GLOBAL WELL-POSEDNESS OF THE 3D PRIMITIVE EQUATIONS WITH PARTIAL VERTICAL TURBULENCE MIXING HEAT DIFFUSION

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GLOBAL WELL–POSEDNESS OF THE $3D$ PRIMITIVE EQUATIONS WITH PARTIAL VERTICAL TURBULENCE MIXING HEAT DIFFUSION

CHONGSHENG CAO AND EDRISS S. TITI

Abstract. The three–dimensional incompressible viscous Boussinesq equations, under the assumption of hydrostatic balance, govern the large scale dynamics of atmospheric and oceanic motion, and are commonly called the primitive equations. To overcome the turbulence mixing a partial vertical diffusion is usually added to the temperature advection (or density stratification) equation. In this paper we prove the global regularity of strong solutions to this model in a three-dimensional infinite horizontal channel, subject to periodic boundary conditions in the horizontal directions, and with no-penetration and stress-free boundary conditions on the solid, top and bottom, boundaries. Specifically, we show that short time strong solutions to the above problem exist globally in time, and that they depend continuously on the initial data.

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1. Introduction

The partial differential equation model that describes convective flow in ocean dynamics is known to be the Boussinesq equations, which are the Navier–Stokes equations (NSE) of incompressible flows with rotation coupled to the heat (or density stratification) and salinity transport equations. The questions of the global well–posedness of the $3D$ Navier–Stokes equations are considered to be among the most challenging mathematical problems. In the context of the atmosphere and the ocean circulation dynamics geophysicists take advantage of the shallowness of the oceans and the atmosphere to simplify the Boussinesq equations by modeling the vertical motion with the hydrostatic balance. This leads to the well-known primitive equations for ocean and atmosphere dynamics (see, e.g., [24], [25], [29], [31], [32], [37], [38] and references therein). A vertical heat diffusivity is usually added as a leading order approximation to the effect of micro-scale turbulence mixing (cf., e.g., [16], [17], [24], [31]). As a result one arrives to the following dimensionless $3D$ variant of the primitive equations (Boussinesq equations):

$$\frac{\partial v}{\partial t} + (v\cdot\nabla_H)v + w\frac{\partial v}{\partial z} + f_0\vec{k}\times v + \nabla_H p + L_1 v = 0 \tag{1}$$

$$\partial_z p + T = 0, \tag{2}$$

$$\nabla_H\cdot v + \partial_z w = 0, \tag{3}$$

$$\frac{\partial T}{\partial t} + v\cdot\nabla_H T + w\frac{\partial T}{\partial z} + L_2 T = Q, \tag{4}$$

where the horizontal velocity vector field $v = (v_1, v_2)$, the velocity vector field (v_1, v_2, w), the temperature T and the pressure p are the unknowns. f_0 is the Coriolis parameter, Q is a given heat source. For simplicity, we drop the coupling with the salinity equation, which is an advection diffusion equation, but the results reported here will be equally valid with the addition of the coupling with the salinity. Moreover, we also assume for simplicity that Q is time independent. The viscosity and the heat vertical diffusion operators L_1 and L_2, respectively, are

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given by

$$L_1 = -\frac{1}{R_1}\Delta_H - \frac{1}{R_2}\,\frac{\partial^2}{\partial z^2}, \tag{5}$$

$$L_2 = -\frac{1}{R_3}\,\frac{\partial^2}{\partial z^2}, \tag{6}$$

where R_1, R_2 are positive constants representing the horizontal and vertical dimensionless Reynolds numbers, respectively, and R_3 is positive constant which stands for the vertical dimensionless eddy heat diffusivity turbulence mixing coefficient (cf., e.g., [16], [17]). We set $\nabla_H = (\partial_x, \partial_y)$ to be the horizontal gradient operator and $\Delta_H = \partial_x^2 + \partial_y^2$ to be the horizontal Laplacian. We denote by

$$\Gamma_u = \{(x, y, 0) \in \mathbb{R}^3\}, \tag{7}$$

$$\Gamma_b = \{(x, y, -h) \in \mathbb{R}^3\}, \tag{8}$$

the upper and lower solid boundaries, respectively. We equip system (1)–(4), on the physical top and bottom boundaries, with the following no–normal flow and stress free boundary conditions for the flow velocity vector field (v, w), namely,

$$\text{on } \Gamma_u : \frac{\partial v}{\partial z} = 0,\ w = 0, \tag{9}$$

$$\text{on } \Gamma_b : \frac{\partial v}{\partial z} = 0,\ w = 0, \tag{10}$$

and for simplicity, we set the Dirichlet boundary condition for T:

$$T|_{z=0} = 0, \quad T|_{z=-h} = 1. \tag{11}$$

Horizontally, we set (v, w) and T to satisfy periodic boundary conditions:

$$v(x+1, y, z) = v(x, y+1, z) = v(x, y, z); \tag{12}$$

$$w(x+1, y, z) = w(x, y+1, z) = w(x, y, z); \tag{13}$$

$$T(x+1, y, z) = T(x, y+1, z) = T(x, y, z). \tag{14}$$

We will denote by

$$M = (0, 1)^2 \qquad \text{and} \quad \Omega = M \times (-h, 0).$$

In addition, we supply the system with the initial condition:

$$v(x, y, z, 0) = v_0(x, y, z), \tag{15}$$

$$T(x, y, z, 0) = T_0(x, y, z). \tag{16}$$

System (1)–(16) is a modified form of the rotational Rayleigh–Bénard convection problem taking into consideration the geophysical situation of the shallowness of oceans and atmosphere. The original three-dimensional Rayleigh–Bénard convection model (which is identical, in the absence of heat diffusion, to the Boussinesq model of stratified fluid) has been a subject to study for many years, numerically, experimentally and analytically (see, e.g., [3], [8], [12], [18], [28], [29], [30], [37], and references therein). However, the question of global regularity is still open and is as challenging as the $3D$ NSE. Recently, the authors of [9] and [20] have shown the global well-posedness to the $2D$ Boussinesq equations without diffusivity in the heat transport equation (see also, [11], for recent improvement). In [7] it is observed that thanks to hydrostatic balance (2) the unknown pressure is essentially a function of the two–dimensional horizontal variables. We take advantage of this observation in [7] to establish the L^6 estimates for the velocity vector field which allows us to prove the global well–posedness of the $3D$ primitive equations under the geophysical boundary conditions. In [22] the authors take advantage of this observation as well, and proved the global well–posedness of the $3D$ primitive equations with the Dirichlet boundary conditions by dealing directly with the "pressure" which is a function of two variables. In this paper we study system (1)–(16), exploring again the hydrostatic balance which leads to an unknown "pressure" that is a function of only two variables, and use the techniques and ideas introduced in [7], [9] and [20], to show in section 3 that strong solutions exist globally in time provided they exist for a short interval of time. Furthermore, we show in section 4 the uniqueness and continuous dependence on initial data of these strong solutions. The short time existence of strong solutions to this model will be reported in a forthcoming paper.

This paper is organized as follows. In section 2, we reformulate system (1)–(16) and introduce our notations and recall some well-known inequalities. Section 3 is the main section in which we establish the required estimates for proving the global existence in time for any initial data. In section 4 we prove the uniqueness of the solutions and their continuous dependence on initial data.

2. Functional setting and Formulation

2.1. **Equivalent Formulation.** We denote by

$$\overline{\phi}(x,y) = \frac{1}{h}\int_{-h}^{0} \phi(x,y,z)dz, \qquad \forall\ (x,y) \in M; \tag{17}$$

and denote the fluctuation by

$$\widetilde{\phi} = \phi - \overline{\phi}. \tag{18}$$

Notice that

$$\overline{\widetilde{\phi}} = 0. \tag{19}$$

Similar to [7], by integrating (2) and (3) vertically, we get

$$w(x,y,z,t) = -\int_{-h}^{z} \nabla_H \cdot v(x,y,\xi,t)d\xi, \tag{20}$$

and

$$p(x,y,z,t) = -\int_{-h}^{z} T(x,y,\xi,t)d\xi + p_s(x,y,t), \tag{21}$$

where p_s is the pressure on the bottom $z = -h$. Essentially, $p_s(x,y,t)$ is the unknown pressure, and we observe, as before, that it is a function of two spatial variables (x,y). As we mentioned in the introduction we explore this property as in [7] (see also [22]) to prove our global regularity result.

Replacing T by $T + \frac{z}{h}$, we have the following equivalent formulation for system (1)–(16):

$$\begin{aligned} &\frac{\partial v}{\partial t} + L_1 v + (v\cdot\nabla_H)v - \left(\int_{-h}^{z} \nabla_H \cdot v(x,y,\xi,t)d\xi\right)\frac{\partial v}{\partial z} \\ &\quad + f_0\vec{k}\times v + \nabla_H p_s(x,y,t) - \nabla_H \int_{-h}^{z} T(x,y,\xi,t)d\xi = 0, \end{aligned} \tag{22}$$

$$\nabla_H \cdot \overline{v} = 0, \tag{23}$$

$$\frac{\partial T}{\partial t} + L_2 T + v\cdot\nabla_H T - \left(\int_{-h}^{z} \nabla_H \cdot v(x,y,\xi,t)d\xi\right)\left(\frac{\partial T}{\partial z} + \frac{1}{h}\right) = Q, \tag{24}$$

$$\left.\frac{\partial v}{\partial z}\right|_{z=0} = \left.\frac{\partial v}{\partial z}\right|_{z=-h} = 0, \quad v(x+1,y,z) = v(x,y+1,z) = v(x,y,z), \tag{25}$$

$$T|_{z=0} = T|_{z=-h} = 0, \quad T(x+1,y,z) = T(x,y+1,z) = T(x,y,z), \tag{26}$$

$$v(x,y,z,0) = v_0(x,y,z), \tag{27}$$

$$T(x,y,z,0) = T_0(x,y,z) - \frac{z}{h}. \tag{28}$$

In addition, $\overline{v}$ and $\widetilde{v}$ satisfy the following coupled system of equations:

$$\frac{\partial \overline{v}}{\partial t} - \frac{1}{R_1}\Delta_H \overline{v} + (\overline{v}\cdot\nabla_H)\overline{v} + \overline{[(\widetilde{v}\cdot\nabla_H)\widetilde{v} + (\nabla_H\cdot\widetilde{v})\,\widetilde{v}]}$$

$$+f_0\vec{k}\times\overline{v} + \nabla_H\left[p_s(x,y,t) - \frac{1}{h}\int_{-h}^{0}\int_{-h}^{z} T(x,y,\xi,t)\,d\xi\,dz\right] = 0, \tag{29}$$

$$\nabla_H\cdot\overline{v} = 0, \tag{30}$$

$$\frac{\partial \widetilde{v}}{\partial t} + L_1\widetilde{v} + (\widetilde{v}\cdot\nabla_H)\widetilde{v} - \left(\int_{-h}^{z}\nabla_H\cdot\widetilde{v}(x,y,\xi,t)d\xi\right)\frac{\partial\widetilde{v}}{\partial z} + (\widetilde{v}\cdot\nabla_H)\overline{v} + (\overline{v}\cdot\nabla_H)\widetilde{v}$$

$$-\overline{[(\widetilde{v}\cdot\nabla_H)\widetilde{v} + (\nabla_H\cdot\widetilde{v})\,\widetilde{v}]} + f_0\vec{k}\times\widetilde{v} - \nabla_H\left(\int_{-h}^{z}T(x,y,\xi,t)d\xi - \frac{1}{h}\int_{-h}^{0}\int_{-h}^{z}T(x,y,\xi,t)d\xi dz\right) = 0. \tag{31}$$

2.2. **Functional spaces and inequalities.** Let us denote by $L^q(\Omega), L^q(M)$ and $W^{m,q}(\Omega), W^{m,q}(M)$, and $H^m(\Omega) =: W^{m,2}(\Omega), H^m(M) =: W^{m,2}(M)$, the usual L^q-Lebesgue and Sobolev spaces, respectively ([1]). We denote by

$$\|\phi\|_q = \begin{cases} \left(\int_\Omega |\phi(x,y,z)|^q\; dxdydz\right)^{\frac{1}{q}}, & \text{for every } \phi\in L^q(\Omega)\\ \left(\int_M |\phi(x,y)|^q\; dxdy\right)^{\frac{1}{q}}, & \text{for every } \phi\in L^q(M). \end{cases} \tag{32}$$

For convenience, we recall the following Sobolev and Ladyzhenskaya type inequalities in $(\mathbb{R}/\mathbb{Z})^2$ and in Ω (see, e.g., [1], [10], [15], [23])

$$\|\phi\|_{L^4(M)} \le C_0\|\phi\|_{L^2}^{1/2}\|\phi\|_{H^1(M)}^{1/2}, \qquad \forall\phi\in H^1(M), \tag{33}$$

$$\|\phi\|_{L^8(M)} \le C_0\|\phi\|_{L^6(M)}^{3/4}\|\phi\|_{H^1(M)}^{1/4}, \qquad \forall\phi\in H^1(M), \tag{34}$$

$$\|\nabla_H\phi\|_{L^4(M)} \le C_0\|\phi\|_{\infty}^{1/2}\|\phi\|_{H^2(M)}^{1/2}, \qquad \forall\phi\in H^2(M), \tag{35}$$

$$\|\nabla_H\phi\|_{L^4(M)} \le C_0\|\phi\|_{L^2(M)}^{1/2}\|\nabla_H\phi\|_{\infty}^{1/2} + \|\phi\|_{L^2(M)}, \qquad \forall\ \phi \text{ such that } \nabla_H\phi\in L^\infty(M), \tag{36}$$

and

$$\|\psi\|_{L^3(\Omega)} \le C_0\|\psi\|_{L^2(\Omega)}^{1/2}\|\psi\|_{H^1(\Omega)}^{1/2}, \tag{37}$$

$$\|\psi\|_{L^6(\Omega)} \le C_0\|\psi\|_{H^1(\Omega)}, \tag{38}$$

for every $\psi\in H^1(\Omega)$. Here C_0 is a positive scale invariant constant. Also, we recall the following version of Helmholtz-Weyl decomposition Theorem (cf. for example, [26], [15], [39])

$$\|\nabla_H\phi\|_{W^{m,q}(M)} \le C\left(\|\nabla_H\cdot\phi\|_{W^{m,q}(M)} + \|\nabla_H^\perp\cdot\phi\|_{W^{m,q}(M)}\right), \tag{39}$$

for every $\vec{\phi}\in\left(W^{m,q}(M)\right)^2$. Moreover, we recall the following Brezis–Gallouet or, Brezis–Wainger inequality (see, e.g., [26], [4], [5], [13])

$$\|\phi\|_{L^\infty(M)} \le C\|\phi\|_{H^1(M)}\;\left(1+\log^+\|\phi\|_{H^2(M)}\right)^{1/2}, \tag{40}$$

for every $\phi\in H^2(M)$, where $\log^+ r = \log r$, when $r\ge 1$, and $\log^+ r = 0$, when $r\le 1$. Also, we recall the following inequality (see, e.g., [2] and [21])

$$\|\nabla_H\phi\|_{L^\infty(M)} \le C(\|\nabla_H\cdot\phi\|_{L^\infty(M)} + \|\nabla_H\times\phi\|_{L^\infty(M)})\;\left(1+\log^+\|\nabla_H\phi\|_{H^2(M)}\right), \tag{41}$$

for every $\nabla_H\phi\in H^2(M)$. Moreover, by (33) we get

$$\|\phi\|_{L^{4q}(M)}^{4q} = \|\;|\phi|^q\;\|_{L^4(M)}^4 \le C\|\;|\phi|^q\;\|_{L^2(M)}^2\|\;|\phi|^q\;\|_{H^1(M)}^2$$

$$\le C_q\|\phi\|_{2q}^{2q}\;\left(\int_M|\phi|^{2q-2}\,|\nabla_H\phi|^2\;dxdy\right) + \|\phi\|_{2q}^{4q}, \tag{42}$$

for every ϕ satisfying $\int_M |\phi|^{2q-2} |\nabla_H \phi|^2 \; dxdy < \infty$ and $q \geq 1$. Also, we recall the integral version of Minkowsky inequality for the L^p spaces, $p \geq 1$. Let $\Omega_1 \subset \mathbb{R}^{m_1}$ and $\Omega_2 \subset \mathbb{R}^{m_2}$ be two measurable sets, where m_1 and m_2 are two positive integers. Suppose that $f(\xi,\eta)$ is a measurable function over $\Omega_1 \times \Omega_2$. Then,

$$\left[\int_{\Omega_1} \left(\int_{\Omega_2} |f(\xi,\eta)| d\eta\right)^p d\xi\right]^{1/p} \leq \int_{\Omega_2} \left(\int_{\Omega_1} |f(\xi,\eta)|^p d\xi\right)^{1/p} d\eta. \tag{43}$$

Finally, we recall the following inequality from Proposition 2.2 in [6]

$$\begin{aligned} &\left|\int_M \left(\int_{-h}^0 \psi_1(x,y,z)\, dz\right) \left(\int_{-h}^0 \psi_2(x,y,z)\, \psi_3(x,y,z)\, dz\right) dxdy\right| \\ &\leq C\|\psi_1\|_2^{1/2} \|\nabla_H \psi_1\|_2^{1/2} \|\psi_2\|_2^{1/2} \|\nabla_H \psi_2\|_2^{1/2} \|\psi_3\|_2 + \|\psi_1\|_2 \|\psi_2\|_2 \|\psi_3\|_2, \end{aligned} \tag{44}$$

for every $\psi_1, \psi_2 \in H^1(\Omega)$ and $\psi_3 \in L^2(\Omega)$, and

$$\begin{aligned} &\left|\int_M \left(\int_{-h}^0 \psi_1(x,y,z)\, dz\right) \left(\int_{-h}^0 |\nabla_H \psi_2(x,y,z)|\, \psi_3(x,y,z)\, dz\right) dxdy\right| \\ &\leq C\|\psi_1\|_2^{1/2} \|\nabla_H \psi_1\|_2^{1/2} \|\psi_2\|_\infty^{1/2} \|\nabla_H \nabla_H \psi_2\|_2^{1/2} \|\psi_3\|_2 + \|\psi_1\|_2 \|\psi_2\|_2 \|\psi_3\|_2, \end{aligned} \tag{45}$$

for every $\psi_1 \in H^1(\Omega)$, $\nabla_H \psi_2 \in H^1(\Omega)$ and $\psi_3 \in L^2(\Omega)$.

3. Global existence of strong solutions

In the previous section we have reformulated system (1)–(16) to be equivalent to system (22)–(28). In this section we will show that strong solutions to system (22)–(28) exist globally in time provided they exist in short time intervals.

Theorem 1. *Let $Q \in H^2(\Omega), v_0 \in H^4(\Omega), T_0 \in H^2(\Omega)$ and $\mathcal{T} > 0$. Suppose that there exists a strong solution $(v(t), T(t))$ of system (22)–(28) on $[0,\mathcal{T}]$ corresponding to the initial data (v_0, T_0) such that*

$$\begin{aligned} &\Delta_H v_z, \;\; \nabla_H T \in C([0,\mathcal{T}], H^1(\Omega)), \\ &v_{zz}, \Delta_H \nabla_H v_z, \;\; \nabla_H T_z \in L^2([0,\mathcal{T}], H^1(\Omega)). \end{aligned}$$

Then this strong solution $(v(t), T(t))$ exists globally in time.

Remark 1. Notice that one can recover the pressure p_s from system (29)–(30) in the same way as in $2D$ NSE (see, e.g., [10], [34], [35]).

Proof. Let $[0,\mathcal{T}_*)$ be the maximal interval of existence of a strong solution $(v(t), T(t))$. In order to establish the global existence, we need to show That $\mathcal{T}_* = \infty$. If $\mathcal{T}_* < \infty$ we will show $\|\Delta_H v_z(t)\|_{H^1(\Omega)}$, $\|\nabla_H T(t)\|_{H^1(\Omega)}$, $\int_0^t \|\Delta_H \nabla_H v_z(s)\|_{H^1(\Omega)}^2 \, ds$, $\int_0^t \|\nabla_H T_z(s)\|_{H^1(\Omega)}^2 \, ds$, and $\int_0^t \|v_{zz}(s)\|_{H^1(\Omega)}^2 \, ds$ are all bounded uniformly in time, for $t \in [0,\mathcal{T}_*)$. As a result the interval $[0,\mathcal{T}_*)$ can not be a maximal interval of existence, and consequently the strong solution $(v(t), T(t))$ exists globally in time.

Therefore, we focus our discussion below on the interval $[0,\mathcal{T}_*)$.

3.1. $\|v\|_2^2 + \|T\|_2^2$ estimates.

By taking the inner product of equation (24) with T, in $L^2(\Omega)$, we get

$$\begin{aligned} &\frac{1}{2} \frac{d\|T\|_2^2}{dt} + \frac{1}{R_3} \|T_z\|_2^2 \\ &= \int_\Omega QT \; dxdydz - \int_\Omega \left[v \cdot \nabla_H T - \left(\int_{-h}^z \nabla_H \cdot v(x,y,\xi,t) d\xi\right) \left(\frac{\partial T}{\partial z} + \frac{1}{h}\right)\right] T \; dxdydz. \end{aligned} \tag{46}$$

Integrating by parts and using the boundary condition (26), we get

$$-\int_\Omega \left(v \cdot \nabla_H T - \left(\int_{-h}^z \nabla_H \cdot v(x,y,\xi,t) d\xi\right) \frac{\partial T}{\partial z}\right) T \; dxdydz = 0. \tag{47}$$

As a result of the above we conclude that

$$\begin{aligned}&\frac{1}{2}\frac{d\|T\|_2^2}{dt}+\frac{1}{R_3}\ \|T_z\|_2^2\\&=\int_\Omega\left[Q-\frac{1}{h}\left(\int_{-h}^z\nabla_H\cdot v(x,y,\xi,t)d\xi\right)\right]\ T\ dxdydz\le\|Q\|_2\ \|T\|_2+\|\nabla_H v\|_2\|T\|_2.\end{aligned}\tag{48}$$

Moreover, by taking the inner product of equation (22) with v, in $L^2(\Omega)$, we reach

$$\begin{aligned}&\frac{1}{2}\frac{d\|v\|_2^2}{dt}+\frac{1}{R_1}\|\nabla_H v\|_2^2+\frac{1}{R_2}\|v_z\|_2^2\\&=-\int_\Omega\left[(v\cdot\nabla_H)v-\left(\int_{-h}^z\nabla_H\cdot v(x,y,\xi,t)d\xi\right)\frac{\partial v}{\partial z}\right]\cdot v\ dxdydz\\&\quad-\int_\Omega\left(f_0\vec{k}\times v\right)\cdot v\ dxdydz-\int_\Omega\left(\nabla_H p_s-\nabla_H\left(\int_{-h}^z T(x,y,\xi,t)d\xi\right)\right)\cdot v\ dxdydz.\end{aligned}\tag{49}$$

First, we notice that

$$\left(f_0\vec{k}\times v\right)\cdot v=0.\tag{50}$$

Next, by integration by parts and using the boundary conditions (25), in particular, the horizontal periodic boundary conditions, we get

$$\int_\Omega\left[(v\cdot\nabla_H)v-\left(\int_{-h}^z\nabla_H\cdot v(x,y,\xi,t)d\xi\right)\frac{\partial v}{\partial z}\right]\cdot v\ dxdydz=0.\tag{51}$$

Thanks to (30) and, again, the horizontal periodic boundary conditions, we also have

$$\int_\Omega\nabla_H p_s(x,y,t)\cdot v(x,y,z,t)\ dxdydz=h\int_M\nabla_H p_s\cdot\overline{v}\ dxdy=-h\int_\Omega p_s(\nabla_H\cdot\overline{v})\ dxdy=0.\tag{52}$$

By integration by parts, the periodic boundary conditions (25), and Cauchy–Schwarz inequality, we obtain

$$\left|\int_\Omega\nabla_H\left(\int_{-h}^z T(x,y,\xi,t)d\xi\right)\cdot v\ dxdydz\right|\le h\,\|T\|_2\ \|\nabla_H v\|_2.\tag{53}$$

Thus, by (48)–(53) we have

$$\begin{aligned}&\frac{1}{2}\frac{d(\|v\|_2^2+\|T\|_2^2)}{dt}+\frac{1}{R_1}\|\nabla_H v\|_2^2+\frac{1}{R_2}\|v_z\|_2^2+\frac{1}{R_3}\|T_z\|_2^2\\&\le\|Q\|_2\|T\|_2+(1+h)\|T\|_2\ \|\nabla_H v\|_2.\end{aligned}\tag{54}$$

By Cauchy–Schwarz inequality, we obtain

$$\frac{d(\|v\|_2^2+\|T\|_2^2)}{dt}+\frac{1}{R_1}\|\nabla_H v\|_2^2+\frac{1}{R_2}\|v_z\|_2^2+\frac{1}{R_3}\|T_z\|_2^2\tag{55}$$

$$\le\|Q\|_2^2+(1+R_1)(1+h)^2\|T\|_2^2.\tag{56}$$

Thanks to Gronwall's inequality we get

$$\|v(t)\|_2^2+\|T(t)\|_2^2\le C\left(\|v_0\|_2^2+\|T_0\|_2^2\right)e^{(1+R_1)(1+h)^2\,t}+C\|Q\|_2^2;\tag{57}$$

and

$$\int_0^t\left[\frac{1}{R_1}\|\nabla_H v(s)\|_2^2+\frac{1}{R_2}\|v_z(s)\|_2^2+\frac{1}{R_3}\|T_z(s)\|_2^2\right]\ ds\le C\left[\left(\|v_0\|_2^2+\|T_0\|_2^2\right)e^{(1+R_1)(1+h)^2\,t}+\|Q\|_2^2\ t\right].\tag{58}$$

Therefore, for every $t \in [0, \mathcal{T}_*)$, we have

$$\|v(t)\|_2^2 + \|T(t)\|_2^2 + \int_0^t \left[\|\nabla_H v(s)\|_2^2 + \|v_z(s)\|_2^2 + \|T_z(s)\|_2^2\right] \, ds \leq K_1, \tag{59}$$

where

$$K_1 = C\left[\left(\|v_0\|_2^2 + \|T_0\|_2^2\right) e^{(1+R_1)(1+h)^2\, t} + \|Q\|_2^2 \, t\right]. \tag{60}$$

3.2. $\|T\|_\infty$ **estimates.** We follow here the idea of Stampaccia for proving the Maximum Principle. The proof we present here is also similar to the one in [14] (see also [36]). Denote by $\tau(t) = T(t) - (1 + \|T_0\|_\infty + \|Q\|_\infty \, t)$. It is clear that τ satisfies:

$$\frac{\partial \tau}{\partial t} + v \cdot \nabla_H \tau + w \frac{\partial \tau}{\partial z} + L_2 \tau = Q - \|Q\|_\infty. \tag{61}$$

Let $\tau^+ = \max\{0, \tau\}$ which belongs to $H^1(\Omega)$ and satisfies

$$\tau^+(z=0) = \tau^+(z=-h) = 0. \tag{62}$$

Taking the inner product of the equation (61) with τ^+ in $L^2(\Omega)$ and applying the boundary conditions (62), we get

$$\frac{1}{2}\frac{d\|\tau^+\|_2^2}{dt} + \frac{1}{R_3}\|\partial_z \tau^+\|_2^2 = \int_\Omega (Q - \|Q\|_\infty)\tau^+ \, dxdydz - \int_\Omega \left[v \cdot \nabla_H \tau + w\, \partial_z \tau\right] \, \tau^+ \, dxdydz. \tag{63}$$

By integration by parts and using the boundary conditions (26) and (62), we get

$$\int_\Omega \left[v \cdot \nabla_H \tau + w\, \partial_z \tau\right] \, \tau^+ \, dxdydz = 0. \tag{64}$$

Thus,

$$\frac{1}{2}\frac{d\|\tau^+\|_2^2}{dt} + \frac{1}{R_3}\|\partial_z \tau^+\|_2^2 \, dxdydz = \int_\Omega (Q - \|Q\|_\infty)\tau^+ \, dxdydz \leq 0. \tag{65}$$

Therefore, we obtain

$$\|\tau^+(t)\|_2^2 \leq \|\tau^+(t=0)\|_2^2 = 0. \tag{66}$$

Thus, $\tau^+(t) \equiv 0$. As a result, we have

$$T(t) \leq 1 + \|T_0\|_\infty + \|Q\|_\infty \, t. \tag{67}$$

By applying similar arguments, we also have

$$T(t) \geq -(1 + \|T_0\|_\infty + \|Q\|_\infty \, t). \tag{68}$$

Therefore, T satisfies the following $L^\infty-$estimate:

$$\|T(t)\|_\infty \leq K_2 = 1 + \|T_0\|_\infty + \|Q\|_\infty \, t. \tag{69}$$

3.3. $\|\widetilde{v}\|_6$ **estimates.** Taking the inner product of the equation (31) with $|\widetilde{v}|^4\widetilde{v}$ in $L^2(\Omega)$ and using the boundary conditions (25), we get

$$\frac{1}{6}\frac{d\|\widetilde{v}\|_6^6}{dt}+\frac{1}{R_1}\int_\Omega\left(|\nabla_H\widetilde{v}|^2|\widetilde{v}|^4+\left|\nabla_H|\widetilde{v}|^2\right|^2|\widetilde{v}|^2\right)\,dxdydz+\frac{1}{R_2}\int_\Omega\left(|\widetilde{v}_z|^2|\widetilde{v}|^4+\left|\partial_z|\widetilde{v}|^2\right|^2|\widetilde{v}|^2\right)\,dxdydz$$
$$=-\int_\Omega\Bigg\{(\widetilde{v}\cdot\nabla_H)\widetilde{v}-\left(\int_{-h}^z\nabla_H\cdot\widetilde{v}(x,y,\xi,t)d\xi\right)\frac{\partial\widetilde{v}}{\partial z}+(\widetilde{v}\cdot\nabla_H)\overline{v}+(\overline{v}\cdot\nabla_H)\widetilde{v}-\overline{[(\widetilde{v}\cdot\nabla_H)\widetilde{v}+(\nabla_H\cdot\widetilde{v})\,\widetilde{v}]}$$
$$+f_0\vec{k}\times\widetilde{v}-\nabla_H\left(\int_{-h}^z T(x,y,\xi,t)d\xi-\frac{1}{h}\int_{-h}^0\int_{-h}^z T(x,y,\xi,t)d\xi dz\right)\Bigg\}\cdot|\widetilde{v}|^4\widetilde{v}\,dxdydz.$$

Observe, again, that

$$\left(f_0\vec{k}\times\widetilde{v}\right)\cdot|\widetilde{v}|^4\widetilde{v}=0. \tag{70}$$

Moreover, by integration by parts and the boundary conditions (25), we also get

$$-\int_\Omega\left[(\widetilde{v}\cdot\nabla_H)\widetilde{v}-\left(\int_{-h}^z\nabla_H\cdot\widetilde{v}(x,y,\xi,t)d\xi\right)\frac{\partial\widetilde{v}}{\partial z}\right]\cdot|\widetilde{v}|^4\widetilde{v}\,dxdydz=0. \tag{71}$$

Furthermore, by virtue of (30) and by the boundary conditions (25), in particular the horizontal periodic boundary conditions, we have

$$\int_\Omega(\overline{v}(x,y,t)\cdot\nabla_H)\widetilde{v}(x,y,z,t)\cdot|\widetilde{v}(x,y,z,t)|^4\widetilde{v}(x,y,z,t)\,dxdydz=0. \tag{72}$$

Thus, (70)–(72) imply

$$\frac{1}{6}\frac{d\|\widetilde{v}\|_6^6}{dt}+\frac{1}{R_1}\int_\Omega\left(|\nabla_H\widetilde{v}|^2|\widetilde{v}|^4+\left|\nabla_H|\widetilde{v}|^2\right|^2|\widetilde{v}|^2\right)\,dxdydz+\frac{1}{R_2}\int_\Omega\left(|\widetilde{v}_z|^2|\widetilde{v}|^4+\left|\partial_z|\widetilde{v}|^2\right|^2|\widetilde{v}|^2\right)\,dxdydz$$
$$=-\int_\Omega\Bigg\{(\widetilde{v}\cdot\nabla_H)\overline{v}-\overline{(\widetilde{v}\cdot\nabla_H)\widetilde{v}+(\nabla_H\cdot\widetilde{v})\,\widetilde{v}}$$
$$-\nabla_H\left(\int_{-h}^z T(x,y,\xi,t)d\xi-\frac{1}{h}\int_{-h}^0\int_{-h}^z T(x,y,\xi,t)d\xi dz\Bigg\}\right)\cdot|\widetilde{v}|^4\widetilde{v}\,dxdydz. \tag{73}$$

Notice that by integration by parts and using the boundary conditions (25), in particular the horizontal periodic boundary conditions, we have

$$-\int_\Omega\Bigg[(\widetilde{v}\cdot\nabla_H)\overline{v}-\overline{[(\widetilde{v}\cdot\nabla_H)\widetilde{v}+(\nabla_H\cdot\widetilde{v})\,\widetilde{v}]} \tag{74}$$
$$-\nabla_H\left(\int_{-h}^z T(x,y,\xi,t)d\xi-\frac{1}{h}\int_{-h}^0\int_{-h}^z T(x,y,\xi,t)d\xi dz\right)\Bigg]\cdot|\widetilde{v}|^4\widetilde{v}\,dxdydz$$
$$=\int_\Omega\Bigg[(\nabla_H\cdot\widetilde{v})\,\overline{v}\cdot|\widetilde{v}|^4\widetilde{v}+(\widetilde{v}\cdot\nabla_H)(|\widetilde{v}|^4\widetilde{v})\cdot\overline{v}-\sum_{k=1}^2\sum_{j=1}^3\overline{\widetilde{v}^k\widetilde{v}^j}\,\partial_{x_k}(|\widetilde{v}|^4\widetilde{v}^j)$$
$$-\left(\int_{-h}^z T(x,y,\xi,t)d\xi-\frac{1}{h}\int_{-h}^0\int_{-h}^z T(x,y,\xi,t)d\xi dz\right)\nabla_H\cdot(|\widetilde{v}|^4\widetilde{v})\Bigg]\,dxdydz. \tag{75}$$

As a result, we obtain

$$\begin{aligned}
&\frac{1}{6}\frac{d\|\widetilde{v}\|_6^6}{dt}+\frac{1}{R_1}\int_\Omega\left(|\nabla_H\widetilde{v}|^2|\widetilde{v}|^4+\left|\nabla_H|\widetilde{v}|^2\right|^2|\widetilde{v}|^2\right)\,dxdydz+\frac{1}{R_2}\int_\Omega\left(|\widetilde{v}_z|^2|\widetilde{v}|^4+\left|\partial_z|\widetilde{v}|^2\right|^2|\widetilde{v}|^2\right)\,dxdydz\\
&\le C\int_M\left[|\overline{v}|\int_{-h}^0|\nabla_H\widetilde{v}|\,|\widetilde{v}|^5\,dz\right]\,dxdy+C\int_M\left[\left(\int_{-h}^0|\widetilde{v}|^2\,dz\right)\left(\int_{-h}^0|\nabla_H\widetilde{v}|\,|\widetilde{v}|^4\,dz\right)\right]\,dxdy\\
&\quad+C\int_M\left[\overline{|T|}\int_{-h}^0|\nabla_H\widetilde{v}|\,|\widetilde{v}|^4\,dz\right]\,dxdy. \qquad (76)
\end{aligned}$$

Therefore, by the Cauchy–Schwarz inequality and Hölder inequality we reach

$$\begin{aligned}
&\frac{1}{6}\frac{d\|\widetilde{v}\|_6^6}{dt}+\frac{1}{R_1}\int_\Omega\left(|\nabla_H\widetilde{v}|^2|\widetilde{v}|^4+\left|\nabla_H|\widetilde{v}|^2\right|^2|\widetilde{v}|^2\right)\,dxdydz+\frac{1}{R_2}\int_\Omega\left(|\widetilde{v}_z|^2|\widetilde{v}|^4+\left|\partial_z|\widetilde{v}|^2\right|^2|\widetilde{v}|^2\right)\,dxdydz\\
&\le C\int_M\left[|\overline{v}|\left(\int_{-h}^0|\nabla_H\widetilde{v}|^2\,|\widetilde{v}|^4\,dz\right)^{1/2}\left(\int_{-h}^0|\widetilde{v}|^6\,dz\right)^{1/2}\right]\,dxdy\\
&\quad+C\int_M\left[\left(\int_{-h}^0|\widetilde{v}|^2\,dz\right)\left(\int_{-h}^0|\nabla_H\widetilde{v}|^2\,|\widetilde{v}|^4\,dz\right)^{1/2}\left(\int_{-h}^0|\widetilde{v}|^4\,dz\right)^{1/2}\right]\,dxdy\\
&\quad+C\int_M\left[\overline{|T|}\left(\int_{-h}^0|\nabla_H\widetilde{v}|^2\,|\widetilde{v}|^4\,dz\right)^{1/2}\left(\int_{-h}^0|\widetilde{v}|^4\,dz\right)^{1/2}\right]\,dxdy \qquad (77)
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\frac{1}{6}\frac{d\|\widetilde{v}\|_6^6}{dt}+\frac{1}{R_1}\int_\Omega\left(|\nabla_H\widetilde{v}|^2|\widetilde{v}|^4+\left|\nabla_H|\widetilde{v}|^2\right|^2|\widetilde{v}|^2\right)\,dxdydz+\frac{1}{R_2}\int_\Omega\left(|\widetilde{v}_z|^2|\widetilde{v}|^4+\left|\partial_z|\widetilde{v}|^2\right|^2|\widetilde{v}|^2\right)\,dxdydz\\
&\le C\|\overline{v}\|_{L^4(M)}\left(\int_\Omega|\nabla_H\widetilde{v}|^2\,|\widetilde{v}|^4\,dxdydz\right)^{1/2}\left(\int_M\left(\int_{-h}^0|\widetilde{v}|^6\,dz\right)^2\,dxdy\right)^{1/4}\\
&\quad+C\left(\int_M\left(\int_{-h}^0|\widetilde{v}|^2\,dz\right)^4\,dxdy\right)^{1/4}\left(\int_\Omega|\nabla_H\widetilde{v}|^2\,|\widetilde{v}|^4\,dxdydz\right)^{1/2}\left(\int_M\left(\int_{-h}^0|\widetilde{v}|^4\,dz\right)^2\,dxdy\right)^{1/4}\\
&\quad+C\|\,\overline{|T|}\,\|_{L^4(M)}\left(\int_\Omega|\nabla_H\widetilde{v}|^2\,|\widetilde{v}|^4\,dxdydz\right)^{1/2}\left(\int_M\left(\int_{-h}^0|\widetilde{v}|^4\,dz\right)^2\,dxdy\right)^{1/4}. \qquad (78)
\end{aligned}$$

Using the Minkowsky inequality (43) we get

$$\left(\int_M\left(\int_{-h}^0|\widetilde{v}|^6\,dz\right)^2\,dxdy\right)^{1/2}\le C\int_{-h}^0\left(\int_M|\widetilde{v}|^{12}\,dxdy\right)^{1/2}\,dz. \qquad (79)$$

Thanks to (42),

$$\int_M|\widetilde{v}|^{12}\,dxdy\le C_0\left(\int_M|\widetilde{v}|^6\,dxdy\right)\left(\int_M|\widetilde{v}|^4|\nabla_H\widetilde{v}|^2\,dxdy\right)+\left(\int_M|\widetilde{v}|^6\,dxdy\right)^2. \qquad (80)$$

Thus, by the Cauchy–Schwarz inequality we obtain

$$\left(\int_M\left(\int_{-h}^0|\widetilde{v}|^6\,dz\right)^2\,dxdy\right)^{1/2}\le C\|\widetilde{v}\|_{L^6(\Omega)}^3\left(\int_\Omega|\widetilde{v}|^4|\nabla_H\widetilde{v}|^2\,dxdydz\right)^{1/2}+\|\widetilde{v}\|_{L^6(\Omega)}^6. \qquad (81)$$

Similarly, by (43) and (34), we also obtain

$$\left(\int_M \left(\int_{-h}^0 |\widetilde{v}|^4 \, dz\right)^2 \, dxdy\right)^{1/2} \le C \int_{-h}^0 \left(\int_M |\widetilde{v}|^8 \, dxdy\right)^{1/2} \, dz$$

$$\le C \int_{-h}^0 \|\widetilde{v}\|_{L^6(M)}^3 \left(\|\nabla_H \widetilde{v}\|_{L^2(M)} + \|\widetilde{v}\|_{L^2(M)}\right) \, dz \le C\|\widetilde{v}\|_6^3 \left(\|\nabla_H \widetilde{v}\|_2 + \|\widetilde{v}\|_2\right), \tag{82}$$

and

$$\left(\int_M \left(\int_{-h}^0 |\widetilde{v}|^2 \, dz\right)^4 \, dxdy\right)^{1/4} \le C \int_{-h}^0 \left(\int_M |\widetilde{v}|^8 \, dxdy\right)^{1/4} \, dz$$

$$\le C \int_{-h}^0 \|\widetilde{v}\|_{L^6(M)}^{3/2} \left(\|\nabla_H \widetilde{v}\|_{L^2(M)}^{1/2} + \|\widetilde{v}\|_{L^2(M)}^{1/2}\right) \, dz \le C\|\widetilde{v}\|_6^{3/2} \left(\|\nabla_H \widetilde{v}\|_2 + \|\widetilde{v}\|_2\right)^{1/2}. \tag{83}$$

Therefore, using (81)–(83) and (33), we reach to

$$\frac{1}{6}\frac{d\|\widetilde{v}\|_6^6}{dt} + \frac{1}{R_1}\int_\Omega \left(|\nabla_H \widetilde{v}|^2 |\widetilde{v}|^4 + \left|\nabla_H |\widetilde{v}|^2\right|^2 |\widetilde{v}|^2\right) \, dxdydz + \frac{1}{R_2}\int_\Omega \left(|\widetilde{v}_z|^2 |\widetilde{v}|^4 + \left|\partial_z |\widetilde{v}|^2\right|^2 |\widetilde{v}|^2\right) \, dxdydz$$

$$\le C\|\overline{v}\|_2^{1/2} \, \|\nabla_H \overline{v}\|_2^{1/2} \|\widetilde{v}\|_6^{3/2} \left(\int_\Omega |\nabla_H \widetilde{v}|^2 \, |\widetilde{v}|^4 \, dxdydz\right)^{3/4} + C\|\widetilde{v}\|_6^3 \left(\|\nabla_H \widetilde{v}\|_2 + \|\widetilde{v}\|_2\right) \left(\int_\Omega |\nabla_H \widetilde{v}|^2 \, |\widetilde{v}|^4 \, dxdydz\right)^{1/2}$$

$$+C\|\overline{v}\|_2^{1/2} \, \|\nabla_H \overline{v}\|_2^{1/2} \|\widetilde{v}\|_6^6 + C\|T\|_\infty \, \|\widetilde{v}\|_6^{3/2} \left(\|\nabla_H \widetilde{v}\|_2^{1/2} + \|\widetilde{v}\|_2^{1/2}\right) \left(\int_\Omega |\nabla_H \widetilde{v}|^2 \, |\widetilde{v}|^4 \, dxdydz\right)^{1/2}.$$

By Young's inequality and Cauchy–Schwarz inequality we have

$$\frac{d\|\widetilde{v}\|_6^6}{dt} + \frac{1}{R_1}\int_\Omega \left(|\nabla_H \widetilde{v}|^2 |\widetilde{v}|^4 + \left|\nabla_H |\widetilde{v}|^2\right|^2 |\widetilde{v}|^2\right) \, dxdydz + \frac{1}{R_2}\int_\Omega \left(|\widetilde{v}_z|^2 |\widetilde{v}|^4 + \left|\partial_z |\widetilde{v}|^2\right|^2 |\widetilde{v}|^2\right) \, dxdydz$$

$$\le C\|\overline{v}\|_2^2 \, \|\nabla_H \overline{v}\|_2^2 \|\widetilde{v}\|_6^6 + C\|\widetilde{v}\|_6^6 \|\nabla_H \widetilde{v}\|_2^2 + C\|T\|_\infty^4 + C\|\widetilde{v}\|_2^2 \|\widetilde{v}\|_6^6.$$

Thanks to (59), (69) and Gronwall inequality, we get

$$\|\widetilde{v}(t)\|_6^6 + \int_0^t \left(\frac{1}{R_1}\int_\Omega |\nabla_H \widetilde{v}|^2 |\widetilde{v}|^4 \, dxdydz + \frac{1}{R_2}\int_\Omega |\widetilde{v}_z|^2 |\widetilde{v}_z|^4 \, dxdydz\right) \le K_3, \tag{84}$$

where

$$K_3 = e^{K_1^2 t}\left[\|v_0\|_{H^1(\Omega)}^6 + K_2^4 \, t\right]. \tag{85}$$

3.4. $\|\nabla_H \overline{v}\|_2$ **estimates.** By taking the inner product of equation (29) with $-\Delta_H \overline{v}$ in $L^2(M)$, and applying (30), and using the boundary conditions (25), we reach

$$\frac{1}{2}\frac{d\|\nabla_H \overline{v}\|_2^2}{dt} + \frac{1}{R_1}\|\Delta_H \overline{v}\|_2^2 = \int_M \left\{(\overline{v}\cdot\nabla_H)\overline{v} + \overline{\left[(\widetilde{v}\cdot\nabla_H)\widetilde{v} + (\nabla_H\cdot\widetilde{v})\,\widetilde{v}\right]} + f_0 \vec{k}\times\overline{v}\right\}\cdot\Delta_H \overline{v} \, dxdy. \tag{86}$$

Following the situation for the $2D$ Navier–Stokes equations (cf. e.g., [10], [32]) we have

$$\left|\int_M (\overline{v}\cdot\nabla_H)\overline{v}\cdot\Delta_H \overline{v} \, dxdy\right| \le C\|\overline{v}\|_2^{1/2}\|\nabla_H \overline{v}\|_2 \, \|\Delta_H \overline{v}\|_2^{3/2}. \tag{87}$$

By the Cauchy–Schwarz and the Hölder inequalities, we have

$$\begin{aligned}
&\left|\int_M \overline{(\widetilde{v}\cdot\nabla_H)\widetilde{v}+(\nabla_H\cdot\widetilde{v})\,\widetilde{v}}\cdot\Delta_H\overline{v}\,dxdy\right| \le C\int_M\int_{-h}^0 |\widetilde{v}|\,|\nabla_H\widetilde{v}|\,dz\,|\Delta_H\overline{v}|\,dxdy\\
&\le C\int_M\left[\left(\int_{-h}^0|\widetilde{v}|^2\,|\nabla_H\widetilde{v}|\,dz\right)^{1/2}\left(\int_{-h}^0|\nabla_H\widetilde{v}|\,dz\right)^{1/2}|\Delta_H\overline{v}|\right]dxdy\\
&\le C\left[\int_M\left(\int_{-h}^0|\widetilde{v}|^2\,|\nabla_H\widetilde{v}|\,dz\right)^2dxdy\right]^{1/4}\left[\int_M\left(\int_{-h}^0|\nabla_H\widetilde{v}|\,dz\right)^2dxdy\right]^{1/4}\left[\int_M|\Delta_H\overline{v}|^2\,dxdy\right]^{1/2}\\
&\le C\|\nabla_H\widetilde{v}\|_2^{1/2}\left(\int_\Omega|\widetilde{v}|^4|\nabla_H\widetilde{v}|^2\,dxdydz\right)^{1/4}\|\Delta_H\overline{v}\|_2,
\end{aligned}\tag{88}$$

and

$$\left|\int_M f_0\times\overline{v}\cdot\Delta_H\overline{v}\,dxdy\right|\le C\|\overline{v}\|_2\|\Delta_H\overline{v}\|_2.\tag{89}$$

Thus, by Young's inequality and the Cauchy–Schwarz inequality, we have

$$\frac{d\|\nabla_H\overline{v}\|_2^2}{dt}+\frac{1}{R_1}\|\Delta_H\overline{v}\|_2^2\le C\|\overline{v}\|_2^2\|\nabla_H\overline{v}\|_2^4+C\|\nabla_H\widetilde{v}\|_2^2+C\int_\Omega|\widetilde{v}|^4|\nabla_H\widetilde{v}|^2\,dxdydz+C\|\overline{v}\|_2^2.\tag{90}$$

By (59), (84) and thanks to Gronwall inequality we obtain

$$\|\nabla_H\overline{v}\|_2^2+\frac{1}{R_1}\int_0^t|\Delta_H\overline{v}|_2^2\,ds\le K_4,\tag{91}$$

where

$$K_4=e^{K_2^2t}\left[\|v_0\|_{H^1(\Omega)}^2+K_2+K_3\right].\tag{92}$$

3.5. $\|v_z\|_6$ **estimates.** The *a priori* estimates (59)–(91) are essentially similar to those obtained in [7]. From now on, we will get new *a priori* estimates of various norms.

Denote by $u=v_z$. It is clear that u satisfies

$$\begin{aligned}
&\frac{\partial u}{\partial t}+L_1u+(v\cdot\nabla_H)u-\left(\int_{-h}^z\nabla_H\cdot v(x,y,\xi,t)d\xi\right)\frac{\partial u}{\partial z}\\
&\qquad+(u\cdot\nabla_H)v-(\nabla_H\cdot v)u+f_0\vec{k}\times u-\nabla_HT=0.
\end{aligned}\tag{93}$$

$$u|_{z=0}=u|_{z=-h}=0.\tag{94}$$

Taking the inner product of the equation (93) with $u|u|^4$ in L^2, we get

$$\begin{aligned}
&\frac{1}{6}\frac{d\|u\|_6^6}{dt}+\frac{5}{R_1}\|\,|u|^2\,|\nabla_Hu|\,\|_2^2+\frac{5}{R_2}\|\,|u|^2\,|\partial_zu|\,\|_2^2\\
&=-\int_\Omega\left((v\cdot\nabla_H)u-\left(\int_{-h}^z\nabla_H\cdot v(x,y,\xi,t)d\xi\right)\frac{\partial u}{\partial z}\right)\cdot u|u|^4\,dxdydz\\
&\quad-\int_\Omega\left((u\cdot\nabla_H)v-(\nabla_H\cdot v)u+f_0\vec{k}\times u-\nabla_HT\right)\cdot u|u|^4\,dxdydz.
\end{aligned}\tag{95}$$

Notice again that

$$f_0\vec{k}\times u\cdot u|u|^4=0.\tag{96}$$

Integrating by parts and using the boundary conditions, in particular (94), give

$$-\int_\Omega \left((v\cdot\nabla_H)u - \left(\int_{-h}^z \nabla_H\cdot v(x,y,\xi,t)d\xi\right)\frac{\partial u}{\partial z}\right)\cdot u|u|^4\, dxdydz = 0. \tag{97}$$

Thus, by (96), (97) and Hölder inequality, we have

$$\begin{aligned}
&\frac{1}{6}\frac{d\|u\|_6^6}{dt} + \frac{5}{R_1}\|\,|u|^2\,|\nabla_H u|\,\|_2^2 + \frac{5}{R_2}\|\,|u|^2\,|\partial_z u|\,\|_2^2 \\
&= -\int_\Omega ((u\cdot\nabla_H)v - (\nabla_H\cdot v)u - \nabla_H T)\cdot u|u|^4\, dxdydz \\
&\le C\int_\Omega |v|\,|u|^5\,|\nabla_H u|\, dxdydz + C\int_\Omega |T|\,|u|^4\,|\nabla_H u|\, dxdydz \\
&\le C\left(\|v\|_6\|u^3\|_3\|\nabla_H u^2\|_2 + \|T\|_6\,\|u^2\|_3\|\nabla_H u^3\|_2\right) \\
&\le C\left(\|v\|_6\|u\|_6^{3/2}\|\nabla_H u^3\|_2^{3/2} + \|T\|_6\,\|u\|_6^2\|\nabla_H u^3\|_2\right).
\end{aligned} \tag{98}$$

Thanks to the Cauchy–Schwarz inequality, we have

$$\begin{aligned}
&\frac{d\|u\|_6^6}{dt} + \frac{1}{R_1}\|\,|u|^2\,|\nabla_H u|\,\|_2^2 + \frac{1}{R_2}\|\,|u|^2\,|\partial_z u|\,\|_2^2 \\
&\le C\left(1+\|v\|_6^4\right)\,\|u\|_6^6 + \|T\|_6^6 &(99)\\
&\le C\left(1+\|\nabla_H\overline{v}\|_2^4 + \|\widetilde{v}\|_6^4\right)\,\|u\|_6^6 + \|T\|_6^6. &(100)
\end{aligned}$$

Using (59), (84), (91), and Gronwall inequality, we get

$$\|u\|_6^6 + \int_0^t\left[\frac{1}{R_1}\|\,|u|^2\,|\nabla_H u|\,\|_2^2 + \frac{1}{R_2}\|\,|u|^2\,|\partial_z u|\,\|_2^2\right] ds \le K_5, \tag{101}$$

where

$$K_5 = e^{(1+K_3^{2/3}+K_4^2)t}\left[\|\partial_z v_0\|_{H^1(\Omega)}^6 + K_2^6\, t\right]. \tag{102}$$

3.6. $\|v_{zz}\|_2$ **estimates.** Taking the inner product of the equation (93) with $-u_{zz}$ in $L^2(\Omega)$ and recalling that $u = v_z$, which satisfies the boundary condition (94), we get

$$\begin{aligned}
&\frac{1}{2}\frac{d\|u_z\|_2^2}{dt} + \frac{1}{R_1}\|\nabla_H u_z\|_2^2 + \frac{1}{R_2}\|u_{zz}\|_2^2 \\
&= \int_\Omega \left((v\cdot\nabla_H)u - \left(\int_{-h}^z \nabla_H\cdot v(x,y,\xi,t)d\xi\right)\frac{\partial u}{\partial z}\right)\cdot u_{zz}\, dxdydz \\
&+ \int_\Omega \left((u\cdot\nabla_H)v - (\nabla_H\cdot v)u + f_0\vec{k}\times u - \nabla_H T\right)\cdot u_{zz}\, dxdydz \\
&= -\int_\Omega [(u\cdot\nabla_H)u + (u_z\cdot\nabla_H)v + (u\cdot\nabla_H)u - (\nabla_H\cdot u)u - (\nabla_H\cdot v)u_z]\cdot u_z\, dxdydz - \int_\Omega T_z(\nabla_H\cdot u_z)\, dxdydz \\
&\le C\|u\|_6\|\nabla_H u\|_2\|u_z\|_3 + C\|v\|_6\|\nabla_H u_z\|_2\|u_z\|_3 + \|T_z\|_2\|\nabla_H u_z\|_2 \\
&\le C\left[\|u\|_6\|\nabla_H u\|_2 + \|v\|_6\|\nabla_H u_z\|_2\right]\|u_z\|_2^{1/2}\left(\|\nabla_H u_z\|_2^{1/2} + \|u_{zz}\|_2^{1/2}\right) + \|T_z\|_2\|\nabla_H u_z\|_2.
\end{aligned}$$

By the Cauchy–Schwarz and Young's inequalities, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d\|u_z\|_2^2}{dt} + \frac{1}{R_1}\|\nabla_H u_z\|_2^2 + \frac{1}{R_2}\|u_{zz}\|_2^2 \\
&\le C\left(\|v\|_6^4 + \|u\|_6^4\right)\,\|u_z\|_2^2 + C\|\nabla_H u\|_2^2 + C\|T_z\|_2^2.
\end{aligned}$$

Applying (59), (84), (101), and Gronwall inequality yield

$$\|u_z\|_2^2 + \int_0^t \left[\frac{1}{R_1}\|\nabla_H u_z\|_2^2 + \frac{1}{R_2}\|u_{zz}\|_2^2 \right] ds \le K_6, \tag{103}$$

where

$$K_6 = Ce^{(K_3^{2/3}+K_5^{2/3})t}\left[\|v_0\|_{H^1(\Omega)}^2 + K_1\right]. \tag{104}$$

3.7. $\|\nabla_H \times v_z\|_2^2 + \|\nabla_H \cdot v_z + R_1 T\|_2^2$ **estimates.** Let β be the solution of the following two–dimensional elliptic problem with periodic boundary conditions:

$$\Delta_H \beta = \nabla_H T, \qquad \int_M \beta \, dxdy = 0, \tag{105}$$

where z is considered as a parameter. Roughly speaking, β is like the potential vorticity. Notice that

$$\nabla_H \cdot \beta = T, \qquad \nabla_H \times \beta = 0. \tag{106}$$

Recall that $u = v_z$. We denote by

$$\zeta = u + R_1\beta, \tag{107}$$

$$\eta = \left(\nabla_H^\perp \cdot \zeta\right) = \nabla_H^\perp \cdot u = \partial_x u_2 - \partial_y u_1, \tag{108}$$

$$\theta = (\nabla_H \cdot \zeta) = \nabla_H \cdot u + R_1 T = \partial_x u_1 + \partial_y u_2 + R_1 T. \tag{109}$$

Applying (39) for ζ and β with $m \ge 0, 1 < q < \infty$, using (106)–(109), we obtain

$$\begin{aligned} \|\nabla_H u\|_{W^{m,q}(M)} &\le C\left(\|\nabla_H \zeta\|_{W^{m,q}(M)} + R_1\|\nabla_H \beta\|_{W^{m,q}(M)}\right) \\ &\le C\left(\|\eta\|_{W^{m,q}(M)} + \|\theta\|_{W^{m,q}(M)} + \|T\|_{W^{m,q}(M)}\right), \end{aligned} \tag{110}$$

where, again, we consider z as a parameter; consequently, the constant C is independent of z. By applying the operator $\nabla_H^\perp \cdot$ to equation (93) we obtain

$$\begin{aligned} &\frac{\partial \eta}{\partial t} + L_1\eta + \nabla_H^\perp \cdot \left[(v\cdot\nabla_H)u - \left(\int_{-h}^z \nabla_H \cdot v(x,y,\xi,t)d\xi\right)\frac{\partial u}{\partial z} + (u\cdot\nabla_H)v - (\nabla_H\cdot v)u\right] \\ &-f_0\left(R_1 T - \theta\right) = 0. \end{aligned} \tag{111}$$

Then, multiplying equation (24) by R_1 and adding to the above equation we reach

$$\begin{aligned} &\frac{\partial \theta}{\partial t} + L_1\theta + \nabla_H \cdot \left[(v\cdot\nabla_H)u - \left(\int_{-h}^z \nabla_H \cdot v(x,y,\xi,t)d\xi\right)\frac{\partial u}{\partial z} + (u\cdot\nabla_H)v - (\nabla_H\cdot v)u\right] - f_0\,\eta \\ &+R_1\left[v\cdot\nabla_H T - \left(\int_{-h}^z \nabla_H\cdot v(x,y,\xi,t)d\xi\right)\left(\frac{\partial T}{\partial z} + \frac{1}{h}\right)\right] = R_1 Q + R_1\left(\frac{1}{R_3} - \frac{1}{R_2}\right)T_{zz}. \end{aligned} \tag{112}$$

Taking the inner product of equation (111) with η in $L^2(\Omega)$ and the equation (112) with θ in $L^2(\Omega)$, integrating by parts and observing that $\eta|_{z=0} = \eta|_{z=-h} = \theta|_{z=0} = \theta|_{z=-h} = 0$, we get

$$
\begin{aligned}
&\frac{1}{2}\frac{d\left(\|\eta\|_2^2 + \|\theta\|_2^2\right)}{dt} + \frac{1}{R_1}\left(\|\nabla_H\eta\|_2^2 + \|\nabla_H\theta\|_2^2\right) + \frac{1}{R_2}\left(\|\partial_z\eta\|_2^2 + \|\partial_z\theta\|_2^2\right) \\
&= -\int_\Omega \left\{\nabla_H^\perp \cdot \left[(v\cdot\nabla_H)u - \left(\int_{-h}^z \nabla_H\cdot v(x,y,\xi,t)d\xi\right)\frac{\partial u}{\partial z} + (u\cdot\nabla_H)v - (\nabla_H\cdot v)u\right]\ \eta - f_0\, R_1 T\,\eta\right\}\ dxdydz \\
&\quad - \int_\Omega \left\{\nabla_H \cdot \left[(v\cdot\nabla_H)u - \left(\int_{-h}^z \nabla_H\cdot v(x,y,\xi,t)d\xi\right)\frac{\partial u}{\partial z} + (u\cdot\nabla_H)v - (\nabla_H\cdot v)u\right]\ \theta\right\}\ dxdydz \\
&\quad - \int_\Omega \left\{R_1\left[v\cdot\nabla_H T - \left(\int_{-h}^z \nabla_H\cdot v(x,y,\xi,t)d\xi\right)\left(\frac{\partial T}{\partial z} + \frac{1}{h}\right) - Q + \left(\frac{1}{R_3} - \frac{1}{R_2}\right)T_{zz}\right]\ \theta\right\}\ dxdydz \\
&\le C\|Q\|_2\|\theta\|_2 + C\int_\Omega \left(|v|\,|\nabla_H u|\ + |u|\,|\nabla_H v| + |u_z|\,\int_{-h}^0 |\nabla_H\cdot v|\ d\xi\right)(|\nabla_H\eta| + |\nabla_H\theta|)\ dxdydz + C\|T\|_2\|\eta\|_2 \\
&\quad + C\int_\Omega \left[|\nabla_H v||T||\theta| + |v||T||\nabla_H\theta| + |T_z|\ \left(\int_{-h}^0 |\nabla_H\cdot v|\ d\xi\right)\ |\theta|\right]\ dxdydz + C\|v\|_2\|\nabla_H\theta\|_2 + C\|T_z\|_2\|\theta_z\|_2 \\
&\le C\|Q\|_2\|\theta\|_2 + C\int_\Omega \left(|v|\,|\nabla_H u|\ + |u|\,|\nabla_H v| + |u_z|\,\int_{-h}^0 (|\theta| + |T|)\ d\xi\right)(|\nabla_H\eta| + |\nabla_H\theta|)\ dxdydz + C\|T\|_2\|\eta\|_2 \\
&\quad + C\int_\Omega \left[|\nabla_H v||T||\theta| + |v||T||\nabla_H\theta| + |T_z|\ \left(\int_{-h}^0 (|\theta| + |T|)\ d\xi\right)|\theta|\right]\ dxdydz + C\|v\|_2\|\nabla_H\theta\|_2 + C\|T_z\|_2\|\theta_z\|_2.
\end{aligned}
$$

Using Hölder inequality, and inequalities (44) and (110), we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d\left(\|\eta\|_2^2 + \|\theta\|_2^2\right)}{dt} + \frac{1}{R_1}\left(\|\nabla_H\eta\|_2^2 + \|\nabla_H\theta\|_2^2\right) + \frac{1}{R_2}\left(\|\partial_z\eta\|_2^2 + \|\partial_z\theta\|_2^2\right) \\
&\le C\left(\|v\|_6\|\nabla_H u\|_3 + \|u\|_6\|\nabla_H v\|_3 + \|u_z\|_2^{\frac{1}{2}}\|\nabla_H u_z\|_2^{\frac{1}{2}}\|\theta\|_2^{\frac{1}{2}}\|\nabla_H\theta\|_2^{\frac{1}{2}} + \|T\|_\infty\|u_z\|_2\right)(\|\nabla_H\eta\|_2 + \|\nabla_H\theta\|_2) \\
&\quad + C\|T\|_2\|\eta\|_2 + C\|\nabla_H v\|_2\|T\|_\infty\|\theta\|_2 + C\|v\|_2\|T\|_\infty\|\nabla_H\theta\|_2 + C\|T_z\|_2\|\theta\|_2\|\nabla_H\theta\|_2 + C\|T\|_\infty\|T_z\|_2\|\theta\|_2 \\
&\quad + C\|Q\|_2\|\theta\|_2 + C\|v\|_2\|\nabla_H\theta\|_2 + C\|T_z\|_2\|\theta_z\|_2 \\
&\le C(\|T\|_3 + \|\eta\|_2^{\frac{1}{2}}\|\nabla_H\eta\|_2^{\frac{1}{2}} + \|\theta\|_2^{\frac{1}{2}}\|\nabla_H\theta\|_2^{\frac{1}{2}})\ (\|v\|_6 + \|u\|_6)\ (\|\nabla_H\eta\|_2 + \|\nabla_H\theta\|_2) + \\
&\quad + C\left(\|u_z\|_2^{1/2}\|\nabla_H u_z\|_2^{1/2}\|\theta\|_2^{1/2}\|\nabla_H\theta\|_2^{1/2} + \|T\|_\infty\|u_z\|_2 + \|T_z\|_2\|\theta\|_2 + \|v\|_2\|T\|_\infty\right)(\|\nabla_H\eta\|_2 + \|\nabla_H\theta\|_2) \\
&\quad + C\|T\|_2\|\eta\|_2 + C\left(\|\nabla_H v\|_2\ \|T\|_\infty + \|T\|_\infty\|T_z\|_2\right)\|\theta\|_2 + C\|Q\|_2\|\theta\|_2 + C\|v\|_2\|\nabla_H\theta\|_2 + C\|T_z\|_2\|\theta_z\|_2.
\end{aligned}
$$

By Young's and the Cauchy–Schwarz inequalities, we have

$$
\begin{aligned}
&\frac{d\left(\|\eta\|_2^2 + \|\theta\|_2^2\right)}{dt} + \frac{1}{R_1}\left(\|\nabla_H\eta\|_2^2 + \|\nabla_H\theta\|_2^2\right) + \frac{1}{R_2}\left(\|\partial_z\eta\|_2^2 + \|\partial_z\theta\|_2^2\right) \\
&\le C\left(1 + \|v\|_6^4 + \|u\|_6^4 + \|T_z\|_2^2 + \|u_z\|_2^2\|\nabla_H u_z\|_2^2\right)\ \left(\|\eta\|_2^2 + \|\theta\|_2^2\right) \\
&\quad + C\|T\|_2^2 + C\|Q\|_2^2 + C\left(\|v\|_6^2 + \|u\|_6^2 + \|u_z\|_2^2 + \|v\|_2^2 + \|\nabla_H v\|_2^2 + \|T_z\|_2^2\right)\left(1 + \|T\|_\infty^2\right).
\end{aligned}
$$

Thanks to (59), (69), (84), (101), (103), and Gronwall inequality, we have

$$
\|\eta\|_2^2 + \|\theta\|_2^2 + \int_0^t \left[\frac{1}{R_1}\left(\|\nabla_H\eta\|_2^2 + \|\nabla_H\theta\|_2^2\right) + \frac{1}{R_2}\left(\|\partial_z\eta\|_2^2 + \|\partial_z\theta\|_2^2\right)\right]\ ds \le K_7, \tag{113}
$$

where

$$
K_7 = Ce^{(K_1 + K_3^{2/3} + K_5^{2/3} + K_6^2)t}\left[\|v_0\|_{H^1(\Omega)}^2 + K_1 + \|Q\|_2^2 + K_2^2\ (K_2 + K_3^{1/3} + K_5^{1/3} + K_6)\right]. \tag{114}
$$

3.8. $\|\Delta_H\overline{v}\|^2_{H^1(M)}+\|\nabla_H(\nabla_H^\perp\cdot v_z)\|^2_{H^1(\Omega)}+\|\nabla_H\left(\nabla_H\cdot v_z+R_1T\right)\|^2_{H^1(\Omega)}+\|\nabla_H T\|^2_{H^1(\Omega)}$ **estimates.** First, let us observe that

$$|\nabla_H v(x,y,z)|\le|\nabla_H\overline{v}(x,y)|+\int_{-h}^0|\nabla_H v_z(x,y,z)|\,dz.$$

Therefore, from the above and (107), we have

$$\begin{aligned}\|\nabla_H v\|_\infty&\le\|\nabla_H\overline{v}\|_\infty+\left\|\int_{-h}^0|\nabla_H u|\,dz\right\|_\infty\\&\le\|\nabla_H\overline{v}\|_\infty+R_1\int_{-h}^0\|\nabla_H\beta\|_\infty\,dz+\left\|\int_{-h}^0|\nabla_H\zeta|\,dz\right\|_\infty.\end{aligned}$$

By applying inequality (40) to $\nabla_H\overline{v}$ and $\int_{-h}^0|\nabla_H\zeta|\,dz$ we reach

$$\|\nabla_H\overline{v}\|_\infty\le C\|\nabla_H\overline{v}\|_{H^1(M)}\left(1+\log^+\|\nabla_H\overline{v}\|_{H^2(M)}\right)^{1/2},\tag{115}$$

$$\left\|\int_{-h}^0|\nabla_H\zeta|\,dz\right\|_\infty\le C\left\|\int_{-h}^0|\nabla_H\zeta|\,dz\right\|_{H^1(M)}\left(1+\log^+\left\|\int_{-h}^0|\nabla_H\zeta|\,dz\right\|_{H^2(M)}\right)^{1/2}\tag{116}$$

$$\tag{117}$$

Applying inequality (41) to $\nabla_H\beta$, also by (105) and (106), we reach

$$\|\nabla_H\beta\|_\infty\le C\left(\|\nabla_H\cdot\beta\|_\infty+\|\nabla_H^\perp\cdot\beta\|_\infty\right)\left(1+\log^+\|\nabla_H\beta\|_{H^2(M)}\right)\le C\|T\|_\infty\left(1+\log^+\|T\|_{H^2(M)}\right)\tag{118}$$

Therefore, by (115)–(118), we infer that

$$\begin{aligned}\|\nabla_H v\|_\infty&\le C\|\nabla_H\overline{v}\|_{H^1(M)}\left(1+\log^+\|\nabla_H\overline{v}\|_{H^2(M)}\right)^{1/2}+C\int_{-h}^0\left[\|T\|_\infty\left(1+\log^+\|T\|_{H^2(M)}\right)\right]\,dz\\&+C\left\|\int_{-h}^0|\nabla_H\zeta|\,dz\right\|_{H^1(M)}\left(1+\log^+\left\|\int_{-h}^0|\nabla_H\zeta|\,dz\right\|_{H^2(M)}\right)^{1/2}\\&\le C\|\nabla_H\overline{v}\|_{H^1(M)}\left(1+\log^+\|\nabla_H\overline{v}\|_{H^2(M)}\right)^{1/2}+C\|T\|_\infty\left(1+\log^+\int_{-h}^0\|T\|_{H^2(M)}\,dz\right)\\&+C\left\|\int_{-h}^0|\nabla_H\zeta|\,dz\right\|_{H^1(M)}\left(1+\log^+\left\|\int_{-h}^0|\nabla_H\zeta|\,dz\right\|_{H^2(M)}\right)^{1/2}\\&\le C\|\nabla_H\overline{v}\|_{H^1(M)}\left(1+\log^+\|\nabla_H\overline{v}\|_{H^2(M)}\right)^{1/2}+C\,\|T\|_\infty\left(1+\log^+\|\Delta_H T\|_{L^2(\Omega)}\right)\\&+C\left(\|\eta\|_{H^1(\Omega)}+\|\theta\|_{H^1(\Omega)}\right)\left[1+\log^+\left(\|\Delta_H\eta\|_{L^2(\Omega)}+\|\Delta_H\theta\|_{L^2(\Omega)}\right)\right]^{1/2}.\end{aligned}\tag{119}$$

3.8.1. $\|\nabla_H\Delta_H\overline{v}\|_2^2$ **estimates**. By taking the Δ_H to equation (29) and then taking the inner product of equation (29) with $-\Delta_H^2\overline{v}$ in $L^2(M)$, we reach

$$\begin{aligned}&\frac12\frac{d\|\nabla_H\Delta_H\overline{v}\|_2^2}{dt}+\frac1{R_1}\|\Delta_H^2\overline{v}\|_2^2\\&=\int_M\Delta_H\left\{(\overline{v}\cdot\nabla_H)\overline{v}+\overline{\left[(\widetilde{v}\cdot\nabla_H)\widetilde{v}+(\nabla_H\cdot\widetilde{v})\,\widetilde{v}\right]}+f_0\vec{k}\times\overline{v}\right\}\cdot\Delta_H^2\overline{v}\,dxdy.\end{aligned}$$

Integrating by parts and applying (30), we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d\|\nabla_H\Delta_H\overline{v}\|_2^2}{dt}+\frac{1}{R_1}\|\Delta_H^2\overline{v}\|_2^2 \\
&\leq C\int_M\left\{|\overline{v}|\;|\nabla_H^3\overline{v}|+|\nabla_H\overline{v}|\;|\nabla_H^2\overline{v}|+\int_{-h}^0\left(|\widetilde{v}|\;|\nabla_H^3\widetilde{v}|+|\nabla_H\widetilde{v}|\;|\nabla_H^2\widetilde{v}|\right)\,dz+|\Delta_H\overline{v}|\right\}|\Delta_H^2\overline{v}|\;dxdy \\
&\leq C\int_M\left\{|\overline{v}|\;|\nabla_H^3\overline{v}|+|\nabla_H\overline{v}|\;|\nabla_H^2\overline{v}|+\left(\int_{-h}^0|\widetilde{v}|\;dz\right)\left(\int_{-h}^0(|\nabla_H^3\zeta|+|\nabla_H^3\beta|)\,dz\right)\right. \\
&\quad\left.+\left(\int_{-h}^0(|\nabla_H\zeta|+|\nabla_H\beta|)\,dz\right)\left(\int_{-h}^0(|\nabla_H^2\zeta|+|\nabla_H^2\beta|)\,dz\right)+|\Delta_H\overline{v}|\right\}|\Delta_H^2\overline{v}|\;dxdy.
\end{aligned}
$$

By applying Hölder inequality, (33), (36), 39, (44) and (110) to the above estimate, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d\|\nabla_H\Delta_H\overline{v}\|_2^2}{dt}+\frac{1}{R_1}\|\Delta_H^2\overline{v}\|_2^2 \\
&\leq C\left\{\|\overline{v}\|_4\;\|\nabla_H^3\overline{v}\|_4+\|\nabla_H\overline{v}\|_4\;\|\nabla_H^2\overline{v}\|_4+\int_{-h}^0\|\widetilde{v}\|_\infty\;dz\left\|\int_{-h}^0(|\nabla_H^3\zeta|+|\nabla_H^3\beta|)\,dz\right\|_2\right. \\
&\quad\left.+\left\|\int_{-h}^0(|\nabla_H\zeta|+|\nabla_H\beta|)\,dz\right\|_4\left\|\int_{-h}^0(|\nabla_H^2\zeta|+|\nabla_H^2\beta|)\,dz\right\|_4+\|\Delta_H\overline{v}\|_2\right\}\|\Delta_H^2\overline{v}\|_2 \\
&\leq C\left\{\|\overline{v}\|_2^{\frac12}\|\nabla_H\overline{v}\|_2^{\frac12}\;\|\nabla_H^3\overline{v}\|_2^{\frac12}\|\Delta_H^2\overline{v}\|_2^{\frac12}+\|\nabla_H\overline{v}\|_2\;\|\nabla_H^3\overline{v}\|_2\right. \\
&\quad+\|\widetilde{v}\|_2^{\frac12}\|\nabla_H\widetilde{v}\|_\infty^{\frac12}\left(\|\Delta_H\eta\|_2+\|\Delta_H\theta\|_2+\|\Delta_HT\|_2\right) \\
&\quad\left.+\left(\|\eta\|_2+\|\theta\|_2+\|T\|_2\right)\left(\|\nabla_H^2\eta\|_2+\|\nabla_H^2\theta\|_2+\|\nabla_H^2T\|_2\right)+\|\Delta_H\theta\|_2\;\right\}\|\Delta_H^2\overline{v}\|_2.
\end{aligned}
$$

Thus, by Young's and the Cauchy–Schwarz inequalities, we have

$$
\begin{aligned}
&\frac{d\|\nabla_H\Delta_H\overline{v}\|_2^2}{dt}+\frac{1}{R_1}\|\Delta_H^2\overline{v}\|_2^2 \\
&\leq C\left(\|\overline{v}\|_2^2\|\nabla_H\overline{v}\|_2^2+\|\nabla_H\overline{v}\|_2^2\right)\;\|\nabla_H\Delta_H\overline{v}\|_2^2 \\
&+\left(\|\widetilde{v}\|_2\|\nabla_H\widetilde{v}\|_\infty+\|\eta\|_2^2+\|\theta\|_2^2+\|T\|_2^2\right)\left(\|\Delta_H\eta\|_2^2+\|\Delta_H\theta\|_2^2+\|\Delta_HT\|_2^2\right).
\end{aligned}
\tag{120}
$$

3.8.2. $\|\Delta_H T\|_2 + \|\nabla_H T_z\|_2$ **estimates**. By applying the operator Δ_H to equation (24), and then taking the inner product of equation (24) with $\Delta_H T + T_{zz}$ in $L^2(\Omega)$, we get

$$
\begin{aligned}
&\frac{1}{2}\frac{d(\|\Delta_H T\|_2^2 + \|\nabla_H T_z\|_2^2)}{dt} + \frac{1}{R_3}\left(\|\Delta_H T_z\|_2^2 + \|\nabla_H T_{zz}\|_2^2\right) \\
&= -\int_\Omega \Delta_H \left[v\cdot\nabla_H T - \left(\int_{-h}^{z} \nabla_H \cdot v(x,y,\xi,t)d\xi\right)\left(\frac{\partial T}{\partial z} + \frac{1}{h}\right) - Q\right]\ \Delta_H T\ dxdydz \\
&\quad -\int_\Omega \nabla_H \left[v\cdot\nabla_H T_z - \left(\int_{-h}^{z} \nabla_H \cdot v(x,y,\xi,t)d\xi\right)\frac{\partial^2 T}{\partial z^2} + u\cdot\nabla_H T\right. \\
&\quad \left. -\left(\nabla_H\cdot v\right)\left(\frac{\partial T}{\partial z} + \frac{1}{h}\right) - Q_z\right]\cdot\nabla_H T_z\ dxdydz \\
&= -\int_\Omega \left[\Delta_H v\cdot\nabla_H T + 2\nabla_H v\cdot\nabla_H^2 T - \left(\int_{-h}^{z}\Delta_H(\nabla_H\cdot v(x,y,\xi,t))d\xi\right)\left(\frac{\partial T}{\partial z}+\frac{1}{h}\right)\right. \\
&\quad \left. -2\left(\int_{-h}^{z}\nabla_H(\nabla_H\cdot v(x,y,\xi,t))d\xi\right)\nabla_H T_z - \Delta_H Q\right]\ \Delta_H T\ dxdydz \\
&\quad -\int_\Omega \left[\nabla_H v\cdot\nabla_H T_z - \left(\int_{-h}^{z}\nabla_H(\nabla_H\cdot v(x,y,\xi,t))d\xi\right)\frac{\partial^2 T}{\partial z^2}\right. \\
&\quad \left. +\nabla_H u\cdot\nabla_H T + u\cdot\nabla_H^2 T - \nabla_H\left(\nabla_H\cdot v\right)\left(\frac{\partial T}{\partial z}+\frac{1}{h}\right) - \left(\nabla_H\cdot v\right)\nabla_H T_z - \nabla_H Q_z\right]\cdot\nabla_H T_z\ dxdydz \\
&= -\int_\Omega \left[\Delta_H v\cdot\nabla_H T + 2\nabla_H v\cdot\nabla_H^2 T - \frac{1}{h}\left(\int_{-h}^{z}\Delta_H(\nabla_H\cdot v(x,y,\xi,t))d\xi\right) + \Delta_H(\nabla_H\cdot v)T\right. \\
&\quad \left. +2\nabla_H(\nabla_H\cdot v)\cdot\nabla_H T - \Delta_H Q\right]\ \Delta_H T\ dxdydz \\
&\quad -\int_\Omega \left[\int_{-h}^{z}\Delta_H(\nabla_H\cdot v(x,y,\xi,t))d\xi\ T + 2\int_{-h}^{z}\nabla_H(\nabla_H\cdot v(x,y,\xi,t))d\xi\cdot\nabla_H T\right]\cdot\Delta_H T_z\ dxdydz \\
&\quad -\int_\Omega \left[\nabla_H v\cdot\nabla_H T_z + (\nabla_H(\nabla_H\cdot v))\frac{\partial T}{\partial z} + \nabla_H u\cdot\nabla_H T + u\cdot\nabla_H^2 T\right. \\
&\quad \left. -\frac{1}{h}\nabla_H\left(\nabla_H\cdot v\right) + \nabla_H\left(\nabla_H\cdot u\right)T + \left(\nabla_H\cdot u\right)\nabla_H T - \nabla_H Q_z\right]\cdot\nabla_H T_z\ dxdydz \\
&\quad -\int_\Omega \left[\left(\int_{-h}^{z}\nabla_H(\nabla_H\cdot v(x,y,\xi,t))d\xi\right)\frac{\partial T}{\partial z} + \nabla_H\left(\nabla_H\cdot v\right)T + \left(\nabla_H\cdot v\right)\nabla_H T\right]\cdot\nabla_H T_{zz}\ dxdydz.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\frac{1}{2}\frac{d(\|\Delta_H T\|_2^2+\|\nabla_H T_z\|_2^2)}{dt}+\frac{1}{R_3}\left(\|\Delta_H T_z\|_2^2+\|\nabla_H T_{zz}\|_2^2\right)\\
&\le C\int_\Omega \Big\{|\nabla_H v||\nabla_H^2 T|+|\Delta_H v|\,|\nabla_H T|+\overline{|\Delta_H(\nabla_H\cdot v)|}+|\nabla_H^2(\nabla_H\cdot v)|\;|T|+|\nabla_H(\nabla_H\cdot v)|\;|\nabla_H T|\\
&\quad +|\Delta_H Q|\big\}\;|\Delta_H T|\;dxdydz\\
&\quad +C\int_\Omega \left\{\left(\int_{-h}^0 |\Delta_H(\nabla_H\cdot v)|\;dz|T|+\int_{-h}^0|\nabla_H(\nabla_H\cdot v)|\;dz|\nabla_H T|\right)|\Delta_H T_z|\right\}\;dxdydz\\
&\quad +C\int_\Omega \big\{|\nabla_H v||\nabla_H T_z|+|\nabla_H(\nabla_H\cdot v)|\,|T_z|+|\nabla_H u|\,|\nabla_H T|+|u|\;|\nabla_H^2 T|\\
&\quad +|\nabla_H(\nabla_H\cdot v)|+|\nabla_H(\nabla_H\cdot u)|\,|T|+|\nabla_H Q_z|\big\}\;|\nabla_H T_z|\;dxdydz\\
&\quad +C\int_\Omega\left[\left(\int_{-h}^0|\nabla_H(\nabla_H\cdot v)|\;dz\right)|T_z|+|\nabla_H(\nabla_H\cdot v)||T|+|\nabla_H\cdot v||\nabla_H T|\right]|\nabla_H T_{zz}|\;dxdydz.
\end{aligned}
$$

Thanks to (107)–(109), we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d(\|\Delta_H T\|_2^2+\|\nabla_H T_z\|_2^2)}{dt}+\frac{1}{R_3}\left(\|\Delta_H T_z\|_2^2+\|\nabla_H T_{zz}\|_2^2\right)\\
&\le C\int_\Omega\Big\{|\nabla_H v||\nabla_H^2 T|+\int_{-h}^0(|\Delta_H\zeta|+|\nabla_H T|)\;dz\,|\nabla_H T|+\int_{-h}^0(|\Delta_H\theta|+|\Delta_H T|)\;dz\\
&\quad +\int_{-h}^0(|\nabla_H^2\theta|+|\nabla_H^2 T|)\,dz\;|T|+\int_{-h}^0(|\nabla_H\theta|+|\nabla_H T|)\;dz|\nabla_H T|+|\Delta_H Q|\Big\}\;|\Delta_H T|\;dxdydz\\
&\quad +C\int_\Omega\left\{\left(\int_{-h}^0(|\nabla_H^2\theta|+|\nabla_H^2 T|)\;dz|T|+\int_{-h}^0(|\nabla_H\theta|+|\nabla_H T|)\;dz|\nabla_H T|\right)|\Delta_H T_z|\right\}\;dxdydz\\
&\quad +C\int_\Omega\Big\{|\nabla_H v||\nabla_H T_z|+\int_{-h}^0(|\nabla_H\theta|+|\nabla_H T|)\,dz\;|T_z|+|\nabla_H u|\,|\nabla_H T|\\
&\quad +|u|\;|\nabla_H^2 T|+\int_{-h}^0(|\nabla_H\theta|+|\nabla_H T|)\;dz+(|\nabla_H\theta|+|\nabla_H T|)\,|T|+|\nabla_H Q_z|\Big\}\;|\nabla_H T_z|\;dxdydz\\
&\quad +C\int_\Omega\Bigg[\left(\int_{-h}^0(|\nabla_H\theta|+|\nabla_H T|)\;dz\right)|T_z|+(|\nabla_H\theta|+|\nabla_H T|)|T|\\
&\quad +\int_{-h}^0(|\theta|+|T|)\;dz|\nabla_H T|\Bigg]\,|\nabla_H T_{zz}|\;dxdydz.
\end{aligned}
$$

Using Hölder inequality, and inequalities (44), (45) and (110), we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d(\|\Delta_H T\|_2^2+\|\nabla_H T_z\|_2^2)}{dt}+\frac{1}{R_3}\left(\|\Delta_H T_z\|_2^2+\|\nabla_H T_{zz}\|_2^2\right)\\
&\leq C\|\nabla_H v\|_\infty\|\Delta_H T\|_2^2+C\|\Delta_H\zeta\|_2^{1/2}\|\nabla_H\Delta_H\zeta\|_2^{1/2}\|T\|_\infty^{1/2}\|\nabla_H^2 T\|_2^{3/2}+C\|T\|_\infty\|\nabla_H^2 T\|_2^2\\
&\quad+C\|\nabla_H^2\theta\|_2\,(1+\|T\|_\infty)\|\Delta_H T\|_2+C\|\nabla_H\theta\|_2^{1/2}\|\nabla_H^2\theta\|_2^{1/2}\|T\|_\infty^{1/2}\|\nabla_H^2 T\|_2^{3/2}+C\|\Delta_H Q\|_2\,\|\Delta_H T\|_2\\
&\quad+C(\|\nabla_H^2\theta\|_2+\|\nabla_H^2 T\|_2)\,\|T\|_\infty\|\Delta_H T_z\|_2+C\|\nabla_H\theta\|_2^{1/2}\|\nabla_H^2\theta\|_2^{1/2}\|T\|_\infty^{1/2}\|\nabla_H^2 T\|_2^{1/2}\|\Delta_H T_z\|_2\\
&\quad+C\|\nabla_H v\|_\infty\|\nabla_H T_z\|_2^2+C\|\nabla_H\theta\|_2^{1/2}\|\nabla_H^2\theta\|_2^{1/2}\|T_z\|_2^{1/2}\|\nabla_H T_z\|_2^{3/2}\\
&\quad+C\|T\|_\infty^{1/2}\|\nabla_H^2 T\|_2^{1/2}\|T_z\|_2^{1/2}\|\nabla_H T_z\|_2^{3/2}+C\|\nabla_H u\|_3\,\|\nabla_H T\|_6\|\nabla_H T_z\|_2+C\|u\|_\infty\,\|\nabla_H^2 T\|_2\|\nabla_H T_z\|_2\\
&\quad+C(\|\nabla_H\theta\|_2+\|\nabla_H T\|_2)(1+\|T\|_\infty)\\
&\quad+C\left(\|\nabla_H\theta\|_2^{1/2}\|\nabla_H^2\theta\|_2^{1/2}+C\|T\|_\infty^{1/2}\|\nabla_H^2 T\|_2^{1/2}\right)\|T_z\|_2^{1/2}\|\nabla_H T_z\|_2^{1/2}\|\nabla_H T_{zz}\|_2\\
&\quad+\|\nabla_H Q_z\|_2\,\|\nabla_H T_z\|_2+C(\|\nabla_H\theta\|_2+\|\nabla_H T\|_2)\|T\|_\infty\|\nabla_H T_{zz}\|_2\\
&\quad+C\|\theta\|_2^{1/2}\|\nabla_H\theta\|_2^{1/2}\|T\|_\infty^{1/2}\|\nabla_H^2 T\|_2^{1/2}\|\nabla_H T_{zz}\|_2\\
&\leq C\|\nabla_H v\|_\infty(\|\Delta_H T\|_2^2+\|\nabla_H T_z\|_2^2)+C\left(\|\nabla_H\eta\|_2\;+\|\nabla_H\theta\|_2\right)^{1/2}\left(\|\Delta_H\eta\|_2\;+\|\Delta_H\theta\|_2\right)^{1/2}\|T\|_\infty^{1/2}\|\nabla_H^2 T\|_2^{3/2}\\
&\quad+C\left(\|\Delta_H\eta\|_2+\|\Delta_H\theta\|_2+\|\Delta_H T\|_2\right)(1+\|T\|_\infty)\;\|\Delta_H T\|_2+C\|\Delta_H Q\|_2\,\|\Delta_H T\|_2+C\|\nabla_H Q_z\|_2\,\|\nabla_H T_z\|_2\\
&\quad+\left(\|\nabla_H\eta\|_2^{1/2}\|\nabla_H^2\eta\|_2^{\frac12}+\|\nabla_H\theta\|_2^{\frac12}\|\nabla_H^2\theta\|_2^{\frac12}+\|T\|_\infty^{\frac12}\|\Delta_H T\|_2^{\frac12}\right)\;\|T\|_\infty^{\frac12}\|\Delta_H T\|_2^{\frac12}\left(\|\Delta_H T\|_2+\|\Delta_H T_z\|_2\right)\\
&\quad+C\|u\|_6\|\nabla_H T_z\|_2^{\frac12}\|\nabla_H^2 T_z\|_2^{\frac12}\|\Delta_H T\|+C\Big[\|\nabla_H\theta\|_2^{\frac12}\|\nabla_H^2\theta\|_2^{\frac12}\|T_z\|_2^{\frac12}\|\nabla_H T_z\|_2^{\frac12}\\
&\quad+\left(\|T\|_4+\|\theta\|_2^{1/2}\|\nabla_H\theta\|_2^{1/2}+\|T_z\|_2^{1/2}\|\nabla_H T_z\|_2^{1/2}\right)\;\|T\|_\infty^{1/2}\|\Delta_H T\|_2^{1/2}\Big]\|\nabla_H T_{zz}\|_2.
\end{aligned}
$$

By the Cauchy–Schwarz and Young's inequalities, we reach

$$
\begin{aligned}
&\frac{d(\|\Delta_H T\|_2^2+\|\nabla_H T_z\|_2^2)}{dt}+\frac{1}{R_3}\left(\|\Delta_H T_z\|_2^2+\|\nabla_H T_{zz}\|_2^2\right)\\
&\leq C\|\nabla_H v\|_\infty\left(\|\Delta_H T\|_2^2+\|\nabla_H T_z\|_2^2\right)+C\left(\|\nabla_H\eta\|_2\;+\|\nabla_H\theta\|_2\right)^{1/2}\left(\|\Delta_H\eta\|_2\;+\|\Delta_H\theta\|_2\right)^{1/2}\|T\|_\infty^{1/2}\|\Delta_H T\|_2^{3/2}\\
&\quad+C\left(\|\Delta_H\eta\|_2+\|\Delta_H\theta\|_2+\|\Delta_H T\|_2\right)\|T\|_\infty\;\|\Delta_H T\|_2+C\|\Delta_H Q\|_2\,\|\Delta_H T\|_2+C\|\nabla_H Q_z\|_2\,\|\nabla_H T_z\|_2\\
&\quad+\left(\|\nabla_H\eta\|_2^{1/2}\|\Delta_H\eta\|_2^{\frac12}+\|\nabla_H\theta\|_2^{\frac12}\|\Delta_H\theta\|_2^{\frac12}+\|T\|_\infty^{\frac12}\|\Delta_H T\|_2^{\frac12}\right)\;\|T\|_\infty^{\frac12}\|\Delta_H T\|_2^{\frac12}\|\Delta_H T\|_2\\
&\quad+\left(\|\nabla_H\eta\|_2\|\Delta_H\eta\|_2+\|\nabla_H\theta\|_2\|\Delta_H\theta\|_2+\|T\|_\infty\|\Delta_H T\|_2\right)\;\|T\|_\infty\|\Delta_H T\|_2+C\|u\|_6^3\left(\|\nabla_H T_z\|_2^2+\|\Delta_H T\|^2\right)\\
&\quad+C\left[\|\nabla_H\theta\|_2\|\Delta_H\theta\|_2\|T_z\|_2\|\nabla_H T_z\|_2+\left(\|T\|_4^2+\|\theta\|_2\|\nabla_H\theta\|_2+\|T_z\|_2\|\nabla_H T_z\|_2\right)\;\|T\|_\infty\|\Delta_H T\|_2\right]\\
&\leq C\|\Delta_H Q\|_2^2+C\|\nabla_H Q_z\|_2^2+C\|T\|_\infty^4+C\|\nabla_H v\|_\infty\left(\|\Delta_H T\|_2^2+\|\nabla_H T_z\|_2^2\right)\\
&\quad+C\left(1+\|\nabla_H\eta\|_2^2+\|\nabla_H\theta\|_2^2+\|T\|_\infty^2+\|u\|_6^3+\|T_z\|_2^2\right)\left(\|\Delta_H T\|_2^2+\|\nabla_H T_z\|_2^2+\|\Delta_H\eta\|_2^2+\|\Delta_H\theta\|_2^2\right).
\end{aligned}
$$

Thus, we get

$$
\begin{aligned}
&\frac{d(\|\Delta_H T\|_2^2+\|\nabla_H T_z\|_2^2)}{dt}+\frac{1}{R_3}\left(\|\Delta_H T_z\|_2^2+\|\nabla_H T_{zz}\|_2^2\right)\\
&\leq C\|\Delta_H Q\|_2^2+C\|\nabla_H Q_z\|_2^2+C\|T\|_\infty^4+C\|\nabla_H v\|_\infty\left(\|\Delta_H T\|_2^2+\|\nabla_H T_z\|_2^2\right)\\
&\quad+C\left(1+\|\nabla_H\eta\|_2^2+\|\nabla_H\theta\|_2^2+\|T\|_\infty^2+\|u\|_6^3+\|T_z\|_2^2\right)\left(\|\Delta_H T\|_2^2+\|\nabla_H T_z\|_2^2+\|\Delta_H\eta\|_2^2+\|\Delta_H\theta\|_2^2\right).
\end{aligned}
\tag{121}
$$

3.8.3. $\|\nabla_H(\nabla_H^\perp\cdot v_z)\|_{H^1(\Omega)}^2+\|\nabla_H\left(\nabla_H\cdot v_z+R_1 T\right)\|_{H^1(\Omega)}^2$ **estimates**. By acting with Δ_H on equation (111) and equation (112), then taking the inner product of equation (111) with $\Delta_H\eta+\eta_{zz}$ in L^2 and equation (112) with

$\Delta_H\theta+\theta_{zz}$ in L^2, respectively, we get

$$
\begin{aligned}
&\frac{1}{2}\frac{d\left(\|\Delta_H\eta\|_2^2+\|\nabla_H\eta_z\|_2^2+\|\Delta_H\theta\|_2^2+\|\nabla_H\theta_z\|_2^2\right)}{dt}\\
&\quad+\frac{1}{R_1}\left(\|\nabla_H\Delta_H\eta\|_2^2+\|\nabla_H\Delta_H\eta_z\|_2^2+\|\nabla_H\Delta_H\theta\|_2^2+\|\Delta_H\theta_z\|_2^2\right)\\
&\quad+\frac{1}{R_2}\left(\|\Delta_H\eta_z\|_2^2+\|\nabla_H\eta_{zz}\|_2^2+\|\Delta_H\theta_z\|_2^2+\|\nabla_H\theta_{zz}\|_2^2\right)\\
&=\int_\Omega\nabla_H\left\{\nabla_H^\perp\cdot\left[(v\cdot\nabla_H)u-\left(\int_{-h}^z\nabla_H\cdot v(x,y,\xi,t)d\xi\right)\frac{\partial u}{\partial z}\right.\right.\\
&\quad\left.\left.+(u\cdot\nabla_H)v-(\nabla_H\cdot v)u\right]+f_0\left(R_1\,T-\theta\right)\right\}\cdot\nabla_H(\Delta_H\eta+\eta_{zz})\;dxdydz\\
&\quad+\int_\Omega\nabla_H\left\{\nabla_H\cdot\left[(v\cdot\nabla_H)u-\left(\int_{-h}^z\nabla_H\cdot v(x,y,\xi,t)d\xi\right)\frac{\partial u}{\partial z}\right.\right.\\
&\quad\quad\left.\left.+(u\cdot\nabla_H)v-(\nabla_H\cdot v)u\right]\;+f_0\,\eta\right\}\cdot\nabla_H(\Delta_H\theta+\theta_{zz})\;dxdydz\\
&\quad-\int_\Omega\left\{R_1\left[\nabla_Hv\cdot\nabla_HT+v\cdot\nabla_H^2T-\left(\int_{-h}^z\nabla_H(\nabla_H\cdot v)(x,y,\xi,t)d\xi\right)\left(\frac{\partial T}{\partial z}+\frac{1}{h}\right)\right.\right.\\
&\quad\left.\left.-\nabla_HQ+\left(\frac{1}{R_3}-\frac{1}{R_2}\right)\nabla_HT_{zz}\right]\right\}\cdot\nabla_H(\Delta_H\theta+\theta_{zz})\;dxdydz\\
&\quad-\int_\Omega\left[\left(\int_{-h}^z\nabla_H(\nabla_H\cdot v)(x,y,\xi,t)d\xi\right)\nabla_HT_z\;\Delta_H\theta+\left(\int_{-h}^z\nabla_H\cdot v(x,y,\xi,t)d\xi\right)\nabla_HT_z\cdot\nabla_H\theta_z)\right]\;dxdydz\\
&\le C\int_\Omega\left\{\left(|u|\;|\nabla_H^3v|+|\nabla_Hu|\;|\nabla_H^2v|+|\nabla_H^2u|\;|\nabla_Hv|+|\nabla_H^3u||v|+|u_z|\;\int_{-h}^0|\nabla_H^2(\nabla_H\cdot v)|\;dz\right.\right.\\
&\quad\left.+|\nabla_Hu_z|\;\int_{-h}^0|\nabla_H(\nabla_H\cdot v)|\;dz+|\nabla_H^2u_z|\;\int_{-h}^0|\nabla_H\cdot v|\;dz\right)\;(|\nabla_H\Delta_H\eta|+|\nabla_H\eta_{zz}|+|\nabla_H\Delta_H\theta|+|\nabla_H\theta_{zz}|)\\
&\quad+C\left[|\nabla_Hv||\nabla_HT|+|v||\nabla_H^2T|+(1+|T_z|)\;\int_{-h}^0|\nabla_H(\nabla_H\cdot v)|\;dz+|\nabla_HT_z|\;\int_{-h}^0|\nabla_H\cdot v|\;dz\right.\\
&\quad\left.\left.+|\nabla_HQ|+\left|\frac{R_1}{R_3}-\frac{R_1}{R_2}\right|\,|\nabla_HT_{zz}|\right](|\nabla_H\Delta_H\theta|+|\nabla_H\theta_{zz}|)\right\}\;dxdydz+C\left(\|\Delta_HT\|_2^2+\;\|\Delta_H\eta\|_2^2+\;\|\Delta_H\theta\|_2^2\right)\\
&\le C\int_\Omega\left\{\left[|u|\;\int_{-h}^0(|\nabla_H^3\zeta|+|\nabla_H^3\beta|)\;dz+(|\nabla_H\zeta|+|\nabla_H\beta|)\;\int_{-h}^0(|\nabla_H^2\zeta|+|\nabla_H^2\beta|)\;dz+(|\nabla_H^3\zeta|+|\nabla_H^3\beta|)|v|\right.\right.\\
&\quad+(|\nabla_H^2\zeta|+|\nabla_H^2\beta|)\;\int_{-h}^0(|\nabla_H\zeta|+|\nabla_H\beta|)\;dz+|u_z|\;\int_{-h}^0(|\nabla_H^2\theta|+|\nabla_H^2T|)\;dz\\
&\quad+(|\nabla_H\zeta_z|+|\nabla_H\beta_z|)\;\int_{-h}^0(|\nabla_H\theta|+|\nabla_HT|)\;dz\\
&\quad\left.+(|\nabla_H^2\zeta_z|+|\nabla_H^2\beta_z|)\;\int_{-h}^0(|\theta|+|T|)\;dz\right]\;(|\nabla_H\Delta_H\eta|+|\nabla_H\eta_{zz}|+|\nabla_H\Delta_H\theta|+|\nabla_H\theta_{zz}|)\\
&\quad+C\left[|\nabla_HT|\;\int_{-h}^0(|\nabla_H\zeta|+|\nabla_H\beta|)\;dz+|v||\nabla_H^2T|\right.\\
&\quad+(1+|T_z|)\;\int_{-h}^0(|\nabla_H\theta|+|\nabla_HT|)\;dz+|\nabla_HT_z|\;\int_{-h}^0(|\theta|+|T|)\;dz\\
&\quad\left.\left.+|\nabla_HQ|+\left|\frac{R_1}{R_3}-\frac{R_1}{R_2}\right|\;|\nabla_HT_{zz}|\right](|\nabla_H\Delta_H\theta|+|\nabla_H\theta_{zz}|)\right\}\;dxdydz+C\left(\|\Delta_HT\|_2^2+\;\|\Delta_H\eta\|_2^2+\;\|\Delta_H\theta\|_2^2\right).
\end{aligned}
$$

Using Hölder inequality, and inequalities (44), (45) and (110), we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d\left(\|\Delta_H\eta\|_2^2+\|\nabla_H\eta_z\|_2^2+\|\Delta_H\theta\|_2^2+\|\nabla_H\theta_z\|_2^2\right)}{dt}\\
&\quad+\frac{1}{R_1}\left(\|\nabla_H\Delta_H\eta\|_2^2+\|\nabla_H\Delta_H\eta_z\|_2^2+\|\nabla_H\Delta_H\theta\|_2^2+\|\Delta_H\theta_z\|_2^2\right)\\
&\quad+\frac{1}{R_2}\left(\|\Delta_H\eta_z\|_2^2+\|\nabla_H\eta_{zz}\|_2^2+\|\Delta_H\theta_z\|_2^2+\|\nabla_H\theta_{zz}\|_2^2\right)\\
&\le C\left(\|\nabla_H\Delta_H\eta\|_2+\|\nabla_H\eta_{zz}\|_2+\|\nabla_H\Delta_H\theta\|_2+\|\nabla_H\theta_{zz}\|_2\right)\Big[\left(\|u\|_\infty+\|v\|_\infty\right)\ \left(\|\nabla_H^2\eta\|_2+\|\nabla_H^2\theta\|_2+\|\nabla_H^2T\|_2\right)\\
&\quad+\left(\|\eta\|_2^{\frac12}\|\nabla_H\eta\|_2^{\frac12}+\|\theta\|_2^{\frac12}\|\nabla_H\theta\|_2^{\frac12}+\|T\|_\infty\right)\ \left(\|\nabla_H\eta\|_2^{\frac12}\|\nabla_H^2\eta\|_2^{\frac12}+\|\nabla_H\theta\|_2^{\frac12}\|\nabla_H^2\theta\|_2^{\frac12}+\|T\|_\infty^{\frac12}\|\nabla_H^2T\|_2^{\frac12}\right)\\
&\quad+\left(\|\nabla_H\eta\|_2^{\frac12}\|\nabla_H^2\eta\|_2^{\frac12}+\|\nabla_H\theta\|_2^{\frac12}\|\nabla_H^2\theta\|_2^{\frac12}+\|T\|_\infty^{\frac12}\|\nabla_H^2T\|_2^{\frac12}\right)\left(\|\eta\|_2^{\frac12}\|\nabla_H\eta\|_2^{\frac12}+\|\theta\|_2^{\frac12}\|\nabla_H\theta\|_2^{\frac12}+\|T\|_\infty\right)\\
&\quad+\left(\|\theta_z\|_2^{\frac12}\|\nabla_H\theta_z\|_2^{\frac12}+\|T_z\|_2^{\frac12}\|\nabla_HT_z\|_2^{\frac12}\right)\left(\|\nabla_H\theta\|_2^{\frac12}\|\nabla_H^2\theta\|_2^{\frac12}+\|T\|_\infty^{\frac12}\|\nabla_H^2T\|_2^{\frac12}\right)\\
&\quad+\|u_z\|_2^{\frac12}\|\nabla_Hu_z\|_2^{\frac12}\ \left(\|\nabla_H\theta\|_2^{\frac12}\|\nabla_H^2\theta\|_2^{\frac12}+\|T\|_\infty^{\frac12}\|\nabla_H^2T\|_2^{\frac12}\right)\\
&\quad+\left(\|\nabla_H\theta_z\|_2^{\frac12}\|\nabla_H^2\theta_z\|_2^{\frac12}+\|\nabla_HT_z\|_2^{\frac12}\|\nabla_H^2T_z\|_2^{\frac12}\right)\left(\|\theta\|_2^{\frac12}\|\nabla_H\theta\|_2^{\frac12}+\|T\|_\infty\right)\Big]\\
&\quad+C\Big[\left(\|\eta\|_2^{\frac12}\|\nabla_H\eta\|_2^{\frac12}+\|\theta\|_2^{\frac12}\|\nabla_H\theta\|_2^{\frac12}+\|T\|_4\right)\ \|T\|_\infty^{\frac12}\|\Delta_HT\|_2^{\frac12}+\|v\|_\infty\|\Delta_HT\|_2\\
&\quad+\left(\|\nabla_H\theta\|_2^{\frac12}\|\nabla_H^2\theta\|_2^{\frac12}+\|T\|_\infty^{\frac12}\|\Delta_HT\|_2^{\frac12}\right)\ \|T_z\|_2^{\frac12}\|\nabla_HT_z\|_2^{\frac12}\\
&\quad+\left(\|\theta\|_2^{\frac12}\|\nabla_H\theta\|_2^{\frac12}+\|T\|_\infty\right)\ \|\nabla_HT_z\|_2^{\frac12}\|\Delta_HT_z\|_2^{\frac12}+\|\nabla_H\theta\|_2+\|\nabla_HT\|_2+\|\nabla_HQ\|_2\\
&\quad+\left|\frac{R_1}{R_3}-\frac{R_1}{R_2}\right|\|\nabla_HT_{zz}\|_2\Big]\left(\|\nabla_H\Delta_H\theta\|_2+\|\nabla_H\theta_{zz}\|_2\right)+C\left(\|\Delta_HT\|_2^2+\|\Delta_H\eta\|_2^2+\|\Delta_H\theta\|_2^2\right)
\end{aligned}
$$

By Young's inequality and Cauchy–Schwarz inequality, we have

$$
\begin{aligned}
&\frac{d\left(\|\Delta_H\eta\|_2^2+\|\nabla_H\eta_z\|_2^2+\|\Delta_H\theta\|_2^2+\|\nabla_H\theta_z\|_2^2\right)}{dt}+\frac{1}{R_1}\left(\|\nabla_H\Delta_H\eta\|_2^2+\|\nabla_H\Delta_H\eta_z\|_2^2+\|\nabla_H\Delta_H\theta\|_2^2+\|\Delta_H\theta_z\|_2^2\right)\\
&\quad+\frac{1}{R_2}\left(\|\Delta_H\eta_z\|_2^2+\|\nabla_H\eta_{zz}\|_2^2+\|\Delta_H\theta_z\|_2^2+\|\nabla_H\theta_{zz}\|_2^2\right)\\
&\le C\left(\|\Delta_HT\|_2^2+\|\nabla_HT_z\|_2^2+\|\Delta_H\eta\|_2^2+\|\nabla_H\eta_z\|_2^2+\|\Delta_H\theta\|_2^2+\|\nabla_H\theta_z\|_2^2\right)\big[1+\|T\|_\infty^4+\|\nabla_Hu_z\|_2^2+\|u_{zz}\|_2^2\\
&\quad+\|\nabla_H\eta\|_2^2+\|\eta_z\|_2^2+\|\nabla_H\theta\|_2^2+\|\theta_z\|_2^2+\|T_z\|_2^2+\|\eta\|_2^2\|\nabla_H\eta\|_2^2+\|\theta\|_2^2\|\nabla_H\theta\|_2^2\big]\\
&\quad+\|\eta\|_2^2\|\nabla_H\eta\|_2^2+\|\theta\|_2^2\|\nabla_H\theta\|_2^2+\|T\|_\infty^4+\|u_z\|_2^2\|\nabla_Hu_z\|_2^2+\|\nabla_HQ\|_2^2\\
&\quad+\frac{R_1^2(R_1+R_2)(R_2-R_3)^2}{R_2^2R_3^2}\|\nabla_HT_{zz}\|_2^2.
\end{aligned}
\tag{122}
$$

Next, we will obtain an estimate for

$$\|\nabla_H\Delta_H\overline{v}\|_2^2+\|\nabla_H(\nabla_H^\perp\cdot v_z)\|_{H^1(\Omega)}^2+\|\nabla_H\left(\nabla_H\cdot v_z+R_1T\right)\|_{H^1(\Omega)}^2+C_R\|\nabla_HT\|_{H^1(\Omega)}^2,$$

where $C_R=\frac{2R_1^2(R_1+R_2)(R_2-R_3)^2}{R_2^2R_3}$. Denote by

$$
\begin{aligned}
&\mathcal{X}=1+\|\nabla_H\Delta_H\overline{v}\|_2^2+C_R\,\|\Delta_HT\|_2^2+C_R\,\|\nabla_HT_z\|_2^2+\|\Delta_H\eta\|_2^2+\|\nabla_H\eta_z\|_2^2+\|\Delta_H\theta\|_2^2+\|\nabla_H\theta_z\|_2^2,\\
&\mathcal{Y}=\|\Delta_H^2\overline{v}\|_2^2+\|\Delta_HT_z\|_2^2+\|\nabla_HT_{zz}\|_2^2+\|\nabla_H\Delta_H\eta\|_2^2+\|\Delta_H\eta_z\|_2^2+\|\nabla_H\Delta_H\theta\|_2^2+\|\Delta_H\theta_z\|_2^2.
\end{aligned}
$$

Thus, by (119), we get

$$\|\nabla_Hv\|_\infty\le C\left(\|\nabla_H\overline{v}\|_{H^1(M)}+\|T\|_\infty+\|\eta\|_{H^1(\Omega)}+\|\theta\|_{H^1(\Omega)}\right)\ \log\mathcal{X}.$$

By virtue of (120), (121), (122), Young's inequality, Cauchy–Schwarz inequality and the above, we obtain

$$\begin{aligned}
&\frac{d\mathcal{X}}{dt} + C\mathcal{Y} \leq C\|v\|_2 \left(\|\nabla_H \overline{v}\|_{H^1(M)} + \|T\|_\infty + \|\eta\|_{H^1(\Omega)} + \|\theta\|_{H^1(\Omega)}\right) \mathcal{X} \log \mathcal{X} \\
&\quad + \Big[1 + \|T\|_\infty^4 + \|T_z\|_2^2 + \|\overline{v}\|_2^2 \left(1 + \|\overline{v}\|_{H^1(M)}^2\right) \\
&\quad + \|u_z\|_{H^1}^2 + \left(1 + \|\eta\|_2^2\right) \|\eta\|_{H^1}^2 + \left(1 + \|\theta\|_2^2\right) \|\theta\|_{H^1}^2 \Big] \mathcal{X} \\
&\quad + C\left\{\|\Delta_H Q\|_2^2 + \|\nabla_H Q_z\|_2^2 + \|\eta\|_2^2 \|\nabla_H \eta\|_2^2 + \|\theta\|_2^2 \|\nabla_H \theta\|_2^2 + \|T\|_\infty^4 + \|u_z\|_2^2 \|\nabla_H u_z\|_2^2\right\}.
\end{aligned}$$

Let $\mathcal{X} = e^{\mathcal{Z}}$. Then, $\frac{d\mathcal{X}}{dt} = e^{\mathcal{Z}} \, \frac{d\mathcal{Z}}{dt}$. As a result we have

$$\begin{aligned}
&\frac{d\mathcal{Z}}{dt} \leq C\|v\|_2 \left(\|\nabla_H \overline{v}\|_{H^1(M)} + \|T\|_\infty + \|\eta\|_{H^1(\Omega)} + \|\theta\|_{H^1(\Omega)}\right) \mathcal{Z} \\
&\quad + \Big[1 + \|\Delta_H Q\|_2^2 + \|\nabla_H Q_z\|_2^2 + \|T\|_\infty^4 + \|T_z\|_2^2 + \|\overline{v}\|_2^2 \left(1 + \|\overline{v}\|_{H^1(M)}^2\right) \\
&\quad + \left(1 + \|u_z\|_2^2\right) \|u_z\|_{H^1}^2 + \left(1 + \|\eta\|_2^2\right) \|\eta\|_{H^1}^2 + \left(1 + \|\theta\|_2^2\right) \|\theta\|_{H^1}^2 \Big]. \qquad (123)
\end{aligned}$$

Thanks to Gronwall inequality, and the estimates established in the previous subsections, we get

$$\mathcal{Z} \leq K, \qquad \mathcal{X} \leq e^K \tag{124}$$

where

$$K = e^{C(K_1+K_2+K_7)} \left[1 + \|v_0\|_{H^4}^2 + \|T_0\|_{H^2}^2 + t + \|\Delta_H Q\|_2^2 \, t + \|\nabla_H Q_z\|_2^2 \, t\right]. \tag{125}$$

Moreover, we have

$$\int_0^t \mathcal{Y} \, ds \leq K_8, \tag{126}$$

where

$$K_8 = e^{C(K_1+K_2+K_7)} \left[1 + \|v_0\|_{H^4}^2 + \|T_0\|_{H^2}^2 + t + \|\Delta_H Q\|_2^2 \, t + \|\nabla_H Q_z\|_2^2 \, t\right]. \tag{127}$$

Thanks to (103) and the above we conclude that the quantities $\int_0^t \|v_{zz}(s)\|_{H^1(\Omega)}^2 \, ds$, $\int_0^t \|\Delta_H \nabla_H v_z(s)\|_{H^1(\Omega)}^2 \, ds$, $\|\Delta_H v_z(t)\|_{H^1(\Omega)}, \|\nabla_H T(t)\|_{H^1(\Omega)}$, and $\int_0^t \|\nabla_H T_z(s)\|_{H^1(\Omega)}^2 \, ds$ are all bounded uniformly in time, t, over the interval $[0, \mathcal{T}_*)$. Therefore, the strong solution $(v(t), T(t))$ exists globally in time. ☐

4. Uniqueness of the Solutions

In this section we state and prove the global existence and uniqueness of the strong solution of system (22)–(28).

Theorem 2. *Suppose that $Q \in H^2(\Omega)$. Then for every $v_0 \in H^4(\Omega)$, $T_0 \in H^2(\Omega)$ and $\mathcal{T} > 0$, there is a unique solution (v, p_s, T) of system* (22)–(28) *with*

$$\begin{aligned}
&\Delta_H v_z, \;\; \nabla_H T \in C([0, \mathcal{T}], H^1(\Omega)), \\
&v_{zz}, \;\; \Delta_H \nabla_H v_z, \;\; \nabla_H T_z \in L^2([0, \mathcal{T}], H^1(\Omega)),
\end{aligned}$$

Proof. In Theorem 2 of the previous section we proved that the strong solutions exist globally in time. Therefore, it remains to prove the uniqueness of strong solutions, and their continuous dependence on initial data, in the sense specified by equation (140) below. Let $(v_1, (p_s)_1, T_1)$ and $(v_2, (p_s)_2, T_2)$ be two strong solutions of system (22)–(28) with initial data $((v_0)_1, (T_0)_1)$ and $((v_0)_2, (T_0)_2)$, respectively. Denote by $\phi = v_1 - v_2, q_s = (p_s)_1 - (p_s)_2, \psi =$

$T_1 - T_2$. It is clear that

$$\frac{\partial \phi}{\partial t} + L_1\phi + (v_1 \cdot \nabla_H)\phi + (\phi \cdot \nabla_H)v_2 - \left(\int_{-h}^{z} \nabla_H \cdot v_1(x,y,\xi,t)d\xi\right)\frac{\partial \phi}{\partial z} - \left(\int_{-h}^{z} \nabla_H \cdot \phi(x,y,\xi,t)d\xi\right)\frac{\partial v_2}{\partial z}$$
$$+ f_0\vec{k} \times \phi + \nabla_H q_s - \nabla_H\left(\int_{-h}^{z} \psi(x,y,\xi,t)d\xi\right) = 0, \tag{128}$$
$$\frac{\partial \psi}{\partial t} - \frac{1}{R_3}\psi_{zz} + v_1 \cdot \nabla_H\psi + \phi \cdot \nabla_H T_2$$
$$-\left(\int_{-h}^{z} \nabla_H \cdot v_1(x,y,\xi,t)d\xi\right)\frac{\partial \psi}{\partial z} - \left(\int_{-h}^{z} \nabla_H \cdot \phi(x,y,\xi,t)d\xi\right)\left(\frac{\partial T_2}{\partial z} + \frac{1}{h}\right) = 0, \tag{129}$$

with initial data

$$\phi(x,y,z,0) = (v_0)_1 - (v_0)_2, \tag{130}$$
$$\psi(x,y,z,0) = (T_0)_1 - (T_0)_2. \tag{131}$$

Taking the inner product of equation (128) with ϕ in $L^2(\Omega)$, and equation (129) with ψ in $L^2(\Omega)$, we get

$$\frac{1}{2}\frac{d\|\phi\|_2^2}{dt} + \frac{1}{R_1}\|\nabla_H\phi\|_2^2 + \frac{1}{R_2}\|\phi_z\|_2^2$$
$$= -\int_\Omega \left[(v_1 \cdot \nabla_H)\phi + (\phi \cdot \nabla_H)v_2 - \left(\int_{-h}^{z} \nabla_H \cdot v_1(x,y,\xi,t)d\xi\right)\frac{\partial \phi}{\partial z} - \left(\int_{-h}^{z} \nabla_H \cdot \phi(x,y,\xi,t)d\xi\right)\frac{\partial v_2}{\partial z}\right] \cdot \phi \, dxdydz$$
$$+ \int_\Omega \left[f_0\vec{k} \times \phi + \nabla_H q_s - \nabla_H\left(\int_{-h}^{z} \psi(x,y,\xi,t)d\xi\right)\right] \cdot \phi \, dxdydz,$$

and

$$\frac{1}{2}\frac{d\|\psi\|_2^2}{dt} + \frac{1}{R_3}\|\psi_z\|_2^2 = -\int_\Omega [v_1 \cdot \nabla_H\psi + \phi \cdot \nabla_H T_2$$
$$-\left(\int_{-h}^{z} \nabla_H \cdot v_1(x,y,\xi,t)d\xi\right)\frac{\partial \psi}{\partial z} - \left(\int_{-h}^{z} \nabla_H \cdot \phi(x,y,\xi,t)d\xi\right)\left(\frac{\partial T_2}{\partial z} + \frac{1}{h}\right)\Bigg] \; \psi \, dxdydz.$$

Notice that

$$f_0\vec{k} \times \phi \cdot \psi = 0. \tag{132}$$

Integrating by parts, and using the boundary conditions (25) and (26), we have

$$-\int_\Omega \left((v_1 \cdot \nabla_H)\phi - \left(\int_{-h}^{z} \nabla_H \cdot v_1(x,y,\xi,t)d\xi\right)\frac{\partial \phi}{\partial z}\right) \cdot \phi \, dxdydz = 0, \tag{133}$$
$$-\int_\Omega \left(v_1 \cdot \nabla_H\psi - \left(\int_{-h}^{z} \nabla_H \cdot v_1(x,y,\xi,t)d\xi\right)\frac{\partial \psi}{\partial z}\right) \cdot \psi \, dxdydz = 0. \tag{134}$$

Integrating by parts, and using the boundary conditions (25) and (26), we get

$$\int_\Omega \left[\nabla_H q_s - \nabla_H\left(\int_{-h}^{z} \psi(x,y,\xi,t)d\xi\right)\right] \cdot \phi \, dxdydz$$
$$= \int_\Omega \left(\int_{-h}^{z} \psi(x,y,\xi,t)d\xi\right) \; (\nabla_H \cdot \phi) \; dxdydz. \tag{135}$$

Thus, by (132)–(135) we have

$$\frac{1}{2}\frac{d\|\phi\|_2^2}{dt} + \frac{1}{R_1}\|\nabla_H\phi\|_2^2 + \frac{1}{R_2}\|\phi_z\|_2^2 = -\int_\Omega (\phi\cdot\nabla_H)v_2\cdot\phi\, dxdydz$$
$$+\int_\Omega\int_{-h}^z \nabla_H\cdot\phi(x,y,\xi,t)d\xi\frac{\partial v_2}{\partial z}\cdot\phi\, dxdydz + \int_\Omega\left(\int_{-h}^z\psi(x,y,\xi,t)d\xi\right)\,(\nabla_H\cdot\phi)\,\,dxdydz.$$

and

$$\frac{1}{2}\frac{d\|\psi\|_2^2}{dt} + \frac{1}{R_3}\|\psi_z\|_2^2 = -\int_\Omega(\phi\cdot\nabla_H T_2)\psi\, dxdydz$$
$$-\int_\Omega(\nabla_H\cdot\phi)\,T_2\psi\, dxdydz - \int_\Omega\int_{-h}^z\nabla_H\cdot\phi(x,y,\xi,t)d\xi T_2\psi_z\, dxdydz.$$

Notice that by Hölder inequality and (44)

$$\left|\int_\Omega(\phi\cdot\nabla_H)v_2\cdot\phi\, dxdydz\right| \le \left|\int_\Omega |v_2|\,|\phi|\,|\nabla_H\phi|\, dxdydz\right| \le \|v_2\|_6\|\phi\|_2^{\frac12}\|\nabla_H\phi\|_2^{3/2}, \tag{136}$$

$$\left|\int_\Omega \phi\cdot\nabla_H T_2\,\psi\, dxdydz\right| \le \|\nabla_H T_2\|_3\|\psi\|_2\|\phi\|_6 \le C\|\nabla_H T_2\|_{H^1}\|\psi\|_2\|\nabla_H\phi\|_2; \tag{137}$$

$$\begin{aligned}&\left|\int_\Omega\int_{-h}^z\nabla_H\cdot\phi(x,y,\xi,t)d\xi\frac{\partial v_2}{\partial z}\cdot\phi\, dxdydz\right|\\ &\le C\|\nabla_H\phi\|_2\left\|\frac{\partial v_2}{\partial z}\right\|_6\|\phi\|_3 \le C\left\|\frac{\partial v_2}{\partial z}\right\|_6\|\phi\|_2^{\frac12}\|\nabla_H\phi\|_2^{3/2};\end{aligned} \tag{138}$$

$$\begin{aligned}&\left|\int_\Omega\int_{-h}^z\nabla_H\cdot\phi(x,y,\xi,t)d\xi\frac{\partial T_2}{\partial z}\psi\, dxdydz\right| \le \left|\int_\Omega\int_{-h}^0|\nabla_H\phi|dz\int_{-h}^0\left|\frac{\partial T_2}{\partial z}\psi\right|dz\, dxdy\right|\\ &\le C\left|\int_\Omega\int_{-h}^0|\nabla_H\phi|dz\left(\int_{-h}^0\left|\frac{\partial T_2}{\partial z}\right|^2\,dz\right)^{1/2}\left(\int_{-h}^0|\psi|^2\,dz\right)^{1/2}\,dxdy\right|\\ &\le C\left\|\int_{-h}^0\left|\frac{\partial T_2}{\partial z}\right|^2\,dz\right\|_\infty^{1/2}\|\nabla_H\phi\|_2\|\psi\|_2 \le C\left\|\frac{\partial\Delta_H T_2}{\partial z}\right\|_2\|\nabla_H\phi\|_2\|\psi\|_2.\end{aligned} \tag{139}$$

Therefore, by estimates (136)–(139), we reach

$$\begin{aligned}&\frac{1}{2}\frac{d\left(\|\phi\|_2^2+\|\psi\|_2^2\right)}{dt} + \frac{1}{R_1}\|\nabla_H\phi\|_2^2 + \frac{1}{R_2}\|\phi_z\|_2^2 + \frac{1}{R_3}\|\psi_z\|_2^2\\ &\le C\left(\|v_2\|_6\|\phi\|_2^{\frac12}\|\nabla_H\phi\|_2^{3/2} + \|\nabla_H T_2\|_{H^1}\|\psi\|_2\|\nabla_H\phi\|_2 + \|\partial_z v_2\|_6\|\phi\|_2^{\frac12}\|\nabla_H\phi\|_2^{3/2}\right) + C\left\|\frac{\partial\Delta_H T_2}{\partial z}\right\|_2\|\nabla_H\phi\|_2\|\psi\|_2.\end{aligned}$$

By Young's inequality and the Cauchy–Schwarz inequality, we get

$$\frac{d\|\phi\|_2^2+\|\psi(t)\|_2^2}{dt} \le C\left(\|v_2\|_6^4 + \|\nabla_H T_2\|_{H^1}^2 + \|\partial_z v_2\|_6^4 + \left\|\frac{\partial\Delta_H T_2}{\partial z}\right\|_2^2\right)\left(\|\phi\|_2^2+\|\psi\|_2^2\right).$$

Thanks to Gronwall inequality, we obtain

$$\begin{aligned}&\|\phi(t)\|_2^2+\|\psi(t)\|_2^2 \le \left(\|\phi(t=0)\|_2^2+\|\psi(t=0)\|_2^2\right)\times\\ &\exp\left\{C\int_0^t\left(\|v_2(s)\|_6^4 + \|\nabla_H T_2(s)\|_{H^1}^2 + \|\partial_z v_2(s)\|_6^4 + \left\|\frac{\partial\Delta_H T_2}{\partial z}\right\|_2^2\right)\,ds\right\}.\end{aligned}$$

As a result of (124), we have

$$\|\phi(t)\|_2^2 + \|\psi(t)\|_2^2 \le \left(\|\phi(t=0)\|_2^2 + \|\psi(t=0)\|_2^2\right) \exp\left\{C\left((K_3^{2/3} + K + K_4^{2/3})\, t + K_8\right)\right\}. \tag{140}$$

The above inequality proves the continuous dependence of the solutions on the initial data, and in particular, when $\phi(t=0)=0$ and $\psi(t=0)=0$, we have $\phi(t)=0$ and $\psi(t)=0$, for all $t\ge 0$. Therefore, the strong solution is unique.

□

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