TRAFFIC LIGHT CONTROL IN AN AVENUE

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Traffic Light Control in an Avenue

Ezio Marchi *

In honor to the memory of Professor Ewald Burger

Abstract

In this short paper we solve the general problem of synchronization of light in a street of avenue having two directions assuming that the all vehicles have a constant speed. The general approach is elementary and the case for practical application is vast.

Key words: traffic control two ways.

Introduction

The bibliography in traffic flows is very vast. Let us mention the pioneering work of Charnes & Cooper [3] which provides the basic ideas for the application of the mathematical models in the study of traffic flow. On the other hand, recently A. and M. Papageorgion: The importance of traffic flow modeling for motorways, Traffic control, Networks and Spatial Economics 1:2001 pp 179-203, Barceló J. et al [1], Marchi E. [9]. Here are brief explanations of the aims of some of these papers.

Deganzo in [5] considers a importante study for the general theory of transportation and traffic operation. He presents several cases where the vehicles or platoons go in a straight line.

Further Barceló, J. et al [1] in a very recent article published in the SIAM News Nov. 2007, says that quantitative decision-making relies on appropiate mathematical models of the system about which decisions are to be made. Intelligent transport systems (ITS), which apply combined advanced detection, communication and computer sciences technologies to traffic and transport systems, are important examples of such quantitative decisions support systems.

Other studies related to the mathematics of this papers are engaged with the development of simple indices in order to characterize the two flows of vehicles that on an avenue in two directions that have traffic light systems.

Finally we in [9] introduce study called "The Manhattan problem", called this way by Chapad, Duponts & Luthi in [4].

The green wave problem, "The Manhattan problem", called in engineering literature can be seen as to find compatibility conditions between systems of equations among equations which take into account the overall flow of the vehicles. The compatibility conditions will determine the phase difference between the lights and the cycles of each light.

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To my knowledge there are very few studies that deal with the topics we have studied here (see E. Mancinelli et al. [7]). However, in this simple paper we present one of the most general cases of cooordinations in a avenue with two streets in which the vehicles moves in different directions.

We would like to emphazise from an elementary point of view that in the papers of Marchi [8] and Marchi & Tarazaga [9] we have obtained an implicit solution of this problem in one and two dimensions.

However, in this simple paper we write an illustrate the positive aspects of the LAUMAR systems.

The Model

Consider an avenue having bidirectional traffic flow. This might be viewed as a segment of the straight line from 0 to $n+1$ as it is shown in the figure:

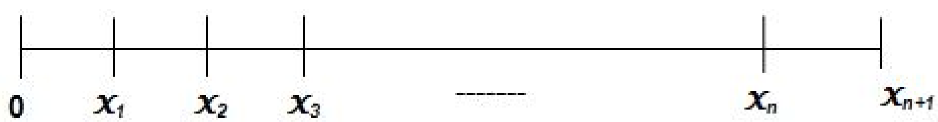


Fig. 1

The first block in the avenue is described as having the starting point at 0 and the end at x_1. The second block is described from x_1 to x_2 and so on, untill the last one wich stops at x_{n+1}. At x_i it is not necessary that we assume that the perpendicular street has a non zero wideness. Here we consider that this assumptions is only in order to reduce the complexity of the theory. We assume that the perpendicular street have zero wich. At the end of this paper we discuss how one overcomes this trivial obstacle.

Now the light at each intersection $x_i, i = 1, 2, ..., n$ can be introduced in different ways. One of them is mechanism wich at the intersections i from the times a_{ij} to a_{ij+1} with j odd a green light in the positive direction, which is to say from left to right. On the reverse direction right to left it is also green. During the times from a_{ij} to a_{ij+1} whith j even the light is red.

From a physical engineering and practical point of view, a light can be described by a vector.

$$v_i(t) = \begin{cases} 1 & \text{if } a_{ij+1}, \quad j \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

where t is the time variable. Here the 1 means that the vehicle in front the lights sees it green and 0 red, respectively.

In the figure 1 we consider that at each intersection i, the crossing street has zero. This fact is not a restriction in our study since in the last part of this work we say how to handle the general situation and, we present the study (in) this way only for simplicity.

Furthermore a light can be viewed in the corresponding axis t as a partition of segments shown as in the next figure:

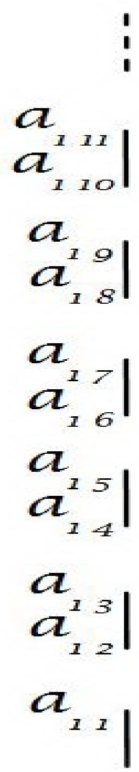


Fig. 2

where we have solid lines indicating that the light is red and gaps where the light is green.

In this way we can write in the coortinate system (x, t) the following arrangement with the different lights. We have taken the system (x, t) in the same way as that considered in Deganzo [5].

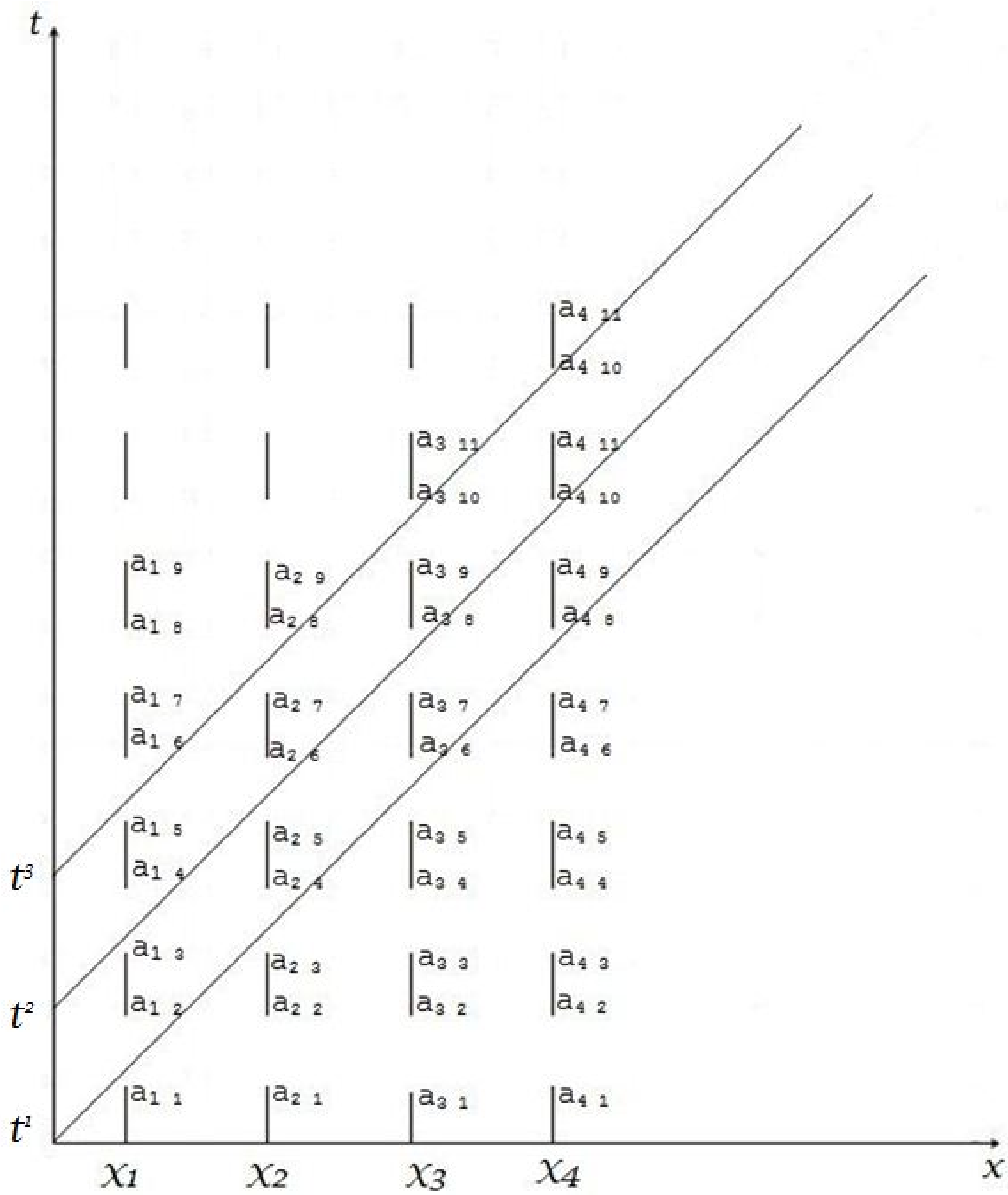


Fig. 3

Thus, we obtain the fundamental diagram, from which we are going to work for our synchronization system.

Now we will consider a first vehicle, which for reasons of simplicity will be considered a point going from left to right, having the easiest movement law, wich is to say,

$$x = at + b \qquad a > 0$$

We arrange it in a suitable way for the (t, x) axis system

$$t = \frac{1}{a}x - \frac{b}{a} = \alpha x + \beta \qquad \alpha = \frac{1}{a} \qquad \beta = \frac{b}{a} \qquad a > 0$$

This fact from a real point of view is important since we consider it as a stationary regime. The vehicles or cars, if not stopped by a light, move in a straight line.

Consider a first vehicle moving by the equation

$$t^1 = \alpha^1 x + \beta^1$$

where α^1 is the velocity of a given vehicle in the corresponding block.

Then it passed at $x_1$at time

$$t_1^1 = \alpha^1 x_1 + \beta^1$$

Therefore a necessary and sufficient condition that such a vehicle passes the first light on green is given by

$$a_{1\ 1} < t_1^1 = \alpha^1 x_1 + \beta^1 < a_{1\ 2}$$

For the second and subsequents lights

$$\begin{aligned}
a_{2\ 3} &< t_2^1 = \alpha^1 x_2 + \beta^1 < a_{2\ 4} \\
a_{3\ 5} &< t_3^1 = \alpha^1 x_3 + \beta^1 < a_{3\ 6} \\
a_{4\ 7} &< t_4^1 = \alpha^1 x_4 + \beta^1 < a_{4\ 8}
\end{aligned}$$

and in general

$$a_{i,2i-1} < t_i^1 = \alpha^1 x_i + \beta^1 < a_{i2i} \qquad i = 1, 2, ...$$

At this point it is important to emphazise that we have considered the model that from light i to light $i+1$ the law of passing the cars is calling jumping the two time periods, since we consider that the first car and the subsequents are going through from a_{11} to a_{23}. This assumption is restrictive in this model and it will derive rather a theory for slow cars. The reason that we study it is for having a simpler understanding for the existence and real computation of the green wave. In a separate study we will consider it, elsewhere.

For the next car indexed with 2 as shown in the figure we will have for the constrained of the passing through the lights without stopping

$$\begin{aligned}
a_{1\ 3} &< t_1^2 = \alpha^2 x_1 + \beta^2 < a_{1\ 4} \\
a_{2\ 5} &< t_2^2 = \alpha^2 x_2 + \beta^2 < a_{2\ 6} \\
a_{3\ 7} &< t_3^2 = \alpha^2 x_3 + \beta^2 < a_{3\ 8} \\
a_{4\ 9} &< t_4^2 = \alpha^2 x_4 + \beta^2 < a_{4\ 10}
\end{aligned}$$

and in general we will have

$$a_{i,2i+3} < t_i^2 = \alpha^2 x_i + \beta^2 < a_{i,2i+2}$$

In order to keep the material and the presentation in the paper elemental, we will consider the third car and the fourth and then will derive the general relation among the constraints for the validation of the green wave:

$$\begin{aligned}
a_{1\ 5} &< t_1^3 = \alpha^3 x_1 + \beta^3 < a_{1\ 6} \\
a_{2\ 7} &< t_2^3 = \alpha^3 x_2 + \beta^3 < a_{2\ 8} \\
a_{3\ 9} &< t_3^3 = \alpha^3 x_3 + \beta^3 < a_{3\ 10} \\
a_{4\ 11} &< t_4^3 = \alpha^3 x_4 + \beta^3 < a_{4\ 12}
\end{aligned}$$

we obtain for the fourth vehicle the following inequalities

$$\begin{aligned}
a_{i\ 2i+5} &< t_1^4 = \alpha^4 x_i + \beta^4 < a_{i\ 2i+6} \\
a_{i\ 2i+7} &< t_1^5 = \alpha^5 x_i + \beta^5 < a_{i\ 2i+8} \\
a_{i\ 2i+9} &< t_1^6 = \alpha^6 x_i + \beta^6 < a_{i\ 2i+10} \\
a_{i\ 2i+11} &< t_1^7 = \alpha^7 x_i + \beta^7 < a_{i\ 2i+12} \\
a_{i\ 2i+13} &< t_1^8 = \alpha^8 x_i + \beta^8 < a_{i\ 2i+14}
\end{aligned}$$

and from here in general we will have

$$a_{i,2i+3} < t_i^3 = \alpha^3 x_i + \beta^3 < a_{i,2i+4}$$

Then the general law for an arbitrary i and j is given by

$$a_{i,2i+2j-3} < t_i^j = \alpha^j x_i + \beta^j < a_{i,2i+2j-2} \qquad i,j = 1,2,3,... \tag{1}$$

The inequality (1) resume all the movement for all cars going from left to right with the right of way , and we call it the fundamental solution for the positive direction.

Now we will present a simple example of the general of the fundamental solution.

Consider for example the following case:

As an example we provide when the times a_{ij} are given as follow

$$a_{ij} = r_j + s_i \qquad r, s_i > 0$$

then replacing in (1) these quantities and also in order to have the terms only in a given speed a for all the cars and b^j, then we have

$$a_{i,2i+2j-3} < t_i^j = \alpha^j x_i + \beta^j = \frac{x_i}{a} - \frac{b_j}{a} < a_{i,2i+2j-2}$$

and taking

$$a_{ij} = r_j + s_i \qquad \beta^j = -\frac{b_j}{a}$$

then it follows with $x_i = 100i$ and $b^j = -(j-1)\frac{100}{a}$

$$s_i + r_{2j+2i-3} < \frac{100i}{a} + (j-1)\frac{100}{a} < s_i + r_{2i+2j-2}$$

As a numerical example we take $a = 5$ wich correspond to 20 km/h, $r_j = 10j + 5$ and $s_i = 0$, then replacing we have

$$20\left(i + j - \frac{3}{2}\right) + 5 < 2(j + i - 1) < 20(i + j - 1) + 5$$

$$\alpha^i = \frac{1}{5} \qquad b^j = -(j-1)\,100$$

$$-\frac{b^j}{a} = \beta^j = (j-1)\,20, \quad a_{ij} = 10j + 5 \quad x_i = 100i$$

$$t_i^j = \alpha^j x_i + \beta^j = 20(i + j - 1)$$

wich describes the lights change from red to green or viceversa is independent of i. Then if you are located in a place x at time t, and you look at the sides into the coordinate x, you will see all the lights at the same color, to the right and to the left.

A further example is when the speed of all the cars of vehicles is 40 km/h which gives $a = 10$, with $x_i = 200i, b^j = -(j-1)\,v, \quad v = 200, \quad s = 5$ and $a_{ij} = 10j + 5$.

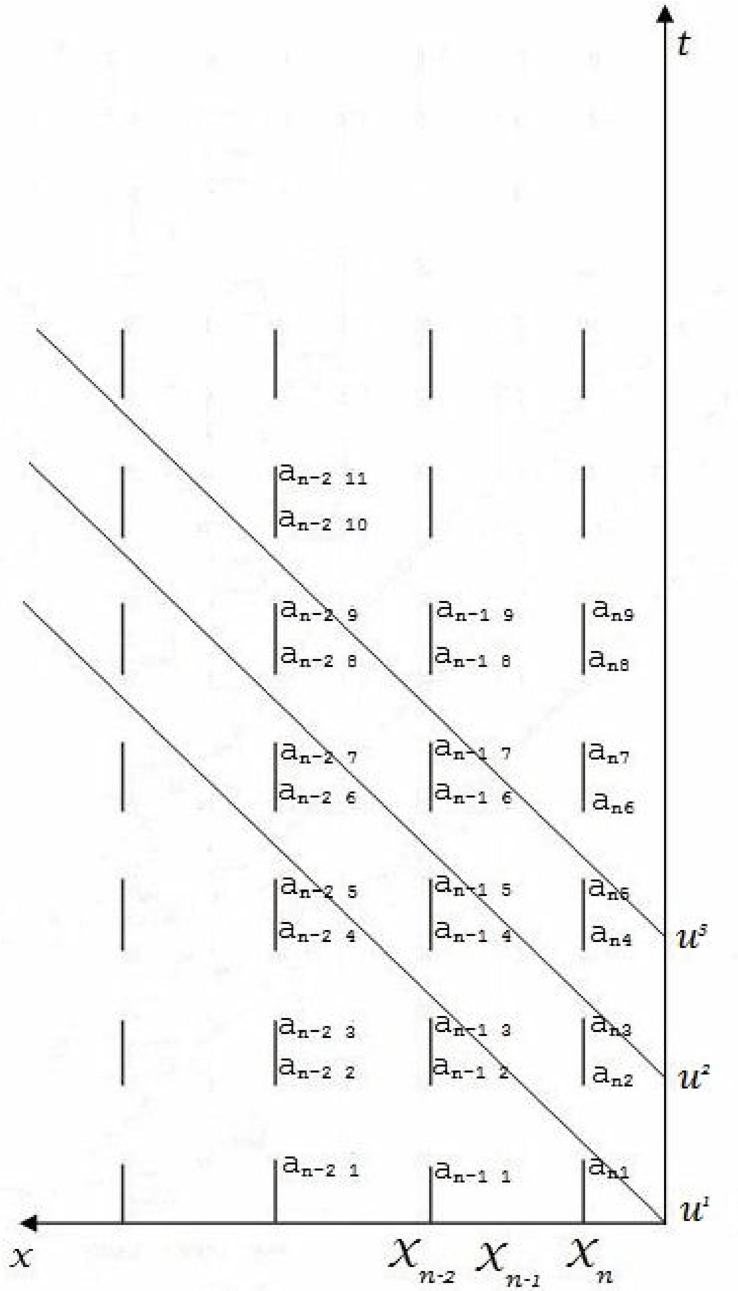
$$10(2i + 2j - 3) + 5 < 20(i + j - 1) < 10(2i + 2j - 3) + 5$$

which is valid.

NEGATIVE DIRECTION

Next we consider the negative direction. That is to say, the movement of the cars coming from the right going to the left and the corresponding inequalities relating the movements of the cars going from right to left. In order to keep this study simple, we begin with the first car from right to left begining with the number one. The second the following one and so on.

Now we present in the next figure the timinig of the corresponding a's in the right part of the avenue.



t
x
$a_{n-2\ 11}$
$a_{n-2\ 10}$
$a_{n-2\ 9}$
$a_{n-2\ 8}$
$a_{n-1\ 9}$
$a_{n-1\ 8}$
a_{n9}
a_{n8}
$a_{n-2\ 7}$
$a_{n-2\ 6}$
$a_{n-1\ 7}$
$a_{n-1\ 6}$
a_{n7}
a_{n6}
$a_{n-2\ 5}$
$a_{n-2\ 4}$
$a_{n-1\ 5}$
$a_{n-1\ 4}$
a_{n5}
a_{n4}
$a_{n-2\ 3}$
$a_{n-2\ 2}$
$a_{n-1\ 3}$
$a_{n-1\ 2}$
a_{n3}
a_{n2}
$a_{n-2\ 1}$
$a_{n-1\ 1}$
a_{n1}
u^3
u^2
u^1
x_{n-2}
x_{n-1}
x_n

Fig. 4

Here we have that for cars coming from the right to the left or in the contrary or negative direction, is written by the equation

$$u = \gamma y + \delta$$

where u here is time and $\gamma < 0$ since the displacement of the cars is from right to left, which it is the inverse of the velocity:

$$y = \frac{u - \delta}{\gamma} = \frac{u}{\gamma} - \frac{\delta}{\gamma} = cu + d, \qquad \frac{1}{\gamma} = c < 0, \quad -\frac{\delta}{\gamma} = d$$

Then we have for the first car the inequalities

$$\begin{array}{rcl} a_{n\ 1} & < & u_1^1 = \gamma^1 x_n + \delta^1 < a_{n\ 2} \\ a_{n-\ 1\ 3} & < & u_2^1 = \gamma^1 x_{n-1} + \delta^1 < a_{n-\ 1\ 4} \\ a_{n-\ 2\ 5} & < & u_3^1 = \gamma^1 x_{n-2} + \delta^1 < a_{n-\ 2\ 6} \\ a_{n-\ 3\ 7} & < & u_4^1 = \gamma^1 x_{n-3} + \delta^1 < a_{n-\ 3\ 8} \\ a_{n-\ 4\ 9} & < & u_5^1 = \gamma^1 x_{n-4} + \delta^1 < a_{n-\ 4\ 10} \end{array}$$

and in general we have

$$a_{n-i+1\ 2i-1} < u_i^1 = \gamma^1 x_{n-i+1} + \delta^1 < a_{n-i+1\ 2i}$$

For the second car we have

$$\begin{array}{rcl} a_{n\ 3} & < & u_1^2 = \gamma^2 x_n + \delta^2 < a_{n\ 4} \\ a_{n-\ 1\ 5} & < & u_2^2 = \gamma^2 x_{n-1} + \delta^2 < a_{n-\ 1\ 6} \\ a_{n-\ 2\ 7} & < & u_3^2 = \gamma^2 x_{n-2} + \delta^2 < a_{n-\ 2\ 8} \\ a_{n-\ 3\ 9} & < & u_4^2 = \gamma^2 x_{n-3} + \delta^2 < a_{n-\ 3\ 10} \\ a_{n-\ 4\ 11} & < & u_5^2 = \gamma^2 x_{n-4} + \delta^2 < a_{n-\ 4\ 12} \end{array}$$

and therefore in general

$$a_{n-i+1\ 2i-1} < u_i^2 = \gamma^2 x_{n-i+1} + \delta^2 < a_{n-i+1\ 2i+2}$$

In this way we have obtained the entire inequality for the second platoon or car. For the third one we will have

$$\begin{aligned}
a_{n\ 5} &< u_1^3 = \gamma^3 x_n + \delta^3 < a_{n\ 6} \\
a_{n-\ 1\ 7} &< u_2^3 = \gamma^3 x_{n-1} + \delta^3 < a_{n-\ 1\ 8} \\
a_{n-\ 2\ 9} &< u_3^3 = \gamma^3 x_{n-2} + \delta^3 < a_{n-\ 2\ 10} \\
a_{n-\ 3\ 11} &< u_4^3 = \gamma^3 x_{n-3} + \delta^3 < a_{n-\ 3\ 12} \\
a_{n-\ 4\ 13} &< u_5^3 = \gamma^3 x_{n-4} + \delta^3 < a_{n-\ 4\ 14}
\end{aligned}$$

and in general we have

$$a_{n-i+1\ 2i-3} < u_i^3 = \gamma^3 x_{n-i+1} + \delta^3 < a_{n-i+1\ 2i+4}$$

For the fourth car we obtain the following system of inequalities

$$\begin{aligned}
a_{n\ 7} &< u_1^4 = \gamma^4 x_n + \delta^4 < a_{n\ 8} \\
a_{n-\ 1\ 9} &< u_2^4 = \gamma^4 x_{n-1} + \delta^4 < a_{n-\ 1\ 10} \\
a_{n-\ 2\ 11} &< u_3^4 = \gamma^4 x_{n-2} + \delta^4 < a_{n-\ 2\ 12} \\
a_{n-\ 3\ 13} &< u_4^4 = \gamma^4 x_{n-3} + \delta^4 < a_{n-\ 3\ 14} \\
a_{n-\ 4\ 15} &< u_5^4 = \gamma^4 x_{n-4} + \delta^4 < a_{n-\ 4\ 16}
\end{aligned}$$

Then, form these inequalities we might extrapolate to

$$a_{n-i+1\ 2i+2j-3} < u_i^j = \gamma^j x_{n-i+1} + \delta^j < a_{n-i+1\ 2i+2j-2} \tag{2}$$

Thus we have got a general necessary and sufficient condition in orden to have the green wave for the cars coming from the right to the left, or the negative direction.

As a general example we will present the following set of numbers.

$$a_{ij} = r_j + s_i \qquad r_j, s_i > 0$$

which remind at this point that we have that the green wave coming in the positive way has to be synchronized with the same a_{ij}'s as those for the negative direction. Both waves have to be coordinated at the intersection points. Then replacing in (2) we have

$$s_{n-i+1} + r_{2i+2j-3} < u_i^j = \gamma^j x_{n-i+1} + \delta^j < s_{n-i+1} + r_{2i+2j-2}$$

Remember that

$$u_i^j = \gamma^j x_i + \delta^j \qquad j > 0$$

and

$$x_i = \frac{u_i^j}{\gamma^j} - \frac{\delta^j}{\gamma^j} = c^j u_i^j + d^j \qquad c^j < 0$$

and we take, as an example, $c = -5$ for the car going to the speed of 18 km/h.

The sign means that it goes to the negative direction, that means that the movement goes from rigth to left. On the other hand, from the figure and for symmetry of the related first part of the material, we will have as an example

$$d - d^j = (n+1)\, i100 + 100j$$

Final Remarks

First of all we would like to point out that in an easy way it is possible to introduce the directions of the perpendicular streets. Until now they were considered with zero wide. The only consideration that we have to do is to consider $x_i + L_i$ where L_i is the wideness of the i-th perpendicular street. The relation $x_i + L_i < x_{i+1}$ is obvious. Now here the different inequalities must be considered accordingly.

On the other hand we have that the squeme that we have considered here is flexible in the sense that it possible to consider the mechanism of the lights in the sense that the velocity in the different regions are different. This is done in such a way that the trajectories are piece ?????.

Another question is the aspect that instead of one single car, we have an entire platoon. In such a case the conditions of the platoons must give for the initial and the final cars of the platoon. This change the squeme but not the essential of the model.

Also it is possible to arrange the timing in order that two platoons separate by same distance become one platoon by a simple fusion.

Finally , it is possible to relate the material presented here with that study how the congestion begins at the intersection points. It is interesting to take into consideration in the future to avoid lights in the city. This might be obtained by the study of cities without lights as the first paper in the subject presented by Marchi [10]. By the way our system in operation satisfies Walrop principle.

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