DYNAMICS AND STABILITIES OF GENERALIZED FORCHHEIMER FLOWS WITH THE FLUX BOUNDARY CONDITION

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DYNAMICS AND STABILITIES OF GENERALIZED FORCHHEIMER FLOWS WITH THE FLUX BOUNDARY CONDITION

LUAN HOANG† AND AKIF IBRAGIMOV

ABSTRACT. We study generalized Forchheimer equations for slightly compressible fluids in porous media subjected to the flux condition on the boundary. We derive estimates for the pressure, its gradient and time derivative in terms of the time-dependent boundary data. For the stability, we establish the continuous dependence of the pressure and pressure gradient on the boundary flux and coefficients of the Forchheimer polynomial in the momentum equation. In particular, we show the asymptotic dependence of the shifted solution on the asymptotic behavior of the boundary data. In order to improve estimates of various types, we prove and utilize suitable Poincaré-Sobolev and nonlinear Gronwall inequalities, as well as obtain Gronwall-type inequalities from a system of coupled differential inequalities. We also introduce additional flux-related quantities as controlling parameters of fluid flows for large time in case of unbounded fluxes.

CONTENTS

1. INTRODUCTION

In this article we study qualitative behaviors of time dependent Forchheimer flows in porous media for large time. It was observed from many experiments and actual field data that Darcy's law is not adequate to describe fluid flows in porous media (c.f. [24, 17, 6, 20]). After Darcy's work [10], Forchheimer (c.f. [17]) proposed three types of nonlinear equations of motion to capture deviations from

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Darcy flows, namely, two-term, three-term and power laws (see also [6, 25, 31, 33]). It is worth to mention that Darcy and Dupuit already observed the deviation from linear relation in there original works (c.f. [10, 13]). In later years these equations were studied from different points of views. Engineers and physicists adopt Forchheimer models for numerical simulations and data analysis of the filtration of the flows with high velocity and fractured porous media (c.f. [12, 5, 14] and references therein). In a number of mathematical papers Forchheimer models were studied to capture dissipation associated with flows in small porous channels. In a series of the papers started by Straughan and Payne [26, 27, 28, 29] and then by [8, 16, 36, 7, 35], Forchheimer and related Brinkman-Forchheimer models for incompressible fluids are investigated to analyze the structural stability, dependence on initial data, global attractors, etc. In those papers particular Forchheimer terms are used. However what terms should be present in Forchheimer equation are still in debate, from theoretical and experimental points of view (see e.g. [4, 23]). To analyze all possible scenarios, generalized Forchheimer equation is proposed in our previous papers [1, 18] (see also [12]). Note that we do not propose new laws of physics but rather do general analysis which is applicable to many cases arising in physics and engineering.

We study a fluid in a porous medium which satisfies a generalized polynomial relation between the velocity and pressure gradient. Such fluid flow is called generalized Forchheimer flow. Under the slightly compressibility condition, the system of equations governing the fluid's motion can be transformed into a degenerate parabolic equation for the pressure. Its degeneracy is defined by the degree of nonlinearity of the Forchheimer polynomial in the momentum equation. We focus on the initial boundary value problem (IBVP) for that parabolic equation. Our previous paper [1] analyzes the stability of the so-called pseudo-steady state solutions and their similar types. These are widely used to estimate productivity indices in reservoir engineering (see [2, 3, 30]) and satisfy very particular boundary conditions. In recent paper [18] we establish the structural stability with respect to the time-dependent boundary data and coefficients of the Forchheimer polynomials for the IBVP with the Dirichlet boundary condition. In the current paper time-dependent Neumann boundary data are considered. Although the general approach is similar to [18], new techniques are used and new flux-related quantities are introduced to analyze more accurately the behavior of the solutions, particularly for large time. These result in sharper estimates for pressure and its derivatives, more precise description of the flow for large time in terms of the asymptotics of the boundary data and the coefficients in the Forchheimer equation, and allowing the analysis for a wider class of unbounded (in time) boundary fluxes. Also new here are the stability results of the pressure gradient. Below we highlight and discuss the results in each section.

In section 2 we formulate the problem and recall relevant results from our previous works. Degeneracy in our parabolic equation can be modeled by $(1+|\nabla p|^a)^{-1}$, where p is the pressure and constant a between 0 and 1 is defined by the degree of the Forchheimer polynomial (see (2.8), (2.10) and (2.9)). In connections with this type of degeneracy and with the structure of the equation, we derive suitable inequalities of Poincaré-Sobolev type in Lemma 2.1 and trace estimates in Lemma 2.2. In section 3 we investigate a class of non-autonomous nonlinear differential inequalities arising from our estimation of the solution to the IBVP. Nonlinear analog of Gronwall inequality is obtained for solutions to those differential inequalities. Moreover we estimate the solutions when time is large, as well as their limits superior when time goes to infinity, with the bounds depending only on the asymptotic behavior of the forcing term. Section 4 consists of bounds for the solution $p(x,t)$ to the IBVP in terms of the initial data and boundary fluxes. In order to derive practically effective estimates for the solution and better characterize its dependence on the data, we use the shifted solution $\bar{p}(x,t)$ which is a simple translation of $p(x,t)$ by, roughly speaking, the total flux and initial datum (c.f. the discussion following (4.8)). One of the features of this section is uniform Gronwall-type inequalities for a system of coupled nonlinear differential inequalities. These uniform Gronwall estimates result in asymptotic bounds for $\bar{p}(x,t)$ in terms of the asymptotics of the boundary data's average on each time interval $(t-1,t)$ as $t\to\infty$, c.f. Theorems 4.4 and 4.5. We use the number β and function $t \to \int_{t-1}^t \tilde{f}(\tau)d\tau$ (see (4.28) and (4.5)), which quantify specific oscillations of the boundary fluxes, as auxiliary parameters to control the fluid flows for large time. (Similar quantities will be used in the stability analysis in section 5.) This is contrasting to [18] where estimates for similar shifted solutions depend on the accumulation of values of the Dirichlet data for all time. Section 5 is divided into four subsections where we establish and present different continuous dependence and structural stability results. In subsection 5.1 we establish the structurally stability for the constant pressure fluid regime by analyzing the dependence of the functionals $J[\bar{p}](t)$, $J_H[p](t)$ and $J[p_t](t)$ (see notations in (4.15)) on the boundary data. Results in this subsection provide non-trivial conditions on boundary fluxes that guarantee the stability, allowing large accumulative flux in time with small oscillation at time infinity. These can be viewed as analysis for "boundary dominated regime" for the Neumann boundary condition. (See [18] for the case of Dirichlet boundary condition.) In subsection 5.2 we prove the continuous dependence of the solution on the initial data and the general boundary flux. In particular, we find asymptotic bounds for L^2-norm of $\bar{p}_1-\bar{p}_2$ with respect to the difference of two boundary fluxes, where p_1 and p_2 are two solutions corresponding to those fluxes. Limit superior estimates are obtained under certain constraints on the asymptotic behavior at time infinity of the individual boundary flux (see Theorem 5.6(iii) and Theorem 5.9). In subsection 5.3 we estimate the L^{2-a}-norm of $\nabla p_1-\nabla p_2$ both for finite time and at time infinity. Though an additional condition is imposed when compared to its counterpart in subsection 5.2, estimates for limits superior are obtained (see Theorems 5.14) and are applicable to fast growing data (see Remark 5.15). In subsection 5.4 are improved results, compared to [18], on the continuous dependence on coefficients of the Forchheimer polynomials for the solution and its gradient. This is carried out by combining our new estimates with the perturbed monotonicity property in [18].

2. Preliminaries and auxiliary results

Consider a fluid in a porous medium occupying a bounded domain U in space $\mathbb{R}^n$. For physics problem $n=3$, but here we consider any $n\geq 2$. Let $x\in\mathbb{R}^n$ and $t\in\mathbb{R}$ be the spatial and time variables. The fluid flow has velocity $u(x,t)\in\mathbb{R}^n$, pressure $p(x,t)\in\mathbb{R}$ and density $\rho(x,t)\in\mathbb{R}^+=[0,\infty)$.

A generalized Forchheimer equation, which is considered as a momentum equation, is studied in [1, 18] and has the form:

$$g(|u|)u = -\nabla p, \tag{2.1}$$

where $g(s) \geq 0$ is a function defined on $[0, \infty)$. When $g(s) = \alpha, \alpha + \beta s, \alpha + \beta s + \gamma s^2, \alpha + \gamma_m s^{m-1}$, where $\alpha, \beta, \gamma, m, \gamma_m$ are empirical constants, we have Darcy's law, Forchheimer's two-term, three-term and power laws, respectively.

In this paper, we study the case when the function g in (2.1) is a generalized polynomial with non-negative coefficients. More precisely, the function $g : \mathbb{R}_+ \to \mathbb{R}_+$ is of the form

$$g(s) = a_0 s^{\alpha_0} + a_1 s^{\alpha_1} + \ldots + a_N s^{\alpha_N}, \ s \geq 0, \tag{2.2}$$

where $N \geq 1$, $\alpha_0 = 0 < \alpha_1 < \ldots < \alpha_N$ are fixed numbers, the coefficients $a_0, a_1, \ldots, a_N$ are non-negative with $a_0 > 0$ and $a_N > 0$.

From (2.1) one can solve u implicitly in terms of ∇p and derives a nonlinear Darcy equation:

$$u = -K(|\nabla p|)\nabla p. \tag{2.3}$$

The function $K : \mathbb{R}_+ \to \mathbb{R}_+$ is defined by

$$K(\xi) = \frac{1}{g(s(\xi))}, \text{ where } s = s(\xi) \geq 0 \text{ satisfies } sg(s) = \xi, \quad \text{for } \xi \geq 0. \tag{2.4}$$

The number α_N is the degree of g and is denoted by $\deg(g)$. The vector of powers in (2.2) is denoted by $\vec{\alpha} = (\alpha_0, \ldots, \alpha_N)$, and the vector $\vec{a} = (a_0, \ldots, a_N)$ is referred to as the coefficient vector. When the dependence on $\vec{a}$ needs be specified, we use notation $g(s, \vec{a})$, $K(\xi, \vec{a})$, $s(\xi, \vec{a})$ to denote respective/corresponding functions in (2.2) and (2.4).

Other equations governing the fluid's motion are the equation of continuity:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0, \tag{2.5}$$

and the equation of state which, for slightly compressible fluids, is

$$\frac{d\rho}{dp} = \kappa \rho, \text{ or } \rho(p) = \rho_0 \exp\left(\frac{p - p_0}{\kappa}\right), \quad \kappa > 0. \tag{2.6}$$

From (2.3), (2.5) and (2.6) one derives a scalar equation for the pressure:

$$\frac{\partial p}{\partial t} = \kappa \nabla \cdot (K(|\nabla p|)\nabla p) + K(|\nabla p|)|\nabla p|^2. \tag{2.7}$$

On the right-hand side (2.7), the constant κ is very large for most slightly compressible fluids in porous media [24, 9], hence we neglect its second term and study the reduced equation

$$\frac{\partial p}{\partial t} = \kappa \nabla \cdot (K(|\nabla p|)\nabla p). \tag{2.8}$$

Note that this reduction is commonly used in engineering. By changing the reference system [1], we obtain a non-dimensional equation which is (2.8) with $\kappa = 1$. Hence hereafter we assume that $\kappa = 1$.

The class of functions $g(s)$ as in (2.2) is denoted by $\mathrm{FP}(N, \vec{\alpha})$, which is the abbreviation of "Forchheimer polynomials". When the function g in (2.1) is one of the $g(s)$ in (2.2), it is referred to as the Forchheimer polynomial.

Let $g = g(s, \vec{a})$ in $\mathrm{FP}(N, \vec{\alpha})$. The following two exponents are frequently used in our calculations:

$$a = \frac{\alpha_N}{1+\alpha_N} \in (0,1), \; b = \frac{a}{2-a} = \frac{\alpha_N}{2+\alpha_N} \in (0,1). \tag{2.9}$$

The function $K(\xi)$ has the following properties (c.f. [1, 18]): it is decreasing in ξ mapping $\xi \in [0,\infty)$ onto $(0, 1/a_0]$, and

$$\frac{C_1}{(1+\xi)^a} \le K(\xi, \vec{a}) \le \frac{C_2}{(1+\xi)^a}, \tag{2.10}$$

$$C_3(\xi^{2-a} - 1) \le K(\xi)\xi^2 \le C_2 \xi^{2-a}, \tag{2.11}$$

$$-aK(\xi) \le K'(\xi)\xi \le 0, \tag{2.12}$$

where C_1, C_2, C_3 are positive constants depending on U and g.

As in [1, 18] the following function is crucial to our estimates. We define

$$H(\xi) = \int_0^{\xi^2} K(\sqrt{s})ds \text{ for } \xi \ge 0. \tag{2.13}$$

The function $H(\xi)$ can be compared with ξ and $K(\xi)$ by

$$K(\xi)\xi^2 \le H(\xi) \le 2K(\xi)\xi^2. \tag{2.14}$$

As a consequence of (2.14) and (2.11), one has

$$C_3(\xi^{2-a} - 1) \le H(\xi) \le 2C_2\xi^{2-a}. \tag{2.15}$$

Degree Condition: $\deg(g) \le \dfrac{4}{n-2}$.

We will assume the Degree Condition very often in this paper, but not always. Whenever this condition is met, the Sobolev space $W^{1,2-a}(U)$ is continuously embedded into $L^2(U)$. The Poincaré-Sobolev inequality [22] is modified into the following form which takes into account the nonlinearity in our equation.

Lemma 2.1. *Assume the Degree Condition. Let u belong to $W^{1,2-a}(U)$ and satisfy either $\int_U u dx = 0$ or $u\big|_\Gamma = 0$. Then*

$$\int_U u^2 dx \le \varphi_{c_0,\gamma_0}\left(\int_U H(|\nabla u|)dx\right), \tag{2.16}$$

where $\gamma_0 = 2/(2-a)$, $c_0 > 0$, and

$$\varphi_{c,\gamma}(z) = cz + c^\gamma z^\gamma, \; \text{for } c > 0, \; \gamma > 0, \; z \ge 0. \tag{2.17}$$

Consequently,

$$\int_U H(|\nabla u|)dx \ge \varphi^{-1}_{c_0,\gamma_0}\left(\int_U u^2 dx\right). \tag{2.18}$$

Proof. For any $v \in W^{1,2-a}(U)$, it is proved in Lemma 2.4 of [18] that

$$\int_U u^2 dx \le C_4 \Big(\int_U K(|\nabla v|)|\nabla u|^2 dx\Big)\Big(1 + \int_U H(|\nabla v|)dx\Big)^{\frac{a}{2-a}}. \tag{2.19}$$

Taking $v = u$, then (2.19) and (2.14) yield

$$\int_U u^2 dx \le C_4 \int_U H(|\nabla u|)dx + C_4\Big(\int_U H(|\nabla u|)dx\Big)^{1+\frac{a}{2-a}}.$$

hence (2.16) follows with $c_0 = \max\{C_4, C_4^{1/\gamma_0}\}$. $\square$

Note that, by Hölder's inequality, it follows from (2.16) that

$$\Big(\int_U u^2 dx\Big)^{\frac{2-a}{2}} \le C_5\Big(\int_U H(|\nabla u|)dx + 1\Big). \tag{2.20}$$

This is the inequality we used to estimate the solution in our previous works [1, 18]. We will see that (2.18) leads to improved estimates in this paper.

Next we derive trace estimates suitable to our nonlinear problem.

Lemma 2.2. *(i) If $u \in W^{1,2}(U)$, $v \in W^{1,2-a}(U)$ and $\int_U u dx = 0$ then*

$$\int_\Gamma |u| d\sigma \le \varepsilon \int_U K(|\nabla v|)|\nabla u|^2 dx + C\varepsilon^{-1}\Big(1 + \int_U H(|\nabla v|)dx\Big)^b, \tag{2.21}$$

$$\begin{aligned}\int_\Gamma |u| d\sigma &\le \varepsilon \int_U K(|\nabla v|)|\nabla u|^2 dx + \delta \int_U H(|\nabla v|)dx \\ &\quad + C\varepsilon^{-1} + C\varepsilon^{-\frac{2-a}{2-2a}}\,\delta^{-\frac{a}{2-2a}},\end{aligned} \tag{2.22}$$

for any $\varepsilon, \delta > 0$, where C is a positive constant.

(ii) If $u \in W^{1,2-a}(U)$ satisfies $\int_U u dx = 0$ then

$$\int_\Gamma |u| d\sigma \le \varepsilon \int_U H(|\nabla u|)dx + C\varepsilon^{-1} + C\varepsilon^{-\frac{1}{1-a}}, \tag{2.23}$$

for any $\varepsilon > 0$, where C is a positive constant.

Proof. (i) By the trace theorem, Poincaré-Sobolev and Young inequalities, we have

$$\begin{aligned}\int_\Gamma |u| d\sigma &\le C\int_U |\nabla u| dx \\ &\le \varepsilon \int_U K(|\nabla v|)|\nabla u|^2 dx + \frac{C}{4\varepsilon}\int_U K(|\nabla v|)^{-1}dx \\ &\le \varepsilon \int_U K(|\nabla v|)|\nabla u|^2 dx + C\varepsilon^{-1}\int_U (1 + |\nabla v|^a)dx \\ &\le \varepsilon \int_U K(|\nabla v|)|\nabla u|^2 dx + C\varepsilon^{-1}\int_U 1 + H(|\nabla v|)^{\frac{a}{2-a}} dx.\end{aligned} \tag{2.24}$$

Applying Hölder's inequality to the last integral on the right-hand side of (2.24) we have

$$\int_\Gamma |u| d\sigma \le \varepsilon \int_U K(|\nabla v|)|\nabla u|^2 dx + C\varepsilon^{-1} + C\varepsilon^{-1}\Big(\int_U H(|\nabla v|)dx\Big)^{\frac{a}{2-a}}. \tag{2.25}$$

Then (2.21) follows. Next, applying Young's inequality to the last term of (2.25) we get (2.22).

(ii) Following the proof in part (i), we can apply (2.22) with $v = u$ and ε, δ having value $\varepsilon/2$, hence obtain (2.23). □

For our applications, we use the following direct consequences of Lemma 2.2.

Lemma 2.3. *Let w belong to $L^\infty(\Gamma)$.*

(i) If $u \in W^{1,2}(U)$, $v \in W^{1,2-a}(U)$, $\int_U u dx = 0$ and $\varepsilon, \delta > 0$ then

$$\int_\Gamma |w\,u| d\sigma \le \varepsilon \int_U K(|\nabla v|)|\nabla u|^2 dx + C\varepsilon^{-1}\|w\|^2_{L^\infty(\Gamma)}\Big(1 + \int_U H(|\nabla v|)dx\Big)^b, \tag{2.26}$$

$$\begin{aligned}(2.27)\qquad \int_\Gamma |w\,u| d\sigma &\le \varepsilon \int_U K(|\nabla v|)|\nabla u|^2 dx + \delta \int_U H(|\nabla v|)dx\\ &\quad + C\varepsilon^{-1}\|w\|^2_{L^\infty(\Gamma)} + C\varepsilon^{-\frac{2-a}{2-2a}}\,\delta^{-\frac{a}{2-2a}}\,\|w\|^{\frac{2-a}{1-a}}_{L^\infty(\Gamma)},\end{aligned}$$

(ii) If $u \in W^{1,2-a}(U)$ satisfies $\int_U u dx = 0$ and $\varepsilon > 0$ then

$$(2.28)\qquad \int_\Gamma |w\,u| d\sigma \le \varepsilon \int_U H(|\nabla u|)dx + C\varepsilon^{-1}\|w\|^2_{L^\infty(\Gamma)} + C\varepsilon^{-\frac{1}{1-a}}\,\|w\|^{\frac{2-a}{1-a}}_{L^\infty(\Gamma)}.$$

Above, C is a positive constant.

Proof. Without loss of generality, we assume $\|w\|_{L^\infty(\Gamma)} \ne 0$. By (2.21), we have

$$\begin{aligned}&\int_\Gamma |w\,u| d\sigma \le \|w\|_{L^\infty(\Gamma)} \int_\Gamma |u| d\sigma\\ &\le \varepsilon'\|w\|_{L^\infty(\Gamma)} \int_U K(|\nabla v|)|\nabla u|^2 dx + C\varepsilon'^{-1}\|w\|_{L^\infty(\Gamma)}\Big(1 + \int_U H(|\nabla v|)dx\Big)^b,\end{aligned}$$

for any $\varepsilon' > 0$. Taking $\varepsilon' = \varepsilon\|w\|^{-1}_{L^\infty(\Gamma)}$ yields (2.26). Similarly, we obtain (2.27) and (2.28) from (2.22) and (2.25), respectively. □

For the monotonicity of the differential operator in (2.8), we define

$$(2.29)\qquad \Phi(y, y') = (K(|y'|)y' - K(|y|)y)\cdot(y' - y), \quad \text{for} \quad y, y' \in \mathbb{R}^n.$$

We recall Lemma III.11 in [1] (see also Lemma 2.3 in [18]).

Lemma 2.4 (c.f. [1])**.** *Let the function g belong to FP($N, \vec{\alpha}$). If p_1 and p_2 are two functions in $W^{1,2}(U)$, then*

$$\begin{aligned}(2.30)\qquad \int_U \Phi(\nabla p_1, \nabla p_2)dx &\ge C_6 \left(\int_U |\nabla(p_1 - p_2)|^{2-a} dx\right)^{\frac{2}{2-a}}\\ &\quad \cdot \Big(1 + \int_U |\nabla p_1|^{2-a} dx + \int_U |\nabla p_2|^{2-a} dx\Big)^{-b}.\end{aligned}$$

where C_6 is a positive constant.

For explicit dependence on $\vec{a}$ of positive constants appearing above, see [18].

Notation. *We will denote C a generic positive constant which may change from line to line, may depend on the domain U, dimension n and the Forchheimer polynomial, but independent of the initial and boundary data. The Lebesgue norm $\|\cdot\|_{L^2}, \|\cdot\|_{L^{2-a}}$ are understood over the domain U. The partial derivative with respect to time t can be denoted differently, for instance, by $\frac{\partial p}{\partial t}$, $\partial_t p$, or p_t. For a function $\psi(x,t)$ defined on $\Gamma\times[0,\infty)$, we use the short-hand notation $\|\psi(t)\|_{L^\infty} = \|\psi(\cdot,t)\|_{L^\infty(\Gamma)}$, $\|\psi_t(t)\|_{L^\infty} = \|\psi_t(\cdot,t)\|_{L^\infty(\Gamma)}$.*

3. Nonlinear differential inequalities

In this section, we derive bounds for solutions of nonlinear differential inequalities. They will be used to establish estimates, especially the asymptotic ones, of the solution to the IBVP (4.1), (4.2) and (4.3) in next section.

We consider a differential inequality of the form

$$(3.1)\qquad y'(t) \le -\phi^{-1}(y(t)) + f(t),$$

where the forcing term $f(t)$ is a non-negative continuous function on $[0,\infty)$, and $\phi(z)$ is a continuous, strictly increasing function from $[0,\infty)$ onto $[0,\infty)$. Note that $\phi(0)=0$ and $\phi(\infty)\stackrel{\rm def}{=\!=}\lim_{z\to\infty}\phi(z)=\infty$.

Let $y(t)$ be a non-negative solution of (3.1) that belongs to $C([0,\infty))$ and $C^1((0,\infty))$.

First, we obtain an estimate of $y(t)$ for all $t\ge 0$. The proof is the same as in Lemma A.2 of [18].

Lemma 3.1 (c.f. [18])**.** *For all* $t\ge 0$

$$y(t)\le \max\{y(0),\phi(F(t))\}\le y(0)+\phi(F(t)), \tag{3.2}$$

where $F(t)$ *is a continuous, increasing majorant of* $f(t)$ *on* $[0,\infty)$*.*

Here $F(t)$ being a majorant of $f(t)$ on $[0,\infty)$ means that $F(t)\ge f(t)$ for all $t\in[0,\infty)$.

The function $F(t)$ can be $F(t)=\max_{[0,t]}f(t)$, or in case $f(t)$ is of class $C^1([0,\infty))$,

$$F(t)=f(0)+\int_0^t (f')^+(s)ds.$$

Throughout, for a function $g(t)$ we denote the positive and negative parts by $g^+(t)=\max\{0,g(t)\}$ and $g^-(t)=\max\{0,-g(t)\}$, respectively.

Our next goal is to find an estimate for $\limsup_{t\to\infty}y(t)$ which only depends on the asymptotic behavior of $f(t)$ as $t\to\infty$. First we need the following simple, technical result.

Lemma 3.2. *Let* $g(t)$ *be a continuous majorant of* $f(t)$ *on* $[0,\infty)$*. If* $T\ge 0$ *and* $\varepsilon>0$ *satisfy*

$$y(T)\ge \phi(g(T)+2\varepsilon), \tag{3.3}$$

then there exists $t_0>T$ *such that*

$$\phi(g(t_0)+\varepsilon)<y(t_0)<\phi(g(t_0)+2\varepsilon). \tag{3.4}$$

Proof. Suppose $y(t)>\phi(g(t)+\varepsilon)$ for all $t>T$. Then from the differential inequality (3.1) we have $y'(t)\le -\varepsilon-g(t)+f(t)\le -\varepsilon$ for all $t>T$, hence

$$y(t)\le y(T)-\varepsilon(t-T)$$

for all $t>T$. Thus $y(t)\to-\infty$ as $t\to\infty$ which contradicts $y(t)$ being non-negative. Therefore there is $t_1>T$ such that $y(t_1)\le\phi(g(t_1)+\varepsilon)$. This and (3.3) imply

$$y(T)-\phi(g(T)+3\varepsilon/2)>0 \text{ and } y(t_1)-\phi(g(t_1)+3\varepsilon/2)<0.$$

By the intermediate value theorem, there exists t_0 between T and t_1 such that $y(t_0)=\phi(g(t_0)+3\varepsilon/2)$, and hence (3.4) follows. ◻

In the following, we assume additionally that $\phi\in C^1(0,\infty)$. As a consequence $\phi'(z)\ge 0$ for all $z>0$.

Lemma 3.3. *Let* $g\in C([0,\infty))\cap C^1((0,\infty))$ *be a majorant of* $f(t)$ *on* $[0,\infty)$*. Given* $\varepsilon>0$*, if*

$$\limsup_{t\to\infty}\frac{\left[\frac{d}{dt}\phi(g(t)+\varepsilon)\right]^-}{g(t)-f(t)+\varepsilon/2}<1, \tag{3.5}$$

then there is $T>0$ such that

$$y(t)\le \phi(g(t)+\varepsilon),\qquad \textit{for all } t>T. \tag{3.6}$$

Proof. By (3.5), there are $\delta>0$ and $t_0\ge 0$ such that

$$\left[\frac{\frac{d}{dt}\phi(g(t)+\varepsilon)}{g(t)-f(t)+\varepsilon/2}\right]^-\le (1-\delta),\quad \forall t\ge t_0. \tag{3.7}$$

If (3.6) holds for $T=t_0$, then there is nothing else to prove. Otherwise, we will show that (3.6) holds for some $T>t_0$. In this case, there is $t_0'\ge t_0$ such that

$$y(t_0')>\phi(g(t_0')+\varepsilon). \tag{3.8}$$

Applying Lemma 3.2 for $g(t)=f(t)$ and ε replaced by $\varepsilon/2$, we find $T>t_0'$ such that

$$\phi(g(T)+\varepsilon/2)<y(T)<\phi(g(T)+\varepsilon). \tag{3.9}$$

Claim: (3.6) holds true for such T.

Suppose not. Then there is $T'>T$ such that

$$y(T')>\phi(g(T')+\varepsilon). \tag{3.10}$$

Thanks to (3.9) and (3.10) there are t_1 and t_2 that satisfy $T\le t_1<t_2\le T'$ and

$$\phi(g(t)+\varepsilon/2)\le y(t)<\phi(g(t)+\varepsilon),\quad \forall t\in[t_1,t_2), \tag{3.11}$$

$$y(t_2)=\phi(g(t_2)+\varepsilon). \tag{3.12}$$

For $t\in(t_1,t_2)$, it follows from (3.1) and (3.11) that $y'(t)\le -\varepsilon/2-g(t)+f(t)$, hence $y(t_2)-y(t_1)\le -\int_{t_1}^{t_2} g(s)-f(s)+\varepsilon/2ds$. Thus

$$\begin{aligned}
y(t_1)&\ge y(t_2)+\int_{t_1}^{t_2} g(s)-f(s)+\varepsilon/2ds\\
&=\phi(g(t_2)+\varepsilon)+\int_{t_1}^{t_2} g(s)-f(s)+\varepsilon/2ds\\
&=\phi(g(t_1)+\varepsilon)+\int_{t_1}^{t_2}\frac{d}{ds}\phi(g(s)+\varepsilon)ds+\int_{t_1}^{t_2} g(s)-f(s)+\varepsilon/2ds.
\end{aligned}$$

By (3.7), note that

$$\begin{aligned}
\int_{t_1}^{t_2}\frac{d}{ds}\phi(g(s)+\varepsilon)ds&=\int_{t_1}^{t_2}\left[\frac{\frac{d}{ds}\phi(g(s)+\varepsilon)}{g(s)-f(s)+\varepsilon/2}\right]\big(g(s)-f(s)+\varepsilon/2\big)ds\\
&\ge -\int_{t_1}^{t_2}\left[\frac{\frac{d}{ds}\phi(g(s)+\varepsilon)}{g(s)-f(s)+\varepsilon/2}\right]^-\big(g(s)-f(s)+\varepsilon/2\big)ds\\
&\ge -(1-\delta)\int_{t_1}^{t_2} g(s)-f(s)+\varepsilon/2ds.
\end{aligned}$$

Therefore

$$y(t_1)\ge\phi(g(t_1)+\varepsilon)+\delta\int_{t_1}^{t_2} g(s)-f(s)+\varepsilon/2ds>\phi(g(t_1)+2\varepsilon).$$

This contradicts (3.11), hence the *Claim* is true. The proof of (3.6) is complete. $\square$

Corollary 3.4. *Assume $f \in C^1((0,\infty))$.*

(i) Given $\varepsilon > 0$, if

$$\lim_{t\to\infty} \Big[\frac{d}{dt}\phi(f(t)+\varepsilon)\Big]^- = 0 \tag{3.13}$$

then there is $T > 0$ such that

$$y(t) \le \phi(f(t)+\varepsilon), \text{ for all } t > T. \tag{3.14}$$

(ii) If $\lim_{t\to\infty}[f'(t)]^- = 0$ then

$$\limsup_{t\to\infty} y(t) \le \phi\big(\limsup_{t\to\infty} f(t)\big). \tag{3.15}$$

Proof. (i) Let $g(t) = f(t)$, then (3.5) is satisfied. Applying Lemma (3.3), we obtain (3.14).

(ii) Let $A = \limsup_{t\to\infty} f(t)$. If $A = \infty$, then (3.15) holds true. Consider $A < \infty$. The continuity of f on $[0,\infty)$ implies that f is uniformly bounded. Suppose $|f(t)| \le B$, for all $t \ge 0$. Let ε be any given number in $(0,1)$. Let $M = \sup\{\phi'(z) : \varepsilon \le z \le B+1\}$. Then

$$\Big[\frac{d}{dt}\phi(f(t)+\varepsilon)\Big]^- = \phi'\big(f(t)+\varepsilon\big)[f'(t)]^- \le M[f'(t)]^- \to 0,$$

as $t \to \infty$. Applying Lemma 3.3, there is $T > 0$ such that $y(t) \le \phi(f(t)+\varepsilon)$, for all $t > T$. Taking $t \to \infty$ and using the fact that $\phi(\cdot)$ is increasing and continuous, we have $\limsup_{t\to\infty} y(t) \le \phi(\limsup_{t\to\infty} f(t) + \varepsilon)$. Letting $\varepsilon \to 0$, we obtain (3.15). □

In general cases, we have the following estimate.

Proposition 3.5. *If $A \stackrel{\text{def}}{=\!=} \limsup_{t\to\infty} f(t) < \infty$ then*

$$\limsup_{t\to\infty} y(t) \le \phi(A). \tag{3.16}$$

Proof. Given $0 < \varepsilon < 1$, there is $T > 0$ such that $f(t) < A + \varepsilon$ for all $t > T$. There is a C^1-majorant F_ε of f on $[0,\infty)$ such that $F_\varepsilon(t) = A + \varepsilon$ for all $t > 2T$. Applying Corollary 3.4 to the differential inequality $y'(t) \le -\phi^{-1}(y) + F_\varepsilon(t)$, we have $\limsup_{t\to\infty} y(t) \le \phi(A+\varepsilon)$. Letting $\varepsilon \to 0$ yields (3.16). □

When $f(t)$ is unbounded ($A = \infty$), the estimate (3.16) is trivial and provides no information about the asymptotic behavior of $y(t)$. In this case, estimate (3.14) is more useful. Using Lemma 3.3, we derive a similar estimate to (3.14) with less stringent condition than (3.13).

Proposition 3.6. *Suppose $f \in C^1((0,\infty))$ satisfies $\lim_{t\to\infty} f(t) = \infty$. Let*

$$\alpha = \limsup_{z\to\infty} \phi'(z)/z \text{ and } \beta = \limsup_{t\to\infty} [f'(t)]^-.$$

If $4\alpha\beta < 1$, then for any given $\varepsilon > 0$, there is $T > 0$ such that

$$y(t) \le \phi(2f(t)+\varepsilon), \text{ for all } t > T. \tag{3.17}$$

Proof. Take $g(t) = 2f(t)$ in Lemma 3.3. Then

$$\limsup_{t\to\infty} \frac{\big[\frac{d}{dt}\phi(g(t)+\varepsilon)\big]^-}{g(t) - f(t) + \varepsilon/2} = \limsup_{t\to\infty} \frac{4\phi'(2f(t)+\varepsilon)[f'(t)]^-}{2f(t)+\varepsilon} = 4\alpha\beta < 1;$$

above we used the fact $\lim_{t\to\infty}(2f(t)+\varepsilon) = \infty$. Hence (3.5) is satisfied and (3.17) follows from (3.6). □

For our main applications in this paper, we derive estimates when the function ϕ has a particular form $\varphi_{c,\gamma}$ defined in (2.17).

Proposition 3.7. *Let $\phi = cz + c^\gamma z^\gamma$, for $z \ge 0$, where $c > 0$, $\gamma \in (1, 2]$.*
(i) Then

$$y(t) \le y(0) + C_\gamma F(t) \text{ for all } t \ge 0, \tag{3.18}$$

where $C_\gamma = c + \max\{2, c^\gamma\}$.
(ii) Let $A = \limsup_{t\to\infty} f(t)$ and $\beta = \limsup_{t\to\infty}[f'(t)]^-$. Then

$$\limsup_{t\to\infty} y(t) \le C_\gamma A, \tag{3.19}$$

and there is $T > 0$ such that for $t \ge T$,

$$y(t) \le \max\{3, c, c^2\}\big(1 + \beta + f(t)\big). \tag{3.20}$$

Proof. (i) By Young's inequality: $c^\gamma z^\gamma \le cz + c^2 z^2$, hence $\phi(z) \le \varphi(z) \overset{\text{def}}{=\!=} 2cz + c^2 z^2$. Thus for $y \ge 0$,

$$\phi^{-1}(y) \ge \varphi^{-1}(y) = \frac{y^2}{c(1+\sqrt{1+y^2})} \ge \frac{y^2}{c(2+y)} = c^{-1}\Big(y - 2 + \frac{4}{2+y}\Big),$$

which gives $\phi^{-1}(y) \ge c^{-1}(y-2)$. Therefore $y(t)$ satisfies

$$y' \le -c^{-1}y + 2c^{-1} + f(t), \quad t > 0. \tag{3.21}$$

By Gronwall's inequality we obtain

$$y(t) \le e^{-t/c} y(0) + 2 + \int_0^t e^{-(t-\tau)/c} f(\tau) d\tau, \tag{3.22}$$

and consequently,

$$y(t) \le y(0) + 2 + cF(t), \tag{3.23}$$

for all $t \ge 0$. Also, by Lemma 3.1, we have

$$y(t) \le y(0) + cF(t) + c^\gamma F(t)^\gamma, \quad t \ge 0. \tag{3.24}$$

Combining (3.23) and (3.24), for $t \ge 0$, we have

$$y(t) \le y(0) + cF(t) + \max\{2, c^\gamma\} \min\{1, F(t)^\gamma\}.$$

Note that $\min\{1, F^\gamma\}$ is 1 if $F \ge 1$, and is F^γ if $F < 1$. Since $\gamma > 1$, we have $\min\{1, F^\gamma\} \le F$ in both cases. Therefore,

$$y(t) \le y(0) + cF(t) + \max\{2, c^\gamma\} F(t) \le y(0) + C_\gamma F(t).$$

Thus we obtain (3.18).

(ii) One can easily prove (or use similar reasoning in Lemma 3.9 below to prove) that

$$\limsup_{t\to\infty} \int_1^t e^{-(t-\tau)/c} f(\tau)\tau \le c \limsup_{t\to\infty} f(t) = cA. \tag{3.25}$$

Hence (3.22) implies $\limsup_{t\to\infty} y(t) \le 2 + cA$. By Lemma 3.1 we have another estimate $\limsup_{t\to\infty} y(t) \le cA + c^\gamma A^\gamma$. Using the same arguments as in part (i), we obtain (3.19).

It suffices to prove (3.20) when $\beta < \infty$. By (3.21) we have for $t \ge 1$ that

$$y(t) \le e^{-(t-1)/c} y(1) + 2 + \int_1^t e^{-(t-\tau)/c} f(\tau) d\tau.$$

Integration by parts yields

$$\begin{aligned} y(t) &\le e^{-(t-1)/c}y(1) + 2 + cf(t) - c\int_1^t e^{-(t-\tau)/c}f'(\tau)d\tau \\ &\le e^{-(t-1)/c}y(1) + 2 + cf(t) + c\int_1^t e^{-(t-\tau)/c}[f'(\tau)]^- d\tau. \end{aligned} \tag{3.26}$$

Similar to (3.25) we have

$$\limsup_{t\to\infty}\int_1^t e^{-(t-\tau)/c}[f'(\tau)]^- d\tau \le c\limsup_{t\to\infty}[f'(\tau)]^- = c\beta.$$

Hence we obtain from (3.26) that $y(t) \le 3 + cf(t) + c^2\beta$ for sufficient large t, and therefore (3.20) follows. □

Remark 3.8. *(a) Ineq. (3.18) combines the nonlinear estimate (3.2), which is without an additive constant and is suitable for small $F(t)$, with the linear estimate (3.23) suitable for large $F(t)$, hence it unifies both of them and is applicable to general $F(t)$.*

(b) In case $\gamma < 2$, similar to (3.17), one can bound $y(t)$ by

$$y(t) \le \phi\big(2f(t) + (16c\beta\gamma)^{1/(2-\gamma)}\big), \text{ for all } t > T,$$

where $T > 0$ depends on $y(t)$. Since $f(t)$ is considered large in this case, inequality (3.20) is simpler and more suitable to our application.

In the same spirit of Proposition 3.5, we have the following linear version which will replace the L'Hôpital Rule in our asymptotic estimates.

Lemma 3.9. *Let $y(t) \ge 0$, $h(t) > 0$, $f(t) \ge 0$ be continuous on $[0,\infty)$ and satisfy*

$$y'(t) \le -h(t)y(t) + f(t) \text{ for all } t > 0. \tag{3.27}$$

If $\int_0^\infty h(t)dt = \infty$ then

$$\limsup_{t\to\infty} y(t) \le \limsup_{t\to\infty}\frac{f(t)}{h(t)}. \tag{3.28}$$

Proof. From (3.27) and Gronwall's inequality, we have

$$y(t) \le e^{-\int_0^t h(\tau)d\tau}y(0) + e^{-\int_0^t h(\tau)d\tau}\int_0^t e^{-\int_0^\tau h(s)ds}f(\tau)d\tau.$$

It suffices to prove (3.28) when $A \overset{\text{def}}{=\!=} \limsup_{t\to\infty}\frac{f(t)}{h(t)}$ is finite. Given $\delta > 0$, let $T > 0$ such that $\frac{f(t)}{h(t)} \le A + \delta$ for all $t > T$. Let $M = M(T) = y(0) + \int_0^T e^{\int_0^\tau h(s)ds}f(\tau)d\tau$. For $t > T$, we have

$$\begin{aligned} y(t) &\le e^{-\int_0^t h(\tau)d\tau}y(0) + e^{-\int_0^t h(\tau)d\tau}\int_0^T e^{\int_0^\tau h(s)ds}f(\tau)d\tau \\ &\quad + e^{-\int_0^t h(\tau)d\tau}\int_T^t h(\tau)e^{\int_0^\tau h(s)ds}\frac{f(\tau)}{h(\tau)}d\tau \\ &\le Me^{-\int_0^t h(\tau)d\tau} + e^{-\int_0^t h(\tau)d\tau}\int_T^t h(\tau)e^{\int_0^\tau h(s)ds}(A+\delta)d\tau \\ &= Me^{-\int_0^t h(\tau)d\tau} + (A+\delta)e^{-\int_0^t h(\tau)d\tau}\Big(e^{\int_0^t h(\tau)d\tau} - e^{\int_0^T h(\tau)d\tau}\Big). \end{aligned}$$

Hence

$$y(t) \le A + \delta + Me^{-\int_0^t h(\tau)d\tau}, \text{ for all } t > T. \tag{3.29}$$

Because $\int_0^\infty h(t)dt = \infty$, letting $t \to \infty$ in (3.29) yields $\limsup_{t\to\infty} y(t) \le A + \delta$. Then letting $\delta \to 0$, we obtain (3.28). □

Remark 3.10. *In Lemma 3.9, if* $\int_0^\infty h(t)dt < \infty$ *then*

$$\limsup_{t\to\infty} y(t) \le e^{-\int_0^\infty h(t)dt} y(0) + \int_0^\infty f(\tau)d\tau. \tag{3.30}$$

However, unlike (3.28), estimate (3.30) depends on the initial value $y(0)$ *and accumulation of values of* $f(t)$ *over* $(0, \infty)$.

4. Bounds of solutions

Consider the generalized Forchheimer equation (2.1) with a fixed $g(s) = g(s, \vec{a}) \in FP(N, \vec{\alpha})$. We study the resulting parabolic equation for the pressure $p = p(x, t)$:

$$\frac{\partial p}{\partial t} = \nabla \cdot (K(|\nabla p|)\nabla p), \quad x \in U,\ t > 0, \tag{4.1}$$

where U is a bounded, open, connected subset of $\mathbb{R}^n$ with boundary Γ of class C^2.

Recall that the velocity is $u = -K(|\nabla p|)\nabla p$. The flow satisfies the flux condition on the boundary:

$$u \cdot \nu = \psi(x, t), \quad x \in \Gamma,\ t > 0.$$

where ν is the unit outward normal vector on Γ, the flux $\psi(x, t)$ is known. Hence

$$-K(|\nabla p|)\nabla p \cdot \nu = \psi \quad \text{on} \quad \Gamma \times (0, \infty). \tag{4.2}$$

The initial data

$$p(x, 0) \text{ is given.} \tag{4.3}$$

For our estimates, we define for $t \ge 0$,

$$f(t) = \|\psi(t)\|^2_{L^\infty} + \|\psi(t)\|^{\frac{2-a}{1-a}}_{L^\infty}, \tag{4.4}$$

$$\tilde{f}(t) = \|\psi_t(t)\|^2_{L^\infty} + \|\psi_t(t)\|^{\frac{2-a}{1-a}}_{L^\infty}. \tag{4.5}$$

Assume throughout that $\psi(\cdot, t)$ and $\psi_t(\cdot, t)$ belong $C([0, \infty), L^\infty(\Gamma))$, hence $f(t)$ and $\tilde{f}(t)$ belong to $C([0, \infty))$. Whenever $f'(t)$ is mentioned, we implicitly assume that $f \in C^1((0, \infty))$.

Hereafter in this section, $p(x, t)$ is a solution of the IBVP (4.1), (4.2), (4.3).

Throughout this paper, we consider solutions with sufficient regularities both in x *and* t *variables such that our calculations can be performed legitimately. For the existence and regularity theory of degenerate parabolic equations, see e.g.* [19, 21, 11].

Integrating (4.1) over U, we easily find

$$\frac{d}{dt}\int_U p(x, t)dx = \int_\Gamma K(|\nabla p|)\nabla p \cdot \nu d\sigma = -\int_\Gamma \psi(x, t)d\sigma, \quad t > 0.$$

By the continuity of $\int_U p(x, t)dx$ and $\int_\Gamma \psi(x, t)d\sigma$ on $[0, \infty)$, we assert

$$\int_U p(x, t)dx = \int_U p(x, 0)dx - \int_0^t \int_\Gamma \psi(x, \tau)d\sigma d\tau,\ t \ge 0. \tag{4.6}$$

Let $\bar p(x,t) = p(x,t) - \frac{1}{|U|}\int_U p(x,t)dx$, for $x \in U$ and $t \ge 0$, where $|U|$ denotes the volume of U. Then $\bar p$ satisfies the zero average condition

$$\int_U \bar p(x,t)dx = 0 \text{ for all } t \ge 0. \tag{4.7}$$

By (4.6), we have for $t \ge 0$,

$$\bar p(x,t) = p(x,t) - \frac{1}{|U|}\int_U p(x,0)dx + \gamma(t), \text{ where } \gamma(t) = \frac{1}{|U|}\int_0^t \int_\Gamma \psi(x,\tau)d\sigma d\tau. \tag{4.8}$$

Here $d\sigma$ is the surface area element. We call $\bar p(x,t)$ the shifted solution.

The shifted solution $\bar p$ satisfies the following IBVP

$$\frac{\partial \bar p}{\partial t} = \nabla\cdot(K(|\nabla \bar p|)\nabla \bar p) + \gamma'(t), \tag{4.9}$$

$$-K(|\nabla \bar p|)\nabla \bar p \cdot \nu = \psi(x,t) \quad \text{on} \quad \Gamma, \tag{4.10}$$

$$\bar p(x,0) = p(x,0) - \frac{1}{|U|}\int_U p(x,0)dx. \tag{4.11}$$

Note that even in the linear case, i.e. $K(\xi) \equiv const.$, $p(x,t)$ can be unbounded as $t \to \infty$, while $\psi(x,t)$ is uniformly bounded in $(x,t) \in \Gamma \times [0,\infty)$. Therefore instead of estimating $p(x,t)$ directly, we will estimate $\bar p(x,t)$. It has the following advantages. On one hand, function $\bar p(x,t)$ is simply the oscillation of the solution $p(x,t)$ with respect to its average $\frac{1}{|U|}\int_U p(x,t)dx$, hence satisfies the zero average property (4.7), and therefore will have simpler estimates in terms of the initial and boundary data. On another hand, it is a shift of $p(x,t)$ by the average of the initial pressure distribution and the accumulation of the total boundary flux from 0 to t. Thus the obtained results will have explicit physical interpretations, and the estimates for $p(x,t)$ can be easily retrieved from those for $\bar p(x,t)$ by virtue of the following basic relations.

Lemma 4.1. *For $t \ge 0$,*

$$J[\bar p](t) \le J[p](t) \le J[\bar p](t) + 2J[p](0) + 2|U|\gamma^2(t), \tag{4.12}$$

$$J[\bar p_t](t) \le J[p_t](t) \le 2J[\bar p_t](t) + 2|\Gamma|^2|U|^{-1}\|\psi(t)\|^2_{L^\infty}. \tag{4.13}$$

Proof. Let $\hat p(t) = |U|^{-1}\int_U p(x,t)dx$, for $t \ge 0$. Elementary calculations give

$$J[\bar p](t) = J[p](t) - |U|\hat p^2(t), \tag{4.14}$$

which yields the first inequality of (4.12). Note that $\hat p(t) = \hat p(0) - \gamma(t)$, then

$$\hat p^2(t) \le 2\hat p^2(0) + 2\gamma^2(t) \le 2|U|^{-1}J[p](0) + 2\gamma^2(t).$$

Combining with (4.14), we obtain the second inequality of (4.12).

The first inequality of (4.13) is the same as that of (4.12). Next, we have $p_t = \bar p_t - |U|^{-1}\int_\Gamma \psi(x,t)d\sigma$. Then

$$\begin{aligned} J[p_t](t) &\le 2J[\bar p_t](t) + 2|U|^{-2}\int_U \Big(\int_\Gamma \psi(y,t)d\sigma_y\Big)^2 dx \\ &\le 2J[\bar p_t](t) + 2|\Gamma|^2|U|^{-1}\|\psi(t)\|^2_{L^\infty}. \end{aligned}$$

Therefore we obtain (4.13). $\square$

Notation. *For a function $u(x,t)$ defined on $U\times[0,\infty)$ and $H(\xi)$ defined on $[0,\infty)$, we denote*

$$J[u](t)=\int_U u^2(x,t)dx \text{ and } J_H[u](t)=\int_U H(|\nabla u(x,t)|)dx.$$

Hereafter, $H(\xi)$ is the function defined by (2.13).

In this section we will obtain various bounds for the solution which are formulated in the terms of the functionals

$$J[\bar p],\ J[\bar p_t],\ \text{and } J_H[p]=J_H[\bar p]. \tag{4.15}$$

The following are the main differential inequalities for these functionals.

Lemma 4.2. *We have for all $t>0$ that*

$$\frac{d}{dt}J[\bar p](t)\le -J_H[p](t)+Cf(t), \tag{4.16}$$

$$\frac{d}{dt}J_H[p](t)\le -2J[\bar p_t](t)+\varepsilon\int_U K(|\nabla p|)|\nabla \bar p_t|^2dx+\varepsilon J_H[p](t)+C_\varepsilon f(t), \tag{4.17}$$

$$\frac{d}{dt}J_H[p](t)\le -2J[\bar p_t](t)+\varepsilon J_H[p](t)+2\frac{d}{dt}\left(\int_\Gamma \psi\bar p d\sigma\right)+C_\varepsilon \tilde f(t), \tag{4.18}$$

$$\frac{d}{dt}J[\bar p_t](t)\le -(1-a)\int_U K(|\nabla p|)|\nabla \bar p_t|^2dx+\varepsilon J_H[p](t)+C_\varepsilon \tilde f(t), \tag{4.19}$$

where $\varepsilon>0$ is arbitrary and $C_\varepsilon>0$ depends on ε.

Proof. Multiplying equation (4.9) by $\bar p$ and integrating over U give

$$\frac{1}{2}\frac{d}{dt}\int_U \bar p^2dx=-\int_U K(|\nabla \bar p|)|\nabla \bar p|^2dx+\int_\Gamma K(|\nabla p|)\nabla \bar p\cdot \nu \bar p d\sigma+\gamma'(t)\int_U \bar p dx.$$

By (4.7) the last integral vanishes, hence

$$\frac{1}{2}\frac{d}{dt}\int_U \bar p^2dx=-\int_U K(|\nabla \bar p|)|\nabla \bar p|^2dx-\int_\Gamma \psi\bar p d\sigma. \tag{4.20}$$

Applying inequality (2.28) to the boundary term on the right-hand side of (4.20) and using estimate (2.14) yield

$$\frac{1}{2}\frac{d}{dt}J[\bar p](t)\le -(1-\varepsilon)J_H[p](t)+C_\varepsilon\big(\|\psi\|_{L^\infty}+\|\psi\|_{L^\infty}^{\frac{2-a}{1-a}}\big), \tag{4.21}$$

which deduces to (4.16) with $\varepsilon=1/2$.

Proof of (4.17) and (4.18). Multiplying (4.9) by $\bar p_t$ and integrating by parts, similar to (4.20) we obtain

$$\frac{d}{dt}J_H[p](t)=-2J[\bar p_t](t)+2\int_\Gamma \psi\bar p_t d\sigma. \tag{4.22}$$

Applying (2.27) to the second term on the right-hand side with $f=\psi$, $u=\bar p_t$ and $v=p$ and estimate (2.14) yield (4.17).

On the other hand, (4.22) provides

$$\frac{d}{dt}J_H[p](t)=-2J[\bar p_t](t)+2\frac{d}{dt}\int_\Gamma \psi\bar p d\sigma-2\int_\Gamma \psi_t\bar p d\sigma.$$

Applying the inequality (2.28) to the last boundary integral yields

$$\frac{d}{dt}J_H[p](t) \le -2J[\bar p_t](t) + C_\varepsilon(\|\psi_t\|_{L^\infty}^2 + \|\psi_t\|_{L^\infty}^{\frac{2-a}{1-a}}) + \varepsilon J_H[p](t) + 2\frac{d}{dt}\left(\int_\Gamma \psi \bar p d\sigma\right),$$

hence (4.18) follows.

Proof of (4.19). Denote $\bar q = \bar p_t$. Differentiating equation (4.9) in time gives

$$\frac{\partial \bar q}{\partial t} = \nabla\cdot\big(K(|\nabla \bar p|)\nabla \bar p\big)_t + \gamma''(t). \tag{4.23}$$

Multiplying both sides of the equation (4.23) by $\bar q$, integrating by parts and recollecting from (4.7) that $\int_U \bar q dx = 0$, we get

$$\frac{1}{2}\frac{d}{dt}\int_U \bar q^2 = -\int_U K(|\nabla\bar p|)|\nabla \bar q|^2 dx - \int_U \big(K(|\nabla \bar p|)\big)_t \nabla\bar p\cdot\nabla\bar q dx + \int_\Gamma \psi_t(x,t)\bar q(x,t)d\sigma. \tag{4.24}$$

Applying (2.12) with $u = \bar p(x,t)$:

$$\big|\big(K(|\nabla\bar p|)\big)_t\nabla\bar p\cdot\nabla\bar p_t\big| \le |K'(|\nabla\bar p|)|\,|\nabla\bar p|\,|\nabla\bar p_t|^2 \le aK(|\nabla\bar p|)|\nabla\bar p_t|^2. \tag{4.25}$$

Using (4.25) to bound the second integral on the right-hand side of (4.24) gives

$$\frac{1}{2}\frac{d}{dt}J[\bar p_t](t) \le -(1-a)\int_U K(|\nabla\bar p|)|\nabla\bar p_t|^2 dx + \int_\Gamma \psi_t(x,t)\bar p_t(x,t)d\sigma. \tag{4.26}$$

Applying (2.27) to the last term of (4.26) yields

$$\int_\Gamma \psi_t(x,t)\bar p_t(x,t)d\sigma \le \frac{1-a}{2}\int_U K(|\nabla p|)|\nabla\bar p_t|^2 dx + \frac{\varepsilon}{2}J_H[p](t) + C_\varepsilon \tilde f(t). \tag{4.27}$$

Then (4.19) follows (4.26) and (4.27). □

From the basic differential inequalities in Lemma 4.2, we will derive various estimates for each quantity in (4.15).

Let $M_f(t)$ be a continuous, increasing majorant of $f(t)$ on $[0,\infty)$.

We start with estimates for $J[\bar p](t)$. Let

$$A = \limsup_{t\to\infty} f(t) \text{ and } \beta = \limsup_{t\to\infty}[f'(t)]^-. \tag{4.28}$$

Again, whenever β is used in subsequently statements, it is understood that $f(t)\in C^1((0,\infty))$.

Theorem 4.3. *(i) For $t\ge 0$,*

$$J[\bar p](t) + \int_0^t J_H[p](\tau)d\tau \le J[\bar p](0) + C\int_0^t f(\tau)d\tau, \tag{4.29}$$

(ii) Assume the Degree Condition. Then

$$J[\bar p](t) \le J[\bar p](0) + CM_f(t) \text{ for all } t\ge 0. \tag{4.30}$$

If $A<\infty$ then

$$\limsup_{t\to\infty} J[\bar p](t) \le CA. \tag{4.31}$$

If $\beta<\infty$ then there is $T>0$ such that

$$J[\bar p](t) \le C\big(1+\beta+f(t)\big) \text{ for all } t > T. \tag{4.32}$$

Proof. (i) Integrating (4.16) from 0 to t results in (4.29) directly.

(ii) Applying inequality (2.18) to integral $J_H[p](t)$ on the right-hand side of (4.16) gives

$$\frac{d}{dt}J[\bar{p}](t) \le -\varphi^{-1}_{C_1,\frac{2}{2-a}}(J[\bar{p}](t)) + C_2 f(t), \tag{4.33}$$

where $C_1, C_2 > 0$, the function $\varphi_{C_1,\frac{2}{2-a}}$ is defined in (2.17). Let $y(t) = J[\bar{p}](t)$, $\phi = \varphi_{C_1,\frac{2}{2-a}}$, then

$$y'(t) \le \phi^{-1}(y(t)) + C_2 f(t), \text{ for all } t > 0. \tag{4.34}$$

Applying estimates in Proposition 3.7 to the differential inequality (4.34), then (3.18) yields (4.30), (3.19) yields (4.31), and (3.20) yields (4.32). □

We now obtain bounds for $J_H[p](t)$. When t is large they is expressed mainly in terms of the boundary data on the interval $[t-1, t]$. As it will be shown later, these estimates of uniform Gronwall-type are pivotal to our quantitative study of the solution as $t \to \infty$. (The uniform Gronwall inequality originates in the study of Navier–Stokes equations [15], see also [34, 32].)

Theorem 4.4. *(i) We have*

$$\begin{aligned} J_H[p](t) + 4\int_0^t J[\bar{p}_t](\tau)d\tau &\le 4J_H[p](0) + 2J[\bar{p}](0) + Cf(0) \\ &\quad + Cf(t) + C\int_0^t \Big(f(\tau) + \tilde{f}(\tau)\Big)d\tau \quad \textit{for all } t \ge 0, \end{aligned} \tag{4.35}$$

$$\begin{aligned} J_H[p](t) + 2\int_{t-1/2}^t J[\bar{p}_t](\tau)d\tau &\le CJ[\bar{p}](t-1) + Cf(t) \\ &\quad + C\int_{t-1}^t \Big(f(\tau) + \tilde{f}(\tau)\Big)d\tau \quad \textit{for all } t \ge 1. \end{aligned} \tag{4.36}$$

(ii) Assume the Degree Condition. Then

$$J_H[p](t) \le CJ[\bar{p}](0) + CM_f(t) + C\int_{t-1}^t \tilde{f}(\tau)d\tau \ \textit{for all } t \ge 1. \tag{4.37}$$

If $A < \infty$ then

$$\limsup_{t\to\infty} J_H[p](t) \le C\Big(A + \limsup_{t\to\infty}\int_{t-1}^t \tilde{f}(\tau)d\tau\Big), \tag{4.38}$$

and consequently, there is $T > 0$ such that

$$J_H[p](t) \le C\Big(1 + A + \int_{t-1}^t \tilde{f}(\tau)d\tau\Big) \ \textit{for all } t > T. \tag{4.39}$$

If $\beta < \infty$ then there is $T > 0$ such that

$$J_H[p](t) \le C\Big(1 + \beta + f(t-1) + f(t) + \int_{t-1}^t f(\tau) + \tilde{f}(\tau)d\tau\Big) \ \textit{for all } t > T. \tag{4.40}$$

Proof. (i) Integrating (4.18) and after regrouping, we obtain

$$\begin{aligned} J_H[p](t)+2\int_0^t J[\bar p_t](\tau)d\tau \le \varepsilon\int_0^t J_H[p](\tau)d\tau + 2\int_\Gamma \psi(x,t)\bar p(t,x)d\sigma \\ + C_\varepsilon\int_0^t \tilde f(\tau)d\tau + J_H[p](0) - 2\int_\Gamma \psi(x,0)\bar p(x,0)d\sigma. \end{aligned} \tag{4.41}$$

Applying the trace inequality (2.28) to the boundary term on the right-hand side of (4.41) above we get

$$\begin{aligned} J_H[p](t)+2\int_0^t J[\bar p_t](\tau)d\tau \le \varepsilon\int_0^t J_H[p](\tau)d\tau + \Big(\frac12 J_H[p](t)+Cf(t)\Big) \\ + J_H[p](0)+C_\varepsilon\int_0^t \tilde f(\tau)d\tau + \Big(J_H[p](0)+Cf(0)\Big). \end{aligned} \tag{4.42}$$

We then have

$$\begin{aligned} \frac12 J_H[p](t)+2\int_0^t J[\bar p_t](\tau)d\tau \le \varepsilon\int_0^t J_H[p](\tau)d\tau + Cf(t) + C_\varepsilon\int_0^t \tilde f(\tau)d\tau \\ + 2J_H[p](0)+Cf(0). \end{aligned} \tag{4.43}$$

Setting $\varepsilon = 1/2$, then adding (4.29) to inequality (4.43) and multiplying the resulting inequality by 2, we obtain (4.35).

Proof of (4.36). Integrating both sides of the inequality (4.16) from $t-1$ to t, we obtain

$$\int_{t-1}^t J_H[p](\tau)d\tau \le J[\bar p](t-1) + C\int_{t-1}^t f(\tau)d\tau, \text{ for } t\ge 1. \tag{4.44}$$

Let $t\ge 1$ and $t-1<s<t$. Integrating the inequality (4.18) from s to t gives

$$\begin{aligned} J_H[p](t) \le J_H[p](s) - 2\int_s^t J[\bar p_t](t)d\tau + \varepsilon\int_s^t J_H[p](\tau)d\tau \\ + \int_\Gamma p(x,t)\psi(x,t)d\sigma - \int_\Gamma p(x,s)\psi(x,s)d\sigma + C_\varepsilon\int_s^t \tilde f(\tau)d\tau. \end{aligned} \tag{4.45}$$

Next applying the trace inequality (2.28) to the boundary terms, we get

$$\begin{aligned} J_H[p](t) \le J_H[p](s) - 2\int_s^t J[\bar p_t](t)d\tau + \varepsilon\int_s^t J_H[p](\tau)d\tau \\ + \Big\{\varepsilon J_H[p](t)+C_\varepsilon f(t)\Big\} + \Big\{\varepsilon J_H[p](s)+C_\varepsilon f(s)\Big\} + C_\varepsilon\int_s^t \tilde f(\tau)d\tau, \end{aligned} \tag{4.46}$$

hence

$$\begin{aligned} (1-\varepsilon)J_H[p](t) \le (1+\varepsilon)J_H[p](s) - 2\int_s^t J[\bar p_t](\tau)d\tau \\ + \varepsilon\int_{t-1}^t J_H[p](\tau)d\tau + C_\varepsilon f(s) + C_\varepsilon f(t) + C_\varepsilon\int_{t-1}^t \tilde f(\tau)d\tau. \end{aligned} \tag{4.47}$$

Now applying (4.44) to the integral $\int_{t-1}^t J_H[p](\tau)d\tau$ yields

$$(4.48)\quad \begin{aligned}&(1-\varepsilon)J_H[p](t)\le(1+\varepsilon)J_H[p](s)-2\int_s^t J[\bar p_t](\tau)d\tau\\&+\Big\{C\varepsilon J[\bar p](t-1)+C\varepsilon\int_{t-1}^t f(\tau)d\tau\Big\}+C_\varepsilon f(s)+C_\varepsilon f(t)+C_\varepsilon\int_{t-1}^t\tilde f(\tau)d\tau.\end{aligned}$$

Setting $\varepsilon=1/2$, integrating the last inequality from $t-1$ to t with respect to s, applying (4.44) once more and grouping terms, we obtain

$$(4.49)\quad \begin{aligned}&\frac12 J_H[p](t)\le\frac32 J[\bar p](t-1)+C\int_{t-1}^t f(\tau)d\tau-2\int_{t-1}^t\int_s^t J[\bar p_t](\tau)d\tau ds\\&+CJ[\bar p](t-1)+Cf(t)+C\int_{t-1}^t f(\tau)+\tilde f(\tau)d\tau.\end{aligned}$$

Therefore we obtain:

$$(4.50)\quad \begin{aligned}J_H[p](t)+4\int_{t-1}^t\int_s^t J[\bar p_t](\tau)d\tau ds&\le CJ[\bar p](t-1)+Cf(t)\\&\quad+C\int_{t-1}^t\Big(f(\tau)+\tilde f(\tau)\Big)d\tau.\end{aligned}$$

Note that

$$(4.51)\quad \begin{aligned}&4\int_{t-1}^t\int_s^t J[\bar p_t](\tau)d\tau ds=4\int_{t-1}^t\int_{t-1}^\tau J[\bar p_t](\tau)ds d\tau\\&\ge 4\int_{t-1/2}^t J[\bar p_t](\tau)\int_{t-1}^\tau 1\,ds d\tau\ge 4\int_{t-1/2}^t J[\bar p_t](\tau)\cdot(1/2)d\tau\\&=2\int_{t-1/2}^t J[\bar p_t](\tau)d\tau.\end{aligned}$$

Using (4.51) in (4.50) yields (4.36).

(ii) Assume the Degree Condition. In (4.36) dropping the integral on the left-hand side, using (4.30) to bound $J[\bar p](t-1)$, and noting that $f(t)$, $M_f(t-1)$, $\int_{t-1}^t f(\tau)d\tau$ are less than or equal to $M_f(t)$, we obtain (4.37).

Consider $A<\infty$. On the right-hand side of (4.36), taking into account (4.31) we have

$$\limsup_{t\to\infty} J[\bar p](t-1)=\limsup_{t\to\infty} J[\bar p](t)\le CA,$$

$$\limsup_{t\to\infty}\int_{t-1}^t f(\tau)d\tau\le\limsup_{t\to\infty} f(t)=A.$$

Then taking the limit superior of (4.36) yields (4.38), also (4.39) follows for sufficiently large t.

Consider $\beta<\infty$. Using (4.32) to estimate $J[\bar p](t-1)$ in (4.36), we derive Ineq. (4.40). □

Next, we estimate the L^2-norm of $\bar q(x,t)=\bar p_t(x,t)$. For large t, we derive uniform Gronwall estimates for $J[\bar p_t](t)$.

Theorem 4.5. *Assume the Degree Condition.*

(i) For $0 < t_0 < 1$ and $t \ge t_0$, one has

$$J[\bar{p}_t](t) \le C\, t_0^{-1} \left(J[\bar{p}_0] + J_H[p_0] + M_f(t_0) + \int_0^{t_0} \tilde{f}(\tau) d\tau \right) + C \int_0^t f(\tau) + \tilde{f}(\tau) d\tau. \tag{4.52}$$

(ii) For $t \ge 1$, one has

$$J[\bar{p}_t](t) \le C J[\bar{p}](0) + C M_f(t) + C \int_{t-1}^t \tilde{f}(\tau) d\tau. \tag{4.53}$$

(iii) If $A < \infty$ then

$$\limsup_{t\to\infty} J[\bar{p}_t](t) \le C\Big(A + \limsup_{t\to\infty} \int_{t-1}^t \tilde{f}(\tau) d\tau \Big), \tag{4.54}$$

and there is $T > 0$ such that

$$J[\bar{p}_t](t) \le C\Big(1 + A + \int_{t-1}^t \tilde{f}(\tau) d\tau \Big) \text{ for all } t > T. \tag{4.55}$$

(iv) If $\beta < \infty$, then there is $T > 0$ such that for all $t > T$,

$$J[\bar{p}_t](t) \le C\Big(1 + \beta + f(t-1) + f(t) + \int_{t-1}^t f(\tau) + \tilde{f}(\tau) d\tau \Big). \tag{4.56}$$

Proof. (i) From (4.19): for $0 < s < t$ we have

$$J[\bar{p}_t](t) \le J[\bar{p}_t](s) + \int_s^t J_H[p](\tau) d\tau + C \int_s^t \tilde{f}(\tau) d\tau.$$

The first integral can be bounded similar to (4.29) to obtain

$$J[\bar{p}_t](t) \le J[\bar{p}_t](s) + C J[\bar{p}](s) + C \int_s^t f(\tau) d\tau + C \int_s^t \tilde{f}(\tau) d\tau. \tag{4.57}$$

Then estimating $J[\bar{p}](s)$ by (4.30) gives

$$J[\bar{p}_t](t) \le J[\bar{p}_t](s) + C J[\bar{p}](0) + C M_f(s) + C \int_s^t f(\tau) + \tilde{f}(\tau) d\tau. \tag{4.58}$$

Let $0 < t_0 < 1$ and $0 < t_0 \le t$. Integrating (4.58) from 0 to t_0 in s yields

$$t_0 J[\bar{p}_t](t) \le \int_0^{t_0} J[\bar{p}_t](s) ds + C t_0 J[\bar{p}](0) + C t_0 M_f(t_0) + C t_0 \int_0^t f(\tau) + \tilde{f}(\tau) d\tau.$$

Hence

$$J[\bar{p}_t](t) \le t_0^{-1} \left(\int_0^{t_0} J[\bar{p}_t](s) ds \right) + C J[\bar{p}](0) + C M_f(t_0) + C \int_0^t f(\tau) + \tilde{f}(\tau) d\tau.$$

Using (4.35) to estimate $\int_0^{t_0} J[\bar{p}_t](s)ds$ with $t_0 < 1$ gives:

$$\begin{aligned}
J[\bar{p}_t](t) \le t_0^{-1}\Big(J_H[p](0) + \frac{1}{2}J[\bar{p}](0) + Cf(0) + Cf(t_0) + C\int_0^{t_0} f(\tau) + \tilde{f}(\tau)d\tau\Big) \\
+ CJ[\bar{p}](0) + CM_f(t_0) + C\int_0^t f(\tau) + \tilde{f}(\tau)d\tau \\
\le t_0^{-1}\Big(J_H[p](0) + CJ[\bar{p}](0) + CM_f(t_0) + \int_0^{t_0} \tilde{f}(\tau)d\tau\Big) \\
+ C\int_0^t f(\tau) + \tilde{f}(\tau)d\tau,
\end{aligned}$$

thus yields (4.52).

(ii) Let $t \ge 1$. Integrating (4.57) in s from $t - 1/2$ to t gives

$$1/2J[\bar{p}_t](t) \le \int_{t-1/2}^t J[\bar{p}_t](s)ds + C\int_{t-1/2}^t J[\bar{p}](s)ds + C/2\int_{t-1}^t f(\tau) + \tilde{f}(\tau)d\tau,$$

Using (4.36) to estimate $\int_{t-1/2}^t J[\bar{p}_t](s)ds$, we obtain

$$J[\bar{p}_t](t) \le CJ[\bar{p}](t-1) + Cf(t) + C\int_{t-1}^t f(\tau) + \tilde{f}(\tau)d\tau + C\int_{t-1/2}^t J[\bar{p}](s)ds. \tag{4.59}$$

Then applying (4.30) yields

$$\begin{aligned}
J[\bar{p}_t](t) \le C\{J[\bar{p}](0) + M_f(t-1)\} + Cf(t) + C\int_{t-1}^t f(\tau) + \tilde{f}(\tau)d\tau \\
+ C\int_{t-1/2}^t J[\bar{p}](0) + M_f(s)ds \\
\le CJ[\bar{p}](0) + CM_f(t) + Cf(t) + C\int_{t-1}^t f(\tau) + \tilde{f}(\tau)d\tau.
\end{aligned}$$

Since $\int_{t-1}^t f(\tau)d\tau \le f(t) \le M_f(t)$, we obtain (4.53).

(iii) Ineq. (4.55) results from (4.59) and the limit (4.31). Taking limit superior (4.59) as $t \to \infty$, we obtain (4.54).

(iv) By (4.59) and (4.32), for t sufficiently large:

$$\begin{aligned}
J[\bar{p}_t](t) \le C\big(1 + \beta + f(t-1)\big) + Cf(t) + C\int_{t-1}^t f(\tau) + \tilde{f}(\tau)d\tau \\
+ C\int_{t-1/2}^t \big(1 + \beta + f(s)\big)ds \\
\le C\big(1 + \beta + f(t-1) + f(t)\big) + C\int_{t-1}^t f(\tau) + \tilde{f}(\tau)d\tau.
\end{aligned}$$

Therefore we obtain (4.56). □

Note. Although estimates (4.32), (4.39) and (4.56) hold even when $A < \infty$, we usually use them for the case $A = \infty$.

Remark 4.6. *The estimates of $J_H[p](t)$ and $J[\bar{p}_t](t)$, for $t \ge 1$, in Theorems 4.4 and 4.5 are of uniform Gronwall-type and are derived from the system of coupled* *differential inequalities (4.17)–(4.19). This is an essential improvement from our previous paper* [18].

Remark 4.7. *Combining the above estimates for individual quantities $J_H[p](t)$, $J[\bar p](t)$ and $J[\bar p_t](t)$, we can readily bound their sum $I_H[\bar p](t) \stackrel{\text{def}}{=\!=} J[\bar p](t) + J[\bar p_t](t) + J_H[p](t)$. Another way to bound $I_H[\bar p](t)$ is by using its differential inequality directly. For instance, one easily derives from the above*

$$\frac{d}{dt} I_H[\bar p](t) \le -C I_H[\bar p](t) + C J[\bar p](t) + C f(t) + C\tilde f(t). \tag{4.60}$$

In [18] *we used (4.60) in estimation of solutions (see Propositions 3.13 and 3.15 of the referred article) and stability analysis. Though such approach is still valid, we do not repeat it in the current paper.*

5. Stability of solutions

In this section we establish L^2-stability for the pressure and its time derivative, and L^{2-a}-stability for the pressure gradient. For the L^2-stability, we combine new estimates in the previous section with the method used in [18]. Particular attention is paid to the large time and limit superior estimates.

5.1. **Perturbation of zero flux condition.** We will show that $J[\bar p](t)$, $J_H[\bar p](t)$, $J[\bar p_t](t)$ are small when the boundary flux $\psi(x,t)$ and initial data are small. Moreover, we show that the first three quantities are asymptotically small, regardless the size of the initial data, if the flux is asymptotically small.

Theorem 5.1. *Assume the Degree Condition. Given $\varepsilon > 0$. There are positive numbers δ_1, δ_2, δ_3, δ_4 depending on ε such that:*

(i) If $J[\bar p](0) < \delta_1$ and $\sup_{[0,\infty)} f(t) < \delta_2$ then

$$\sup_{[0,\infty)} J[\bar p](t) < \varepsilon. \tag{5.1}$$

(ii) Additionally, if $J_H[p](0) < \delta_3$ and $\sup_{[1,\infty)} \int_{t-1}^t \tilde f(\tau)d\tau < \delta_4$ then

$$\sup_{[0,\infty)} J_H[p](t) < \varepsilon, \tag{5.2}$$

$$\sup_{[t_0,\infty)} J[p_t](t) < \varepsilon\, t_0^{-1}, \ \sup_{[t_0,\infty)} J[\bar p_t](t) < \varepsilon\, t_0^{-1}, \ \textit{for all } t_0 \in (0,1). \tag{5.3}$$

Proof. Assume

$$J[\bar p](0) < \delta_1, \ \sup_{[0,\infty)} f(t) < \delta_2, \ J_H[p](0) < \delta_3, \ \sup_{[1,\infty)} \int_{t-1}^t \tilde f(\tau)d\tau < \delta_4.$$

Set $M_f(t) = \delta_2$ for all t, then $M_f(t)$ is a continuously increasing majorant of $f(t)$ in the inequality (4.30). Then from (4.30) follows that for all $t \ge 0$

$$J[\bar p](t) \le \delta_1 + C\delta_2. \tag{5.4}$$

To estimate $J_H[p](t)$ we use (4.35) for $t \in [0,1)$, and use (4.37) for $t \ge 1$, we obtain

$$J_H[p](t) \le C(\delta_1 + \delta_2 + \delta_3 + \delta_4), \ \text{for all } t \in [0,\infty). \tag{5.5}$$

Similarly, let $t_0 \in (0,1)$ be fixed, we estimate $J[\bar{p}_t](t)$ for $t \in [t_0, \infty)$ by using (4.52) if $t_0 \le t < 1$, and using (4.53) if $t \ge 1$, we obtain

$$J[\bar{p}_t](t) \le C\, t_0^{-1}(\delta_1 + \delta_2 + \delta_3 + \delta_4), \text{ for all } t \in [t_0, \infty). \tag{5.6}$$

Note that $f(t) = \varphi_{1,\frac{2-a}{2-2a}}(\|\psi(t)\|^2_{L^\infty})$, where the function $\varphi_{1,\frac{2-a}{2-2a}}$ is defined by (2.17), hence $\|\psi(t)\|^2_{L^\infty} = \varphi^{-1}_{1,\frac{2-a}{2-2a}}(f(t))$. Then by Lemma 4.1, we have

$$J[p_t](t) \le 2J[\bar{p}_t](t) + 2|\Gamma|^2|U|^{-1}\varphi^{-1}_{1,\frac{2-a}{2-2a}}(f(t)). \tag{5.7}$$

Under our assumptions and with estimate (5.6), we obtain

$$J[p_t](t) \le C\, t_0^{-1}\Big(\delta_1 + \delta_2 + \delta_3 + \delta_4 + \varphi^{-1}_{1,\frac{2-a}{2-2a}}(\delta_2)\Big), \text{ for all } t \in [t_0, \infty). \tag{5.8}$$

Note that $\varphi_{1,\frac{2-a}{2-2a}}$ is continuous at 0 with $\varphi_{1,\frac{2-a}{2-2a}}(0) = 0$. From (5.4), (5.5), (5.6) and (5.8), we can easily select positive numbers δ_1, δ_2, δ_3, and δ_4 depending on ε so that (5.1), (5.2) and (5.3) hold. □

Above theorem provides stability for $\bar{p}$ for all time depending on both *initial* and *boundary* data. Using asymptotic estimates in the section 4, we can obtain stability as $t \to \infty$ with respect to asymptotic behavior of boundary data only, hence such stability is independent of the initial data.

Theorem 5.2. *Assume the Degree Condition. For any $\varepsilon > 0$ there are positive numbers δ_1 and δ_2 such that:*
(i) If $\limsup_{t\to\infty} f(t) < \delta_1$ then

$$\limsup_{t\to\infty} J[\bar{p}](t) < \varepsilon. \tag{5.9}$$

(ii) Additionally, if $\limsup_{t\to\infty} \int_{t-1}^t \tilde{f}(\tau)d\tau < \delta_2$ then

$$\limsup_{t\to\infty} J_H[p](t) < \varepsilon, \ \limsup_{t\to\infty} J[p_t](t) < \varepsilon \ \textit{and} \ \limsup_{t\to\infty} J[\bar{p}_t](t) < \varepsilon. \tag{5.10}$$

Proof. If $\limsup_{t\to\infty} f(t) < \delta_1$ and $\limsup_{t\to\infty} \int_{t-1}^t \tilde{f}(\tau)d\tau < \delta_2$, then we have from (4.31), (4.38) and (4.54) that

$$\limsup_{t\to\infty} J[\bar{p}](t) < C\delta_1, \quad \limsup_{t\to\infty} J_H[p](t) < C(\delta_1+\delta_2),$$
$$\limsup_{t\to\infty} J[\bar{p}_t](t) < C(\delta_1+\delta_2).$$

The last inequality and (5.7) imply

$$\limsup_{t\to\infty} J[p_t](t) < C\big(\delta_1 + \delta_2 + \varphi^{-1}_{1,\frac{2-a}{2-2a}}(\delta_1)\big).$$

Clearly, δ_1 and δ_2 exist so that (5.9) and (5.10) hold. □

As a consequence, $J[\bar{p}](t)$, $J_H[p](t)$ and $J[p_t](t)$ converge to zero as $t \to \infty$ under a priori assumptions that boundary data are vanishing at infinity.

Corollary 5.3. *Assume the Degree Condition.*
(i) If $\lim_{t\to\infty} f(t) = 0$ then $\lim_{t\to\infty} J[\bar{p}](t) = 0$.
(ii) Additionally, if $\lim_{t\to\infty} \int_{t-1}^t \tilde{f}(\tau)d\tau = 0$ then

$$\lim_{t\to\infty} J_H[p](t) = \lim_{t\to\infty} J[p_t](t) = \lim_{t\to\infty} J[\bar{p}_t](t) = 0.$$

In case $\|\psi(t)\|_{L^\infty}$ is large or even unbounded as $t\to\infty$, $J[\bar p_t](t)$ can still be small provided that $\|\psi_t(t)\|_{L^\infty}$ is sufficiently small.

Theorem 5.4. *Assume the Degree Condition. Suppose*

$$\int_1^\infty \Big(1+M_f(t)+\int_{t-1}^t \tilde f(\tau)d\tau\Big)^{-b}dt=\infty. \tag{5.11}$$

Let $A_0=J[\bar p](0)$ *in general case, and* $A_0=\beta$ *in case* $\beta<\infty$. *Then*

$$\limsup_{t\to\infty} J[\bar p_t](t)\le C\limsup_{t\to\infty}\Big\{\|\psi_t(t)\|^2_{L^\infty}\Big(1+A_0+M_f(t)+\int_{t-1}^t \tilde f(\tau)d\tau\Big)^{2b}\Big\}. \tag{5.12}$$

Consequently, if $\|\psi(t)\|_{L^\infty}$ *is uniformly bounded on* $[0,\infty)$ *and* $\|\psi_t(t)\|_{L^\infty}\to 0$ *as* $t\to\infty$ *then* $J[\bar p_t](t)\to 0$ *as* $t\to\infty$.

Proof. We use differential inequality (4.26) for $J[\bar p_t](t)$. Applying inequality (2.26) from Lemma 2.3 with $w=\psi_t$, $v=p$, $u=p_t$, $\varepsilon=(1-a)/2$ to estimate the boundary integral in (4.26), we obtain

$$\frac12\frac{d}{dt}J[\bar p_t](t)\le -\frac{(1-a)}{2}\int_U K(|\nabla p|)|\nabla p_t|^2dx + C\|\psi_t\|^2_{L^\infty}\Big(1+\big(J_H[p](t)\big)^{\frac{a}{2-a}}\Big).$$

Applying weighted Poincaré inequality (2.19) with $u=\bar p_t$ and $v=p$ to the first integral on the right-hand side one gets

$$\frac12\frac{d}{dt}J[\bar p_t](t)\le -CJ[\bar p_t](t)\big(1+J_H[p](t)\big)^{\frac{-a}{2-a}} + C_\varepsilon\|\psi_t\|^2_{L^\infty}\Big(1+\big(J_H[p](t)\big)^{\frac{a}{2-a}}\Big).$$

Let $N(t)=1+A_0+M_f(t)+\int_{t-1}^t \tilde f(\tau)d\tau$. Applying estimate (4.37) in general case, and estimate (4.40) in case $\beta<\infty$, we have $J_H[p](t)\le CN(t)$ for $t>T$, some $T>0$. Therefore

$$\frac12\frac{d}{dt}J[\bar p_t](t)\le -CN(t)^{-b}J[\bar p_t](t)+C\|\psi_t(t)\|^2_{L^\infty}N(t)^b, \tag{5.13}$$

for $t>1$. Then applying Lemma 3.9 for $t\in(1,\infty)$, we obtain (5.12).

In case $\|\psi(t)\|_{L^\infty}$ is uniformly bounded on $[0,\infty)$ and $\|\psi_t(t)\|_{L^\infty}\to 0$ as $t\to\infty$, we have $M_f(t)$ and $\int_{t-1}^t \tilde f(\tau)d\tau$ are uniformly bounded on $[1,\infty)$. Hence (5.11) is satisfied and (5.12) implies that $J[\bar p_t](t)\to 0$ as $t\to\infty$. □

5.2. **Continuous dependence for pressure.** In this section, we establish the continuous dependence of $\bar p(x,t)$ with respect to the L^2-norm. We estimate the difference between two solutions of the IBVP (4.9)–(4.11) in terms of the difference between the boundary and initial data either in finite time intervals, or at time infinity.

Let $p_1(x,t)$ and $p_2(x,t)$ be two solutions having fluxes ψ_1 and ψ_2, and initial data $p_1(x,0)$ and $p_2(x,0)$, respectively.

Let $\Psi=\psi_1-\psi_2$, $P=p_1-p_2$, and $\bar P=P-|U|^{-1}\int_U Pdx$. Hence by (4.8),

$$\begin{aligned}\bar P(x,t)&=\bar p_1(x,t)-\bar p_2(x,t)\\ &=P(x,t)-\frac{1}{|U|}\int_U P(x,0)dx+\frac{1}{|U|}\int_0^t\int_\Gamma \Psi(x,\tau)d\sigma d\tau.\end{aligned} \tag{5.14}$$

From (4.9) follows

$$\partial_t\bar P=\nabla\cdot\big(K(|\nabla\bar p_1|)\nabla\bar p_1\big)-\nabla\cdot\big(K(|\nabla\bar p_2|)\nabla\bar p_2\big)+\frac{1}{|U|}(\gamma_1'(t)-\gamma_2'(t)), \tag{5.15}$$

Multiplying equation above by $\bar{P}=\bar{p}_1-\bar{p}_2$, integrating by part, and taking into account boundary conditions (4.10) for functions $\bar{p}_1$ and $\bar{p}_2$ we have

$$\frac{1}{2}\frac{d}{dt}\int_U \bar{P}^2 dx = -\int_U \Phi(\nabla\bar{p}_1,\nabla\bar{p}_2)dx + \int_\Gamma \Psi\bar{P}d\sigma + \frac{1}{|U|}(\gamma_1'(t)-\gamma_2'(t))\int_U \bar{P}dx.$$

Note that the last integral is zero due to (4.7), hence

$$\frac{1}{2}\frac{d}{dt}\int_U \bar{P}^2 dx = -\int_U \Phi(\nabla p_1,\nabla p_2)dx - \int_\Gamma \Psi\bar{P}d\sigma. \tag{5.16}$$

When $\psi_1=\psi_2$, the last boundary integral vanishes, hence by the monotonicity of Φ (Lemma 2.4), we have $\frac{d}{dt}J[\bar{P}](t)\le 0$. Therefore for $t\ge 0$, $J[\bar{P}](t)\le J[\bar{P}](0)$. Also, in this case, by (5.14), $P=\bar{P}+|U|^{-1}\int_U P(x,0)dx$, hence one easily finds that $J[P](t)\le J[\bar{P}](t)+J[P](0)\le 2J[P](0)$. Consequently, the solution of IBVP is unique and (*a priori*) Lyapunov stable in $L^2(U)$ (with respect to the initial data). We will not go further to analyze the stability of the gradient of p in this case of equal fluxes, but rather turn to the general case.

Now consider two arbitrary fluxes ψ_1 and ψ_2. Similar to Lemma 4.1, we have for all $t\ge 0$ that

$$J[\bar{P}](t)\le J[P](t)\le J[\bar{P}](t)+2J[P](0)+2|U|^{-1}\Big(\int_0^t\int_\Gamma \Psi(x,s)d\sigma ds\Big)^2.$$

We focus on estimating $J[\bar{P}](t)$. Let $\Lambda(t)=1+J_H[\bar{p}_1](t)+J_H[\bar{p}_2](t)$.

Lemma 5.5. *(i) For $t\ge 0$,*

$$J[\bar{P}](t)\le J[\bar{P}](0)+C\int_0^t \|\Psi(\tau)\|_{L^\infty}^2\Lambda(\tau)^b d\tau. \tag{5.17}$$

(ii) Assume the Degree Condition. Then for $t\ge 0$

$$J[\bar{P}](t)\le e^{-C\int_0^t \Lambda^{-b}(\tau)d\tau}J[\bar{P}](0)+\int_0^t e^{-C\int_\tau^t \Lambda^{-b}(\theta)d\theta}\|\Psi(\tau)\|_{L^\infty}^2\Lambda(\tau)^b d\tau. \tag{5.18}$$

Moreover, if $\int_0^\infty \Lambda^{-b}(\tau)d\tau=\infty$ then

$$\limsup_{t\to\infty} J[\bar{P}](t)\le C\limsup_{t\to\infty}\Big\{\|\Psi(t)\|_{L^\infty}^2\Lambda(t)^{2b}\Big\}. \tag{5.19}$$

Proof. (i) Using Ineq. (2.30) in (5.16) we obtain

$$\frac{1}{2}\frac{d}{dt}J[\bar{P}](t)\le -C_1\Big(\int_U|\nabla p_1-\nabla p_2|^{2-a}\Big)^{\frac{2}{2-a}}\Lambda(t)^{-b}+\int_\Gamma|\Psi\bar{P}|d\sigma. \tag{5.20}$$

By the trace theorem and Poincaré's inequality:

$$\begin{aligned}\int_\Gamma|\Psi||\bar{P}|d\sigma &\le C\|\Psi\|_{L^\infty}\int_U|\nabla P|dx\\ &\le C\|\Psi\|_{L^\infty}\Lambda(t)^{-b/2}\Big(\int_U|\nabla P|^{2-a}dx\Big)^{\frac{1}{2-a}}\Lambda(t)^{b/2}\\ &\le \varepsilon\Lambda(t)^{-b}\Big(\int_U|\nabla P|^{2-a}dx\Big)^{\frac{2}{2-a}}+C_\varepsilon\|\Psi\|_{L^\infty}^2\Lambda(t)^b.\end{aligned} \tag{5.21}$$

Set $\varepsilon=C_1/2$, then from (5.20) and (5.21) follows

$$\frac{1}{2}\frac{d}{dt}J[\bar{P}](t)\le -\frac{C_1}{2}\Big(\int_U|\nabla P|^{2-a}\Big)^{\frac{2}{2-a}}\Lambda(t)^{-b}+C\|\Psi\|_{L^\infty}^2\Lambda(t)^b. \tag{5.22}$$

Neglecting the negative term on the right-hand side and integrating (5.22) in time yield (5.17).

(ii) Under the Degree Condition, using Poincaré-Sobolev inequality in (5.22) gives

$$\frac{d}{dt}\int_U \bar{P}^2 dx \le -C\Lambda(t)^{-b}\int_U \bar{P}^2 dx + C\|\Psi\|_{L^\infty}^2 \Lambda(t)^b. \tag{5.23}$$

We then use Gronwall's inequality to obtain (5.18), and apply Lemma 3.9 to obtain (5.19). □

Notation. *Same as (4.4) and (4.5), we define for $i = 1, 2$,*

$$f_i(t) = \|\psi_i(t)\|_{L^\infty}^2 + \|\psi_i(t)\|_{L^\infty}^{\frac{2-a}{1-a}},\ \ \tilde{f}_i(t) = \|\psi_{it}(t)\|_{L^\infty}^2 + \|\psi_{it}(t)\|_{L^\infty}^{\frac{2-a}{1-a}}.$$

For $i = 1, 2$, we assume $f_i(t), \tilde{f}_i(t) \in C([0,\infty))$ and when needed $f_i(t) \in C^1((0,\infty))$; let $A_i = \limsup_{t\to\infty} f_i(t)$ and $\beta_i = \limsup_{t\to\infty}[f_i'(t)]^-$.

Set $\bar{A} = A_1 + A_2$, $\bar{\beta} = \beta_1 + \beta_2$.

Let $F(t) = f_1(t) + f_2(t)$, $M_F(t) = M_{f_1}(t) + M_{f_2}(t)$, where $M_{f_i}(t)$, $i = 1, 2$, is a continuous increasing majorant of $f_i(t)$ on $[0,\infty)$. Let $\tilde{F}(t) = \tilde{f}_1(t) + \tilde{f}_2(t)$.

For initial data, set $A_0 = J[\bar{p}_1](0) + J[\bar{p}_2](0)$ and $B_0 = J_H[p_1](0) + J_H[p_2](0)$.

Using the bounds of $J_H[p](t)$ for individual solutions we obtain explicit continuous dependence of the solution (on bounded time intervals) on the initial and boundary data with respect to L^2-norm.

Theorem 5.6. *(i) If $t \in [m, m+1)$, where m is a non-negative integer, then*

$$\begin{aligned} J[\bar{P}](t) \le J[\bar{P}](0) + C\sum_{i=0}^{m}\Big\{\Big(\int_i^{i+1}\|\Psi(\tau)\|_{L^\infty}^{\frac{2-a}{1-a}}d\tau\Big)^{\frac{2-2a}{2-a}} \\ \cdot\Big(1 + J[\bar{p}_1](0) + J[\bar{p}_2](0) + \int_0^{i+1}\big(f_1(\tau)+f_2(\tau)\big)d\tau\Big)^b\Big\}. \end{aligned} \tag{5.24}$$

(ii) For any $T > 0$,

$$\sup_{[0,T]} J[\bar{P}](t) \le J[\bar{P}](0) + C\cdot M_{1,T}\sup_{[0,T]}\|\Psi(t)\|_{L^\infty}^2, \tag{5.25}$$

where $M_{1,T} = A_0 + T + \int_0^T f_1(\tau) + f_2(\tau)d\tau$.

(iii) Assume the Degree Condition and $\bar{A} < \infty$. If

$$\int_1^\infty \Big(1 + \int_{\tau-1}^{\tau}\tilde{F}(s)ds\Big)^{-b} d\tau = \infty \tag{5.26}$$

then

$$\limsup_{t\to\infty} J[\bar{P}](t) \le C\limsup_{t\to\infty}\Big\{\|\Psi(t)\|_{L^\infty}^2\Big(1 + \bar{A} + \int_{t-1}^{t}\tilde{F}(\tau)d\tau\Big)^{2b}\Big\}. \tag{5.27}$$

Proof. (i) For any given $t \in [m, m+1)$ we rewrite the integral in (5.17) and apply Hölder's inequality:

$$\begin{aligned}\int_0^t \|\Psi(\tau)\|_{L^\infty}^2 \Lambda(\tau)^b d\tau &= \sum_{i=0}^{m} \int_i^{i+1} \|\Psi\|_{L^\infty}^2 \Lambda^b(\tau) d\tau \\ &\le \sum_{i=0}^{m} \Big(\int_i^{i+1} \|\Psi(\tau)\|_{L^\infty}^{\frac{2-a}{1-a}} d\tau \Big)^{\frac{2-2a}{2-a}} \Big(\int_i^{i+1} \Lambda(\tau) d\tau \Big)^b .\end{aligned} \tag{5.28}$$

For each i, using (4.29):

$$\begin{aligned}\int_i^{i+1} \Lambda(\tau) d\tau &\le 1 + \int_0^{i+1} J_H[p_1](\tau) + J_H[p_2](\tau) d\tau \\ &\le 1 + \sum_{j=1}^{2} \Big(J[\bar p_j](0) + C \int_0^{i+1} f_j(\tau) d\tau \Big).\end{aligned} \tag{5.29}$$

Then (5.24) follows (5.28) and (5.29).

(ii) Since $b < 1$, we have $\int_0^T \Lambda^b(\tau) d\tau \le T + \int_0^T \Lambda(\tau) d\tau$. Then applying (4.29) to estimate $\int_0^T \Lambda(\tau) d\tau$, we obtain (5.25) from (5.17).

(iii) From (4.39) follows that for there exist T such that for $t \ge T$

$$\Lambda(t) \le C\Big(1 + A_1 + A_2 + \int_{t-1}^t \tilde F(\tau) d\tau\Big) = C\Big(1 + \bar A + \int_{t-1}^t \tilde F(\tau) d\tau\Big). \tag{5.30}$$

Estimate (5.30) and (5.26) imply $\int_0^\infty \Lambda^{-b}(\tau) d\tau = \infty$. Using (5.30) in (5.19) we obtain (5.27). □

Remark 5.7. *Ineq. (5.24) is an estimate of $J[\bar P](t)$ for all t when the Degree Condition is not imposed. If the sum $\sum_{i=1}^\infty \{\ldots\}$ is finite, where $\{\ldots\}$ denotes the summand in (5.24), then $J[\bar P](t)$ is uniformly bounded on $[0, \infty)$. In that case, the integral $\int_i^{i+1} \|\Psi(\tau)\|_{L^\infty}^{\frac{2-a}{1-a}} d\tau$ must sufficiently diminish the growth of $\int_0^{i+1} (f_1(\tau) + f_2(\tau)) d\tau$, as $i \to \infty$.*

The estimate in Theorem 5.6(iii) requires $\bar A < \infty$, i.e. each individual flux $\psi_i(x,t)$ $(i = 1, 2)$ is bounded at time infinity. In case $\bar A = \infty$, we we can still control the L^2-norm of $\bar P(x,t)$ for large t by using previously obtained estimates for individual solutions and under an integral constraint on the flux and its time derivative.

We use the following functions to quantify the effects of unbounded boundary data on estimates of solutions:

$$W_1(t) = 1 + M_F(t) + \int_{t-1}^t \tilde F(\tau) d\tau, \tag{5.31}$$

$$W_2(t) = 1 + F(t) + F(t-1) + \int_{t-1}^t \tilde F(\tau) d\tau. \tag{5.32}$$

Note that, due to the presence of $M_F(t)$ which needs to be increasing, the function $W_1(t)$ depends on the initial values of the boundary fluxes. In contrast, $W_2(t)$ only involves the data's values on the interval $[t-1, t]$.

Lemma 5.8. *Assume the Degree Condition and $\bar{A}=\infty$.*

(i) There is $T_1>0$ such that

$$\Lambda(t)\le CW_1(t),\ \ J[\bar{p}_{1t}](t)+J[\bar{p}_{2t}](t)\le CW_1(t) \text{ for all } t>T_1. \tag{5.33}$$

(ii) If $\bar{\beta}<\infty$ then there is $T_2>0$ such that

$$\Lambda(t)\le W_2(t),\ \ J[\bar{p_1}_t](t)+J[\bar{p_2}_t](t)\le CW_2(t) \text{ for all } t>T_2. \tag{5.34}$$

Proof. (i) By (4.37) we have

$$\Lambda(t)\le 1+CJ[\bar{p}_1](0)+CJ[\bar{p}_2](0)+CM_F(t)+\int_{t-1}^t \tilde{F}(\tau)d\tau.$$

Since $M_F(t)\to\infty$ as $t\to\infty$, we obtain the first inequality in (5.33) for sufficiently large t. The second inequality of (5.33) is proved similarly by using (4.53).

(ii) Note that $\beta_1,\beta_2<\infty$ and $F(t)\to\infty$ as $t\to\infty$. Using (4.40) to estimate individual $J_H[p_1](t)$ and $J_H[p_2](t)$ we have

$$J_H[p_i](t)\le C\Big(1+F(t-1)+F(t)+\int_{t-1}^t F(\tau)+\tilde{F}(\tau)d\tau\Big),$$

for $i=1,2$, and t sufficiently large. Hence we obtain the estimate of $\Lambda(t)$ in (5.34).

Similarly, using (4.56) to estimate $J[\bar{p_1}_t](t)$, $J[\bar{p_2}_t](t)$ for large t, we obtain the second estimate in (5.34). □

We readily obtain:

Theorem 5.9. *Assume the Degree Condition and $\bar{A}=\infty$. Set $W(t)=W_1(t)$ in general case, or $W(t)=W_2(t)$ in case $\bar{\beta}<\infty$. If $\int_1^\infty W^{-b}(t)dt=\infty$ then*

$$\limsup_{t\to\infty} J[\bar{P}](t)\le C\limsup_{t\to\infty}\big\{\|\Psi(t)\|^2_{L^\infty}W^{2b}(t)\big\}. \tag{5.35}$$

Proof. By Lemma 5.8, for sufficiently large t, we have $\Lambda(t)\le CW(t)$. Then applying the limit superior estimate in Lemma 5.5(ii) yields (5.35). □

5.3. **Continuous dependence for pressure gradient.** We use the same notation as in subsection 5.2. We now turn to estimating the difference $\nabla p_1-\nabla p_2=\nabla P=\nabla\bar{P}$. By virtue of estimates for individual solutions in Theorem 4.4 and relation (2.15), we have bounds for $\int_U|\nabla p_i|^{2-a}dx$, $i=1,2$. Hence it is natural to consider estimation of $\int_U|\nabla p_1-\nabla p_2|^{2-a}dx$ which also equals $\int_U|\nabla\bar{P}|^{2-a}dx$.

First, we connect $\int_U|\nabla\bar{P}(x,t)|^{2-a}dx$ to $\Lambda(t)$, $J[p_{1t}](t)+J[p_{2t}](t)$ and $J[\bar{P}](t)$.

Proposition 5.10. *For $t>0$,*

$$\Big(\int_U|\nabla\bar{P}(x,t)|^{2-a}dx\Big)^{\frac{2}{2-a}}\le C\Big(\Lambda^{2b}(t)\cdot\big(J[\bar{p}_{1t}](t)+J[\bar{p}_{2t}](t)\big)\cdot J[\bar{P}](t)\Big)^{1/2} + C\|\Psi\|^2_{L^\infty}\Lambda(t)^{2b}. \tag{5.36}$$

Proof. Multiplying inequality (5.22) in Lemma 5.5 by $2\Lambda^b(t)/C_1$ we obtain

$$\begin{aligned}\Big(\int_U|\nabla P(x,t)|^{2-a}dx\Big)^{\frac{2}{2-a}} &\le -\frac{C\Lambda(t)^b}{2}\frac{d}{dt}J[\bar{P}](t)+C\|\Psi(t)\|^2_{L^\infty}\Lambda(t)^{2b}\\ &= -C\Lambda(t)^b\int_U\bar{P}_t\bar{P}dx+C\|\Psi(t)\|^2_{L^\infty}\Lambda(t)^{2b}\\ &\le C\Lambda(t)^b\Big(\int_U\bar{P}_t^2(x,t)dx\Big)^{1/2}\Big(\int_U\bar{P}^2(x,t)dx\Big)^{1/2}+C\|\Psi(t)\|^2_{L^\infty}\Lambda(t)^{2b}.\end{aligned}$$

Applying the inequality $\int_U \bar P_t^2(x,t)dx \le 2\int_U (\bar p_{1t}^2(x,t)+\bar p_{2t}^2(x,t))dx$ yields (5.36). □

In order to estimate the L^{2-a}-norm for $\nabla\bar P$, we combine (5.36) with estimates in section 4 for individual solutions to control $\Lambda(t)$ and $J[\bar p_{1t}](t)+J[\bar p_{2t}](t)$, as well as with estimates in subsection 5.2 to control $J[\bar P](t)$. Doing so we obtain various stability results for the pressure gradient (Theorems 5.11, 5.12, 5.14).

Theorem 5.11. *For $0<t_0\le 1$ and $T\ge t_0$,*

$$\begin{aligned}\sup_{[t_0,T]}\Big(\int_U |\nabla\bar P(x,t)|^{2-a}dx\Big)^{\frac{2}{2-a}} &\le Ct_0^{-1/2}\Big(M_{2,T}^{b+1/2}J[\bar P](0)^{1/2}\\ &+M_{2,T}^{b+1}\sup_{[0,T]}\|\Psi\|_{L^\infty}\Big)+M_{2,T}^{2b}\sup_{[0,T]}\|\Psi\|_{L^\infty}^2,\end{aligned}\tag{5.37}$$

where $M_{2,T}=1+A_0+B_0+\sum_{i=1}^2\Big(\sup_{[0,T]}f_i(t)+\int_0^T f_i(\tau)+\tilde f_i(\tau)d\tau\Big)$.

Proof. Using (4.35) and (4.52), we have

$$\sup_{[0,T]}\Lambda(t)\le C\cdot M_{2,T},\tag{5.38}$$

$$\sup_{[t_0,T]}(J[\bar p_{1t}](t)+J[\bar p_{2t}](t))\le C\cdot t_0^{-1}\cdot M_{2,T},\ \text{respectively.}\tag{5.39}$$

Above we can chose $M_F(t)=\sum_{i=1,2}\sup\{f_i(s):s\in[0,t]\}$ for all $t\ge0$, hence $M_{1,T}$ in Theorem 5.6 is bounded by $CM_{2,T}$.

Combining (5.36) with (5.38), (5.39) and (5.25) we obtain

$$\begin{aligned}\sup_{[t_0,T]}\Big(\int_U|\nabla\bar P(x,t)|^{2-a}dx\Big)^{\frac{2}{2-a}}&\le C\Big(\big\{M_{2,T}^{2b}\big\}\big\{t_0^{-1}M_{2,T}\big\}\big\{J[\bar P](0)\\ &+M_{2,T}\sup_{[0,T]}\|\Psi\|_{L^\infty}^2\big\}\Big)^{1/2}+CM_{2,T}^{2b}\sup_{[0,T]}\|\Psi\|_{L^\infty}^2.\end{aligned}$$

Thus (5.37) follows. □

Theorem 5.12. *Assume the Degree Condition.*

(i) For any $t\ge1$

$$\begin{aligned}&\Big(\int_U|\nabla\bar P(x,t)|^{2-a}dx\Big)^{\frac{2}{2-a}}\\ &\le CN(t)^{b+1/2}\cdot\Big(e^{-C\int_1^t N(\theta)^{-b}d\theta}\big(J[\bar P](0)+M_{1,1}\sup_{[0,1]}\|\Psi(t)\|_{L^\infty}^2\big)\\ &\quad+\int_1^t e^{-C\int_\tau^t N(\theta)^{-b}d\theta}\|\Psi(\tau)\|_{L^\infty}^2N(\tau)^bd\tau\Big)^{1/2}+C\|\Psi\|_{L^\infty}^2N(t)^{2b},\end{aligned}\tag{5.40}$$

where $N(t)=A_0+W_1(t)$, and $M_{1,1}$ is $M_{1,T}$, with $T=1$, defined in Theorem 5.6.

(ii) If $\bar A<\infty$ and $\int_{t-1}^t\tilde F(\tau)d\tau$ is uniformly bounded on $[1,\infty)$, then

$$\begin{aligned}\sup_{[1,\infty)}\Big(\int_U|\nabla\bar P(x,t)|^{2-a}dx\Big)^{\frac{2}{2-a}}&\le CM_3^{b+1/2}J[\bar P](0)^{1/2}\\ &+CM_3^{3b/2+1}\sup_{[0,\infty)}\|\Psi\|_{L^\infty}+C(M_3^{b+1}+M_3^{2b+1/2})\sup_{[0,\infty)}\|\Psi\|_{L^\infty}^2,\end{aligned}\tag{5.41}$$

where $M_3 = 1 + A_0 + \sup_{[0,\infty)} f_1(t) + \sup_{[0,\infty)} f_2(t) + \sup_{[1,\infty)} \int_{t-1}^t \tilde{F}(\tau)d\tau$, *and*

$$\limsup_{t\to\infty} \Big(\int_U |\nabla \bar{P}(x,t)|^{2-a}dx\Big)^{\frac{2}{2-a}} \le CM_4^{2b+1/2} \limsup_{t\to\infty} \|\Psi(t)\|_{L^\infty} + CM_4^{2b} \limsup_{t\to\infty} \|\Psi(t)\|^2_{L^\infty}, \tag{5.42}$$

where $M_4 = 1 + \bar{A} + \limsup_{t\to\infty} \int_{t-1}^t \tilde{F}(\tau)d\tau$.

Proof. (i) Note from (4.37) and (4.53) that

$$\Lambda(t), J[\bar{p}_{1t}](t) + J[\bar{p}_{1t}](t) \le CN(t) \text{ for all } t \ge 1. \tag{5.43}$$

Let $t \ge 1$. Replacing $\Lambda(t)$ by $N(t)$ in (5.23) and integrating from 1 to t, we obtain

$$J[\bar{P}](t) \le e^{-C\int_1^t N(\theta)^{-b}d\theta} J[\bar{P}](1) + \int_1^t e^{-C\int_\tau^t N(\theta)^{-b}d\theta} \|\Psi(\tau)\|^2_{L^\infty} N(\tau)^b d\tau.$$

We estimate $J[\bar{P}](1)$ by (5.25). Then combining these with (5.36) gives (5.40).

(ii) Note in this case that $N(t) \le CM_3$ for $t \ge 1$, and also $M_{1,1} \le CM_3$. From (5.40) we have for $t \ge 1$ that

$$\Big(\int_U |\nabla \bar{P}(x,t)|^{2-a}dx\Big)^{\frac{2}{2-a}} \le CM_3^{b+1/2}\Big(J[\bar{P}](0) + M_3 \sup_{[0,\infty)} \|\Psi\|^2_{L^\infty} + M_3^b \sup_{[0,\infty)} \|\Psi\|^2_{L^\infty} \int_1^t e^{-M_3^{-b}(t-\tau)}d\tau\Big)^{1/2} + \sup_{[0,\infty)} \|\Psi\|^2_{L^\infty} M_3^{2b}.$$

Then (5.41) follows.

By (4.39) and (4.55), $\Lambda(t)$ and $J[p_{1t}](t) \cdot J[p_{2t}](t)$ are bounded by $C\big(1 + \bar{A} + \int_{t-1}^t \tilde{F}(\tau)d\tau\big)$ for large t. We estimate $\limsup J[\bar{P}](t)$ by (5.27), hence it follows from (5.36) that

$$\begin{aligned}
&\limsup_{t\to\infty} \left(\int_U |\nabla \bar{P}(x,t)|^{2-a}dx\right)^{\frac{2}{2-a}}\\
&\le CM_4^{(2b+1)/2}\Big(\limsup_{t\to\infty} \|\Psi(t)\|^2_{L^\infty} M_4^{2b}\Big)^{1/2} + CM_4^{2b} \limsup_{t\to\infty} \|\Psi(t)\|^2_{L^\infty}\\
&= CM_4^{2b+1/2} \limsup_{t\to\infty} \|\Psi(t)\|_{L^\infty} + CM_4^{2b} \limsup_{t\to\infty} \|\Psi(t)\|^2_{L^\infty}.
\end{aligned}$$

Then (5.42) follows. □

In the following we deal with unbounded data. For that we need to estimate the group $\Lambda^{2b}(t) \cdot \big(J[\bar{p}_{1t}](t) + J[\bar{p}_{2t}](t)\big) \cdot J[\bar{P}](t)$ on the right-hand side of (5.36). In case $\Lambda(t)$ and $J[\bar{p}_{1t}](t) + J[\bar{p}_{2t}](t)\big)$ are bounded by the same function $Z(t)$, we then need to estimate $Z^{2b+1}(t)J[\bar{P}](t)$.

Lemma 5.13. *Assume the Degree Condition. Let* $Z(t) \in C([0,\infty)) \cap C^1(0,\infty))$ *be a majorant of* $\Lambda(t)$ *on* $(0,\infty)$. *If*

$$\lim_{t\to\infty} Z'(t)Z^{b-1}(t) = 0 \text{ and } \int_0^\infty Z^{-b}(t)dt = \infty, \tag{5.44}$$

then

$$\limsup_{t\to\infty} \Big(Z^{2b+1}(t)J[\bar{P}](t)\Big) \le C \limsup_{t\to\infty} \Big(\|\Psi(t)\|^2_{L^\infty} Z^{4b+1}(t)\Big). \tag{5.45}$$

Proof. First, we find a differential inequality for $Z^{1+2b}(t)J[\bar{P}](t)$. From (5.23) follows

$$\frac{d}{dt}J[\bar{P}](t) \le -CZ^{-b}(t)J[\bar{P}](t) + C\|\Psi\|^2_{L^\infty}Z^b(t).$$

Then

$$\begin{aligned}\frac{d}{dt}\Big(Z^{2b+1}(t)J[\bar{P}](t)\Big) &= (2b+1)J[\bar{P}](t)Z^{2b}\frac{d}{dt}Z(t) + Z^{2b+1}(t)\frac{d}{dt}J[\bar{P}](t)\\ &\le CJ[\bar{P}](t)Z^{2b}(t)\frac{d}{dt}Z(t) - CZ^{b+1}(t)J[\bar{P}](t) + C\|\Psi(t)\|^2_{L^\infty}Z^{3b+1}(t)\\ &= -CZ(t)^{b+1}J[\bar{P}](t)\Big(1 - Z'(t)Z^{b-1}(t)\Big) + C\|\Psi(t)\|^2_{L^\infty}Z^{3b+1}(t).\end{aligned}$$

By virtue of the first condition in (5.44) we obtain the differential inequality:

(5.46)
$$\frac{d}{dt}\Big(Z(t)^{2b+1}J[\bar{P}](t)\Big) \le -CZ^{-b}(t)\Big(Z(t)^{2b+1}J[\bar{P}](t)\Big) + C\|\Psi(t)\|^2_{L^\infty}Z^{3b+1}(t),$$

for all $t > T$, some $T > 0$. Since $\int_0^\infty Z^{-b}(t)dt = \infty$ applying Lemma 3.9 to (5.46) we obtain (5.45). □

Theorem 5.14. *Assume the Degree Condition and $\bar{A} = \infty$.*

(i) Let $W(t)$ be either $W_1(t)$ in general case, or $W_2(t)$ in case $\bar{\beta} < \infty$. If

$$\lim_{t\to\infty} W'(t)W^{b-1}(t) = 0 \text{ and } \int_1^\infty W^{-b}(\tau)d\tau = \infty, \tag{5.47}$$

then

$$\begin{aligned}\limsup_{t\to\infty}\left(\int_U |\nabla P(x,t)|^{2-a}dx\right)^{\frac{2}{2-a}} &\le C\limsup_{t\to\infty}\Big(W^{2b+1/2}(t)\|\Psi(t)\|_{L^\infty}\Big)\\ &\quad + C\limsup_{t\to\infty}\Big(W^{2b}(t)\|\Psi(t)\|^2_{L^\infty}\Big).\end{aligned}\tag{5.48}$$

(ii) Consequently, if $\int_{t-1}^t \tilde{F}(\tau)d\tau$ is uniformly bounded on $[1,\infty)$, then one can replace $W(t)$ by $M_F(t)$ in (5.47) and (5.48).

Proof. (i) By Lemma 5.8, there is $T > 0$ such that for all $t > T$, we have $J[\bar{p}_{1t}](t) + J[\bar{p}_{2t}](t), \Lambda(t) \le CW(t)$, and hence by Proposition 5.10

$$\left(\int_U |\nabla\bar{P}(x,t)|^{2-a}dx\right)^{\frac{2}{2-a}} \le C\Big(W^{2b+1}(t)J[\bar{P}](t)\Big)^{1/2} + C\|\Psi(t)\|^2_{L^\infty}W(t)^{2b}. \tag{5.49}$$

Regarding the first term on the right-hand side of (5.49), applying Lemma 5.13 with $Z(t) = CW(t)$ on (T,∞) (instead of $(0,\infty)$), we have

$$\limsup_{t\to\infty}\Big(W^{2b+1}(t)J[\bar{P}](t)\Big) \le C\limsup_{t\to\infty}\Big(\|\Psi(t)\|^2_{L^\infty}W^{4b+1}(t)\Big).$$

Then taking limit superior of (5.49) yields (5.48).

(ii) Here we have $\Lambda(t), J[\bar{p}_{1t}](t) + J[\bar{p}_{2t}](t) \le CM_F(t)$ for large t. The proof follows the same arguments as in part (i) above with $W(t)$ being replaced by $M_F(t)$. □

Remark 5.15. *Comparing Theorem 5.14 to Theorem 5.9, only additional condition $\lim_{t\to\infty} W'(t)W^{b-1}(t) = 0$ is required. Since $b < 1$, this condition can be met even when both $W(t)$ and $W'(t)$ become unbounded as $t \to \infty$. For instance, $W(t) = (1+t)^\alpha$, for any $1 < \alpha < 1/b$, will satisfy both conditions in (5.47).*

5.4. **Continuous dependence on Forchheimer polynomials.** Let the exponent vector $\vec{\alpha} = (0, \alpha_1, \ldots, \alpha_N)$ and the boundary data $\psi(x,t)$ be fixed. For each coefficient vector $\vec{a}$, we denote by $p(x,t;\vec{a})$ the solution of (4.1), (4.2) with $K = K(\xi, \vec{a})$ and initial data $p(x,0;\vec{a})$. Denote

$$R(N) = \{\vec{a} = (a_0, a_1, \ldots, a_N) : a_0, a_N > 0, a_1, \ldots, a_{N-1} \geq 0\}.$$

For $\vec{a} \in R(N)$, we define

$$\chi(\vec{a}) = \max\left\{a_0, a_1, \ldots, a_N, \frac{1}{a_0}, \frac{1}{a_N}\right\} \in [1, \infty).$$

Let D be a compact set in $R(N)$ and denote $\hat{\chi}(D) = \max\{\chi(\vec{a}) : \vec{a} \in D\}$, then $\hat{\chi}(D) \in [1, \infty)$. As shown in [18], for $\vec{a} \in D$, all constants appearing in estimates in the previous sections can be made independent of $\vec{a}$, and depend on U, $\hat{\chi}(D)$, and $\vec{\alpha}$ only. We denote them by C_* in this subsection.

Let $g_1(s) = g(s, \vec{a}^{(1)})$ and $g_2(s) = g(s, \vec{a}^{(2)})$ be two functions of class $\mathrm{FP}(N, \vec{\alpha})$, where $\vec{a}^{(1)}$ and $\vec{a}^{(2)}$ belong to D. Let $p_k = p_k(x,t;\vec{a}^{(k)})$ for $k = 1, 2$.

Define γ, $\bar{p}_1$ and $\bar{p}_2$ as in section 4. Let $P = p_1 - p_2$, $\bar{P} = \bar{p}_1 - \bar{p}_2$. Then

$$\bar{P}(x,t) = p_1(x.t) - p_2(x,t) - \frac{1}{|U|}\int_U p_1(x,0) - p_2(x,0)dx = P(x,t) - \frac{1}{|U|}\int_U P(x,0)dx.$$

Since $\int_U \bar{P}(x,t)dx = 0$ we have for all $t \geq 0$ that

$$J[\bar{P}](t) \leq J[P](t) \leq J[\bar{P}](t) + J[P](0).$$

Our goal is to estimate $J[\bar{P}](t)$ and $\int_U |\nabla \bar{P}(x,t)|^{2-a}dx$. The function $\bar{P}$ satisfies equation

$$\frac{\partial \bar{P}}{\partial t} = \nabla \cdot \big(K(|\nabla \bar{p}_1|, \vec{a}^{(1)})\nabla \bar{p}_1\big) - \nabla \cdot \big(K(\nabla \bar{p}_2, \vec{a}^{(2)})\nabla \bar{p}_2\big), \quad t > 0. \tag{5.50}$$

Multiplying (5.50) by $\bar{P}$ and integrating over U, then using the perturbed monotonicity ([18], Lemma 5.2), and the same calculations as in Proposition 5.3 of [18], we obtain

$$\frac{1}{2}\frac{d}{dt}\int_U \bar{P}^2 dx \leq -C_*\Big(\int_U |\nabla \bar{P}(x,t)|^{2-a}dx\Big)^{\frac{2}{2-a}}\Lambda^{-b}(t) + C_*|\vec{a}^{(1)} - \vec{a}^{(2)}|\int_U \big(|\nabla p_1|^{2-a} + |\nabla p_2|^{2-a}\big)dx,$$

where $\Lambda(t) = 1 + J_{H_1}[p_1](t) + J_{H_2}[p_2](t)$. Here $H_i(\xi)$, $i = 1, 2$, is defined by (2.13) with $K = K_i$. Hence

$$\frac{1}{2}\frac{d}{dt}\int_U \bar{P}^2 dx \leq -C_*\Big(\int_U |\nabla \bar{P}(x,t)|^{2-a}dx\Big)^{\frac{2}{2-a}}\Lambda^{-b}(t) + C_*|\vec{a}^{(1)} - \vec{a}^{(2)}|\Lambda(t). \tag{5.51}$$

Under the Degree Condition, by Poincaré's inequality, (5.51) yields

$$\frac{1}{2}\frac{d}{dt}\int_U \bar{P}^2 dx \leq -C_*\Lambda^{-b}(t)\int_U \bar{P}^2 dx + C_*|\vec{a}^{(1)} - \vec{a}^{(2)}|\Lambda(t). \tag{5.52}$$

Regarding the boundary data, we use the notations $f(t), \tilde{f}(t), M_f(t), \bar{A}, \beta$ as defined in section 4. Also, set

$$A_0 = J[\bar{p}_1](0) + J[\bar{p}_2](0) \text{ and } B_0 = J_{H_1}[p_1](0) + J_{H_2}[p_2](0).$$

Theorem 5.16. *(i) For $0<T<\infty$, we have*

$$\sup_{[0,T]} J[\bar P](t) \le J[\bar P](0) + C_* M_{5,T} |\vec a^{(1)} - \vec a^{(2)}|, \tag{5.53}$$

where $M_{5,T} = A_0 + T + \int_0^T f(\tau)d\tau$.

(ii) Assume the Degree Condition. Then

$$\sup_{[1,\infty)} J[\bar P](t) \le J[\bar P](0) + C_* M_6^{b+1} |\vec a^{(1)} - \vec a^{(2)}|, \tag{5.54}$$

$$\begin{aligned}\sup_{[1,\infty)} \Big(\int_U |\nabla P(x,t)|^{2-a}dx\Big)^{\frac{2}{2-a}} &\le C_* M_6^{b+1/2} J[\bar P](0)^{1/2}\\ &\quad + C_* M_6^{3b/2+1} |\vec a^{(1)} - \vec a^{(2)}|^{1/2} + C_* M_6^{b+1} |\vec a^{(1)} - \vec a^{(2)}|,\end{aligned} \tag{5.55}$$

where $M_6 = 1 + A_0 + \sup_{[0,\infty)} f(t) + \sup_{[1,\infty)} \int_{t-1}^t \tilde f(\tau)d\tau$.

Also, if $0<t_0<1$ and $T\ge t_0$, then

$$\begin{aligned}\sup_{[t_0,T]} \Big(\int_U |\nabla P(x,t)|^{2-a}dx\Big)^{\frac{2}{2-a}} &\le C_* t_0^{-1/2} M_{7,T}^{b+1/2} J[\bar P](0)^{1/2}\\ &\quad + C_* t_0^{-1/2} M_{7,T}^{b+1} |\vec a^{(1)} - \vec a^{(2)}|^{1/2} + C_* M_{7,T}^{b+1} |\vec a^{(1)} - \vec a^{(2)}|,\end{aligned} \tag{5.56}$$

where $M_{7,T} = (1+T)\big(1+\sup_{[0,T]} f(t)\big) + A_0 + B_0 + \int_0^T \tilde f(\tau)d\tau$.

Proof. In (5.51) neglecting the negative term on the right-hand side and integrating in time, we obtain for $t\ge 0$ that

$$J[\bar P](t) \le J[\bar P](0) + C_* |\vec a^{(1)} - \vec a^{(2)}| \int_0^t \Lambda(\tau)d\tau. \tag{5.57}$$

Also from (5.51) we derive

$$\begin{aligned}\Big(\int_U |\nabla P(x,t)|^{2-a}dx\Big)^{\frac{2}{2-a}} &\le C_*\Lambda(t)^b \Big|\int_U \bar P_t \bar P dx\Big| + C_*\Lambda(t)^b\big(1 + J_{H_1}[p_1](t) + J_{H_2}[p_2](t)\big)|\vec a^{(1)} - \vec a^{(2)}|\\ &\le C_*\Lambda(t)^b \Big(\int_U \bar P_t^2 dx\Big)^{1/2}\Big(\int_U \bar P^2 dx\Big)^{1/2} + C_*\Lambda(t)^{b+1}|\vec a^{(1)} - \vec a^{(2)}|,\end{aligned}$$

therefore

$$\begin{aligned}\Big(\int_U |\nabla P(x,t)|^{2-a}dx\Big)^{\frac{2}{2-a}} &\le C_*\Lambda(t)^b \big(J[\bar P_t](t)\big)^{1/2}\big(J[\bar P](t)\big)^{1/2}\\ &\quad + C_*\Lambda(t)^{b+1}|\vec a^{(1)} - \vec a^{(2)}|.\end{aligned} \tag{5.58}$$

(i) Using (4.29) to estimate $\int_0^t \Lambda(\tau)d\tau$ in (5.57), we immediately obtain relation (5.53).

(ii) Under the Degree Condition, it follows from (5.52) and Gronwall's inequality that for $t\ge 1$,

$$\begin{aligned}J[\bar P](t) &\le e^{-\int_0^t \Lambda(\tau)^{-b}d\tau} J[\bar P](1)\\ &\quad + C_*|\vec a^{(1)} - \vec a^{(2)}| \int_1^t e^{-\int_\tau^t \Lambda(\theta)^{-b}d\theta}\Lambda(\tau)d\tau.\end{aligned} \tag{5.59}$$

Since $\Lambda(t) \le C_* M_6$ for $t \ge 1$, using (5.53) to estimate $J[\bar P](1)$, we obtain (5.54) from (5.59).

For proofs of (5.55), resp. (5.56), we take corresponding supremum of (5.58), with the use of estimates (4.35), resp. (4.52), combined with (5.54), resp. (5.53). We omit details. $\square$

For asymptotic estimates we have:

Theorem 5.17. *Assume the Degree Condition. If $f(t)$ $(t \ge 0)$ and $\int_{t-1}^t \tilde f(\tau)d\tau$ $(t \ge 1)$ are uniformly bounded, then*

$$\limsup_{t\to\infty} J[\bar P](t) \le C_* M_8^{b+1} |\vec a^{(1)} - \vec a^{(2)}|, \tag{5.60}$$

$$\limsup_{t\to\infty} \Big(\int_U |\nabla \bar P(x,t)|^{2-a} dx \Big)^{\frac{2}{2-a}} \le C_* M_8^{3b/2+1} |\vec a^{(1)} - \vec a^{(2)}|^{1/2} + C_* M_8^{b+1} |\vec a^{(1)} - \vec a^{(2)}|, \tag{5.61}$$

where $M_8 = 1 + A + \limsup_{t\to\infty} \int_{t-1}^t \tilde f(\tau)d\tau$.

Proof. Let $W_3(t) = 1 + A + \int_{t-1}^t \tilde f(\tau)d\tau$, then $M_8 = \limsup_{t\to\infty} W_3(t)$. By estimates (4.39) and (4.55), we have $\Lambda(t), J[\bar P_t](t)$ are bounded by $C_* W_3(t)$ for large t. Hence

$$\limsup_{t\to\infty} \Lambda(t), \limsup_{t\to\infty} J[\bar P_t](t) \le C_* \limsup_{t\to\infty} W_3(t) = C_* M_8. \tag{5.62}$$

By (5.52) and Lemma 3.9:

$$\limsup_{t\to\infty} J[\bar P](t) \le C_* |\vec a^{(1)} - \vec a^{(2)}| \limsup_{t\to\infty} \Lambda(t)^{b+1} \le C_* M_8^{b+1} |\vec a^{(1)} - \vec a^{(2)}|,$$

thus we obtain (5.60). Using (5.62) and (5.60) in (5.58), we easily obtain (5.61). $\square$

Remark 5.18. *By the trace theorem and Poincaré's inequality, we have*

$$\int_\Gamma |\bar p(x,t)| d\sigma \le C \Big(\int_U |\nabla p(x,t)|^{2-a} dx \Big)^{\frac{1}{2-a}}, \tag{5.63}$$

where $p(x,t)$ is as in section 4. Similar inequality applies to $\bar P$, where $\bar P$ is either in this current subsection or in subsection 5.2. Thus previous analysis for the gradients directly yield corresponding results for the pressure on the boundary (with respect to L^1-norm).

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