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II. AN EXPLICIT POINCARÉ-DULAC NORMAL FORM

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ASYMPTOTIC INTEGRATION OF NAVIER–STOKES EQUATIONS WITH POTENTIAL FORCES. II. AN EXPLICIT POINCARÉ–DULAC NORMAL FORM

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Abstract. We study the incompressible Navier–Stokes equations with potential body forces on the three-dimensional torus. We show that the normalization introduced in the paper *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 4(1):1–47, 1987, produces a Poincaré–Dulac normal form which is obtained by an explicit change of variable. This change is the formal power series expansion of the inverse of the normalization map. Each homogeneous term of a finite degree in the series is proved to be well-defined in appropriate Sobolev spaces and is estimated recursively by using a family of homogeneous gauges which is suitable for estimating homogeneous polynomials in infinite dimensional spaces.

Contents

1. Introduction and preliminaries

The initial value problem for the incompressible Navier–Stokes equations in the three-dimensional space $\mathbb{R}^3$ with a potential body force is

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\Delta\mathbf{u} = -\nabla p - \nabla\phi, \\ \text{div } \mathbf{u} = 0, \\ \mathbf{u}(\mathbf{x},0) = \mathbf{u}^0(\mathbf{x}), \end{cases} \tag{1.1}$$

where $\nu > 0$ is the kinematic viscosity, $\mathbf{u} = \mathbf{u}(\mathbf{x},t)$ is the unknown velocity field, p is the unknown pressure, $(-\nabla\phi)$ is the body force specified by a given function ϕ and $\mathbf{u}^0(\mathbf{x})$ is the known initial velocity field. We consider only solutions $\mathbf{u}(\mathbf{x},t)$ such that for any $t \geq 0$, $\mathbf{u}(\mathbf{x}) = \mathbf{u}(\mathbf{x},t)$ satisfies

$$\mathbf{u}(\mathbf{x} + L\mathbf{e}_j) = \mathbf{u}(\mathbf{x}) \quad \text{for all} \quad \mathbf{x}\in\mathbb{R}^3, \quad j = 1,2,3, \tag{1.2}$$

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(where $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is the canonical orthonormal basis in $\mathbb{R}^3$,) and

$$\int_\Omega \mathbf{u}(\mathbf{x}) d\mathbf{x} = 0 \tag{1.3}$$

where $L > 0$ is fixed and $\Omega = (-L/2, L/2)^3$. We call the functions satisfying (1.2) L-periodic functions. Throughout this paper we take $L = 2\pi$ and $\nu = 1$. The general case is easily recovered by a change of scale. For the theory of Navier–Stokes equations, the reader is referred to the pioneering works by J. Leray [12–14] as well as the books [3,11,17,19]. For the dynamical point of view of Navier–Stokes equations, see [16,18].

Let $\mathcal{V}$ be the set of all L-periodic trigonometric polynomials on Ω with values in $\mathbb{R}^3$ which are divergence-free as well as satisfy the condition (1.3). We define H, resp. V, the closure of $\mathcal{V}$ in $L^2(\Omega)^3$, resp. $H^1(\Omega)^3$, where $H^l(\Omega)$, $l = 0, 1, 2, \ldots$, denote the Sobolev spaces $W^{l,2}(\Omega)$.

Let $\mathbf{a} \cdot \mathbf{b}$ denote the standard scalar product of vectors $\mathbf{a}, \mathbf{b}$ in $\mathbb{R}^3$ and $|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$. The inner product and norm in $L^2(\Omega)^3$ are given by

$$\langle u, v\rangle = \int_\Omega \mathbf{u}(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x}) d\mathbf{x}, \quad |u| = \langle u, u\rangle^{1/2}, \quad u = \mathbf{u}(\cdot), v = \mathbf{v}(\cdot) \in L^2(\Omega)^3.$$

Though the notation $|\cdot|$ denotes the length of vectors in $\mathbb{R}^3$ as well as the L^2-norm of vector fields in $L^2(\Omega)^3$, its meaning will be clear in the context.

On V we consider the norm $\|\cdot\|$ defined by

$$\|u\| = \left(\sum_{j,k=1}^{3} \int_\Omega \left| \frac{\partial u_j(\mathbf{x})}{\partial x_k} \right|^2 d\mathbf{x} \right)^{1/2}, \text{ for } u = \mathbf{u}(\cdot) = (u_1, u_2, u_3) \in V.$$

Define the Stokes operator A by

$$Au = -\Delta u \quad \text{for all} \quad u \in \mathcal{D}(A),$$

where $\mathcal{D}(A)$ is the closure of $\mathcal{V}$ in $H^2(\Omega)^3$. The condition (1.3) implies that on $\mathcal{D}(A)$, the norm $|Aw|$, $w \in \mathcal{D}(A)$, is equivalent to the usual Sobolev norm of $H^2(\Omega)^3$.

We recall that the spectrum $\sigma(A)$ of the Stokes operator A consists of the eigenvalues $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$ of the form $\lambda_j = |\mathbf{k}|^2$ for some $\mathbf{k} \in \mathbb{Z}^3 \setminus \{0\}$ where $j = 1, 2, 3, \ldots$. Note that the additive semi-group generated by $\sigma(A)$ coincides with the set $\mathbb{N} = \{1, 2, 3, \ldots\}$ of all natural numbers. We denote by R_n the orthogonal projection of H onto the eigenspace of A associated to n if n is an eigenvalue of A, otherwise $R_n = 0$. Define $P_n = R_1 + R_2 + \cdots + R_n$.

Fractional powers of A: For $\alpha \geq 0$, define

$$A^\alpha u = \sum_{\mathbf{k} \neq 0} |\mathbf{k}|^{2\alpha} \hat{u}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}, \text{ for } u = \sum_{\mathbf{k} \neq 0} \hat{u}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} \in H.$$

The domain of A^α is $\mathcal{D}(A^\alpha) = \{u \in H : |A^\alpha u| < \infty\}$. Note that $\mathcal{D}(A^0) = H$, $\mathcal{D}(A^{1/2}) = V$ and $\|u\| = |A^{1/2} u|$ for $u \in V$. Also, $|A^\alpha \xi| \leq |A^\beta \xi|$ and hence $\mathcal{D}(A^\beta) \subset \mathcal{D}(A^\alpha)$ for all $\beta \geq \alpha \geq 0$.

We also define the bilinear mapping associated with the nonlinear term in the Navier–Stokes equations by

$$B(u, v) = P_L(u \cdot \nabla v) \quad \text{for all} \quad u, v \in \mathcal{D}(A), \tag{1.4}$$

where P_L denotes the orthogonal projection in $L^2(\Omega)^3$ onto H.

We denote by $\mathcal{R}$ the set of all initial value $u^0 \in V$ such that there is a (unique) solution $u(t), t > 0$, satisfying the functional form of (1.1):

$$\frac{du(t)}{dt} + Au(t) + B(u(t), u(t)) = 0, \quad t > 0, \tag{1.5}$$

with the initial data $u(0) = u^0$, where the equation holds in H, and $u(t)$ is continuous from $[0, \infty)$ into V. In other words, $\mathcal{R}$ is the set of all initial data $u^0 \in V$ such that the solution $u(t)$ of the Navier–Stokes equations (1.5) is regular on $[0, \infty)$. Note that $\mathcal{R}$ is an open set in V that contains infinitely many unbounded closed linear manifolds of infinite dimension, see Remark 7 of [7] (and also Proposition 6.4 of [4]).

Let us recall some known results on the asymptotic expansions and the normal form of the regular solutions to the Navier–Stokes equations (see [8, 9] for more details). First, for any $u^0 \in \mathcal{R}$ the solution $u(t)$ has the asymptotic expansion

$$u(t) \sim q_1(t)e^{-t} + q_2(t)e^{-2t} + q_3(t)e^{-3t} + \cdots, \tag{1.6}$$

where $q_j(t)$, $j \geq 1$, is a polynomial in t with values in $\mathcal{V}$. This means that for any $N \in \mathbb{N}$ the correction term $\tilde{u}_{N+1}(t) = u(t) - \sum_{j=1}^{N} q_j(t)e^{-jt}$ satisfies

$$|\tilde{u}_{N+1}(t)| = O\big(e^{-(N+\varepsilon)t}\big) \quad \text{as} \quad t \to \infty \quad \text{for some} \quad \varepsilon = \varepsilon_N > 0. \tag{1.7}$$

In fact, $\tilde{u}_{N+1}(t)$ belongs to $C^1\big([0, \infty), V\big) \cap C^\infty\big((0, \infty), C^\infty(\mathbb{R}^3)\big)$, and for each $m \in \mathbb{N}$ relation (1.7) holds for the Sobolev norm $\|\tilde{u}_{N+1}(t)\|_{H^m(\Omega)}$ and $\varepsilon = \varepsilon_{N,m} > 0$.

Define the normalization map W by $W(u^0) = W_1(u^0) \oplus W_2(u^0) \oplus \cdots$, where $W_j(u^0) = R_j q_j(0)$ for $j \in \mathbb{N}$. Then W is an one-to-one analytic mapping from $\mathcal{R}$ to the Fréchet space $S_A = R_1 H \oplus R_2 H \oplus \cdots$ endowed with the topology induced by the family of semi-norms $|R_j u|$ $(u \in S_A)$, $j \in \mathbb{N}$.

If $u^0 \in \mathcal{R}$ then the polynomials $q_j(t)$ are the unique polynomial solutions to the following equations

$$q_j'(t) + (A - j)q_j(t) + \beta_j(t) = 0, \quad t \in \mathbb{R}, \tag{1.8}$$

with $R_j q_j(0) = W_j(u^0)$, where the terms $\beta_j(t)$ are defined by

$$\beta_1(t) = 0 \quad \text{and} \quad \beta_j(t) = \sum_{k+l=j} B(q_k(t), q_l(t)) \quad \text{for} \quad j > 1. \tag{1.9}$$

Given arbitrary $\xi = (\xi_n)_{n=1}^\infty \in S_A$, the polynomial solutions $q_j(t, \xi)$ of (1.8) satisfying the initial condition $R_j q_j(0) = \xi_j$, are explicitly given by the recursive formula

$$q_j(t, \xi) = \xi_j - \int_0^t R_j \beta_j(\tau) d\tau + \sum_{n \geq 0} (-1)^{n+1} (A - j)^{-n-1} \frac{d^n}{dt^n} (I - R_j)\beta_j, \tag{1.10}$$

for $j \in \mathbb{N}$. Here $(A - j)^{-n-1}$ is used to denote $[(A - j)\big|_{(I-R_j)H}]^{-n-1}$ and is defined by

$$(A - j)^{-n-1} u = \sum_{|\mathbf{k}|^2 \neq j} \frac{a_\mathbf{k}}{(|\mathbf{k}|^2 - j)^{n+1}} e^{i\mathbf{k}\cdot\mathbf{x}}$$

for $u = \sum_{|\mathbf{k}|^2 \neq j} a_\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{x}} \in \mathcal{V}$. Above I denotes the identity map on H.

The S_A-valued function $\xi(t) = (\xi_j(t))_{j=1}^\infty = \big(W_j(u(t))\big)_{j=1}^\infty = W(u(t))$ satisfies the following system of differential equations

$$(1.11)\qquad \begin{cases} \dfrac{d\xi_1(t)}{dt} + A\xi_1(t) = 0, \\[2ex] \dfrac{d\xi_j(t)}{dt} + A\xi_j(t) + \displaystyle\sum_{k+l=j} R_j B\big(q_k(0,\xi(t)), q_l(0,\xi(t))\big) = 0,\ j > 1. \end{cases}$$

For $\xi \in S_A$, denote

$$(1.12)\qquad \mathcal{P}_j(\xi) = q_j(0,\xi), \quad j \geq 1,$$

and $\mathcal{B} = (\mathcal{B}_j)_{j=1}^\infty$ where

$$(1.13)\qquad \mathcal{B}_1(\xi) = 0, \quad \mathcal{B}_j(\xi) = \sum_{k+l=j} R_j B\big(\mathcal{P}_k(\xi), \mathcal{P}_l(\xi)\big) = 0,\ j > 1.$$

Note that $\mathcal{P}_j(\xi)$ is a polynomial in ξ; here and throughout, concerning the polynomials between vector spaces, we refer to Chapter IV in [10] (with the caveat that we consider the zero map a homogeneous polynomial of any degree). In fact, $\mathcal{P}_j(\xi)$, $\xi = (\xi_n)_{n=1}^\infty$, is a $\mathcal{V}$-valued polynomial in the variables $\xi_1, \xi_2, \ldots, \xi_j$, each belonging to a finite dimensional space. For example, $\mathcal{P}_1(\xi) = \xi_1$, $\mathcal{P}_2(\xi) = \xi_2 - (A-2)^{-1}(I - R_2)B(\xi_1, \xi_1)$.

Regarding the notation, hereafter, for any polynomial Q in ξ regardless if it depends on t, we denote $Q^{[d]}$, for $d \geq 0$, the sum of all its monomials of degree d, i.e., the homogeneous part of degree d of Q.

The system (1.11) written in the vector form in S_A is

$$(1.14)\qquad \frac{d\xi}{dt} + A\xi + \mathcal{B}(\xi) = 0.$$

This system is the normal form in S_A of the Navier–Stokes equations (1.5) associated with the asymptotic expansion (1.6). It is easy to check that the solution of (1.11) with initial data $\xi^0 = (\xi_j^0)_{j=1}^\infty \in S_A$ is precisely $\big(R_j q_j(t, \xi^0) e^{-jt}\big)_{j=1}^\infty$. Thus, formula (1.10) yields the normal form and its solutions.

It was proved by G. Minea in [15] that this type of normalization for ordinary differential equations in the finite dimensional case "coincides with the distinguished normalization in the sense of A. D. Brjuno" [15]. However, whether this is true for Navier–Stokes equations is still an open question.

In previous works [5, 6] the normalization map W and the normal form (1.11) are studied and well understood in a Banach space of the following type:

$$(1.15)\qquad S_A^\star = \{\, \xi = (\xi_n)_{n=1}^\infty \in S_A : \|\xi\|_\star \stackrel{\text{def}}{=\!=} \sum_{n=1}^\infty \rho_n \|\xi_n\| < \infty \,\},$$

where the sequence $(\rho_n)_{n=1}^\infty$ of positive weights decays suitably fast to zero. However these spaces are much too large to easily connect to the concrete approach of the classical Poincaré–Dulac theory (see e.g. [1], and also our relevant summary of the subject in Sect. 4 below). In this paper we will show that the Poincaré–Dulac theory can be extended to the Navier–Stokes equations (1.5) by using the normal form (1.14) restricted to the space $E^\infty = C^\infty(\mathbb{R}^3, \mathbb{R}^3) \cap V$ which is continuously embedded in S_A. In fact we prove that if $\mathcal{P}_j^{[d]}(\xi)$ and $\mathcal{B}_j^{[d]}(\xi)$ denote the sum of all homogeneous monomials of degree d of $\mathcal{P}_j(\xi)$ and $\mathcal{B}_j(\xi)$, respectively, then the series $\sum_j \mathcal{P}_j^{[d]}(\xi)$ and $\sum_j \mathcal{B}_j^{[d]}(\xi)$ converge in E^∞ to continuous polynomials $\mathcal{P}^{[d]}(\xi)$ and $\mathcal{B}^{[d]}(\xi)$, respectively, such that

$$\frac{d}{dt}\xi + A\xi + \sum_{d=2}^{\infty} \mathcal{B}^{[d]}(\xi) = 0 \tag{1.16}$$

is a Poincaré–Dulac normal form for the Navier–Stokes equations obtained by the formal change of variable

$$u = \xi + \sum_{d=2}^{\infty} \mathcal{P}^{[d]}(\xi). \tag{1.17}$$

In contrast to [5,6] where we relied on the recursive relations between the polynomials $q_j(t,\xi)$ $(j \in \mathbb{N})$, here we rely on the recursive formulas of the homogeneous terms in the normal form. Our main tool in estimating their Sobolev norms is a family of homogeneous gauges $[[\,\xi\,]]_{d,n}$ introduced in Sect. 2 (see (2.8)). These gauges have useful multiplicative properties (Lemma 2.1) as well as estimates in terms of Sobolev norms (Lemma 2.2). Also in Sect. 2, we recall some necessary estimates for the bilinear operator $B(u,v)$ (Lemma 2.3). In Sect. 3, by using those gauges we establish the absolute convergence of the series $\sum_{j=1}^{\infty} \mathcal{P}_j^{[d]}(\xi)$ and $\sum_{j=1}^{\infty} \mathcal{B}_j^{[d]}(\xi)$ in $\mathcal{D}(A^\alpha)$, for $\xi \in \mathcal{D}(A^{\alpha+3d/2})$ and $\alpha \geq 1/2$. The main result of the paper (Theorem 4.9), the rigorous definition of its framework, and its full proof (necessitating in several steps) are given in Sect. 4.

2. Homogeneous gauges

In this section, we give several estimates which are of independent interest as well as needed in our next sections.

Hereafter, we identify $\xi \in S_A$ with $u \sim \sum_j \xi_j$, and hence can define fractional power A^α, for $\alpha \geq 0$, on S_A as well as its domain $\mathcal{D}(A^\alpha)$ as a subspace of S_A.

In working with polynomials in infinite dimensional spaces, it is convenient to introduce the set of general multi-indices $GI = \bigcup_{n=1}^{\infty} GI(n)$ where for $n \geq 1$,

$$GI(n) = \big\{\bar\alpha = (\alpha_k)_{k=1}^{\infty},\ \alpha_k \in \{0,1,2,\dots\},\ \alpha_k = 0 \text{ for } k > n \text{ or } k \notin \sigma(A)\big\}.$$

For $\bar\alpha \in GI$, define

$$|\bar\alpha| = \sum_{k=1}^{\infty} \alpha_k \text{ and } \|\bar\alpha\| = \sum_{k=1}^{\infty} k\alpha_k. \tag{2.1}$$

For $d, n \geq 1$, define the set of special multi-indices (see Lemma 3.1 for its motivation):

$$SI(d,n) = \Big\{\bar\alpha = (\alpha_k)_{=1}^{\infty} \in GI, |\bar\alpha| = \sum_{k=1}^{\infty} \alpha_k = d,\ \|\bar\alpha\| = \sum_{k=1}^{\infty} k\alpha_k = n\Big\}; \tag{2.2}$$

note $1 \leq d \leq n$ hence $SI(d,n) \subset GI(n)$. Also, for $n \geq d \geq 1$ and $n' \geq d' \geq 1$ we have

$$SI(d,n) + SI(d',n') \subset SI(d+d', n+n'). \tag{2.3}$$

Let $\xi = (\xi_k)_{k=1}^{\infty} \in S_A$ and $\bar\alpha = (\alpha_k)_{k=1}^{\infty} \in GI$, define

$$\big[\,\xi\,\big]^{\bar\alpha} = \prod_{\alpha_k > 0} |\xi_k|^{\alpha_k}.$$

We have the following properties

$$\big[\,\xi\,\big]^{\bar\alpha}\big[\,\xi\,\big]^{\bar\alpha'}=\big[\,\xi\,\big]^{\bar\alpha+\bar\alpha'}, \tag{2.4}$$

$$\big[\,\xi\,\big]^{r\bar\alpha}=\big(\big[\,\xi\,\big]^{\bar\alpha}\big)^r \text{ for } r=0,1,2,\dots, \tag{2.5}$$

$$\sum_{|\bar\alpha|=d}\big[\,\xi\,\big]^{2\bar\alpha}=|\xi|^{2d}. \tag{2.6}$$

For $n\ge d\ge 1$, we define the (d,n)-gauge of $\xi\in S_A$ by

$$[[\,\xi\,]]_{d,n}=\Big(\sum_{\bar\alpha\in SI(d,n)}\big[\,\xi\,\big]^{2\bar\alpha}\Big)^{1/2}=\Big(\sum_{|\bar\alpha|=d,\|\bar\alpha\|=n}\big[\,\xi\,\big]^{2\bar\alpha}\Big)^{1/2}. \tag{2.7}$$

The family of gauges referred to in the abstract and introduction is $\big\{\ [[\,\cdot\,]]_{d,n}: n\ge d\ge 1\,\big\}$.

Three useful properties of these gauges are given below. First, note that

$$[[\,\xi\,]]_{d,n}\le\Big(\sum_{\bar\alpha\in GI(n),|\bar\alpha|=d}\big[\,\xi\,\big]^{2\bar\alpha}\Big)^{1/2}\le|P_n\xi|^d. \tag{2.8}$$

Next is a multiplicative inequality:

Lemma 2.1. *Let $\xi\in S_A$, $n\ge d\ge 1$ and $n'\ge d'\ge 1$. Then*

$$[[\,\xi\,]]_{d,n}\cdot[[\,\xi\,]]_{d',n'}\le e^{d+d'}\,[[\,\xi\,]]_{d+d',n+n'}\,, \tag{2.9}$$

Proof. By (2.4),

$$\begin{aligned}[[\,\xi\,]]^2_{d,n}\cdot[[\,\xi\,]]^2_{d',n'}&=\Big(\sum_{\bar\alpha\in SI(d,n)}\big[\,\xi\,\big]^{2\bar\alpha}\Big)\Big(\sum_{\bar\alpha'\in SI(d',n')}\big[\,\xi\,\big]^{2\bar\alpha'}\Big)\\&=\sum_{\substack{\bar\alpha\in SI(d,n)\\ \bar\alpha'\in SI(d',n')}}\big[\,\xi\,\big]^{2(\bar\alpha+\bar\alpha')}\end{aligned} \tag{2.10}$$

By (2.3) the index $\bar\gamma=\bar\alpha+\bar\alpha'$ belongs to $SI(d+d',n+n')$, where $\bar\alpha,\bar\alpha'$ are as in (2.10). We need to compare the above sum to $\sum_{\bar\gamma\in SI(d+d',n+n')}\big[\,\xi\,\big]^{2\bar\gamma}$. For that we estimate the number of times $\big[\,\xi\,\big]^{2\bar\gamma}$ is summed up as $\big[\,\xi\,\big]^{2(\bar\alpha+\bar\alpha')}$ in (2.10).

Fix $\bar\gamma=(\gamma_k)_{k=1}^\infty\in SI(d,n)+SI(d',n')$. We count the number of ways to write each $\bar\gamma$ as the sum $\bar\alpha+\bar\alpha'$. If $k>n$ or $k>n'$ then $\alpha_k=0,\alpha'_k=\gamma_k$ or $\alpha'_k=0,\alpha_k=\gamma_k$, hence one way. Let $k\le\min\{n,n'\}$. Counting via α_k: the set of possible values for α_k is $\{0,1,2,\dots,\gamma_k\}$, hence at most γ_k+1 values. Thus the number of repetition of $\bar\gamma$ as the sum $\bar\alpha+\bar\alpha'$ is at most

$$N(\bar\gamma)=(\gamma_1+1)(\gamma_2+1)\dots(\gamma_n+1)\le(\gamma_1+1)(\gamma_2+1)\dots(\gamma_{n+n'}+1).$$

By generalized Young's inequality:

$$\begin{aligned}N(\bar\gamma)&\le\Big(\frac{(\gamma_1+1)+(\gamma_2+1)+\dots+(\gamma_{n+n'}+1)}{n+n'}\Big)^{n+n'}\\&=\Big(\frac{d+d'+n+n'}{n+n'}\Big)^{n+n'}=\Big(1+\frac{d+d'}{n+n'}\Big)^{n+n'}.\end{aligned}$$

Since the function $f(x) = (1 + a/x)^x$ is increasing for $x \geq a > 0$ we have $N(\bar{\gamma}) \leq e^{d+d'}$. It follows that

$$\sum_{\substack{\bar{\alpha}\in SI(d,n)\\ \bar{\alpha}'\in SI(d',n')}} \big[\,\xi\,\big]^{2(\bar{\alpha}+\bar{\alpha}')} \leq \sum_{\bar{\gamma}\in SI(d+d',n+n')} N(\bar{\gamma})\big[\,\xi\,\big]^{2\bar{\gamma}} \leq e^{d+d'} \sum_{\bar{\gamma}\in SI(d+d',n+n')} \big[\,\xi\,\big]^{2\bar{\gamma}}.$$

Combining with (2.10), we obtain (2.9). □

It is noteworthy that the factor $e^{d+d'}$ on the right-hand side of (2.9) depends on neither n nor n'.

Lemma 2.2. *For any $\xi \in S_A$, any numbers $\alpha, s \geq 0$ and $n \geq d \geq 1$, one has*

$$[[\,A^{\alpha}\xi\,]]_{d,n} \leq \left(\frac{d}{n}\right)^{s} \big[\big[\,A^{\alpha+s}\xi\,\big]\big]_{d,n} \leq \left(\frac{d}{n}\right)^{s} |P_n A^{\alpha+s}\xi|^{d}. \tag{2.11}$$

Proof. For $|\bar{\alpha}| = d$ and $\|\bar{\alpha}\| = n$ we have

$$\begin{aligned}\big[\,\xi\,\big]^{2\bar{\alpha}} &= \prod_{\alpha_k>0} |\xi_k|^{2\alpha_k} = \prod_{\alpha_k>0} \frac{|k^s\xi_k|^{2\alpha_k}}{k^{2\alpha_k s}} \\ &= \frac{\prod_{\alpha_k>0} |k^s\xi_k|^{2\alpha_k}}{(\prod_{\alpha_k>0} k^{\alpha_k})^{2s}} = \frac{\big[\,A^s\xi\,\big]^{2\bar{\alpha}}}{(\prod_{\alpha_k>0} k^{\alpha_k})^{2s}}.\end{aligned}$$

Let $k_0 = \max\{k : \alpha_k \neq 0\}$. Then $n = \sum k\alpha_k \leq k_0(\sum \alpha_k) = k_0 d$. Hence $k_0 \geq n/d$ and

$$\prod_{\alpha_k>0} k^{\alpha_k} \geq k_0^{\alpha_{k_0}} \geq k_0 \geq n/d. \tag{2.12}$$

Therefore

$$\big[\,\xi\,\big]^{2\bar{\alpha}} \leq (d/n)^{2s}\big[\,A^s\xi\,\big]^{2\bar{\alpha}}. \tag{2.13}$$

Summing over $\bar{\alpha} \in SI(n,d)$ one obtains

$$[[\,\xi\,]]_{d,n} \leq (d/n)^{s}\,[[\,A^s\xi\,]]_{d,n}\,. \tag{2.14}$$

Replacing ξ by $A^{\alpha}\xi$ in (2.14) yields the first inequality of (2.11). The second one results from (2.8). □

We now turn to the estimates of the bilinear mapping $B(u,v)$. We will use the following simple inequalities.

Lemma 2.3. *For $\alpha \geq 0$ one has*

$$\begin{aligned}|A^{\alpha}B(u,v)| \leq 4^{\alpha}C(&|A^{1/2}u|^{1/2}|Au|^{1/2}|A^{\alpha+1/2}v| \\ &+ |A^{\alpha}u|^{1/2}|A^{\alpha+1/2}u|^{1/2}|Av|),\end{aligned} \tag{2.15}$$

$$\begin{aligned}|A^{\alpha}B(u,v)| \leq 4^{\alpha}C(&|A^{1/2}u|^{1/2}|Au|^{1/2}|A^{\alpha+1/2}v| \\ &+ |A^{\alpha+1/2}u||A^{1/2}v|^{1/2}|Av|^{1/2}),\end{aligned} \tag{2.16}$$

where $C > 0$ is an absolute constant defined by (2.25).

In particular, when $\alpha \geq 1/2$ one has

$$|A^{\alpha}B(u,v)| \leq K^{\alpha}|A^{\alpha+1/2}u|\,|A^{\alpha+1/2}v|, \tag{2.17}$$

for all $u, v \in \mathcal{D}(A^{\alpha+1/2})$, where $K = 4(\max\{2C,1\})^2$.

Proof. Suppose $u, v, w \in H$ with

$$u = \sum_{\mathbf{k}\neq 0} \hat{\mathbf{u}}(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}},\ v = \sum_{\mathbf{k}\neq 0} \hat{\mathbf{v}}(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}},\ w = \sum_{\mathbf{k}\neq 0} \hat{\mathbf{w}}(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}}. \tag{2.18}$$

Let

$$u_* = \sum_{\mathbf{k}\neq 0} |\hat{\mathbf{u}}(\mathbf{k})| e^{-i\mathbf{k}\cdot\mathbf{x}},\ v_* = \sum_{\mathbf{k}\neq 0} |\hat{\mathbf{v}}(k)| e^{-i\mathbf{k}\cdot\mathbf{x}},\ w_* = \sum_{\mathbf{k}\neq 0} |\hat{\mathbf{w}}(\mathbf{k})| e^{-i\mathbf{k}\cdot\mathbf{x}}. \tag{2.19}$$

The relation between the Sobolev norms of u and u_* is:

$$|A^\alpha u| = |(-\Delta)^\alpha u_*| \text{ for all } \alpha \geq 0. \tag{2.20}$$

We have

$$\Big| \int_\Omega A^\alpha B(u,v)\cdot w d\mathbf{x} \Big| = 8\pi^3 \Big| \sum_{\mathbf{k}+\mathbf{l}+\mathbf{m}=0} |\mathbf{m}|^{2\alpha} (\hat{\mathbf{u}}(\mathbf{k})\cdot \mathbf{l})\,(\hat{\mathbf{v}}(\mathbf{l})\cdot \hat{\mathbf{w}}(\mathbf{m})) \Big|.$$

Using the inequality $|\mathbf{m}|^\alpha \leq 2^\alpha(|\mathbf{k}|^\alpha + |\mathbf{l}|^\alpha)$ we estimate

$$\begin{aligned}\Big| \int_\Omega A^\alpha B(u,v)\cdot w d\mathbf{x} \Big| &\leq 8\pi^3 4^\alpha \sum_{\mathbf{k}+\mathbf{l}+\mathbf{m}=0} |\mathbf{k}|^{2\alpha} |\hat{\mathbf{u}}(\mathbf{k})|\,|\mathbf{l}|\,|\hat{\mathbf{v}}(\mathbf{l})||\hat{\mathbf{w}}(\mathbf{m})| \\ &\quad + 8\pi^3 4^\alpha \sum_{\mathbf{k}+\mathbf{l}+\mathbf{m}=0} |\hat{\mathbf{u}}(\mathbf{k})|\,|\mathbf{l}|^{2\alpha+1}\,|\hat{\mathbf{v}}(\mathbf{l})||\hat{\mathbf{w}}(\mathbf{m})|.\end{aligned}$$

Rewriting the last two sums in terms of u_*, v_* and w_* we have

$$\begin{aligned}\Big| \int_\Omega A^\alpha B(u,v)\cdot w d\mathbf{x} \Big| &\leq 8\pi^3 4^\alpha \Big| \int_\Omega (-\Delta)^\alpha u_* (-\Delta)^{1/2} v_* w_* d\mathbf{x} \Big| \\ &\quad + 8\pi^3 4^\alpha \Big| \int_\Omega u_* (-\Delta)^{\alpha+1/2} v_* w_* d\mathbf{x} \Big|.\end{aligned} \tag{2.21}$$

For the first integral on the right-hand side of (2.21), we have two options for Hölder's inequality using powers $3, 6, 2$ or $6, 3, 2$, and then use interpolation and Sobolev inequalities and the relation (2.20):

$$\begin{aligned}&\Big| \int_\Omega (-\Delta)^\alpha u_* (-\Delta)^{1/2} v_* w_* d\mathbf{x} \Big| \\ &\qquad \leq C_1^{3/2} |(-\Delta)^\alpha u_*|^{1/2} |(-\Delta)^{\alpha+1/2} u_*|^{1/2} |(-\Delta) v_*||w_*| \\ &\qquad \leq C_1^{3/2} |A^\alpha u|^{1/2} |A^{\alpha+1/2} u|^{1/2} |Av||w|,\end{aligned} \tag{2.22}$$

$$\begin{aligned}&\Big| \int_\Omega (-\Delta)^\alpha u_* (-\Delta)^{1/2} v_* w_* d\mathbf{x} \Big| \\ &\qquad \leq C_1^{3/2} |(-\Delta)^{\alpha+1/2} u_*||(-\Delta)^{1/2} v_*|^{1/2} |(-\Delta) v_*|^{1/2} |w_*| \\ &\qquad \leq C_1^{3/2} |A^{\alpha+1/2} u||A^{1/2} v|^{1/2} |Av|^{1/2} ||w|,\end{aligned} \tag{2.23}$$

where $C_1 > 0$ is the Sobolev constant for the embedding of V into $L^6(\Omega)$.

For the second integral on the right-hand side of (2.21), applying the Hölder inequality and then using the Agmon inequality for the embedding of $\mathcal{D}(A)$ into

$L^\infty(\Omega)$, we obtain

$$
\begin{aligned}
&\Big|\int_\Omega u_*(-\Delta)^{\alpha+1/2}v_*w_*d\mathbf{x}\Big|\\
&\quad\le \|u_*\|_{L^\infty(\Omega)}|(-\Delta)^{\alpha+1/2}v_*||w_*|\\
&\quad\le C_2|(-\Delta)^{1/2}u_*|^{1/2}|(-\Delta)u_*|^{1/2}|(-\Delta)^{\alpha+1/2}v_*||w_*|\\
&\quad\le C_2|A^{1/2}u|^{1/2}|Au|^{1/2}|A^{\alpha+1/2}v||w|.
\end{aligned}
\tag{2.24}
$$

Combining (2.21) with (2.24) and (2.22), resp. (2.23), yields (2.15), resp. (2.16) with

$$C = 8\pi^3\max\{C_1^{3/2}, C_2\}. \tag{2.25}$$

For $\alpha\ge 1/2$, by either (2.15) or (2.16):

$$|A^\alpha B(u,v)|\le 2C4^\alpha|A^{\alpha+1/2}u|\,|A^{\alpha+1/2}v|,$$

and hence (2.17) follows. □

3. Homogeneous polynomials in the normal form

In this section we show that the homogeneous polynomials $\mathcal{P}^{[d]}(\xi)$ in (1.17) and $\mathcal{B}^{[d]}(\xi)$ in (1.16) are well-defined.

Let $\xi=(\xi_k)_{k=1}^\infty\in S_A$ and $q_j(t,\xi)$ be the polynomial solutions given by (1.10). Using the explicit formula (1.10), one can easily verify the following properties by induction:

(a) Each $q_j(t,\xi)$ is a polynomial in t and in $\xi_1,\xi_2,\dots,\xi_j$, hence is a $\mathcal{V}$-valued polynomial on the finite dimensional space $\mathbb{R}\times P_jH$.
(b) The degree in t: $\deg_t q_j(t,\xi)\le j-1$.
(c) The degree in ξ: $\deg_\xi q_j(t,\xi)\le j$.

Thus we can write

$$q_j(t,\xi)=\sum_{m=0}^{j-1}q_{j,m}(\xi)t^m=\sum_{m=0}^{j-1}\sum_{d=1}^{j}q_{j,m}^{[d]}(\xi)t^m=\sum_{d=1}^{j}q_j^{[d]}(t,\xi), \tag{3.1}$$

where $q_{j,m}(\xi)$ is a polynomial in ξ, and $q_{j,m}^{[d]}(\xi)$ and $q_j^{[d]}(t,\xi)$ are homogeneous polynomials in ξ of degree d.

We also write $q_j^{[d]}(t,\xi)=\sum_{|\bar\alpha|=d}q_j^{[d],(\bar\alpha)}(t,\xi)$, where $\bar\alpha=(\alpha_k)_{k=1}^\infty\in GI$ and $q_j^{[d],(\bar\alpha)}(t,\xi)$ is the sum of all monomials (in ξ) of $q_j^{[d]}(t,\xi)$ having degree α_k in ξ_k for all $k\ge1$. Similarly, we write $q_{j,m}^{[d]}(\xi)=\sum_{|\bar\alpha|=d}q_{j,m}^{[d],(\bar\alpha)}(\xi)$. We also have

$$\beta_j(t,\xi)=\sum_{m=0}^{j-2}\beta_{j,m}(\xi)t^m=\sum_{m=0}^{j-2}\sum_{d=1}^{j}\beta_{j,m}^{[d]}(\xi)t^m=\sum_{d=1}^{j}\beta_j^{[d]}(t,\xi),$$

where $\beta_{1,m}(\xi)=\beta_{1,m}^{[d]}(\xi)=\beta_1^{[d]}(t,\xi)=0$ for all m,d,t and ξ,

$$\beta_{j,m}(\xi)=\sum_{l+l'=j}\ \sum_{r+r'=m}B(q_{l,r}(\xi),q_{l',r'}(\xi)), \tag{3.2}$$

$$\beta_{j,m}^{[d]}(\xi)=\sum_{l+l'=j}\ \sum_{r+r'=m}\ \sum_{s+s'=d}B(q_{l,r}^{[s]}(\xi),q_{l',r'}^{[s']}(\xi)), \tag{3.3}$$

for $j \geq 2$ and $0 \leq m \leq j-2$.

We need the following properties of the degrees in t and in ξ of $q_j^{[d]}(t,\xi)$ and $q_j^{[d],(\bar\alpha)}(t,\xi)$.

Lemma 3.1. *(i) The degree in* t *of the polynomial* $q_j^{[d]}(t,\xi)$ *is less than or equal to* $d-1$*, i.e.,* $\deg_t q_j^{[d]}(t,\xi) \leq d-1$*.*

(ii) If $q_j^{[d],(\bar\alpha)} \neq 0$ *then* $\bar\alpha \in SI(d,j)$*.*

(iii) Consequently, for each (non-zero) monomial of $\mathcal{P}_j(\xi)$*,* $j \geq 1$*, having degree* α_k *in* ξ_k*,* $k \geq 1$*, one has* $\bar\alpha = (\alpha_k)_{k=1}^\infty$ *belongs to* $SI(d,j)$ *where* $d = |\bar\alpha|$*. Also, for each (non-zero) monomial of* $B(\mathcal{P}_m(\xi), \mathcal{P}_n(\xi))$*, having degree* α_k *in* ξ_k*,* $k \geq 1$*, one has* $\bar\alpha = (\alpha_k)_{k=1}^\infty$ *belongs to* $SI(d, m+n)$ *where* $d = |\bar\alpha|$ *.*

Proof. We prove (i) and (ii) by induction in j and d. Since $q_1(t,\xi) = \xi_1$ and $q_j^{[1]}(t,\xi) = \xi_j$, the statements (i) and (ii) hold for $j=1$ and all d's, as well as for $d=1$ and all j's. Let $j \geq 2$, $d \geq 2$ and assume that for all $1 \leq j' < j$ and $1 \leq d' < d$, we have $\deg_t q_{j'}^{[d']}(t,\xi) \leq d'-1$ and $\bar\alpha \in SI(d',j')$ whenever $q_{j'}^{[d'],(\bar\alpha)} \neq 0$.

Observe from (3.3) and the induction hypothesis that $m = r+r' \leq (s-1)+(s'-1) = d-2$ whenever $B(q_{l,r}^{[s]}, q_{l',r'}^{[s']}) \neq 0$, hence $\deg_t \beta_j^{[d]} \leq d-2$. From the integral in the formula (1.10), $\deg_t q_j^{[d]} \leq \deg_t \beta_j^{[d]} + 1 \leq d-1$, hence (i) is verified for j and d.

For (ii), we also have that $\beta_j^{[d]}(t,\xi) = \sum_{|\bar\alpha|=d} \beta_j^{[d],(\bar\alpha)}(t,\xi)$, where

$$\beta_j^{[d],(\bar\alpha)}(t,\xi) = \sum_{l+l'=j} \sum_{k+k'=d} \sum_{\bar\gamma+\bar\gamma'=\bar\alpha} B(q_l^{[k],(\bar\gamma)}(t,\xi), q_{l'}^{[k'],(\bar\gamma')}(t,\xi)). \tag{3.4}$$

By the induction hypothesis $\bar\gamma \in SI(k,l)$ and $\bar\gamma' \in SI(k',l')$ whenever the summand $B(q_l^{[k],(\bar\gamma)}, q_{l'}^{[k'],(\bar\gamma')})$ in (3.4) is not identically zero, hence by (2.3) $\bar\alpha \in SI(k+k', l+l') = SI(d,j)$. This completes the proof of (ii).

The first statement of (iii) is a direct consequence of (ii) since $P_j(\xi) = q_j(0,\xi)$. Each monomial in the second statement of (iii) is of the form $\beta_{m+n}^{[d],(\bar\alpha)}(0,\xi)$ as in (3.4), hence the argument in (ii) readily shows $\bar\alpha \in SI(d, m+n)$. $\square$

In the following calculations we will use the convention $0/0 = 0$ as well as the shorthand notation

$$j\big|_d = \min\{j, d-1\} \text{ for all } j, d.$$

It is clear from Lemma 3.1 that $q_{j,m}^{[d]} = 0$ for $m > (j-1)\big|_d$, and $\beta_{j,m}^{[d]} = 0$ for $m > (j-2)\big|_{d-1}$.

Applying the projection R_k to (1.10) we have

$$\begin{aligned}
R_k q_j(t,\xi) &= R_k\xi_j - \int_0^t \sum_{m=0}^{j-2} R_k R_j \beta_{j,m}\tau^m d\tau \\
&\quad + \sum_{n=0}^{j-2} \frac{(-1)^{n+1}}{(k-j)^{n+1}} \sum_{m=0}^{j-2} \frac{d^n}{dt^n}\Big(R_k(I-R_j)\beta_{j,m}t^m\Big) \\
&= R_k\xi_j - \sum_{m=1}^{j-1} \frac{R_k R_j\beta_{j,m-1}}{m} t^m \\
&\quad + \sum_{n=0}^{j-2} \frac{(-1)^{n+1}}{(k-j)^{n+1}} \sum_{m=n}^{j-2} \frac{m!}{(m-n)!}\Big(R_k(I-R_j)\beta_{j,m}t^{m-n}\Big).
\end{aligned}$$

By a suitable relabelling we obtain

$$\begin{aligned}
R_k q_j(t,\xi) &= R_k\xi_j - \sum_{m=1}^{j-1} \frac{R_k R_j\beta_{j,m-1}}{m} t^m \\
&\quad + \sum_{n=0}^{j-2} \frac{(-1)^{n+1}}{(k-j)^{n+1}} \sum_{m=0}^{j-2-n} \frac{(m+n)!}{m!}\Big(R_k(I-R_j)\beta_{j,m+n}t^m\Big) \\
&= R_k\xi_j - \sum_{m=1}^{j-1} \frac{R_k R_j\beta_{j,m-1}}{m} t^m \\
&\quad + \sum_{m=0}^{j-2} \sum_{n=0}^{j-2-m} \Big(\frac{(-1)^{n+1}}{(k-j)^{n+1}} \frac{(m+n)!}{m!} R_k(I-R_j)\beta_{j,m+n}t^m\Big).
\end{aligned}$$

Hence for $m=0$:

$$R_k q_{j,0} = R_k\xi_j + \sum_{n=0}^{j-2}\Big(\frac{(-1)^{n+1}n!}{(k-j)^{n+1}} R_k(I-R_j)\beta_{j,n}\Big); \tag{3.5}$$

for $m=1,\ldots,j-2$:

$$R_k q_{j,m} = -\frac{R_k R_j \beta_{j,m-1}}{m} + \sum_{n=0}^{j-2-m}\Big(\frac{(-1)^{n+1}}{(k-j)^{n+1}}\frac{(m+n)!}{m!} R_k(I-R_j)\beta_{j,m+n}\Big); \tag{3.6}$$

and for $m=j-1$:

$$R_k q_{j,j-1} = -\sum_{m=1}^{j-1} \frac{R_k R_j \beta_{j,j-2}}{j-1}. \tag{3.7}$$

Collecting the homogeneous components of degree $d \geq 1$ in ξ of the identities (3.5), (3.6) and (3.7) yields

$$\begin{aligned}
R_k q_{j,0}^{[d]} &= R_k\xi_j^{[d]} + \sum_{n=0}^{j-2}\Big(\frac{(-1)^{n+1}n!}{(k-j)^{n+1}} R_k(I-R_j)\beta_{j,n}^{[d]}\Big) \\
&= R_k\xi_j^{[d]} + \sum_{n=0}^{(j-2)|_{d-1}}\Big(\frac{(-1)^{n+1}n!}{(k-j)^{n+1}} R_k(I-R_j)\beta_{j,n}^{[d]}\Big),
\end{aligned} \tag{3.8}$$

$$\begin{aligned}
R_k q^{[d]}_{j,m} &= -\frac{R_k R_j \beta^{[d]}_{j,m-1}}{m}\\
&\quad + \sum_{n=0}^{j-2-m} \Big(\frac{(-1)^{n+1}}{(k-j)^{n+1}} \frac{(m+n)!}{m!} R_k (I-R_j)\beta^{[d]}_{j,m+n} \Big)\\
&= -\frac{R_k R_j \beta^{[d]}_{j,m-1}}{m} + \sum_{n=m}^{(j-2)|_{d-1}} \Big(\frac{(-1)^{n-m+1}}{(k-j)^{n-m+1}} \frac{n!}{m!} R_k (I-R_j)\beta^{[d]}_{j,n} \Big)
\end{aligned} \tag{3.9}$$

for $m = 1, \dots, (j-2)\big|_d$, and

$$R_k q^{[d]}_{j,j-1} = -\frac{R_k R_j \beta^{[d]}_{j,j-2}}{j-1}. \tag{3.10}$$

After this preparation, we will establish recursive estimates for the norms of $q^{[d]}_{j,m}$.

Lemma 3.2. *For $j \ge 2$, $d \ge 1$, $\alpha \ge 0$ and $\xi \in S_A$, one has*

$$|A^\alpha q^{[d]}_{j,0}(\xi)|^2 \le 2(d!)(d-1)! \Big(|A^\alpha \xi^{[d]}_j|^2 + \sum_{n=0}^{(j-2)|_{d-1}} |A^\alpha (I-R_j)\beta^{[d]}_{j,n}(\xi)|^2 \Big); \tag{3.11}$$

$$\begin{aligned}
|A^\alpha q^{[d]}_{j,m}(\xi)|^2 \le (d!)(d-1)! \Big(&\frac{|A^\alpha R_j \beta^{[d]}_{j,m-1}(\xi)|^2}{m^2}\\
&+ \frac{1}{m!^2} \sum_{n=0}^{(j-2)|_{d-1}} |A^\alpha (I-R_j)\beta^{[d]}_{j,n}(\xi)|^2 \Big)
\end{aligned} \tag{3.12}$$

for $m = 1, \dots, (j-2)\big|_d$; and

$$|A^\alpha q^{[d]}_{j,j-1}(\xi)|^2 = \frac{|A^\alpha R_j \beta^{[d]}_{j,j-2}(\xi)|^2}{(j-1)^2}. \tag{3.13}$$

Proof. For $d = 1$, the inequality (3.11) is trivially true, because $q^{[1]}_j(t,\xi) = \xi_j$ for all j, and the sum on the right-hand side of (3.11) is missing; also (3.12) and (3.13) trivially hold since all polynomials involved are zero.

Let $d \ge 2$. For $m = 0$ we apply the Cauchy-Schwarz inequality to the sum on the right-hand side of (3.8), which consists at most d terms, to obtain

$$\begin{aligned}
|A^\alpha R_k q^{[d]}_{j,0}|^2 &\le d(|A^\alpha R_k \xi^{[d]}_j|^2 + \sum_{n=0}^{(j-2)|_{d-1}} \Big(\frac{n!^2}{|k-j|^{2(n+1)}} |A^\alpha R_k (I-R_j)\beta^{[d]}_{j,n}|^2 \Big)\\
&\le d\Big(|A^\alpha R_k \xi^{[d]}_j|^2 + \sum_{n=0}^{(j-2)|_{d-1}} n!^2 |A^\alpha R_k (I-R_j)\beta^{[d]}_{j,n}|^2 \Big)\\
&\le (d-1)!^2 d \Big(|A^\alpha R_k \xi^{[d]}_j|^2 + \sum_{n=0}^{(j-2)|_{d-1}} |A^\alpha R_k (I-R_j)\beta^{[d]}_{j,n}|^2 \Big).
\end{aligned}$$

Then summing this estimate over k yields (3.11). Similarly, for $m = 1, \dots, (j-2)\big|_d$, we have from (3.9) that

$$|A^\alpha R_k q^{[d]}_{j,m}|^2 \le ((d-1)!)^2 d\Big(\frac{|A^\alpha R_k R_j \beta^{[d]}_{j,m-1}|^2}{m^2} + \frac{1}{m!^2}\sum_{n=0}^{(j-2)|_{d-1}} |A^\alpha R_k (I-R_j)\beta^{[d]}_{j,n}|^2\Big),$$

which gives (3.12) after summing over k. Finally, the formula (3.10) yields (3.13) directly. □

Our next aim is to prove that the following two formal series

$$\mathcal{P}^{[d]}(\xi) = \sum_{j=d}^{\infty} \mathcal{P}^{[d]}_j(\xi) = \sum_{j=d}^{\infty} q^{[d]}_{j,0}(\xi) = \sum_{j=d}^{\infty} q^{[d]}_j(0,\xi), \ d \ge 1, \tag{3.14}$$

(3.15)

$$\mathcal{B}^{[d]}(\xi) = \sum_{j=1}^{\infty} \mathcal{B}^{[d]}_j(\xi) = \sum_{j=1}^{\infty}\Big(\sum_{k+l=j}\ \sum_{m+n=d} R_j B\big(\mathcal{P}^{[m]}_k(\xi), \mathcal{P}^{[n]}_l(\xi)\big)\Big), \ d \ge 2,$$

are in fact convergent in Sobolev spaces and define homogeneous polynomials acting between appropriate Sobolev spaces.

We note that when $d = 1$,

$$\mathcal{P}^{[1]}_j(\xi) = \xi_j = R_j\xi \quad \text{for all } j, \text{ hence } \mathcal{P}^{[1]}(\xi) = \xi \quad \text{for } \xi \in H. \tag{3.16}$$

First, we estimate the norms $|A^\alpha q^{[d]}_{j,m}(\xi)|$ and $|A^\alpha \mathcal{P}^{[d]}_j(\xi)|$ by using the (d,j)-gauge.

Proposition 3.3. *For $j \ge d \ge 1$ and $0 \le m \le (j-1)\big|_d$, one has*

$$|A^\alpha q^{[d]}_{j,m}(\xi)| \le c(\alpha,d)\left[\left[A^{\alpha+\frac{3}{2}(d-1)}\xi\right]\right]_{d,j}, \tag{3.17}$$

for all $\xi \in S_A$ and $\alpha \ge 1/2$, where the positive number $c(\alpha,d)$ is

$$c(\alpha,d) = (M_d)^{(\alpha+\tau_d)(d-1)}, \tag{3.18}$$

with

$$M_d = K^2 + d^6 e^{2d}(d!)^2 \quad \text{and} \quad \tau_d = (d-1)/2. \tag{3.19}$$

In particular, when $m = 0$ one has

$$|A^\alpha \mathcal{P}^{[d]}_j(\xi)| \le c(\alpha,d)\left[\left[A^{\alpha+\frac{3}{2}(d-1)}\xi\right]\right]_{d,j}. \tag{3.20}$$

Proof. For this proof, we rewrite (3.17) in the following convenient form

$$|A^\alpha q^{[d]}_{j,m}(\xi)| \le c(\alpha,d)\left[\left[A^{\alpha+h\delta(d-1)}\xi\right]\right]_{d,j} \text{ with } \delta = 1/2, \ h = \frac{\delta+1}{\delta} = 3. \tag{3.21}$$

We will prove (3.21) by induction in j.

For $j = 1$, we have $q_1(t,\xi) = \xi_1$; it is clear that $d = 1$, $m = 0$ and (3.21) holds with $c(\alpha,1) = 1$ for all $\alpha \ge 0$.

Let $j \ge 2$. Suppose

$$|A^{\alpha'} q^{[d']}_{j',m'}(\xi)| \le c(\alpha',d')\left[\left[A^{\alpha'+h\delta(d'-1)}\xi\right]\right]_{d',j'} \tag{3.22}$$

for all $1 \le j' < j$, $1 \le d' \le j'$, and $0 \le m' \le (j'-1)\big|_{d'-1}$, all $\xi \in S_A$ and $\alpha' \ge 1/2$.

For $d = 1$, $q^{[1]}_{j,m}(\xi)$ is ξ_j for $m = 0$ and is 0 otherwise; hence (3.17) holds for all $\alpha \ge 0$.

Consider $d \ge 2$. We will use the recursive estimates in Lemma 3.2. First, we estimate $|A^\alpha \beta^{[d]}_{j,n}|$. Note that $M_d > 1$ and both M_d and τ_d are increasing in d. Using formula (3.3), inequality (2.17) and the induction hypothesis (3.22) we have

$$\begin{aligned}|A^\alpha \beta^{[d]}_{j,m}(\xi)| &\le \sum_{l+l'=j}\sum_{r+r'=m}\sum_{s+s'=d} K^\alpha |A^{\alpha+\delta} q^{[s]}_{l,r}(\xi)|\,|A^{\alpha+\delta} q^{[s']}_{l',r'}(\xi))|\\ &\le \sum_{l+l'=j}\sum_{r+r'=m}\sum_{s+s'=d} K^\alpha c(\alpha+\delta,s)c(\alpha+\delta,s')\\ &\quad\times \Big[\Big[A^{\alpha+\delta+h\delta(s-1)}\xi\Big]\Big]_{s,l}\Big[\Big[A^{\alpha+\delta+h\delta(s'-1)}\xi\Big]\Big]_{s',l'}.\end{aligned}$$

Applying the multiplicative inequality (2.9) yields

$$\begin{aligned}|A^\alpha \beta^{[d]}_{j,m}(\xi)| &\le \sum_{l+l'=j}\sum_{r+r'=m}\sum_{s+s'=d} K^\alpha M_s^{(\alpha+\delta+\tau_s)(s-1)} M_{s'}^{(\alpha+\delta+\tau_{s'})(s'-1)}\\ &\quad\times e^{s+s'}\Big[\Big[A^{\alpha+\delta+h\delta(s+s'-2)}\xi\Big]\Big]_{s+s',l+l'}\\ &\le \sum_{l+l'=j}\sum_{r+r'=m}\sum_{s+s'=d} K^\alpha M_d^{(\alpha+\delta+\tau_d)[(s-1)+(s'-1)]} e^d \Big[\Big[A^{\alpha+\delta+h\delta(d-2)}\xi\Big]\Big]_{d,j}\\ &\le \sum_{l+l'=j}\sum_{r+r'=m}\sum_{s+s'=d} K^\alpha M_d^{(\alpha+\delta+\tau_d)(d-2)} e^d \Big[\Big[A^{\alpha+\delta+h\delta(d-2)}\xi\Big]\Big]_{d,j}\\ &\le d^2 K^\alpha \sum_{l+l'=j} M_d^{(\alpha+\delta+\tau_d)(d-2)} e^d \Big[\Big[A^{\alpha+\delta+h\delta(d-2)}\xi\Big]\Big]_{d,j},\end{aligned}$$

thus

$$|A^\alpha \beta^{[d]}_{j,m}(\xi)| \le K^\alpha d^2 e^d M_d^{(\alpha+\delta+\tau_d)(d-2)} \cdot j \Big[\Big[A^{\alpha+\delta+h\delta(d-2)}\xi\Big]\Big]_{d,j}.$$

Applying the inequality (2.11) with $s = 1$, we obtain

$$\begin{aligned}|A^\alpha \beta^{[d]}_{j,m}(\xi)| &\le K^\alpha d^2 e^d M_d^{(\alpha+\delta+\tau_d)(d-2)} \cdot j \cdot \Big(\frac{d}{j}\Big)\Big[\Big[A^{\alpha+\delta+h\delta(d-2)+1}\xi\Big]\Big]_{d,j}\\ &\le K^\alpha d^3 e^d M_d^{(\alpha+\delta+\tau_d)(d-2)}\Big[\Big[A^{\alpha+\delta+h\delta(d-2)+1}\xi\Big]\Big]_{d,j}.\end{aligned}$$

Note that h and δ satisfy $\delta + h\delta(d-2)+1 = h\delta(d-1)$. Setting $N = N(\alpha,d) = K^\alpha d^3 e^d M_d^{(\alpha+\delta+\tau_d)(d-2)}$, we have

$$|A^\alpha \beta^{[d]}_{j,m}(\xi)| \le N \Big[\Big[A^{\alpha+h\delta(d-1)}\xi\Big]\Big]_{d,j}. \tag{3.23}$$

For $m = 1, 2, \dots, ..., (j-2)\big|_d$, it follows from (3.12) and (3.23) that

$$\begin{aligned}|A^\alpha q^{[d]}_{j,m}|^2 &\le (d-1)!d!\Big(N^2\Big[\Big[A^{\alpha+h\delta(d-1)}\xi\Big]\Big]^2_{d,j} + \sum_{n=0}^{(j-2)|_{d-1}} N^2\Big[\Big[A^{\alpha+h\delta(d-1)}\xi\Big]\Big]^2_{d,j}\Big)\\ &\le d!^2 N^2\Big[\Big[A^{\alpha+h\delta(d-1)}\xi\Big]\Big]^2_{d,j}\\ &= K^{2\alpha}(d!)^2 d^6 e^{2d} M_d^{2(\alpha+\delta+\tau_d)(d-2)}\Big[\Big[A^{\alpha+h\delta(d-1)}\xi\Big]\Big]^2_{d,j}.\end{aligned}$$

Since the numbers M_d and τ_d in (3.19) satisfy

$$K^2, (d!)^2 d^6 e^{2d} \le M_d \text{ and } \alpha + 1 + 2(\alpha + \delta + \tau_d)(d-2) \le 2(\alpha + \tau_d)(d-1),$$

it follows that

$$\begin{aligned} |A^\alpha q^{[d]}_{j,m}|^2 &\le M_d^{\alpha+1+2(\alpha+\delta+\tau_d)(d-2)} \left[\left[A^{\alpha+h\delta(d-1)}\xi \right]\right]^2_{d,j} \\ &\le M_d^{2(\alpha+\tau_d)(d-1)} \left[\left[A^{\alpha+h\delta(d-1)}\xi \right]\right]^2_{d,j}. \end{aligned}$$

Therefore (3.21) is readily obtained for j, d and $m \neq 0, j-1$. Similarly, we obtain (3.21) for $|A^\alpha q^{[d]}_{j,0}(\xi)|$ and $|A^\alpha q^{[d]}_{j,j-1}|$, resp., by using (3.11) and (3.13), resp., together with (3.23). This concludes the induction step and completes our proof. □

Theorem 3.4. *Let $\alpha \ge 1/2$, $d \ge 1$ and $\xi \in \mathcal{D}(A^{\alpha+3d/2})$. Then $\mathcal{P}^{[d]}(\xi)$ defined in (3.14) converges absolutely in $\mathcal{D}(A^\alpha)$ and satisfies*

$$|A^\alpha \mathcal{P}^{[d]}(\xi)| \le \sum_{j=d}^{\infty} |A^\alpha \mathcal{P}^{[d]}_j(\xi)| \le M(\alpha, d) |A^{\alpha+3d/2}\xi|^d, \tag{3.24}$$

where $M(\alpha, d) = (1+2d)c(\alpha, d)$ with $c(\alpha, d)$ given by (3.18). Moreover, $\mathcal{P}^{[d]}(\xi)$ is a continuous homogeneous polynomial of degree d from $\mathcal{D}(A^{\alpha+3d/2})$ to $\mathcal{D}(A^\alpha)$.

Proof. Let $r > 1$. From Proposition 3.3 and inequality (2.11) we have

$$\begin{aligned} \sum_{j=1}^{\infty} |A^\alpha q^{[d]}_{j,m}(\xi)| &\le \sum_{j=d}^{\infty} c(\alpha, d) \left[\left[A^{\alpha+h\delta(d-1)}\xi \right]\right]_{d,j} \\ &\le \sum_{j=d}^{\infty} c(\alpha, d) \left(\frac{d}{j}\right)^r |A^{\alpha+h\delta(d-1)+r}\xi|^d \\ &= c_0(d, r) c(\alpha, d) |A^{\alpha+h\delta(d-1)+r}\xi|^d, \end{aligned}$$

where $c_0(d, r) = d^r \sum_{j=d}^{\infty} 1/j^r$. Particularly, when $m = 0$ we have

$$|A^\alpha \mathcal{P}^{[d]}(\xi)| \le \sum_{j=1}^{\infty} |A^\alpha \mathcal{P}^{[d]}_j(\xi)| \le c_0(d, r) c(\alpha, d) |A^{\alpha+h\delta(d-1)+r}\xi|^d. \tag{3.25}$$

The constant $c_0(d, r)$ can be bounded by

$$c_0(d, r) \le d^r \left(\frac{1}{d^r} + \int_d^\infty \frac{1}{t^r} dt \right) = 1 + \frac{d}{r-1}.$$

Taking $r = 3/2$ and recalling that $\delta h = 3/2$, we have

$$\alpha + h\delta(d-1) + r = \alpha + 3d/2 \text{ and } c_0(d, 3/2) \le 1 + 2d. \tag{3.26}$$

Hence (3.24) follows (3.25) and (3.26).

By the uniform estimate (3.24) and Theorem 4.2.9 in [10], the limit function $\mathcal{P}^{[d]}(\xi)$ is a continuous homogeneous polynomial of degree d. □

Theorem 3.5. *Let $\alpha \ge 1/2$, $d \ge 2$ and $\xi \in \mathcal{D}(A^{\alpha+3d/2})$. Then one has for all $m \ge 0$ that*

$$\sum_{j=2}^{\infty} |A^\alpha \beta^{[d]}_{j,m}(\xi)| \le C(\alpha, d) |A^{\alpha+3d/2}\xi|^d, \tag{3.27}$$

where $C(\alpha,d)=(2d+1)N(\alpha,d)$ with $N(\alpha,d)$ being the same number as in (3.23).

Consequently, $\mathcal{B}^{[d]}(\xi)$, $d\ge 2$, defined in (3.15) is a continuous homogeneous polynomial of degree d from $\mathcal{D}(A^{\alpha+3d/2})$ to $\mathcal{D}(A^{\alpha})$ for all $\alpha\ge 1/2$, and satisfies

$$|A^\alpha \mathcal{B}^{[d]}(\xi)|\le \sum_{n=1}^\infty |A^\alpha \mathcal{B}_n^{[d]}(\xi)|\le C(\alpha,d)|A^{\alpha+3d/2}\xi|^d. \tag{3.28}$$

Proof. By (3.23), and with $r>1$ we have

$$\begin{aligned}\sum_{j=1}^\infty |A^\alpha \beta_{j,m}^{[d]}(\xi)| &\le \sum_{j=1}^\infty N(\alpha,d)(d/j)^r|A^{\alpha+h\delta(d-1)+r}\xi|^d\\ &\le c_0(d,r)N(\alpha,d)|A^{\alpha+h\delta(d-1)+r}\xi|^d.\end{aligned}$$

Taking $r=3/2$ and using (3.26), we obtain (3.27). The relation (3.28) is obtained by taking $m=0$ in (3.27). The same argument as in Theorem 3.4 shows that $\mathcal{B}^{[d]}$ is a continuous homogeneous polynomial of degree d from $\mathcal{D}(A^{\alpha+3d/2})$ to $\mathcal{D}(A^\alpha)$. □

4. A Poincaré–Dulac normal form

The normal form (1.14), originally constructed in S_A, now can be rewritten formally as (1.16). Concerning the framework for (1.16) where $\mathcal{B}^{[d]}(\xi)$ is well-defined for all $d\ge 2$, Theorem 3.5 with the estimate (3.28) of $|A^\alpha\mathcal{B}^{[d]}(\xi)|$ in terms of $|A^{\alpha+3d/2}\xi|$ for all $\alpha\ge 1/2$ and $d\ge 1$ suggests an appropriate framework, namely, the Fréchet space $E^\infty=C^\infty(\mathbb{R}^3,\mathbb{R}^3)\cap V$. We recall that the classical L. Schwartz's topology on E^∞ is the one given by the family of norms $|A^\alpha u|$ ($u\in E^\infty$), $\alpha\ge 0$. Theorems 3.4 and 3.5 directly imply

Lemma 4.1. *The series $\mathcal{P}^{[d]}(\xi)=\sum_{j=d}^\infty \mathcal{P}_j^{[d]}(\xi)$ and $\mathcal{B}^{[d]}(\xi)=\sum_{j=d}^\infty \mathcal{B}_j^{[d]}(\xi)$, for $d\ge 2$, converge in E^∞. Moreover $\mathcal{P}^{[d]}(\xi)$ and $\mathcal{B}^{[d]}(\xi)$, for $d\ge 2$, are continuous homogeneous polynomials of degree d from E^∞ to E^∞.*

We denote by $\mathcal{H}^{[d]}(E^\infty)$, $d\ge 1$, the linear space of continuous homogeneous polynomials of degree d from E^∞ to E^∞. For $Q\in\mathcal{H}^{[d]}(E^\infty)$ we denote by $\hat{Q}$ the continuous symmetric d-linear map from $(E^\infty)^d$ to E^∞ representing Q, that is, $Q(\xi)=\hat{Q}(\xi,\xi,\dots,\xi)$ for $\xi\in E^\infty$.

We recall from [9] that the asymptotic expansion (1.6) can be written formally as

$$u(t)=\sum_{j=1}^\infty \mathcal{P}_j(W(u(t))=\sum_{j=1}^\infty \mathcal{P}_j(\xi(t)). \tag{4.1}$$

Therefore it is reasonable to consider that $u=\sum_{j=1}^\infty \mathcal{P}_j(\xi)$ is the formal inverse of the normalization map $\xi=W(u)$.

We write $u=\sum_{j=1}^\infty \mathcal{P}_j(\xi)=\sum_{j=1}^\infty\sum_{d=1}^j \mathcal{P}_j^{[d]}(\xi)$, then by formally interchanging the order of summations, we have the formal relation

$$u=\sum_{j=1}^\infty\sum_{d=1}^\infty \mathcal{P}_j^{[d]}(\xi)=\sum_{d=1}^\infty\sum_{j=1}^\infty \mathcal{P}_j^{[d]}(\xi)=\sum_{d=1}^\infty \mathcal{P}^{[d]}(\xi).$$

Therefore we formally write

$$u=\mathcal{P}(\xi)\overset{\text{def}}{=\!=}\sum_{d=1}^\infty \mathcal{P}^{[d]}(\xi)=\xi+\sum_{d=2}^\infty \mathcal{P}^{[d]}(\xi). \tag{4.2}$$

Similarly, for the nonlinear term in (1.14), we have

$$\mathcal{B}(\xi) = \sum_{j=1}^{\infty} \mathcal{B}_j(\xi) = \sum_{j=1}^{\infty}\sum_{d=2}^{\infty} \mathcal{B}_j^{[d]}(\xi) = \sum_{d=2}^{\infty}\sum_{j=1}^{\infty} \mathcal{B}_j^{[d]}(\xi) = \sum_{d=2}^{\infty} \mathcal{B}^{[d]}(\xi). \tag{4.3}$$

We aim to prove that (1.16) is a Poincaré–Dulac normal form in E^∞ which can be obtained from the Navier–Stokes equations (1.5) by using the explicit formal power series (4.2).

The power series (4.2) has its formal right inverse of the form

$$\xi = \tilde{\mathcal{P}}(u) \overset{\text{def}}{=\!=} \sum_{d=1}^{\infty} \tilde{\mathcal{P}}^{[d]}(u) = u + \sum_{d=2}^{\infty} \tilde{\mathcal{P}}^{[d]}(u), \tag{4.4}$$

where each $\tilde{\mathcal{P}}^{[d]}(u)$, $d \geq 1$, is a homogeneous polynomial of degree d, particularly, $\tilde{\mathcal{P}}^{[1]}(u) = \mathcal{P}^{[1]}(u) = u$. To compute this inverse we introduce (4.4) into (4.2) and formally obtain

$$u = u + \sum_{d=2}^{\infty} \tilde{\mathcal{P}}^{[d]}(u) + \sum_{d=2}^{\infty} \mathcal{P}^{[d]}\Big(\sum_{k=1}^{\infty} \tilde{\mathcal{P}}^{[k]}(u)\Big),$$

here $\mathcal{P}^{[d]}(v) = \hat{\mathcal{P}}^{[d]}(v, v, \ldots, v)$, where $\hat{\mathcal{P}}^{[d]}$ is the continuous symmetric d-linear mapping from $(E^\infty)^d$ into E^∞ that represents $\mathcal{P}^{[d]}$, hence

$$\sum_{d=2}^{\infty} \tilde{\mathcal{P}}^{[d]}(u) + \sum_{d=2}^{\infty} \hat{\mathcal{P}}^{[d]}\Big(\sum_{k_1=1}^{\infty} \tilde{\mathcal{P}}^{[k_1]}u, \ldots, \sum_{k_d=1}^{\infty} \tilde{\mathcal{P}}^{[k_d]}u\Big) = 0.$$

Then collecting the homogeneous terms of the same degree d, we have

$$\begin{aligned} \tilde{\mathcal{P}}^{[d]}(u) &= -\sum_{m=2}^{d} \Big(\sum_{k_1+\ldots+k_m=d} \hat{\mathcal{P}}^{[m]}(\tilde{\mathcal{P}}^{[k_1]}u, \ldots, \tilde{\mathcal{P}}^{[k_m]}u)\Big) \\ &= -\mathcal{P}^{[d]}(u) - \sum_{m=2}^{d-1} \Big(\sum_{k_1+\ldots+k_m=d} \hat{\mathcal{P}}^{[m]}(\tilde{\mathcal{P}}^{[k_1]}u, \ldots, \tilde{\mathcal{P}}^{[k_m]}u)\Big) \end{aligned} \tag{4.5}$$

for $d > 2$, and $\tilde{\mathcal{P}}^{[2]}(u) = -\mathcal{P}^{[2]}(u)$ when $d = 2$.

Using formula (4.5), we define recursively all homogeneous polynomials $\tilde{\mathcal{P}}^{[d]}(u)$ which are continuous from E^∞ to E^∞. Obviously $(\mathcal{P} \circ \tilde{\mathcal{P}})(u) = u$ for all $u \in E^\infty$. Similarly, one can find the formal right inverse $\tilde{\tilde{\mathcal{P}}}$ of (4.4) and can verify that $\tilde{\tilde{\mathcal{P}}} = \mathcal{P}$, hence $(\tilde{\mathcal{P}} \circ \mathcal{P})(\xi) = \xi$ for all $\xi \in E^\infty$.

We make the formal change of variable (4.2) in the Navier–Stokes equations and obtain the following formal relation

$$\begin{aligned} \frac{du}{dt} &= -Au - B(u, u) \\ &= -A\xi - \sum_{d=2}^{\infty} A\mathcal{P}^{[d]}(\xi) - \sum_{k,l=1}^{\infty} B(\mathcal{P}^{[k]}(\xi), \mathcal{P}^{[l]}(\xi))). \end{aligned} \tag{4.6}$$

Letting $u = u(t)$ and $\xi = \xi(t)$ in (4.4) and formally taking the derivative in t, we obtain

$$
\begin{aligned}
\frac{d\xi}{dt} &= \frac{du}{dt} + \sum_{d=2}^{\infty} \Big(D\tilde{\mathcal{P}}^{[d]}(u) \Big) \frac{du}{dt} \\
&= -A\xi - \sum_{d=2}^{\infty} A\mathcal{P}^{[d]}(\xi) - \sum_{k,l=1}^{\infty} B(\mathcal{P}^{[k]}(\xi), \mathcal{P}^{[l]}(\xi))) \\
&\quad - \sum_{d=2}^{\infty} \Big(D\tilde{\mathcal{P}}^{[d]}(u)\Big|_{u=\sum_{k=1}^{\infty} \mathcal{P}^{[k]}(\xi)} \Big) \Big(\sum_{l=1}^{\infty} A\mathcal{P}^{[l]}(\xi) \Big) \\
&\quad - \sum_{d=2}^{\infty} \Big(D\tilde{\mathcal{P}}^{[d]}(u)\Big|_{u=\sum_{m=1}^{\infty} \mathcal{P}^{[m]}(\xi)} \Big) \Big(\sum_{k,l=1}^{\infty} B(\mathcal{P}^{[k]}(\xi), \mathcal{P}^{[l]}(\xi)) \Big).
\end{aligned} \tag{4.7}
$$

Above, D applied to a polynomial $Q \in \mathcal{H}^{[d]}(E^\infty)$ indicates the (Fréchet) derivation of Q, that is

$$
(DQ(\xi))v = d\,\hat{Q}(v, \xi, \dots, \xi), \quad \xi \in E^\infty, \ v \in E^\infty. \tag{4.8}
$$

Using (4.8), we rewrite the sums in (4.7). For instance,

$$
\begin{aligned}
&\sum_{d=2}^{\infty} \Big(D\tilde{\mathcal{P}}^{[d]}(u)\Big|_{u=\sum_{m=1}^{\infty} \mathcal{P}^{[m]}(\xi)} \Big) \Big(\sum_{k,l=1}^{\infty} B(\mathcal{P}^{[k]}(\xi), \mathcal{P}^{[l]}(\xi)) \Big) \\
&= \sum_{d=2}^{\infty} d\,\widehat{\tilde{\mathcal{P}}^{[d]}} \Big(\sum_{k,l=1}^{\infty} B(\mathcal{P}^{[k]}(\xi), \mathcal{P}^{[l]}(\xi)), \sum_{m_2=1}^{\infty} \mathcal{P}^{[m_2]}(\xi), \dots, \sum_{m_d=1}^{\infty} \mathcal{P}^{[m_d]}(\xi) \Big) \\
&= \sum_{d=2}^{\infty} \sum_{k,l=1}^{\infty} \sum_{m_2,\dots,m_d=1}^{\infty} d\,\widehat{\tilde{\mathcal{P}}^{[d]}} \Big(B(\mathcal{P}^{[k]}(\xi), \mathcal{P}^{[l]}(\xi)), \mathcal{P}^{[m_2]}(\xi), \dots, \mathcal{P}^{[m_d]}(\xi) \Big).
\end{aligned} \tag{4.9}
$$

Note that each summand in the last expression has degree $k + l + m_2 + \ldots + m_d$. Hence each homogeneous component of a finite degree in (4.9) is only a finite sum of those summands. Similar calculations apply to other simpler sums in (4.7). By collecting the homogeneous terms of the same degree in (4.7), we then derive

$$
\frac{d\xi}{dt} + \sum_{d=1}^{\infty} Q^{[d]}(\xi) = 0, \tag{4.10}
$$

where $Q^{[d]}(\xi) \in \mathcal{H}^{[d]}(E^\infty)$, $d \geq 1$; however the series in (4.10) is still formal.

Although $Q^{[d]}(\xi)$ can be computed recursively in terms of $\mathcal{P}^{[m]}(\xi)$ and $\tilde{\mathcal{P}}^{[m]}(\xi)$, we will express it in terms of $\mathcal{P}^{[m]}(\xi)$ only. This will be helpful in our later considerations.

Obviously, we have $Q^{[1]}(\xi) = A\xi$. Up to degree $d \geq 2$ in ξ we formally calculate from (4.2), knowing that (4.10) already holds,

$$
\begin{aligned}
\frac{du}{dt} &= \frac{d\xi}{dt} + \sum_{m \geq 2} (D\mathcal{P}^{[m]}(\xi)) \frac{d\xi}{dt} \\
&= -A\xi - \sum_{d \geq 2} Q^{[d]}(\xi) - \sum_{k \geq 2} (D\mathcal{P}^{[k]}(\xi)) \Big(A\xi + \sum_{l \geq 2} Q^{[l]}(\xi) \Big).
\end{aligned}
$$

We substitute this into the left-hand side of (4.6) and collect the homogeneous polynomials of the same degree to derive $Q^{[d]}(\xi)$. Note that either $\mathcal{P}^{[m]} = 0$ or

$\deg\big((D\mathcal{P}^{[m]}(\xi))Q(\xi)\big) = m + n - 1$ for any $m \geq 1$ and $Q \in \mathcal{H}^{[n]}(E^\infty)$. Hence

$$Q^{[d]}(\xi) = A\mathcal{P}^{[d]}(\xi) - D\mathcal{P}^{[d]}(\xi)A\xi + \sum_{k+l=d} B(\mathcal{P}^{[k]}(\xi), \mathcal{P}^{[l]}(\xi)) - \sum_{\substack{k,l\geq 2\\ k+l=d+1}} (D\mathcal{P}^{[k]}(\xi))Q^{[l]}(\xi).$$

Therefore we obtain the recursive formula for $d \geq 2$:

$$Q^{[d]}(\xi) = H_A^{(d)}\mathcal{P}^{[d]}(\xi) + \sum_{k+l=d} B(\mathcal{P}^{[k]}(\xi), \mathcal{P}^{[l]}(\xi)) - \sum_{\substack{2\leq k,l\leq d-1\\ k+l=d+1}} (D\mathcal{P}^{[k]}(\xi))Q^{[l]}(\xi); \tag{4.11}$$

above $H_A^{(d)} : Q(\xi) \to AQ(\xi) - (DQ(\xi))A\xi$, where $Q \in \mathcal{H}^{[d]}(E^\infty)$, is the analogue of Poincaré homology operator of order d (c.f. see e.g. [1]) in the present framework. Clearly $H_A^{(d)}(\mathcal{H}^{[d]}(E^\infty)) \subset \mathcal{H}^{[d]}(E^\infty)$, for all $d \in \mathbb{N}$.

Now the equation (4.10) can be rewritten as

$$\frac{d\xi}{dt} + A\xi + \sum_{d=2}^{\infty} Q^{[d]}(\xi) = 0, \tag{4.12}$$

where the polynomials $Q^{[d]}(\xi)$, $d \geq 2$, are given in more explicit way by (4.11).

We will prove that $Q^{[d]}(\xi) = \mathcal{B}^{[d]}(\xi)$ for all $d \geq 2$ and $\xi \in E^\infty$.

The Poincaré homology operators have the following property.

Lemma 4.2. *For $d \geq 1$ the homology operator $H_A^{(d)}$ satisfies*

$$H_A^{(d)}\mathcal{P}^{[d]}(\xi) = \sum_{j=1}^{\infty}(A - j)\mathcal{P}_j^{[d]}(\xi) \text{ for all } \xi \in E^\infty. \tag{4.13}$$

For the proof we need the following preliminaries, which will also be useful later.

Definition 4.3. *Let $Q \in \mathcal{H}^{[d]}(E^\infty)$. Then $Q(\xi)$ ($\xi \in E^\infty$ and $\xi_j = R_j\xi$, $j \in \mathbb{N}$), is a monomial of degree $\alpha_{k_i} > 0$ in ξ_{k_i} where $i = 1, 2, \ldots, N$, $\alpha_{k_1} + \ldots + \alpha_{k_N} = d$ and $k_1 < k_2 < \ldots < k_N$, if it can be represented as*

$$Q(\xi) = \tilde{Q}(\underbrace{\xi_{k_1}, \ldots, \xi_{k_1}}_{\alpha_{k_1}}, \underbrace{\xi_{k_2}, \ldots, \xi_{k_2}}_{\alpha_{k_2}}, \ldots, \underbrace{\xi_{k_N}, \ldots, \xi_{k_N}}_{\alpha_{k_N}}), \tag{4.14}$$

where $\tilde{Q}(\xi^{(1)}, \xi^{(2)}, \ldots, \xi^{(d)})$ is a continuous d-linear map from $(E^\infty)^d$ to E^∞.

The monomial $Q(\xi)$ defined by (4.14), with degree $d \geq 2$, is called resonant *if $\sum_{i=1}^{N} \alpha_{k_i}k_i = j$ whenever $R_jQ \neq 0$.*

In later calculations, it will be convenient to have α_k defined for $Q(\xi)$ for all $k \in \mathbb{N}$, by defining the degree α_k in ξ_k to be zero if $k \notin \{k_1, \ldots, k_N\}$.

For $j \in \mathbb{N}$, let $N_j = \dim R_jH$ and $\{\varphi_{j,1}, \varphi_{j,2}, \ldots, \varphi_{j,N_j}\}$ be a fixed orthonormal basis of R_jH. Let $J(A) = \{(j,i) : j \in \sigma(A),\, 1 \leq i \leq N_j\}$. For any $\xi \in E^\infty$ we have the unique expansion

$$\xi = \sum_{(j,i)\in J(A)} x_{j,i}\varphi_{j,i}, \text{ where } x_{j,i} = \langle \xi, \varphi_{j,i}\rangle \in \mathbb{R}, \tag{4.15}$$

and the series is absolutely convergent in $\mathcal{D}(A^\alpha)$ for all $\alpha \ge 0$ hence converges in E^∞. Here $x_{j,i}$, with $(j,i) \in J(A)$, are the coordinates of ξ with respect to the orthonormal basis $\{\varphi_{j,i} : (j,i) \in J(A)\}$ of H. Then any function of ξ can be viewed as a function of $x = (x_{j,i})$.

Let $Q(\xi)$ be a monomial with representation (4.14). Using the preceding notation, we have

$$\begin{aligned} Q(\xi) &= \tilde{Q}(\underbrace{\xi_{k_1}, \dots, \xi_{k_1}}_{\alpha_{k_1}}, \underbrace{\xi_{k_2}, \dots, \xi_{k_2}}_{\alpha_{k_2}}, \dots, \underbrace{\xi_{k_d}, \dots, \xi_{k_d}}_{\alpha_{k_d}}) \\ &= \tilde{Q}\Big(\underbrace{\sum_{i=1}^{N_{k_1}} x_{k_1,i}\varphi_{k_1,i}, \dots, \sum_{i=1}^{N_{k_1}} x_{k_1,i}\varphi_{k_1,i}}_{\alpha_{k_1}}, \underbrace{\sum_{i=1}^{N_{k_2}} x_{k_2,i}\varphi_{k_2,i}, \dots, \sum_{i=1}^{N_{k_2}} x_{k_2,i}\varphi_{k_2,i}}_{\alpha_{k_2}}, \\ &\qquad \dots, \underbrace{\sum_{i=1}^{N_{k_d}} x_{k_d,i}\varphi_{k_d,i}, \dots, \sum_{i=1}^{N_{k_d}} x_{k_d,i}\varphi_{k_d,i}}_{\alpha_{k_d}}\Big). \end{aligned}$$

By the multi-linearity of $\tilde{Q}$:

$$\begin{aligned} Q(\xi) = &\sum_{i_1^{(1)},\dots,i_1^{(\alpha_{k_1})}=1}^{N_{k_1}} \ \sum_{i_2^{(1)},\dots,i_2^{(\alpha_{k_2})}=1}^{N_{k_2}} \dots \sum_{i_d^{(1)},\dots,i_d^{(\alpha_{k_d})}=1}^{N_{k_d}} \\ &\Big(x_{k_1,i_1^{(1)}} \dots x_{k_1,i_1^{(\alpha_{k_1})}}\, x_{k_2,i_2^{(1)}} \dots x_{k_2,i_2^{(\alpha_{k_2})}} \dots x_{k_d,i_d^{(1)}} \dots x_{k_d,i_d^{(\alpha_{k_d})}} \\ &\times \tilde{Q}(\varphi_{k_1,i_1^{(1)}}, \dots, \varphi_{k_1,i_1^{(\alpha_{k_1})}}, \varphi_{k_2,i_2^{(1)}}, \dots, \varphi_{k_2,i_2^{(\alpha_{k_2})}}, \dots, \varphi_{k_d,i_d^{(1)}}, \dots, \varphi_{k_d,i_d^{(\alpha_{k_d})}})\Big). \end{aligned}$$

Applying the expansion (4.15) to the term $\tilde{Q}(\dots)$ of the last expression, we have

$$\begin{aligned} Q(\xi) = &\sum_{(k,l)\in J(A)} \ \sum_{i_1^{(1)},\dots,i_1^{(\alpha_{k_1})}=1}^{N_{k_1}} \ \sum_{i_2^{(1)},\dots,i_2^{(\alpha_{k_2})}=1}^{N_{k_2}} \dots \sum_{i_d^{(1)},\dots,i_d^{(\alpha_{k_d})}=1}^{N_{k_d}} \\ &\Big(x_{k_1,i_1^{(1)}} \dots x_{k_1,i_1^{(\alpha_{k_1})}}\, x_{k_2,i_2^{(1)}} \dots x_{k_2,i_2^{(\alpha_{k_2})}} \dots x_{k_d,i_d^{(1)}} \dots x_{k_d,i_d^{(\alpha_{k_d})}} \\ &\times \Big\langle \tilde{Q}\big(\varphi_{k_1,i_1^{(1)}}, \dots, \varphi_{k_1,i_1^{(\alpha_{k_1})}}, \varphi_{k_2,i_2^{(1)}}, \dots, \varphi_{k_2,i_2^{(\alpha_{k_2})}}, \dots, \\ &\quad \varphi_{k_d,i_d^{(1)}}, \dots, \varphi_{k_d,i_d^{(\alpha_{k_d})}}\big), \varphi_{k,l} \Big\rangle \varphi_{k,l}\Big). \end{aligned} \tag{4.16}$$

Each summand in (4.16) above has the form

$$c\langle \xi, \varphi_{j_1,h_1}\rangle^{\alpha_{j_1,h_1}} \langle \xi, \varphi_{j_2,h_2}\rangle^{\alpha_{j_2,h_2}} \dots \langle \xi, \varphi_{j_n,h_n}\rangle^{\alpha_{j_n,h_n}} \varphi_{j_{n+1},h_{n+1}}, \quad \xi \in E^\infty, \tag{4.17}$$

where $c \in \mathbb{R}$, $(j_i, h_i) \in J(A)$ for $i = 1, 2, \dots, n+1$, $\alpha_{j_i,h_i} > 0$ for $i = 1, \dots, n$, and (j_i, h_i), $i = 1, 2, \dots, n$, are distinct.

Let $q(\xi)$ be the function of ξ defined by (4.17). Obviously $q(\xi)$ as a function of x is a monomial having degree α_{j_i,h_i} in x_{j_i,h_i}, for $i = 1, 2, \dots, n$. Again we define the degree $\alpha_{j,h}$ in $x_{j,h}$ ($(j,h) \in J(A)$) to be zero if $(j,h) \notin \{(j_i,h_i) : i = 1, \dots, n\}$. Since this type of monomials is used in the classical theory of normal form (see again [1]), we call the monomial in (4.17) classical monomial.

Also, $q(\xi)$ can be written as

$$q(\xi) = c\langle \xi_{j_1}, \varphi_{j_1,h_1}\rangle^{\alpha_{j_1,h_1}} \langle \xi_{j_2}, \varphi_{j_2,h_2}\rangle^{\alpha_{j_2,h_2}} \ldots \langle \xi_{j_n}, \varphi_{j_n,h_n}\rangle^{\alpha_{j_n,h_n}} \varphi_{j_{n+1},h_{n+1}}, \tag{4.18}$$

hence $q(\xi)$ is a monomial of degree

$$\alpha_j = \sum_{\{i: j_i = j\}} \alpha_{j_i,h_i} \text{ in } \xi_j \text{ for } j \in \{j_1, \ldots, j_n\}. \tag{4.19}$$

Therefore it is easy to prove that:

Lemma 4.4. *The classical monomial (4.17) is resonant if and only if*

$$j_{n+1} = \alpha_{j_1,h_1} j_1 + \alpha_{j_2,h_2} j_2 + \ldots + \alpha_{j_n,h_n} j_n. \tag{4.20}$$

Remark 4.5. Lemma 4.4 implies that for classical monomials, being resonant (in the sense of Definition 4.3) is the same as being resonant in the classical theory [1].

Lemma 4.6. *Let $Q(\xi)$ be a monomial. Then it is the sum of a convergent series of classical monomials. Moreover, if $Q(\xi)$ is resonant then those classical monomials can be chosen to be resonant.*

Proof. The first statement results directly from (4.16). Assume $Q(\xi)$ is resonant. Let $q(\xi) \neq 0$ be the summand in (4.16) which is also expressed as in (4.17) with $(j_{n+1}, h_{n+1}) = (k, l)$. Assume $q(\xi)$ has degree $\alpha_{j,i}$ in $x_{j,i}$ and has degree α_j in ξ_j $(j \in \sigma(A))$. Let

$$q_{k,l}(\xi) = \langle Q(\xi), \varphi_{k,l}\rangle \varphi_{k,l}, \quad \xi \in E^\infty. \tag{4.21}$$

We have from (4.19) that $\alpha_j = \sum_{i=1}^{N_j} \alpha_{j,i}$ which, by expression (4.16), equals the degree in ξ_j of the monomial $q_{k,l}(\xi)$ and hence, by expression (4.21), is also the degree in ξ_j of $Q(\xi)$. Since $q(\xi) \neq 0$, we have $\langle R_k Q(\xi), \varphi_{k,l}\rangle \neq 0$, thus $R_k Q \neq 0$. Then $Q(\xi)$ being resonant implies that $\sum_{j\in\sigma(A)} \alpha_j\, j = k$, therefore $q(\xi)$ is resonant. □

We now return to Lemma 4.2.

Proof of Lemma 4.2. Let $j \geq 1$. Since $\mathcal{P}_j^{[d]}(\xi)$ is a polynomial of $\xi_1, \ldots, \xi_j$ only, it can be expressed as a finite sum $\mathcal{P}_j^{[d]}(\xi) = \sum_{l=1}^s \mathcal{M}_l(\xi)$, for some $s > 0$, where for each l, the summand $\mathcal{M}_l(\xi)$ is a monomial in $\xi_1, \xi_2, \ldots, \xi_j$. Let $1 \leq l \leq s$ and $\mathcal{M}(\xi) = \mathcal{M}_l(\xi)$. Suppose $\mathcal{M}(\xi)$ has degree α_k in ξ_k, $k = 1, 2, \ldots, j$ and $\{k : \alpha_k \neq 0\} = \{k_1, k_2, \ldots, k_N\}$ for some $1 \leq N \leq j$ and $0 < k_1 < k_2 < \ldots < k_N$. Then $\mathcal{M}(\xi)$ can be represented as

$$\mathcal{M}(\xi) = \tilde{\mathcal{M}}(\underbrace{\xi_{k_1}, \ldots, \xi_{k_1}}_{\alpha_{k_1}}, \underbrace{\xi_{k_2}, \ldots, \xi_{k_2}}_{\alpha_{k_2}}, \ldots, \underbrace{\xi_{k_N}, \ldots, \xi_{k_N}}_{\alpha_{k_N}}), \tag{4.22}$$

where $\alpha_{k_1} + \ldots + \alpha_{k_N} = d$ and $\tilde{\mathcal{M}}(\xi^{(1)}, \xi^{(2)}, \ldots, \xi^{(d)})$ is a continuous d-linear map from $(E^\infty)^d$ to E^∞. We have

$$\begin{aligned}(D\mathcal{M}(\xi))A\xi &= \sum_{m=1}^{N} \Big(\tilde{\mathcal{M}}(\xi_{k_1}, \ldots, \xi_{k_1}, \ldots, A\xi_{k_m}, \xi_{k_m}, \ldots, \xi_{k_m}, \ldots, \xi_{k_N}, \ldots, \xi_{k_N})\\ &\quad + \ldots + \tilde{\mathcal{M}}(\xi_{k_1}, \ldots, \xi_{k_1}, \ldots, \xi_{k_m}, \ldots, \xi_{k_m}, A\xi_{k_m}, \ldots, \xi_{k_N}, \ldots, \xi_{k_N})\Big)\\ &= \sum_{m=1}^{N} \Big(\tilde{\mathcal{M}}(\xi_{k_1}, \ldots, \xi_{k_1}, \ldots, k_m\xi_{k_m}, \xi_{k_m}, \ldots, \xi_{k_m}, \ldots, \xi_{k_N}, \ldots, \xi_{k_N})\\ &\quad + \ldots + \tilde{\mathcal{M}}(\xi_{k_1}, \ldots, \xi_{k_1}, \ldots, \xi_{k_m}, \ldots, \xi_{k_m}, k_m\xi_{k_m}, \ldots, \xi_{k_N}, \ldots, \xi_{k_N})\Big)\\ &= \sum_{m=1}^{N} \alpha_{k_m} k_m \tilde{\mathcal{M}}(\xi_{k_1}, \ldots, \xi_{k_1}, \ldots, \xi_{k_m}, \ldots, \xi_{k_m}, \ldots, \xi_{k_N}, \ldots, \xi_{k_N}).\end{aligned}$$

Therefore $(D\mathcal{M}(\xi))A\xi = \sum_{k=1}^{j} k\alpha_k \mathcal{M}(\xi) = j\mathcal{M}(\xi)$, thanks to Lemma 3.1(ii). Thus $(DP_j^{[d]}(\xi))A\xi = jP_j^{[d]}(\xi)$ and consequently we have

$$A\mathcal{P}_j^{[d]}(\xi) - D\mathcal{P}_j^{[d]}(A\xi) = (A-j)\mathcal{P}_j^{[d]}(\xi). \tag{4.23}$$

By virtue of Theorem 3.4, summing up (4.23) in j yields (4.13). □

Our key result is the following relation.

Proposition 4.7. *One has*

$$Q^{[d]} = \mathcal{B}^{[d]} \text{ on } E^\infty \text{ for all } d \geq 2. \tag{4.24}$$

Proof. We prove (4.24) in two steps.

Step 1: Let $d \geq 2$. It is proved in Proposition 2.2(ii) of [9] that

$$(A-j)\mathcal{P}_j(\xi) + \sum_{k+l=j} B(\mathcal{P}_k(\xi), \mathcal{P}_l(\xi)) = (D\mathcal{P}_j(\xi))\Big(\sum_{k=2}^{j} \mathcal{B}_k(\xi)\Big). \tag{4.25}$$

Collecting the homogeneous terms of degree d in ξ from (4.25) gives

$$\begin{aligned}&(A-j)\mathcal{P}_j^{[d]}(\xi) + \sum_{k+l=j} \sum_{m+n=d} B(\mathcal{P}_k^{[m]}(\xi), \mathcal{P}_l^{[n]}(\xi)))\\ &\qquad = \sum_{k=2}^{j} \sum_{\substack{n\geq 2\\ m+n=d+1}} (D\mathcal{P}_j^{[m]}(\xi))\mathcal{B}_k^{[n]}(\xi) = \sum_{\substack{n\geq 2\\ m+n=d+1}} (D\mathcal{P}_j^{[m]}(\xi))\mathcal{B}^{[n]}(\xi).\end{aligned}$$

The last identity is due to the fact that $\mathcal{P}_j$ is a polynomial of $\xi_1, \xi_2, \ldots, \xi_j$ only. In the last sum, $(D\mathcal{P}_j^{[m]}(\xi))\mathcal{B}^{[n]}(\xi)$ equals $R_j\mathcal{B}^{[d]}(\xi)$ when $m = 1$, $n = d$ (see (3.16)), hence

$$\begin{aligned}R_j\mathcal{B}^{[d]}(\xi) = (A-j)\mathcal{P}_j^{[d]}(\xi) &+ \sum_{m+n=d} \sum_{k+l=j} B(\mathcal{P}_k^{[m]}(\xi), \mathcal{P}_l^{[n]}(\xi)))\\ &- \sum_{\substack{2\leq m,n\leq d-1\\ m+n=d+1}} (D\mathcal{P}_j^{[m]}(\xi))\mathcal{B}^{[n]}(\xi).\end{aligned}$$

Summing up in j and applying Lemma 4.2, Theorems 3.4 and 3.5, we obtain

$$\mathcal{B}^{[d]}(\xi) = H_A\mathcal{P}^{[d]}(\xi) + \sum_{m+n=d} B(\mathcal{P}^{[m]}(\xi), \mathcal{P}^{[n]}(\xi))) - \sum_{\substack{2\le m,n\le d-1\\ m+n=d+1}} (D\mathcal{P}^{[m]}(\xi))\mathcal{B}^{[n]}(\xi). \tag{4.26}$$

Step 2: We now prove (4.24) by induction in d. First, by (4.26) and (4.11), we have

$$\mathcal{B}^{[2]}(\xi) = H_A\mathcal{P}^{[d]}(\xi) + B(\mathcal{P}^{[1]}(\xi), \mathcal{P}^{[1]}(\xi))) = Q^{[2]}(\xi).$$

Let $d \ge 3$. Suppose $\mathcal{B}^{[k]}(\xi) = Q^{[k]}(\xi)$ for all $2 \le k < d$. Then $\mathcal{B}^{[n]}(\xi) = Q^{[n]}(\xi)$ in (4.26), and therefore by comparing (4.26) with (4.11) we obtain $\mathcal{B}^{[d]}(\xi) = Q^{[d]}(\xi)$.

The proof is now complete. □

We recall that the Poincaré–Dulac theory first deals with the equations in $\mathbb{R}^n$ of the form

$$\frac{dx}{dt} + Ax + \Phi^{[2]}(x) + \Phi^{[3]}(x) + \ldots = 0,\ x \in \mathbb{R}^n, \tag{4.27}$$

here A is a linear operator from $\mathbb{R}^n$ to $\mathbb{R}^n$, each $\Phi^{[d]}$ is a homogeneous polynomial of degree d from $\mathbb{R}^n$ to $\mathbb{R}^n$, and the series is formal. In this case using an ingenious iteration of particular formal changes of variable H. Poincaré proved that there exists a formal series $y = x+\sum_{d=1}^{\infty} \Psi^{[d]}(x)$, where $\Psi^{[d]}$ is a homogeneous polynomial of degree d from $\mathbb{R}^n$ to $\mathbb{R}^n$, which transforms (4.27) into an equation

$$\frac{dy}{dt} + Ay + \Theta^{[2]}(y) + \Theta^{[3]}(y) + \ldots = 0,\ y \in \mathbb{R}^n, \tag{4.28}$$

where all monomials of each $\Theta^{[d]}$ are resonant (see [1] for the definition of these concepts for this case). The following remarks are now in order. First, in this case the concepts of monomial for (4.28) and our classical monomial coincide. Second, the definition of resonant monomial for (4.28) coincides with that of our resonant classical monomial (see Remark 4.5). Third, today (4.28) is called a Poincaré–Dulac normal form of (4.27).

Based on these considerations, we set the following definition.

Definition 4.8. *A differential equation in* E^∞

$$\frac{d\xi}{dt} + A\xi + \sum_{d=2}^{\infty} \Phi^{[d]}(\xi) = 0 \tag{4.29}$$

is a Poincaré–Dulac normal form for the Navier–Stokes equations (1.5) if

(i) Each $\Phi^{[d]} \in \mathcal{H}^{[d]}(E^\infty)$ *and* $\Phi^{[d]}(\xi) = \sum_{k=1}^{\infty} \Phi_k^{[d]}(\xi)$, *where all* $\Phi_k^{[d]} \in \mathcal{H}^{[d]}(E^\infty)$ *are resonant monomials,*

(ii) Equation (4.29) is obtained from (1.5) by a formal change of variable $u = \sum_{d=1}^{\infty} \Psi^{[d]}(\xi)$ *where* $\Psi^{[d]} \in \mathcal{H}^{[d]}(E^\infty)$.

We can now synthesize the previous results in the following:

Theorem 4.9. *The formal power series change of variable (4.2) reduces the Navier–Stokes equations (1.5) to a Poincaré–Dulac normal form, namely (1.16).*

Proof. The formal change of variable (4.2) transforms the Navier–Stokes equations into (4.10) which coincides with (1.16), by Proposition 4.7. For $d \geq 2$, we have from Lemma 4.1 that $\mathcal{B}^{[d]} \in \mathcal{H}^{[d]}(E^\infty)$ and

$$\mathcal{B}^{[d]}(\xi) = \sum_{j \in \sigma(A)} \sum_{k+l=j} \sum_{m+n=d} R_j B(\mathcal{P}_k^{[m]}(\xi), \mathcal{P}_l^{[n]}(\xi)). \tag{4.30}$$

Each $R_j B(\mathcal{P}_k^{[m]}(\xi), \mathcal{P}_l^{[n]}(\xi))$ above, in turn, is a finite sum of monomials, say, having degree α_k in ξ_k for $k \in \sigma(A)$, $1 \leq k \leq j$. By virtue of Lemma 3.1(iii),

$$\sum_{k \in \sigma(A), k \leq j, \alpha_k > 0} \alpha_k \cdot k = j,$$

hence each monomial of such is resonant. Therefore (1.16) is a Poincaré–Dulac normal form for the Navier–Stokes equations (1.5). □

Remark 4.10. The classical Poincaré–Dulac theory has a second part concerning the convergence of the series which give the change of variable, and the convergence of the series in the normal form (c.f. [1]). We have not yet been able to extend this part of the theory to the Navier–Stokes equations.

Remark 4.11. Our particular Poincaré–Dulac normal form (1.16) is uniquely determined by the asymptotic expansion (1.6) and its associated normal form (1.14) in S_A. (Indeed, each summand $\mathcal{B}^{[d]}$ in (1.16) is obtained by collecting the homogeneous terms of degree d in (1.14), see (4.30).) It is still open whether (1.16) is Brjuno's distinguished normal form [2].

Remark 4.12. The methods developed in this paper can be applied to other dissipative partial differential equations with quadratic nonlinearity and a singleton global attractor. In particular, they apply directly to the 2D Navier–Stokes equations, since the latter are the restriction of the 3D ones to an appropriate invariant linear manifold.

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