

Approaching conformality in lattice models

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- 1 QCD motivations, overview
- 2 The Tensor Renormalization Group (TRG) method
 - Exact blocking formulas (spin and gauge lattice models)
 - Fixed points and exponents (Ising)
 - Chemical potential ($O(2)$, superfluid-BKT phase)
 - Entanglement entropy ($O(2)$: $c = 1$ CFT)
 - The Polyakov's loop in the Abelian Higgs model
- 3 Finite size scaling (Fisher's zeros)
 - Spin models (TRG compared to MC reweighting)
 - $SU(3)$ with 4 and 12 flavors (MC only)
- 4 Conclusions



Important Lattice QCD problems/questions

- A common problem to practical lattice QCD calculations: large box size/small lattice spacing = many lattice sites
- The problem gets more acute for many flavors with small masses (composite Higgs models?). Existence of non-trivial IR fixed points for enough flavors (e.g. $SU(3)$ with 12 massless quarks)? What are the remnants of the expected CFT on the lattice? Is there a topological picture?
- Finite density calculations (sign problem, silver blaze ..)

Significant progress could be made with **accurate blocking methods**

I hope to convince you that recent developments with the **Tensor Renormalization Group (TRG) method** represent **continuous advances** towards these goals

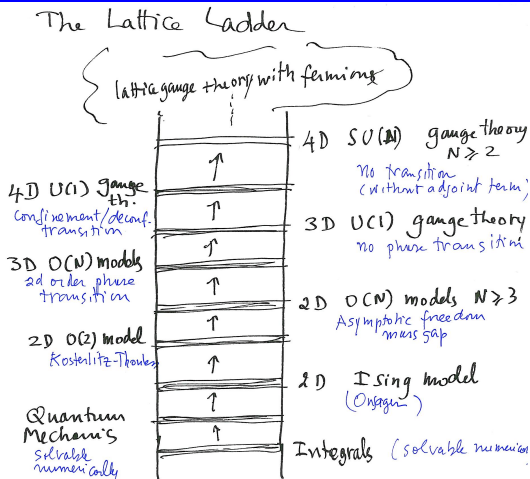


The Tensor Renormalization Group (TRG) Method

- **Exact** blocking (spin and gauge, PRD 88 056005)
- Applies to many lattice models: Ising model, $O(2)$ model, $O(3)$ model, Principal chiral models, Abelian and $SU(2)$ gauge theories
- **Solution of sign problems** (PRD 89, 016008)
- Can be checked with (worm) sampling methods (Prokofiev, Svistunov, Banerjee, Chandrasekharan, Gattringer ...)
- Critical exponents (Y.M. PRB 87, 064422, Kadanoff et al. RMP 86)
- Connects easily to the **Hamiltonian picture** and provides spectra
- Used to design quantum simulators: $O(2)$ model with a chemical potential (PRA 90, 063603), Abelian Higgs model (PRD 92 076003) on optical lattices (not discussed in this talk)
- Schwinger model: Y. Shimizu and Y. Kuramashi



Climbing the lattice ladder: a "dialogue" between the TRG and sampling methods



Block Spining in Configuration Space is difficult!

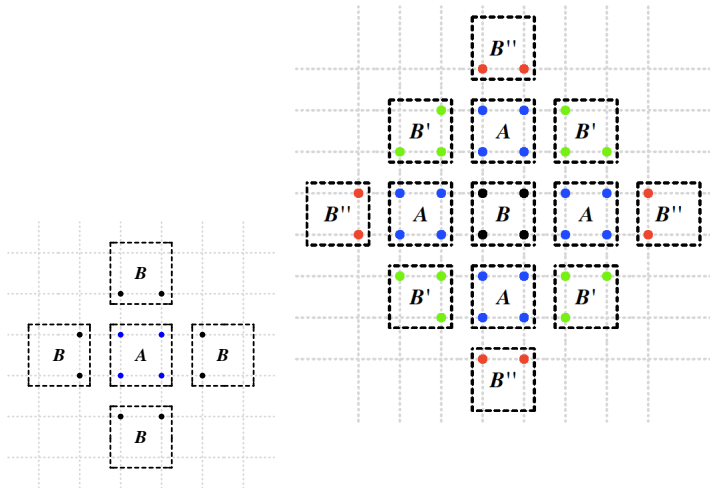


Figure: Ising 2, Step 1, Step 2,write the formula!



TRG: simple and exact! (Levin, Wen, Xiang ..)

Ising model: for each link, we use the Z_2 character expansion:

$$\begin{aligned} \exp(\beta\sigma_1\sigma_2) &= \cosh(\beta)(1 + \sqrt{\tanh(\beta)}\sigma_1\sqrt{\tanh(\beta)}\sigma_2) = \\ \cosh(\beta) \sum_{n_{12}=0,1} & (\sqrt{\tanh(\beta)}\sigma_1\sqrt{\tanh(\beta)}\sigma_2)^{n_{12}}. \end{aligned}$$

Regroup the four terms involving a given spin σ_i and sum over its two values ± 1 . The results can be expressed in terms of a tensor: $T_{xx'yy'}^{(i)}$ which can be visualized as a cross attached to the site i with the four legs covering half of the four links attached to i . The horizontal indices x, x' and vertical indices y, y' take the values 0 and 1 as the index n_{12} .

$$T_{xx'yy'}^{(i)} = f_x f_{x'} f_y f_{y'} \delta(\text{mod}[x + x' + y + y', 2]) ,$$

where $f_0 = 1$ and $f_1 = \sqrt{\tanh(\beta)}$. The delta symbol is 1 if $x + x' + y + y'$ is zero modulo 2 and zero otherwise.



New form of the partition function:

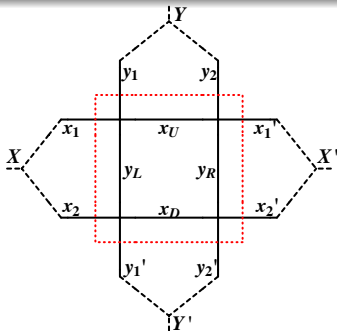
$$Z = (\cosh(\beta))^{2V} \text{Tr} \prod_i T_{xx'yy'}^{(i)}.$$

Tr means contractions (sums over 0 and 1) over the link indices.
Reproduces the closed paths of the HT expansion.

Important feature of the TRG blocking:

It separates the degrees of freedom inside the block (integrated over), from those kept to communicate with the neighboring blocks.

Graphically :
(isotropic blocking)



TRG Blocking defines a new rank-4 tensor $T'_{XX'YY'}$

Exact blocking formula (isotropic):

$$T'_{X(x_1, x_2)X'(x'_1, x'_2)Y(y_1, y_2)Y'(y'_1, y'_2)} = \sum_{X_U, X_D, X_R, X_L} T_{X_1 X_U Y_1 Y_L} T_{X_U X'_1 Y_2 Y_R} T_{X_D X'_2 Y_R Y'_2} T_{X_2 X_D Y_L Y'_1} ,$$

where $X(x_1, x_2)$ is a notation for the product states e. g. ,
 $X(0, 0) = 1$, $X(1, 1) = 2$, $X(1, 0) = 3$, $X(0, 1) = 4$.

The partition function can again be written as

$$Z = \text{Tr} \prod_{2i} T'^{(2i)}_{XX'YY'} ,$$

where $2i$ denotes the sites of the coarser lattice with twice the lattice spacing of the original lattice.



Anisotropic blocking and projection

The truncation method of T. Xiang et al. (PhysRevB.86.045139) relies on an anisotropic blocking involving two sites. This provides a new rank-4 tensor:

$$M_{X(x_1, x_2)X'(x'_1, x'_2)yy'}^{<ij>} = \sum_{y''} T_{x_1, x'_1, y, y''}^{(i)} T_{x_2, x'_2, y'', y'}^{(j)},$$

which can be put in a canonical form by using a Higher Order Singular Value Decomposition defined by a unitary transformation on each of the indices. Only the d highest eigenvalues of the “metric” $G_{XX'}$

$$G_{XX'} = \sum_{X''yy'} M_{XX''yy'} M_{X'X''yy'}^*$$

are kept. This generates a new tensor with d states associated with each of the 4 indices.



Two state approximations (2D Ising)

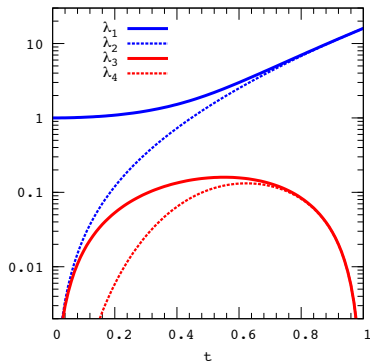


Figure: The four eigenvalues of $G_{XX'}$ at the first step, on a logarithmic scale, as a function of $t = th\beta$. After iterations the gap sharpens as if going to smaller t for $t < t_c$ or larger t for $t > t_c$. (YM, PRB 87, 064422)



Accurate exponents from approximate tensor renormalizations (YM, Phys. Rev. B 87, 064422 2013)

- For the Ising model on square lattice, truncation method (HOSVD) sharply singles out a surprisingly small subspace of dimension two.
- In the two states limit, the transformation can be handled analytically yielding a value 0.964 for the critical exponent ν much closer to the exact value 1 than 1.338 obtained in Migdal-Kadanoff approximations. Alternative blocking procedures that preserve the isotropy can improve the accuracy to $\nu = 0.987$ and 0.993 respectively.
- Two states for 3D Ising: $\nu = 0.74$.
- Few states improvement not well-understood.



More than two states

When a few more states are added, the quality of the approximation does not immediately improve. One first observes oscillations, false bifurcations, approximate degeneracies. Same observations were made by Leo Kadanoff (1937-2015) and collaborators.

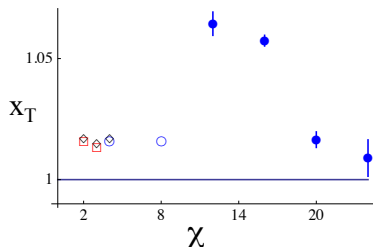
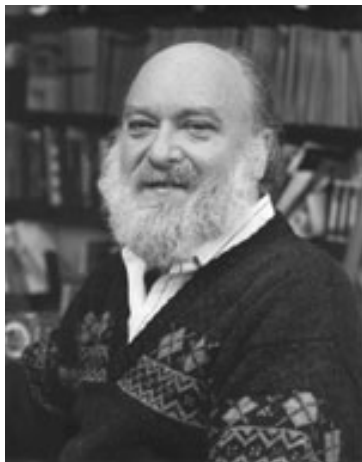


Figure: Thermal x-value $1/\nu$ versus the truncation size χ in various calculations summarized in by Efi Efrati, Zhe Wang, Amy Kolan, Leo P. Kadanoff, Rev. Mod. Phys. 86, 647 May 2014



Leo Kadanoff (1937-2015)



Kaufman exact expression of the transfer matrix for Ising 2 and TRG recursion (Y. M. work in progress)

- For a finite spatial size N_x , Kaufman uses a Clifford algebra with $2 * N_x$ gamma-matrices (2^{N_x} dimensional matrices). The transfer Matrix is a representation of product of rotations in $2N_x$ dimensions which can be diagonalized (Phys. Rev. 76, 1232 (1949)).
- “Chirality”: L-R states ($\gamma_5 \rightarrow \Gamma_1 \Gamma_2 \dots \Gamma_{2N_x}$) is even-odd in TRG
- In progress: exact Grassmann representation of the transfer matrix with recursion relations, comparison with numerical treatment ...
- CFT at β_c : is there a remnant of $z \rightarrow -1/z$ transformation on a finite lattice? Can we obtain the continuum expression of Itzykson et al. for the correlation functions (in terms of modular functions) from the fixed point? (S. Smirnov work on CFT for the critical Ising model, Field medal 2010, may help)



Partition function of the $O(2)$ model

$$Z = \int \prod_{(x,t)} \frac{d\theta_{(x,t)}}{2\pi} e^{-S}.$$

$$\begin{aligned} S = & - \beta_{\hat{t}} \sum_{(x,t)} \cos(\theta_{(x,t+1)} - \theta_{(x,t)} - i\mu) \\ & - \beta_{\hat{x}} \sum_{(x,t)} \cos(\theta_{(x+1,t)} - \theta_{(x,t)}). \end{aligned} \quad (1)$$

$$\begin{aligned} Z = & \sum_{\{n\}} \prod_{(x,t)} I_{n_{(x,t),\hat{x}}}(\beta_{\hat{x}}) I_{n_{(x,t),\hat{t}}}(\beta_{\hat{t}}) e^{\mu n_{(x,t),\hat{t}}} \\ & \times \delta_{n_{(x-1,t),\hat{x}} + n_{(x,t-1),\hat{t}}, n_{(x,t),\hat{x}} + n_{(x,t),\hat{t}}}. \end{aligned} \quad (2)$$



Worm configurations

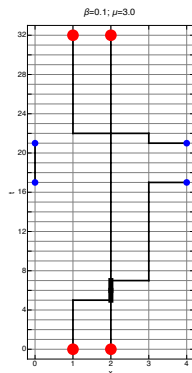


Figure: Allowed configuration of $\{n\}$ for a 4 by 32 lattice. The uncovered links on the grid have $n=0$, the more pronounced dark lines have $|n|=1$ and the wider lines have $n=2$. The dots need to be identified periodically. The time slice 5, represents a transition between $|1100\rangle$ and $|0200\rangle$. Statistical sampling of these configurations (worm algorithm) has been used to check the TRG calculations.



TRG approach of the transfer matrix

The partition function can be expressed in terms of a transfer matrix:

$$Z = \text{Tr } \mathbb{T}^{L_t} .$$

The matrix elements of \mathbb{T} can be expressed as a product of tensors associated with the sites of a time slice (fixed t) and traced over the space indices (PhysRevA.90.063603)

$$\mathbb{T}_{(n_1, n_2, \dots, n_{L_x})(n'_1, n'_2, \dots, n'_{L_x})} = \sum_{\tilde{n}_1 \tilde{n}_2 \dots \tilde{n}_{L_x}} T_{\tilde{n}_{L_x} \tilde{n}_1 n_1 n'_1}^{(1,t)} T_{\tilde{n}_1 \tilde{n}_2 n_2 n'_2}^{(2,t)} \dots T_{\tilde{n}_{L_x-1} \tilde{n}_{L_x} n_{L_x} n'_{L_x}}^{(L_x,t)}$$

with

$$T_{\tilde{n}_{x-1} \tilde{n}_x n_x n'_x}^{(x,t)} = \sqrt{I_{n_x}(\beta_{\hat{t}}) I_{n'_x}(\beta_{\hat{t}}) I_{\tilde{n}_{x-1}}(\beta_{\hat{x}}) I_{\tilde{n}_x}(\beta_{\hat{x}}) e^{\mu(n_x + n'_x)}} \delta_{\tilde{n}_{x-1} + n_x, \tilde{n}_x + n'_x}$$

The Kronecker delta function reflects the existence of a conserved current, a good quantum number ("particle number").



Coarse-graining of the transfer matrix

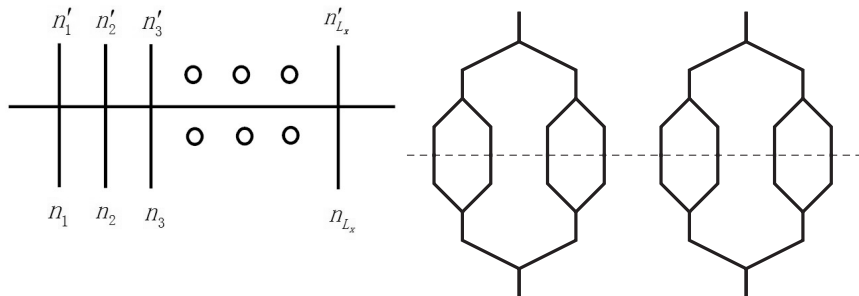


Figure: Graphical representation of the transfer matrix (left) and its successive coarse graining (right). See PRD 88 056005 and PRA 90, 063603 for explicit formulas.



Phase diagram

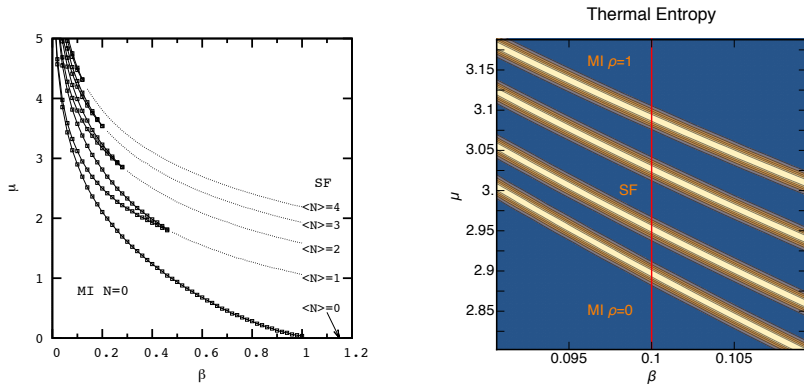


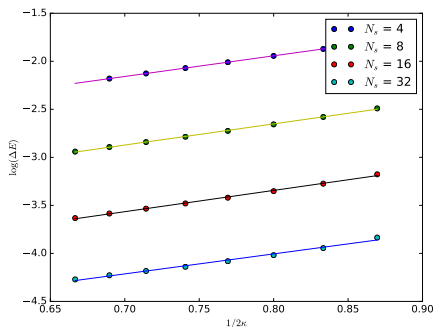
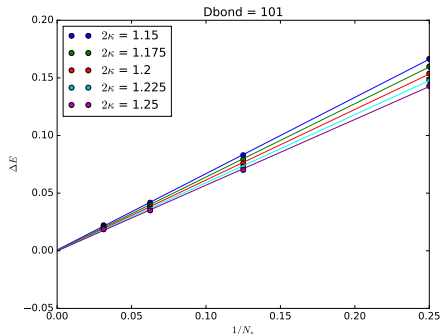
Figure: Mott Insulating “tongues” and Thermal entropy in a small region of the $\beta - \mu$ plane. Intensity plot for the thermal entropy of the classical XY model on a 4×128 lattice in the $\beta - \mu$ plane. The dark (blue) regions are close to zero and the light (yellow ochre) regions peak near $\ln 2$ (level crossing).



A numerical study by J. Unmuth-Yockey

Spatial size dependence: the critical line $\mu_c(n=1)$ of level crossing “pinches” $\mu = 0$ in the BKT phase as N_x increases.

$$\mu_c(n=1) = \Delta E = E_0(n=1) - E_0(n=0) \simeq A \exp(B/\kappa) / N_x$$



Entanglement entropy S_E (PRE 93, 012138 (2016))

We consider the subdivision of AB into A and B (two halves in our calculation) as a subdivision of the spatial indices.

$$\hat{\rho}_A \equiv \text{Tr}_B \hat{\rho}_{AB}; \quad S_E = - \sum_i \rho_{A_i} \ln(\rho_{A_i}). \quad (3)$$

We use blocking methods until A and B are each reduced to a single site.

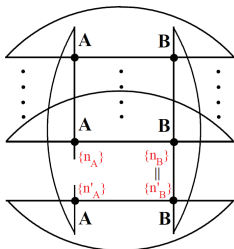


Figure: The horizontal lines represent the traces on the space indices. There are L_t of them, the missing ones being represented by dots. The two vertical lines represent the traces over the blocked time indices in A and B .

The fine structure for $L_x = 4$, $L_t = 256$

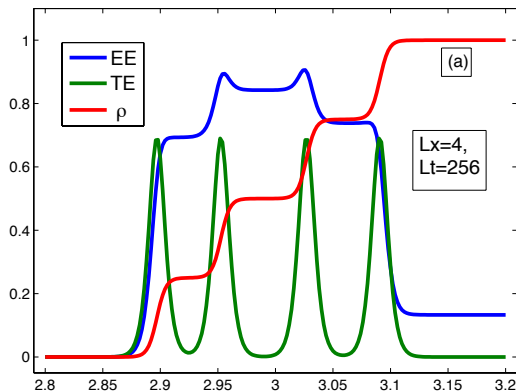


Figure: Entanglement entropy (EE, blue), thermal entropy (TE, green) and particle density ρ (red) versus the chemical potential μ . The thermal entropy has $L_x = 4$ peaks culminating near $\ln 2 \simeq 0.69$; ρ goes from 0 to 1 in $L_x = 4$ steps and the entanglement entropy has an approximate mirror symmetry near half fillings where it peaks.



QCD with chemical potential on $S_1 \times S_3$

QCD at finite chemical potential in a small hyperspherical box

Joyce C. Myers

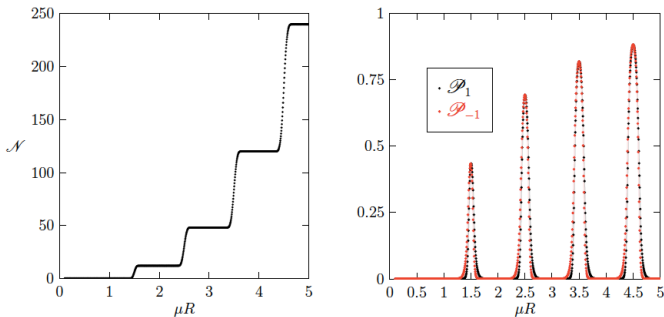


Figure 1: Quark number (Left) and Polyakov lines (Right) as a function of the chemical potential for QCD on $S^1 \times S^3$. (Right). $N = 3, N_f = 1, m = 0, \beta/R = 30$ (low T).

Figure: From: Simon Hands, Timothy J. Hollowood, Joyce C. Myers, arxiv 1012.0192, Lattice 2010.



Similar features for $L_x = 16$ with $L_t = 1024$

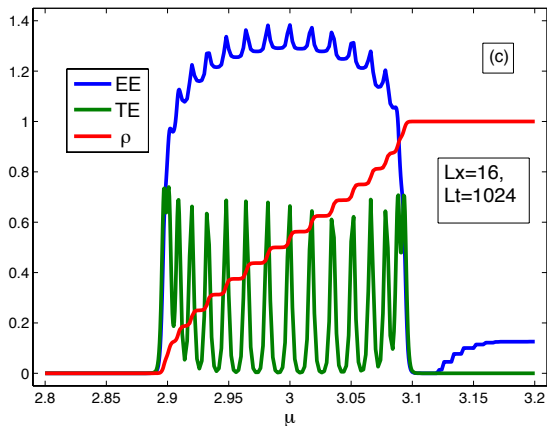


Figure: Entanglement entropy (EE, blue), thermal entropy (TE, green) and particle density ρ (red). The thermal entropy has $L_x = 16$ peaks culminating near $\ln 2 \simeq 0.69$; ρ goes from 0 to 1 in $L_x = 16$ steps and the entanglement entropy has an approximate mirror symmetry near half fillings where it peaks.

Cardy scaling (with Li-Ping Yang): $c=1$?

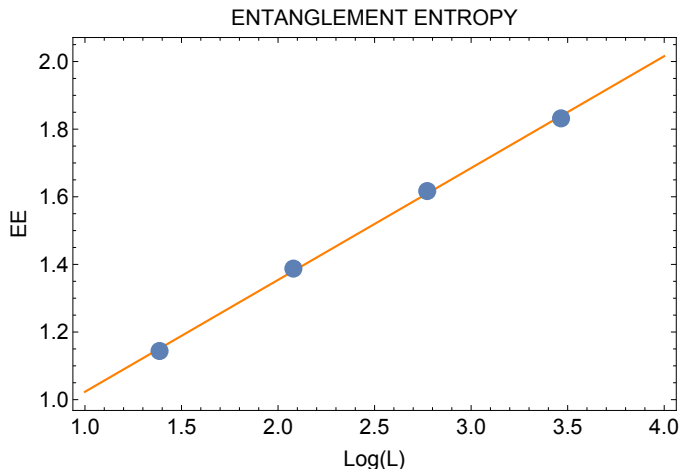


Figure: Entanglement entropy for $O(2)$, $\mu = 0$, $\beta_{spin} = 1.2$. The fit is $0.692 + 0.331 \text{ Log}(L)$. Cardy CFT prediction is $cst. + (c/3)\text{Log}(L)$.



Abelian Higgs model : $Z = \text{Tr}[\prod T]$ (PRD.92.076003)

$$Z = (I_0(\beta_{pl}) I_0(2\kappa_s) I_0(2\kappa_\tau))^V \times \text{Tr} \left[\prod_{h, v, \square} A_{m_{up} m_{down}}^{(s)} A_{m_{right} m_{left}}^{(\tau)} B_{m_1 m_2 m_3 m_4}^{(\square)} \right].$$

The traces are performed by contracting the indices as shown

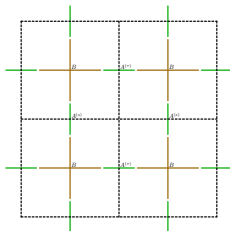


Figure: The basic B and A tensors (in brown and green, respectively, colors online). The $A^{(s)}$ are associated with the vertical tensors, and the horizontal (spatial) links of the lattice. The $A^{(\tau)}$ are associated with the horizontal tensors, and the vertical (temporal) links of the lattice.



TRG calculations of the Polyakov's loop

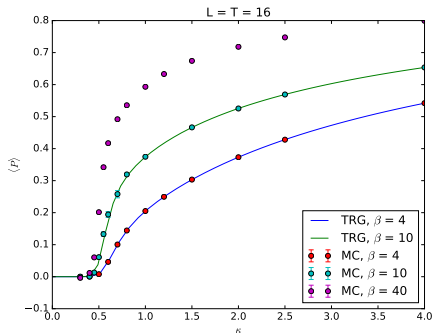
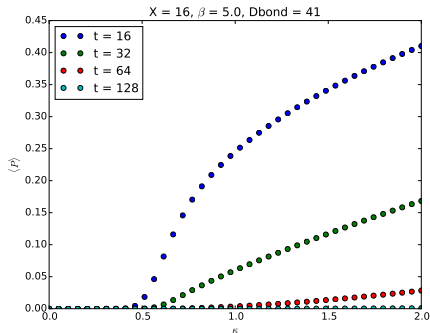
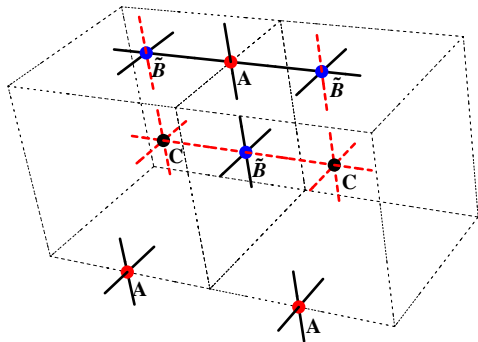


Figure: Polyakov's loop as a function of κ ($2\kappa = \beta_{O(2)}$) for various N_τ (left) and various $\beta_{\text{plaq.}} = 1/g^2$ (right). The results compare well with our MC study and existing results by Poppitz and Shang arXiv:0801.0587. At finite N_X , sufficiently large κ , it seems possible to get a non-zero Polyakov's loop in a suitable limit $N_\tau \rightarrow \infty$ and $\beta_{\text{plaq.}} \rightarrow \infty$.



3D gauge theories

A blocking procedure can be constructed by sequentially combining two cubes into one in each of the directions (PRD 88 056005)



Fisher's zeros: TRG and MC for Ising and O(2)

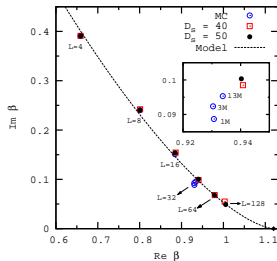
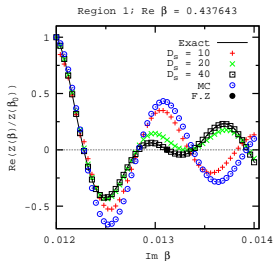


Figure: Left: the real part of the normalized partition function $\text{Re}[Z(\beta)/Z(\beta_0)]$ for β near the Fisher zero $0.437643 + i0.01312$ (the big filled circle on the horizontal axis): result from the HOTRG with $D_s = 10, 20, 40$ ($D_s = 30$ result is not shown as it is close to the $D_s = 40$ case), MC, and exact solution. Right: zeros of XY model with linear size $L = 4, 8, 16, 32, 64, 128$ (from up-left to down-right) calculated from HOTRG with $D_s = 40, 50$ and zeros with $L = 4, 8, 16, 32$ from MC. The curve is a model for trajectory of the lowest zeros. Fit: $\text{Im}\beta_z = 1.27986 \times (1.1199 - \text{Re}\beta_z)^{1.49944}$.



Non trivial fixed points for $SU(3)$ with $N_f=12$?

Irrelevant directions can be slow: problem for small volumes. Finite size scaling (Fisher zeros)?

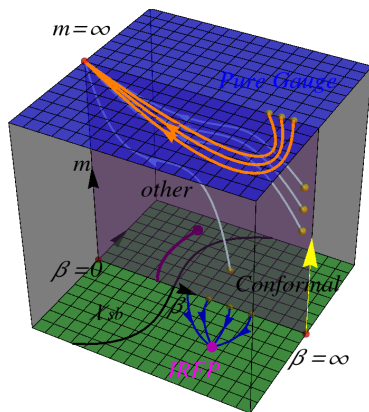


Figure: Schematic flows for $SU(3)$ with 12 flavors (picture by Yuzhi Liu).



Fisher zeros for $SU(3)$ with $N_f=4$ and 12 (Yuzhi Liu)

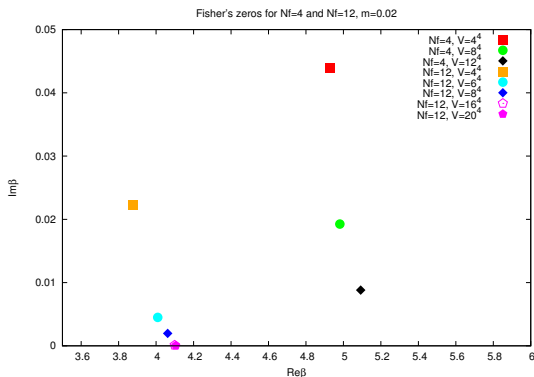


Figure: Zeros for $N_f = 4$ and $N_f = 12$ for L^4 lattices ($L = 16, 20$). 12 flavors pinch the real axis, the zeros scale like L^{-4} (first order) transition.



Search for the end point in (m, β) (Zech Gelzer)

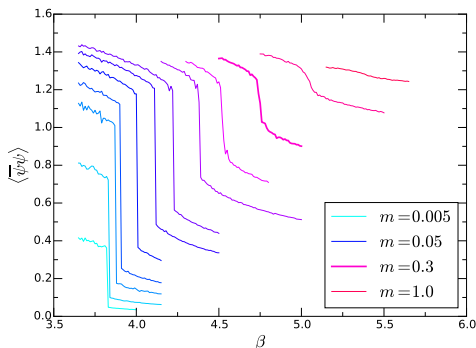


Figure: Unimproved HMC. Chiral condensate vs. $\beta = 6/g^2$ for increasing m , with $N_f = 12$, $V = 4^4$. The masses included (from left to right) are as follows: 0.0050, 0.0105, ..., 0.5000, 0.9999. Will the Fisher's zero pinch the real axis like L^{-2} ($\nu=1/2$, mean field for a free scalar) instead of L^{-4} near the end point (for $m = m_c$)? Does first order scaling changes when $m \rightarrow 0$?



Combining TRG and new perturbative methods?

- The divergence of QFT perturbative series can be traced to the large field configurations. For suitably chosen field cuts, converging perturbative series provide good approximation of results that can be obtained by independent numerical methods. The method can be combined with blocking for the hierarchical model. (YM, PRL 88, 141601 (2002)).
- In many of the TRG calculations, the microscopic tensor is constructed in terms of $I_n(\beta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta e^{\beta \cos(\theta) + in\theta}$. In the known asymptotic expansions of the $I_n(\beta)$, one adds tails of integration to the compact range in order to get Gaussian integrals. Keeping the range of integration finite leads to converging weak coupling expansion (L. Li and YM PRD 71 054509 (2005)). Hopefully this can be connected to resurgence ideas.
- Understanding the connection between topology and the perturbative expansion for the 1D O(2) model on a lattice is easy (Poisson summation), but a challenging problem in 2D.



Conclusions

- The Tensor Renormalization Group provides exact blocking formulas for a large class of lattice models.
- Numerical implementations in 2 and 3 dimensions.
- Good progress has been made in studying 2D CFT models (Ising 2 exponents and $O(2)$ Cardy scaling in the BKT phase).
- Not discussed in this talk: TRG inspired quantum simulators on optical lattices.
- I hope I have convinced you that continuous advances for the continuum limit of lattice models have been made!
- Thanks!

