)^k \ni (\tau_j) =: \tau' \to \Psi(\tau') := V_1 e^{\tau_1 L_1} ... V_k e^{\tau_k L_k} \in \mathcal{B}(W^{s,p}_a, W^{s-4k,p}_{a-|b|})$$

is continuous.

Corollary 2.2 gives that the map $\tau_0 \to e^{\tau_0 L_0} \in \mathcal{B}(W^{s-4k,p}_{a-|b|}, W^{r,p}_{a-|b|})$ is continuous for $\tau_0 \ge 1/(k+2)$(If $j = k$, we shall use Corollary (2.5)). This proves the continuity of Φ on $\mathcal{V}_0$ and completes the proof of the lemma. $\square$

In particular,in the above lemma if $L_j = L_0, j = 1, 2, \cdots, k$, namely, a second order strongly elliptic constant-coefficient operator, and the coefficients of V_j are of polynomial growth, an immediate result of lemma (2.6) is

Corollary 2.7. *If $L(t)$ is defined by* (0.1)*, L_0 is a second order strongly elliptic constant-coefficient operator, and the coefficients of L_j are polynomials in x. Then for some $b = (b_1, \dots, b_k) \in \mathbb{R}^k_+$, $k \in \mathbb{Z}_+$,*

$$\Phi(\tau) = e^{\tau_0 L_0} L_1 e^{\tau_1 L_0} \dots e^{\tau_{k-1} L_0} L_k E(\tau_k), \qquad \tau \in \Sigma_k,$$

defines a continuous function $\Phi : \Sigma_k \to \mathcal{B}(W^{s,p}_{a,z}(\mathbb{R}^N), W^{r,p}_{a-|b|}(\mathbb{R}^N))$ for any $a \in \mathbb{R}$ $r \ge s$, and $1 < p < \infty$, where either $E(\tau_k) = e^{\tau_k L_k}$ or $E(\tau_k) = U(\tau_k)$.

Later on, in the expansion of the Green's function of $L(t)$, the operators will fit in the conditions of the above corollary.

Proposition 2.8. *Let $d \in \mathbb{Z}_+$, and $L(t)$ is split as in equation* (2.2)*, then*

$$\begin{aligned} U(1) = {} & e^{L_0} + \int_{\Sigma_1} e^{\tau_0 L_0} V(\tau_1) e^{\tau_1 L_0} d\tau \\ & + \int_{\Sigma_2} e^{\tau_0 L_0} V(\tau_1) e^{\tau_1 L_0} V(\tau_2) e^{\tau_2 L_0} d\tau + \cdots + \\ & + \int_{\Sigma_d} e^{\tau_0 L_0} V(\tau_1) e^{\tau_1 L_0} \dots e^{\tau_{d-1} L_0} V(\tau_d) e^{\tau_d L_0} d\tau \\ & + \int_{\Sigma_{d+1}} e^{\tau_0 L_0} V(\tau_1) e^{\tau_1 L_0} \dots e^{\tau_d L_0} V(\tau_{d+1}) U(\tau_{d+1}) d\tau, \end{aligned} \tag{2.5}$$

and each integral is a well-defined Bochner integral.

The positive integer d will be called the *iteration level* of the approximation. Later on, V will be replaced by a Taylor approximation of L, so that V will have polynomial coefficients in x and t.

Proof. Recall that Lemma 2.3 reads

$$U(t) - e^{tL_0} = \int_0^t e^{(t-\tau)L_0} V(\tau) U(\tau) d\tau.$$

Setting $t = 1$ gives the desired result for $d = 0$. The result for any d then follows by induction using the above formula repeatedly. Explicitly,

$$\begin{aligned}
(2.6)\quad U(1) = e^{L_0} &+ \int_{\Sigma_1} e^{(1-\sigma_1)L_0} V(\sigma_1) e^{\sigma_1 L_0} d\sigma \\
&+ \int_{\Sigma_2} e^{(1-\sigma_1)L_0} V(\sigma_1) e^{(\sigma_1-\sigma_2)L_0} V(\sigma_2) e^{\sigma_2 L_0} d\sigma \\
&+ \cdots + \int_{\Sigma_{d-1}} e^{(1-\sigma_1)L_0} V(\sigma_1) \ldots e^{(\sigma_{d-2}-\sigma_{d-1})L_0} V(\sigma_{d-1}) U(\sigma_{d-1}) d\sigma \\
= e^{L_0} &+ \int_{\Sigma_1} e^{(1-\sigma_1)L_0} V(\sigma_1) e^{\sigma_1 L_0} d\sigma + \int_{\Sigma_2} e^{(1-\sigma_1)L_0} V(\sigma_1) e^{(\sigma_1-\sigma_2)L_0} V(\sigma_2) e^{\sigma_2 L_0} d\sigma \\
&+ \cdots + \int_{\Sigma_{d-1}} e^{(1-\sigma_1)L_0} V(\sigma_1) \ldots V(\sigma_{d-1}) e^{\sigma_{d-1} L_0} d\sigma \\
&+ \int_{\Sigma_{d-1}} \int_0^{\sigma_{d-1}} e^{(1-\sigma_1)L_0} V(\sigma_1) \ldots V(\sigma_{d-1}) e^{(\sigma_{d-1}-\sigma_d)L_0} V(\sigma_d) U(\sigma_d) d\sigma d\sigma_d \\
= e^{L_0} &+ \int_{\Sigma_1} e^{(1-\sigma_1)L_0} V(\sigma_1) e^{\sigma_1 L_0} d\sigma + \int_{\Sigma_2} e^{(1-\sigma_1)L_0} V(\sigma_1) e^{(\sigma_1-\sigma_2)L_0} V(\sigma_2) e^{\sigma_2 L_0} d\sigma \\
&+ \cdots + \int_{\Sigma_d} e^{(1-\sigma_1)L_0} V(\sigma_1) e^{(\sigma_1-\sigma_2)L_0} \ldots e^{(\sigma_{d-1}-\sigma_d)L_0} V(\sigma_d) U(\sigma_d) d\sigma,
\end{aligned}$$

where each integral is well defined as a Bochner integral by the Lemma. □

3. Dilation of the operator

For any function $v(t,x)$ and $f(x)$, we choose an arbitrary but fixed basepoint z and dilate them in the following sense

$$\begin{aligned}
v^s(t,x) &= v(s^2 t, z + s(x-z)) \\
f^s(x) &= f(z + s(x-z)).
\end{aligned}$$

For the operator L, we set

$$(3.1)\quad L^s = \sum a_{ij}^s(s^2 t, z + s(x-z)) \partial_i \partial_j + s \sum b_i^s(s^2 t, z + s(x-z)) \partial_i + s^2 c^s(s^2 t, z + s(x-z)).$$

It is not difficult to show that if $u(t,x)$ is a solution of the equation (0.3), then $u^s(t,x)$ is a solution of the following equation

$$(3.2)\quad \begin{cases} \partial_t u^s(t,x) - L^s u^s(t,x) = 0 & \text{in } (0,\infty) \times \mathbb{R}^N \\ u^s(0,x) = f^s(x), \quad f \in \mathcal{C}^\infty(\mathbb{R}^n) & \text{on } \{0\} \times \mathbb{R}^N, \end{cases}$$

Clearly, L^s satisfies all the conditions we assumed above. Denote the evolution system generated by L^s by $U^{L^s}(t)$. Also, let $\mathcal{G}_t^L(x,y)$ and $\mathcal{G}_t^{L^s}(x,y)$ be the Green functions or fundamental solutions for the operator $\partial_t - L$ and $\partial_t - L^s$ respectively. We want to relate $\mathcal{G}_t^L(x,y)$ and $\mathcal{G}_t^{L^s}(x,y)$.

Lemma 3.1. *Let z be a fixed but arbitrary point in $\mathbb{R}^N$. Then for any $s > 0$, we have*

$$\mathcal{G}_t^L(x,y) = s^{-N}\mathcal{G}_{s^{-2}t}^{L^s}(z + \frac{x-z}{s}, z + \frac{y-z}{s}).$$

In particular, when $s = \sqrt{t}$,

$$\mathcal{G}_t^L(x,y) = t^{-N/2}\mathcal{G}_1^{L^{\sqrt{t}}}(z + \frac{x-z}{\sqrt{t}}, z + \frac{y-z}{\sqrt{t}}). \tag{3.3}$$

Proof. Without loss of generality, we assume $z = 0$. On one hand, by definition of Green's function,

$$\begin{aligned} u^s(t,x) &= \int_{\mathbb{R}^N} \mathcal{G}_t^{L^s}(x,y)f^s(y)dy = \int_{\mathbb{R}^N} \mathcal{G}_t^{L^s}(x,y)f(sy)dy \\ &= s^{-N}\int_{\mathbb{R}^N} \mathcal{G}_t^{L^s}(x,\frac{y}{s})f(y)dy \end{aligned}$$

On the other hand,

$$u^s(t,x) = u(s^2t, sx) = \int_{\mathbb{R}^N} \mathcal{G}_{s^2t}^L(sx,y)f(y)dy.$$

Therefore,

$$s^{-N}\int_{\mathbb{R}^N} \mathcal{G}_t^{L^s}(x,\frac{y}{s})f(y)dy = \int_{\mathbb{R}^N} \mathcal{G}_{s^2t}^L(sx,y)f(y)dy,$$

which implies,

$$s^{-N}\mathcal{G}_t^{L^s}(x,\frac{y}{s}) = \mathcal{G}_{s^2t}^L(sx,y).$$

After a change of variable, we get

$$\mathcal{G}_t^L(x,y) = s^{-N}\mathcal{G}_{s^{-2}t}^{L^s}(\frac{x}{s}\frac{y}{s}).$$

$\square$

With this lemma in hand, in order to approximate $\mathcal{G}_t^L(x,y)$, it suffices to approximate $\mathcal{G}_1^{L^{\sqrt{t}}}(x,y)$. We shall apply the perturbation technique illustrated as follows.

3.1. **Formal expansion of the operator L^s.** Suppose L^s is given by (3.1). Then we Taylor-expand it with respect to the parameter s to obtain

$$L^s = L_0 + \sum_{m=1}^{n} s^m L_m + V_{n+1}^s,$$

where

$$L_m = \frac{1}{m!}\left(\frac{d^m}{ds^m}L^s\right)\bigg|_{s=0}. \tag{3.4}$$

The operators L_m are independent of s. However, V_{n+1}^s does depend on s, and all of the terms depend on z even it does not appear in the notation. We also use another way to denote V_{n+1}^s, that is,

$$V_{n+1}^s = s^{n+1}L_{n+1}^{s,z}.$$

We shall look for a general formula for L_m. For a function $f(t,x)$ smooth enough, we can Taylor expand it around $(0,z)$ as

$$f(t,x)=\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\frac{t^l(x-z)^k}{l!k!}\frac{\partial^l}{\partial t^l}\frac{\partial^k}{\partial x^k}f(0,z)$$

Therefore,

$$f(s^2t,z+s(x-z))=\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\frac{(s^2t)^l s^k(x-z)^k}{l!k!}\frac{\partial^l}{\partial t^l}\frac{\partial^k}{\partial x^k}f(0,z)$$

$$\frac{1}{m!}\frac{d^m}{ds^m}f(s^2t,z+s(x-z))\Big|_{s=0}=\sum_{l=0}^{[\frac{m}{2}]}\frac{t^l(x-z)^{m-2l}}{l!(m-2l)!}\frac{\partial^l}{\partial t^l}\frac{\partial^{m-2l}}{\partial x^{m-2l}}f(0,z) \tag{3.5}$$

Combine this with (3.4), we can explicitly write L_m as

$$\begin{aligned} L_m=&\sum_{i,j=1}^{d}\sum_{l=0}^{[\frac{m}{2}]}\frac{t^l(x-z)^{m-2l}}{l!(m-2l)!}\frac{\partial^l}{\partial t^l}\frac{\partial^{m-2l}}{\partial x^{m-2l}}a_{ij}(0,z)\frac{\partial^2}{\partial x_i\partial x_j}\\ &+\sum_{i}^{d}\sum_{l=0}^{[\frac{m-1}{2}]}\frac{t^l(x-z)^{m-1-2l}}{l!(m-1-2l)!}\frac{\partial^l}{\partial t^l}\frac{\partial^{m-1-2l}}{\partial x^{m-1-2l}}b_i(0,z)\frac{\partial}{\partial x_i}\\ &+\sum_{l=0}^{[\frac{m-2}{2}]}\frac{t^l(x-z)^{m-2-2l}}{l!(m-2-2l)!}\frac{\partial^l}{\partial t^l}\frac{\partial^{m-2-2l}}{\partial x^{m-2-2l}}c(0,z)\end{aligned} \tag{3.6}$$

where $m\geq 2$. So L_m is a second order differential operator with polynomial coefficients with degree at most m with respect to $x-z$ and $[\frac{m}{2}]$ with respect to t. We can write the first few terms in the Taylor expansion explicitly,

$$\begin{aligned} L_0 &= \sum_{i,j=1}^{d}a_{ij}(0,z)\frac{\partial^2}{\partial x_i\partial x_j}\\ L_1 &= \sum_{i,j=1}^{d}(x-z)\nabla a_{i,j}(0,z)\frac{\partial^2}{\partial x_i\partial x_j}+\sum_{i=1}^{d}b_i(0,z)\frac{\partial}{\partial x_i}\\ L_2 &= \sum_{i,j=1}^{d}\left(\frac{1}{2}(x-z)^T\nabla^2a_{ij}(0,z)(x-z)+t\partial_t a_{ij}(0,z)\right)\frac{\partial^2}{\partial x_i\partial x_j}\\ &\quad+\sum_{i=1}^{d}((x-z)\nabla b_i(0,z))\frac{\partial}{\partial x_i}+c(0,z)\end{aligned}$$

Remark 3.2. Clearly, L_0 is an operator with constant coefficients. By our assumption (0.2), L_0 generates an analytic semigroup, explicitly, we have

$$e^{tL_0}=\frac{1}{\sqrt{(4\pi t)^N\det(A^0)}}e^{-\frac{(x-y)^t(A^0)^{-1}(x-y)}{4t}}, \tag{3.7}$$

where $A^0:=A(0,z)=[a_{ij}(0,z)]$ and N is the dimension.

3.2. **Asymptotic expansion of the evolution system.** Recall proposition 2.8, we want to explicitly express $U(t)$ in a nice way, and now $L^s - L_0 = \sum_{m=1}^{n} s^m L_m + V_{n+1}^s$ will serve the role as V does in proposition 2.8, and L_0 is what we introduced above.

To compute the integrals in the above lemma, we need

Lemma 3.3. *(Baker-Campbell-Hausdorf formula) A and B are two operators, then*

$$[e^A, B] = \left([A, B] + \frac{1}{2!}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, B]]] + \dots \right) e^A. \tag{3.8}$$

In general this formula is an infinite series. But in our later application, it will be a finite series. This formula tells us how to commute e^A and B:

$$e^A B = \left(B + [A, B] + \frac{1}{2!}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, B]]] + \dots \right) e^A.$$

If we apply this repeatedly in proposition 2.8, we can reduce the integrals.

Now let's give some definitions as those in [5].

Definition 3.4 (Spaces of Differentials)**.** *For any nonnegative integers a, b we denote by $\mathcal{D}(a, b)$ the vector space of all differentiations of degree at most a and order at most b. We extend this definition to negative indices by defining $\mathcal{D}(a, b) = \{0\}$ if either a or b is negative. By degree of A we mean the highest power of the polynomials appearing as coefficients in A.*

Definition 3.5 (Adjoint Representation)**.** *For any two differentiations $A_1 \in \mathcal{D}(a_1, b_1)$ and $A_2 \in \mathcal{D}(a_2, b_2)$ we define $\mathrm{ad}_{A_1}(A_2)$ by*

$$\mathrm{ad}_{A_1}(A_2) := [A_1, A_2] = A_1 A_2 - A_2 A_1 \tag{3.9}$$

and for any integer $j \geq 1$ we define $\mathrm{ad}^j_{A_1}(A_2)$ recursively by

$$\mathrm{ad}^j_{A_1}(A_2) := \mathrm{ad}_{A_1}(\mathrm{ad}^{j-1}_{A_1}(A_2)) \tag{3.10}$$

Proposition 3.6. *Suppose $A_1 \in \mathcal{D}(a_1, b_1)$ and $A_2 \in \mathcal{D}(a_2, b_2)$. Then for any integer $k \geq 1$,*

$$\mathrm{ad}^k_{A_1}(A_2) \in \mathcal{D}(k(a_1 - 1) + a_2, k(b_1 - 1) + b_2). \tag{3.11}$$

Proof. We first notice that

$$\mathrm{ad}_{A_1}(A_2) \in \mathcal{D}(a_1 - 1 + a_2, b_1 - 1 + b_2). \tag{3.12}$$

Next, from (3.10) we have

$$\mathrm{ad}^k_{A_1}(A_2) = \mathrm{ad}_{A_1}(\mathrm{ad}_{A_1}(\mathrm{ad}_{A_1}(\mathrm{ad}_{A_1}(\dots)))) \tag{3.13}$$

so that an application of (3.12) k times yields the result. $\square$

Lemma 3.7. *Let m, k be fixed integers ≥ 1. Let $L^z_0 \in \mathcal{D}(0, 2)$ be the constant coefficient operator and $L^z_m \in \mathcal{D}(m, 2)$ be the operator given above, Then,*

$$\mathrm{ad}^k_{L_0}(L_m) \in \mathcal{D}(m - k, k + 2). \tag{3.14}$$

In particular,

$$\mathrm{ad}^k_{L_0}(L_m) = 0 \quad \forall k > m. \tag{3.15}$$

Proof. Applying Lemma 3.7 we see that $\mathrm{ad}^k_{L_0^z}(L^z_m) \in \mathcal{D}(m-k, m+2)$. If $k > m$, then by definition $\mathcal{D}(m-k, m+2) = \{0\}$ and we obtain (3.15). $\square$

Lemma 3.8. *Let L_0 and L_m be defined above, then for any $\theta \in (0,1)$,*

$$e^{(1-\theta)L_0} L_m(\theta) = P_m(\theta, x-z, \partial) e^{(1-\theta)L_0}$$

where

$$P_m(\theta, x-z, \partial) := L_m(\theta) + \sum_{i=1}^{m} \frac{(1-\theta)^i}{i!} \mathrm{ad}^i_{L_0}(L_m(\theta)) \in \mathcal{D}(m, m+2)$$

is a finite sum of terms with the form $a(z)(1-\theta)^i \theta^j (x-z)^k \partial_x^\alpha$, in which $a(z)$ and all its derivatives are bounded, α is a multi-index.

Proof. Setting $A = (1-\theta)L_0$ and $B = L_m(\theta)$ in Baker-Campbell-Hausdorff formula yields the results. $\square$

Next, we shall rewrite equation (2.6) in a more computable and explicit form. In abuse of notations, it is convenient to write $L^{s,z}_{n+1} = L_{n+1}$. Recall that $L_m = L_m(t)$ is a function of t, thus so is V. Plug $V = \sum_{m=1}^{n+1} s^m L_m(t)$ into (2.6) and expand it, we obtain

(3.16)

$$\begin{aligned}
U(1) = {} & e^{L_0} + \sum_{k=1}^{d} \sum_{1 \le \alpha_i \le n+1} \int_{\Sigma_k} e^{(1-\sigma_1)L_0} s^{\alpha_1} L_{\alpha_1}(\sigma_1) e^{(\sigma_1-\sigma_2)L_0} \dots e^{(\sigma_{k-1}-\sigma_k)L_0} s^{\alpha_k} L_{\alpha_k}(\sigma_k) e^{\sigma_k L_0} d\sigma \\
& + \sum_{1 \le \alpha_i \le n+1} \int_{\Sigma_{d+1}} e^{(1-\sigma_1)L_0} s^{\alpha_1} L_{\alpha_1}(\sigma_1) e^{(\sigma_1-\sigma_2)L_0} \dots e^{(\sigma_d-\sigma_{d+1})L_0} s^{\alpha_{d+1}} L_{\alpha_{d+1}}(\sigma_{d+1}) U(\tau_{d+1}) d\sigma, \\
= {} & e^{L_0} + \sum_{k=1}^{d} \sum_{1 \le \alpha_i \le n+1} s^{\alpha_1 + \dots + \alpha_k} \int_{\Sigma_k} e^{(1-\sigma_1)L_0} L_{\alpha_1}(\sigma_1) e^{(\sigma_1-\sigma_2)L_0} \dots e^{(\sigma_{k-1}-\sigma_k)L_0} L_{\alpha_k}(\sigma_k) e^{\sigma_k L_0} d\sigma \\
& + \sum_{1 \le \alpha_i \le n+1} s^{\alpha_1 + \dots + \alpha_{d+1}} \int_{\Sigma_{d+1}} e^{(1-\sigma_1)L_0} L_{\alpha_1}(\sigma_1) e^{(\sigma_1-\sigma_2)L_0} \dots e^{(\sigma_d-\sigma_{d+1})L_0} L_{\alpha_{d+1}}(\sigma_{d+1}) U(\sigma_{d+1}) d\sigma,
\end{aligned}$$

To simplify the above formula, we first introduce the notations as follows

Definition 3.9. *For any integers $1 \le k \le d+1$ and ℓ, we shall denote by $\mathfrak{A}_{k,\ell}$ the set of multi-indexes $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k) \in \mathbb{N}^k$, such that $|\alpha| := \sum \alpha_j = \ell$. Furthermore, we denote $\mathfrak{A}_\ell := \bigcup_{k=1}^{\ell} \mathfrak{A}_{k,\ell}$. For symmetry, it will be convenient to set $\mathfrak{A}_0 = \{\emptyset\}$.*

Clearly, since $\alpha_i \ge 1$, the set $\mathfrak{A}_{k,\ell}$ is empty if $\ell < k$. The meaning of ℓ is that of the corresponding power of s and the meaning of k is that of the expansion order. For each $\alpha \in \mathfrak{A}_{k,\ell}$, we denote if $k < d+1$

$$\Lambda_{\alpha,z} \quad = \quad \int_{\Sigma_k} e^{(1-\sigma_1)L_0} L_{\alpha_1}(\sigma_1) e^{(\sigma_1-\sigma_2)L_0} \dots e^{(\sigma_{k-1}-\sigma_k)L_0} L_{\alpha_k}(\sigma_k) e^{\sigma_k L_0} d\sigma$$

if $k = d+1$,

$$\begin{aligned}
\Lambda_{\alpha,z} &= \int_{\Sigma_{d+1}} e^{(1-\sigma_1)L_0} L_{\alpha_1}(\sigma_1) e^{(\sigma_1-\sigma_2)L_0} \dots e^{(\sigma_d-\sigma_{d+1})L_0} L_{\alpha_{d+1}}(\sigma_{d+1}) U(\sigma_{d+1}) d\sigma \\
&= \int_{\Sigma_{d+1}} P_{\alpha_1}(\sigma_1, x-z, \partial) e^{(1-\sigma_2)L_0} \cdots e^{(\sigma_d-\sigma_{d+1})L_0} L_{\alpha_{d+1}}(\sigma_{d+1}) U(\sigma_{d+1}) d\sigma \\
&= \cdots\cdots \\
&= \int_{\Sigma_{d+1}} P_{\alpha_1}(\sigma_1, x-z, \partial) \cdots P_{\alpha_{d+1}}(\sigma_d, x-z, \partial) e^{(1-\sigma_{d+1})L_0} U(\sigma_{d+1}) d\sigma
\end{aligned}$$

A simple but useful lemma about $\Lambda_{\alpha,z}$ is the following, which we record for later use

Lemma 3.10. *For any given multi-index $\alpha \in \mathfrak{A}_{k,\ell}$ with $k \le d$, then*

$$\Lambda_{\alpha,z} = \mathcal{P}_\alpha(x, z, \partial) e^{L_0}$$

where the product is the composition of operators and $\mathcal{P}_\alpha(x, z, \partial)$ is a differential operator of order $2k+\ell$ and polynomial degree$\le \ell = |\alpha|$. More precisely, we have

$$\begin{aligned}
\mathcal{P}_\alpha(x, z, \partial) &= \int_{\Sigma_k} P_{\alpha_1}(\sigma_1, x-z, \partial) P_{\alpha_2}(\sigma_2, x-z, \partial) \cdots P_{\alpha_k}(\sigma_k, x-z, \partial) d\sigma \\
&= \sum_{|\beta|\le \ell} \sum_{|\gamma| \le \ell+2k} a_{\beta,\gamma}(z)(x-z)^\beta \partial_x^\gamma
\end{aligned} \tag{3.17}$$

where $a_{\beta,\gamma}(z) \in \mathcal{C}_b^\infty(\mathbb{R})$.

Proof. Applying Lemma (3.8) repeatedly, we have

$$\begin{aligned}
\Lambda_{\alpha,z} &= \int_{\Sigma_k} e^{(1-\sigma_1)L_0} L_{\alpha_1}(\sigma_1) e^{(\sigma_1-\sigma_2)L_0} \dots e^{(\sigma_{k-1}-\sigma_k)L_0} L_{\alpha_k}(\sigma_k) e^{\sigma_k L_0} d\sigma \\
&= \int_{\Sigma_k} P_{\alpha_1}(\sigma_1, x-z, \partial) e^{(1-\sigma_2)L_0} \cdots e^{(\sigma_{k-1}-\sigma_k)L_0} L_{\alpha_k}(\sigma_k) e^{\sigma_k L_0} d\sigma \\
&= \cdots\cdots \\
&= \int_{\Sigma_k} P_{\alpha_1}(\sigma_1, x-z, \partial) P_{\alpha_2}(\sigma_2, x-z, \partial) \cdots P_{\alpha_k}(\sigma_k, x-z, \partial) e^{L_0} d\sigma \\
&= \left(\int_{\Sigma_k} P_{\alpha_1}(\sigma_1, x-z, \partial) P_{\alpha_2}(\sigma_2, x-z, \partial) \cdots P_{\alpha_k}(\sigma_k, x-z, \partial) d\sigma \right) e^{L_0}.
\end{aligned}$$

where $\sigma_j = \tau_j + \tau_{j+1} + \cdots + \tau_k$. By Lemma (3.8) and Lemma (3.7), we know that each operator $P_{\alpha_i}(\sigma_i, x-z, \partial) \in \mathcal{D}(\alpha_i, \alpha_i+2), i = 1, 2, \cdots, k$. Thus $\mathcal{P}_\alpha(x, z, \partial) \in \mathcal{D}(|\alpha|, |\alpha|+2k) = \mathcal{D}(\ell, \ell+2k)$. Also notice that each $P_{\alpha_i}(\sigma_i, x-z, \partial)$ has polynomial coefficients in σ_i, so the integration with respect to σ will be exact, and $\mathcal{P}_\alpha(x, z, \partial)$ is of the desired form. The proof is complete. □

We also set

$$\Lambda_z^{k,\ell} = \sum_{\alpha \in \mathfrak{A}_{k,\ell}} \Lambda_{\alpha,z} \tag{3.18}$$

and

$$\Lambda_z^\ell = \sum_{k=1}^{\min(\ell,d+1)} \Lambda_z^{k,\ell} \tag{3.19}$$

For convenience, let $\Lambda_z^0 = e^{L_0}$.

Now we record the above calculation as the following main theorem of this section

Theorem 3.11. *Let $M = (d+1)(n+1)$. Suppose $U(t)$ is the one parameter evolution system, and $d \geq n$, then it has the expansion*

$$U(1) = e^{L_0} + \sum_{\ell=1}^{m} s^\ell \Lambda_z^\ell + s^{m+1} E_{d,n}^{s,z}, \tag{3.20}$$

where $E_{d,n}^{s,z} = \sum_{\ell=m+1}^{M} s^{\ell-m-1}\Lambda_z^\ell$ is the error term. Recall that d is the iteration level and n is the expansion order of $L(t)$.

Proof. The proof is straightforward. Rewrite (3.16) with the above notations, it becomes

$$U(1) = \sum_{\ell=0}^{M} \sum_{k=1}^{\min(d+1,\ell)} \sum_{\alpha \in \mathfrak{A}_{k,\ell}} s^\ell \Lambda_{\alpha,z}. \tag{3.21}$$

Picking up the terms with powers of s less than $m+1$, and putting all the other higher order terms in the last term $E_{d,n}^{s,z}$ completes the proof. $\square$

Remark 3.12. If $\ell \leq n$, we mention that $\Lambda_{\alpha,z}$ and Λ_z^ℓ are both independent of d, the iteration level, as long as $d \geq n$. Moreover, Λ_z^ℓ is independent of n also as long as $n \geq \max(\alpha_i)$. These facts will be useful in our error analysis. If these conditions are satisfied, $E_{d,n}^{s,z}$ will depend only on m, and then we will write also $E_m^{s,z} = E_{d,n}^{s,z}$.

4. Error Analysis

In this section we shall mainly apply the pseudodifferential operator techniques to justify that our approximation yields accurate solution to arbitrary prescribed order in time. For all relevant properties of pseudodifferential operators, we refer to [26]. We start from the operator L_m in the expansion (3.4). As we mentioned before the differential operators L_m, $0 \leq m \leq n+1$, are second order differential operator with polynomial coefficients. Moreover, L_m has coefficients of degree at most m in $x - z$. An immediate consequence of this fact is recorded in the following lemma.

Lemma 4.1. *The family*

$$\{\langle x\rangle_z^{-j} L_j^z,\ \langle x\rangle_z^{-n-1} L_{n+1}^{s,z};\ s \in (0,1],\ z \in \mathbb{R}^N,\ j = 0, \ldots, n+1\}$$

defines a bounded subset of $\mathbb{L}$.

Recall that for convenience, we also denote $L_{n+1}^{s,z}$ by L_{n+1}^z, which actually depends on s and the dilation center z as well.

Lemma 4.2. *For each given $\epsilon > 0$, the family*

$$\{e^{-\epsilon < z-w >} e^{-\epsilon\langle x\rangle_w} L_j^z,\ s \in (0,1],\ z \in \mathbb{R}^N,\ j = 0, \ldots, n+1\}$$

is a bounded subset of $\mathbb{L}$.

Proof. If $w = z$, then the desired result follows directly from Lemma 4.1 and the simple observation that $\langle x\rangle_z^j e^{-\epsilon\langle x\rangle_z} \leq C$, with C independent of z and j.

If $w \neq z$, then

$$
\begin{aligned}
&< x - z > - < x - w > = \sqrt{1 + |x - z|^2} - \sqrt{1 + |x - w|^2} \\
&= \frac{(|x - z| - |x - w|)(|x - z| + |x - w|)}{\sqrt{1 + |x - z|^2} + \sqrt{1 + |x - w|^2}} \qquad (4.1)\\
&\leq |w - z| \leq < w - z > \quad \text{(triangle inequality)}.
\end{aligned}
$$

Therefore $e^{\epsilon(<x-z>-<x-w>-<w-z>)} \leq 1$, and the family

$$
e^{\epsilon(<x-z>-<x-w>-<w-z>)} e^{-\epsilon<x>_z} L_j^z = e^{-\epsilon<z-w>} e^{-\epsilon\langle x\rangle_w} L_j^z
$$

is bounded for $s \in (0, 1]$ and $j = 0, 1, 2, \cdots, n + 1$ as claimed. $\square$

Lemma 2.6 and lemma (4.2) then give

Corollary 4.3. *For any $\alpha_1, \alpha_2, \cdots, \alpha_k$ with $\sum_{i=1}^k \alpha_i = \ell$, the operators*

$$
\Lambda_{\alpha,\ell} = \int_{\Sigma_k} e^{\tau_0 L_0} L_{\alpha_1}(\tau_1) e^{\tau_1 L_0} \cdots e^{\tau_{k-1} L_0} L_{\alpha_k}(\tau_k) e^{\tau_k L_0} d\tau, \quad k \leq d
$$

and

$$
\Lambda_{\alpha,\ell} = \int_{\Sigma_{d+1}} e^{\tau_0 L_0} L_{\alpha_1}(\tau_1) e^{\tau_1 L_0} \cdots e^{\tau_d L_0} L_{\alpha_{d+1}}(\tau_{d+1}) U(\tau_{d+1}) d\tau
$$

are bounded linear operators from $W^{s,p}_{a,z}$ to $W^{r,p}_{a-\epsilon}$ for any $z \in \mathbb{R}^N$, $r, s \in \mathbb{R}$, $1 < p < \infty$, and $\epsilon > 0$. Moreover, we have that

$$
\|\Lambda_{\alpha,\ell}\|_{W^{s,p}_{a,z} \to W^{r,p}_{a-\epsilon}} \leq C_{s,r,p,a,\epsilon} e^{k\epsilon<z-w>},
$$

for a constant $C_{s,r,p,a,\epsilon}$ that does not depend on z. In particular, each $\Lambda_{\alpha,\ell}$ is an operator with smooth kernel $\Lambda_{\alpha,\ell}(x, y)$.

Therefore, the above corollary gives

$$
\Lambda_{\alpha,\ell} f(x) = \int_{\mathbb{R}^N} \Lambda_{\alpha,\ell}(x, y) f(y) dy. \qquad (4.2)
$$

From now on, we shall denote by $T(x, y)$ the kernel of an operator T with smooth kernel. Then in terms of kernels, theorem (3.11) becomes

$$
U(1)(x, y) = e^{L_0}(x, y) + \sum_{\ell=1}^m s^\ell \Lambda_z^\ell(x, y) + s^{m+1} E^{s,z}_{d,n}(x, y)
$$

By lemma (3.1), if we do the substitution $x = z + s^{-1}(x - z)$ and $y = z + s^{-1}(y - z)$ in the above equation, we have

$$
\begin{aligned}
\mathcal{G}_t^L(x, y) &= s^{-N} \Big(e^{L_0}(z + s^{-1}(x - z), z + s^{-1}(y - z)) \\
&+ \sum_{\ell=1}^m s^\ell \Lambda_z^\ell(z + s^{-1}(x - z), z + s^{-1}(y - z)) \qquad (4.3)\\
&+ s^{m+1} E^{s,z}_{d,n}(z + s^{-1}(x - z), z + s^{-1}(y - z)) \Big) \\
&=: \mathcal{G}_t^{[m,z]}(x, y) + s^{m+1} E^{t,z}_{d,n}(x, y),
\end{aligned}
$$

where $s = \sqrt{t}$, and recall $\mathcal{G}_t^L(x, y)$ is the Green function of the operator $\partial_t - L(t)$. We can compute $\mathcal{G}_t^{[m,z]}(x, y)$ explicitly, then

$$
s^{m+1} E^{t,z}_{d,n}(x, y) = \mathcal{G}_t^L(x, y) - \mathcal{G}_t^{[m,z]}(x, y)
$$

is the error term which we need to bound. We define the error operator as

$$\mathcal{E}_{t,d,n}^{[m,z]} f(x) = \int E_{d,n}^{t,z}(x,y) f(y) dy \tag{4.4}$$

In abuse of notations, if $\mathcal{G}_t^{[m,z]}$ denotes the operator with kernel $\mathcal{G}_t^{[m,z]}(x,y)$, then

$$U(t,0) = \mathcal{G}_t^{[m,z]} + s^{m+1} \mathcal{E}_{t,d,n}^{[m,z]}.$$

For d and n large, $\mathcal{E}_{t,d,n}^{[m,z]}$ is independent of d and n, so from now on we shall drop d and n from the notation and write simply $\mathcal{E}_t^{[m,z]}$.

By the definition of the error operator (4.4) and equation (3.21), we have

$$\mathcal{E}_t^{[m,z]} = \sum_{\ell=m+1}^{M} \sum_{k=1}^{\min(d+1,\ell)} \sum_{\alpha \in \mathfrak{A}_{k,\ell}} s^{\ell-m-1} \Lambda_{\alpha,z} \tag{4.5}$$

Later on we shall estimate the error by splitting $\mathcal{E}_t^{[m,z]}$ into two parts, namely

$$\mathcal{E}_t^{[m,z]} = \sum_{\ell=m+1}^{h} \sum_{k=1}^{\ell} \sum_{\alpha \in \mathfrak{A}_{k,\ell}} s^{\ell-m-1} \Lambda_{\alpha,z} + s^{h-m} \mathcal{E}_t^{[h,z]}. \tag{4.6}$$

In all the above formulas, we do not specify the dilation center z, which in general may be a function of x and y: $z = z(x,y)$. For our error analysis, we need to specify the dilation center

Definition 4.4. *A function $z : \mathbb{R}^{2N} \to \mathbb{R}^N$ will be called* admissible *if*

(i) $z(x,x) = x$, for all $x \in \mathbb{R}^N$.

(ii) All derivatives of z are bounded.

A typical example is $z(x,y) = \lambda x + (1-\lambda)y$, for some fixed parameter λ. A simple application of the mean value theorem gives that $\langle z - x \rangle \leq C \langle y - x \rangle$ for some constant $C > 0$. From the application point of view, $z(x,y) = x$ will give us the simplest formula to approximate the Green function [3]. However, as discussed in that paper, this may not be the best choice

4.1. **Bounds for the principle term.** In this subsection, we consider the desired term

$$\mathcal{G}_t^{[m,z]}(x,y) = \sum_{\ell=0}^{m} s^{\ell} \Lambda_z^{\ell}(z + s^{-1}(x-z), z + s^{-1}(y-z))$$

and we shall fix the function $z = z(x,y)$ which is admissible. Recall that

$$\Lambda_z^{\ell} = \sum_{k=1}^{\min(d+1,\ell)} \sum_{\alpha \in \mathfrak{A}_{k,\ell}} s^{\ell} \Lambda_{\alpha,z}$$

We treat each operator $\Lambda_{\alpha,z}$ in one time. Define the operator

$$\mathcal{L}_{s,\alpha} f(x) = s^{-N} \int_{\mathbb{R}^N} \Lambda_{\alpha,z}(z + s^{-1}(x-z), z + s^{-1}(y-z)) f(y) dy, \tag{4.7}$$

We will show below that for an admissible function z, and $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathfrak{A}_{k,\ell}$, $k \leq n$, $\alpha_i \leq n$, the operator $\mathcal{L}_{s,\alpha}$ is a pseudodifferential operator whose symbol is well behaved. We shall then use symbol calculus to derive the desired error estimates. Let's first recall some standard definitions and results from pseudodifferential calculus. Let $m \in \mathbb{R}$. We define $S_{1,0}^m$ to be the set of all functions $\sigma(x,\xi)$

in $\mathcal{C}^\infty(\mathbb{R}^N \times \mathbb{R}^N)$ such that for any two multi-indices α and β, there is positive constant $C_{\alpha,\beta}$, depending on α and β only, such that

$$\left|\left(D_x^\alpha D_\xi^\beta \sigma(x,\xi)\right)\right| \leq C_{\alpha,\beta}(1+|\xi|)^{m-|\beta|}$$

then we call any function σ in $\bigcup_{m\in\mathbb{R}} S^m_{1,0}$ a symbol, and we denote $S^{-\infty} = \bigcap_{m\in\mathbb{R}} S^m_{1,0}$. Any operator whose symbol in $S^{-\infty}$ is a smoothing operator. Now if $\sigma(x,\xi)$ is a symbol. Then the pseudodifferential operator $\sigma(x,D)$ associated to $\sigma(x,\xi)$ is defined by

$$(\sigma(x,D)\psi)(x) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{ix\cdot\xi}\sigma(x,\xi)\hat{\psi}(\xi)d\xi$$

where $D = \frac{1}{i}\partial$ and

$$\mathcal{F}\psi(x) = \hat{\psi}(\xi) := \int_{\mathbb{R}^N} e^{-\imath\xi\cdot x}\psi(x)dx \tag{4.8}$$

the usual Fourier transform of ψ. Next let's relate the operator $\sigma(x,D)$ with its distributional kernel, actually we can recover one from the other under some conditions. Denote by $\mathcal{F}_2$ the Fourier transform in the second variable of a function of two variables. For $\sigma(x,\xi) \in S^{-\infty}$, the operator $\sigma(x,D)$ is smoothing with distribution kernel

$$\sigma(x,D)(x,y) = (2\pi)^{-N}\int_{\mathbb{R}^N} e^{\imath(x-y)\cdot\xi}\sigma(x,\xi)d\xi = (\mathcal{F}_2^{-1}\sigma)(x,x-y).$$

Let us denote by K a smooth function on $\mathbb{R}^N \times \mathbb{R}^N$, if the integral (smoothing) operator defined by K is in fact a pseudodifferential operator $\sigma(x,D)$, then we can recover σ from K by the formula $\mathcal{F}_2^{-1}\sigma(x,y) = K(x,x-y)$, so

$$\sigma(x,\xi) = \int_{\mathbb{R}^N} e^{-\imath\xi\cdot y}K(x,x-y)dy. \tag{4.9}$$

Concerning the class $S^{-\infty}$, the following result is also standard and we are going to use it later on.

Lemma 4.5. *(i) The Fourier transform in the second variable establishes an isomorphism $\mathcal{F}_2 : S^{-\infty} := S^{-\infty}(\mathbb{R}^N \times \mathbb{R}^N) \to S^{-\infty}$.*
(ii) Multiplication defines a continuous map $S^m_{(1,0)} \times S^{-\infty} \to S^{-\infty}$.

For more about pseudodifferential calculus, we refer to the works of Taylor [27, 26] and Wong [31].

With this tool in hand, we move on to do the analysis. Recall that the function $G(z;x) = (4\pi)^{-N/2}\det(A(z))^{-1/2}e^{-x^TA(z)^{-1}x/4}$ introduced in Equation (3.7): Then the distribution kernel of $e^{L_0^z}$ is given by

$$e^{L_0^z}(x,y) = G(z;x-y), \tag{4.10}$$

A direct computation gives the following lemma, which coincides with the fact that $e^{L_0^z}$ is a convolution operator.

Lemma 4.6. *Let $z \in \mathbb{R}^N$ be a parameter and let us consider the operator $T = a(z)(x-z)^\beta\partial_x^\gamma e^{L_0^z}$, where β and γ are multi-indices and $a \in \mathcal{C}_b^\infty(\mathbb{R}^N)$. Then the distribution kernel of T is given by*

$$T(x,y) = a(z)(x-z)^\beta(\partial_x^\gamma G)(z;x-y).$$

The next theorem characterizes the symbol of $\mathcal{L}_{s,\alpha}$.

Theorem 4.7. *Let $\alpha \in \mathfrak{A}_{k,\ell}$, $k \le n$, $\alpha_i \le n$. Assume that $z : \mathbb{R}^N \times \mathbb{R}^N$ satisfies $z(x,x) = x$ and $\partial^\alpha z$ is bounded for all $\alpha \neq 0$. Then there exists a uniformly bounded family $\{\varrho_s\}_{s\in(0,1]}$ in $S^{-\infty}$ such that*

$$\mathcal{L}_{s,\alpha} = \sigma_s(x,D) := \varrho_s(x,sD), \quad \sigma_s(x,\xi) = \varrho_s(x,s\xi).$$

Proof. By Lemma 3.10, we know that $\Lambda_{\alpha,z}$ is a finite sum of terms of the form $a(z)(x-z)^\beta \partial_x^\gamma e^{L_0^z}$. We recall that $a(z)$ is a function that itself and all its derivatives are bounded. Suppose $k_z(x,y)$ is the distribution kernel of $a(z)(x-z)^\beta \partial_x^\gamma e^{L_0^z}$ and let

$$K_s(x,y) := s^{-N} k_z(z + s^{-1}(x-z), z + s^{-1}(y-z)), \quad z = z(x,y).$$

By abuse of notation, we shall denote also by K_s the integral operator defined by K_s. It suffices to prove our theorem for K_s. Namely, it is enough to show that there exists a uniformly bounded family $\{\varrho_s\}_{s\in(0,1]}$ in $S^{-\infty}$ such that

$$K_s = \varrho_s(x,sD).$$

By lemma 4.6, we have that the distribution kernel of $\partial_x^\gamma e^{L_0^z}$ is of the form $\varsigma(z, x-y)$ for some $\varsigma \in S^{-\infty}$. More precisely $\varsigma(z,x)$ is the Fourier tranform of the function $(\imath\xi)^\gamma e^{\xi\cdot A(z)\cdot\xi}$. This gives

$$K_s(x,y) = a(z(x,y)) s^{-|\beta|-N}(x - z(x,y))^\beta \varsigma(z(x,y), s^{-1}(x-y)) =: \\ a(z) s^{-|\beta|-N}(x-z)^\beta \varsigma(z, s^{-1}(x-y)), \quad z = z(x,y).$$

Then by (4.9), the symbol of K_s, $\sigma_s(x,\xi)$ is given by

$$\sigma_s(x,\xi) = \int_{\mathbb{R}^N} e^{-\imath y\cdot\xi} a(z) s^{-|\beta|-N}(x-z)^\beta \varsigma(z, s^{-1}y)dy, \quad z = z(x, x-y).$$

Let us substitute y with sy and let us denote

$$\varrho_s(x,\xi) = \int_{\mathbb{R}^N} e^{-\imath y\cdot\xi} a(z) s^{-|\beta|}(x-z)^\beta \varsigma(z,y)dy, \quad z = z(x, x-sy).$$

Then $\sigma_s(x,\xi) = \varrho_s(x,s\xi)$, so we need to show that ϱ_s is a bounded family in $S^{-\infty}$.

Notice that $a(z) \in S^0_{(1,0)}$ and $s^{-1}(x_j - z_j(x, x-sy)) \in S^0_{(1,0)}$ and they form bounded families for $s \in [0,1]$, then by Lemma (4.5) the proof is complete. □

The next lemma is obvious

Lemma 4.8. *Let $\varrho(x,\xi)$ be a symbol in $S^{-\infty}$, then $s^k\varrho(x,s\xi)$ is a symbol in $S^{-k}_{1,0}$ uniformly bounded in $(0,1]$ with respect to s.*

Proof. Denote ∂_1 and ∂_2 the derivatives of $\varrho(x,\xi)$ with respect to the first and second variable respectively. Since $\varrho(x,\xi) \in S^{-\infty}$, of course $\varrho(x,\xi) \in S^{-k}_{1,0}$, thus for any α and β we have

$$|\partial_x^\alpha \partial_\xi^\beta \varrho(x,\xi)| \le C(1+|\xi|)^{-k-|\beta|}.$$

Therefore,

$$|\partial_x^\alpha \partial_\xi^\beta (s^k \varrho(x,s\xi))| = |s^{k+\beta} \partial_1^\alpha \partial_2^\beta \varrho(x,s\xi)| \\ \le C s^{k+\beta}(1+|s\xi|)^{-k-|\beta|} \le \tilde{C}(1+|\xi|)^{-k-|\beta|}$$

where $\tilde{C}$ does not depend on s. Thus $s^k\varrho(x,s\xi)$ is uniformly bounded in $S^{-k}_{1,0}$ for $s \in (0,1]$. □

We now obtain the main result of this subsection, the desired refined mapping property estimate by standard results from pseudodifferential operators theory.

Theorem 4.9. *For any $1 < p < \infty$, any $r \in \mathbb{R}$,*

$$s^k \|\mathcal{L}_{s,\alpha}\|_{W^{r,p} \to W^{r+k,p}} \leq C_{k,r,p}, \tag{4.11}$$

for a constant $C_{k,r,p}$ independent of $t \in (0,1]$.

From (4.3), we immediately obtain the desired estimate on the principal part of the asymptotic expansion.

Corollary 4.10. *For each $1 < p < \infty$, $r \in \mathbb{R}$, and any $f \in W^{r,p}$*

$$\int_{\mathbb{R}^N} \mathcal{G}_t^{[m,z]}(x,y) f(y)\, dy,$$

is uniformly bounded in $W^{r,p}$ for $t \in (0,1]$.

4.2. **Bounds for the error term.** In this subsection, we shall bound the error term $E^{t,z}_{d,n}$, which is the sum of two kinds of operators. The first is the one we discussed in the last subsection, which is actually a pseudodifferential operator and behaves well(Theorem (4.9)). The second is the operator $\Lambda_{\alpha,l}$ with either $\alpha \in \mathfrak{A}_{n+1,\ell}$ or for some $\alpha_i = n+1$. In order to bounded $E^{t,z}_{d,n}$, it suffices to obtain mapping properties of the latter operator. In this case, the operator will depend on t also. Generally, we do not know whether $\Lambda_{\alpha,l}$ is a pseudodifferential operator or not. However, we are going to show that $\Lambda_{\alpha,l}$ also behaves well, and has a similar mapping property with Theorem (4.9) but a little bit rougher. It turns out that this rough estimate is enough to give us the desired error control. In stead of pseudodifferential calculus applied in the last subsection, the main technique we shall use is the so called Riesz's Lemma. (See for example [28].)

Lemma 4.11. *(Riesz) Assume K is an integral operator with kernel $k(x,y)$, that is,*

$$Ku(x) = \int_X k(x,y) u(y) d\mu(y),$$

where (X,μ) is a mearsure space. If $k(x,y)$ is measurable on $X \times X$ and

$$\int_X |k(x,y)| d\mu(x) \leq C_1, \int_X |k(x,y)| d\mu(y) \leq C_2 \tag{4.12}$$

for all y and for all x respectively. Then K is a bounded operator on $L^p(X,\mu)$ for each $p \in [1,\infty]$. Moreover,

$$\|K\| \leq C_1^{1/p} C_2^{1/q},$$

where q is the conjugate of p.

The main result of this subsection is as follows

Theorem 4.12. *Let z be admissible, $r \geq 0$. Then we have*

$$s^{k+r} \|\mathcal{L}_{s,\alpha}\|_{W^{r,p} \to W^{r+k,p}} \leq C_{k,r,p}. \tag{4.13}$$

Note that the main difference between Theorem (4.9) and Theorem (4.12) is the additional r. This result is rougher, but as we mentioned before, it is enough to give us the main theorem of this paper.

Before we prove this theorem, we first recall some notations and prove a lemma. We shall denote by $W^{r,p}_a = W^{r,p}_{a,w}$ as before, where w is the center of the weight

$\langle x\rangle_w = \langle x-w\rangle$ used to define our exponentially weighted Sobolev spaces (1.19). We shall write $L^p_{a,z} = W^{0,p}_{a,z}$. The following lemma is a special case of Theorem (4.12).

Lemma 4.13. *Assume that $z : \mathbb{R}^N \times \mathbb{R}^N$ is admissible. For any α, any $1 < p < \infty$, $k \in \mathbb{Z}_+$, $r \geq 0$, and $a \in \mathbb{R}$*

$$s^k \|\mathcal{L}_{s,\alpha}\|_{L^p_a \to W^{k,p}_a} \leq C_{k,p}, \tag{4.14}$$

for a constant $C_{k,p}$ independent of $t \in (0,1]$, independent of a in a bounded set, and independent of the center of the weight that defines the weighted Sobolev spaces.

Proof. The proof will be mainly an application of the Riesz's Lemma. Because of the reason we mentioned before, we may assume that $a = 0$. Recall that

$$\mathcal{L}_{s,\alpha}(x,y) = s^{-N}\Lambda_{\alpha,z}(z + s^{-1}(x-z), z + s^{-1}(y-z))$$

is the kernel of the operator $\mathcal{L}_{s,\alpha}$, where $z = z(x,y)$. Then by Riesz's Lemma it suffices to show that for any multi-index γ with $|\gamma| \leq k$,

$$\int_{\mathbb{R}^N} s^{|\gamma|}|\partial_x^\gamma \mathcal{L}_{s,\alpha}(x,y)|dy \leq C_1, \int_{\mathbb{R}^N} s^{|\gamma|}|\partial_x^\gamma \mathcal{L}_{s,\alpha}(x,y)|dx \leq C_2, \tag{4.15}$$

where the constants C_1 and C_2 should be independent of x and y respectively. Generally, we need to estimate the growth rate of $s^{-N}\partial_x^\gamma \Lambda_{\alpha,z}(z + s^{-1}(x-z), z + s^{-1}(y-z))$. with respect to x and y. We need to use weighted Sobolev spaces introduced in (1.19). Recall that the mapping properties between the weighted Sobolev spaces are uniform in terms of the weight center, thus we can choose z as the weight center. Notice that $\partial_x^\gamma \mathcal{L}_{s,\alpha}(x,y)$ is the sum of terms of the form

$$s^{-N-j}\partial_x^\beta \partial_z^{\beta'} \partial_y^{\beta''} \Lambda_{\alpha,z}(z + s^{-1}(x-z), z + s^{-1}(y-z)) \cdot \xi(z), \tag{4.16}$$

where $j \leq |\gamma|$ and $\xi(z)$ is the product of derivatives of z with respect to x, it is bounded as z is admissible. While

$$\begin{aligned} |\partial_x^\beta \partial_z^{\beta'} \partial_y^{\beta''} \Lambda_{\alpha,z}(x,y)| &= |< \partial^\beta \delta_x, \partial_z^{\beta'} \Lambda_{\alpha,z} \partial^{\beta''} \delta_y >| \\ &\leq C \|\partial^\beta \delta_x\|_{H^{-q}_{-a-\epsilon}} \|\partial_z^{\beta'} \Lambda_{\alpha,z}\|_{H^{-q}_{-a} \to H^q_{-a-\epsilon}} \|\partial^{\beta''} \delta_y\|_{H^{-q}_{-a}}. \end{aligned} \tag{4.17}$$

Next we shall estimate the three norms at the right hand side of the above estimate. For each multi-index β, $\partial_x^\beta \in H^{-q}(\mathbb{R}^N)$ as long as $q > N + |\beta|$. Therefore, if we choose z as the base point and $q > N + |\beta|$. Then for all $a \in \mathbb{R}$ and $\epsilon > 0$

$$\|\partial^\beta \delta_x\|_{H^{-q}_{-a-\epsilon}} := \|e^{-(a+\epsilon)<x-z>} \partial^\beta \delta_x\|_{H^{-q}} \leq Ce^{-(a+\epsilon)<x-z>}$$

and

$$\|\partial^{\beta''} \delta_y\|_{H^{-q}_{-a}} := \|e^{-a<y-z>} \partial^{\beta''} \delta_y\|_{H^{-q}} \leq Ce^{-a<y-z>}$$

For the second term $\|\partial_z^{\beta'} \Lambda_{\alpha,z}\|_{H^{-q}_{-a} \to H^q_{-q-\epsilon}}$, since all the coefficients and their derivatives of $L(t)$ are bounded, $\partial_z^{\beta'} \Lambda_{\alpha,z}$ will satisfy the same mapping properties as $\Lambda_{\alpha,z}$. Thus by Corollary (4.3),

$$\|\partial_z^{\beta'} \Lambda_{\alpha,z}\|_{H^{-q}_{-a} \to H^q_{-a-\epsilon}} \leq Ce^{\epsilon<z-x>}$$

Now get back to (4.17), we have

$$|\partial_x^\beta \partial_z^{\beta'} \partial_y^{\beta''} \Lambda_{\alpha,z}(x,y)| \leq Ce^{\epsilon<z-x>-a<y-z>-(a+\epsilon)<x-z>} = Ce^{-a<y-z>-a<x-z>}$$

Therefore, we obtain

$$
\begin{aligned}
&|s^{-N-j}\partial_x^{\beta}\partial_z^{\beta'}\partial_y^{\beta''}\Lambda_{\alpha,z}(z+s^{-1}(x-z),z+s^{-1}(y-z))\cdot\xi(z)| \\
&\leq Cs^{-N-|\gamma|}e^{-a<s^{-1}(y-z)>-a<s^{-1}(x-z)>} \\
&\leq Cs^{-N-|\gamma|}e^{-a<s^{-1}(y-x)>}
\end{aligned} \tag{4.18}
$$

In the last inequality, we have used the triangle inequality $< y-z > + < x-z > \geq < y-x >$. Then after the change of variable $\lambda = \frac{y-x}{s}$, we find that (4.15) holds. The proof is complete. □

Proof of Theorem (4.12)*:* Notice that $W^{r,p} \subset L^p$ for any $r \geq 0$ (for non-integer r, it is a consequence of the interpolation argument). Then if we consider $\mathcal{L}_{s,\alpha}$ as an operator from L^p to $W^{r+k,p}$ in stead of from $W^{r,p}$ to $W^{r+k,p}$, the result follows directly from Lemma (4.13). □

Recall that $\mathcal{E}_t^{[m,z]}$ is the sum of two kinds of operators we mentioned before, then a direct corollary of Theorem (4.12) and Theorem (4.9) is the following

Corollary 4.14. *Assume z is admissible and $r \geq 0$, then*

$$
\|\mathcal{E}_t^{[m,z]}\|_{W^{r,p}\to W^{r+k,p}} \leq Cs^{-r-k}. \tag{4.19}
$$

Surprisingly, it turns out that the r at the right hand side of equation (4.19) is redundant, we can get rid of it to obtain a more refined estimate.

Theorem 4.15. *Assume z is admissible and $r \geq 0$, then $\mathcal{E}_t^{[m,z]}$ satisfies the following mapping property*

$$
\|\mathcal{E}_t^{[m,z]}\|_{W^{r,p}\to W^{r+k,p}} \leq Cs^{-k} \tag{4.20}
$$

Proof. Recall that as long as $d \geq n > m$, $\mathcal{G}_t^{[m]}(x,y)$ does not depend on d and n. Thus as the difference

$$
\mathcal{E}_t^{[m,z]}(x,y) = U(1)(z+\frac{x-z}{\sqrt{t}}, z+\frac{y-z}{\sqrt{t}}) - \mathcal{G}_t^{[m,z]}(x,y)
$$

also does not depend on n and d. In the expansion (3.21), we expand it to much more terms, specifically, such that $M \geq m+r-1$. Then by Theorem (4.9) and Corollary (4.14)

$$
\begin{aligned}
\|\mathcal{E}_t^{[m,z]}\|_{W^{r,p}\to W^{r+k,p}} \leq &\sum_{\ell=m+1}^{M} s^{\ell-m-1}\sum_{k=m+1}^{\ell}\sum_{\alpha\in\mathfrak{A}_{k,\ell}}\|\mathcal{L}_{\alpha,z}\|_{W^{r,p}\to W^{r+k,p}} \\
&+ s^{M+1-m}\|\mathcal{E}_t^{[M,z]}\|_{W^{r,p}\to W^{r+k,p}} \leq Cs^{-k}(1+s^{M+1-m}s^{-r}) \leq Cs^{-k}.
\end{aligned}
$$

This completes the proof. □

This completes the proof of the main Theorem (0.1).

5. Invariance under affine transformations

Recall that in Section 3 we considered a new equation (3.2) obtained by parabolically scaling the original equation (0.3). As in Lemma 3.1, the Green functions to the original equation and the dilated equation are explicity related to each other, and this precise relation is one of the key steps of our *Dyson-Taylor commutator* *method.* In this section, we investigate the role of affine transformations on our approximations, which is needed in some applications.

Throughout this section, we assume that

$$F(\xi) = D\xi + Q, \quad D \in \mathbb{R}^{\mathbb{N}\times\mathbb{N}},\ Q \in \mathbb{R}^{\mathbb{N}}$$

is a linear transformation with $\det(D) \neq 0$. For any function $u(t,x)$, we also define its transformation as $u^F(t,x) = u(t, F(x))$. Therefore, if $u(t,x)$ is the solution to the equation (0.3) with $g = 0$. Then it is easy to see that $u^F(t,x)$ satisfies the following equation

$$\begin{cases} \partial_t u^F(t,x) - L^F(t)u^F(t,x) = 0 & \text{in } (0,\infty)\times\mathbb{R}^N, \\ \ u^F(0,x) = f^F(x), & \text{on } \{0\}\times\mathbb{R}^N\,, \end{cases} \tag{5.1}$$

where $L^F(t)$ is defined as $(L(t)u(t,x))^F = L^F(t)u^F(t,x)$. Assume $\mathcal{G}_t^L(x,y)$ and $\mathcal{G}_t^{L^F}(x,y)$ are the Green functions to equations (0.3) and (5.1). We first explore the relationship between these two kernels. On one hand, by definition,

$$u(t,x) = \int \mathcal{G}_t^L(x,y)f(y)dy.$$

Therefore,

$$u^F(t,x) = u(t,F(x)) = \int \mathcal{G}_t^L(F(x),y)f(y)dy.$$

On the other hand,

$$\begin{aligned} u^F(t,x) &= \int \mathcal{G}_t^{L^F}(x,y)f^F(y)dy = \int \mathcal{G}_t^{L^F}(x,y)f(F(y))dy \\ &= \det(D^{-1})\int \mathcal{G}_t^{L^F}(x,F^{-1}(y))f(y)dy. \end{aligned}$$

Comparison of the above two equations leads to

$$\mathcal{G}_t^L(F(x),y) = \det(D^{-1})\mathcal{G}_t^{L^F}(x,F^{-1}(y)).$$

After a change of variable we obtain

$$\mathcal{G}_t^L(x,y) = \det(D^{-1})\mathcal{G}_t^{L^F}(F^{-1}(x),F^{-1}(y)). \tag{5.2}$$

Similar to Lemma 3.1, we also have

Proposition 5.1. *Suppose $\mathcal{G}_t^L, \mathcal{G}_t^{L^s}, \mathcal{G}_t^{L^{F,s}}$ are the Green's functions to the operator $L, L^s, L^{F,s}$ respectively. Then they are related by*

$$\begin{aligned} &\mathcal{G}_t^L(x,y) = s^{-\mathbb{N}}\mathcal{G}_{s^{-2}t}^{L^s}\left(z+\frac{x-z}{s}, z+\frac{y-z}{s}\right) \\ = \quad & s^{-\mathbb{N}}\mathcal{G}_{s^{-2}t}^{L^{F,s}}\left(F^{-1}(z)+\frac{F^{-1}(x)-F^{-1}(z)}{s}, F^{-1}(z)+\frac{F^{-1}(y)-F^{-1}(z)}{s}\right)\left|D^{-1}\right|. \end{aligned}$$

Proof. For convenience, we denote $v(t,\xi) = u^F(t,x) = u(t,F(x)), \xi = F(x)$. By Lemma 3.1

$$\begin{aligned} u^s(t,x) &= s^{-N}\int \mathcal{G}_t^{L^s}\left(x, z+\frac{y-z}{s}\right)f(y)dy \\ v^s(t,\xi) &= s^{-N}\int \mathcal{G}_t^{L^{F,s}}(\xi, \tilde{z}+\frac{y-\tilde{z}}{s})f(F(y))dy. \end{aligned}$$

Therefore,

$$
\begin{aligned}
&v^s(t,F^{-1}(x)) = s^{-N}\int \mathcal{G}_t^{L^{F,s}}(F^{-1}(x),\tilde z + \frac{y-\tilde z}{s})f(F(y))dy\\
&= v(s^2t,\tilde z + s(F^{-1}(x)-\tilde z)) = u(s^2t, F(\tilde z + s[F^{-1}(x)-\tilde z]))\\
&= u(s^2t, D(\tilde z + s[D^{-1}(x-Q)-\tilde z]) + Q) = u(s^2t,[D\tilde z + Q] + s(x-[D\tilde z+Q]))\\
&= u^s(t,x;z=D\tilde z + Q) = u^s(t,x;z=F(\tilde z))\\
&= s^{-N}\int \mathcal{G}_t^{L^s}\left(x,z+\frac{y-z}{s}\right)f(y)dy\,\big|_{z=F(\tilde z)}\ ,
\end{aligned}
$$

which gives

$$
\begin{aligned}
&s^{-N}\int \mathcal{G}_t^{L^{F,s}}(F^{-1}(x),\tilde z + \frac{y-\tilde z}{s})f(F(y))dy\\
=\ &s^{-N}\int \mathcal{G}_t^{L^s}\left(x,z+\frac{y-z}{s}\right)f(y)dy\,\big|_{z=F(\tilde z)}\ .
\end{aligned}
$$

After a change of variable, this relation reads

$$
\begin{aligned}
&s^{-N}\int \mathcal{G}_t^{L^{F,s}}(F^{-1}(x),\tilde z + \frac{F^{-1}(y)-\tilde z}{s})f(y)dy\,\left|D^{-1}\right|\\
=\ &s^{-N}\int \mathcal{G}_t^{L^s}\left(x,z+\frac{y-z}{s}\right)f(y)dy\,\big|_{z=F(\tilde z)}
\end{aligned}
$$

Thus we must prove

$$
\mathcal{G}_t^{L^{F,s}}(F^{-1}(x),F^{-1}(z) + \frac{F^{-1}(y)-F^{-1}(z)}{s})\left|D^{-1}\right| = \mathcal{G}_t^{L^s}\left(x,z+\frac{y-z}{s}\right), \tag{5.3}
$$

which follows from Lemma 3.1, where we have shown that

$$
\mathcal{G}^L_{s^2t}(z+s(x-z),y) = s^{-\mathbb{N}}\mathcal{G}_t^{L^s}(x,z+\frac{y-z}{s}) \tag{5.4}
$$

A simple algebra on equations (5.3) and (5.4) yields our result. □

Note that L^F still satisfies all of our assumptions, therefore by our Main Theorem 0.1 we have the following asymptotic expansion

$$
\mathcal{G}_t^{L^F}(\xi,y) = \mathcal{G}_t^{[m,z],L^F}(\xi,y) + s^{m+1}E^{t,z}_{d,n}(\xi,y) \tag{5.5}
$$

and if we define the error operator as

$$
\mathcal{E}^{[t,z]}_m f(x) = \int E^{t,z}_{d,n}(\xi,y)f(y)dy,
$$

we have

$$
\|\mathcal{E}^{[t,z]}_m f\|_{W^{r+k,p}_{a,z}} \leq Ct^{-r/2}\|f\|_{W^{k,p}_{a,z}}, \tag{5.6}
$$

but we know that $\mathcal{G}_t^L(x,y)$ and $\mathcal{G}_t^{L^F}(\xi,y)$ are related by

$$
\mathcal{G}_t^L(x,y) = \mathcal{G}_t^{L^F}(F^{-1}(x),F^{-1}(y))|D^{-1}|. \tag{5.7}
$$

Combining (5.5) and (5.7) yields

(5.8)

$$
\mathcal{G}_t^L(x,y) = \mathcal{G}_t^{[m,z],L^F}(F^{-1}(x),F^{-1}(y))|D^{-1}| + s^{m+1}E^{t,z}_{d,n}(F^{-1}(x),F^{-1}(y))|D^{-1}|.
$$

Let us now define the operator

$$\bar{\mathcal{E}}_m^{t,z} f(x) = \int E_{d,n}^{t,z}(F^{-1}(x), F^{-1}(y))|D^{-1}|f(y)dy.$$

A simple calculation shows that $\bar{\mathcal{E}}_m^{t,z}$ satisfies the same inequality as $\mathcal{E}_m^{t,z}$ does in (5.6) with a difference constant C. It is worth noting that it is crucial to assume $\det(D) \neq 0$.

The conclusion is that the approximation introduced in Theorem 0.1 is invariant under affine transformations.

6. Applications

In this section, we shall numerically test our short time approximation of the Green function of parabolic equations. We then extend our method to approximate solutions of parabolic equations for large time. We also give concrete, simple examples to illustrate our approach.

6.1. **Numerical implementation.** Given the approximation of the Green function $\mathcal{G}_t^{[m]}(x,y)$, then $\int_{\mathbb{R}^N} \mathcal{G}_t^{[m]}(x,y)f(y)dy$ is an approximation of the solution to the non-autonomous Cauchy problem (0.3) with $g = 0$. In practice we do not have explicit formulas for the integral in general, thus numerical integration is required. In this case two other sources of errors will be introduced. One is the numerical quadrature error, and the other is the truncation error, for the integration domain is infinite. In the following of this subsection, we aim to control the overall error.

It is reasonable to assume that one needs to compute the the numerical solution of $u(t,x)$ on the interval $I = [\underline{x}, \bar{x}]$. As mentioned before, we need to truncate the domain first. Also, we split the integral into two parts,

$$\int \mathcal{G}_t^{[m]}(x,y)f(y)dy = \int_{|y|<d} \mathcal{G}_t^{[m]}(x,y)f(y)dy + \int_{|y|>d} \mathcal{G}_t^{[m]}(x,y)f(y)dy \tag{6.1}$$

for the first part we shall apply numerical quadrature rules. The second part is simply ignored and considered as the truncation error. Since the approximated Green's function $\mathcal{G}_t^{[m]}(x,y)$ decays exponentially, for fixed I, we can always choose d big enough such that the truncation error is arbitrarily small, *i.e.,*

$$\left| \int_{|y|>d} \mathcal{G}_t^{[m]}(x,y)f(y)dy \right| < \epsilon, \tag{6.2}$$

where ϵ is small and dependent on d.

Define $I_t f$ as the function obtained by numerical integration of $\int \mathcal{G}_t^{[m]}(x,y)f(y)dy$. For the integral $\int_{|y|<d} \mathcal{G}_t^{[m]}(x,y)f(y)dy$ we can choose a k^{th} order quadrature rule with mesh size h. Then for any $x \in I$,

$$\|I_t f - \mathcal{G}_t^{[m]} f\|_{L^\infty(I)} < \epsilon + Ch^k. \tag{6.3}$$

Combining this estimate with Equation (6.2) and the Sobolev embedding theorem, we obtain, for p big enough,

$$\begin{aligned} \|(I_t - e^{tL})f\|_{L^\infty(I)} &\leq \|(I_t - \mathcal{G}_t^{[m]})f\|_{L^\infty(I)} + \|(\mathcal{G}_t^{[m]} - e^{tL})f\|_{L^\infty(\mathbb{R}^N)} \\ &\leq \epsilon + Ch^k + C_3\|(\mathcal{G}_t^{[m]} - e^{tL})f\|_{W_{a,z}^{k,p}(\mathbb{R}^N)} \\ &\leq \left(\epsilon + Ch^k + Ct^{(m+1)/2}\right)\|f\|_{W_{a,z}^{k,p}(\mathbb{R}^N)}. \end{aligned} \tag{6.4}$$

We formulate our discussion as the following

Theorem 6.1. *Consider the non-autonomous Cauchy problem* (0.3) *with* $g = 0$*, the operator* I_t *is defined as above. Then for any interval* $I = [\underline{x}, \bar{x}]$ *and a* k^{th} *order quadrature rule, we have*

$$\|(I_t - e^{tL})f\|_{L^\infty(I)} \le \left(\epsilon + Ch^k + Ct^{(m+1)/2}\right) \|f\|_{W^{k,p}_{a,z}(\mathbb{R}^{\mathbb{N}})},$$

where ϵ *is dependent on* I *and can be arbitrarily small,* C *is constant independent of* t *and* h.

6.2. **The bootstrap scheme.** Note that our approximation is only accurate provided t is small. In applications, it is not necessarily in this case, *i.e.,* , we would have relatively large t. Even if t is small, to improve the accuracy we need to compute high order approximations of the Green function. However, this is not always feasible in practice. As the order of accuracy goes up, computing the Green function becomes more expensive. In this subsection, we shall introduce the bootstrap scheme to extend our results to relatively large t. We also show that high precision can be obtained by using as low as a second order approximation of the Green function.

Before we start our analysis, we first give some preliminaries. In our error estimate, we need the following useful lemmas:

Lemma 6.2. *Assume* $U(t, s)$ *is the evolution system generated by the operator* $L(t)$ *on* X *with norm* $\|\cdot\|$*. Then we can find an equivalent norm* $|||\cdot|||_t$ *which is time dependent, such that*

$$|||U(t, s)x|||_t \le e^{\omega(t-s)}|||x|||_s,$$

i.e., $U(t, s)$ *is a contraction between* $|||\cdot|||_t$ *and* $|||\cdot|||_s$.

Proof. By Lemma 2.4, $\|U(t, s)\| \le Me^{\omega(t-s)}$. Set $V(t, s) = e^{-\omega(t-s)}U(t, s)$, then it is clear that $V(t, s)$ is uniformly bounded by M. We define a new norm as

$$\|x\|_{X_s} = |||x|||_s = \sup_{t \ge s} \|V(t, s)f\|. \tag{6.5}$$

Obviously, we have $\|x\| \le |||x||| \le M\|x\|$. Thus $|||\cdot|||$ is equivalent to $\|\cdot\|$ on X. Note that by our definition

$$\begin{aligned} |||V(t, s)x|||_t &= \sup_{r \ge t} \|V(r, t)V(t, s)x\| = \sup_{r \ge t} \|V(r, s)x\| \\ &\le \sup_{r \ge s} \|V(r, s)x\| = \|x\|_s. \end{aligned}$$

Plugging in $V(t, s) = e^{-\omega(t-s)}U(t, s)$, we obtain the desired estimate

$$|||U(t, s)x|||_t \le e^{\omega(t-s)}|||x|||_s. \tag{6.6}$$

□

Thereafter, we denote $|||\cdot|||_{t,s} = \|\cdot\|_{X_s \to X_t}$. Next, we shall investigate the effect of the bootstrap strategy in estimating the solution to the equation (0.3) with $g = 0$. By the evolution property, $U(t, 0) = U(t, \frac{n-1}{n}t)U(\frac{n-1}{n}t, \frac{n-2}{n}t) \cdot \cdots \cdot U(\frac{t}{n}, 0)$. Then in the bootstrap scheme, we approximate $U(\frac{k+1}{n}t, \frac{k}{n}t)$ by $\mathcal{G}^{[m]}_{t/n}$. We need to bound the error $U(t, 0) - \left(\mathcal{G}^{[m]}_{t/n}\right)^n$ in some norm.

Theorem 6.3. *Assume that $U(t,s)$ is the evolution system generated by $L(t)$, and $\mathcal{G}_t^{[m]}$ is the m^{th} order approximation of the Green function of $L(t)$. Then*

$$|||U(t,0) - \left(\mathcal{G}_{t/n}^{[m]}\right)^n |||_{t,0} \leq M \frac{t^{(m+1)/2}}{n^{(m-1)/2}} e^{\bar{\omega} t}.$$

Proof. Notice that we have the identity

$$U(t,0) - \left(\mathcal{G}_{t/n}^{[m]}\right)^n = \sum_{k=0}^{n-1} U(t, \frac{k+1}{n}) \left(U(\frac{k+1}{n}t, \frac{k}{n}t) - \mathcal{G}_{t/n}^{[m]} \right) \left(\mathcal{G}_{t/n}^{[m]}\right)^k. \tag{6.7}$$

Therefore,

$$\begin{aligned} &|||U(t,0) - \left(\mathcal{G}_{t/n}^{[m]}\right)^n |||_{t,0} \\ &= ||| \sum_{k=0}^{n-1} U(t, \frac{k+1}{n}) \left(U(\frac{k+1}{n}t, \frac{k}{n}t) - \mathcal{G}_{t/n}^{[m]} \right) \left(\mathcal{G}_{t/n}^{[m]}\right)^k |||_{t,0} \\ &\leq \sum_{k=0}^{n-1} |||U(t, \frac{k+1}{n})|||_{t, \frac{k+1}{n}} ||| \left(U(\frac{k+1}{n}t, \frac{k}{n}t) - \mathcal{G}_{t/n}^{[m]} \right) |||_{\frac{k+1}{n}, \frac{k}{n}} ||| \left(\mathcal{G}_{t/n}^{[m]}\right)^k |||_{\frac{k}{n}, 0} \end{aligned} \tag{6.8}$$

By Lemma 6.2 and our main Theorem 0.1, we have

$$|||U(\frac{k+1}{n}t, \frac{k}{n}t) - \mathcal{G}_{t/n}^{[m]}|||_{\frac{k+1}{n}, \frac{k}{n}} \leq M \|U(\frac{k+1}{n}t, \frac{k}{n}t) - \mathcal{G}_{t/n}^{[m]}\| \leq M \left(\frac{t}{n}\right)^{\frac{m+1}{2}}. \tag{6.9}$$

The triangle inequality and Lemma 6.2 then lead to

$$|||\mathcal{G}_{t/n}^{[m]}|||_{\frac{k+1}{n}, \frac{k}{n}} \leq e^{\omega \frac{t}{n}} + M \left(\frac{t}{n}\right)^{(m+1)/2}$$

Therefore,

$$||| \left(\mathcal{G}_{t/n}^{[m]}\right)^k |||_{\frac{k}{n}, 0} \leq \left[e^{\omega \frac{t}{n}} + M \left(\frac{t}{n}\right)^{(m+1)/2} \right]^k \tag{6.10}$$

Using Equations (6.9) and (6.10) in (6.8), we obtain

(6.11)

$$\begin{aligned} &|||U(t,0) - \left(\mathcal{G}_{t/n}^{[m]}\right)^n |||_{t,0} \leq \sum_{k=0}^{n-1} e^{\omega \frac{n-k-1}{n} t} M \left(\frac{t}{n}\right)^{(m+1)/2} \left[e^{\omega \frac{t}{n}} + M \left(\frac{t}{n}\right)^{(m+1)/2} \right]^k \\ &\leq Mn \left(\frac{t}{n}\right)^{(m+1)/2} e^{\omega t} \left[e^{\omega \frac{t}{n}} + M \left(\frac{t}{n}\right)^{(m+1)/2} \right]^n \\ &\leq M \frac{t^{(m+1)/2}}{n^{(m-1)/2}} e^{\bar{\omega} t} \end{aligned}$$

In the last inequality we used the fact that the limit

$$\lim_{n\to\infty} \left(e^{\omega \frac{t}{n}} + M \left(\frac{t}{n}\right)^{(m+1)/2} \right)^n$$

exists and it grows at most exponentially with respect to t. $\square$

Remark 6.4. Theorem 6.3 is of great interest in practice. It says that we do not need to compute high order approximations of the Green function which in general becomes more complicated as the order goes higher, but a second order approximation ($m = 2$) will be enough. As the number of bootstrap steps n goes to infinity, the error will go to zero. Thus one can use many steps to reduce the error.

Remark 6.5. If the operator $L(t)$ is independent of t, then it generates a semigroup. In this case, we are able to make the equivalent norm independent of time. Actually, if $\|e^{tL}\| \le Me^{\omega t}$, then we can define $V(t) = e^{-\omega t}e^{tL}$ and a new norm $|||x||| = \sup_{t\ge 0}\|V(t)x\|$. It is easy to check that this new norm is equivalent to the original norm and $V(t)$ is a contraction semigroup under the norm $|||\cdot|||$.

6.3. **An example.** In this subsection, we illustrate our *Dyson-Taylor commutator method* and its accuracy by a simple example. We shall consider the operator

$$L = a(t)\partial_x^2 + b(t)\partial_x + c(t), t \in [0,\infty), x \in \mathbb{R}. \tag{6.12}$$

and assume that L satisfies our general conditions. On one hand, we can apply our method to find an explicit formula of the n^{th} order approximation of the Green function of the operator L, and thus obtain an approximation of the solution of the equation

$$\begin{cases} \partial_t u(t,x) = Lu(t,x), t > 0, x \in \mathbb{R}, \\ u(0,x) = f(x) \end{cases} \tag{6.13}$$

On the other hand, notice that the coefficients of L are space-independent, and hence we can use the Fourier transfom to find the solution to the above equation explicitly. This enables us to compare our approximation with the true solution.

Without loss of generality, we can assume that $c(t) = 0$. Otherwise, we apply the change of variable $v(t,x) = e^{-\int c(t)dt}u(t,x)$, then $v(t,x)$ will satisfy the same equation with $c(t) = 0$.

Following the procedures in Section 3, we first dilate this operator to get

$$L^{s,z} = a(s^2t)\partial_x^2 + sb(s^2t)\partial_x. \tag{6.14}$$

The Taylor expansion of $L^{s,z}$ with respect to the parameter s reads

$$L^{s,z} = \sum_{m=0}^{\infty} s^m L_m, \tag{6.15}$$

where by (3.6)

$$L_0 = a(0)\partial_x^2$$

and

$$L_m = \begin{cases} \frac{t^k}{k!}\partial_t^k a(0)\partial_x^2, \text{ if } m = 2k \\ \frac{t^k}{k!}\partial_t^k b(0)\partial_x, \text{ if } m = 2k+1 \end{cases} \tag{6.16}$$

we also denote $L_m = t^k\tilde{L}_m$.

Note that for every operator L_m, the coefficients are independent of the space variable x. Thus the semigroup $e^{\tau L_0}$ commutes with L_m for every m. In this case, formula (3.16) can be greatly simplified. As a matter of fact,

$$U(1,0) = e^{L_0} + sL_1 e^{L_0} + s^2\left(\frac{1}{2}L_1^2 + \frac{1}{2}\tilde{L}_2\right)e^{L_0}$$
$$+ s^3\left(\frac{1}{6}L_1^3 + \frac{1}{2}L_1\tilde{L}_2 + \frac{1}{2}\tilde{L}_3\right)e^{L_0} \tag{6.17}$$
$$+ s^4\left(\frac{1}{24}L_1^4 + \frac{1}{2}L_1\tilde{L}_3 + \frac{1}{8}\tilde{L}_2^2 + \frac{1}{3}\tilde{L}_4\right)e^{L_0} + \cdots,$$

where $e^{L_0} = \frac{1}{2\sqrt{\pi a(0)}} e^{-\frac{(x-y)^2}{4a(0)}}$.

Here are now some details for (6.17):

Note: L_0, L_1 are independent on time, but L_2, L_3 are dependent on time by (3.16),

(6.18)

$$U(1,0) = e^{L_0} + s\int_0^1 e^{(1-\sigma)L_0} L_1 e^{\sigma L_0} d\sigma$$
$$+ s^2\left[\int_0^1\int_0^{\sigma_1} e^{(1-\sigma_1)L_0} L_1 e^{(\sigma_1-\sigma_2)L_0} L_1 e^{\sigma_2 L_0} d\sigma_2 d\sigma_1 + \int_0^1 e^{(1-\sigma)L_0} L_2(\sigma) e^{\sigma L_0} d\sigma\right]$$
$$+ s^3\left[\int_0^1\int_0^{\sigma_1}\int_0^{\sigma_2} e^{(1-\sigma_1)L_0} L_1 e^{(\sigma_1-\sigma_2)L_0} L_1 e^{(\sigma_2-\sigma_3)L_0} L_1 e^{\sigma_3 L_0} d\sigma_3 d\sigma_2 d\sigma_1\right.$$
$$+ \int_0^1\int_0^{\sigma_1} e^{(1-\sigma_1)L_0} L_1 e^{(\sigma_1-\sigma_2)L_0} L_2(\sigma_2) e^{\sigma_2 L_0} d\sigma_2 d\sigma_1$$
$$\left.+ \int_0^1\int_0^{\sigma_1} e^{(1-\sigma_1)L_0} L_2(\sigma_1) e^{(\sigma_1-\sigma_2)L_0} L_1 e^{\sigma_2 L_0} d\sigma_2 d\sigma_1 + \int_0^1 e^{(1-\sigma)L_0} L_3(\sigma) e^{\sigma L_0} d\sigma\right] + O(s^4)$$
$$= e^{L_0} + sL_1 e^{L_0} + s^2\left(L_1^2\int_0^1\int_0^{\sigma_1} d\sigma_2 d\sigma_1 + \tilde{L}_2\int_0^1 \sigma d\sigma\right)$$
$$+ s^3\left(L_1^3\int_0^1\int_0^{\sigma_1}\int_0^{\sigma_2} d\sigma_3 d\sigma_2 d\sigma_1 + L_1\tilde{L}_2\int_0^1\int_0^{\sigma_1}\sigma_2 d\sigma_2 d\sigma_1 + L_1\tilde{L}_2\int_0^1\int_0^{\sigma_1}\sigma_1 d\sigma_2 d\sigma_1\right.$$
$$\left.+ L_3\int_0^1 \sigma d\sigma\right) + O(s^4)$$
$$= (6.17)$$

Next, as we mentioned above, we use Fourier Transform to find the Green's function to equation (6.13). Applying the Fourier Transform to (6.13), it then reads

$$\partial_t \hat{u}(t,\xi) = \left(i\xi b(t) - \xi^2 a(t)\right)\hat{u}(t,\xi), \quad \hat{u}(0) = \hat{f}(\xi),$$

where $\hat{u} = \mathcal{F}(u)$ is the Fourier Transform. The above ODE is easy to solve. Denote $\alpha(t) = \int_0^t a(\tau)d\tau, \beta(t) = \int_0^t b(\tau)d\tau$, then

$$\hat{u}(t,\xi) = e^{\int_0^t \left(i\xi b(\tau) - \xi^2 a(\tau)\right)d\tau}\hat{f}(\xi) = e^{\left(i\xi\beta(t) - \xi^2\alpha(t)\right)}\hat{f}(\xi).$$

Therefore,

$$\begin{aligned}
u(t,x) &= \mathcal{F}^{-1}(\hat{u}(t,\xi)) = \frac{1}{\sqrt{2\pi}}\int e^{\left(i\xi(\beta(t)+x)-\xi^2\alpha(t)\right)}\hat{f}(\xi)d\xi \\
&= \frac{1}{2\pi}\int\int e^{\left(i\xi(\beta(t)+x-y)-\xi^2\alpha(t)\right)}f(y)dyd\xi \\
&= \frac{1}{2\sqrt{\pi\alpha(t)}}\int e^{-\frac{(\beta(t)+x-y)^2}{4\alpha(t)}}f(y)dy,
\end{aligned}$$

which gives the exact Green's function

$$\mathcal{G}_t^L(x,y) = \frac{1}{2\sqrt{\pi\alpha(t)}}e^{-\frac{(\beta(t)+x-y)^2}{4\alpha(t)}}.$$

By Lemma 3.1

$$U(1,0;z=x)(x,y) = s\mathcal{G}_t^L(x,x+s(y-x)), s=\sqrt{t}.$$

actually, in this case the Green function does not depend on z.

Our *Taylor-Commutator method* suggests that if we expand $s\mathcal{G}_t^L(x,x+s(y-x))$ with respect to s, we should be able to recover equation (6.17).

Notice that $\lim_{t\to 0}\frac{\alpha(t)}{t} = a(0), \lim_{t\to 0}\frac{\beta(t)}{t} = b(0)$, thus

$$s\mathcal{G}_t^L(x,x+s(y-x))\big|_{s=0} = \frac{s}{2\sqrt{\pi\alpha(s^2)}}e^{-\frac{(\beta(s^2)+s(x-y))^2}{4\alpha(s^2)}}\Bigg|_{s=0} = \frac{1}{2\sqrt{\pi a(0)}}e^{-\frac{(x-y)^2}{4a(0)}}$$

This is exactly e^{L_0}.

Next, we shall expand $s\mathcal{G}_{s^2}^L(x,x+s(y-x)) = \frac{s}{2\sqrt{\pi\alpha(s^2)}}e^{-\frac{(\beta(s^2)+s(x-y))^2}{4\alpha(s^2)}}$ in powers of s and compare it with (6.17) term by term. To carry out the comparison, the scheme we shall apply is to expand $\frac{s}{2\sqrt{\pi\alpha(s^2)}}$ and $e^{-\frac{(\beta(s^2)+s(x-y))^2}{4\alpha(s^2)}}$ with respect to s respectively and then take the product. For the later exponential, in order to obtain its power expansion, we expand its power term, namely $\frac{(\beta(s^2)+s(x-y))^2}{4\alpha(s^2)}$ first. More precisely, notice that

$$\begin{aligned}
\alpha(s^2) &= a(0)s^2 + \frac{1}{2}a'(0)s^4 + \frac{1}{6}a''(0)s^6 + O(s^8) \\
\beta(s^2) &= b(0)s^2 + \frac{1}{2}b'(0)s^4 + \frac{1}{6}b''(0)s^6 + O(s^8)
\end{aligned}$$

then plugging in the expansion of $\alpha(s)$ and $\beta(s)$ to the two functions as we mentioned above, we have

$$\begin{aligned}
&\frac{s}{\sqrt{\alpha(s^2)}} = \frac{1}{\sqrt{a(0)}\sqrt{1+\frac{a'(0)}{2a(0)}s^2+\frac{a''(0)}{6a(0)}s^4+O(s^6)}} \\
&= \frac{1}{\sqrt{a(0)}}\left(1-\frac{a'(0)}{4a(0)}s^2-\frac{a''(0)}{12a(0)}s^4+\frac{3a'(0)^2}{32a(0)^2}s^4+O(s^6)\right)
\end{aligned} \tag{6.19}$$

$$\frac{\left(\beta(s^2)+s(x-y)\right)^2}{4\alpha(s^2)} = \frac{1}{4}\frac{\left(b(0)s^2+\frac{1}{2}b'(0)s^4+\frac{1}{6}b''(0)s^6+O(s^8)+s(x-y)\right)^2}{a(0)s^2+\frac{1}{2}a'(0)s^4+\frac{1}{6}a''(0)s^6+O(s^8)}$$

$$= \frac{1}{4a(0)}\left(b(0)s+\frac{1}{2}b'(0)s^3+\frac{1}{6}b''(0)s^5+O(s^7)+(x-y)\right)^2 \frac{1}{1+\frac{a'(0)}{2a(0)}s^2+\frac{a'(0)}{6a(0)}s^4+O(s^6)}$$

$$= \frac{1}{4a(0)}\left(b(0)s+\frac{1}{2}b'(0)s^3+\frac{1}{6}b''(0)s^5+O(s^7)+(x-y)\right)^2$$

$$\left(1-\frac{a'(0)}{2a(0)}s^2-\frac{a'(0)}{6a(0)}s^4+\frac{a'(0)^2}{4a(0)^2}s^4+O(s^6)\right)$$

$$= \frac{1}{4a(0)}\left((x-y)^2+2b(0)(x-y)s+(b(0)^2-\frac{a'(0)}{2a(0)}(x-y)^2)s^2\right.$$

$$\left.+\frac{a(0)b'(0)-a'(0)b(0)}{a(0)}(x-y)s^3+O(s^4)\right)$$

Therefore,

(6.20)

$$e^{-\frac{\left(\beta(s^2)+s(x-y)\right)^2}{4\alpha(s^2)}} = e^{-\frac{(x-y)^2}{4a(0)}}\left(1-\frac{b(0)}{2a(0)}(x-y)s-\left(\frac{b(0)^2}{4a(0)}-\frac{a'(0)}{8a(0)^2}(x-y)^2\right)s^2\right.$$

$$-\frac{a(0)b'(0)-a'(0)b(0)}{4a(0)^2}(x-y)s^3+\frac{b(0)^2}{8a(0)^2}(x-y)^2s^2$$

$$\left.+\frac{b(0)}{2a(0)}\left(\frac{b(0)^2}{4a(0)}-\frac{a'(0)}{8a(0)^2}(x-y)^2\right)(x-y)s^3-\frac{b(0)^3}{48a(0)^3}(x-y)^3s^3+O(s^4)\right)$$

$$= e^{-\frac{(x-y)^2}{4a(0)}}\left(1-\frac{b(0)}{2a(0)}(x-y)s-\left(\frac{b(0)^2}{4a(0)}-\frac{b(0)^2+a'(0)}{8a(0)^2}(x-y)^2\right)s^2\right.$$

$$\left.+\left[\frac{b(0)^3-2a(0)b'(0)+2a'(0)b(0)}{8a(0)^2}(x-y)-\frac{b(0)(b(0)^2+3a'(0))}{48a(0)^3}(x-y)^3\right]s^3+O(s^4)\right).$$

Taking the product of (6.19) and (6.20), we obtain

(6.21)

$$\frac{s}{2\sqrt{\pi\alpha(s^2)}}e^{-\frac{(\beta(s^2)+s(x-y))^2}{4\alpha(s^2)}} = \frac{1}{2\sqrt{\pi a(0)}}e^{-\frac{(x-y)^2}{4a(0)}}\left[1-\frac{b(0)}{2a(0)}(x-y)s\right.$$

$$+\left(\frac{(x-y)^2}{8a(0)^2}-\frac{1}{4a(0)}\right)\left(b(0)^2+a'(0)\right)s^2+$$

$$\left.\left(\frac{b(0)^3-2a(0)b'(0)+3a'(0)b(0)}{8a(0)^2}(x-y)-\frac{b(0)(b(0)^2+3a'(0))}{48a(0)^3}(x-y)^3\right)s^3+O(s^4)\right]$$

A routine check shows that (6.21) is exactly the same as (6.17), which confirms our *Dyson-Taylor Commutator method.*

6.4. **Applications to Stochastic Volatility models.** In this subsection we will apply our *Dyson-Taylor commutator method* to stochastic volatility models that appear in option pricing theory. Though our method is applicable to very general types of financial derivatives, for simplicity we shall only consider European style options. A European call option is a financial contract that gives the option holder the right (not obligation) to buy the underlying asset, which we assume to be a stock throughout this section, at a predetermined future date T (the maturity or expiry date) for a predetermined price K (the strike). At time T, if the stock price X_T is greater than K, then the option will be exercised with realized payoff $X_T - K$, because the option holder buy the stock with price K and he or she can immediately sell it in the market with price X_T. But at time T if $X_T < K$ then nothing will happen and the option will just go expired, so the payoff will be zero. As a result, the general payoff function is $(X_T - K)^+ := \max\{X_T - K, 0\}$ for a European call.

To price such a call option, stochastic volatility models have been intensively studied in the literature ([7, 13]). One major difficulty in such models is that we do not have tractable or explicitly computable solutions for the opton prices (for instance, the SABR model ([12, 10]) we will discuss shortly), thus numerical methods need to be applied. An alternative is to asymptotically approximate the option prices as in many papers ([12, 8, 4, 11]). In this section we shall carry out this idea and show how our method can be used to price European style options under stochastic volatility models. We shall first take the Heston model for example, the reason is that this model admits a semi-closed form formula for European call options ,so we are able to compare our results with the exact solutions numerically. Again for simplicity we assume the interest rate $r = 0$, then the stock price X_t satisfies the following stochastic process

$$dX_t = \sqrt{Y_t} X_t dW_t^1$$

and the volatility term Y_t itself satisfies another CIR process given by

$$dY_t = (a - bY_t)dt + \sigma\sqrt{Y_t} dW_t^2,$$

where W_t^1 and W_t^2 are standard Brownian motions with correlation $dW_t^1 dW_t^2 = \rho dt$.

Using the same notations as we elaborated above, a call option price $u(t, x, y)$ is the solution to a parabolic PDE

$$\begin{aligned} &u_t(t,x,y) = \\ &\frac{1}{2}yx^2 u_{xx}(t,x,y) + \rho\sigma yx u_{xy}(t,x,y) + \frac{1}{2}y\sigma^2 u_{yy}(t,x,y) + (a-by)u_y(t,x,y) \end{aligned} \tag{6.22}$$

with initial condition $u(0, x, y) = (x - K)^+$ by Feymann-Kac formula [25]. In this PDE we abused the notations and actually did the change of variable $t \to T - t$.

Then by exactly the same procedure as we did in previous sections, we are able to obtain the zeroth order approximated Green's function of the Heston PDE

$$\begin{aligned} &\mathcal{H}_t^{[0]}(x,y,w,z) = \\ &\frac{1}{2\pi t\sigma xy\sqrt{1-\rho^2}} \exp\left(-\frac{\sigma^2(x-w)^2 - 2\rho\sigma x(x-w)(y-z) + x^2(y-z)^2}{2t\sigma^2 x^2 y(1-\rho^2)}\right) \end{aligned} \tag{6.23}$$

Next we compute the first order approximation of the Green's function for the Heston model. By our main result, the first order approximation is

$$\mathcal{H}_t^{[1]} = (1 + sQ_1)\, e^{L_0},$$

where

$$Q_1 = (L_1 + \frac{1}{2}[L_0, L_1]).$$

We can compute it explicitly as follows:

$$
\begin{aligned}
Q_1 e^{L_0}(x,y) = {} & C_{3,0}\left(\frac{\partial^3}{\partial x^3} e^{L_0}(x,y)\right) + C_{2,1}\left(\frac{\partial^3}{\partial x^2 \partial y} e^{L_0}(x,y)\right) \\
& + C_{1,2}\left(\frac{\partial^3}{\partial x \partial y^2} e^{L_0}(x,y)\right) + C_{0,3}\left(\frac{\partial^3}{\partial y^3} e^{L_0}(x,y)\right) \\
& + C_{0,1}\left(\frac{\partial}{\partial y} e^{L_0}(x,y)\right),
\end{aligned} \tag{6.24}
$$

where

$$
\begin{aligned}
C_{3,0}(x,y) &= \left(\frac{1}{2}y + \frac{1}{4}\rho\sigma\right) x^3 y \\
C_{2,1}(x,y) &= \left(\rho\sigma y + \frac{1}{2}\rho^2\sigma^2 + \frac{1}{4}\sigma^2\right) x^2 y \\
C_{1,2}(x,y) &= \frac{1}{2}\rho^2\sigma^2 x y^2 + \frac{1}{4}\rho\sigma^3 x y + \frac{1}{2}\rho\sigma^3 x y \\
C_{0,3}(x,y) &= \frac{1}{4}\sigma^4 y \\
C_{0,1}(x,y) &= a - by.
\end{aligned}
$$

Denote

$$
x_t = \frac{x-w}{\sqrt{t}}, \quad y_t = \frac{y-z}{\sqrt{t}}
$$

$$
e^{L_0}(x,y) := const(x_0,y_0) \cdot e^{\alpha x^2 + \beta x y + \gamma y^2},
$$

$$
D_{i,j}(x,y) e^{L_0}(x,y) = \frac{\partial^{i+j}}{\partial x^i y^j} e^{L_0}(x,y),
$$

where

$$
const(x_0,y_0) = \frac{1}{2\pi\sigma x_0 y_0 \sqrt{1-\rho^2}}.
$$

Then

$$
\begin{aligned}
D_{3,0}(x,y) &= 12\alpha^2 x + 6\alpha\beta y + 8\alpha^3 x^3 + 12\alpha^2\beta x^2 y + 6\alpha\beta^2 x y^2 + \beta^3 y^3 \\
D_{2,1}(x,y) &= 6\alpha\beta x + (4\alpha\gamma + 2\beta^2) y + 4\alpha^2\beta x^3 + (8\alpha^2\gamma + 4\alpha\beta^2) x^2 y \\
&\quad + (8\alpha\beta\gamma + \beta^3) x y^2 + 2\beta^2\gamma y^3 \\
D_{1,2}(x,y) &= (2\beta^2 + 4\alpha\gamma) x + 6\beta\gamma y + 2\alpha\beta^2 x^3 + (8\alpha\beta\gamma + \beta^3) x^2 y \\
&\quad + (8\alpha\gamma^2 + 4\beta^2\gamma) x y^2 + 4\beta\gamma^2 y^3 \\
D_{0,3}(x,y) &= 6\beta\gamma x + 12\gamma^2 y + \beta^3 x^3 + 6\beta^2\gamma x^2 y + 12\beta\gamma^2 x y^2 + 8\gamma^3 y^3 \\
D_{0,1}(x,y) &= \beta x + 2\gamma y.
\end{aligned}
$$

In the Heston case, we have

$$
\begin{aligned}
\alpha(x_0,y_0) &= -\frac{1}{2(1-\rho^2)x_0^2 y_0} \\
\beta(x_0,y_0) &= \frac{\rho}{\sigma(1-\rho^2)x_0 y_0} \\
\gamma(x_0,y_0) &= -\frac{1}{2\sigma^2(1-\rho^2)y_0}.
\end{aligned}
$$

Note that in all the Ds, α, β and γ are evaluated at (x, y). Finally, the first order approximation of the Green function for the Heston model reads

(6.25)

$$\mathcal{H}_t^{[1]}(x,y,w,z) =$$
$$\mathcal{H}_t^{[0]}(x,y,w,z)\Big(1+\sqrt{t}\sum_{j=0}^{3}(C_{3-j,j}(x,y)\cdot D_{3-j,j}(x_t,y_t)+C_{0,1}(x,y)\cdot D_{0,1}(x_t,y_t))\Big).$$

Once we have the Green's functions of the Heston PDE, we can obtain the call option price by integrating the Green's function against the payoff function (initial data). In in a work in progress, we plan to compare numerically our approximation with Heston's formula, and we will show that our approximations are extremely accurate. We plan to actually give general formulas of the first and second order approximations of the Green's function of a general parabolic PDE. In the above Heston example, the final formula we obtained is just a special case of our general first order approximation. If we want to get more accurate results, then the second order approximation should be caught out. It is worth indicating that our general formulas actually can be applied to any other option pricing models. As another example, we consider a min/max options under the Bivariate Black-Scholes-Merton (BSM) model. In this type of options there are two underlying assets, and under the BSM framework the prices of two assets S_t and S_2 satisfies

$$\begin{aligned} dS_1 &= rS_1dt + \sigma_1 S_1 dW_t^1 \\ dS_2 &= rS_2dt + \sigma_2 S_2 dW_t^2, \end{aligned}$$

where again W_t^1 and W_t^2 are Brownian motions with correlation ρ. For the European min option written on these two assets with strike K, the payoff at maturity is $\max(\min(S_1(T), S_2(T)) - K, 0)$. A routine argument show that when time to maturity is t, the min option price $u(t, S_1, S_2)$ is the solution to the following Cauchy problem

$$(6.26)\qquad \begin{cases} u_t = \frac{1}{2}\sigma_1^2 S_1^2 \partial_{S_1}^2 u + \rho\sigma_1\sigma_2 S_1 S_2 \partial_{S_1}\partial_{S_2} u + \frac{1}{2}\sigma_2^2 S_2^2 \partial_{S_2}^2 u \\ \qquad + r\sigma_1 S_1 \partial_{S_1} u + r\sigma_2 S_2 \partial_{S_2} u - ru \\ u(0, S_1, S_2) = \max(\min(S_1, S_2) - K, 0). \end{cases}$$

This is also a second order parabolic PDE and it fits into our setting. Therefore, the same approximation procedure as we did in the Heston case applies to European min/man options as well. Similarly, bivariate BSM models are also tractable. Johnson (1987) and Stulz (1982) show that the min option price has a closed form representation. So we are also able to compare our approximations of the European min option prices with the exact prices.

So far we have considered two tractable option pricing models. Though closed form exact formulas are available in these models, they can not accurately capture the features of option prices. To overcome this difficulty, complicated models have been developed. But the tradeoff is that these models do not admit closed form solutions. Take the SABR model ([10]) for example, like the Heston model, it is also a stochastic volatility model, and the two processes that drive the stock price and the volatility are

$$
\begin{aligned}
dX_t &= Y_t X_t^{\beta} dW_t^1 \\
dY_t &= \sigma Y_t dW_t^2,
\end{aligned}
\tag{6.27}
$$

where σ, β are constant and $0 \leq \beta \leq 1$. This model has been intensively used in fixed income and commodities modeling. If we want, we can also add mean-reverting feature in the volatility pricess. Unless $\beta = 0, 1$, no explicit pricing formulas for this model are available in the literature to the authors' knowledge. Therefore, numerical methods like finite difference or finite element have to be used. But these numerical methods are in general very slow and there are also other issues. However, our approximation methods are still applicable to this case, and we are able to obtain accurate first order and second order formulas for the SABR model. More detailed numerical tests and discussions will be included in work in progress.

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