

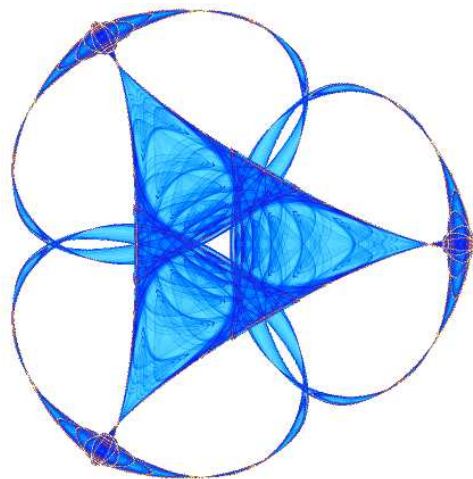
**UNIFORM CONVERGENCE OF LOCAL MULTIGRID METHODS FOR
THE TIME-HARMONIC MAXWELL EQUATION**

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UNIFORM CONVERGENCE OF LOCAL MULTIGRID METHODS FOR THE TIME-HARMONIC MAXWELL EQUATION

HUANGXIN CHEN*, RONALD H.W. HOPPE†, AND XUEJUN XU*

Abstract. For the efficient numerical solution of indefinite linear systems arising from curl conforming edge element approximations of the time-harmonic Maxwell equation, we consider local multigrid methods (LMM) on adaptively refined meshes. The edge element discretization is done by the lowest order edge elements of Nédélec's first family. The LMM features local hybrid Hiptmair smoothers of Jacobi and Gauss-Seidel type which are performed only on basis functions associated with newly created edges/nodal points or those edges/nodal points where the support of the corresponding basis function has changed during the refinement process. The adaptive mesh refinement is based on Dörfler marking for residual-type a posteriori error estimators and the newest vertex bisection strategy. Using the abstract Schwarz theory of multilevel iterative schemes, quasi-optimal convergence of the LMM is shown, i.e., the convergence rates are independent of mesh sizes and mesh levels provided the coarsest mesh is chosen sufficiently fine. The theoretical findings are illustrated by the results of some numerical examples.

Key words. Maxwell equations, Nédélec edge elements, indefinite, multigrid methods, local Hiptmair smoothers, adaptive edge finite element methods, optimality

1. Introduction. In this paper, we develop, analyze, and implement local multigrid methods for indefinite algebraic systems arising from adaptive curl-conforming edge element approximations of the time-harmonic Maxwell equation. In particular, we consider a lossless medium occupying a bounded Lipschitz polyhedron $\Omega \subset \mathbb{R}^3$ with a perfectly conducting boundary $\partial\Omega$. Given a solenoidal current density \mathbf{f} , the problem is to compute a time-harmonic electric field \mathbf{u} in Ω with wave number $\kappa > 0$ such that

$$\mathbf{curl} \mathbf{curl} \mathbf{u} - \kappa^2 \mathbf{u} = \mathbf{f} \quad \text{in } \Omega, \tag{1.1a}$$

$$\mathbf{u} \times \mathbf{n} = \mathbf{0} \quad \text{on } \partial\Omega, \tag{1.1b}$$

where \mathbf{n} stands for the unit outward normal on $\partial\Omega$. The choice of the boundary condition (1.1b) is made for ease of presentation only. Similar results are valid for other types of boundary conditions as well. Under the assumption that κ^2 is not a Maxwell eigenvalue, i.e., κ^2 is not an eigenvalue of the **curl-curl** operator, it is well-known that (1.1a),(1.1b) has a unique solution (cf. e.g., [28], [31, Chap. 4]).

Curl-conforming edge elements, originally known as Whitney forms [40] and designed to study the multiplicity of zero as an eigenvalue of the Hodge Laplacian, have been introduced to the numerical analysis community by Nédélec [32, 33] and since then have become a standard tool in computational electromagnetism (cf. [8, 21, 31] and the references therein). An intrinsic difficulty in the numerical solution of edge element discretized PDEs involving the **curl-curl** operator is the non-trivial kernel of the discrete **curl** operator which is given by the gradients of the standard nodal basis functions. Within the multigrid iterative solution, this has been taken care of by

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hybrid smoothers, namely the Hiptmair smoother [20] and the Arnold-Falk-Winther smoother [3] which have been originally designed for $\mathbf{H}(\mathbf{curl}; \Omega)$ -elliptic problems. Adaptive edge finite element methods (AEFEM) on the basis of residual-type a posteriori error estimators have been developed first in [6, 7, 30] and further studied in [12, 14, 25, 45]. Quasi-optimal convergence of AEFEM for the time-harmonic Maxwell equations has been established recently in [46].

In this paper, we are interested in local multigrid methods (LMM) on adaptively refined meshes obtained from the application of AEFEM to the time-harmonic Maxwell equation (1.1a),(1.1b) and to prove uniform convergence of the LMM which together with [46] results in an overall quasi-optimal algorithm. LMM on adaptively refined meshes feature hybrid smoothing only on new edges/nodes and those old edges/nodes where the support of the associated edge/nodal basis function has changed. This strategy makes the computational cost on each level of the LMM proportional to the number of elements appearing in the local refinement. The idea can be traced back to multilevel adaptive techniques (MLAT) studied in [5, 11, 34] and multigrid methods for locally refined finite element meshes [1, 2, 16, 36]. However, these locally refined meshes obey restrictive conditions which are not satisfied by the newest vertex bisection algorithm which will be used for adaptivity in this paper. The uniform convergence theory of LMM for 2D and 3D $H^1(\Omega)$ -elliptic problems has been studied in [22, 41, 43, 44]. The hierarchy of meshes used in the LMM can be obtained either by successive adaptive refinement of an initial coarse mesh or by successive coarsening of a fine mesh. Recently, Hiptmair, Zheng et al. [22, 23], and Xu, Chen and Nochetto [42] have developed LMM based on a different strategy for the construction of hierarchies of meshes and have succeeded to establish uniform convergence in case of $\mathbf{H}(\mathbf{curl}; \Omega)$ -elliptic problems. We emphasize that in our algorithms we do not reconstruct a virtual refinement hierarchy of meshes, but use the hierarchy generated by the AEFEM. For time-harmonic Maxwell problems, LMM with hybrid smoothers have been studied numerically in [6, 15, 26, 37]. The computational results in these papers indicate efficiency and robustness of the approach. But so far, there does not exist any theoretical investigation in the literature. In this paper, using the methodology developed in [13, 19], we present a convergence analysis which is based on a perturbation of the estimates for $\mathbf{H}(\mathbf{curl}; \Omega)$ -elliptic problems. In our analysis, we apply the techniques from [18, 19] and show that LMM with additive local Hiptmair-Jacobi smoothers or multiplicative local Hiptmair-Gauss-Seidel smoothers converge uniformly provided that the coarsest grid is chosen sufficiently fine, a condition that seems to be unavoidable in the current numerical solution of time-harmonic Maxwell equations. The main difficulties in the convergence analysis are

- how to apply the perturbation analysis and a $\mathbf{H}(\mathbf{curl}; \Omega)$ -elliptic stable multilevel decomposition of the edge element space to obtain the estimate **(A1)** in section 3;
- how to apply a global strengthened Cauchy-Schwarz inequality with respect to this decomposition to get the estimate **(A3)**;
- how to get a global spectral estimate.

The remainder of this paper is organized as follows. In section 2, we introduce the weak formulation and the edge finite element approximation of (1.1a),(1.1b), and address the LMM featuring additive and multiplicative local Hiptmair smoothers. The convergence theory of the LMM is developed in section 3 within the abstract framework of the Schwarz theory of multilevel iterative schemes, whereas section 4 is devoted to the verification of the assumptions required by the abstract theory for

the local Hiptmair smoothers. In the final section 5, we present the results of some numerical experiments illustrating the performance of the LMM and exemplifying our theoretical results.

2. Edge element approximation and local multigrid methods. Throughout this paper, we adopt standard notation from Lebesgue and Sobolev space theory (cf., e.g., [38]). In particular, we refer to $L^2(\Omega)$ and $H^m(\Omega)$, $m \in \mathbb{N}$, as the Hilbert space of Lebesgue integrable functions in Ω and the Sobolev space of L^2 -functions with L^2 -integrable weak derivatives up to order m . For broken $s \in \mathbb{R}_+$, the Sobolev space $H^s(\Omega)$ is defined by interpolation. Likewise, $\mathbf{L}^2(\Omega)$ and $\mathbf{H}^s(\Omega)$ stand for the corresponding Hilbert spaces of vector-valued functions. In both cases, the inner products and associated norms will be denoted by $(\cdot, \cdot)_{s,\Omega}$ and $\|\cdot\|_{s,\Omega}$, respectively. For a function $v \in H^s(\Omega)$, we denote by $v|_{\partial\Omega}$ the trace of v on $\partial\Omega$ and define $H_0^s(\Omega) := \{v \in H^s(\Omega) \mid v|_{\partial\Omega} = 0\}$. Moreover, we denote by $\mathbf{H}(\mathbf{curl}; \Omega) := \{\mathbf{v} \in \mathbf{L}^2(\Omega) \mid \mathbf{curl} \mathbf{v} \in \mathbf{L}^2(\Omega)\}$ and $\mathbf{H}(\text{div}; \Omega) := \{\mathbf{v} \in \mathbf{L}^2(\Omega) \mid \text{div} \mathbf{v} \in L^2(\Omega)\}$ the Hilbert spaces of vector-valued functions with the inner products $(\cdot, \cdot)_{\text{curl},\Omega}$, $(\cdot, \cdot)_{\text{div},\Omega}$ and associated norms $\|\cdot\|_{\text{curl},\Omega}$, $\|\cdot\|_{\text{div},\Omega}$. We further refer to $\mathbf{H}_0(\mathbf{curl}; \Omega) := \{\mathbf{v} \in \mathbf{H}(\mathbf{curl}; \Omega) \mid (\mathbf{v} \times \mathbf{n})|_{\partial\Omega} = 0\}$ as the subspace of vector fields with vanishing tangential trace on $\partial\Omega$ and to $\mathbf{H}(\text{div}^0; \Omega) := \{\mathbf{v} \in \mathbf{H}(\text{div}; \Omega) \mid \text{div} \mathbf{v} = 0\}$ as the subspace of solenoidal vector fields.

For a given $\mathbf{f} \in \mathbf{H}(\text{div}^0; \Omega)$, the weak formulation of (1.1a) and (1.1b) is to find $\mathbf{u} \in \mathbf{H}_0(\mathbf{curl}; \Omega)$ such that

$$\mathbf{a}(\mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}), \quad \mathbf{v} \in \mathbf{H}_0(\mathbf{curl}; \Omega), \quad (2.1)$$

where the bilinear form $\mathbf{a} : \mathbf{H}_0(\mathbf{curl}; \Omega) \times \mathbf{H}_0(\mathbf{curl}; \Omega) \rightarrow \mathbb{R}$ is given by

$$\mathbf{a}(\mathbf{u}, \mathbf{v}) = (\mathbf{curl} \mathbf{u}, \mathbf{curl} \mathbf{v})_{0,\Omega} - \kappa^2(\mathbf{u}, \mathbf{v})_{0,\Omega}, \quad \mathbf{u}, \mathbf{v} \in \mathbf{H}_0(\mathbf{curl}; \Omega).$$

We further introduce a symmetric positive definite form $\hat{\mathbf{a}}(\cdot, \cdot)$ according to

$$\hat{\mathbf{a}}(\mathbf{u}, \mathbf{v}) := (\mathbf{u}, \mathbf{v})_{\text{curl},\Omega}, \quad \mathbf{u}, \mathbf{v} \in \mathbf{H}_0(\mathbf{curl}; \Omega).$$

For any subdomain $D \subset \Omega$, we define the associated energy norm by $\|\cdot\|_{\hat{\mathbf{a}},D} := \hat{\mathbf{a}}(\cdot, \cdot)|_D$. We omit the subscript D when $D = \Omega$.

Let $\{\mathcal{T}_l, l = 0, 1, \dots, L\}$ be a shape regular family of nested geometrically conforming simplicial triangulations of the computational domain Ω obtained by successive refinement of a sufficiently fine initial coarse mesh \mathcal{T}_0 using newest vertex bisection. The initial mesh size is scaled such that $h_0 < 1$. We define \mathcal{E}_l as the set of edges on \mathcal{T}_l and \mathcal{N}_l as the set of interior nodes of \mathcal{T}_l . We further refer to $\Omega_l^{\mathcal{P}}$ as the union of elements in \mathcal{T}_l containing $\mathbf{p} \in \mathcal{N}_l$ and to $\Omega_l^{\mathcal{E}}$ as the union of elements in \mathcal{T}_l containing $E \in \mathcal{E}_l$. $\mathcal{T}_l(\Omega_l^{\mathcal{P}}) \subset \mathcal{T}_l$ and $\mathcal{T}_l(\Omega_l^{\mathcal{E}}) \subset \mathcal{T}_l$ denote the sets of elements contained in $\Omega_l^{\mathcal{P}}$ and $\Omega_l^{\mathcal{E}}$, respectively. The quantities $h_{l,\mathbf{p}}, h_{l,E}$ stand for the diameters of the subdomains $\Omega_l^{\mathcal{P}}, \Omega_l^{\mathcal{E}}$, and for any tetrahedron $T \in \mathcal{T}_l$, h_T refers to the diameter of T . Moreover, $\mathcal{G}(T)$ is the number of bisections needed to generate T from an element in \mathcal{T}_0 . It is reasonable to assume [4]

$$C_d \theta^m \leq h_T \leq C_u \theta^m, \quad m = \mathcal{G}(T), \quad \forall T \in \bigcup_{l=0}^L \mathcal{T}_l, \quad (2.2)$$

where $0 < \theta < 1$, and C_d, C_u are positive constants that only depend on \mathcal{T}_0 and the shape regularity of the meshes. Throughout this paper, $\#S$ denotes the cardinality of

a set S , and C , with or without subscript, denotes a generic positive constant. This constant, depending on the wave number κ and the shape regularity of the meshes, can take on different values in different occurrences but will always be independent of mesh sizes and mesh levels.

For $l = 0, \dots, L$, let \mathbf{U}_l denote the curl-conforming edge element space generated by the lowest order edge elements of Nédélec's first family [32] with respect to the mesh \mathcal{T}_l . Since the meshes are nested, we have a sequence of nested edge element spaces $\mathbf{U}_0 \subset \mathbf{U}_1 \subset \dots \subset \mathbf{U}_L$. The finite element approximation of (2.1) is to find $\mathbf{u}_l \in \mathbf{U}_l$ such that

$$\mathbf{a}(\mathbf{u}_l, \mathbf{v}_l) = (\mathbf{f}, \mathbf{v}_l), \quad \mathbf{v}_l \in \mathbf{U}_l. \quad (2.3)$$

Under the assumption that $\max_{T \in \mathcal{T}_l} h_T$ is sufficiently small, existence and uniqueness of the solution \mathbf{u}_l are well-known [21, 31]. In particular, the projector

$$\mathbf{a}(\mathbf{P}_l \mathbf{v}, \mathbf{w}) = \mathbf{a}(\mathbf{v}, \mathbf{w}), \quad \mathbf{v} \in \mathbf{U}_L, \mathbf{w} \in \mathbf{U}_l, 0 \leq l \leq L,$$

is well defined. We further denote by $\mathbf{Q}_l : \mathbf{L}^2(\Omega) \rightarrow \mathbf{U}_l$ the L^2 -projector onto \mathbf{U}_l . By a similar technique as in [18, Lemma 4.3], the imbedding results in [31, Theorem 3.50 and Corollary 3.51], and the estimate in [31, Lemma 7.6], for discrete divergence-free vector fields we have the following estimate:

LEMMA 2.1. *Let Ω be a bounded Lipschitz polyhedral domain. If the initial mesh size h_0 is sufficiently small, there exists a constant $s \in (1/2, 1]$, depending only on the domain Ω , such that for any $\mathbf{v} \in \mathbf{U}_L$ and $\mathbf{w}_0 \in \mathbf{U}_0$ there holds*

$$(\mathbf{v} - \mathbf{P}_0 \mathbf{v}, \mathbf{w}_0) \leq Ch_0^s \|\mathbf{v} - \mathbf{P}_0 \mathbf{v}\|_{\tilde{\mathbf{a}}} \|\mathbf{w}_0\|_{\tilde{\mathbf{a}}}. \quad (2.4)$$

For $0 \leq l \leq L$, we define the level l operator $\mathbf{A}_l : \mathbf{U}_l \rightarrow \mathbf{U}_l$ according to

$$(\mathbf{A}_l \mathbf{v}, \mathbf{w}) = \mathbf{a}(\mathbf{v}, \mathbf{w}), \quad \mathbf{v}, \mathbf{w} \in \mathbf{U}_l.$$

and refer to $\mathbf{f}_l \in \mathbf{U}_l$ as the \mathbf{L}^2 -projection of \mathbf{f} onto \mathbf{U}_l . The level l edge element approximation of (2.1) reads as follows: Find $\mathbf{u}_l \in \mathbf{U}_l$ such that

$$\mathbf{A}_l \mathbf{u}_l = \mathbf{f}_l. \quad (2.5)$$

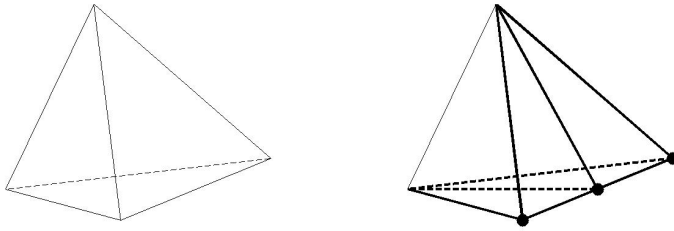


FIG. 2.1. The left figure is a tetrahedron in \mathcal{T}_{l-1} to be refined. The right figure shows that the tetrahedron is bisected into two tetrahedra in \mathcal{T}_l . The big vertices in the right figure are the local smoothing vertices contained in \tilde{N}_l , and the boldfaced edges are the local smoothing edges contained in $\tilde{\mathcal{E}}_l$.

We now consider local Hiptmair smoothers which smooth with respect to both edge basis functions and the gradients of nodal basis functions. The local smoothers are generalized Jacobi or Gauss-Seidel iterations with respect to appropriate subspace decompositions (cf. [22, 23]). For $1 \leq l \leq L$, denoting by \mathbf{b}_l^E the edge basis function associated with $E \in \mathcal{E}_l$ and by $\varphi_l^{\mathbf{p}}$ the nodal basis function with supporting point $\mathbf{p} \in \mathcal{N}_l$, we define the set of edges $\tilde{\mathcal{E}}_l$ and the set of vertices $\tilde{\mathcal{N}}_l$ (see Fig. 2.1) on which the local smoothers are performed as follows:

$$\begin{aligned}\tilde{\mathcal{E}}_l &= \{E \in \mathcal{E}_l : E \in \mathcal{E}_l \setminus \mathcal{E}_{l-1} \text{ or } E \in \mathcal{E}_{l-1} \text{ but } \text{supp}(\mathbf{b}_l^E) \neq \text{supp}(\mathbf{b}_{l-1}^E)\}, \\ \tilde{\mathcal{N}}_l &= \{\mathbf{p} \in \mathcal{N}_l : \mathbf{p} \in \mathcal{N}_l \setminus \mathcal{N}_{l-1} \text{ or } \mathbf{p} \in \mathcal{N}_{l-1} \text{ but } \text{supp}(\varphi_l^{\mathbf{p}}) \neq \text{supp}(\varphi_{l-1}^{\mathbf{p}})\}.\end{aligned}$$

In fact, $\tilde{\mathcal{E}}_l$ ($\tilde{\mathcal{N}}_l$) is the set of new edges (vertices) and those local edges (vertices) where the support of the basis function has changed.

For the edge element space \mathbf{U}_L we consider the following subspace decomposition (cf. [22, 23]):

$$\mathbf{U}_L = \mathbf{U}_0 + \sum_{l=1}^L \left(\sum_{\mathbf{p} \in \tilde{\mathcal{N}}_l} \text{Span}\{\nabla \varphi_l^{\mathbf{p}}\} + \sum_{E \in \tilde{\mathcal{E}}_l} \text{Span}\{\mathbf{b}_l^E\} \right), \quad (2.6)$$

which follows from the discrete Helmholtz decomposition of the edge element space and the corresponding subspace decompositions (cf. [24]). For ease of notation, for $1 \leq l \leq L$ we write

$$\bigcup_{i=1}^{N_l} \text{Span}\{\nabla \varphi_l^i\} = \bigcup_{\mathbf{p} \in \tilde{\mathcal{N}}_l} \text{Span}\{\nabla \varphi_l^{\mathbf{p}}\} \quad \text{and} \quad \bigcup_{i=1}^{M_l} \text{Span}\{\mathbf{b}_l^i\} = \bigcup_{E \in \tilde{\mathcal{E}}_l} \text{Span}\{\mathbf{b}_l^E\},$$

where $N_l = \#\tilde{\mathcal{N}}_l$ and $M_l = \#\tilde{\mathcal{E}}_l$. We define the local subspace \mathbf{U}_l^i according to

$$\mathbf{U}_l^i = \begin{cases} \text{Span}\{\nabla \varphi_l^i\}, & i = 1, \dots, N_l, \\ \text{Span}\{\mathbf{b}_l^j\}, & i = N_l + j, j = 1, \dots, M_l. \end{cases}$$

Let Ω_l^i be the support of the basis function spanning \mathbf{U}_l^i and $h_{l,i} = \text{diam}(\Omega_l^i)$. Then, for any $\mathbf{w} \in \mathbf{U}_l^i$, $i = N_l + 1, \dots, N_l + M_l$, there holds (cf. [19, Lemma 3.1])

$$\|\mathbf{w}\|_{0,\Omega} \leq Ch_{l,i} \|\mathbf{curl} \mathbf{w}\|_{0,\Omega}. \quad (2.7)$$

We further introduce the local projector $\mathbf{P}_l^i : \mathbf{U}_L \rightarrow \mathbf{U}_l^i$ according to

$$\mathbf{a}(\mathbf{P}_l^i \mathbf{v}, \psi_l^i) = \mathbf{a}(\mathbf{v}, \psi_l^i), \quad \mathbf{v} \in \mathbf{U}_L, \psi_l^i \in \mathbf{U}_l^i. \quad (2.8)$$

For $i = N_l + 1, \dots, N_l + M_l$, (2.7) and (2.8) imply that \mathbf{P}_l^i is well defined for sufficiently small $h_{l,i}$ and satisfies (cf. [19, Proposition 3.1])

$$\|\mathbf{P}_l^i \mathbf{v}\|_{\tilde{\mathbf{a}}, \Omega_l^i} \leq C \|\mathbf{v}\|_{\tilde{\mathbf{a}}, \Omega_l^i}, \quad \mathbf{v} \in \mathbf{U}_L. \quad (2.9)$$

In particular, for $i = 1, \dots, N_l$ we have $\|\mathbf{P}_l^i \mathbf{v}\|_0^2 = (v, \mathbf{P}_l^i \mathbf{v})$ and hence, (2.9) also holds true. Moreover, for sufficiently small $h_{l,i}$ the following estimate is valid (cf. [19, Lemma 4.4]):

$$(v - \mathbf{P}_l^i v, \mathbf{w}_l^i) \leq Ch_{l,i} \|v - \mathbf{P}_l^i v\|_{0, \Omega_l^i} \|\mathbf{curl} \mathbf{w}_l^i\|_{0, \Omega_l^i}, \quad v \in \mathbf{U}_L, \mathbf{w}_l^i \in \mathbf{U}_l^i. \quad (2.10)$$

Obviously, (2.10) follows from (2.7) and the Cauchy-Schwarz inequality, if $i = N_l + 1, \dots, N_l + M_l$, whereas for $i = 1, \dots, N_l$ both sides of (2.10) are zero.

We define $\mathbf{A}_l^i : \mathbf{U}_l^i \rightarrow \mathbf{U}_l^i$ by

$$(\mathbf{A}_l^i \mathbf{v}, \psi_l^i) = \mathbf{a}(\mathbf{v}, \psi_l^i), \quad \psi_l^i, \mathbf{v} \in \mathbf{U}_l^i,$$

and refer to $\mathbf{Q}_l^i : \mathbf{U}_L \rightarrow \mathbf{U}_l^i$ as the local L^2 projector. Clearly, (2.8) and (2.9) also imply the invertibility of \mathbf{A}_l^i . We note that $\hat{\mathbf{A}}_l, \hat{\mathbf{P}}_l, \hat{\mathbf{A}}_l^i$ and $\hat{\mathbf{P}}_l^i$ can be defined analogously by using $\hat{\mathbf{a}}(\cdot, \cdot)$ instead of $\mathbf{a}(\cdot, \cdot)$. For notational ease we set $\mathbf{U}_0 := \mathbf{U}_0^{N_0+M_0}$.

Let $\mathbf{R}_l^J : \mathbf{U}_l \rightarrow \mathbf{U}_l$ be the local Hiptmair-Jacobi smoother which performs Jacobi relaxations on the edges in $\tilde{\mathcal{E}}_l$ and at the vertices in $\tilde{\mathcal{N}}_l$, and let $\mathbf{R}_l^G : \mathbf{U}_l \rightarrow \mathbf{U}_l$ be the local Hiptmair-Gauss-Seidel smoother which performs Gauss-Seidel relaxations on the edges in $\tilde{\mathcal{E}}_l$ and at the vertices in $\tilde{\mathcal{N}}_l$, $1 \leq l \leq L$. Moreover, we set $\mathbf{R}_0^J = \mathbf{R}_0^G = \mathbf{A}_0^{-1}$ on the initial mesh \mathcal{T}_0 . Then \mathbf{R}_l^J defines an additive smoother (cf. [9])

$$\mathbf{R}_l^J := \gamma \sum_{i=1}^{N_l+M_l} (\mathbf{A}_l^i)^{-1} \mathbf{Q}_l^i, \quad 1 \leq l \leq L, \quad (2.11)$$

with a scaling factor $\gamma > 0$, whereas \mathbf{R}_l^G defines a multiplicative smoother

$$\mathbf{R}_l^G := (\mathbf{I} - \mathcal{E}_l) \mathbf{A}_l^{-1}, \quad \mathcal{E}_l := (\mathbf{I} - \mathbf{P}_l^{N_l+M_l}) \dots (\mathbf{I} - \mathbf{P}_l^1), \quad 1 \leq l \leq L, \quad (2.12)$$

where \mathbf{I} stands for the identity operator.

Remark 2.1. In case of nonsymmetric and indefinite linear second order elliptic boundary value problems, the associated bilinear forms become coercive on a subdomain for sufficiently small subdomain size and hence, the local symmetric and definite smoothers work well for LMM (cf. [13]). However, for the time-harmonic Maxwell problem the associated bilinear form is never positive on a subdomain. Therefore, in contrast to the algorithms presented in [15] where definite smoothers are used, we consider local smoothers based on the original indefinite problem (cf. also Remark 3.2 in [18]).

With \mathbf{R}_l^J and \mathbf{R}_l^G at hand, the LMM for the AEFEM approximation of (1.1a)-(1.1b) read as follows:

ALGORITHM 2.1. *Local Multigrid Methods (LMM).* Given an initial iterate $\mathbf{u}_l^0 \in \mathbf{U}_l$, a sequence of approximations of the solution of (2.5) can be generated according to

$$\mathbf{u}_l^{n+1} = \mathbf{u}_l^n + \mathbf{B}_l(\mathbf{f}_l - \mathbf{A}_l \mathbf{u}_l^n).$$

Here, for any $\mathbf{g} \in \mathbf{U}_l$ the multigrid operator $\mathbf{B}_l : \mathbf{U}_l \rightarrow \mathbf{U}_l$: $l \geq 0$ is recursively defined by means of:

$\mathbf{B}_0 = \mathbf{A}_0^{-1}$ and $\mathbf{B}_l \mathbf{g} = \mathbf{x}_2$, where

(i) Correction: $\mathbf{x}_1 = \mathbf{B}_{l-1} \mathbf{Q}_{l-1} \mathbf{g}$;

(ii) Post-smoothing: $\mathbf{x}_2 = \mathbf{x}_1 + \mathbf{R}_l(\mathbf{g} - \mathbf{A}_l \mathbf{x}_1)$,

and the smoother can be either a local Hiptmair-Jacobi smoother $\mathbf{R}_l = \mathbf{R}_l^J$ or a local Hiptmair-Gauss-Seidel smoother $\mathbf{R}_l = \mathbf{R}_l^G$.

We point out that the local multigrid operator \mathbf{B}_l can be treated as a preconditioner for GMRES applied to (2.5) as it will be used in the numerical computations in section 5.

3. The abstract Schwarz theory. In this section, we present an abstract framework for the convergence theory of LMM. The abstract theory depends on two important properties of the space decomposition of \mathbf{U}_L which have been established in [22] (see also [23]) for $\mathbf{H}(\mathbf{curl}; \Omega)$ -elliptic problems, i.e., a stable multilevel decomposition of \mathbf{U}_L and an associated global strengthened Cauchy-Schwarz inequality. We simply state the two properties as follows:

(S1) Stability of multilevel decomposition. For any function $\mathbf{v} \in \mathbf{U}_L$, there exists a decomposition

$$\mathbf{v} = \mathbf{v}_0 + \sum_{l=1}^L \sum_{i=1}^{N_l+M_l} \mathbf{v}_l^i, \quad \mathbf{v}_0 \in \mathbf{U}_0, \mathbf{v}_l^i \in \mathbf{U}_l^i, \quad (3.1)$$

and a positive constant C_{stab} , independent of mesh sizes and mesh levels, such that

$$\|\mathbf{v}_0\|_{\hat{\mathbf{a}}}^2 + \sum_{l=1}^L \sum_{i=1}^{N_l+M_l} \|\mathbf{v}_l^i\|_{\hat{\mathbf{a}}}^2 \leq C_{\text{stab}} \|\mathbf{v}\|_{\hat{\mathbf{a}}}^2. \quad (3.2)$$

(S2) Global strengthened Cauchy-Schwarz inequality. For any functions

$$\mathbf{v}_l^i, \mathbf{w}_l^i \in \mathbf{U}_l^i, \quad 1 \leq i \leq N_l + M_l, 0 \leq l \leq L,$$

there exists a positive constant C_{orth} , independent of mesh sizes and mesh levels, such that

$$\begin{aligned} & \sum_{l=0}^L \sum_{i=1}^{N_l+M_l} \sum_{k=0}^{l-1} \sum_{j=1}^{N_k+M_k} \hat{\mathbf{a}}(\mathbf{v}_l^i, \mathbf{w}_k^j) \leq \\ & C_{\text{orth}} \left(\sum_{l=0}^L \sum_{i=1}^{N_l+M_l} \|\mathbf{v}_l^i\|_{\hat{\mathbf{a}}}^2 \right)^{\frac{1}{2}} \left(\sum_{l=0}^L \sum_{i=1}^{N_l+M_l} \|\mathbf{w}_l^i\|_{\hat{\mathbf{a}}}^2 \right)^{\frac{1}{2}}. \end{aligned} \quad (3.3)$$

Set $\mathbf{T}_l = \mathbf{R}_l \mathbf{A}_l \mathbf{P}_l$, $l = 0, 1, \dots, L$, and $\mathbf{T} = \sum_{l=0}^L \mathbf{T}_l$. The abstract theory provides an estimate for the error operator

$$\mathbf{E} = (\mathbf{I} - \mathbf{T}_L) \cdots (\mathbf{I} - \mathbf{T}_1) (\mathbf{I} - \mathbf{T}_0) = \prod_{l=0}^L (\mathbf{I} - \mathbf{T}_l),$$

which can be deduced from the following statements:

(A1) There exist constants C_0 and C_1 such that

$$\hat{\mathbf{a}}(\mathbf{v}, \mathbf{v}) \leq C_0 \hat{\mathbf{a}}(\mathbf{T}\mathbf{v}, \mathbf{v}) + C_1 h_0^{2s} \hat{\mathbf{a}}(\mathbf{v}, \mathbf{v}), \quad \mathbf{v} \in \mathbf{U}_L.$$

(A2) Global spectral estimate. There exist constants $\omega \in (0, 2)$ and $C_2 > 0$ such that for any $\mathbf{v} \in \mathbf{U}_L$,

$$\sum_{l=0}^L \hat{\mathbf{a}}(\mathbf{T}_l \mathbf{E}_{l-1} \mathbf{v}, \mathbf{T}_l \mathbf{E}_{l-1} \mathbf{v}) \leq \omega \sum_{l=0}^L \hat{\mathbf{a}}(\mathbf{T}_l \mathbf{E}_{l-1} \mathbf{v}, \mathbf{E}_{l-1} \mathbf{v}) + C_2 h_0^{2s} \hat{\mathbf{a}}(\mathbf{v}, \mathbf{v}),$$

where $\mathbf{E}_l = (\mathbf{I} - \mathbf{T}_l) \cdots (\mathbf{I} - \mathbf{T}_0)$ and $\mathbf{E}_{-1} = \mathbf{I}$.

(A3) There exist positive constants C_3 , C_4 and C_5 such that for any $\mathbf{v} \in \mathbf{U}_L$

$$\begin{aligned} \sum_{l=0}^L \sum_{k=0}^{l-1} \hat{\mathbf{a}}(\mathbf{T}_l \mathbf{v}, \mathbf{T}_k \mathbf{E}_{k-1} \mathbf{v}) &\leq C_3 \left(\sum_{l=0}^L \hat{\mathbf{a}}(\mathbf{T}_l \mathbf{v}, \mathbf{v}) + C_4 h_0^{2s} \hat{\mathbf{a}}(\mathbf{v}, \mathbf{v}) \right)^{\frac{1}{2}} \\ &\quad \cdot \left(\sum_{l=0}^L \hat{\mathbf{a}}(\mathbf{T}_l \mathbf{E}_{l-1} \mathbf{v}, \mathbf{E}_{l-1} \mathbf{v}) + C_5 h_0^{2s} \hat{\mathbf{a}}(\mathbf{v}, \mathbf{v}) \right)^{\frac{1}{2}}. \end{aligned}$$

(A4) There exist positive constants C_6 , C_7 and C_8 such that for any $\mathbf{v} \in \mathbf{U}_L$

$$\begin{aligned} \sum_{l=0}^L \hat{\mathbf{a}}(\mathbf{T}_l \mathbf{v}, \mathbf{E}_{l-1} \mathbf{v}) &\leq C_6 \left(\sum_{l=0}^L \hat{\mathbf{a}}(\mathbf{T}_l \mathbf{v}, \mathbf{v}) + C_7 h_0^{2s} \hat{\mathbf{a}}(\mathbf{v}, \mathbf{v}) \right)^{\frac{1}{2}} \\ &\quad \cdot \left(\sum_{l=0}^L \hat{\mathbf{a}}(\mathbf{T}_l \mathbf{E}_{l-1} \mathbf{v}, \mathbf{E}_{l-1} \mathbf{v}) + C_8 h_0^{2s} \hat{\mathbf{a}}(\mathbf{v}, \mathbf{v}) \right)^{\frac{1}{2}}. \end{aligned}$$

Based on the properties (S1) and (S2), in the next section we will apply a perturbation analysis to verify (A1)-(A3) for LMM with additive and multiplicative local Hiptmair smoothers. We note that (A4) can be derived similarly, and we thus do not give details. Combining (A3),(A4) leads to

$$\sum_{l=0}^L \hat{\mathbf{a}}(\mathbf{T}_l \mathbf{v}, \mathbf{v}) \leq C_9 \sum_{l=0}^L \hat{\mathbf{a}}(\mathbf{T}_l \mathbf{E}_{l-1} \mathbf{v}, \mathbf{E}_{l-1} \mathbf{v}) + C_{10} h_0^{2s} \hat{\mathbf{a}}(\mathbf{v}, \mathbf{v}).$$

The main result of this paper reads as follows:

THEOREM 3.1. *For sufficiently small h_0 , (A1)-(A4) are satisfied and the norm of the error operator \mathbf{E} can be bounded as follows (cf. [41]):*

$$\hat{\mathbf{a}}(\mathbf{E}\mathbf{v}, \mathbf{E}\mathbf{v}) \leq \delta \hat{\mathbf{a}}(\mathbf{v}, \mathbf{v}), \quad \mathbf{v} \in \mathbf{U}_L,$$

where $\delta = 1 + K_0 h_0^{2s} - 1/K_1$. The positive constants K_0 and K_1 only depend on the shape regularity of the meshes and the wave number κ .

The theorem shows uniform convergence of LMM for (1.1a)-(1.1b) provided that the coarsest mesh is sufficiently fine. Similar to the estimates in [13], we can deduce uniform convergence of GMRES preconditioned by LMM.

4. Application to LMM with local Hiptmair smoothers. In this section, we verify (A1)-(A3) for both the additive local Hiptmair-Jacobi smoother and the multiplicative local Hiptmair-Gauss-Seidel smoother by a perturbation analysis which has been originally developed in [10] within a multigrid analysis for nonsymmetric and indefinite elliptic problems and has been also used in [13] and [19].

4.1. Local Hiptmair-Jacobi smoother. Due to the definition of the local Hiptmair-Jacobi smoother $\mathbf{R}_l^J, l \geq 1$ in (2.11) and $\mathbf{R}_0^J = \mathbf{A}_0^{-1}$, we have

$$\mathbf{T}_0 = \mathbf{P}_0, \quad \mathbf{T}_l = \mathbf{R}_l^J \mathbf{A}_l \mathbf{P}_l = \gamma \sum_{i=1}^{N_l+M_l} \mathbf{P}_l^i, \quad l = 1, \dots, L. \quad (4.1)$$

For any $\mathbf{u}, \mathbf{v} \in \mathbf{U}_L$, let $\mathbf{N}(\mathbf{u}, \mathbf{v}) = \mathbf{a}(\mathbf{u}, \mathbf{v}) - \hat{\mathbf{a}}(\mathbf{u}, \mathbf{v}) = -(\kappa^2 + 1)(\mathbf{u}, \mathbf{v})$. Note that $\hat{\mathbf{A}}_l, \hat{\mathbf{P}}_l, \hat{\mathbf{A}}_l^i, \hat{\mathbf{P}}_l^i$ are defined based on the bilinear form $\hat{\mathbf{a}}(\cdot, \cdot)$. By definition

$$\hat{\mathbf{a}}(\mathbf{P}_l \mathbf{u}, \mathbf{v}) = \mathbf{a}(\mathbf{u}, \hat{\mathbf{P}}_l \mathbf{v}) - \mathbf{N}(\mathbf{P}_l \mathbf{u}, \hat{\mathbf{P}}_l \mathbf{v}) = \hat{\mathbf{a}}(\hat{\mathbf{P}}_l \mathbf{u}, \mathbf{v}) + \mathbf{N}((\mathbf{I} - \mathbf{P}_l) \mathbf{u}, \hat{\mathbf{P}}_l \mathbf{v}).$$

and hence,

$$\hat{\mathbf{a}}((\mathbf{P}_l - \hat{\mathbf{P}}_l)\mathbf{u}, \mathbf{v}) = \mathbf{N}((\mathbf{I} - \mathbf{P}_l)\mathbf{u}, \hat{\mathbf{P}}_l\mathbf{v}), \quad l = 0, 1, \dots, L. \quad (4.2)$$

For the subspaces in the decomposition (2.6) spanned by the gradients of nodal basis functions, we have

$$\mathbf{P}_l^i = \hat{\mathbf{P}}_l^i, \quad i = 1, \dots, N_l, \quad l = 1, \dots, L. \quad (4.3)$$

Similar to (4.2), for $l = 1, \dots, L$ we also obtain

$$\hat{\mathbf{a}}((\mathbf{P}_l^i - \hat{\mathbf{P}}_l^i)\mathbf{u}, \mathbf{v}) = \mathbf{N}((\mathbf{I} - \mathbf{P}_l^i)\mathbf{u}, \hat{\mathbf{P}}_l^i\mathbf{v}), \quad i = N_l + 1, \dots, N_l + M_l. \quad (4.4)$$

4.1.1. Verification of (A1). Applying the stability of the multilevel decomposition (S1) for $\mathbf{H}(\mathbf{curl}; \Omega)$ -elliptic problems and using similar arguments as in [43, Lemma 4.2, Lemma 4.7] results in

$$\hat{\mathbf{a}}(\mathbf{v}, \mathbf{v}) \leq C\hat{\mathbf{a}}(\hat{\mathbf{T}}\mathbf{v}, \mathbf{v}), \quad \mathbf{v} \in \mathbf{U}_L, \quad (4.5)$$

where $\hat{\mathbf{T}} = \sum_{l=0}^L \hat{\mathbf{T}}_l$, $\hat{\mathbf{T}}_l = \hat{\mathbf{R}}_l \hat{\mathbf{A}}_l \hat{\mathbf{P}}_l$, and $\hat{\mathbf{R}}_l$ is the associated local Hiptmair-Jacobi or local Hiptmair-Gauss-Seidel smoother.

LEMMA 4.1. *Let \mathbf{R}_l be given by (2.11). Then (A1) holds true.*

Proof. An application of (4.5) yields

$$\hat{\mathbf{a}}(\mathbf{v}, \mathbf{v}) \leq C\hat{\mathbf{a}}(\hat{\mathbf{T}}\mathbf{v}, \mathbf{v}) = C\hat{\mathbf{a}}(\mathbf{T}\mathbf{v}, \mathbf{v}) + C \sum_{l=0}^L \hat{\mathbf{a}}((\hat{\mathbf{T}}_l - \mathbf{T}_l)\mathbf{v}, \mathbf{v}). \quad (4.6)$$

Combining (4.2)-(4.4) with (2.4) and (2.10), we deduce that

$$\begin{aligned} \sum_{l=0}^L \hat{\mathbf{a}}((\hat{\mathbf{T}}_l - \mathbf{T}_l)\mathbf{v}, \mathbf{v}) &= \hat{\mathbf{a}}((\hat{\mathbf{P}}_0 - \mathbf{P}_0)\mathbf{v}, \mathbf{v}) + \gamma \sum_{l=1}^L \sum_{i=1}^{N_l+M_l} \hat{\mathbf{a}}((\hat{\mathbf{P}}_l^i - \mathbf{P}_l^i)\mathbf{v}, \mathbf{v}) \\ &= (\kappa^2 + 1) \left(((\mathbf{I} - \mathbf{P}_0)\mathbf{v}, \hat{\mathbf{P}}_0\mathbf{v}) + \gamma \sum_{l=1}^L \sum_{i=N_l+1}^{N_l+M_l} ((\mathbf{I} - \mathbf{P}_l^i)\mathbf{v}, \hat{\mathbf{P}}_l^i\mathbf{v}) \right) \\ &\leq Ch_0^s \|(\mathbf{I} - \mathbf{P}_0)\mathbf{v}\|_{\hat{\mathbf{a}}} \|\hat{\mathbf{P}}_0\mathbf{v}\|_{\hat{\mathbf{a}}} + C\gamma \sum_{l=1}^L \sum_{i=N_l+1}^{N_l+M_l} h_{l,i} \|(\mathbf{I} - \mathbf{P}_l^i)\mathbf{v}\|_{\hat{\mathbf{a}}, \Omega_i^E} \|\hat{\mathbf{P}}_l^i\mathbf{v}\|_{\hat{\mathbf{a}}} \\ &\leq Ch_0^s \|\mathbf{v}\|_{\hat{\mathbf{a}}} \|\hat{\mathbf{P}}_0\mathbf{v}\|_{\hat{\mathbf{a}}} + C\gamma \sum_{l=1}^L \sum_{i=N_l+1}^{N_l+M_l} h_{l,i} \|\mathbf{v}\|_{\hat{\mathbf{a}}, \Omega_i^E} \|\hat{\mathbf{P}}_l^i\mathbf{v}\|_{\hat{\mathbf{a}}}. \end{aligned} \quad (4.7)$$

Applying (4.6),(4.7), the Cauchy-Schwarz inequality, and Young's inequality gives

$$\hat{\mathbf{a}}(\hat{\mathbf{T}}\mathbf{v}, \mathbf{v}) \leq C\hat{\mathbf{a}}(\mathbf{T}\mathbf{v}, \mathbf{v}) + Ch_0^{2s} \|\mathbf{v}\|_{\hat{\mathbf{a}}}^2 + C\gamma \sum_{l=1}^L \sum_{i=N_l+1}^{N_l+M_l} h_{l,i}^2 \|\mathbf{v}\|_{\hat{\mathbf{a}}, \Omega_i^E}^2. \quad (4.8)$$

Let $\hat{\mathcal{T}}_m^\varepsilon = \bigcup_{l=1}^L \{T \in \mathcal{T}_l(\Omega_l^E) : E \in \tilde{\mathcal{E}}_l, \mathcal{G}(T) = m\}$. In view of (2.2) we have

$$\begin{aligned} \sum_{l=1}^L \sum_{i=N_l+1}^{N_l+M_l} h_{l,i}^2 \|\mathbf{v}\|_{\hat{\mathbf{a}}, \Omega_i^E}^2 &\leq C \sum_{l=1}^L \sum_{E \in \tilde{\mathcal{E}}_l} \sum_{m=0}^{\infty} \sum_{\substack{T \in \mathcal{T}_l(\Omega_l^E), \\ \mathcal{G}(T)=m}} h_0^2 \theta^{2m} \|\mathbf{v}\|_{\hat{\mathbf{a}}, T}^2 \\ &= Ch_0^2 \sum_{m=0}^{\infty} \theta^{2m} \sum_{T \in \hat{\mathcal{T}}_m^\varepsilon} \sum_{l \in \sigma(m, T)} \|\mathbf{v}\|_{\hat{\mathbf{a}}, T}^2, \end{aligned}$$

where $\sigma(m, T) = \{l : T \in \widehat{\mathcal{T}}_m^\mathcal{E}, T \in \mathcal{T}_l, 1 \leq l \leq L\}$. The shape regularity of the meshes implies $\#\sigma(m, T) \leq C$. Observing that the elements in $\widehat{\mathcal{T}}_m^\mathcal{E}$ are nonintersecting and that the union of these elements is also a subset of Ω , it follows that

$$\sum_{l=1}^L \sum_{i=N_l+1}^{N_l+M_l} h_{l,i}^2 \|\mathbf{v}\|_{\tilde{\mathbf{a}}, \Omega_i^i}^2 \leq Ch_0^2 \sum_{m=0}^{\infty} \theta^{2m} \|\mathbf{v}\|_{\tilde{\mathbf{a}}}^2 \leq Ch_0^2 \|\mathbf{v}\|_{\tilde{\mathbf{a}}}^2. \quad (4.9)$$

Combining (4.6), (4.8), (4.9), and the fact that $h_0^2 \leq h_0^{2s}$ concludes the proof. \square

4.1.2. Verification of (A2). As a prerequisite to verify (A2) we provide the following key estimate.

LEMMA 4.2. *For any functions $\mathbf{v}_l^i \in \mathbf{U}_l^i$, $1 \leq i \leq N_l + M_l$, $1 \leq l \leq L$, there holds*

$$\sum_{l=1}^L \sum_{i=1}^{N_l+M_l} h_{l,i}^2 \|\sum_{k=1}^{l-1} \sum_{j=1}^{N_k+M_k} \mathbf{v}_k^j\|_{\tilde{\mathbf{a}}, \Omega_i^i}^2 \leq C \sum_{k=1}^L \sum_{j=1}^{N_k+M_k} h_{k,j}^2 \|\mathbf{v}_k^j\|_{\tilde{\mathbf{a}}}^2. \quad (4.10)$$

Proof. Let $N_k^{\mathbf{q}}$ be the number of elements in $\mathcal{T}_k \setminus \mathcal{T}_{k-1}$ which share $\mathbf{q} \in \tilde{\mathcal{N}}_k$ and let M_k^E be the number of elements in $\mathcal{T}_k \setminus \mathcal{T}_{k-1}$ which share $E \in \tilde{\mathcal{E}}_k$. Then, we have

$$\sum_{k=1}^{l-1} \sum_{j=1}^{N_k+M_k} \mathbf{v}_k^j = \sum_{k=1}^{l-1} \sum_{m=0}^{\infty} \sum_{\substack{K \in \mathcal{T}_k \setminus \mathcal{T}_{k-1} \\ \mathcal{G}(K)=m}} \left(\sum_{\mathbf{q} \in \mathcal{N}(K)} \tilde{\mathbf{v}}_k^{\mathbf{q}} + \sum_{E \in \mathcal{E}(K)} \tilde{\mathbf{v}}_k^E \right),$$

where $\mathcal{E}(T)$ and $\mathcal{N}(T)$ are the sets of edges and vertices in T , respectively, and

$$\tilde{\mathbf{v}}_k^{\mathbf{q}} = \begin{cases} \mathbf{v}_k^{\mathbf{q}} / N_k^{\mathbf{q}}, & \text{if } \mathbf{q} \in \tilde{\mathcal{N}}_k, \\ 0, & \text{otherwise,} \end{cases} \quad \tilde{\mathbf{v}}_k^E = \begin{cases} \mathbf{v}_k^E / M_k^E, & \text{if } E \in \tilde{\mathcal{E}}_k, \\ 0, & \text{otherwise.} \end{cases}$$

For any $K \in \mathcal{T}_k$, let $D_k^K = \{K' \in \mathcal{T}_k : \overline{K'} \cap \overline{K} \neq \emptyset\}$. For an element $T \in \mathcal{T}_l(\Omega_l^i)$, we assume that $\mathcal{G}(T) = n$. On this tetrahedron T , we find

$$\begin{aligned} & \left\| \sum_{k=1}^{l-1} \sum_{j=N_k+1}^{N_k+M_k} \mathbf{v}_k^j \right\|_{\tilde{\mathbf{a}}, T}^2 = \left\| \sum_{k=1}^{l-1} \sum_{m=0}^{\infty} \sum_{\substack{K \in \mathcal{T}_k \setminus \mathcal{T}_{k-1} \\ \mathcal{G}(K)=m}} \sum_{E \in \mathcal{E}(K)} \tilde{\mathbf{v}}_k^E \right\|_{\tilde{\mathbf{a}}, T}^2 \\ & \leq C \left(\sum_{k=1}^{l-1} \sum_{m=0}^{\infty} \sum_{\substack{K \in \mathcal{T}_k \setminus \mathcal{T}_{k-1} \\ T \subset D_k^K, \mathcal{G}(K)=m}} \sum_{E \in \mathcal{E}(K)} (\|\mathbf{curl} \tilde{\mathbf{v}}_k^E\|_{0,K} + \|\tilde{\mathbf{v}}_k^E\|_{0,K}) \frac{|T|^{1/2}}{|K|^{1/2}} \right)^2. \end{aligned}$$

We note that

$$\begin{aligned} & \left(\sum_{k=1}^{l-1} \sum_{m=0}^{\infty} \sum_{\substack{K \in \mathcal{T}_k \setminus \mathcal{T}_{k-1} \\ T \subset D_k^K, \mathcal{G}(K)=m}} \sum_{E \in \mathcal{E}(K)} (\|\mathbf{curl} \tilde{\mathbf{v}}_k^E\|_{0,K} + \|\tilde{\mathbf{v}}_k^E\|_{0,K}) \frac{|T|^{1/2}}{|K|^{1/2}} \right)^2 \\ & \leq C \left(\sum_{k=1}^{l-1} \sum_{m=0}^{\infty} \sum_{\substack{K \in \mathcal{T}_k \setminus \mathcal{T}_{k-1} \\ T \subset D_k^K, \mathcal{G}(K)=m}} \sum_{E \in \mathcal{E}(K)} (\|\mathbf{curl} \tilde{\mathbf{v}}_k^E\|_{0,K}^2 + \|\tilde{\mathbf{v}}_k^E\|_{0,K}^2) \frac{|T|^{2/3}}{|K|^{2/3}} \right) \\ & \cdot \left(\sum_{k=1}^{l-1} \sum_{m=0}^{\infty} \sum_{\substack{K \in \mathcal{T}_k \setminus \mathcal{T}_{k-1} \\ T \subset D_k^K, \mathcal{G}(K)=m}} \sum_{E \in \mathcal{E}(K)} \frac{|T|^{1/3}}{|K|^{1/3}} \right). \end{aligned}$$

Since $T \subset D_k^K$, we know that $\mathcal{G}(K) = m$ for any $K \in \mathcal{T}_k \setminus \mathcal{T}_{k-1}$, whence $m \leq n + s_0$, where the integer s_0 only depends on the shape regularity of the meshes. We set

$$\eta(m, T) := \{K : K \in \mathcal{T}_k \setminus \mathcal{T}_{k-1}, \mathcal{G}(K) = m, T \subset D_k^K, 1 \leq k \leq L\}.$$

The local overlapping of $\{D_k^K : K \in \mathcal{T}_k\}$ implies $\#\eta(m, T) \leq C$. It follows that

$$\begin{aligned} & \sum_{k=1}^{l-1} \sum_{m=0}^{\infty} \sum_{\substack{K \in \mathcal{T}_k \setminus \mathcal{T}_{k-1} \\ T \subset D_k^K, \mathcal{G}(K)=m}} \sum_{E \in \mathcal{E}(K)} \frac{|T|^{1/3}}{|K|^{1/3}} \leq C \sum_{k=1}^{l-1} \sum_{m=0}^{n+s_0} \sum_{\substack{K \in \mathcal{T}_k \setminus \mathcal{T}_{k-1} \\ T \subset D_k^K, \mathcal{G}(K)=m}} \theta^{n-m} \\ & \leq C \sum_{m=0}^{n+s_0} \sum_{K \in \eta(m, T)} \theta^{n-m} \leq C \sum_{m=0}^{n+s_0} \theta^{n-m} \leq C. \end{aligned}$$

Hence,

$$\left\| \sum_{k=1}^{l-1} \sum_{j=N_k+1}^{N_k+M_k} \mathbf{v}_k^j \right\|_{\tilde{\mathbf{a}}, T}^2 \leq C \sum_{k=1}^{l-1} \sum_{m=0}^{\infty} \sum_{\substack{K \in \mathcal{T}_k \setminus \mathcal{T}_{k-1} \\ T \subset D_k^K, \mathcal{G}(K)=m}} \sum_{E \in \mathcal{E}(K)} \|\tilde{\mathbf{v}}_k^E\|_{\tilde{\mathbf{a}}, K}^2 \frac{|T|^{2/3}}{|K|^{2/3}}. \quad (4.11)$$

Similar arguments yield

$$\left\| \sum_{k=1}^{l-1} \sum_{j=1}^{N_k} \mathbf{v}_k^j \right\|_{\tilde{\mathbf{a}}, T}^2 \leq C \sum_{k=1}^{l-1} \sum_{m=0}^{\infty} \sum_{\substack{K \in \mathcal{T}_k \setminus \mathcal{T}_{k-1} \\ T \subset D_k^K, \mathcal{G}(K)=m}} \sum_{\mathbf{q} \in \mathcal{N}(K)} \|\tilde{\mathbf{v}}_k^{\mathbf{q}}\|_{\tilde{\mathbf{a}}, K}^2 \frac{|T|^{2/3}}{|K|^{2/3}}. \quad (4.12)$$

An application of (4.11) and (4.12) shows

$$\begin{aligned} & \sum_{l=1}^L \sum_{i=N_l+1}^{N_l+M_l} h_{l,i}^2 \left\| \sum_{k=1}^{l-1} \sum_{j=1}^{N_k+M_k} \mathbf{v}_k^j \right\|_{\tilde{\mathbf{a}}, \Omega_l^i}^2 \leq C \sum_{l=1}^L \sum_{n=0}^{\infty} \sum_{E \in \tilde{\mathcal{E}}_l} \sum_{\substack{T \in \mathcal{T}_l(\Omega_l^E) \\ \mathcal{G}(T)=n}} h_T^2 \\ & \cdot \left(\sum_{k=1}^{l-1} \sum_{m=0}^{\infty} \sum_{\substack{K \in \mathcal{T}_k \setminus \mathcal{T}_{k-1} \\ T \subset D_k^K, \mathcal{G}(K)=m}} \left(\sum_{E \in \mathcal{E}(K)} \|\tilde{\mathbf{v}}_k^E\|_{\tilde{\mathbf{a}}, K}^2 + \sum_{\mathbf{q} \in \mathcal{N}(K)} \|\tilde{\mathbf{v}}_k^{\mathbf{q}}\|_{\tilde{\mathbf{a}}, K}^2 \right) \frac{|T|^{2/3}}{|K|^{2/3}} \right) \\ & \leq C \sum_{k=1}^{L-1} \sum_{m=0}^{\infty} \sum_{\substack{K \in \mathcal{T}_k \setminus \mathcal{T}_{k-1} \\ \mathcal{G}(K)=m}} |K|^{2/3} \left(\sum_{E \in \mathcal{E}(K)} \|\tilde{\mathbf{v}}_k^E\|_{\tilde{\mathbf{a}}, K}^2 + \sum_{\mathbf{q} \in \mathcal{N}(K)} \|\tilde{\mathbf{v}}_k^{\mathbf{q}}\|_{\tilde{\mathbf{a}}, K}^2 \right) \\ & \cdot \left(\sum_{l=k+1}^L \sum_{n=0}^{\infty} \sum_{E \in \tilde{\mathcal{E}}_l} \sum_{\substack{T \in \mathcal{T}_l(\Omega_l^E) \\ T \subset D_k^K, \mathcal{G}(T)=n}} \frac{|T|^{4/3}}{|K|^{4/3}} \right). \quad (4.13) \end{aligned}$$

Let

$$\tilde{\eta}(n, K, k) = \{T : T \in \mathcal{T}_l(\Omega_l^E), T \subset D_k^K, \mathcal{G}(T) = n, E \in \tilde{\mathcal{E}}_l, 1 \leq k < l \leq L\}.$$

Since $\mathcal{G}(K) = m$ for any $T \in \tilde{\eta}(n, K, k)$, $T \subset D_k^K$, we have $m - s_0 \leq n$, and the cardinality of the set $\tilde{\eta}(n, K, k)$ can be bounded as follows:

$$\#\tilde{\eta}(n, K, k) \leq C \frac{|K|}{|T|} \leq C \theta^{3(m-n)}. \quad (4.14)$$

Furthermore, from (4.14) we deduce

$$\begin{aligned} & \sum_{l=k+1}^L \sum_{n=0}^{\infty} \sum_{E \in \tilde{\mathcal{E}}_l} \sum_{\substack{T \in \mathcal{T}_l(\Omega_l^E) \\ T \subset D_k^K, \mathcal{G}(T)=n}} \frac{|T|^{4/3}}{|K|^{4/3}} \leq \\ & C \sum_{n=0}^{\infty} \sum_{T \in \tilde{\eta}(n, K, k)} \frac{|T|^{4/3}}{|K|^{4/3}} \leq C \sum_{n=m-s_0}^{\infty} \theta^{n-m} \leq C. \end{aligned} \quad (4.15)$$

Combining (4.13) and (4.15) gives

$$\begin{aligned} & \sum_{l=1}^L \sum_{i=N_l+1}^{N_l+M_l} h_{l,i}^2 \left\| \sum_{k=1}^{l-1} \sum_{j=1}^{N_k+M_k} \mathbf{v}_k^j \right\|_{\hat{\mathbf{a}}, \Omega_l^i}^2 \\ & \leq C \sum_{k=1}^{L-1} \sum_{m=0}^{\infty} \sum_{\substack{K \in \mathcal{T}_k \setminus \mathcal{T}_{k-1} \\ \mathcal{G}(K)=m}} |K|^{2/3} \left(\sum_{E \in \mathcal{E}(K)} \|\tilde{\mathbf{v}}_k^E\|_{\hat{\mathbf{a}}, K}^2 + \sum_{\mathbf{q} \in \mathcal{N}(K)} \|\tilde{\mathbf{v}}_k^{\mathbf{q}}\|_{\hat{\mathbf{a}}, K}^2 \right) \\ & \leq C \sum_{k=1}^L \sum_{j=1}^{N_k+M_k} h_{k,j}^2 \|\mathbf{v}_k^j\|_{\hat{\mathbf{a}}}^2. \end{aligned} \quad (4.16)$$

By similar arguments, we find

$$\sum_{l=1}^L \sum_{i=1}^{N_l} h_{l,i}^2 \left\| \sum_{k=1}^{l-1} \sum_{j=1}^{N_k+M_k} \mathbf{v}_k^j \right\|_{\hat{\mathbf{a}}, \Omega_l^i}^2 \leq C \sum_{k=1}^L \sum_{j=1}^{N_k+M_k} h_{k,j}^2 \|\mathbf{v}_k^j\|_{\hat{\mathbf{a}}}^2. \quad (4.17)$$

Summing up (4.16) and (4.17) completes the proof. \square

The above lemma allows to verify **(A2)**.

LEMMA 4.3. *Let \mathbf{R}_l be given by (2.11). For sufficiently small initial mesh size h_0 and scaling parameter γ , there exists a constant $\omega \in (0, 2)$ such that **(A2)** is satisfied.*

Proof. In view of (2.4) and (4.2), it is obvious that

$$\begin{aligned} \hat{\mathbf{a}}(\mathbf{P}_0 \mathbf{v}, \mathbf{P}_0 \mathbf{v}) &= \hat{\mathbf{a}}(\mathbf{P}_0 \mathbf{v}, \mathbf{v}) - (\kappa^2 + 1)((\mathbf{I} - \mathbf{P}_0) \mathbf{v}, \hat{\mathbf{P}} \mathbf{P}_0 \mathbf{v}) \\ &\leq \hat{\mathbf{a}}(\mathbf{P}_0 \mathbf{v}, \mathbf{v}) + Ch_0^s \|(\mathbf{I} - \mathbf{P}_0) \mathbf{v}\|_{\hat{\mathbf{a}}} \|\mathbf{P}_0 \mathbf{v}\|_{\hat{\mathbf{a}}} \\ &\leq \hat{\mathbf{a}}(\mathbf{P}_0 \mathbf{v}, \mathbf{v}) + \frac{1}{4} \|\mathbf{P}_0 \mathbf{v}\|_{\hat{\mathbf{a}}}^2 + Ch_0^{2s} \|\mathbf{v}\|_{\hat{\mathbf{a}}}^2, \end{aligned}$$

which yields

$$\hat{\mathbf{a}}(\mathbf{P}_0 \mathbf{v}, \mathbf{P}_0 \mathbf{v}) \leq \frac{4}{3} \hat{\mathbf{a}}(\mathbf{P}_0 \mathbf{v}, \mathbf{v}) + Ch_0^{2s} \|\mathbf{v}\|_{\hat{\mathbf{a}}}^2. \quad (4.18)$$

The local overlapping of $\{\Omega_l^i : 1 \leq i \leq N_l + M_l\}$ on each level implies

$$\sum_{l=1}^L \hat{\mathbf{a}}(\mathbf{T}_l \mathbf{E}_{l-1} \mathbf{v}, \mathbf{T}_l \mathbf{E}_{l-1} \mathbf{v}) \leq C \gamma^2 \sum_{l=1}^L \sum_{i=1}^{N_l+M_l} \hat{\mathbf{a}}(\mathbf{P}_l^i \mathbf{E}_{l-1} \mathbf{v}, \mathbf{P}_l^i \mathbf{E}_{l-1} \mathbf{v}). \quad (4.19)$$

By definition of \mathbf{P}_l^i , $\hat{\mathbf{P}}_l^i$, and taking (4.3),(4.4) into account, we have

$$\begin{aligned} & \sum_{l=1}^L \sum_{i=1}^{N_l+M_l} \hat{\mathbf{a}}(\mathbf{P}_l^i \mathbf{E}_{l-1} \mathbf{v}, \mathbf{P}_l^i \mathbf{E}_{l-1} \mathbf{v}) \\ &= \sum_{l=1}^L \sum_{i=1}^{N_l+M_l} \hat{\mathbf{a}}(\mathbf{P}_l^i \mathbf{E}_{l-1} \mathbf{v}, \mathbf{E}_{l-1} \mathbf{v}) - (\kappa^2 + 1) \sum_{l=1}^L \sum_{i=N_l+1}^{N_l+M_l} ((\mathbf{I} - \mathbf{P}_l^i) \mathbf{E}_{l-1} \mathbf{v}, \hat{\mathbf{P}}_l^i \mathbf{P}_l^i \mathbf{E}_{l-1} \mathbf{v}). \end{aligned}$$

Note that

$$\begin{aligned} & (\kappa^2 + 1) \sum_{l=1}^L \sum_{i=N_l+1}^{N_l+M_l} ((\mathbf{I} - \mathbf{P}_l^i) \mathbf{E}_{l-1} \mathbf{v}, \hat{\mathbf{P}}_l^i \mathbf{P}_l^i \mathbf{E}_{l-1} \mathbf{v}) \\ & \leq C \sum_{l=1}^L \sum_{i=N_l+1}^{N_l+M_l} h_{l,i} \|\mathbf{E}_{l-1} \mathbf{v}\|_{\hat{\mathbf{a}}, \Omega_l^i} \|\mathbf{P}_l^i \mathbf{E}_{l-1} \mathbf{v}\|_{\hat{\mathbf{a}}} \\ & \leq C \sum_{l=1}^L \sum_{i=N_l+1}^{N_l+M_l} h_{l,i}^2 \|\mathbf{E}_{l-1} \mathbf{v}\|_{\hat{\mathbf{a}}, \Omega_l^i}^2 + \frac{1}{2} \sum_{l=1}^L \sum_{i=N_l+1}^{N_l+M_l} \|\mathbf{P}_l^i \mathbf{E}_{l-1} \mathbf{v}\|_{\hat{\mathbf{a}}}^2. \end{aligned}$$

Hence,

$$\begin{aligned} & \sum_{l=1}^L \sum_{i=1}^{N_l+M_l} \hat{\mathbf{a}}(\mathbf{P}_l^i \mathbf{E}_{l-1} \mathbf{v}, \mathbf{P}_l^i \mathbf{E}_{l-1} \mathbf{v}) \\ & \leq C \sum_{l=1}^L \sum_{i=1}^{N_l+M_l} \hat{\mathbf{a}}(\mathbf{P}_l^i \mathbf{E}_{l-1} \mathbf{v}, \mathbf{E}_{l-1} \mathbf{v}) + C \sum_{l=1}^L \sum_{i=N_l+1}^{N_l+M_l} h_{l,i}^2 \|\mathbf{E}_{l-1} \mathbf{v}\|_{\hat{\mathbf{a}}, \Omega_l^i}^2. \quad (4.20) \end{aligned}$$

Obviously, we have

$$\mathbf{I} - \mathbf{E}_{l-1} = \sum_{k=1}^{l-1} \mathbf{T}_k \mathbf{E}_{k-1} + \mathbf{P}_0 = \gamma \sum_{k=1}^{l-1} \sum_{j=1}^{N_k+M_k} \mathbf{P}_k^j \mathbf{E}_{k-1} + \mathbf{P}_0. \quad (4.21)$$

An application of (4.21), Lemma 4.2, the Cauchy-Schwarz inequality, and (4.9) gives

$$\begin{aligned} & \sum_{l=1}^L \sum_{i=N_l+1}^{N_l+M_l} h_{l,i}^2 \|\mathbf{E}_{l-1} \mathbf{v}\|_{\hat{\mathbf{a}}, \Omega_l^i}^2 \leq 2 \sum_{l=1}^L \sum_{i=N_l+1}^{N_l+M_l} h_{l,i}^2 \|\mathbf{v}\|_{\hat{\mathbf{a}}, \Omega_l^i}^2 \\ & + 4\gamma^2 \sum_{l=1}^L \sum_{i=N_l+1}^{N_l+M_l} h_{l,i}^2 \sum_{k=1}^{l-1} \sum_{j=1}^{N_k+M_k} \|\mathbf{P}_k^j \mathbf{E}_{k-1} \mathbf{v}\|_{\hat{\mathbf{a}}, \Omega_l^i}^2 + 4 \sum_{l=1}^L \sum_{i=N_l+1}^{N_l+M_l} h_{l,i}^2 \|\mathbf{P}_0 \mathbf{v}\|_{\hat{\mathbf{a}}, \Omega_l^i}^2 \\ & \leq Ch_0^2 \|\mathbf{v}\|_{\hat{\mathbf{a}}}^2 + C \sum_{l=1}^L \sum_{i=1}^{N_l+M_l} h_{l,i}^2 \|\mathbf{P}_l^i \mathbf{E}_{l-1} \mathbf{v}\|_{\hat{\mathbf{a}}}^2 + Ch_0^2 \|\mathbf{P}_0 \mathbf{v}\|_{\hat{\mathbf{a}}}^2. \quad (4.22) \end{aligned}$$

Combining (4.20) and (4.22), for sufficiently small h_0 we have

$$\begin{aligned} & \sum_{l=1}^L \sum_{i=1}^{N_l+M_l} \hat{\mathbf{a}}(\mathbf{P}_l^i \mathbf{E}_{l-1} \mathbf{v}, \mathbf{P}_l^i \mathbf{E}_{l-1} \mathbf{v}) \leq C \sum_{l=1}^L \sum_{i=1}^{N_l+M_l} \hat{\mathbf{a}}(\mathbf{P}_l^i \mathbf{E}_{l-1} \mathbf{v}, \mathbf{E}_{l-1} \mathbf{v}) \\ & + Ch_0^2 \|\mathbf{v}\|_{\hat{\mathbf{a}}}^2 + Ch_0^2 \left(\frac{4}{3} \hat{\mathbf{a}}(\mathbf{P}_0 \mathbf{v}, \mathbf{v}) + Ch_0^{2s} \|\mathbf{v}\|_{\hat{\mathbf{a}}}^2 \right). \quad (4.23) \end{aligned}$$

Now, in view of (4.18),(4.19), and (4.23), it follows that

$$\begin{aligned} \sum_{l=0}^L \hat{\mathbf{a}}(\mathbf{T}_l \mathbf{E}_{l-1} \mathbf{v}, \mathbf{T}_l \mathbf{E}_{l-1} \mathbf{v}) &\leq C\gamma^2 \sum_{l=1}^L \sum_{i=1}^{N_l+M_l} \hat{\mathbf{a}}(\mathbf{P}_l^i \mathbf{E}_{l-1} \mathbf{v}, \mathbf{E}_{l-1} \mathbf{v}) \\ &\quad + \frac{4}{3}(1 + C\gamma^2 h_0^2) \hat{\mathbf{a}}(\mathbf{P}_0 \mathbf{v}, \mathbf{v}) + C(\gamma^2 h_0^2(1 + h_0^{2s}) + h_0^{2s}) \hat{\mathbf{a}}(\mathbf{v}, \mathbf{v}). \end{aligned}$$

We set $\omega := \max\{C\gamma, \frac{4}{3}(1 + C\gamma^2 h_0^2)\}$. Then, **(A2)** follows by choosing h_0 and γ sufficiently small such that $\omega \in (0, 2)$. \square

4.1.3. Verification of (A3). Based on the global strengthened Cauchy-Schwarz inequality (3.3) and **(A2)**, we now verify **(A3)**.

LEMMA 4.4. *Let \mathbf{R}_l be given by (2.11). For sufficiently small h_0 , there exist positive constants C_3 , C_4 and C_5 , which only depend on the shape regularity of the meshes, the wave number κ , the scaling parameter γ , and the initial mesh size h_0 , such that **(A3)** is satisfied.*

Proof. We first note that the global strengthened Cauchy-Schwarz inequality (3.3) directly gives rise to

$$\begin{aligned} \sum_{l=0}^L \sum_{k=0}^{l-1} \hat{\mathbf{a}}(\mathbf{T}_l \mathbf{v}, \mathbf{T}_k \mathbf{E}_{k-1} \mathbf{v}) &\leq C \left(\gamma^2 \sum_{l=1}^L \sum_{i=1}^{N_l+M_l} \hat{\mathbf{a}}(\mathbf{P}_l^i \mathbf{v}, \mathbf{P}_l^i \mathbf{v}) + \hat{\mathbf{a}}(\mathbf{P}_0 \mathbf{v}, \mathbf{P}_0 \mathbf{v}) \right)^{\frac{1}{2}} \\ &\quad \cdot \left(\gamma^2 \sum_{l=1}^L \sum_{i=1}^{N_l+M_l} \hat{\mathbf{a}}(\mathbf{P}_l^i \mathbf{E}_{l-1} \mathbf{v}, \mathbf{P}_l^i \mathbf{E}_{l-1} \mathbf{v}) + \hat{\mathbf{a}}(\mathbf{P}_0 \mathbf{v}, \mathbf{P}_0 \mathbf{v}) \right)^{\frac{1}{2}}. \end{aligned}$$

By definition of \mathbf{P}_l^i and $\hat{\mathbf{P}}_l^i$, (2.10), and (4.3),(4.4), similar arguments as in (4.18) show

$$\begin{aligned} &\sum_{l=1}^L \sum_{i=1}^{N_l+M_l} \hat{\mathbf{a}}(\mathbf{P}_l^i \mathbf{v}, \mathbf{P}_l^i \mathbf{v}) \\ &= \sum_{l=1}^L \sum_{i=1}^{N_l+M_l} \hat{\mathbf{a}}(\mathbf{P}_l^i \mathbf{v}, \mathbf{v}) - \sum_{l=1}^L \sum_{i=N_l+1}^{N_l+M_l} (\kappa^2 + 1) ((\mathbf{I} - \mathbf{P}_l^i) \mathbf{v}, \hat{\mathbf{P}}_l^i \mathbf{P}_l^i \mathbf{v}) \\ &\leq \sum_{l=1}^L \sum_{i=1}^{N_l+M_l} \hat{\mathbf{a}}(\mathbf{P}_l^i \mathbf{v}, \mathbf{v}) + C \sum_{l=1}^L \sum_{i=N_l+1}^{N_l+M_l} h_{l,i} \|\mathbf{v}\|_{\hat{\mathbf{a}}, \Omega_l^i} \|\mathbf{P}_l^i \mathbf{v}\|_{\hat{\mathbf{a}}}. \end{aligned}$$

Applying the Cauchy-Schwarz inequality, Young's inequality, and (4.9) gives

$$\begin{aligned} \sum_{l=1}^L \sum_{i=1}^{N_l+M_l} \hat{\mathbf{a}}(\mathbf{P}_l^i \mathbf{v}, \mathbf{P}_l^i \mathbf{v}) &\leq C \sum_{l=1}^L \sum_{i=1}^{N_l+M_l} \hat{\mathbf{a}}(\mathbf{P}_l^i \mathbf{v}, \mathbf{v}) + C \sum_{l=1}^L \sum_{i=N_l+1}^{N_l+M_l} h_{l,i}^2 \|\mathbf{v}\|_{\hat{\mathbf{a}}, \Omega_l^i}^2 \\ &\leq C \sum_{l=1}^L \sum_{i=1}^{N_l+M_l} \hat{\mathbf{a}}(\mathbf{P}_l^i \mathbf{v}, \mathbf{v}) + Ch_0^2 \hat{\mathbf{a}}(\mathbf{v}, \mathbf{v}). \end{aligned} \quad (4.24)$$

The assertion now follows from (4.18),(4.23), and (4.24). \square

4.2. Local Hiptmair-Gauss-Seidel smoother. In this subsection, we consider the convergence of algorithm 2.1 with the local Hiptmair-Gauss-Seidel smoother \mathbf{R}_l^G , $l \geq 1$ as given by (2.12). Observing $\mathbf{R}_0^G = \mathbf{A}_0^{-1}$, we have

$$\mathbf{T}_0 = \mathbf{P}_0, \quad \mathbf{T}_l = \mathbf{R}_l^G \mathbf{A}_l \mathbf{P}_l = (\mathbf{I} - \mathcal{E}_l) \mathbf{P}_l = \mathbf{I} - \mathcal{E}_l, \quad l = 1, \dots, L. \quad (4.25)$$

Setting

$$\mathcal{E}_l^0 = \mathbf{I} \text{ and } \mathcal{E}_l^i = (\mathbf{I} - \mathbf{P}_l^i) \cdots (\mathbf{I} - \mathbf{P}_l^1), \quad i = 1, \dots, N_l + M_l, \quad l \geq 1,$$

it follows that $\mathcal{E}_l = \mathcal{E}_l^{N_l + M_l}$. Obviously, we have

$$\mathcal{E}_l^{i-1} - \mathcal{E}_l^i = \mathbf{P}_l^i \mathcal{E}_l^{i-1} \text{ and } \mathbf{I} - \mathcal{E}_l = \sum_{i=1}^{N_l + M_l} \mathbf{P}_l^i \mathcal{E}_l^{i-1}.$$

From the identities (4.3) and (4.4), we further deduce (cf. [13, Section 4.2])

$$\hat{\mathbf{a}}(\mathcal{E}_l^{i-1} \mathbf{v}, \mathcal{E}_l^{i-1} \mathbf{v}) = \hat{\mathbf{a}}(\mathcal{E}_l^i \mathbf{v}, \mathcal{E}_l^i \mathbf{v}) + \hat{\mathbf{a}}(\mathbf{P}_l^i \mathcal{E}_l^{i-1} \mathbf{v}, \mathbf{P}_l^i \mathcal{E}_l^{i-1} \mathbf{v}), \quad i = 1, \dots, N_l, \quad \mathbf{v} \in \mathbf{U}_L.$$

Moreover, for $i = N_l + 1, \dots, N_l + M_l$ and $\mathbf{v} \in \mathbf{U}_L$ there holds

$$\begin{aligned} \hat{\mathbf{a}}(\mathcal{E}_l^{i-1} \mathbf{v}, \mathcal{E}_l^{i-1} \mathbf{v}) &= \hat{\mathbf{a}}(\mathcal{E}_l^i \mathbf{v}, \mathcal{E}_l^i \mathbf{v}) + \hat{\mathbf{a}}(\mathbf{P}_l^i \mathcal{E}_l^{i-1} \mathbf{v}, \mathbf{P}_l^i \mathcal{E}_l^{i-1} \mathbf{v}) \\ &\quad - 2\mathbf{N}((\mathbf{I} - \mathbf{P}_l^i) \mathcal{E}_l^{i-1} \mathbf{v}, \hat{\mathbf{P}}_l^i \mathbf{P}_l^i \mathcal{E}_l^{i-1} \mathbf{v}). \end{aligned}$$

Summing the above two identities over all i results in

$$\begin{aligned} \sum_{i=1}^{N_l + M_l} \hat{\mathbf{a}}(\mathbf{P}_l^i \mathcal{E}_l^{i-1} \mathbf{v}, \mathbf{P}_l^i \mathcal{E}_l^{i-1} \mathbf{v}) &= 2 \sum_{i=N_l + 1}^{N_l + M_l} \mathbf{N}((\mathbf{I} - \mathbf{P}_l^i) \mathcal{E}_l^{i-1} \mathbf{v}, \hat{\mathbf{P}}_l^i \mathbf{P}_l^i \mathcal{E}_l^{i-1} \mathbf{v}) \\ &\quad + 2\hat{\mathbf{a}}(\mathbf{T}_l \mathbf{v}, \mathbf{v}) - \hat{\mathbf{a}}(\mathbf{T}_l \mathbf{v}, \mathbf{T}_l \mathbf{v}). \end{aligned} \quad (4.26)$$

4.2.1. Verification of (A1). We recall that in Lemma 4.1 we have obtained the estimate

$$\hat{\mathbf{a}}(\mathbf{v}, \mathbf{v}) \leq C(\hat{\mathbf{a}}(\mathbf{P}_0 \mathbf{v}, \mathbf{v}) + \sum_{l=1}^L \sum_{i=1}^{N_l + M_l} \hat{\mathbf{a}}(\mathbf{P}_l^i \mathbf{v}, \mathbf{v}) + h_0^{2s} \hat{\mathbf{a}}(\mathbf{v}, \mathbf{v})), \quad \mathbf{v} \in \mathbf{U}_L, \quad (4.27)$$

which is a basic tool in the verification of (A1).

LEMMA 4.5. *Let \mathbf{R}_l be given by (2.12). For sufficiently small h_0 , (A1) holds true.*

Proof. Obviously, in view of (2.10), (4.3), (4.4), and (4.9) we have

$$\begin{aligned} \sum_{l=1}^L \sum_{i=1}^{N_l + M_l} \hat{\mathbf{a}}(\mathbf{P}_l^i \mathbf{v}, \mathbf{v}) &= \sum_{l=1}^L \sum_{i=1}^{N_l + M_l} \hat{\mathbf{a}}(\mathbf{P}_l^i \mathbf{v}, \mathbf{P}_l^i \mathbf{v}) - \sum_{l=1}^L \sum_{i=N_l + 1}^{N_l + M_l} \mathbf{N}((\mathbf{I} - \mathbf{P}_l^i) \mathbf{v}, \hat{\mathbf{P}}_l^i \mathbf{P}_l^i \mathbf{v}) \\ &\leq \sum_{l=1}^L \sum_{i=1}^{N_l + M_l} \hat{\mathbf{a}}(\mathbf{P}_l^i \mathbf{v}, \mathbf{P}_l^i \mathbf{v}) + C \sum_{l=1}^L \sum_{i=N_l + 1}^{N_l + M_l} h_{l,i} \|\mathbf{v}\|_{\hat{\mathbf{a}}, \Omega_l^i} \|\mathbf{P}_l^i \mathbf{v}\|_{\hat{\mathbf{a}}} \\ &\leq C \sum_{l=1}^L \sum_{i=1}^{N_l + M_l} \hat{\mathbf{a}}(\mathbf{P}_l^i \mathbf{v}, \mathbf{P}_l^i \mathbf{v}) + C \sum_{l=1}^L \sum_{i=N_l + 1}^{N_l + M_l} h_{l,i}^2 \|\mathbf{v}\|_{\hat{\mathbf{a}}, \Omega_l^i}^2 \\ &\leq C \sum_{l=1}^L \sum_{i=1}^{N_l + M_l} \hat{\mathbf{a}}(\mathbf{P}_l^i \mathbf{v}, \mathbf{P}_l^i \mathbf{v}) + Ch_0^2 \hat{\mathbf{a}}(\mathbf{v}, \mathbf{v}). \end{aligned} \quad (4.28)$$

Taking

$$I - \mathcal{E}_l^{i-1} = \sum_{j=1}^{i-1} \mathbf{P}_l^j \mathcal{E}_l^{j-1}, \quad (4.29)$$

into account, it follows that

$$\hat{\mathbf{a}}(\mathbf{P}_l^i \mathbf{v}, \mathbf{P}_l^i \mathbf{v}) = \hat{\mathbf{a}}(\mathbf{P}_l^i \mathbf{v}, \mathbf{P}_l^i \mathcal{E}_l^{i-1} \mathbf{v}) + \hat{\mathbf{a}}(\mathbf{P}_l^i \mathbf{v}, \sum_{j=1}^{i-1} \mathbf{P}_l^i \mathbf{P}_l^j \mathcal{E}_l^{j-1} \mathbf{v}).$$

Due to the above identity, the local overlapping of $\{\Omega_l^i : 1 \leq i \leq N_l + M_l\}$ on each level implies (cf. [9, Lemma 5.1])

$$\sum_{i=1}^{N_l+M_l} \hat{\mathbf{a}}(\mathbf{P}_l^i \mathbf{v}, \mathbf{P}_l^i \mathbf{v}) \leq C \sum_{i=1}^{N_l+M_l} \hat{\mathbf{a}}(\mathbf{P}_l^i \mathcal{E}_l^{i-1} \mathbf{v}, \mathbf{P}_l^i \mathcal{E}_l^{i-1} \mathbf{v}). \quad (4.30)$$

Using (2.10) and (4.26), we get

$$\begin{aligned} \sum_{i=1}^{N_l+M_l} \hat{\mathbf{a}}(\mathbf{P}_l^i \mathcal{E}_l^{i-1} \mathbf{v}, \mathbf{P}_l^i \mathcal{E}_l^{i-1} \mathbf{v}) &\leq 2\hat{\mathbf{a}}(\mathbf{T}_l \mathbf{v}, \mathbf{v}) - \hat{\mathbf{a}}(\mathbf{T}_l \mathbf{v}, \mathbf{T}_l \mathbf{v}) \\ &\quad + C \sum_{i=N_l+1}^{N_l+M_l} h_{l,i} \|\mathcal{E}_l^{i-1} \mathbf{v}\|_{\hat{\mathbf{a}}, \Omega_l^i} \|\mathbf{P}_l^i \mathcal{E}_l^{i-1} \mathbf{v}\|_{\hat{\mathbf{a}}}. \end{aligned}$$

Then, an application of Young's inequality results in

$$\sum_{i=1}^{N_l+M_l} \hat{\mathbf{a}}(\mathbf{P}_l^i \mathcal{E}_l^{i-1} \mathbf{v}, \mathbf{P}_l^i \mathcal{E}_l^{i-1} \mathbf{v}) \leq C\hat{\mathbf{a}}(\mathbf{T}_l \mathbf{v}, \mathbf{v}) + C \sum_{i=N_l+1}^{N_l+M_l} h_{l,i}^2 \|\mathcal{E}_l^{i-1} \mathbf{v}\|_{\hat{\mathbf{a}}, \Omega_l^i}^2. \quad (4.31)$$

In order to deal with the second term on the right-hand side of (4.31), the limited interaction property and (4.29) imply

$$\begin{aligned} \sum_{i=N_l+1}^{N_l+M_l} h_{l,i}^2 \|\mathcal{E}_l^{i-1} \mathbf{v}\|_{\hat{\mathbf{a}}, \Omega_l^i}^2 &\leq 2 \sum_{i=N_l+1}^{N_l+M_l} h_{l,i}^2 \|\mathbf{v}\|_{\hat{\mathbf{a}}, \Omega_l^i}^2 + 2 \sum_{i=N_l+1}^{N_l+M_l} h_{l,i}^2 \left\| \sum_{j=1}^{i-1} \mathbf{P}_l^j \mathcal{E}_l^{j-1} \mathbf{v} \right\|_{\hat{\mathbf{a}}, \Omega_l^i}^2 \\ &\leq 2 \sum_{i=N_l+1}^{N_l+M_l} h_{l,i}^2 \|\mathbf{v}\|_{\hat{\mathbf{a}}, \Omega_l^i}^2 + C \sum_{j=1}^{N_l+M_l} \sum_{\substack{i=N_l+1 \\ \Omega_l^i \cap \Omega_l^j \neq \emptyset}}^{N_l+M_l} h_{l,i}^2 \|\mathbf{P}_l^j \mathcal{E}_l^{j-1} \mathbf{v}\|_{\hat{\mathbf{a}}, \Omega_l^i}^2 \\ &\leq 2 \sum_{i=N_l+1}^{N_l+M_l} h_{l,i}^2 \|\mathbf{v}\|_{\hat{\mathbf{a}}, \Omega_l^i}^2 + C \sum_{j=1}^{N_l+M_l} h_{l,j}^2 \|\mathbf{P}_l^j \mathcal{E}_l^{j-1} \mathbf{v}\|_{\hat{\mathbf{a}}}^2. \end{aligned} \quad (4.32)$$

Obviously, for sufficiently small h_0 , (4.31) and (4.32) give rise to

$$\sum_{i=1}^{N_l+M_l} \hat{\mathbf{a}}(\mathbf{P}_l^i \mathcal{E}_l^{i-1} \mathbf{v}, \mathbf{P}_l^i \mathcal{E}_l^{i-1} \mathbf{v}) \leq C\hat{\mathbf{a}}(\mathbf{T}_l \mathbf{v}, \mathbf{v}) + C \sum_{i=N_l+1}^{N_l+M_l} h_{l,i}^2 \|\mathbf{v}\|_{\hat{\mathbf{a}}, \Omega_l^i}^2.$$

Summing over l , it follows that

$$\begin{aligned} \sum_{l=1}^L \sum_{i=1}^{N_l+M_l} \hat{\mathbf{a}}(\mathbf{P}_l^i \mathcal{E}_l^{i-1} \mathbf{v}, \mathbf{P}_l^i \mathcal{E}_l^{i-1} \mathbf{v}) &\leq C \sum_{l=1}^L \hat{\mathbf{a}}(\mathbf{T}_l \mathbf{v}, \mathbf{v}) + C \sum_{l=1}^L \sum_{i=N_l+1}^{N_l+M_l} h_{l,i}^2 \|\mathbf{v}\|_{\hat{\mathbf{a}}, \Omega_i}^2 \\ &\leq C \sum_{l=1}^L \hat{\mathbf{a}}(\mathbf{T}_l \mathbf{v}, \mathbf{v}) + Ch_0^2 \hat{\mathbf{a}}(\mathbf{v}, \mathbf{v}), \end{aligned} \quad (4.33)$$

which together with (4.27), (4.28) and (4.30) allows to conclude. \square

4.2.2. Verification of (A2). For the verification of (A2), Lemma 4.2 also plays a key role in the analysis.

LEMMA 4.6. *Let \mathbf{R}_i be given by (2.12). For sufficiently small h_0 , there exists a constant $\omega \in (0, 2)$ such that (A2) holds true.*

Proof. In view of (4.18), it suffices to deal with the remaining terms on the left-hand side of (A2). By means of the limited interaction property on each level we have

$$\sum_{l=1}^L \hat{\mathbf{a}}(\mathbf{T}_l \mathbf{E}_{l-1} \mathbf{v}, \mathbf{T}_l \mathbf{E}_{l-1} \mathbf{v}) \leq C \sum_{l=1}^L \sum_{i=1}^{N_l+M_l} \hat{\mathbf{a}}(\mathbf{P}_l^i \mathcal{E}_l^{i-1} \mathbf{E}_{l-1} \mathbf{v}, \mathbf{P}_l^i \mathcal{E}_l^{i-1} \mathbf{E}_{l-1} \mathbf{v}). \quad (4.34)$$

Using (2.10) and (4.26) again, we find

$$\begin{aligned} \sum_{i=1}^{N_l+M_l} \hat{\mathbf{a}}(\mathbf{P}_l^i \mathcal{E}_l^{i-1} \mathbf{E}_{l-1} \mathbf{v}, \mathbf{P}_l^i \mathcal{E}_l^{i-1} \mathbf{E}_{l-1} \mathbf{v}) &\leq 2\hat{\mathbf{a}}(\mathbf{T}_l \mathbf{E}_{l-1} \mathbf{v}, \mathbf{E}_{l-1} \mathbf{v}) \\ -\hat{\mathbf{a}}(\mathbf{T}_l \mathbf{E}_{l-1} \mathbf{v}, \mathbf{T}_l \mathbf{E}_{l-1} \mathbf{v}) + C \sum_{i=N_l+1}^{N_l+M_l} h_{l,i} \|\mathcal{E}_l^{i-1} \mathbf{E}_{l-1} \mathbf{v}\|_{\hat{\mathbf{a}}, \Omega_i} &\|\mathbf{P}_l^i \mathcal{E}_l^{i-1} \mathbf{E}_{l-1} \mathbf{v}\|_{\hat{\mathbf{a}}}. \end{aligned} \quad (4.35)$$

Young's inequality and (4.32) imply

$$\begin{aligned} &C \sum_{i=N_l+1}^{N_l+M_l} h_{l,i} \|\mathcal{E}_l^{i-1} \mathbf{E}_{l-1} \mathbf{v}\|_{\hat{\mathbf{a}}, \Omega_i} \|\mathbf{P}_l^i \mathcal{E}_l^{i-1} \mathbf{E}_{l-1} \mathbf{v}\|_{\hat{\mathbf{a}}} \\ &\leq C \sum_{i=N_l+1}^{N_l+M_l} h_{l,i}^2 \|\mathcal{E}_l^{i-1} \mathbf{E}_{l-1} \mathbf{v}\|_{\hat{\mathbf{a}}, \Omega_i}^2 + \frac{1}{2} \sum_{i=N_l+1}^{N_l+M_l} \|\mathbf{P}_l^i \mathcal{E}_l^{i-1} \mathbf{E}_{l-1} \mathbf{v}\|_{\hat{\mathbf{a}}}^2 \\ &\leq C \sum_{i=N_l+1}^{N_l+M_l} h_{l,i}^2 \|\mathbf{E}_{l-1} \mathbf{v}\|_{\hat{\mathbf{a}}, \Omega_i}^2 + C \sum_{i=1}^{N_l+M_l} h_{l,i}^2 \|\mathbf{P}_l^i \mathcal{E}_l^{i-1} \mathbf{E}_{l-1} \mathbf{v}\|_{\hat{\mathbf{a}}, \Omega_i}^2 \\ &\quad + \frac{1}{2} \sum_{i=N_l+1}^{N_l+M_l} \|\mathbf{P}_l^i \mathcal{E}_l^{i-1} \mathbf{E}_{l-1} \mathbf{v}\|_{\hat{\mathbf{a}}}^2. \end{aligned} \quad (4.36)$$

Consequently, in view of (4.35), (4.36), by summation over l we find that for sufficiently small initial mesh size h_0 there holds

$$\begin{aligned} \sum_{l=1}^L \sum_{i=1}^{N_l+M_l} \hat{\mathbf{a}}(\mathbf{P}_l^i \mathcal{E}_l^{i-1} \mathbf{E}_{l-1} \mathbf{v}, \mathbf{P}_l^i \mathcal{E}_l^{i-1} \mathbf{E}_{l-1} \mathbf{v}) &\leq 2C \sum_{l=1}^L \hat{\mathbf{a}}(\mathbf{T}_l \mathbf{E}_{l-1} \mathbf{v}, \mathbf{E}_{l-1} \mathbf{v}) \\ -C \sum_{l=1}^L \hat{\mathbf{a}}(\mathbf{T}_l \mathbf{E}_{l-1} \mathbf{v}, \mathbf{T}_l \mathbf{E}_{l-1} \mathbf{v}) + C_{11} \sum_{l=1}^L \sum_{i=N_l+1}^{N_l+M_l} h_{l,i}^2 &\|\mathbf{E}_{l-1} \mathbf{v}\|_{\hat{\mathbf{a}}, \Omega_i}^2. \end{aligned} \quad (4.37)$$

Observing

$$\mathbf{I} - \mathbf{E}_{l-1} = \sum_{k=1}^{l-1} \mathbf{T}_k \mathbf{E}_{k-1} + \mathbf{P}_0 = \sum_{k=1}^{l-1} \sum_{j=1}^{N_k+M_k} \mathbf{P}_k^j \mathcal{E}_k^{j-1} \mathbf{E}_{k-1} + \mathbf{P}_0,$$

by Lemma 4.2 and similar arguments as in (4.22) we obtain

$$\begin{aligned} & \sum_{l=1}^L \sum_{i=N_l+1}^{N_l+M_l} h_{l,i}^2 \|\mathbf{E}_{l-1} \mathbf{v}\|_{\hat{\mathbf{a}}, \Omega_i^i}^2 \leq 2 \sum_{l=1}^L \sum_{i=N_l+1}^{N_l+M_l} h_{l,i}^2 \|\mathbf{v}\|_{\hat{\mathbf{a}}, \Omega_i^i}^2 \\ & + 2 \sum_{l=1}^L \sum_{i=N_l+1}^{N_l+M_l} h_{l,i}^2 \left\| \sum_{k=1}^{l-1} \sum_{j=1}^{N_k+M_k} \mathbf{P}_k^j \mathcal{E}_k^{j-1} \mathbf{E}_{k-1} \mathbf{v} + \mathbf{P}_0 \mathbf{v} \right\|_{\hat{\mathbf{a}}, \Omega_i^i}^2 \\ & \leq Ch_0^2 \hat{\mathbf{a}}(\mathbf{v}, \mathbf{v}) + Ch_0^2 \hat{\mathbf{a}}(\mathbf{P}_0 \mathbf{v}, \mathbf{P}_0 \mathbf{v}) + C \sum_{l=1}^L \sum_{i=1}^{N_l+M_l} h_{l,i}^2 \|\mathbf{P}_l^i \mathcal{E}_l^{i-1} \mathbf{E}_{l-1} \mathbf{v}\|_{\hat{\mathbf{a}}}^2, \end{aligned}$$

which together with (4.37) implies that for sufficiently small h_0 there holds

$$\begin{aligned} & \sum_{l=1}^L \sum_{i=1}^{N_l+M_l} \hat{\mathbf{a}}(\mathbf{P}_l^i \mathcal{E}_l^{i-1} \mathbf{E}_{l-1} \mathbf{v}, \mathbf{P}_l^i \mathcal{E}_l^{i-1} \mathbf{E}_{l-1} \mathbf{v}) \leq 2C \sum_{l=1}^L \hat{\mathbf{a}}(\mathbf{T}_l \mathbf{E}_{l-1} \mathbf{v}, \mathbf{E}_{l-1} \mathbf{v}) \\ & - C \sum_{l=1}^L \hat{\mathbf{a}}(\mathbf{T}_l \mathbf{E}_{l-1} \mathbf{v}, \mathbf{T}_l \mathbf{E}_{l-1} \mathbf{v}) + C_{12} \left(h_0^2 \hat{\mathbf{a}}(\mathbf{v}, \mathbf{v}) + h_0^2 \hat{\mathbf{a}}(\mathbf{P}_0 \mathbf{v}, \mathbf{P}_0 \mathbf{v}) \right). \quad (4.38) \end{aligned}$$

Hence, due to (4.34)

$$\begin{aligned} & \sum_{l=1}^L \hat{\mathbf{a}}(\mathbf{T}_l \mathbf{E}_{l-1} \mathbf{v}, \mathbf{T}_l \mathbf{E}_{l-1} \mathbf{v}) \leq 2C \sum_{l=1}^L \hat{\mathbf{a}}(\mathbf{T}_l \mathbf{E}_{l-1} \mathbf{v}, \mathbf{E}_{l-1} \mathbf{v}) \\ & - C \sum_{l=1}^L \hat{\mathbf{a}}(\mathbf{T}_l \mathbf{E}_{l-1} \mathbf{v}, \mathbf{T}_l \mathbf{E}_{l-1} \mathbf{v}) + C_{13} \left(h_0^2 \hat{\mathbf{a}}(\mathbf{v}, \mathbf{v}) + h_0^2 \hat{\mathbf{a}}(\mathbf{P}_0 \mathbf{v}, \mathbf{P}_0 \mathbf{v}) \right), \end{aligned}$$

which can be written as follows

$$\begin{aligned} \sum_{l=1}^L \hat{\mathbf{a}}(\mathbf{T}_l \mathbf{E}_{l-1} \mathbf{v}, \mathbf{T}_l \mathbf{E}_{l-1} \mathbf{v}) & \leq \frac{2C}{1+C} \sum_{l=1}^L \hat{\mathbf{a}}(\mathbf{T}_l \mathbf{E}_{l-1} \mathbf{v}, \mathbf{E}_{l-1} \mathbf{v}) \\ & + \frac{C_{13}}{1+C} \left(h_0^2 \hat{\mathbf{a}}(\mathbf{v}, \mathbf{v}) + h_0^2 \hat{\mathbf{a}}(\mathbf{P}_0 \mathbf{v}, \mathbf{P}_0 \mathbf{v}) \right). \end{aligned}$$

Taking (4.18) into account and setting $\omega := \max\{\frac{2C}{1+C}, \frac{4}{3}(\frac{C_{13}h_0^2}{1+C} + 1)\}$, it follows that $\omega \in (0, 2)$ for sufficiently small h_0 . \square

4.2.3. Verification of (A3). An application of the global strengthened Cauchy-Schwarz inequality (3.3) and arguing as in Lemma 4.5 and Lemma 4.6, we can easily prove (A3).

LEMMA 4.7. *Let \mathbf{R}_l be given by (2.12). For sufficiently small h_0 , (A3) holds true with positive constants C_3 , C_4 , and C_5 that only depend on the shape regularity of the meshes, the wave number κ , and the initial mesh size h_0 .*

Proof. Using (3.3), we obtain

$$\begin{aligned} \sum_{l=0}^L \sum_{k=0}^{l-1} \hat{\mathbf{a}}(\mathbf{T}_l \mathbf{v}, \mathbf{T}_k \mathbf{E}_{k-1} \mathbf{v}) &\leq C \left(\sum_{l=1}^L \sum_{i=1}^{N_l+M_l} \hat{\mathbf{a}}(\mathbf{P}_l^i \mathcal{E}_l^{i-1} \mathbf{v}, \mathbf{P}_l^i \mathcal{E}_l^{i-1} \mathbf{v}) + \hat{\mathbf{a}}(\mathbf{P}_0 \mathbf{v}, \mathbf{P}_0 \mathbf{v}) \right)^{\frac{1}{2}} \\ &\quad \cdot \left(\sum_{l=1}^L \sum_{i=1}^{N_l+M_l} \hat{\mathbf{a}}(\mathbf{P}_l^i \mathcal{E}_l^{i-1} \mathbf{E}_{l-1} \mathbf{v}, \mathbf{P}_l^i \mathcal{E}_l^{i-1} \mathbf{E}_{l-1} \mathbf{v}) + \hat{\mathbf{a}}(\mathbf{P}_0 \mathbf{v}, \mathbf{P}_0 \mathbf{v}) \right)^{\frac{1}{2}}. \end{aligned}$$

The assertion follows directly from (4.18), (4.33), and (4.38). \square

5. Numerical results. We illustrate the theoretical convergence results and test the performance of LMM by the results of two numerical examples. For the solution of the algebraic systems resulting from the curl-conforming edge element discretization of the time-harmonic Maxwell equation, LMM (Algorithm 2.1) is used as a preconditioner for GMRES (PGMRES). The adaptive mesh refinement has been done by Dörfler marking [17] on the basis of the residual-type a posteriori error estimators from [15] and [46], and the refinement itself has been realized by the newest vertex bisection algorithm. Since the computations at the l -th level involve only local nodes corresponding to the components in $\tilde{\mathcal{E}}_l$ and $\tilde{\mathcal{N}}_l$, the computational cost of the local multigrid algorithm is proportional to the number of degrees of freedom (DOF).

At the l -th level, the discrete problem reads $\mathbb{A}_l \mathbf{u}_l = \mathbf{F}_l$. We denote by \mathbf{u}_l^0 the prolonged coarse grid correction, i.e., $\mathbf{u}_l^0 = \mathbf{I}_{l-1} \mathbf{u}_{l-1}$ where \mathbf{I}_{l-1} is the transfer matrix realizing the prolongation from the coarse to the fine edge element space. We further refer to $\mathbf{r}_l^n = \mathbf{F}_l - \mathbb{A}_l \mathbf{u}_l^n$ as the residual with respect to the n -th iteration. The PGMRES algorithm terminates when

$$\|\mathbf{r}_l^n\| / \|\mathbf{r}_l^0\| \leq 10^{-6},$$

where $\|\cdot\|$ stands for the Euclidean norm. The number of iteration steps required to achieve the desired accuracy is denoted by **iter**.

The theoretical results obtained in the previous section predict an initial mesh size h_0 satisfying $\kappa^2 h_0^s \leq C$. As will be seen in the following examples, in actual computations, however, h_0 can be chosen slightly coarser than predicted by theory.

EXAMPLE 5.1. We consider the time-harmonic Maxwell equation (1.1a),(1.1b) on the L-shaped domain

$$\begin{aligned} \Omega &= (-1, 1) \times (-1, 1) \times (0, 2) \setminus (\Omega_1 \cup \Omega_2 \cup \Omega_3), \quad \Omega_1 = (-1, 0) \times (-1, 0) \times (0, 2), \\ \Omega_2 &= (-1, -0.5) \times (0, 0.5) \times (0, 2), \quad \Omega_3 = (0, 0.5) \times (-1, -0.5) \times (0, 2), \end{aligned}$$

and choose the right-hand side \mathbf{f} according to $\mathbf{f} = (1, 1, 1)^T$.

We test the cases $\kappa^2 = 9$ and $\kappa^2 = 100$. The scaling factor is chosen as $\gamma = 0.6$ in the local Hiptmair-Jacobi smoother. For $\kappa^2 = 9$, the restriction $\kappa^2 h_0^s \leq C$ requires $h_0 \approx 0.01 \sim 0.1$ in theory, but $h_0 \approx 0.5$ worked well in our experiments. Fig. 5.1 (left) shows the locally refined mesh with 249231 DOF at the 21-st refinement level, whereas Fig. 5.1 (right) displays the associated discrete solution vector on the boundary. We observe that the singularity of the solution is near the corner lines $l_1 : x_1 = x_2 = 0$, $l_2 : x_1 = -0.5, x_2 = 0.5$ and $l_3 : x_1 = 0.5, x_2 = -0.5$. Table 5.1 shows that the iteration steps of PGMRES with the local Hiptmair-Jacobi smoother or the local Hiptmair-Gauss-Seidel smoother are almost uniform and bounded independently of mesh sizes and mesh levels. We find that even for the finer initial mesh with $h_0 \approx 0.25$,

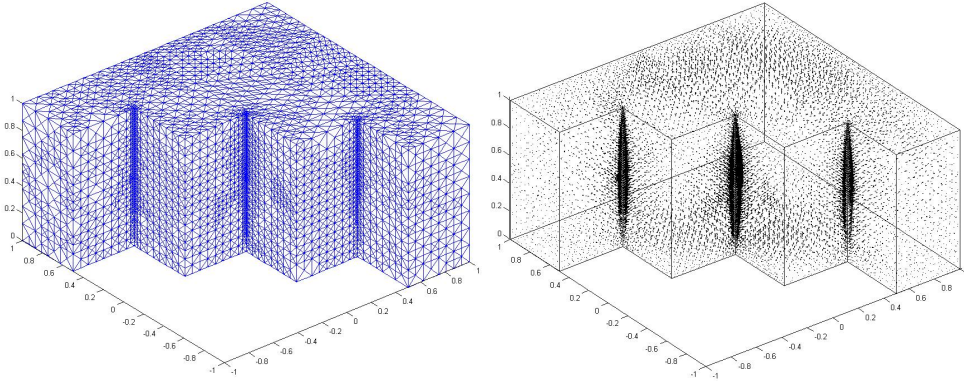


FIG. 5.1. *Example 5.1* ($\kappa^2 = 9$): *Locally refined mesh* (left) with 249231 DOF and the *discrete solution vector* (right) on the boundary at the 21-st refinement level.

TABLE 5.1

Example 5.1: The number of PGMRES iterations with the local Hiptmair-Jacobi smoother and the local Hiptmair-Gauss-Seidel smoother for $\kappa^2 = 9$ and $\kappa^2 = 100$.

$\kappa^2 = 9$				$\kappa^2 = 100$			
Level	DOF	iter		Level	DOF	iter	
		GS	Jacobi			GS	Jacobi
13	21911	12	19	5	37803	19	27
15	40893	13	20	7	74471	23	34
17	75948	12	19	9	125880	27	42
19	138427	12	19	11	174715	26	39
21	249231	12	19	13	298580	27	42
23	445534	13	20	15	474480	27	43
25	768161	11	19	17	817911	27	44
27	1342894	11	19	19	1422440	26	43

the convergence properties of PGMRES with the two types of local smoothers are almost the same as for $h_0 \approx 0.5$.

For $\kappa^2 = 100$, we have chosen $h_0 \approx 0.16$ which is also coarser than predicted by theory. Table 5.1 shows that in case of this initial mesh size LMM is indeed a good preconditioner for GMRES.

Fig. 5.2 shows that for $\kappa^2 = 9$ and $\kappa^2 = 100$ the CPU times (in seconds) of each PGMRES iteration with different types of local smoothers is almost linear in terms of DOF which, together with the almost uniform convergence, implies quasi-optimality of the PGMRES algorithm.

The next example deals with the application of LMM to the time-harmonic Maxwell equations on a non-Lipschitz domain.

EXAMPLE 5.2. We test the time-harmonic problem (1.1a-1.1b) on a non-Lipschitz domain with two cubic cavities. The computational domain reads

$$\Omega = (-1, 1) \times (-1, 1) \times (0, 2) \setminus (Q_1 \cup Q_2),$$

$$Q_1 = (-0.4, 0) \times (-0.4, 0) \times (0.6, 1), \quad Q_2 = (0, 0.4) \times (0, 0.4) \times (1, 1.4),$$

and the right-hand side \mathbf{f} is again given by $\mathbf{f} = (1, 1, 1)^T$.

We test $\kappa^2 = 1, 9, 25, 49, 100$ and only use the local Hiptmair-Gauss-Seidel smoother in this experiment. In order to illustrate the convergence property of PGMRES, we

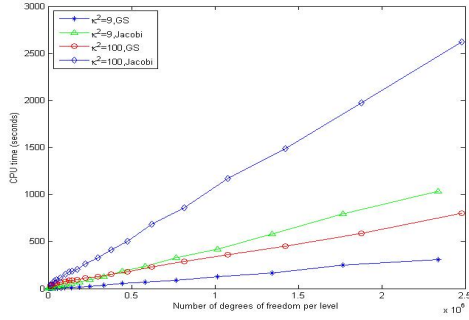


FIG. 5.2. Example 5.1: CPU times of PGMRES iterations with different smoothers for $\kappa^2 = 9$ and $\kappa^2 = 100$.

TABLE 5.2

Example 5.2: The number of PGMRES iterations with the local Hiptmair-Gauss-Seidel smoother for $\kappa^2 = 1, 9, 25, 49, 100$ based on the same initial mesh with $h_0 \approx 0.3$.

$\kappa^2 = 1$	Level	5	7	9	11	13	15	17
	DOF	20419	39336	77847	152017	288629	535882	991044
	iter	11	12	12	12	12	12	12
$\kappa^2 = 9$	Level	5	7	9	11	13	15	17
	DOF	16935	33359	66451	127286	240873	450916	821193
	iter	12	12	13	13	13	13	12
$\kappa^2 = 25$	Level	5	7	9	11	13	15	17
	DOF	18042	34848	69961	137471	262696	487616	873067
	iter	15	16	17	18	18	18	18
$\kappa^2 = 49$	Level	6	8	10	12	14	16	18
	DOF	19546	37918	73673	124783	229763	422671	756569
	iter	28	31	34	34	33	34	34
$\kappa^2 = 100$	Level	7	9	11	15	19	23	27
	DOF	18548	27041	38480	83130	148381	304108	857085
	iter	128	149	165	169	154	153	150

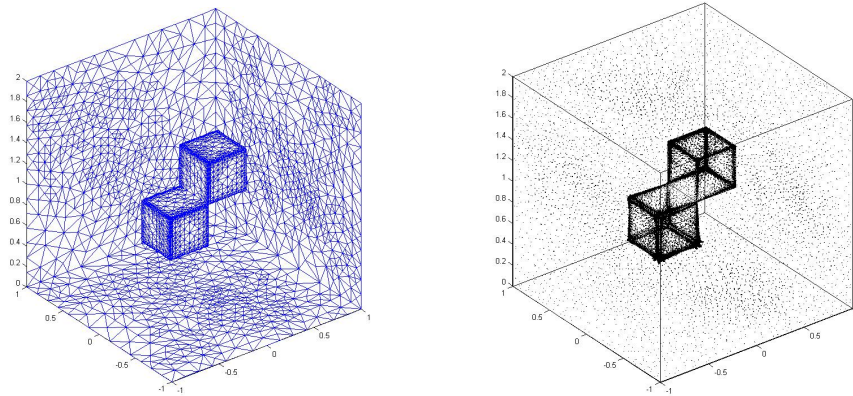


FIG. 5.3. Example 5.2 ($\kappa^2 = 9$): Locally refined mesh (left, part of surface mesh on the boundary) with 331369 DOF, and the discrete solution vector (right) on the boundary at the 14-th refinement level.

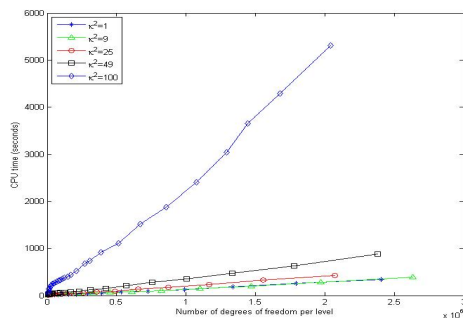


FIG. 5.4. *Example 5.2: CPU times of PGMRES iterations with the local Hiptmair-Gauss-Seidel smoother for $\kappa^2 = 1, 9, 25, 49, 100$.*

have first chosen an initial mesh of mesh size $h_0 \approx 0.3$. Fig. 5.3 (left) shows a part of the locally refined mesh with 331369 DOF on the boundary for $\kappa^2 = 9$. We observe that the mesh is always locally refined near the boundary of the cavities. Fig. 5.3 (right) displays the associated discrete solution vector on the boundary. Fig. 5.4 shows that for each wave number the CPU time of the PGMRES iteration is almost linear in terms of DOF. Table 5.2 indicates that in each case the iteration steps remain almost uniform on different levels.

TABLE 5.3

Example 5.2: The number of PGMRES iterations for $\kappa^2 = 100$ based on different initial meshes.

$h_0 \approx 0.25$	Level	8	10	12	15	18	21	24
	DOF	36866	54596	78258	138683	254987	514008	876407
	iter	104	108	108	111	107	112	111
$h_0 \approx 0.1$	Level	8	10	12	14	17	20	23
	DOF	259677	324188	392348	481689	613173	740370	951532
	iter	16	18	20	22	28	28	28

For $\kappa^2 = 100$, the approximation is far from the true solution, if the initial mesh size is too coarse. Indeed, we observe that for this higher wave number the convergence of PGMRES is much better for the finer initial mesh. This is reflected by the results displayed in Table 5.2 and Table 5.3.

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