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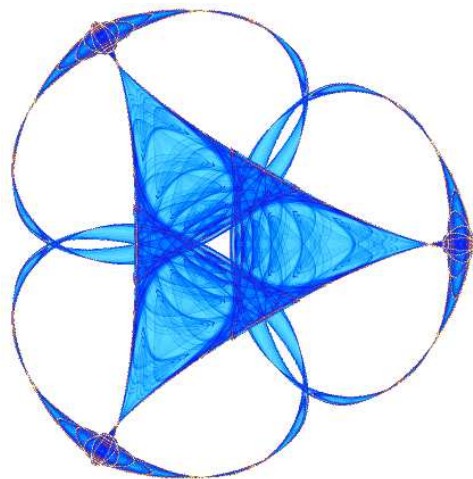
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NEW POISSON-BOLTZMANN TYPE EQUATIONS: ONE-DIMENSIONAL SOLUTIONS

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Abstract. The Poisson-Boltzmann (PB) equation is conventionally used to model the equilibrium of *bulk* ionic species in different media and solvents. In this paper we study a new Poisson-Boltzmann type (PB_n) equation with a small dielectric parameter ϵ^2 and nonlocal nonlinearity which takes into consideration of the preservation of the total amount of each individual ion. This equation can be derived from the original Poisson-Nernst-Planck (PNP) system. Under Robin type boundary conditions with various coefficient scales, we demonstrate the asymptotic behaviors of one dimensional solutions of PB_n equations as the parameter ϵ approaches to zero. In particular, we show that in case of electro-neutrality, i.e., $\alpha = \beta$, solutions of 1-D PB_n equations have the similar asymptotic behavior as those of 1-D PB equations. However, as $\alpha \neq \beta$ (local non-electroneutrality), solutions of 1-D PB_n equations may have blow-up behavior which can not be found in 1-D PB equations. Such a difference between 1-D PB and PB_n equations can also be verified by numerical simulations.

Key words. Poisson-Nernst-Planck system, Renormalized, Poisson-Boltzmann equations

AMS subject classifications. 76A05, 76M99, 65C30

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1. Introduction.

Understanding ion transport is crucial in the study of many physical and biological problems, such as semiconductors [31], electro-kinetic fluids [28, 33, 34, 35], transport of electrochemical systems [32] and ion channels in cell membranes [5, 13, 14, 22, 24, 27, 29, 30]. One of the fundamental models for the ionic transport is the time dependent system consisting coupled diffusion-convection equations, the Poisson-Nernst-Planck (PNP) system.

In this paper, we confine ourselves to the physical domain as a one-dimensional interval $(-1, 1)$, as was done in [1]. The PNP system is given by

$$n_t = -\partial_x J_n, \quad p_t = -\partial_x J_p, \quad (1.1)$$

$$J_n = -D_n \left(n_x - \frac{z_n e}{k_B T} n \phi_x \right), \quad J_p = -D_p \left(p_x + \frac{z_p e}{k_B T} p \phi_x \right), \quad (1.2)$$

$$\epsilon^2 \phi_{xx} = -\rho + z_n e n - z_p e p, \quad \text{for } x \in (-1, 1), t > 0 \quad (1.3)$$

where ϕ is the electrostatic potential, n is the density of anions, p is the density of cations, ρ is the permanent (fixed) charge density in the domain, z_n, z_p are the valence

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of ions, e is the elementary charge, k_B is the Boltzmann constant, T is temperature, J_n, J_p are the ionic flux densities and D_n, D_p are their diffusion coefficients. The parameter $\epsilon = (\epsilon_0 U_T / (d^2 e S))^{1/2} > 0$, where ϵ_0 is the dielectric constant of the electrolyte, U_T is the thermal voltage, d is the length of the domain $(-1, 1)$, and S is the appropriate concentration scale (cf. [16]). Furthermore, ϵd is known as the Debye length and ϵ is of order 10^{-2} for the physiological cases of interest (cf. [2]). Thus we may assume ϵ as a small parameter tending to zero. For simplicity, we consider monovalent ions, that is, $z_n = z_p = 1$ with $e/k_B T = 1$, $\rho = 0$, $D_n = D_p = 1$. Here we reuse the notation, ϵ again. Then the PNP system (1.1)-(1.3) becomes

$$n_t = -\partial_x J_n, \quad p_t = -\partial_x J_p, \quad (1.4)$$

$$J_n = -(n_x - n\phi_x), \quad J_p = -(p_x + p\phi_x), \quad (1.5)$$

$$\epsilon^2 \phi_{xx} = n - p, \quad \text{for } x \in (-1, 1), t > 0. \quad (1.6)$$

The equation (1.4) represents the conservation of total charges of the individual ions (similar to the conservation of mass equation in [15]). These so-called Nernst-Planck equations describe electro-diffusion and electrophoresis according to Fick's and Kohlrausch's laws, respectively. The equation (1.6) depicts the electro-static Poisson's law. We assume no-flux boundary conditions on the boundary to describe the insulated domain boundaries:

$$J_n = J_p = 0 \quad \text{for } x = \pm 1, t > 0. \quad (1.7)$$

Hence it is easy to conclude that the total charges of both negative and positive ions are conserved in time by (1.4) and (1.7):

$$\begin{aligned} \frac{d}{dt} \int_{-1}^1 n dx &= - \int_{-1}^1 \partial_x J_n dx = -J_n|_{x=-1}^{x=1} = 0, \\ \frac{d}{dt} \int_{-1}^1 p dx &= - \int_{-1}^1 \partial_x J_p dx = -J_p|_{x=-1}^{x=1} = 0, \quad \text{for } t > 0. \end{aligned}$$

Consequently, we have the normalization condition given by

$$\int_{-1}^1 n dx = \alpha, \quad \int_{-1}^1 p dx = \beta, \quad \text{for } t > 0 \quad (1.8)$$

where α and β are positive constants only determined by the initial conditions.

The PNP system (1.4)-(1.6) can be derived by the energetic variational approaches [33, 36]. The energy dissipation law of the PNP system (1.4)-(1.6) is given as follows:

$$\frac{d}{dt} \int_{-1}^1 \left(n \log n + p \log p + \frac{\epsilon^2}{2} |\phi_x|^2 \right) dx = - \int_{-1}^1 \left(n \left| \frac{n_x}{n} - \phi_x \right|^2 + p \left| \frac{p_x}{p} + \phi_x \right|^2 \right) dx.$$

Here the integral of $n \log n$ and $p \log p$ is the entropy related to Brownian motion of anions and cations, and the integral of $\frac{\epsilon^2}{2} |\phi_x|^2$ is the electrostatic potential of the coulomb interaction between charged ions.

To see the equilibrium of the PNP system (1.4)-(1.6), it is standard to consider the steady state i.e. time-independent solutions of the PNP system (1.4)-(1.6). Such

a concept is well-known in many fields including biological systems and the electrophysiology. Hence we may set $n_t \equiv p_t \equiv 0$ and (1.4)-(1.6) can be transformed into the following system

$$\partial_x(n_x - n\phi_x) = 0, \quad \partial_x(p_x + p\phi_x) = 0, \quad \text{for } x \in (-1, 1), \quad (1.9)$$

$$\epsilon^2 \phi_{xx} + p - n = 0, \quad \text{for } x \in (-1, 1), \quad (1.10)$$

where $n = n(x)$, $p = p(x)$, $\phi = \phi(x)$ are real-valued functions. On the other hand, the no-flux boundary condition (1.7) and the normalization condition (1.8) become

$$(n_x - n\phi_x)(\pm 1) = 0, \quad (p_x + p\phi_x)(\pm 1) = 0, \quad (1.11)$$

and $\int_{-1}^1 n dx = \alpha$, $\int_{-1}^1 p dx = \beta$ where α and β are positive constants.

REMARK 1.1. *We want to point out, although most of the physical and biological systems possess the (overall) electroneutrality, it may not be the case in local domains (cf. [6, 12, 17]), which are the considerations of this paper. Such non-electroneutrality phenomena are ubiquitous and can be extremely subtle when systems involves empirical boundary conditions to take into account of electrochemical reactions on the boundaries, such as those in [?, ?, ?, ?]. For instance, the non-electroneutrality may hold near the electrodes when the Faradaic current is driven by redox reactions occurring at the electrodes (cf. [42]).*

Boundary conditions play a crucial role on solutions of the system (1.9), (1.10). The boundary conditions (1.11) guarantee the solutions of (1.9) have the following forms:

$$n = n(x) = \tilde{\alpha} e^{\phi(x)}, \quad p = p(x) = \tilde{\beta} e^{-\phi(x)}, \quad \text{for } x \in (-1, 1) \quad (1.12)$$

where $\tilde{\alpha}$ and $\tilde{\beta}$ are constants. Note that from (1.9), we have $n_x - n\phi_x \equiv c_1$ and $p_x + p\phi_x \equiv c_2$ for some constants c_j 's. Then the conditions (1.11) imply $c_1 = c_2 = 0$ which gives (1.12) by solving $n_x - n\phi_x \equiv 0$ and $p_x + p\phi_x \equiv 0$, respectively. To fulfill the total charge condition, $\tilde{\alpha}$ and $\tilde{\beta}$ must satisfy

$$\tilde{\alpha} = \frac{\alpha}{\int_{-1}^1 e^{\phi} dx}, \quad \tilde{\beta} = \frac{\beta}{\int_{-1}^1 e^{-\phi} dx} \quad (1.13)$$

having the nonlocal dependence on ϕ . Consequently, (1.10), (1.12) and (1.13) give the new Poisson-Boltzmann type (PB_n) equation,

$$\epsilon^2 \phi_{xx} = \frac{\alpha e^{\phi}}{\int_{-1}^1 e^{\phi} dx} - \frac{\beta e^{-\phi}}{\int_{-1}^1 e^{-\phi} dx} \quad \text{for } x \in (-1, 1). \quad (1.14)$$

Conventionally, the Poisson-Boltzmann (PB) equation for electrolytes with two species' densities n and p occupying $(-1, 1)$ is the combination of the Poisson equation

$$\epsilon^2 \phi_{xx} = n - p \quad \text{in } (-1, 1),$$

and the Boltzmann distribution

$$n = \alpha_1 e^{\phi} \quad \text{and} \quad p = \beta_1 e^{-\phi} \quad \text{in } (-1, 1),$$

where α_1 and β_1 are positive constants (cf. [4], [8] and [9]). Then the PB equation can be written as

$$\epsilon^2 \phi_{xx} = \alpha_1 e^\phi - \beta_1 e^{-\phi}, \quad \text{for } x \in (-1, 1). \quad (1.15)$$

Note that the PB equation (1.15) has only (spatially) local nonlinearity which is very different from the nonlocal nonlinearity of the PB_n equation (1.14). We may derive the PB equation (1.15) from the steady state PNP system (1.9)-(1.10) with another boundary conditions as follows:

$$(n_x - n\phi_x)(-1) = (p_x + p\phi_x)(-1) = 0, \quad (1.16)$$

$$n(1) = n_1, \quad p(1) = p_1, \quad \phi(1) = \phi_1, \quad (1.17)$$

where n_1 , p_1 and ϕ_1 are positive constants satisfying $\alpha_1 = n_1 e^{-\phi_1}$ and $\beta_1 = p_1 e^{\phi_1}$. One may remark that both the PB_n equation (1.14) and the PB equation (1.15) come from the same PNP system (1.9)-(1.10) but with respect to the different boundary conditions (1.11) and (1.16)-(1.17), respectively. Physically, the boundary conditions (1.11) provide no-flux at the boundary points $x = \pm 1$ but the boundary conditions (1.16)-(1.17) have the Dirichlet boundary condition which may give nonzero flux at the one end $x = 1$ and no-flux at the other end $x = -1$. Since various boundary conditions may result in distinct solution behaviors, it would be natural to believe that the PB_n equation (1.14) and the PB equation (1.15) have different solution behaviors.

In many applications, especially those in biological systems, the interfacial boundary condition is extremely important. Here we may consider the interval $(-1, 1)$ as the ion channel with boundaries at $x = \pm 1$. To describe more general physical phenomena, a Robin-type boundary condition (cf. [7]) for the electrostatic potential ϕ at $x = \pm 1$ is given by

$$\eta_\epsilon \frac{\partial \phi}{\partial \nu} = \phi_{extra} - \phi_{intra} \quad (1.18)$$

where ν is the unit outer normal vector, and ϕ_{intra} and ϕ_{extra} are the intrachannel and extrachannel electrostatic potentials at the channel boundaries, respectively. The coefficient $\eta_\epsilon \sim \frac{\epsilon_0}{\epsilon_m}$ is governed by the ratio of ϵ_0 the dielectric constant of the electrolyte and ϵ_m the dielectric constant of the membrane (cf. [40, 41]). Due to $\epsilon_0 \sim \epsilon^2$, η_ϵ depends on the parameter ϵ and satisfies $\eta_\epsilon/\epsilon \sim \epsilon/\epsilon_m$. Generically, one may not know the asymptotic limit of the ratio η_ϵ/ϵ as the parameter ϵ goes to zero. Here we assume the limit $\lim_{\epsilon \rightarrow 0} \eta_\epsilon/\epsilon$ to be zero, a positive constant γ and infinity, respectively.

The boundary condition (1.18) may also represent the capacitance effect of cell membrane [19, 20, 21] and can be found in other physical systems e.g. [33, 34, 35]. Setting $\phi_{extra} = \phi_0$ and $\phi_{intra} = \phi$, we may transform the condition (1.18) into

$$\begin{cases} (\phi + \eta_\epsilon \phi_x)(x) = \phi_0(x), & \text{if } x = 1, \\ (\phi - \eta_\epsilon \phi_x)(x) = \phi_0(x), & \text{if } x = -1, \end{cases} \quad (1.19)$$

where $\phi_0(1), \phi_0(-1)$ are constants and η_ϵ is a nonnegative constant, with scale of length, and related to the surface capacitance. Since we want to study the properties of the solutions with respect to the relative relation of this parameter and the dielectric constant, we assume that η_ϵ depends on the parameter ϵ . Notice, as $\eta_\epsilon = 0$, (1.19)

becomes the Dirichlet boundary condition. As $\eta_\epsilon \rightarrow \infty$, (1.19) tends to the Neumann boundary condition. It is obvious that the Robin boundary condition (1.19) has more general feature rather than Dirichlet and Neumann boundary conditions.

To see the difference of the PB_n equation (1.14) and the PB equation (1.15), we assume $\alpha \neq \beta$ and remark the difference by contradiction. Suppose the solutions for the PB_n equation (1.14) and the PB equation (1.15) have the same asymptotic behavior as the parameter ϵ goes to zero. Then by the PB equation (1.15), the solution ϕ satisfies $\forall x \in (-1, 1)$, $\phi(x) \sim C$ as $\epsilon \rightarrow 0$, where C is a (finite) constant determined by $\alpha_1 e^C = \beta_1 e^{-C}$. Hence $\frac{\alpha e^\phi}{\int_{-1}^1 e^\phi dx} \sim \frac{\alpha}{2}$ and $\frac{\beta e^\phi}{\int_{-1}^1 e^\phi dx} \sim \frac{\beta}{2}$ as $\epsilon \rightarrow 0$. However, by the PB_n equation (1.14), we have

$$0 \sim \epsilon^2 \phi_{xx} = \frac{\alpha e^\phi}{\int_{-1}^1 e^\phi dx} - \frac{\beta e^\phi}{\int_{-1}^1 e^\phi dx} \sim \frac{\alpha}{2} - \frac{\beta}{2}$$

which contradicts to $\alpha \neq \beta$. Therefore, the solutions for PB_n equation (1.14) and the PB equation (1.15) have different asymptotic behaviors as the parameter ϵ goes to zero.

The main goal of this paper is to compare solutions of the PB_n equation (1.14) and the PB equation (1.15) under the general Robin boundary condition (1.19) for the electrostatic potential. For the coefficients α, β of the PB_n equation (1.14), two cases are considered as follows: One case is $\alpha = \beta$ for the global electroneutrality, which means the total amounts of the positive charge and the negative charge are the same. The other case is $\alpha \neq \beta$ for the non-electroneutrality which means the total positive and negative charge densities are not equal to each other. For the later case, we want emphasize again, although most of the physical and biological systems possess the (overall) electroneutrality, it may not be the case in local domains (cf. [6, 12, 17]), which are the considerations of this paper. For these two cases, we prove rigorously interior and boundary estimates for the PB_n equation (1.14) with the boundary condition (1.19):

- Under the boundary condition (1.19), if $\alpha = \beta$, then the solution of (1.14) asymptotically close to the solution of (1.15) with $\alpha_1 = \alpha/2 = \beta/2 = \beta_1$ as ϵ goes to zero. This shows that as $\alpha = \beta$, solutions of the PB_n equation (1.14) and the PB equation (1.15) may have the same asymptotic behavior.
- If $\alpha \neq \beta$, then the solution of (1.14) with the boundary condition (1.19) may tend to infinity as ϵ goes to zero. Such a blow-up behavior is absent for the solutions of (1.15) with the boundary condition (1.19). This provides one of the key different behaviors of the solutions for PB_n equation (1.14) and the PB equation (1.15).

Now we summarize our results as follows:

(a) In the electroneutral case ($\alpha = \beta$):

- (a1) If $\lim_{\epsilon \downarrow 0} \frac{\epsilon}{\eta_\epsilon} = 0$, the solution ϕ approaches zero in $[-1, 1]$ as $\epsilon \downarrow 0$. However, ϕ has slope of order $O(1/\eta_\epsilon)$ on the boundary (cf. Theorem 1.3(i)).
- (a2) When $\frac{\epsilon}{\eta_\epsilon} \geq C$ for some positive constant C independent of ϵ , the solution ϕ possesses boundary layers with thickness ϵ (cf. Theorem 1.3(ii) and (iii) and (1.25)).

(b) In the non-electroneutral case ($\alpha \neq \beta$):

The solution ϕ has boundary layers with thickness ϵ^2 and $\phi(x) - \phi(\pm 1)$ tends to infinity with the leading order term $\log(\epsilon^{-2})$ as $\epsilon \downarrow 0$ for $x \in (-1, 1)$ (see Theorem 1.5 – 1.7). The values $\phi(\pm 1)$ can be estimated as follows:

(b1) If $\frac{\eta\epsilon}{\epsilon^2} \leq C$, $\phi(1)$ and $\phi(-1)$ converge to different finite values as $\epsilon \downarrow 0$, where C is a positive constant independent of ϵ (cf. Theorem 1.6(i) and (ii)).

(b2) If $\lim_{\epsilon \downarrow 0} \frac{\eta\epsilon}{\epsilon^2} = \infty$, both $\phi(1)$ and $\phi(-1)$ diverge to ∞ , but $|\phi(1) - \phi(-1)|$ converges to zero as $\epsilon \downarrow 0$ (cf. Theorem 1.6(iii)).

(c) The difference between the solutions to the PB_n equation (1.14) and the PB equation (1.26) can be stated as follows:

(c1) When $\alpha = \beta$, the solution of the PB_n equation (1.14) may converge to the solution of the PB equation (1.26) (cf. Theorem 1.4). Namely, in the case of $\alpha = \beta$, the solution of the PB_n equation (1.14) have the same asymptotic behavior as that of the PB equation (1.26).

(c2) When $\alpha \neq \beta$, the solution of the PB equation (1.26) remain bounded for $\epsilon > 0$ (cf. Theorem 4.2). However, as $\alpha \neq \beta$, the solution of the PB_n equation (1.14) may tend to infinity as ϵ goes to zero (see **(b)**). This may provide the difference between the solutions to the PB_n equation (1.14) and the PB equation (1.26).

Although people may believe that the non-electroneutrality i.e. $\alpha \neq \beta$ is impossible for most biological systems, it seems difficult to specify what kind of biological systems making the non-electroneutrality impossible. Here we show that if the non-electroneutrality holds, then the blow-up behavior of the electrical potential ϕ occurs by solving the new model PB_n equation (1.14). Thus the non-electroneutrality is impossible for those biological systems without the blow-up behavior of the electrical potential. To avoid the non-electroneutrality, the new model PB_n equation (1.14) provides a new way which can not be found in the conventional PB equation (1.15).

Finally, we want to point out that in our ongoing projects [10, 11], we studied the renormalized Poisson Boltzmann equations with multiple species, such as those of Na^+ , K^+ , Ca^{2+} , Cl^- , as in the real biological environment. We also study the corresponding results for higher dimensional cases.

1.1. Main Theorems. Hereafter, we set $\phi = \phi(x)$ as the solution of the equation (1.14) with the boundary condition (1.19) [18, 23, 25, 26]. Actually, the solution ϕ may depend on the parameter ϵ and should be denoted as ϕ_ϵ but we denote it as ϕ for simplicity. The equation (1.14) with the boundary condition (1.19) can be regarded as the Euler-Lagrange equation of the energy functional written as

$$E_\epsilon[u] = \frac{\epsilon^2}{2} \int_{-1}^1 |u'|^2 dx + \alpha \log \left(\int_{-1}^1 e^u dx \right) + \beta \log \left(\int_{-1}^1 e^{-u} dx \right) + \frac{\epsilon^2}{2\eta\epsilon^2} [(\phi_0(1) - u(1))^2 + (\phi_0(-1) - u(-1))^2], \quad (1.20)$$

for $u \in H^1((-1, 1))$. Using the standard Direct method, we get the energy minimizer of E_ϵ over $H^1((-1, 1))$. Applying the standard elliptic regularity theory (cf. [3]), we obtain the classical solution of (1.14) with the boundary condition (1.19). The uniqueness theorem is stated as follows:

THEOREM 1.1. (*Uniqueness Theorem*) *There exists at most one $\phi \in C^\infty((-1, 1)) \cap C^2([-1, 1])$ solution of the equation (1.14) with the boundary condition (1.19).*

We will give the proof of this theorem in Appendix (Section 7).

Suppose $\alpha = \beta$. When $\phi_0(-1) = \phi_0(1) = A$, one gets $\phi \equiv A$ trivially by Theorem 1.1. Note that the equation (1.14) is invariant under the replacement of ϕ by $\phi + C$, for any nonzero constant C . Hence for the case of $\phi_0(-1) \neq \phi_0(1)$, without loss of generality, we may assume $\phi_0(-1) = -\phi_0(1) > 0$ by adjusting a suitable constant C . Such an assumption and Theorem 1.1 can assure the solution ϕ is an odd function on $[-1, 1]$ i.e., $\phi(-x) = -\phi(x), \forall x \in [-1, 1]$. Consequently, we have $\int_{-1}^1 e^\phi dx = \int_{-1}^1 e^{-\phi} dx$ which is crucial to study the equation (1.14). Now we state the interior estimate of ϕ as follows:

THEOREM 1.2. (*Interior Estimates*) *Assume $\alpha = \beta > 0$ and $\phi_0(1) = -\phi_0(-1) > 0$. For $\epsilon > 0$, let $\phi \in C^\infty((-1, 1)) \cap C^2([-1, 1])$ be the solution of the equation (1.14) with the boundary condition (1.19). Then the solution ϕ is odd, monotonically increasing on $[-1, 1]$, convex on $(0, 1)$, and concave on $(-1, 0)$. Moreover, there exists $M > 0$ independent of ϵ such that*

$$|\phi(x)| \leq \phi_0(1) \left(e^{-\frac{M}{\epsilon}(1+x)} + e^{-\frac{M}{\epsilon}(1-x)} \right), \quad \forall x \in [-1, 1], \quad \epsilon > 0. \quad (1.21)$$

At the boundary $x = 1$, the asymptotic behavior of $\phi(x)$ may depend on η_ϵ . Here we study three cases for $\eta_\epsilon > 0$: (i) $\frac{\epsilon}{\eta_\epsilon} \rightarrow 0$ as $\epsilon \downarrow 0$, (ii) $\eta_\epsilon = \gamma\epsilon$ and (iii) $\frac{\epsilon}{\eta_\epsilon} \rightarrow \infty$ as $\epsilon \downarrow 0$. The boundary estimates of these cases are presented as follows:

THEOREM 1.3. (*Boundary Estimates*) *Under the same hypotheses as in Theorem 1.2, we have*

(i) *if $\frac{\epsilon}{\eta_\epsilon} \rightarrow 0$ as $\epsilon \downarrow 0$, then $\lim_{\epsilon \downarrow 0} |\phi(x)| = 0$ uniformly in $[-1, 1]$, and $\lim_{\epsilon \downarrow 0} \eta_\epsilon \phi'(1) = \phi_0(1)$,*

(ii) *if $\eta_\epsilon = \gamma\epsilon$, where $\gamma > 0$ is a constant independent of ϵ , then*

$$\lim_{\epsilon \downarrow 0} \phi(1) = \phi^* \quad \text{and} \quad \lim_{\epsilon \downarrow 0} \epsilon \phi'(1) = \sqrt{\alpha} \left(e^{\phi^*/2} - e^{-\phi^*/2} \right), \quad (1.22)$$

where $0 < \phi^* < \phi_0(1)$ is uniquely determined by

$$\phi_0(1) - \phi^* = \gamma \sqrt{\alpha} \left(e^{\phi^*/2} - e^{-\phi^*/2} \right), \quad (1.23)$$

(iii) *if $\frac{\epsilon}{\eta_\epsilon} \rightarrow \infty$ as $\epsilon \downarrow 0$, then*

$$\lim_{\epsilon \downarrow 0} \phi(1) = \phi_0(1) \quad \text{and} \quad \lim_{\epsilon \downarrow 0} \epsilon \phi'(1) = \sqrt{\alpha} \left(e^{\phi_0(1)/2} - e^{-\phi_0(1)/2} \right). \quad (1.24)$$

For the case (ii) and (iii), the asymptotic behavior of ϕ near the boundary $x = 1$ can be represented by

$$\phi_1^\epsilon(x) \leq \phi(x) \leq \phi_2^\epsilon(x) \quad \text{and} \quad \phi_i^\epsilon(x) = 2 \log \left(\frac{a_i^\epsilon + b_i^\epsilon e^{-\frac{\epsilon}{c}(1-x)}}{a_i^\epsilon - b_i^\epsilon e^{-\frac{\epsilon}{c}(1-x)}} \right), \quad (1.25)$$

for $x \in (0, 1)$, $\epsilon > 0$, $i = 1, 2$. Here a_i^ϵ 's, b_i^ϵ 's and c_i^ϵ 's are constants satisfying

$$a_i^\epsilon \rightarrow e^{s/2} + 1, \quad b_i^\epsilon \rightarrow e^{s/2} - 1 \quad \text{and} \quad c_i^\epsilon \rightarrow \sqrt{\alpha} \quad \text{as} \quad \epsilon \downarrow 0, \quad \text{for} \quad i = 1, 2,$$

where

$$s = \begin{cases} \phi^* & \text{if } \eta_\epsilon = \gamma\epsilon, \\ \phi_0(1) & \text{if } \lim_{\epsilon \downarrow 0} \frac{\epsilon}{\eta_\epsilon} = \infty, \end{cases}$$

and ϕ^* is defined in (1.23).

Theorem 1.3 (i) shows that there is no boundary layer if $\frac{\epsilon}{\eta_\epsilon} \rightarrow 0$ as $\epsilon \downarrow 0$. Theorem 1.3 (ii) and (iii) provide the existence of boundary layers having asymptotic behaviors exhibited by (1.25). Consequently, we obtain the boundary estimates of ϕ at $x = -1$ since the solution ϕ is an odd function on $[-1, 1]$. One may refer to Figure 5.1 for the solution profile.

Theorem 1.2 and 1.3 imply $\int_{-1}^1 e^{\pm\phi} dx \rightarrow 2$ as $\epsilon \rightarrow 0$. Hence the PB_n equation (1.14) is asymptotically close to the following PB equation:

$$\epsilon^2 \phi_{xx} = \frac{\alpha}{2} e^\phi - \frac{\beta}{2} e^{-\phi}, \quad \text{for } x \in (-1, 1), \quad (1.26)$$

i.e., the equation (1.15) with $\alpha_1 = \alpha/2 = \beta/2 = \beta_1$. It would be expected that as $\alpha = \beta$, the solutions of the PB_n equation (1.14) and the PB equation (1.26) have the same asymptotic behavior. Such a result can be stated as follows:

THEOREM 1.4. *Assume $\alpha = \beta > 0$ and $\phi_0(-1) = -\phi_0(1) < 0$. Under the boundary condition (1.19), let $\phi, w \in C^\infty((-1, 1)) \cap C^2([-1, 1])$ be the solution of (1.14) and (1.26), respectively. Then w is odd, monotone increasing on $[-1, 1]$, convex on $(0, 1)$, and concave on $(-1, 0)$. Furthermore,*

$$\lim_{\epsilon \downarrow 0} |\phi(x) - w(x)| = 0, \quad \forall x \in [-1, 1]. \quad (1.27)$$

The numerical results of the maximum norms $\|\phi - w\|_\infty$'s are presented in Table ?? which may support Theorem 1.4. To see the difference between the PB_n equation (1.14) and the PB equation (1.26), we set $\alpha < \beta$ and study the asymptotic behaviors of the solution ϕ of (1.14) and the solution w of (1.26) as ϵ goes to zero. The interior estimates of ϕ are stated as follows:

THEOREM 1.5. *(Interior Estimates) Assume $0 < \alpha < \beta$. For $\epsilon > 0$, let $\phi \in C^\infty((-1, 1)) \cap C^2([-1, 1])$ be the solution of (1.14) with the boundary condition (1.19). For any compact subset K of $(-1, 1)$, we have*

- (i) $\sup_{x, y \in K} |\phi(x) - \phi(y)|$ exponentially converges to zero as ϵ goes to zero.
- (ii) There exists positive constant $C_{K, \alpha, \beta}$ depending only on α, β and K such that

$$\left| \phi(x) - \phi(\pm 1) - \log \frac{1}{\epsilon^2} \right| \leq C_{K, \alpha, \beta}, \quad (1.28)$$

for all $x \in K$.

Theorem 1.5 (ii) presents that the interior value $\phi(x) - \phi(\pm 1)$ tends to infinity with the leading order term $\log \frac{1}{\epsilon^2}$ and the second order term $O(1)$ as ϵ approaches zero. Note that this also shows $\lim_{\epsilon \downarrow 0} \frac{\phi(x) - \phi(\pm 1)}{\log \epsilon^{-2}} = 1$ uniformly in any compact subset K of $(-1, 1)$.

When $\phi_0(1) = \phi_0(-1)$, the asymptotic behaviors of $\phi(-1)$, $\phi(1)$ are depicted by

$$\lim_{\epsilon \downarrow 0} \phi(1) = \phi_0(1) \quad \text{if } \lim_{\epsilon \downarrow 0} \frac{\eta_\epsilon}{\epsilon^2} = 0, \quad (1.29)$$

$$\phi(1) = \phi_0(1) + \frac{\gamma(\beta - \alpha)}{2} \quad \text{if } \eta_\epsilon = \gamma\epsilon^2, \quad (1.30)$$

$$\phi(1) = \phi_0(1) + \frac{\eta_\epsilon(\beta - \alpha)}{2\epsilon^2} \rightarrow \infty \quad \text{if } \lim_{\epsilon \downarrow 0} \frac{\eta_\epsilon}{\epsilon^2} = \infty, \quad (1.31)$$

where γ is a positive constant independent of ϵ . These formulas can be proved by Remark 3.1 in Section 3. On the other hand, as $\phi_0(1) \neq \phi_0(-1)$, the asymptotic behaviors of $\phi(-1)$, $\phi(1)$ are given by

THEOREM 1.6. *Assume $0 < \alpha < \beta$ and $\phi_0(1) \neq \phi_0(-1)$. Let $\phi \in C^\infty((-1, 1)) \cap C^2([-1, 1])$ be the solution of the equation (1.14) with the boundary condition (1.19). Then ϕ has the following properties at $x = \pm 1$:*

(i) *If $\frac{\eta_\epsilon}{\epsilon^2} \rightarrow 0$ as $\epsilon \downarrow 0$, then $\lim_{\epsilon \downarrow 0} \phi(1) = \phi_0(1)$ and $\lim_{\epsilon \downarrow 0} \phi(-1) = \phi_0(-1)$ and*

$$\lim_{\epsilon \downarrow 0} \epsilon^2 \phi'(1) = \frac{(\alpha - \beta)e^{\phi_0(-1)/2}}{e^{\phi_0(1)/2} + e^{\phi_0(-1)/2}}, \quad \lim_{\epsilon \downarrow 0} \epsilon^2 \phi'(-1) = \frac{(\beta - \alpha)e^{\phi_0(1)/2}}{e^{\phi_0(1)/2} + e^{\phi_0(-1)/2}}; \quad (1.32)$$

(ii) *If $\eta_\epsilon = \gamma\epsilon^2$ for some positive constant γ independent of ϵ , then $\lim_{\epsilon \downarrow 0} \phi(1) = \phi_1^*$ and $\lim_{\epsilon \downarrow 0} \phi(-1) = \phi_2^*$ satisfy*

$$\begin{cases} \phi_1^* + \phi_2^* = \phi_0(1) + \phi_0(-1) + \gamma(\beta - \alpha), \\ (\phi_1^* - \phi_0(1))e^{\phi_1^*/2} - (\phi_2^* - \phi_0(-1))e^{\phi_2^*/2} = 0, \\ \phi_1^* > \phi_0(1), \phi_2^* > \phi_0(-1), \end{cases} \quad (1.33)$$

and $\lim_{\epsilon \downarrow 0} \epsilon^2 \phi'(1) = \frac{\phi_0(1) - \phi_1^*}{\gamma}$ and $\lim_{\epsilon \downarrow 0} \epsilon^2 \phi'(-1) = -\frac{\phi_0(-1) - \phi_2^*}{\gamma}$;

(iii) *If $\frac{\eta_\epsilon}{\epsilon^2} \rightarrow \infty$ as $\epsilon \downarrow 0$, then $\lim_{\epsilon \downarrow 0} (\phi(-1) - \phi(1)) = 0$ and*

$$\lim_{\epsilon \downarrow 0} \epsilon^2 \phi'(-1) = -\lim_{\epsilon \downarrow 0} \epsilon^2 \phi'(1) = -\frac{\alpha - \beta}{2}. \quad (1.34)$$

As $\phi_0(1) = \phi_0(-1)$, the boundary estimates of ϕ are given by

THEOREM 1.7. *(Boundary Estimates) Under the same hypotheses of Theorem 1.5, if $\phi_0(1) = \phi_0(-1)$, then*

$$\begin{aligned} \phi_1^\epsilon(x) &\leq \phi(x) - \phi(1) - \log \frac{(\alpha - \beta)^2}{2\epsilon^2} \leq \phi_2^\epsilon(x), \\ \phi_1^\epsilon(x) &= 2 \log \frac{1 - d_{1,1}^\epsilon e^{-\frac{c_{1,1}^\epsilon}{\epsilon}(1-x)}}{\sqrt{\alpha + \beta} \left(1 + d_{1,2}^\epsilon e^{-\frac{c_{1,1}^\epsilon}{\epsilon}(1-x)}\right)}, \quad \phi_2^\epsilon(x) = 2 \log \frac{1 - d_{2,1}^\epsilon e^{-\frac{c_{2,2}^\epsilon}{\epsilon}(1-x)}}{1 + d_{2,2}^\epsilon e^{-\frac{c_{2,2}^\epsilon}{\epsilon}(1-x)}}, \end{aligned}$$

for $\epsilon > 0$ sufficiently small and $x \in (y_\epsilon, 1)$, where $0 < y_\epsilon < 1$ is a constant sufficiently close to 1 and $c_{i,j}^\epsilon$'s, $d_{i,j}^\epsilon$'s ($i, j = 1, 2$) are constants satisfying $c_{1,1}^\epsilon \rightarrow \sqrt{2}(\alpha + \beta)$, $c_{2,2}^\epsilon \rightarrow \sqrt{\frac{3}{2}}(\alpha\beta)^{1/4}$, $d_{1,1}^\epsilon, d_{1,2}^\epsilon$, and $d_{2,1}^\epsilon \rightarrow 1$ and $d_{2,2}^\epsilon \rightarrow 2 + \sqrt{3}$ as $\epsilon \downarrow 0$.

Similar results also hold for $\phi_0(1) \neq \phi_0(-1)$. One may refer to Figure 5.2-5.4 for the solution profile.

1.2. Organization of the paper. The rest of this paper is organized as follows: The proof of the electroneutral cases ($\alpha = \beta$), Theorem 1.2 and 1.3 are showed in Section 2. In Section 3, we prove the non-electroneutral cases, Theorem 1.5-1.7. Theorem 1.4 and 4.2 related to the difference/comparison between the PB_n equation (1.14) and the PB equation (1.26) are given in Section 4. In Section 7, we give the proof of Theorem 1.1 and demonstrate some important facts of the PB_n equation (1.14). In Section 5, the numerical computations to solve the PB_n equation (1.14) and the PB equation (1.15) are performed using convex iteration method based on finite element discretization [35].

2. Electroneutral cases: Proof of Theorem 1.2 and 1.3.

2.1. Proof of Theorem 1.2. Let $\psi(x) = -\phi(-x)$ for $x \in [-1, 1]$. Since $\alpha = \beta$ and $\phi_0(1) = -\phi_0(-1) > 0$, it is obvious that ψ is also a solution of the equation (1.14) with the boundary condition (1.19). Hence by Theorem 1.1, $\psi \equiv \phi$ i.e., ϕ is an odd function on $[-1, 1]$. Consequently,

$$\int_{-1}^1 e^{\phi(y)} dy = \int_{-1}^1 e^{-\phi(y)} dy, \phi''(0) = 0, \phi(0) = 0. \quad (2.1)$$

Moreover, (2.1), and $\alpha = \beta$ may transform (1.14) and (7.3) into

$$\epsilon^2 \phi''(x) \phi(x) = \frac{\alpha}{\int_{-1}^1 e^{\phi(y)} dy} \left(e^{\phi(x)} - e^{-\phi(x)} \right) \phi(x) \geq 0, \forall x \in (-1, 1), \quad (2.2)$$

and

$$\epsilon^2 \phi''(x) \phi'(x) \geq \frac{\alpha}{\int_{-1}^1 e^{\phi(y)} dy} \int_0^x \left(e^{\phi(y)} + e^{-\phi(y)} \right) (\phi'(y))^2 dy, \forall x \in [0, 1], \quad (2.3)$$

which imply that ϕ'' , ϕ' and ϕ do not change sign and share the same sign on $[0, 1]$. In particular, we have $\phi(1)\phi'(1) \geq 0$. By the boundary condition (1.19) and $\eta_\epsilon \geq 0$, we get $0 < \phi(1), \eta_\epsilon \phi'(1) \leq \phi_0(1)$. So ϕ, ϕ' and ϕ'' are non-negative on $(0, 1]$ i.e., ϕ is convex and monotonically increasing on $(0, 1]$. On the other hand, since ϕ is odd on $[-1, 1]$, then ϕ is concave and monotonically increasing on $[-1, 0)$. Thus $|\phi(x)| \leq \phi_0(1)$ for $x \in [-1, 1]$ and (2.2) implies

$$\begin{aligned} \frac{\epsilon^2}{2} (\phi^2(x))'' &\geq \epsilon^2 \phi(x) \phi''(x) \\ &= \frac{\alpha}{\int_{-1}^1 e^{\phi(y)} dy} \left(e^{\phi(x)} - e^{-\phi(x)} \right) \phi(x) \geq \frac{\alpha}{e^{\phi_0(1)}} \phi^2(x), \forall x \in (-1, 1). \end{aligned}$$

By the standard comparison theorem, we have

$$\phi^2(x) \leq \phi_0^2(1) \left(e^{-\frac{\tilde{M}}{\epsilon}(1+x)} + e^{-\frac{\tilde{M}}{\epsilon}(1-x)} \right), \forall x \in [-1, 1],$$

where $\tilde{M} = \sqrt{\frac{2\alpha}{e^{\phi_0(1)}}}$.

Therefore, we have (1.21) and complete the proof of Theorem 1.2.

2.2. Proof of Theorem 1.3. In order to prove Theorem 1.3, we need the following lemma:

LEMMA 2.1. *Assume $\alpha = \beta > 0$ and $-\phi_0(-1) = \phi_0(1) > 0$. Let $\phi \in C^\infty((-1, 1)) \cap C^2([-1, 1])$ satisfy (1.14) and (1.19). Then $\int_{-1}^1 e^{\phi(y)} dy = \int_{-1}^1 e^{-\phi(y)} dy \rightarrow 2$ and $C_\epsilon \rightarrow -\alpha$ as $\epsilon \downarrow 0$, where C_ϵ is defined in (7.1).*

Proof. By (1.21) and the dominated convergence theorem, we obtain

$$\lim_{\epsilon \downarrow 0} \int_{-1}^1 e^{\phi(y)} dy = \lim_{\epsilon \downarrow 0} \int_{-1}^1 e^{-\phi(y)} dy = 2. \quad (2.4)$$

Now we claim that $C_\epsilon \rightarrow -\alpha$ as $\epsilon \downarrow 0$. Since $\alpha = \beta$ and $\int_{-1}^1 e^{\phi(y)} dy = \int_{-1}^1 e^{-\phi(y)} dy$, then (7.1) becomes

$$\frac{\epsilon^2}{2} (\phi'(x))^2 = \frac{\alpha}{\int_{-1}^1 e^{\phi(y)} dy} \left(e^{\phi(x)} + e^{-\phi(x)} \right) + C_\epsilon \quad \text{for } |x| < 1. \quad (2.5)$$

Setting $x = 0$ in (2.5) and using the third condition of (2.1), we get

$$C_\epsilon = \frac{-2\alpha}{\int_{-1}^1 e^{\phi(y)} dy} + \frac{\epsilon^2}{2} (\phi'(0))^2 \geq \frac{-2\alpha}{\int_{-1}^1 e^{\phi(y)} dy}. \quad (2.6)$$

On the other hand, due to the third condition of (2.1) and $0 < \phi(1) \leq \phi_0(1)$, there exists $x_* \in (0, 1)$ such that $0 < \phi'(x_*) = \phi(1) - \phi(0) \leq \phi_0(1)$. Letting $x = x_*$ in (2.5), we obtain

$$C_\epsilon \leq \frac{\epsilon^2}{2} \phi_0^2(1) - \frac{\alpha}{\int_{-1}^1 e^{\phi(y)} dy} \left(e^{\phi(x_*)} + e^{-\phi(x_*)} \right) \leq \frac{\epsilon^2}{2} \phi_0^2(1) - \frac{2\alpha}{\int_{-1}^1 e^{\phi(y)} dy}. \quad (2.7)$$

By (2.4), we combine (2.6) and (2.7) to get $C_\epsilon \rightarrow -\alpha$ as $\epsilon \downarrow 0$. \square

Now we want to give the proof of Theorem 1.3. By (2.5) – (2.7), it is easy to check that

$$0 \leq \epsilon^2 (\phi'(x))^2 - \frac{2\alpha}{\int_{-1}^1 e^{\phi(y)} dy} \left(e^{\phi(x)/2} - e^{-\phi(x)/2} \right)^2 \leq \epsilon^2 \phi_0^2(1). \quad (2.8)$$

By Theorem 1.2, ϕ is positive and monotonically increasing in $(0, 1]$. Hence

$$\begin{aligned} & \sqrt{\frac{2\alpha}{\int_{-1}^1 e^{\phi(y)} dy}} \left(e^{\phi(x)/2} - e^{-\phi(x)/2} \right) \\ & \leq \epsilon \phi'(x) \leq \sqrt{\frac{2\alpha}{\int_{-1}^1 e^{\phi(y)} dy}} \left(e^{\phi(x)/2} - e^{-\phi(x)/2} \right) + \epsilon \phi_0(1), \quad \text{for } x \in (0, 1]. \end{aligned} \quad (2.9)$$

Setting $x = 1$ in (2.9) and using the boundary condition (1.19), we get

$$\begin{aligned} & \sqrt{\frac{2\alpha}{\int_{-1}^1 e^{\phi(y)} dy}} \left(e^{\phi(1)/2} - e^{-\phi(1)/2} \right) \leq \frac{\epsilon}{\eta_\epsilon} (\phi_0(1) - \phi(1)) \\ & = \epsilon \phi'(1) \leq \sqrt{\frac{2\alpha}{\int_{-1}^1 e^{\phi(y)} dy}} \left(e^{\phi(1)/2} - e^{-\phi(1)/2} \right) + \epsilon \phi_0(1). \end{aligned} \quad (2.10)$$

Now we deal with (2.10) in following three cases:

Case (i): $\frac{\epsilon}{\eta_\epsilon} \rightarrow 0$ as $\epsilon \downarrow 0$.

We remark that $0 < \phi(1) \leq \phi_0(1)$. Then (2.10) gives

$$\begin{aligned} \sqrt{\frac{2\alpha}{\int_{-1}^1 e^{\phi(y)} dy}} \left(e^{\phi(1)/2} - 1 \right) &\leq \sqrt{\frac{2\alpha}{\int_{-1}^1 e^{\phi(y)} dy}} \left(e^{\phi(1)/2} - e^{-\phi(1)/2} \right) \\ &\leq \frac{\epsilon}{\eta_\epsilon} (\phi_0(1) - \phi(1)) \leq \frac{\epsilon}{\eta_\epsilon} \phi_0(1), \end{aligned}$$

i.e.,

$$0 < \phi(1) \leq 2 \log \left(1 + \frac{\epsilon}{\eta_\epsilon} \phi_0(1) \sqrt{\frac{\int_{-1}^1 e^{\phi(y)} dy}{2\alpha}} \right). \quad (2.11)$$

Since ϕ is a monotone increasing and odd function on $[-1, 1]$, we have

$$|\phi(x)| \leq 2 \log \left(1 + \frac{\epsilon}{\eta_\epsilon} \phi_0(1) \sqrt{\frac{\int_{-1}^1 e^{\phi(y)} dy}{2\alpha}} \right), \quad \forall x \in [-1, 1]. \quad (2.12)$$

Hence (2.4) and (2.12) imply $\lim_{\epsilon \downarrow 0} |\phi(x)| = 0$, uniformly, in $[-1, 1]$. Then the boundary condition (1.19) gives $\lim_{\epsilon \downarrow 0} \eta_\epsilon \phi'(1) = \phi_0(1)$.

Therefore, we complete the proof of Theorem 1.3 (i).

Case (ii): $\eta_\epsilon = \gamma\epsilon$ for some positive constant γ independent of ϵ .

Due to $|\phi(1)| \leq \phi_0(1)$, we assume $\limsup_{\epsilon \downarrow 0} \phi(1) = \phi_s^*$ and $\liminf_{\epsilon \downarrow 0} \phi(1) = \phi_i^*$. Then (2.10) implies ϕ_s^* and ϕ_i^* are the roots of (1.23) with $0 < \phi_i^* \leq \phi_s^* < \phi_0(1)$. It is obvious that the equation (1.23) has only one root in the interval $(0, \phi_0(1))$. Consequently, $\phi_s^* = \phi_i^* \equiv \phi^*$ is the unique root of (1.23) in the interval $(0, \phi_0(1))$.

Therefore, we have $\lim_{\epsilon \downarrow 0} \phi(1) = \phi^* \in (0, \phi_0(1))$ which can be combined with (2.10) to get (1.22) and complete the proof of Theorem 1.3 (ii).

Case (iii): $\frac{\epsilon}{\eta_\epsilon} \rightarrow \infty$ as $\epsilon \downarrow 0$

For this case, we need the following claim:

Claim 1. There exists $C > 0$ independent of ϵ such that $\phi(1) \geq C$ for all $\epsilon > 0$.

Proof. We shall prove the claim by contradiction. Suppose Claim 1 is false. Then there exists a sequence $\{\epsilon_j\}_{j \in \mathbb{N}}$ such that $\lim_{j \rightarrow \infty} \epsilon_j = 0$ and $\lim_{j \rightarrow \infty} \phi_{\epsilon_j}(1) = 0$. Putting $\epsilon = \epsilon_j$ in (2.10) and taking $j \rightarrow \infty$, we may obtain $\lim_{j \rightarrow \infty} \frac{\epsilon_j}{\eta_{\epsilon_j}} = 0$ which contradicts with the assumption $\frac{\epsilon}{\eta_\epsilon} \rightarrow \infty$ as $\epsilon \downarrow 0$. Here we have used Lemma 2.1 and the fact that $\phi_0(1) > 0$. Therefore, we may complete the proof of Claim 1. \square

Claim 1 and Lemma 2.1 imply

$$0 < C_1 \leq C(\epsilon, \alpha, \phi_0(1)) = \sqrt{\frac{2\alpha}{\int_{-1}^1 e^{\phi(y)} dy}} \left(e^{\phi(1)/2} - e^{-\phi(1)/2} \right) \leq C_2, \quad (2.13)$$

for $0 < \epsilon < 1$, where C_1, C_2 are positive constants independent of ϵ . Thus by (2.10), we have

$$C(\epsilon, \alpha, \phi_0(1)) \leq \frac{\epsilon}{\eta_\epsilon} (\phi_0(1) - \phi(1)) = \epsilon \phi'(1) \leq C(\epsilon, \alpha, \phi_0(1)) + \epsilon \phi_0(1), \quad (2.14)$$

$\forall \epsilon \in (0, 1)$ and then

$$(1 - \eta_\epsilon)\phi_0(1) - \frac{\eta_\epsilon}{\epsilon}C(\epsilon, \alpha, \phi_0(1)) \leq \phi(1) \leq \phi_0(1) - \frac{\eta_\epsilon}{\epsilon}C(\epsilon, \alpha, \phi_0(1)). \quad (2.15)$$

Note that $\eta_\epsilon \rightarrow 0$ as $\epsilon \rightarrow 0$ and $\phi_0(1)$ is independent of ϵ . Hence (2.13) and (2.15) imply $\lim_{\epsilon \downarrow 0} \phi(1) = \phi_0(1)$, and

$$\lim_{\epsilon \downarrow 0} C(\epsilon, \alpha, \phi_0(1)) = \sqrt{\alpha} \left(e^{\phi_0(1)/2} - e^{-\phi_0(1)/2} \right). \quad (2.16)$$

Here we have used the assumption $\frac{\eta_\epsilon}{\epsilon} \rightarrow 0$ i.e., $\frac{\epsilon}{\eta_\epsilon} \rightarrow \infty$ as $\epsilon \rightarrow 0$. As a consequence, by (2.14) and (2.16), we obtain $\lim_{\epsilon \downarrow 0} \epsilon \phi'(1) = \sqrt{\alpha} (e^{\phi_0(1)/2} - e^{-\phi_0(1)/2})$ and complete the proof of Theorem 1.3 (iii).

Now we want to prove (1.25). Let $\delta_\epsilon(\alpha) = \sqrt{\frac{2\alpha}{\int_{-1}^1 e^{\phi(y)} dy}}$. Then Lemma 2.1 gives

$$\lim_{\epsilon \rightarrow 0} \delta_\epsilon(\alpha) = \sqrt{\alpha}. \quad (2.17)$$

Using (2.9), and the fact that $\phi > 0$ in $(0, 1]$ we have

$$\frac{\phi'(z)}{e^{\phi(z)/2} - e^{-\phi(z)/2} + \frac{\varepsilon\phi_0(1)}{\delta_\epsilon(\alpha)}} \leq \frac{\delta_\epsilon(\alpha)}{\epsilon} \leq \frac{\phi'(z)}{e^{\phi(z)/2} - e^{-\phi(z)/2}}, \quad \forall z \in (0, 1]. \quad (2.18)$$

For $x \in (0, 1)$, integrating (2.18) over $(x, 1)$, one can derive that

$$\begin{aligned} & \int_x^1 \frac{\phi'(z) dz}{e^{\phi(z)/2} - e^{-\phi(z)/2} + \frac{\varepsilon\phi_0(1)}{\delta_\epsilon(\alpha)}} = \int_{e^{\phi(x)/2}}^{e^{\phi(1)/2}} \frac{2dt}{t^2 + \frac{\varepsilon\phi_0(1)}{\delta_\epsilon(\alpha)}t - 1} \\ & = \int_{e^{\phi(x)/2}}^{e^{\phi(1)/2}} \frac{2dt}{(t - t_+)(t - t_-)} = \frac{2}{t_+ - t_-} \log \frac{(e^{\phi(1)/2} - t_+)(e^{\phi(x)/2} - t_-)}{(e^{\phi(1)/2} - t_-)(e^{\phi(x)/2} - t_+)}, \end{aligned} \quad (2.19)$$

where $t_\pm = \frac{1}{2} \left(-\frac{\varepsilon\phi_0(1)}{\delta_\epsilon(\alpha)} \pm \sqrt{\left(\frac{\varepsilon\phi_0(1)}{\delta_\epsilon(\alpha)}\right)^2 + 4} \right)$.

Similarly, we also have that

$$\int_x^1 \frac{\phi'(z) dz}{e^{\phi(z)/2} - e^{-\phi(z)/2}} = \log \frac{(e^{\phi(1)/2} - 1)(e^{\phi(x)/2} + 1)}{(e^{\phi(1)/2} + 1)(e^{\phi(x)/2} - 1)}, \quad (2.20)$$

Hence (2.18)-(2.20) imply

$$\begin{aligned} & 2 \log \frac{t_+(e^{t_\epsilon/2} - t_-) - t_-(e^{t_\epsilon/2} - t_+)e^{-(t_+ - t_-)\delta_\epsilon(\alpha)(1-x)/2\epsilon}}{(e^{t_\epsilon/2} - t_-) - (e^{t_\epsilon/2} - t_+)e^{-(t_+ - t_-)\delta_\epsilon(\alpha)(1-x)/2\epsilon}} \\ & \leq \phi(x) \leq 2 \log \frac{(e^{t_\epsilon/2} + 1) + (e^{t_\epsilon/2} - 1)e^{-\delta_\epsilon(\alpha) \cdot (1-x)/\epsilon}}{(e^{t_\epsilon/2} + 1) - (e^{t_\epsilon/2} - 1)e^{-\delta_\epsilon(\alpha)(1-x)/\epsilon}}, \quad \forall x \in (0, 1), \end{aligned} \quad (2.21)$$

where $t_\epsilon = \phi(1)$. Note that (2.17) shows $t_\pm \rightarrow \pm 1$ and $\delta_\epsilon(\alpha) \rightarrow \sqrt{\alpha}$ as $\epsilon \downarrow 0$. Thus (1.22), (1.24) and (2.21) give (1.25) provided $\eta_\epsilon = \gamma\epsilon$ or $\frac{\epsilon}{\eta_\epsilon} \rightarrow \infty$ as $\epsilon \downarrow 0$.

Therefore, this complete the proof of Theorem 1.3.

3. Non-electroneutral cases: Proof of Theorem 1.5 – 1.7. In this section, we assume that $0 < \alpha < \beta$ in the whole section. To prove Theorem 1.5 – 1.7, we need the following lemma:

LEMMA 3.1. Let $B(\phi)(x) = \frac{\alpha}{\int_{-1}^1 e^{\phi(y)} dy} e^{\phi(x)} - \frac{\beta}{\int_{-1}^1 e^{-\phi(y)} dy} e^{-\phi(x)}$ for $x \in [-1, 1]$, where ϕ is the solution of (1.14) and (1.19). Then

- (i) $\int_{-1}^1 B(\phi)(x) dx = \alpha - \beta < 0$;
- (ii) $B(\phi)(x)$ is monotonically increasing to $\phi(x)$ in the sense that if $\phi(x) < \phi(z)$, then $B(\phi)(x) < B(\phi)(z)$;
- (iii) $B(\phi)(x)$ has no negative interior minima and positive interior maxima.
- (iv) Assume that

$$\sup_{\epsilon > 0} \int_{-1}^1 e^{\phi(y)} dy \int_{-1}^1 e^{-\phi(y)} dy < \infty. \quad (3.1)$$

Then there exists $C_5 > 0$ independent of ϵ such that $\forall x \in [-1, 1]$

$$B(\phi)(x) \geq \min\{B(\phi)(1), B(\phi)(-1)\} \left(e^{-\frac{C_5(1+x)}{\epsilon}} + e^{-\frac{C_5(1-x)}{\epsilon}} \right). \quad (3.2)$$

Proof. It is easy to get (i) and (ii) immediately from the definition of $B(\phi)(x)$. Since ϕ solves (1.14), then $B(\phi)(x)$ satisfies

$$\epsilon^2 B(\phi)''(x) = \left(\frac{\alpha}{\int_{-1}^1 e^{\phi(y)} dy} e^{\phi(x)} + \frac{\beta}{\int_{-1}^1 e^{-\phi(y)} dy} e^{-\phi(x)} + \epsilon^2 \phi'^2(x) \right) B(\phi)(x), \quad (3.3)$$

for $x \in (-1, 1)$. Thus (iii) is obtained by (3.3) and the standard maximum principle. Now we want to prove (iv). Note that (i), (ii) and (iii) give

$$\min\{B(\phi)(1), B(\phi)(-1)\} < 0.$$

On the other hand, by (3.1), there exists $C_5 > 0$ independent of ϵ such that

$$\begin{aligned} & \frac{\alpha}{\int_{-1}^1 e^{\phi(y)} dy} e^{\phi(x)} + \frac{\beta}{\int_{-1}^1 e^{-\phi(y)} dy} e^{-\phi(x)} + \epsilon^2 \phi'^2(x) \\ & \geq \frac{2\sqrt{\alpha\beta}}{\sqrt{\int_{-1}^1 e^{\phi(y)} dy} \sqrt{\int_{-1}^1 e^{-\phi(y)} dy}} \geq C_5^2. \end{aligned} \quad (3.4)$$

Next we define the auxiliary function, $v(x)$,

$$v(x) = \min\{B(\phi)(1), B(\phi)(-1)\} \left(e^{-\frac{C_5(1+x)}{\epsilon}} + e^{-\frac{C_5(1-x)}{\epsilon}} \right), \quad \forall x \in [-1, 1].$$

Then $v(x) < 0$ for $x \in [-1, 1]$, $v(\pm 1) \leq \min\{B(\phi)(1), B(\phi)(-1)\} \leq B(\phi)(\pm 1)$ and

$$\begin{aligned} & \epsilon^2 v''(x) - \left(\frac{\alpha}{\int_{-1}^1 e^{\phi(y)} dy} e^{\phi(x)} + \frac{\beta}{\int_{-1}^1 e^{-\phi(y)} dy} e^{-\phi(x)} + \epsilon^2 \phi'^2(x) \right) v(x) \\ & = \left(C_5^2 - \frac{\alpha}{\int_{-1}^1 e^{\phi(y)} dy} e^{\phi(x)} - \frac{\beta}{\int_{-1}^1 e^{-\phi(y)} dy} e^{-\phi(x)} - \epsilon^2 \phi'^2(x) \right) v(x) \geq 0, \quad \forall x \in (-1, 1). \end{aligned}$$

By the standard comparison theorem, $v(x) \leq B(\phi)(x)$ for all $x \in (-1, 1)$.

Therefore, we obtain (3.2) and complete the proof of Lemma 3.1. \square

The interior gradient estimate of ϕ is crucial for the proof of Theorem 1.5 and 1.7. We state the result as follows:

THEOREM 3.2. *Assume $0 < \alpha < \beta$. Let $\phi \in C^\infty((-1, 1)) \cap C^2([-1, 1])$ be the solution of (1.14) and (1.19). Then*

(i) *if $\phi_0(-1) = \phi_0(1)$, then ϕ is even, concave on $(-1, 1)$, increasing on $(-1, 0)$, and decreasing on $(0, 1)$,*

(ii) *if $\phi_0(-1) \neq \phi_0(1)$, then ϕ is concave on $(-1, 1)$ and there exists $x_\epsilon \in (-1, 1)$ such that ϕ is increasing on $(-1, x_\epsilon)$ and decreasing on $(x_\epsilon, 1)$.*

(iii)

$$|\phi'(x)| \leq \max\{|\phi'(-1)|, |\phi'(1)|\} \left(e^{-\frac{\sqrt{2\alpha}}{2\epsilon}(1+x)} + e^{-\frac{\sqrt{2\alpha}}{2\epsilon}(1-x)} \right), \quad \forall x \in (-1, 1). \quad (3.5)$$

Proof. Firstly, we prove (i). Suppose $\phi_0(-1) = \phi_0(1)$. Let $\psi(x) = \phi(-x)$ for $x \in [-1, 1]$. Then it is obvious that ψ also satisfies (1.14) and (1.19). By Theorem 1.1, $\phi \equiv \psi$ i.e., ϕ is an even function on $[-1, 1]$. Hence $\phi'(0) = 0$, along with (7.3), we have

$$\phi''(x)\phi'(x) \geq 0, \quad \forall x \in (0, 1). \quad (3.6)$$

By (3.6), it is easy to check that both ϕ'' and ϕ' never change sign on $(0, 1)$. On the other hand, since ϕ is even on $[-1, 1]$ i.e., $\phi(x) = \phi(-x)$ for $x \in [-1, 1]$, then

$$\phi'(x) = -\phi'(-x), \quad \phi''(x) = \phi''(-x), \quad \forall x \in (0, 1). \quad (3.7)$$

Thus by the second condition of (3.7), the sign of ϕ'' on $(-1, 0)$ is exactly same as the sign of ϕ'' on $(0, 1)$. This implies that ϕ'' never changes sign on $(-1, 1)$. By Lemma 3.1(i) and (1.14), we have that $\phi''(x_0) < 0$ for some $x_0 \in (-1, 1)$. Consequently, $\phi'' \leq 0$ on $(-1, 1)$ and ϕ is concave on $(-1, 1)$. Then by (3.6) and the first condition of (3.7), we have that $\phi'(x) \leq 0$ for $x \in (0, 1)$ and $\phi'(x) \geq 0$ for $x \in (-1, 0)$ i.e., ϕ is increasing on $(-1, 0)$ and decreasing on $(0, 1)$. Therefore, we complete the proof of Theorem 3.2 (i).

For the case of $\phi_0(-1) \neq \phi_0(1)$, without loss of generality, we assume that

$$\phi_0(-1) < \phi_0(1). \quad (3.8)$$

Claim 2. There exists $x_\epsilon \in (-1, 1)$ such that $\phi'(x_\epsilon) = 0$.

Proof. We may prove by contradiction. Suppose $\phi'(x) < 0$ for $x \in (-1, 1)$. Then by (1.19) and (3.8), we have

$$\phi(-1) \geq \phi(1) = \phi_0(1) - \eta_\epsilon \phi'(1) \geq \phi_0(1) > \phi_0(-1) \geq \phi_0(-1) + \eta_\epsilon \phi'(-1) = \phi(-1),$$

which gives a contradiction. On the other hand, suppose $\phi'(x) > 0$ for $x \in (-1, 1)$. Then by (1.19), we have

$$\phi_0(-1) \leq \phi(-1) \leq \phi(x) \leq \phi(1) \leq \phi_0(1), \quad \forall x \in [-1, 1]. \quad (3.9)$$

Hence ϕ satisfies the condition (3.1). Since $\phi(-1) \leq \phi(1)$, then Lemma 3.1(ii) implies $B(\phi)(-1) \leq B(\phi)(1)$. Furthermore, Lemma 3.1(i) and (3.2) give

$$\begin{aligned} \frac{2\epsilon}{C_5} (B(\phi)(-1)) \left(1 - e^{-2C_5/\epsilon}\right) &= (B(\phi)(-1)) \int_{-1}^1 \left(e^{-\frac{C_5(1+x)}{\epsilon}} + e^{-\frac{C_5(1-x)}{\epsilon}}\right) dx \\ &\leq \alpha - \beta. \end{aligned} \quad (3.10)$$

Note that (3.9) provides $|B(\phi)(-1)| = O(1)$. Letting $\epsilon \downarrow 0$ on (3.10), we get $\alpha \geq \beta$ which contradicts with the hypothesis $\alpha < \beta$.

Therefore, we complete the proof of Claim 2. \square

As for (7.3), we integrate (7.2) over (x_ϵ, x) so we have $\phi''(x)\phi'(x) > 0$ for $x \in (x_\epsilon, 1)$. Similarly, we obtain $\phi''(x)\phi'(x) < 0$ for $x \in (-1, x_\epsilon)$. Here we have used the fact that the solution ϕ is nontrivial due to the boundary condition (1.19) and the assumption (3.8). Moreover, the proof of Claim 2 also give that $\phi'(x_1)\phi'(x_2) < 0$ for any $x_1 \in (x_\epsilon, 1)$ and $x_2 \in (-1, x_\epsilon)$. Consequently, ϕ'' never changes sign on $(-1, 1)$. Then by Lemma 3.1(i), we obtain $\phi''(x) \leq 0$ for $x \in (-1, 1)$, $\phi'(x) < 0$ for $x \in (x_\epsilon, 1)$ and $\phi'(x) > 0$ for $x \in (-1, x_\epsilon)$.

Therefore, we complete the proof of Theorem 3.2 (ii).

Now we want to prove (3.5). Firstly, we state a crucial estimate as follows:

Claim 3.

$$4 \leq \int_{-1}^1 e^{\phi(y)} dy \int_{-1}^1 e^{-\phi(y)} dy \leq \frac{4\beta}{\alpha}. \quad (3.11)$$

Proof. The left side of the inequality (3.11) is trivial due to the application of Hölder's inequality. From the proof of Theorem 3.2 (i) and (ii), we have $\phi''(x) \leq 0$ for $x \in (-1, 1)$. Hence by (1.14), we obtain $\frac{\alpha e^{\phi(x)}}{\int_{-1}^1 e^{\phi(y)} dy} - \frac{\beta e^{-\phi(x)}}{\int_{-1}^1 e^{-\phi(y)} dy} = \epsilon^2 \phi''(x) \leq 0$ for $x \in (-1, 1)$ which implies $e^{2\phi(x)} \leq \frac{\beta \int_{-1}^1 e^{\phi(y)} dy}{\alpha \int_{-1}^1 e^{-\phi(y)} dy}$ for $x \in (-1, 1)$. Integrating this inequality over $(-1, 1)$ and using Hölder's inequality, we get

$$\frac{1}{2} \left(\int_{-1}^1 e^{\phi(x)} dx \right)^2 \leq \int_{-1}^1 e^{2\phi(x)} dx \leq \frac{2\beta \int_{-1}^1 e^{\phi(y)} dy}{\alpha \int_{-1}^1 e^{-\phi(y)} dy},$$

which implies the right side of the inequality (3.11) and complete the proof of Claim 3. \square

By (1.14) and (7.1), it is easy to calculate that

$$\frac{\epsilon^2}{2} (\phi'^2)'' = \epsilon^2 (\phi'')^2 + \left(\frac{\alpha}{\int_{-1}^1 e^{\phi(y)} dy} e^{\phi(x)} + \frac{\beta}{\int_{-1}^1 e^{-\phi(y)} dy} e^{-\phi(x)} \right) \phi'^2 \quad (3.12)$$

in $(-1, 1)$. Hence by (3.11) and (3.12), we obtain

$$\frac{\epsilon^2}{2} (\phi'^2)'' \geq 2 \left(\frac{\alpha\beta}{\int_{-1}^1 e^{\phi(y)} dy \int_{-1}^1 e^{-\phi(y)} dy} \right)^{1/2} \phi'^2 \geq \alpha \phi'^2 \quad \text{in } (-1, 1). \quad (3.13)$$

Let $v(x) = \max\{(\phi'(-1))^2, (\phi'(1))^2\} \left(e^{-\frac{\sqrt{2\alpha}}{\epsilon}(1+x)} + e^{-\frac{\sqrt{2\alpha}}{\epsilon}(1-x)} \right)$ for $x \in [-1, 1]$. Then $v(\pm 1) \geq (\phi'(\pm 1))^2$ and $\frac{\epsilon^2}{2} v''(x) = \alpha v(x)$.

Therefore, by (3.13) and the standard comparison theorem, $(\phi'(x))^2 \leq v(x)$ for $x \in (-1, 1)$ which implies (3.5) and complete the proof of Theorem 3.2. \square

REMARK 3.1. (i) By (1.19) and Theorem 3.2, $\phi(1) \geq \phi_0(1)$ and $\phi(-1) \geq \phi_0(-1)$. (ii) By (1.14) and Lemma 3.1 (i), we have

$$\epsilon^2 \int_{-1}^1 \phi''(x) dx = \int_{-1}^1 B(\phi)(x) dx = \alpha - \beta,$$

i.e.,

$$\phi'(1) - \phi'(-1) = \frac{\alpha - \beta}{\epsilon^2}. \quad (3.14)$$

Suppose that $\phi_0(1) = \phi_0(-1)$. Then ϕ is even i.e., $\phi(x) = \phi(-x)$ for $x \in [-1, 1]$ which implies $\phi'(1) = -\phi'(-1)$. Hence by (3.14), $\phi'(1) = -\phi'(-1) = \frac{\alpha - \beta}{2\epsilon^2}$. Furthermore, by (1.19), we obtain (1.29)-(1.31). However, as $\phi_0(1) \neq \phi_0(-1)$, the asymptotic behavior of $\phi(1)$ is not obvious. Now we state results of this case as follows:

LEMMA 3.3. Assume that $0 < \alpha < \beta$ and $\phi_0(1) \neq \phi_0(-1)$. Let $\phi \in C^\infty((-1, 1)) \cap C^2([-1, 1])$ be the solution of (1.14) and (1.19). Then there exist $0 < C_6 \leq C_7 \leq \beta - \alpha$ independent of ϵ such that $\frac{C_6}{\epsilon^2} \leq \phi'(-1)$, $-\phi'(1) \leq \frac{C_7}{\epsilon^2}$ as $0 < \epsilon \ll 1$.

Proof. From Theorem 3.2 (ii), we have $\phi'(-1)$, $-\phi'(1) \geq 0$. Without loss of generality, we assume that $0 \leq -\phi'(1) \leq \phi'(-1)$. Then by (3.14), we obtain

$$\frac{\beta - \alpha}{2\epsilon^2} \leq \phi'(-1) \leq \frac{\beta - \alpha}{\epsilon^2}. \quad (3.15)$$

Setting $x = 1, -1, x_\epsilon$ in (7.1) and using $\phi'(x_\epsilon) = 0$ (see Claim 2), one can easily derive that

$$\frac{\epsilon^2}{2} (\phi'(1))^2 = \frac{\alpha (e^{\phi(1)} - e^{\phi(x_\epsilon)})}{\int_{-1}^1 e^{\phi(y)} dy} + \frac{\beta (e^{-\phi(1)} - e^{-\phi(x_\epsilon)})}{\int_{-1}^1 e^{-\phi(y)} dy}, \quad (3.16)$$

$$\frac{\epsilon^2}{2} (\phi'(-1))^2 = \frac{\alpha (e^{\phi(-1)} - e^{\phi(x_\epsilon)})}{\int_{-1}^1 e^{\phi(y)} dy} + \frac{\beta (e^{-\phi(-1)} - e^{-\phi(x_\epsilon)})}{\int_{-1}^1 e^{-\phi(y)} dy}. \quad (3.17)$$

By (7.4), (3.16) and (3.17), we have

$$\frac{\alpha}{\int_{-1}^1 e^{\phi(y)} dy} e^{\phi(x_\epsilon)} + \frac{\beta}{\int_{-1}^1 e^{-\phi(y)} dy} e^{-\phi(x_\epsilon)} + \frac{\epsilon^2}{4} \int_{-1}^1 (\phi'(x))^2 dx = \frac{\alpha + \beta}{2}. \quad (3.18)$$

Note that by Theorem 3.2 (ii),

$$\phi(\pm 1) \leq \phi(x_\epsilon). \quad (3.19)$$

Thus (3.16)-(3.19) give

$$-\frac{\alpha + \beta}{2} + \frac{\beta}{\int_{-1}^1 e^{-\phi(y)} dy} e^{-\phi(1)} \leq \frac{\epsilon^2}{2} (\phi'(1))^2 \leq \frac{\beta}{\int_{-1}^1 e^{-\phi(y)} dy} e^{-\phi(1)}, \quad (3.20)$$

$$-\frac{\alpha + \beta}{2} + \frac{\beta}{\int_{-1}^1 e^{-\phi(y)} dy} e^{-\phi(-1)} \leq \frac{\epsilon^2}{2} (\phi'(-1))^2 \leq \frac{\beta}{\int_{-1}^1 e^{-\phi(y)} dy} e^{-\phi(-1)}. \quad (3.21)$$

Moreover, (3.15) and (3.21) imply

$$\frac{(\beta - \alpha)^2}{8\epsilon^2} e^{\phi(-1)} \leq \frac{\beta}{\int_{-1}^1 e^{-\phi(y)} dy} \leq \left[\frac{(\beta - \alpha)^2}{2\epsilon^2} + \frac{\alpha + \beta}{2} \right] e^{\phi(-1)}. \quad (3.22)$$

Combining (3.20) and (3.22), we obtain

$$\frac{\epsilon^2}{2} (\phi'(1))^2 \geq -\frac{\alpha + \beta}{2} + \frac{(\beta - \alpha)^2}{8\epsilon^2} e^{\phi(-1) - \phi(1)}. \quad (3.23)$$

By (1.19) and the assumption $0 \leq -\phi'(1) \leq \phi'(-1)$, it is trivial that $\phi(-1) - \phi(1) = \phi_0(-1) - \phi_0(1) + \eta_\epsilon(\phi'(-1) + \phi'(1)) \geq \phi_0(-1) - \phi_0(1)$. Hence by (3.23), there exists $\epsilon^* > 0$ such that

$$\frac{\epsilon^2}{2} (\phi'(1))^2 \geq \frac{(\beta - \alpha)^2}{K\epsilon^2} \quad \text{for } 0 < \epsilon < \epsilon^* \quad (3.24)$$

where $K \geq 8$ is a constant independent of ϵ .

Therefore, by (3.15) and (3.24), we complete the proof of Lemma 3.3. \square

3.1. Proof of Theorem 1.6. To prove Theorem 1.6, we need

Claim 4. Assume that there exists $M > 0$ independent of ϵ such that $\frac{\eta_\epsilon}{\epsilon^2} \leq M$ as $0 < \epsilon \ll 1$. Then

$$\lim_{\epsilon \downarrow 0} \epsilon^2 \left(\phi'(1)e^{\phi(1)/2} + \phi'(-1)e^{\phi(-1)/2} \right) = 0. \quad (3.25)$$

Proof. By (3.16) and (3.17) we have

$$\frac{\epsilon^2}{2} \left[(\phi'(1))^2 e^{\phi(1)} - (\phi'(-1))^2 e^{\phi(-1)} \right] = I_1 - I_2, \quad (3.26)$$

where

$$I_1 = \left[\frac{\alpha}{\int_{-1}^1 e^{\phi(y)} dy} \left(e^{\phi(1)} - e^{\phi(x_\epsilon)} \right) - \frac{\beta}{\int_{-1}^1 e^{-\phi(y)} dy} e^{-\phi(x_\epsilon)} \right] e^{\phi(1)},$$

$$I_2 = \left[\frac{\alpha}{\int_{-1}^1 e^{\phi(y)} dy} \left(e^{\phi(-1)} - e^{\phi(x_\epsilon)} \right) - \frac{\beta}{\int_{-1}^1 e^{-\phi(y)} dy} e^{-\phi(x_\epsilon)} \right] e^{\phi(-1)}.$$

By (1.19), (3.18), (3.19), Lemma 3.3 and Remark 3.1 (i), we have

$$|I_1|, |I_2| \leq \frac{\alpha + \beta}{2} e^{\max\{|\phi_0(1)|, |\phi_0(-1)|\} + \frac{\eta_\epsilon}{\epsilon^2} C_7}, \quad (3.27)$$

and

$$\begin{aligned} \epsilon^2 (-\phi'(1)e^{\phi(1)/2} + \phi'(-1)e^{\phi(-1)/2}) &\geq C_6 \left(e^{\phi(1)/2} + e^{\phi(-1)/2} \right) \\ &\geq C_6 \left(e^{\phi_0(1)/2} + e^{\phi_0(-1)/2} \right). \end{aligned} \quad (3.28)$$

Then we use (3.26) and (3.28) to get

$$\epsilon^2 \left| \phi'(1)e^{\phi(1)/2} + \phi'(-1)e^{\phi(-1)/2} \right| \leq \frac{|I_1| + |I_2|}{\left| \phi'(1)e^{\phi(1)/2} - \phi'(-1)e^{\phi(-1)/2} \right|}$$

$$\leq \frac{\epsilon^2(|I_1| + |I_2|)}{C_6 (e^{\phi(1)/2} + e^{\phi(-1)/2})}. \quad (3.29)$$

Therefore, by (3.27) and (3.29), we complete the proof of Claim 4. \square

Suppose $\frac{\eta_\epsilon}{\epsilon^2} \rightarrow 0$ as $\epsilon \downarrow 0$. Then Lemma 3.3 and the boundary condition (1.19) imply $\lim_{\epsilon \downarrow 0} \phi(1) = \phi_0(1)$ and $\lim_{\epsilon \downarrow 0} \phi(-1) = \phi_0(-1)$. Hence by (3.25) and Remark 3.1 (ii) we obtain (1.32) and complete the proof of Theorem 1.6 (i).

Suppose $\eta_\epsilon = \gamma\epsilon^2$, where $\gamma > 0$ is a constant independent of ϵ . By Lemma 3.3 and the boundary condition (1.19), $|\phi(\pm 1)|$ has an upper bound independent of ϵ . Thus we can assume that $\limsup_{\epsilon \downarrow 0} \phi(1) = \phi_{1,s}^*$, $\liminf_{\epsilon \downarrow 0} \phi(1) = \phi_{1,i}^*$, $\limsup_{\epsilon \downarrow 0} \phi(-1) = \phi_{2,s}^*$, and $\liminf_{\epsilon \downarrow 0} \phi(-1) = \phi_{2,i}^*$. By (3.25) and the boundary condition (1.19) with $\eta_\epsilon = \gamma\epsilon^2$, we have

$$\lim_{\epsilon \downarrow 0} \left[(\phi(1) - \phi_0(1))e^{\phi(1)/2} - (\phi(-1) - \phi_0(-1))e^{\phi(-1)/2} \right] = 0. \quad (3.30)$$

On the other hand, (3.14) and the boundary condition (1.19) give

$$\phi(1) + \phi(-1) = \phi_0(1) + \phi_0(-1) + \gamma(\beta - \alpha). \quad (3.31)$$

By Remark 3.1 (i), (3.30) and (3.31), $(\phi_{1,s}^*, \phi_{2,s}^*)$ and $(\phi_{1,i}^*, \phi_{2,i}^*)$ solve the system (1.33).

Claim 5. System (1.33) has a unique solution (ϕ_1^*, ϕ_2^*) .

Proof. Given (ϕ_1^*, ϕ_2^*) a solution of system (1.33). Then by (3.30) and (3.31), ϕ_j^* , $j = 1, 2$ satisfy $\phi_1^* = \phi_0(1) + s^*$, $\phi_2^* = \phi_0(-1) + \gamma(\beta - \alpha) - s^*$ and $s^* \in (0, \gamma(\beta - \alpha))$ is a root of the equation $f(s) = 0$, where

$$f(s) = se^{\frac{\phi_0(1)+s}{2}} - (\gamma(\beta - \alpha) - s)e^{\frac{\phi_0(-1)+\gamma(\beta-\alpha)-s}{2}}.$$

It is easy to check that $f'(s) > 0$ on $(0, \gamma(\beta - \alpha))$ and $f(0) < 0 < f(\gamma(\beta - \alpha))$. Hence $f(s) = 0$ has a unique solution in the interval $(0, \gamma(\beta - \alpha))$.

Therefore, we complete the proof of Claim 5. \square

By Claim 5, $(\phi_{1,s}^*, \phi_{2,s}^*) = (\phi_{1,i}^*, \phi_{2,i}^*)$, and then $\lim_{\epsilon \downarrow 0} \phi(1) = \phi_1^*$, $\lim_{\epsilon \downarrow 0} \phi(-1) = \phi_2^*$, together with the boundary condition (1.19), we have $\lim_{\epsilon \downarrow 0} \epsilon^2 \phi'(1) = \frac{\phi_0(1) - \phi_1^*}{\gamma}$ and $\lim_{\epsilon \downarrow 0} \epsilon^2 \phi'(-1) = -\frac{\phi_0(-1) - \phi_2^*}{\gamma}$.

Therefore, we complete the proof of Theorem 1.6 (ii).

Suppose $\frac{\eta_\epsilon}{\epsilon^2} \rightarrow \infty$ as $\epsilon \downarrow 0$. Subtracting (3.17) from (3.16) and using (3.14) and (1.19), we obtain

$$\begin{aligned} & \frac{\alpha - \beta}{2\eta_\epsilon} (\phi(-1) - \phi(1) - \phi_0(-1) + \phi_0(1)) \\ &= \frac{\alpha}{\int_{-1}^1 e^{\phi(y)} dy} (e^{\phi(1)} - e^{\phi(-1)}) + \frac{\beta}{\int_{-1}^1 e^{-\phi(y)} dy} e^{-\phi(-1)} (e^{\phi(-1) - \phi(1)} - 1). \end{aligned} \quad (3.32)$$

By (3.18) and (3.19), we have

$$\left| \frac{\alpha}{\int_{-1}^1 e^{\phi(y)} dy} (e^{\phi(1)} - e^{\phi(-1)}) \right| \leq \frac{2\alpha}{\int_{-1}^1 e^{\phi(y)} dy} e^{\phi(x_\epsilon)} \leq \alpha + \beta. \quad (3.33)$$

As for the proof of Lemma 3.3, we assume $\phi'(-1) + \phi'(1) \geq 0$. Then the boundary condition (1.19) gives $\phi(-1) - \phi(1) \geq \phi_0(-1) - \phi_0(1)$. Hence by (3.8), we have

$$\left| e^{\phi(-1)-\phi(1)} - 1 \right| \geq e^{\phi_0(-1)-\phi_0(1)} |\phi(-1) - \phi(1)|. \quad (3.34)$$

By (3.8), (3.22), (3.32) and (3.33), we find the following inequality:

$$\begin{aligned} \frac{(\beta - \alpha)^2}{8\epsilon^2} \left| e^{\phi(-1)-\phi(1)} - 1 \right| &\leq (\alpha + \beta) + \frac{\beta - \alpha}{2\eta_\epsilon} (\phi_0(1) - \phi_0(-1)) \\ &+ \frac{\beta - \alpha}{2\eta_\epsilon} |\phi(-1) - \phi(1)|. \end{aligned} \quad (3.35)$$

Hence (3.34) and (3.35) imply

$$\begin{aligned} &\left[e^{\phi_0(-1)-\phi_0(1)} - \frac{4\epsilon^2}{(\beta - \alpha)\eta_\epsilon} \right] |\phi(-1) - \phi(1)| \\ &\leq \frac{8\epsilon^2}{(\beta - \alpha)^2} \left[(\alpha + \beta) + \frac{\beta - \alpha}{2\eta_\epsilon} (\phi_0(1) - \phi_0(-1)) \right]. \end{aligned} \quad (3.36)$$

Since $\frac{\epsilon^2}{\eta_\epsilon} \rightarrow 0$ as $\epsilon \downarrow 0$, then (3.36) gives $\lim_{\epsilon \downarrow 0} (\phi(-1) - \phi(1)) = 0$. By the boundary condition (1.19), we have $\epsilon^2(\phi'(1) + \phi'(-1)) = \frac{\epsilon^2}{\eta_\epsilon} [(\phi_0(1) - \phi_0(-1)) - (\phi(1) - \phi(-1))] \rightarrow 0$ as $\epsilon \downarrow 0$.

Therefore, by (3.14), we get (1.34) and complete the proof of Theorem 1.6.

3.2. Proof of Theorem 1.5. By (3.5) and Lemma 3.3, we have

$$|\phi'(x)| \leq \left(\frac{\beta - \alpha}{\epsilon^2} \right) \left(e^{-\frac{\sqrt{2\alpha}}{2\epsilon}(1+x)} + e^{-\frac{\sqrt{2\alpha}}{2\epsilon}(1-x)} \right), \quad \forall x \in (-1, 1).$$

Consequently,

$$|\phi(x) - \phi(0)| \leq \int_0^x |\phi'(s)| ds \leq \frac{\beta - \alpha}{\epsilon} \sqrt{\frac{2}{\alpha}} \left| e^{-\sqrt{2\alpha}(1+x)/(2\epsilon)} - e^{-\sqrt{2\alpha}(1-x)/(2\epsilon)} \right|, \quad (3.37)$$

for $x \in (-1, 1)$. By (3.37) and $\sup_{x,y \in K} |\phi(x) - \phi(y)| \leq 2 \sup_{x \in K} |\phi(x) - \phi(0)|$ for $K \subset (-1, 1)$, we complete the proof of Theorem 1.5(i).

Now we prove Theorem 1.5(ii). Assume that $\phi_0(1) = \phi_0(-1)$. Then ϕ is even and $\phi(1) = \phi(-1)$. By Remark 3.1 (ii) and Lemma 7.1, we have

$$\frac{\alpha e^{\phi(1)}}{\int_{-1}^1 e^{\phi(y)} dy} + \frac{\beta e^{-\phi(1)}}{\int_{-1}^1 e^{-\phi(y)} dy} + \frac{\epsilon^2}{4} \int_{-1}^1 (\phi'(x))^2 dx = \frac{(\alpha - \beta)^2}{8\epsilon^2} + \frac{\alpha + \beta}{2}. \quad (3.38)$$

Setting $x = 0$ in (7.1) and integrating (7.1) from -1 to 1 , we get

$$\frac{\alpha e^{\phi(0)}}{\int_{-1}^1 e^{\phi(y)} dy} + \frac{\beta e^{-\phi(0)}}{\int_{-1}^1 e^{-\phi(y)} dy} + \frac{\epsilon^2}{4} \int_{-1}^1 (\phi'(x))^2 dx = \frac{\alpha + \beta}{2}. \quad (3.39)$$

Here we have used the fact that $\phi'(0) = 0$. By Theorem 3.2 (i), $\phi(1) \leq \phi(0)$, and then (3.39) gives

$$\frac{\alpha e^{\phi(1)}}{\int_{-1}^1 e^{\phi(y)} dy} + \frac{\epsilon^2}{4} \int_{-1}^1 (\phi'(x))^2 dx \leq \frac{\alpha e^{\phi(0)}}{\int_{-1}^1 e^{\phi(y)} dy} + \frac{\epsilon^2}{4} \int_{-1}^1 (\phi'(x))^2 dx \leq \frac{\alpha + \beta}{2}. \quad (3.40)$$

Hence (3.38) and (3.40) imply

$$\frac{(\alpha - \beta)^2}{8\epsilon^2} \leq \frac{\beta e^{-\phi(1)}}{\int_{-1}^1 e^{-\phi(y)} dy} \leq \frac{(\alpha - \beta)^2}{8\epsilon^2} + \frac{\alpha + \beta}{2}. \quad (3.41)$$

By (3.39), (3.41) and (3.11), we have

$$\begin{aligned} e^{\phi(0)} &\leq \frac{\alpha + \beta}{2\alpha} \int_{-1}^1 e^{\phi(y)} dy, & e^{-\phi(1)} \int_{-1}^1 e^{\phi(y)} dy &\leq \frac{4}{\alpha} \left[\frac{(\alpha - \beta)^2}{8\epsilon^2} + \frac{\alpha + \beta}{2} \right], \\ e^{-\phi(0)} &\leq \frac{\alpha + \beta}{2\beta} \int_{-1}^1 e^{-\phi(y)} dy, & e^{\phi(1)} \int_{-1}^1 e^{-\phi(y)} dy &\leq \frac{8\beta\epsilon^2}{(\alpha - \beta)^2}. \end{aligned}$$

Consequently,

$$\log \frac{(\alpha - \beta)^2}{4\epsilon^2(\alpha + \beta)} \leq \phi(0) - \phi(1) \leq \log \left\{ \frac{2(\alpha + \beta)}{\alpha^2} \left[\frac{(\alpha - \beta)^2}{8\epsilon^2} + \frac{\alpha + \beta}{2} \right] \right\}. \quad (3.42)$$

Therefore, by (3.37) and (3.42), we obtain (1.28).

Now we assume $\phi_0(1) \neq \phi_0(-1)$. By Lemma 3.3, (3.20) and (3.21), we have

$$\frac{C_6^2}{2\epsilon^2} \leq \frac{\beta e^{-\phi(1)}}{\int_{-1}^1 e^{-\phi(y)} dy}, \quad \frac{\beta e^{-\phi(-1)}}{\int_{-1}^1 e^{-\phi(y)} dy} \leq \frac{C_7^2}{2\epsilon^2} + \frac{\alpha + \beta}{2}. \quad (3.43)$$

As for the proof of Theorem 1.5 (ii) for the case of $\phi_0(1) = \phi_0(-1)$, we use (3.11), (3.18) and (3.43) to derive

$$C_8 \leq \phi(x_\epsilon) - \phi(\pm 1) - \log \frac{1}{\epsilon^2} \leq C_9 \quad \text{for } 0 < \epsilon < 1, \quad (3.44)$$

where C_8, C_9 are positive constants independent of ϵ . To complete the proof of Theorem 1.5 (ii), we need the following claim:

Claim 6. There exists $C_{10} > 0$ independent of ϵ , such that $0 \leq \phi(x_\epsilon) - \phi(0) \leq C_{10}$ as $0 < \epsilon \ll 1$.

Proof. Setting $x = 0$ and $x = x_\epsilon$ in (7.1), one can check that

$$\frac{\beta e^{-\phi(x_\epsilon)}}{\int_{-1}^1 e^{-\phi(y)} dy} \left(e^{\phi(x_\epsilon) - \phi(0)} - 1 \right) = \frac{\epsilon^2}{2} (\phi'(0))^2 + \frac{\alpha e^{\phi(x_\epsilon)}}{\int_{-1}^1 e^{\phi(y)} dy} \left(1 - e^{\phi(0) - \phi(x_\epsilon)} \right). \quad (3.45)$$

By (3.5), (3.11), (3.18) and (3.45), we obtain

$$\begin{aligned} \frac{\alpha^2}{4} \left(e^{\phi(x_\epsilon) - \phi(0)} - 1 \right) &\leq \frac{\alpha e^{\phi(x_\epsilon)}}{\int_{-1}^1 e^{\phi(y)} dy} \frac{\beta e^{-\phi(x_\epsilon)}}{\int_{-1}^1 e^{-\phi(y)} dy} \left(e^{\phi(x_\epsilon) - \phi(0)} - 1 \right) \\ &\leq \frac{\alpha e^{\phi(x_\epsilon)}}{\int_{-1}^1 e^{\phi(y)} dy} \left[\frac{\epsilon^2}{2} (\phi'(0))^2 + \frac{\alpha e^{\phi(x_\epsilon)}}{\int_{-1}^1 e^{\phi(y)} dy} \right] \leq \frac{3(\alpha + \beta)^2}{4} \end{aligned}$$

as $0 < \epsilon \ll 1$. This complete the proof of Claim 6. \square

Therefore, by Claim 6, (3.5) and (3.44), we get (1.28) and complete the proof of Theorem 1.5.

3.3. Proof of Theorem 1.7. By (3.11) and (3.41), we have

$$\frac{(\alpha - \beta)^2}{8\epsilon^2} e^{-\phi(x)+\phi(1)} \leq \frac{\beta e^{-\phi(x)}}{\int_{-1}^1 e^{-\phi(y)} dy} \leq \left[\frac{(\alpha - \beta)^2}{8\epsilon^2} + \frac{\alpha + \beta}{2} \right] e^{-\phi(x)+\phi(1)}, \quad (3.46)$$

$$\frac{2\alpha^2\epsilon^2}{(\alpha - \beta)^2 + 4\epsilon^2(\alpha + \beta)} e^{\phi(x)-\phi(1)} \leq \frac{\alpha e^{\phi(x)}}{\int_{-1}^1 e^{\phi(y)} dy} \leq \frac{2\alpha\beta\epsilon^2}{(\alpha - \beta)^2} e^{\phi(x)-\phi(1)}. \quad (3.47)$$

Let $u(x) = \phi(x) - \phi(1)$ for $x \in [-1, 1]$. Then (1.14), (3.46) and (3.47) give

$$\alpha^2 A e^{u(x)} - \frac{1}{4A} e^{-u(x)} \leq \epsilon^2 u''(x) \leq \alpha\beta B e^{u(x)} - \frac{1}{4B} e^{-u(x)}, \quad (3.48)$$

for $x \in (-1, 1)$, where $A = \frac{2\epsilon^2}{(\alpha - \beta)^2 + 4\epsilon^2(\alpha + \beta)}$ and $B = \frac{2\epsilon^2}{(\alpha - \beta)^2}$. Since $\phi_0(1) = \phi_0(-1)$, then by Theorem 3.2 (i), $u' \leq 0$ in $(0, 1)$. Multiplying (3.48) by $u'(x)$, we find that for $x \in (0, 1)$,

$$\left(\alpha\beta B e^{u(x)} + \frac{1}{4B} e^{-u(x)} \right)' \leq \epsilon^2 u''(x) u'(x) \leq \left(\alpha^2 A e^{u(x)} + \frac{1}{4A} e^{-u(x)} \right)'. \quad (3.49)$$

Integrating (3.49) over $(y, 1)$ for $y \in (0, 1)$, we obtain

$$\frac{\epsilon^2}{2} u'^2(y) \leq \left(\sqrt{\alpha\beta} e^{\frac{u(y)+\log B}{2}} - \frac{1}{2} e^{-\frac{u(y)+\log B}{2}} \right)^2 + \sqrt{\alpha\beta} \quad (3.50)$$

and

$$\frac{\epsilon^2}{2} u'^2(y) \geq \alpha^2 e^{u(y)+\log A} + \frac{1}{4} e^{-(u(y)+\log A)} - \frac{\alpha + \beta}{2} - \frac{2\alpha^2\epsilon^2}{(\alpha - \beta)^2}. \quad (3.51)$$

Here we have used the fact that $u(1) = 0$ and $u'(1) = \frac{\alpha - \beta}{2\epsilon^2}$. Note that $\sqrt{\alpha\beta} e^{\frac{u(1)+\log B}{2}} - \frac{1}{2} e^{-\frac{u(1)+\log B}{2}} = \frac{1}{\sqrt{B}} \left(\frac{2\sqrt{\alpha\beta}\epsilon^2}{(\alpha - \beta)^2} - \frac{1}{2} \right)$ is negative for $0 < \epsilon < \frac{\beta - \alpha}{2(\alpha\beta)^{1/4}}$. Hence by the continuity, there exists $y_{1,\epsilon} \in (0, 1)$ sufficiently close to 1 such that

$$\sqrt{\alpha\beta} e^{\frac{u(y)+\log B}{2}} - \frac{1}{2} e^{-\frac{u(y)+\log B}{2}} < 0 \quad \text{for } y \in (y_{1,\epsilon}, 1). \quad (3.52)$$

Consequently, by (3.50), (3.52) and Theorem 3.2(i), we have

$$0 \leq -\frac{\epsilon}{\sqrt{2}} u'(y) \leq -\left(\sqrt{\alpha\beta} e^{\frac{u(y)+\log B}{2}} - \frac{1}{2} e^{-\frac{u(y)+\log B}{2}} \right) + (\alpha\beta)^{1/4}, \quad (3.53)$$

$\forall y \in (y_{1,\epsilon}, 1)$. On the other hand, for $0 < \epsilon < \frac{\beta - \alpha}{4\sqrt{\alpha + \beta}}$, one can check that

$$\begin{aligned} & \alpha^2 e^{u(1)+\log A} + \frac{1}{4} e^{-(u(1)+\log A)} - \frac{\alpha + \beta}{2} - \frac{2\alpha^2\epsilon^2}{(\alpha - \beta)^2} \\ & \quad - \left(\frac{\alpha + \beta}{2} e^{\frac{u(1)+\log A}{2}} - \frac{1}{2} e^{-\frac{u(1)+\log A}{2}} \right)^2 \\ & = \frac{(3\alpha + \beta)(\alpha - \beta)}{2[(\alpha - \beta)^2 + 4(\alpha + \beta)\epsilon^2]} \epsilon^2 - \frac{2\alpha^2\epsilon^2}{(\alpha - \beta)^2} \geq -\frac{1}{2} \left(\frac{\alpha + \beta}{\alpha - \beta} \right)^2 \epsilon^2 \end{aligned}$$

and

$$\frac{\alpha + \beta}{2} e^{\frac{u(1)+\log A}{2}} - \frac{1}{2} e^{-\frac{u(1)+\log A}{2}} + \frac{\alpha + \beta}{\beta - \alpha} \epsilon < -\frac{\beta - \alpha}{16\epsilon} + \frac{\alpha + \beta}{\beta - \alpha} \epsilon < 0.$$

Hence there exists $y_{2,\epsilon} \in (0, 1)$ depends on ϵ such that

$$\begin{aligned} \alpha^2 e^{u(y)+\log A} + \frac{1}{4} e^{-(u(y)+\log A)} - \frac{\alpha + \beta}{2} - \frac{2\alpha^2 \epsilon^2}{(\alpha - \beta)^2} \\ - \left(\frac{\alpha + \beta}{2} e^{\frac{u(y)+\log A}{2}} - \frac{1}{2} e^{-\frac{u(y)+\log A}{2}} \right)^2 \geq - \left(\frac{\alpha + \beta}{\alpha - \beta} \right)^2 \epsilon^2 \end{aligned} \quad (3.54)$$

and

$$\frac{\alpha + \beta}{2} e^{\frac{u(y)+\log A}{2}} - \frac{1}{2} e^{-\frac{u(y)+\log A}{2}} + \frac{\alpha + \beta}{\beta - \alpha} \epsilon < 0, \quad \text{for } y \in (y_{2,\epsilon}, 1). \quad (3.55)$$

By (3.51), (3.54), (3.55) and Theorem 3.2(i), we have, for $y \in (y_{2,\epsilon}, 1)$,

$$0 \leq - \left(\frac{\alpha + \beta}{2} e^{\frac{u(y)+\log A}{2}} - \frac{1}{2} e^{-\frac{u(y)+\log A}{2}} + \frac{\alpha + \beta}{\beta - \alpha} \epsilon \right) \leq -\frac{\epsilon}{\sqrt{2}} u'(y). \quad (3.56)$$

Let $y_\epsilon = \max\{y_{1,\epsilon}, y_{2,\epsilon}\}$, by (3.53) and (3.56) we have, for $y \in (y_\epsilon, 1)$,

$$\frac{u'(y)}{\sqrt{\alpha\beta} e^{\frac{u(y)+\log B}{2}} - \frac{1}{2} e^{-\frac{u(y)+\log B}{2}} - (\alpha\beta)^{1/4}} \leq \frac{\sqrt{2}}{\epsilon} \leq \frac{u'(y)}{\frac{\alpha+\beta}{2} e^{\frac{u(y)+\log A}{2}} - \frac{1}{2} e^{-\frac{u(y)+\log A}{2}} + \frac{\alpha+\beta}{\beta-\alpha} \epsilon}.$$

Note that (3.52) and (3.55). Integrating the above inequality over $(z, 1)$ for $z \in (y_\epsilon, 1)$, then

$$\begin{aligned} 2 \log \frac{t_{1,+}(\sqrt{A} - t_{1,-}) + t_{1,-}(t_{1,+} - \sqrt{A}) e^{-\frac{\sqrt{\frac{1}{2}(\frac{\alpha+\beta}{\beta-\alpha})^2 \epsilon^2 + 2(\alpha+\beta)}}{\epsilon}(1-z)}}{(\sqrt{A} - t_{1,-}) + (t_{1,+} - \sqrt{A}) e^{-\frac{\sqrt{\frac{1}{2}(\frac{\alpha+\beta}{\beta-\alpha})^2 \epsilon^2 + 2(\alpha+\beta)}}{\epsilon}(1-z)}} \\ \leq u(z) - \log \frac{(\alpha - \beta)^2}{2\epsilon^2} \\ \leq 2 \log \frac{t_{2,+}(\sqrt{B} - t_{2,-}) + t_{2,-}(t_{2,+} - \sqrt{B}) e^{-\sqrt{\frac{3}{2}} \frac{(\alpha\beta)^{1/4}}{\epsilon}(1-z)}}{(\sqrt{B} - t_{2,-}) + (t_{2,+} - \sqrt{B}) e^{-\sqrt{\frac{3}{2}} \frac{(\alpha\beta)^{1/4}}{\epsilon}(1-z)}}, \end{aligned} \quad (3.57)$$

where $t_{1,\pm} = \frac{\epsilon}{\beta-\alpha} \pm \sqrt{(\frac{\epsilon}{\beta-\alpha})^2 + \frac{1}{\alpha+\beta}} \rightarrow \pm \frac{1}{\sqrt{\alpha+\beta}}$ as $\epsilon \downarrow 0$ and $t_{2,\pm} = (1 \pm \sqrt{3})(\alpha\beta)^{-1/4}$.

Therefore, by (3.57), we complete the proof of theorem 1.7.

4. Comparison of PB_n and PB equations: Proof of Theorem 1.4. In this section, we compare the PB_n equation (1.14) and the PB equation (1.26) with the boundary condition (1.19). It is easy to prove that the solution $w \in C^\infty((-1, 1)) \cap C^2([-1, 1])$ of (1.26) and (1.19) has uniqueness which can be proved by the same method as Theorem 1.1. Besides, by (1.26), there exists constant C_ϵ such that

$$\frac{\epsilon^2}{2} (w'(x))^2 = \frac{\alpha}{2} e^{w(x)} + \frac{\beta}{2} e^{-w(x)} + C_\epsilon, \quad \forall x \in [-1, 1], \quad (4.1)$$

and

$$\epsilon^2(w''(b)w'(b) - w''(a)w'(a)) \geq \int_a^b \left(\frac{\alpha}{2} e^{w(x)} + \frac{\beta}{2} e^{-w(x)} \right) (w'(x))^2 dx, \quad (4.2)$$

for $-1 \leq a < b \leq 1$.

As for the proof of Theorem 1.2, we can prove that w is odd, monotonically increasing on $[-1, 1]$, convex on $(0, 1)$ and concave on $(-1, 0)$. Setting $s_\epsilon = w(1)$ and following the proof of Theorem 1.3, we obtain $\lim_{\epsilon \downarrow 0} (t_\epsilon - s_\epsilon) = 0$, and

$$\begin{aligned} & 2 \log \frac{s_+(e^{s_\epsilon/2} - s_-) - s_-(e^{s_\epsilon/2} - s_+)e^{-(s_+ - s_-)\sqrt{\alpha}(1-x)/2\epsilon}}{(e^{s_\epsilon/2} - s_-) - (e^{s_\epsilon/2} - s_+)e^{-(s_+ - s_-)\sqrt{\alpha}(1-x)/2\epsilon}} \\ & \leq w(x) \leq 2 \log \frac{(e^{s_\epsilon/2} + 1) + (e^{s_\epsilon/2} - 1)e^{-\sqrt{\alpha}(1-x)/\epsilon}}{(e^{s_\epsilon/2} + 1) - (e^{s_\epsilon/2} - 1)e^{-\sqrt{\alpha}(1-x)/\epsilon}}, \quad \forall x \in (0, 1) \end{aligned} \quad (4.3)$$

where $t_\epsilon = \phi(1)$ and $s_\pm = \frac{1}{2} \left(\frac{\epsilon \phi_0(1)}{\sqrt{\alpha}} \pm \sqrt{\left(\frac{\epsilon \phi_0(1)}{\sqrt{\alpha}} \right)^2 + 4} \right)$. By (2.21), (4.3), and $\lim_{\epsilon \downarrow 0} (t_\epsilon - s_\epsilon) = 0$ we get (1.27) and complete the proof. Here we have used Theorem 1.2, 1.3 and the fact that both ϕ and w are odd functions on $[-1, 1]$.

For $\alpha \neq \beta$ and $\phi_0(-1) = \phi_0(1)$, we obtain the following lemma:

LEMMA 4.1. *Assume that $\alpha \neq \beta$ and $\phi_0(-1) = \phi_0(1)$. Let $w \in C^\infty((-1, 1)) \cap C^2([-1, 1])$ be the solution of (1.26) and (1.19). Then w is an even function on $[-1, 1]$ satisfying*

- (i) *If $\phi_0(-1) = \phi_0(1) > \frac{1}{2} \log \frac{\beta}{\alpha}$, then w is convex on $[-1, 1]$ and $\phi_0(1) \geq w(x) \geq w(0) > \frac{1}{2} \log \frac{\beta}{\alpha}$ for all $x \in [-1, 1]$;*
- (ii) *If $\phi_0(-1) = \phi_0(1) < \frac{1}{2} \log \frac{\beta}{\alpha}$, then w is concave on $[-1, 1]$ and $\phi_0(1) \leq w(x) \leq w(0) < \frac{1}{2} \log \frac{\beta}{\alpha}$ for all $x \in [-1, 1]$.*
- (iii)

$$|w'(x)| \leq |w'(1)| \left(e^{-\frac{(\alpha\beta)^{1/4}}{\epsilon}(1+x)} + e^{-\frac{(\alpha\beta)^{1/4}}{\epsilon}(1-x)} \right), \quad \forall x \in (-1, 1). \quad (4.4)$$

Proof. By the uniqueness of w and $\phi_0(-1) = \phi_0(1)$, w is an even function on $[-1, 1]$. Consequently, w' is odd on $[-1, 1]$ and $w'(0) = 0$. By (4.2), we have

$$w''(x)w'(x) \geq 0, \quad \text{for } x \in (0, 1), \quad w''(x)w'(x) \leq 0, \quad \text{for } x \in (-1, 0). \quad (4.5)$$

Since w' is odd, (4.5) implies that w'' does not change sign on $(-1, 1)$. Hence either $w'' \geq 0$ or $w'' \leq 0$ on $(-1, 1)$. Suppose $w'' \leq 0$ on $(-1, 1)$. Then (1.26) implies $w(x) \leq \frac{1}{2} \log \frac{\beta}{\alpha}$ for all $x \in (-1, 1)$ and (4.5) implies $w'(x) \geq 0$ for $x \in (-1, 0)$ and $w'(x) \leq 0$ for $x \in (0, 1)$. By the boundary condition (1.19), $\phi_0(1) = w(1) + \eta_\epsilon w'(1) \leq \frac{1}{2} \log \frac{\beta}{\alpha}$. Hence we may get (i) by contradiction.

Similarly, we prove (ii). Following the same argument as Theorem 3.2, we obtain (4.4). \square

REMARK 4.1. *If $\phi_0(-1) = \phi_0(1) = \frac{1}{2} \log \frac{\beta}{\alpha}$, then there is only trivial solution $w \equiv \frac{1}{2} \log \frac{\beta}{\alpha}$.*

It is easy to get Remark 4.1 by the uniqueness of w . To see nontrivial solutions, we assume that $\phi_0(-1) = \phi_0(1) \neq \frac{1}{2} \log \frac{\beta}{\alpha}$. Then we have the following theorem:

THEOREM 4.2. *Assume $\alpha \neq \beta$ and $\phi_0(-1) = \phi_0(1) \neq \frac{1}{2} \log \frac{\beta}{\alpha}$, where α and β are positive constants independent of ϵ . Let $w \in C^\infty((-1, 1)) \cap C^2([-1, 1])$ be the solution of (1.26) and (1.19). Then*

$$\min \left\{ \phi_0(1), \frac{1}{2} \log \frac{\beta}{\alpha} \right\} \leq w(x) \leq \max \left\{ \phi_0(1), \frac{1}{2} \log \frac{\beta}{\alpha} \right\}, \quad \forall x \in [-1, 1], \quad (4.6)$$

$$w \rightarrow \frac{1}{2} \log \frac{\beta}{\alpha}, \quad (4.7)$$

uniformly in any compact subset of $(-1, 1)$ as ϵ goes to zero, and

$$\lim_{\epsilon \downarrow 0} w(1) = t, \quad (4.8)$$

where

$$t = \begin{cases} \frac{1}{2} \log \frac{\beta}{\alpha} & \text{if } \lim_{\epsilon \downarrow 0} \frac{\epsilon}{\eta_\epsilon} = 0, \\ \phi_0(1) & \text{if } \lim_{\epsilon \downarrow 0} \frac{\epsilon}{\eta_\epsilon} = \infty, \end{cases}$$

and $\min \left\{ \phi_0(1), \frac{1}{2} \log \frac{\beta}{\alpha} \right\} < t < \max \left\{ \phi_0(1), \frac{1}{2} \log \frac{\beta}{\alpha} \right\}$ is the unique zero of $|\phi_0(1) - t| = \gamma \left| \sqrt{2\alpha\epsilon} e^{t/2} - \sqrt{2\beta\epsilon} e^{-t/2} \right|$ if $\frac{\eta_\epsilon}{\epsilon} = \gamma$ and γ is a positive constant independent of ϵ . Moreover,

$$w_1^\epsilon(x) \leq w(x) - \frac{1}{2} \log \frac{\beta}{\alpha} \leq w_2^\epsilon(x), \quad \forall x \in [0, 1], \quad (4.9)$$

where $w_i^\epsilon(x) = \log \frac{f_i^\epsilon e^{\frac{(h_i^\epsilon)^{1/4}}{\epsilon}(1-x)} + g_i^\epsilon}{f_i^\epsilon e^{\frac{(h_i^\epsilon)^{1/4}}{\epsilon}(1-x)} - g_i^\epsilon}$ for $x \in [0, 1]$ and f_i^ϵ 's, g_i^ϵ 's and h_i^ϵ 's are constants satisfying $f_i^\epsilon \rightarrow \alpha^{1/4} e^{t/2} + \beta^{1/4}$, $g_i^\epsilon \rightarrow \alpha^{1/4} e^{t/2} - \beta^{1/4}$ and $h_i^\epsilon \rightarrow (4\alpha\beta)^{1/4}$ as $\epsilon \downarrow 0^+$, $i = 1, 2$.

Proof. By Lemma 4.1, we have (4.6). Setting $x = 0$ in (4.1) and using $w'(0) = 0$, one may check that

$$\frac{\epsilon^2}{2} (w'(x))^2 = \alpha \left(e^{w(x)} - e^{w(0)} \right) + \beta \left(e^{-w(x)} - e^{-w(0)} \right), \quad \forall x \in [-1, 1]. \quad (4.10)$$

Note that $\epsilon^2 w'(1) = \epsilon^2 \int_0^1 w''(x) dx = \int_0^1 (\alpha e^{w(x)} - \beta e^{-w(x)}) dx$. If $w'' \geq 0$ on $(0, 1)$, then by (4.5), we have $w' \geq 0$ on $(0, 1)$, hence $\alpha e^{w(x)} - \beta e^{-w(x)}$ is increasing on $(0, 1)$, and then $\epsilon^2 w'(1) \geq \alpha e^{w(0)} - \beta e^{-w(0)} \geq 0$.

Similarly, if $w'' \leq 0$ on $(0, 1)$, we get $\epsilon^2 w'(1) \leq \alpha e^{w(0)} - \beta e^{-w(0)} \leq 0$. Consequently,

$$\left(\alpha e^{w(0)} - \beta e^{-w(0)} \right)^2 \leq \epsilon^4 (w'(1))^2. \quad (4.11)$$

By (4.6), (4.10) (at $x = 1$) and (4.11), we have

$$\lim_{\epsilon \downarrow 0} w(0) = \frac{1}{2} \log \frac{\beta}{\alpha} \text{ and } |w'(1)| \leq \frac{C_{13}}{\epsilon}, \quad (4.12)$$

for some positive constant C_{13} independent of ϵ . It is easy to see that (4.4) and (4.12) imply (4.7). By (4.10) and boundary condition (1.19) we have

$$\frac{\epsilon^2}{2\eta_\epsilon^2} (\phi_0(1) - w(1))^2 = \alpha \left(e^{w(1)} - e^{w(0)} \right) + \beta \left(e^{-w(1)} - e^{-w(0)} \right).$$

Using (4.7) and the proof of Theorem 1.3, we prove (4.8). On the other hand, (4.10) can be written as

$$\frac{\epsilon^2}{2} (w'(x))^2 = \left(\sqrt{\alpha} e^{w(x)/2} - \sqrt{\beta} e^{-w(x)/2} \right)^2 - \left(\sqrt{\alpha} e^{w(0)/2} - \sqrt{\beta} e^{-w(0)/2} \right)^2, \quad (4.13)$$

$\forall x \in [-1, 1]$. Note that by (1.26) and (4.5), $w'(x)$ and $\sqrt{\alpha} e^{w(x)/2} - \sqrt{\beta} e^{-w(x)/2}$ share the same sign for $x \in [0, 1]$. Hence (4.13) implies

$$\frac{w'(x)}{\sqrt{\alpha} e^{w(x)/2} - \sqrt{\beta} e^{-w(x)/2}} \leq \frac{\sqrt{2}}{\epsilon} \leq \frac{w'(x)}{\sqrt{\alpha} e^{w(x)/2} - \sqrt{\beta} e^{-w(x)/2} - \delta_\epsilon}, \quad x \in [0, 1], \quad (4.14)$$

where $\delta_\epsilon = \sqrt{\alpha} e^{w(0)/2} - \sqrt{\beta} e^{-w(0)/2}$. Note that by (4.7), we have $\delta_\epsilon \rightarrow 0$ as $\epsilon \downarrow 0$.

Therefore, as for (2.18)-(2.21), we use (4.14) to get (4.9) and complete the proof of Theorem 4.2. \square

5. Numerical simulations. Here we present numerical results to support the theories in this paper. For the electroneutral case i.e., $\alpha = \beta$, the Gummel method [37, 38, 39] is well-known for solving PB type equations. One may apply Gummel method to PB_n equation. However we have empirically observed that this method shows a deviation to the solution in the nonelectroneutral case i.e., $\alpha \neq \beta$ with high electrostatic potential. It may be caused by the inaccurate correction of the exponential fitting for high electrostatic potential. Moreover, since PB_n equation has a normalization of charge density, it is not easy to find a numerical remedy in using Gummel method for the nonelectroneutral case. Due to these reasons, we here employ convex iteration [35] rather than Gummel method to linearize PB_n equation (1.14) and PB equation (1.26) with the Robin boundary condition (1.19). If ϕ_1 satisfying boundary condition (1.19) is initially given, then an iteration scheme of (1.14) can be defined with $0 < c < 1$ as follows:

$$\begin{cases} \epsilon^2 \phi''_{k+\frac{1}{2}} &= \frac{\alpha}{\int e^{\phi_k}} e^{\phi_k} - \frac{\beta}{\int e^{-\phi_k}} e^{-\phi_k}, \\ \phi_{k+1} &= c\phi_{k+\frac{1}{2}} + (1-c)\phi_k, \text{ for } k = 1, 2, \dots \end{cases} \quad (5.1)$$

with boundary conditions

$$\phi_{k+\frac{1}{2}}(-1) - \eta \phi'_{k+\frac{1}{2}}(-1) = \phi_0(-1), \quad \phi_{k+\frac{1}{2}}(1) + \eta \phi'_{k+\frac{1}{2}}(1) = \phi_0(1). \quad (5.2)$$

Let $\phi_{k+\frac{1}{2}} = \phi_k + \delta_k$ with the correction δ_k which satisfies

$$\delta_k(-1) - \eta \delta'_k(-1) = 0, \quad \delta_k(1) + \eta \delta'_k(1) = 0, \quad (5.3)$$

so that $\phi_{k+1} = \phi_k + c\delta_k = \phi_1 + c\sum_{i=1}^k \delta_i$. If $\lim_{k \rightarrow \infty} |\delta_k| = 0$, then the iterative scheme converges.

Define the residual function $\mathcal{R}(\phi_k)$ as

$$\mathcal{R}(\phi_k) = \frac{\alpha}{\int e^{\phi_k}} e^{\phi_k} - \frac{\beta}{\int e^{-\phi_k}} e^{-\phi_k} - \epsilon^2 \phi_k'', \quad (5.4)$$

then we get

$$\epsilon^2 \phi_{k+1}'' - \epsilon^2 \phi_k'' = \epsilon^2 c \delta_k'' = c \mathcal{R}(\phi_k). \quad (5.5)$$

If we integrate $\mathcal{R}(\phi_{k+1}) - \mathcal{R}(\phi_k)$, then we obtain

$$\int_{-1}^1 \mathcal{R}(\phi_{k+1}) = (1-c) \int_{-1}^1 \mathcal{R}(\phi_k). \quad (5.6)$$

REMARK 5.1. *In (5.5), it is clear that the numerical scheme converges when $\lim_{k \rightarrow \infty} |\mathcal{R}(\phi_k)| = 0$ under the boundary conditions (5.3). In case of $c = 1$ in (5.6), we can not guarantee that $\lim_{k \rightarrow \infty} |\mathcal{R}(\phi_k)| = 0$ but $\int \mathcal{R}(\phi_{k+1}) = 0$ for $k = 1, 2, \dots$ so that it may cause an oscillation during the iteration procedure. Moreover, we have empirically observed that when the value of c is compatible to ϵ^2 , the iteration converges well.*

The finite element methods with piecewise linear basis functions are used to solve the linearized equations. The numerical computations are performed with the constants $\epsilon = 2^{-j}$, $j = 1, \dots, 6$ and $\eta_\epsilon = 0, \epsilon^2, \epsilon, \epsilon^{1/2}, \epsilon^0$. The computational domain is $[-1, 1]$ and the mesh size is $1/2048$ uniformly throughout all computations. The value 10^{-6} is applied for stopping criterion of iterative scheme with $|\delta_k| = |(\phi_{k+1} - \phi_k)/c|$.

5.1. Electroneutral case ($\alpha = \beta$). For simplicity, we set $\alpha = \beta = 1$, and $\phi_0(1) = -\phi_0(-1) = 1$. The numerical solutions of the PB_n equation (1.14) with the Robin boundary condition (1.19) are presented in Figure 5.1. The pictures (a)-(b) of Figure 5.1 are corresponding to the case of $\epsilon = 2^{-2}, 2^{-6}$, and the curves 1-5 in each picture are associated with $\eta_\epsilon = 0, \epsilon^2, \epsilon, \epsilon^{1/2}, \epsilon^0$, respectively. Numerical outputs of $\phi(1)$'s are written at the right end of the curves 1-5 of Figure 5.1. According to the numerical results, we observe that as ϵ goes to zero, these $\phi(1)$'s close to $\phi_0(1) = 1$ when $\eta_\epsilon = 0, \epsilon^2$. $\phi(1)$'s close to 0 when $\eta_\epsilon = \epsilon^{1/2}, \epsilon^0$ as ϵ goes to zero. $\phi(1)$ close to $\phi^* = 0.4974$ when $\eta_\epsilon = \epsilon$ as ϵ goes to zero i.e., $\gamma = 1$, where $\phi^* = 0.4974$ is the root of (1.23) numerically solved by Newton's method. These results have excellent consistency with Theorem 1.2 and Theorem 1.3. One may remark Figure 5.1 to see how the solution ϕ tends to 0 in $(-1, 1)$ but may have boundary layers at $x = \pm 1$ as ϵ goes to zero.

The difference between the PB_n equation (1.14) and the PB equation (1.26) is the integral $\int_{-1}^1 e^{\phi(x)} dx$ and $\int_{-1}^1 e^{-\phi(x)} dx$ located at the denominator of right hand side of (1.14). However, the numerical integrals $\int_{-1}^1 e^{\phi(x)} dx$ and $\int_{-1}^1 e^{-\phi(x)} dx$ are quite close to 2 as ϵ goes to zero (see the first table in Table 5.1). Such a numerical result is consistent with Lemma 2.1. Moreover, the second table in Table 5.1 shows that ϕ the solution of (1.14) approaches to w the solution of (1.26) as ϵ goes to zero. This may support Theorem 1.4.

5.2. Non-electroneutral case ($\alpha \neq \beta$). We consider the case of $\alpha = 1, \beta = 2$ with $\epsilon = 2^{-2}, 2^{-4}$. Numerical solutions of the PB_n equation (1.14) are sketched in Figure 5.2 and Figure 5.3 for the case of $\phi_0(-1) = \phi_0(1) = 1$ and $\phi_0(-1) = 1, \phi_0(1) =$

2, respectively. The numerical values of $\phi(1)$ are presented on the right hand side of pictures in Figure 5.2 and Figure 5.3. As for Theorem 1.5, our numerical results may indicate the solution ϕ tend to infinity as ϵ goes to zero. Figure 5.4 is the zoom-in curve of the curve 5 in Figure 5.2 (b) to express the boundary layer of the solution ϕ with huge slopes at $x = \pm 1$ which can also be observed in Figure 5.3. These results are comparable to Theorem 1.6.

For the PB equation (1.26), numerical solutions are sketched in Figure 5.5 for the case of $\phi_0(-1) = \phi_0(1) = 1$. Such a result may support Theorem 4.2. One may compare Figure 5.2 and 5.5 to see the difference between solutions of (1.14) and (1.26).

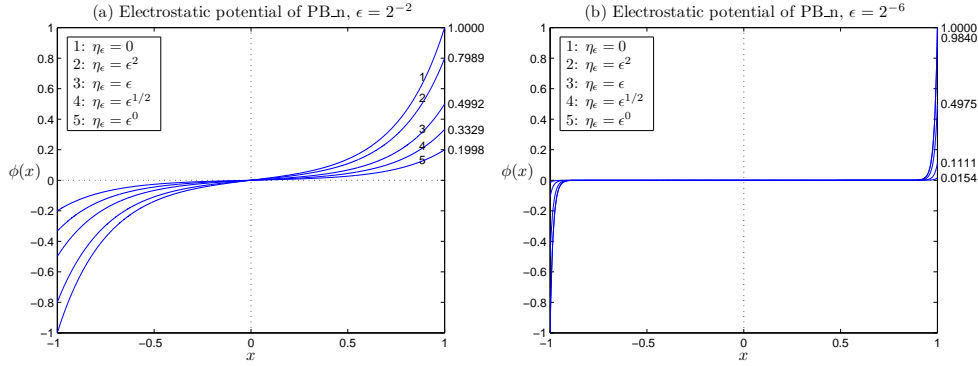


FIG. 5.1. The numerical solutions of electrostatic potential ϕ to PB_n equation with boundary condition $\phi_0(1) = -\phi_0(-1) = 1$ in electroneutral case, $\alpha = \beta = 1$. The numerical values of $\phi(1)$ is written on the right hand side of picture for each curve. Each picture depends on the dielectric constant, (a) $\epsilon = 2^{-2}$, (b) $\epsilon = 2^{-6}$. The label of curves in the picture (b) follows that in the picture (a).

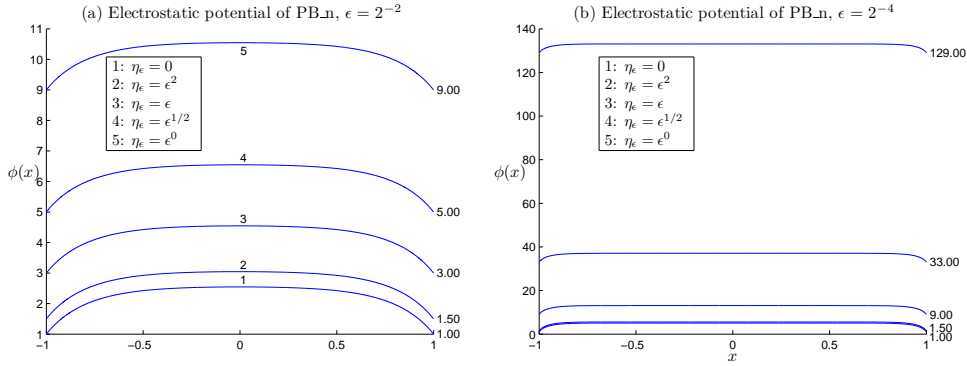


FIG. 5.2. The numerical solutions of electrostatic potential ϕ to PB_n equation with boundary condition $\phi_0(-1) = \phi_0(1) = 1$ in non-electroneutral case, $\alpha = 1, \beta = 2$. The numerical values of $\phi(1)$ is written on the right hand side of picture for each curve. Each picture depends on the dielectric constant, (a) $\epsilon = 2^{-2}$, (b) $\epsilon = 2^{-4}$. The label of curves in the picture (b) follows that in the picture (a).

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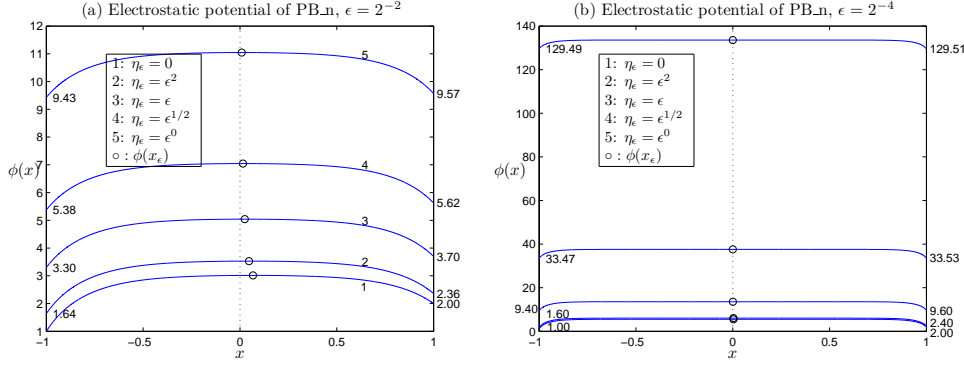


FIG. 5.3. The numerical solutions of electrostatic potential ϕ to PB $_n$ equation with boundary condition $\phi_0(-1) = 1, \phi_0(1) = 2$ in non-electroneutral case, $\alpha = 1, \beta = 2$. The numerical values of $\phi(1)$ is written on the right hand side of picture for each curve. Each picture depends on the dielectric constant, (a) $\epsilon = 2^{-2}$, (b) $\epsilon = 2^{-4}$. The label of curves in the picture (b) follows that in the picture (a).

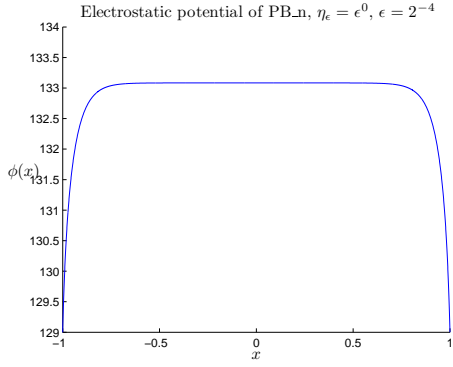


FIG. 5.4. The zoom-in picture of the curve 5 ($\eta_e = \epsilon^0$), which is the top curve in picture (b) of Figure 5.2.

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7. Appendix. In this section, we prove Theorem 1.1 and demonstrate some basic facts of the system (1.14) with the boundary condition (1.19). Let ϕ be the solution of the equation (1.14) with the boundary condition (1.19). Multiplying the equation (1.14) by $\phi'(x)$, it is easy to check that

$$\frac{\epsilon^2}{2}(\phi'(x))^2 = \frac{\alpha}{\int_{-1}^1 e^{\phi(y)} dy} e^{\phi(x)} + \frac{\beta}{\int_{-1}^1 e^{-\phi(y)} dy} e^{-\phi(x)} + C_\epsilon. \quad (7.1)$$

TABLE 5.1

The numerical results of $\int_{-1}^1 e^{\phi(x)} dx$ for PB-n equation with boundary condition $\phi_0(1) = -\phi_0(-1) = 1$ in electroneutral case, $\alpha = \beta = 1$. The values of $\int_{-1}^1 e^{-\phi(x)} dx$ are equivalent to $\int_{-1}^1 e^{\phi(x)} dx$ in stopping criterion. (The first table) The maximum norms $\|\phi - w\|_\infty$ of the solutions of PB-n and PB with the condition $\phi_0(1) = -\phi_0(-1) = 1$ in the case of $\alpha = \beta = 1$. (The second table)

| $\eta_\epsilon \setminus \epsilon$ | 2^{-2} | 2^{-4} | 2^{-6} | $\eta_\epsilon \setminus \epsilon$ | 2^{-2} | 2^{-4} | 2^{-6} |
|------------------------------------|----------|----------|----------|------------------------------------|----------|----------|----------|
| 0 | 2.131 | 2.032 | 2.008 | 0 | 1.14e-02 | 2.90e-03 | 7.25e-04 |
| ϵ^2 | 2.082 | 2.028 | 2.008 | ϵ^2 | 7.06e-03 | 2.55e-03 | 7.01e-04 |
| ϵ | 2.031 | 2.008 | 2.002 | ϵ | 2.32e-03 | 5.83e-04 | 1.46e-04 |
| $\epsilon^{1/2}$ | 2.014 | 2.001 | 2.000 | $\epsilon^{1/2}$ | 8.14e-04 | 5.09e-05 | 2.39e-06 |
| ϵ^0 | 2.005 | 2.000 | 2.000 | ϵ^0 | 2.02e-04 | 1.50e-06 | 7.00e-09 |

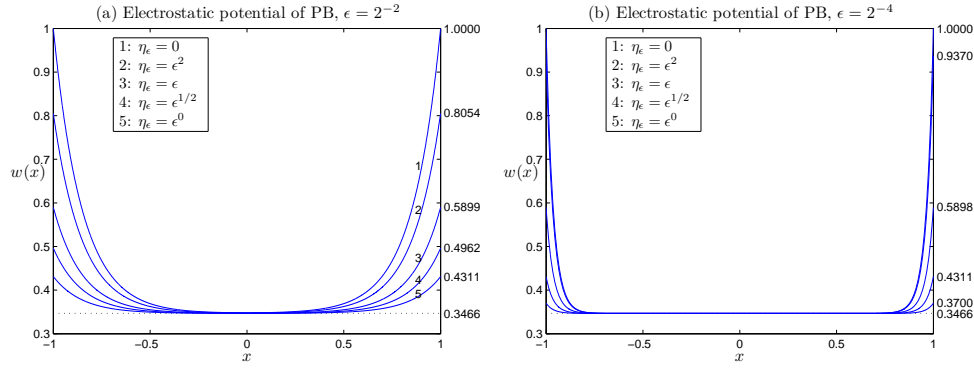


FIG. 5.5. The numerical solutions of electrostatic potential w to PB equation with the condition $\phi_0(-1) = \phi_0(1) = 1$ for the case of $\alpha = 1$, $\beta = 2$. The numerical values of $\phi(1)$ is written on the right hand side of pictures. Each picture depends on the dielectric constant, (a) $\epsilon = 2^{-2}$, (b) $\epsilon = 2^{-4}$. The order of curves corresponding to η_ϵ follows the order in the picture (a).

where C_ϵ is a constant. Differentiating the equation (1.14) to x and then multiplying it by ϕ' , we have

$$\epsilon^2 \phi'''(x) \phi'(x) = \left(\frac{\alpha}{\int_{-1}^1 e^{\phi(y)} dy} e^{\phi(x)} + \frac{\beta}{\int_{-1}^1 e^{-\phi(y)} dy} e^{-\phi(x)} \right) (\phi'(x))^2 \quad (7.2)$$

for $|x| < 1$. Integrating (7.2) over $(0, a)$ (for $0 < a \leq 1$) gives

$$\begin{aligned} & \epsilon^2 (\phi''(a) \phi'(a) - \phi''(0) \phi'(0)) \\ & \geq \int_0^a \left(\frac{\alpha}{\int_{-1}^1 e^{\phi(y)} dy} e^{\phi(x)} + \frac{\beta}{\int_{-1}^1 e^{-\phi(y)} dy} e^{-\phi(x)} \right) (\phi'(x))^2 dx, \quad \forall a \in (0, 1]. \end{aligned} \quad (7.3)$$

This inequality will play a crucial role in this paper.

The following identity is useful in the arguments of this paper:

LEMMA 7.1. *Let $\phi \in C^\infty((-1, 1)) \cap C^2([-1, 1])$ satisfy (1.14). Then*

$$\begin{aligned} & \frac{\epsilon^2}{2} \left[(\phi'(1))^2 + (\phi'(-1))^2 - \int_{-1}^1 (\phi'(x))^2 dx \right] \\ &= \frac{\alpha}{\int_{-1}^1 e^{\phi(y)} dy} \left(e^{\phi(1)} + e^{\phi(-1)} \right) + \frac{\beta}{\int_{-1}^1 e^{-\phi(y)} dy} \left(e^{-\phi(1)} + e^{-\phi(-1)} \right) - (\alpha + \beta). \end{aligned} \quad (7.4)$$

Proof. Multiplying (1.14) by $x\phi'(x)$ and integrating the expression over $(-1, 1)$. Using integration by parts, we get (7.4). \square

Now we state the proof of Theorem 1.1. Suppose $\phi_i \in C^\infty((-1, 1)) \cap C^2([-1, 1])$, $i = 1, 2$, are two solutions of (1.14) and (1.19). Subtracting (1.14) for ϕ_2 from that for ϕ_1 , we have

$$\begin{aligned} \epsilon^2(\phi_1 - \phi_2)''(x) &= \alpha \left(e^{\phi_1(x) - \log \int_{-1}^1 e^{\phi_1}} - e^{\phi_2(x) - \log \int_{-1}^1 e^{\phi_2}} \right) \\ &\quad - \beta \left(e^{-\phi_1(x) - \log \int_{-1}^1 e^{-\phi_1}} - e^{-\phi_2(x) - \log \int_{-1}^1 e^{-\phi_2}} \right). \end{aligned} \quad (7.5)$$

Multiplying (7.5) by $\phi_1(x) - \phi_2(x)$ and integrating the expression from -1 to 1 , we may get

$$\begin{aligned} & -\epsilon^2 \int_{-1}^1 (\phi_1 - \phi_2)''(\phi_1 - \phi_2) dx + \int_{-1}^1 \alpha \left(e^{\phi_1 - \log \int_{-1}^1 e^{\phi_1}} - e^{\phi_2 - \log \int_{-1}^1 e^{\phi_2}} \right) (\phi_1 - \phi_2) dx \\ & - \int_{-1}^1 \beta \left(e^{-\phi_1 - \log \int_{-1}^1 e^{-\phi_1}} - e^{-\phi_2 - \log \int_{-1}^1 e^{-\phi_2}} \right) (\phi_1 - \phi_2) dx = 0. \end{aligned} \quad (7.6)$$

Since both ϕ_1 and ϕ_2 satisfy the same boundary condition (1.19), then we have $(\phi_1 - \phi_2)(1) + \eta_\epsilon(\phi_1 - \phi_2)'(1) = (\phi_1 - \phi_2)(-1) - \eta_\epsilon(\phi_1 - \phi_2)'(-1) = 0$. Consequently, we may use integration by parts to get

$$\begin{aligned} -\epsilon^2 \int_{-1}^1 (\phi_1 - \phi_2)''(\phi_1 - \phi_2) dx &= \epsilon^2 \int_{-1}^1 |(\phi_1 - \phi_2)'|^2 dx \\ &\quad + \frac{\epsilon^2}{\eta_\epsilon} [(\phi_1 - \phi_2)^2(1) + (\phi_1 - \phi_2)^2(-1)]. \end{aligned} \quad (7.7)$$

For the last two terms of (7.6), we set $a(x) = \phi_1(x) - \log \int_{-1}^1 e^{\phi_1}$ and $b(x) = \phi_2(x) - \log \int_{-1}^1 e^{\phi_2}$. Note that $(e^{a(x)} - e^{b(x)})(a(x) - b(x)) \geq 0$ for $x \in [-1, 1]$ and $\int_{-1}^1 e^{a(x)} dx = \int_{-1}^1 e^{b(x)} dx = 1$. Hence we obtain

$$\begin{aligned} & \int_{-1}^1 \alpha \left(e^{\phi_1 - \log \int_{-1}^1 e^{\phi_1}} - e^{\phi_2 - \log \int_{-1}^1 e^{\phi_2}} \right) (\phi_1 - \phi_2) dx \\ &= \int_{-1}^1 \alpha \left(e^{a(x)} - e^{b(x)} \right) \left(a(x) - b(x) + \log \frac{\int_{-1}^1 e^{\phi_1}}{\int_{-1}^1 e^{\phi_2}} \right) dx \\ &\geq \alpha \left(\log \frac{\int_{-1}^1 e^{\phi_1}}{\int_{-1}^1 e^{\phi_2}} \right) \int_{-1}^1 \left(e^{a(x)} - e^{b(x)} \right) dx = 0. \end{aligned} \quad (7.8)$$

Similarly, we have

$$- \int_{-1}^1 \beta \left(e^{-\phi_1 - \log \int_{-1}^1 e^{-\phi_1}} - e^{-\phi_2 - \log \int_{-1}^1 e^{-\phi_2}} \right) (\phi_1 - \phi_2) dx \geq 0. \quad (7.9)$$

Therefore, by (7.6)–(7.9), we have $\phi_1 \equiv \phi_2$ on $[-1, 1]$ and complete the proof of Theorem 1.1.

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