LONG-TIME BEHAVIOR OF HYDRODYNAMIC SYSTEMS MODELING THE NEMATIC LIQUID CRYSTALS

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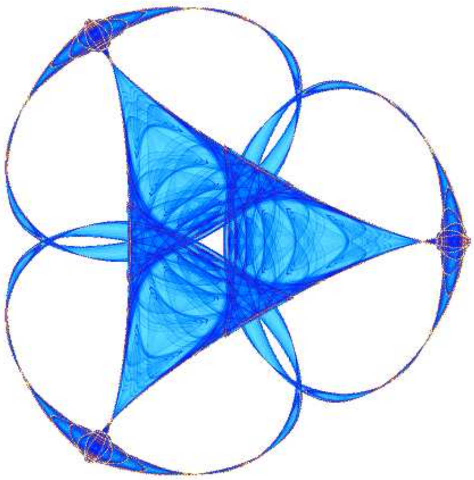

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Long-time Behavior of Hydrodynamic Systems modeling the Nematic Liquid Crystals

Hao Wu [*], Xiang Xu [†] and Chun Liu [‡]

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Abstract

In this paper we study the long-time behavior of the classical solutions to a hydrodynamical system modeling the flow of nematic liquid crystals with rod-like molecules. This system consists of a coupled system of Navier–Stokes equations and kinematic transport equations for the molecular orientations. By using a suitable Łojasiewicz–Simon type inequality, we prove the convergence of global solutions to single steady states as time tends to infinity. Moreover, we provide estimates for the convergence rate.

Keywords: Liquid crystal flows, Navire–Stokes equation, kinematic transport, uniqueness of asymptotic limit, Łojasiewicz–Simon inequality.
AMS Subject Classification: 35B40, 35B41, 35Q35, 76D05.

1 Introduction

We consider the following hydrodynamical system that models the flow of liquid crystal materials (cf. e.g., [2, 23, 34])

$$
\begin{aligned}
v_t + v\cdot\nabla v - \nu\Delta v + \nabla P &= -\lambda\nabla\cdot[\nabla d\odot\nabla d + (\Delta d - f(d))\otimes d], && (1.1)\\
\nabla\cdot v &= 0, && (1.2)\\
d_t + v\cdot\nabla d - d\cdot\nabla v &= \gamma(\Delta d - f(d)), && (1.3)
\end{aligned}
$$

in $Q\times(0,\infty)$. Here, Q is a unit square in $\mathbb{R}^n$, $(n=2,3)$ (the more general case $Q=\Pi_{i=1}^n(0,L_i)$ with different periods L_i in different directions can be treated in a similar way). v is the velocity field of the flow and d represents the averaged macroscopic/continuum molecular orientations in $\mathbb{R}^n$ $(n=2,3)$. P is a scalar function representing the pressure, which includes both the hydrostatic and the induced elastic part from the orientation field. The constants ν,λ and γ stand for viscosity, the competition between kinetic energy and potential energy, and macroscopic elastic relaxation time (Deborah number) for the molecular orientation field, respectively. $f(d)=$

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$\frac{1}{\eta^2}(|d|^2-1)d$ with $0<\eta\le 1$ may be seen as a penalty function to approximate the constraint $|d|=1$, which is due to liquid crystal molecules being of similar size [23], and also fits well with the general theory of Landau's order parameter (cf. [21]). The Ginzburg–Landau type energy is also consistent with the model on variable degree of orientation (cf. [5]). It is obvious that $f(d)$ is the gradient of the scalar valued function $F(d)=\frac{1}{4\eta^2}(|d|^2-1)^2$. $\nabla d\odot\nabla d$ denotes the $n\times n$ matrix whose (i,j)-th entry is given by $\nabla_i d\cdot\nabla_j d$, for $1\le i,j\le n$. $\otimes$ is the usual Kronecker multiplication, e.g., $(a\otimes b)_{ij}=a_ib_j$ for $a,b\in\mathbb{R}^n$.

The hydrodynamic theory of liquid crystals due to Ericksen and Leslie was developed during the period of 1958 through 1968 (cf. e.g., [3,20]). Since then there has been a remarkable research in the field of liquid crystals, both theoretically and experimentally (cf. [1,2,4,5,13,21–28,34,39] and references therein). System (1.1)–(1.3) is a simplified version of the Ericksen–Leslie model for the hydrodynamics of nematic liquid crystals (cf. [3,4,13,20,21,34]). Generally speaking, the system is a macroscopic continuum description of the time evolutions of these materials influenced by both the flow field $v(x,t)$, and the microscopic orientational configuration $d(x,t)$, which can be derived from the coarse graining of the directions of rod-like liquid crystal molecules. Equation (1.1) is the conservation of linear momentum (the force balance equation). It combines a usual equation describing the flow of an isotropic fluid with an extra nonlinear coupling term, which is anisotropic. This extra term is the induced elastic stress from the elastic energy through the transport, which is represented by the third equation. Equation (1.2) represents incompressibility of the fluid. Equation (1.3) is associated with conservation of the angular momentum. The left hand side of (1.3) stands for the kinematic transport by the flow field, while the right hand side represents the internal relaxation due to the elastic energy (cf. e.g., [34]).

The above system was derived from the macroscopic point of view and was very successful in understanding the coupling between the director field and the velocity field, especially in the liquid crystals of nematic type. In many experiments and earlier theories on nematic liquid crystals, the samples are treated as consisting of slow moving particles. Hence, one approach is to study the behavior of the director field d in the absence of the velocity field. Unfortunately, the flow velocity does disturb the alignment of the molecules. More importantly, the converse is also true: a change in the alignment of molecules will induce velocity. This velocity will in turn affect the time evolution of the director field. In this process, we can not assume that the velocity field will remain small even when we start from zero velocity.

In the context of hydrodynamics, the basic variable is the flow map (particle trajectory) $x(X,t)$. X is the original labeling (the Lagrangian coordinate) of the particle, which is also referred to as the material coordinate. x is the current (Eulerian) coordinate, and is also called the reference coordinate. For a given velocity field $v(x,t)$, the flow map is defined by the ODE :

$$x_t=v(x(X,t),t),\quad x(X,0)=X.$$

To incorporate the elastic properties of the material, we need to introduce the deformation tensor

$$\mathcal{F}(X,t)=\frac{\partial x}{\partial X}(X,t).$$

This quantity is defined in the Lagrangian material coordinate and it satisfies

$$\frac{\partial\mathcal{F}(X,t)}{\partial t}=\frac{\partial v(x(X,t),t)}{\partial X}.$$

In Eulerian coordinates, we define $\tilde{\mathcal{F}}(x,t) = \mathcal{F}(X,t)$. By using the chain rule, the above equation can be transformed into the following transport equation for $\tilde{\mathcal{F}}$ (cf. e.g., [9]):

$$\tilde{\mathcal{F}}_t + (v \cdot \nabla)\tilde{\mathcal{F}} \;=\; \nabla v \tilde{\mathcal{F}}.$$

Without ambiguity, in the following text, we will not distinguish the notations $\mathcal{F}$ and $\tilde{\mathcal{F}}$. If the liquid crystal has a rod-like shape, then transport of the direction of d can be expressed as (cf. [15, 19])

$$d(x(X,t),t) = \mathcal{F} d_0(X),$$

where $d_0(X)$ is the initial condition. This equation demonstrates the stretching of the director besides the transport along the trajectory. By taking full time derivative on both sides, we have (cf. e.g., [34])

$$\frac{D}{Dt} d(x(X,t),t) = \dot{\mathcal{F}} d_0(X) = \nabla v \mathcal{F} d_0 = \nabla v d = (d \cdot \nabla) v.$$

Hence, the total transport of the orientation vector d becomes

$$d_t + v \cdot \nabla d - d \cdot \nabla v,$$

which represents the covariant parallel transport with no-slip boundary condition between the rod-like particle and the fluid (cf. [15, 19]). In general, for a molecule of ellipsoidal shape with a finite aspect ratio, the transport of the main axis direction is represented by

$$d(x(X,t),t) = \mathbb{E} d_0(X),$$

where $\mathbb{E}$ is a linear combination of $\mathcal{F}$ and $\mathcal{F}^{-T}$ and satisfies the transport equation:

$$\mathbb{E}_t + (v \cdot \nabla)\mathbb{E} = (\alpha \nabla v + (1-\alpha)(-\nabla^T v))\mathbb{E}.$$

As a consequence, the total transport of d in the general case becomes

$$d_t + v \nabla d - \omega d - (2\alpha - 1) A d,$$

where $A = \frac{\nabla v + \nabla^T v}{2}$, $\omega = \frac{\nabla v - \nabla^T v}{2}$. We note that the spherical, rod-like and disc-like liquid crystal molecules correspond to $\alpha = \frac{1}{2}$, 1 and 0, respectively (cf. e.g., [2, 3, 15, 26, 34]).

In this paper, we shall focus on the rod-like nematic liquid crystal molecules ($\alpha = 1$). We would like to mention that system (1.1)–(1.3) can be derived from an energetic variational point of view. Consider the following action functional in terms of flow map $x(X,t)$:

$$\mathcal{A} = \int_0^T \int_{\Omega_0} \left[\frac{1}{2}|x_t|^2 - \frac{\lambda}{2} |\mathcal{F}^{-T} \nabla_X \mathcal{F} d_0(X)|^2 - \lambda F(\mathcal{F} d_0(X)) \right] J dX dt,$$

where Ω_0 is the region at the initial time and $J = \det \frac{\partial x}{\partial X}$ is the Jacobian. It has been shown that by using the least action principle (Hamilton's principle), $\frac{\delta \mathcal{A}}{\delta x} = 0$, we can recover the system (1.1)–(1.3). Such a derivation using the variation with respect to domain, i.e., least action principle, is equivalent to the principle of virtual work. We refer to [34] for detailed discussion.

We will consider system (1.1)–(1.3) subject to the periodic boundary conditions (namely, v, d are well defined in n-dimensional torus $\mathbb{T}^n$)

$$v(x + e_i) = v(x), \quad d(x + e_i) = d(x), \qquad \text{for } x \in \partial Q, \tag{1.4}$$

where unit vectors e_i $(i = 1, ..., n)$ are the canonical basis of $\mathbb{R}^n$. Besides, we suppose the initial conditions

$$v|_{t=0} = v_0(x) \ \text{ with } \nabla \cdot v_0 = 0, \quad d|_{t=0} = d_0(x), \qquad \text{for } x \in Q. \tag{1.5}$$

As far as the mathematical results are concerned, we notice that the current system (1.1)–(1.3) is a special case of the general Ericksen–Leslie system which describes the flow of liquid crystal material (cf. [1,2,21,25]). However, the general Ericksen–Leslie system was so complicated that only some special cases have been treated theoretically or numerically. A simplified system with suitable boundary and initial conditions were first studied in [23], where the system was highly simplified compared to the general Ericksen–Leslie model but still captured some essential properties of it. For that simplified model in [23], due to the dissipation of energy, the existence of global weak solutions was proved. Moreover, they obtained the existence and uniqueness of global classical solutions in 2D as well as the corresponding result in 3D (provided that the constant viscosity ν was assumed to be large enough). Later on, the authors proved in [24] the partial regularity result that the one dimensional space-time Hausdorff measure of the singular set of the suitable weak solutions was zero. Then, the numerical code using finite element methods was established in [27, 28] to study new features of the system, especially the interaction of the defects and the flow fields. In addition, the results of the existence of solutions to the general Ericksen–Leslie system in a special case, say, when the maximal principle held for the equation of d, were provided in [25].

We can formally derive the basic energy law that governs the dynamics of the system (1.1)–(1.5). Suppose there is a classical solution (v, d), if we multiply equation (1.1) with v, equation (1.3) with $-\Delta d - f(d)$, add them together and integrate over Q, we get

$$\frac{1}{2}\frac{d}{dt}\int_Q \left(|v|^2 + \lambda|\nabla d|^2 + 2\lambda F(d)\right) dx = -\int_Q \left(\nu|\nabla v|^2 + \lambda\gamma|\Delta d - f(d)|^2\right) dx.$$

It is worth pointing out that, for different molecule shapes (and as a result, different kinematic transports), the system indeed possesses the same energy dissipative law (cf. [26]).

Comparing with the small molecule system (cf. [23]), we now have different kinematic transport and accordingly one more stress term $(\Delta d - f(d)) \otimes d$ in the elastic stress in (1.1) and one more transport term $d \cdot \nabla v$ in (1.3). These bring extra technical difficulties to prove the existence result. For instance, $d \cdot \nabla v$ stands for the parallel transport, which includes both rotation and stretching effect of the director d. The stretching effect leads to the loss of maximum principle for the equation of d. On the other hand, the extra stress term $(\Delta d - f(d)) \otimes d$ can not be suitably defined in the weak formulation as in [23], and thus the requirement that $d \in L^\infty(0, T; L^\infty(Q))$ must be imposed so that the problem is well-posed. For detailed discussions, we refer to [34], in which the global existence of weak/classical solutions to the system (1.1)–(1.5) for $n = 2$ or $n = 3$ with large viscosity assumption has been proven. By supposing periodic boundary conditions, one can get rid of the boundary terms when integrating by parts, which is necessary in the derivation of higher-order energy inequalities (cf. [34]). Without this assumption, some boundary terms would remain persistent and undermine the whole estimates. The corresponding initial boundary value problems are still open.

The main results of this paper are as follows.

Theorem 1.1. *When $n = 2$, for any initial data $(v_0, d_0) \in V \times H^2_p(Q)$, the unique global classical solution to problem* (1.1)–(1.5) *has the following property:*

$$\lim_{t\to+\infty}(\|v(t)\|_{H^1} + \|d(t) - d_\infty\|_{H^2}) = 0, \tag{1.6}$$

where d_∞ is a solution to the following nonlinear elliptic periodic boundary value problem:

$$-\Delta d_\infty + f(d_\infty) = 0, \quad x \in Q, \quad d_\infty(x + e_i) = d_\infty(x), \quad x \in \partial Q. \tag{1.7}$$

Moreover, there exists a positive constant C depending on v_0, d_0, Q, d_∞ such that

$$\|v(t)\|_{H^1} + \|d(t) - d_\infty\|_{H^2} \le C(1 + t)^{-\frac{\theta}{(1-2\theta)}}, \quad \forall\ t \ge 0. \tag{1.8}$$

Here, $\theta \in (0, 1/2)$ is the same constant as in the Łojasiewicz–Simon inequality (see Lemma 3.3 below).

In three dimensional case, we have the following results:

Theorem 1.2. *When $n = 3$, for any initial data $(v_0, d_0) \in V \times H^2_p(Q)$, under the large viscosity assumption $\nu \ge \nu_0(\lambda, \gamma, v_0, d_0)$ (cf. (5.24)), the problem* (1.1)–(1.5) *admits a unique global classical solution enjoying the same properties as in Theorem 1.1.*

Theorem 1.3. *When $n = 3$, let $d^* \in H^2_p(Q)$ be an absolute minimizer of the functional*

$$E(d) = \frac{1}{2}\|\nabla d\|^2 + \int_\Omega F(d)dx. \tag{1.9}$$

There exists a constant $\sigma \in (0, 1]$, which may depend on λ, γ, ν, f, Q and d_*, such that for any initial data $(v_0, d_0) \in V \times H^2_p(Q)$ that satisfy $\|v_0\|_{H^1} + \|d_0 - d_*\|_{H^2} < \sigma$, problem (1.1)–(1.5) *admits a unique global classical solution enjoying the same properties as in Theorem 1.1.*

Theorems 1.1–1.3 imply the uniqueness of asymptotic limit of system (1.1)–(1.5) whenever it admits a global bounded solution. The problem on uniqueness of asymptotic limit for nonlinear evolution equations, namely whether the global solution will converge to a single equilibrium as time tends to infinity, has attracted a lot of interests among mathematicians for a long time. If the spacial dimension $n \ge 2$, it is known that the structure of the set of equilibria can be nontrivial and may form a continuum for certain physically reasonable nonlinearities. In particular, in our present case, under the periodic boundary conditions, one may expect that the dimension of the set of stationary solutions is at least n. This is because a shift in each variable should give another steady state (cf. [31]), e.g., if d_* is a steady state, so is $d_*(\cdot\ + \tau e_i)$, $1 \le i \le n$, $\tau \in \mathbb{R}^+$. In this case it is highly nontrivial to decide whether a given trajectory will converge to a single steady state. In 1983, L. Simon [33] made a breakthrough by proving that for a semilinear parabolic equation with a nonlinear term $f(x, u)$ being analytic in the unknown function u, its bounded global solution would converge to an equilibrium solution as $t \to \infty$. Simon's idea relies on a generalization of the Łojasiewicz inequality (cf. [29,30]) for analytic functions defined in the finite dimensional space $\mathbb{R}^m$ (see Section 2 for a brief introduction). Since then, his original approach has been simplified and applied to prove convergence results for various evolution equations (see e.g., [6, 8, 12, 14, 16, 22, 31, 37, 38] and the references therein). In order to apply the Łojasiewicz–Simon approach to our problem (1.1)–(1.5), we need to introduce a suitable Łojasiewicz–Simon type inequality for vector functions with periodic boundary condition (cf. Lemma 3.3). The corresponding result for small molecule system (cf. [23]) was discussed in [39] under various boundary conditions (e.g., Dirichlet boundary conditions/free–slip boundary conditions).

Once we prove the convergence to an equilibrium, a natural question is the convergence rate. It is well known that an estimate in certain (lower–order) norm can usually be obtained directly from the Łojasiewicz–Simon approach (see, e.g., [12, 41]). One can then in a straightforward way, obtain estimates in higher–order norms by using interpolation inequalities (cf. [12]), and consequently, the decay exponent deteriorates. We shall show that by using suitable energy estimates and constructing proper differential inequalities, it is possible to obtain the same estimates on the convergence rate in both higher and lower order norms. Our approach improves the previous results in the literature and can be applied to many other problems (cf. [8, 37–39]).

The remaining part of this paper is organized as follows. In Section 2, we provide an example briefly illustrating the application of Łojasiewicz inequality in finite dimensional space $\mathbb{R}^m$. In Section 3, we introduce the functional setting, some preliminary results as well as some technical lemmas. Section 4 is devoted to the two dimensional case. We prove the convergence of global solutions to single steady states as time tends to infinity and obtain an estimate on convergence rate. In Section 5, we consider the three dimensional case. The same convergence result is proved for two subcases, in which the global existence of classical solution can be obtained. In Section 6, we consider the results for liquid crystal flows with non-vanishing average velocity. In the final Section 7, we briefly discuss our future work.

2 Application of the Finite Dimensional Łojasiewicz Inequality

As mentioned in Section 1, the key ingredient in this paper is the application of the Łojasiewicz–Simon approach. Since it is a generalization of the Łojasiewicz inequality in finite dimensional space $\mathbb{R}^m$ for analytic functions, to understand it better, let us briefly recall the applications in the finite dimensional case first.

In the 1960's, Łojaciewicz proved the following fundamental inequality for gradient systems of analytic functions in finite dimensional Euclidean spaces [29, 30].

Theorem 2.1 (Łojasiewicz inequality). *Suppose that $F : \mathbb{R}^m \to \mathbb{R}$ is an analytic function near its critical point a (i.e., $\nabla F(a) = 0$). Then there is a positive constant σ and $\theta \in (0, \frac{1}{2})$ depending on a, such that when $\|x - a\|_{\mathbb{R}^m} \le \sigma$,*

$$|F(x) - F(a)|^{1-\theta} \le \|\nabla F(x)\|_{\mathbb{R}^m}. \tag{2.1}$$

The Łojasiewicz inequality is a powerful tool to study the asymptotic behavior of solutions to gradient systems. To describe the idea, let us recall a simple example discussed in [22] (for other applications on ODEs, cf. e.g. [7, 11]).

Consider the ODE system

$$\begin{cases} x_t &= -\nabla f(x), \qquad x \in \mathbb{R}^N, \\ x(0) &= x_0. \end{cases} \tag{2.2}$$

We assume that f is analytic in x, $f \ge 0$. We also assume that the ODE system (2.2) admits a bounded smooth solution $x(t)$, defined for all $t \ge 0$. For brevity we denote $F(t) = f(x(t))$,

$t \geq 0$. Multiplying both sides of (2.2) with $x_t(t)$, we know

$$\frac{dF(t)}{dt} = -\|\nabla f(x(t))\|^2_{\mathbb{R}^N} = -\|x_t(t)\|^2_{\mathbb{R}^N} \leq 0, \quad \forall\, t \geq 0. \tag{2.3}$$

Therefore, the nonnegative function $F(t)$ is decreasing on $[0, +\infty)$ and serves as a Lyapunov function for (2.2). Then integrating (2.3) from 0 to t, we have

$$F(t) + \int_0^t \|x_t(\tau)\|^2_{\mathbb{R}^N} d\tau = F(0). \tag{2.4}$$

For the gradient system (2.2), we infer that the ω-limit set of $x(t)$, is nonempty and consists of equilibria (cf. [10]). Namely, there exists an increasing unbounded sequence $\{t_n\}_{n\in N}$ and an equilbrium $x_\infty \in \mathbb{R}^N$, such that

$$\lim_{t_n\to\infty} \|x(t_n) - x_\infty\|_{\mathbb{R}^N} = 0. \tag{2.5}$$

Consequently, $F(t_n) \geq f(x_\infty)$ and

$$\lim_{t_n\to\infty} F(t_n) = F_\infty = f(x_\infty) \geq 0. \tag{2.6}$$

Our goal is to prove

$$\lim_{t_n\to\infty} \|x(t) - x_\infty\|_{\mathbb{R}^N} = 0. \tag{2.7}$$

We can discuss in two subcases.

Case 1. If there exists some $t_0 \geq 0$, such that $F(t_0) = f(x_\infty)$, then we deduce from (2.3) that $\forall\, t \geq t_0$, $x(t) \equiv x(t_0)$.

Case 2. If $\forall\, t > 0$, $F(t) > f(x_\infty)$, due to (2.3), (2.6), we have

$$\lim_{t\to\infty} F(t) = f(x_\infty). \tag{2.8}$$

Let $\varepsilon = (\frac{\sigma\theta}{4})^{\frac{1}{\theta}}$, it follows from (2.5) and (2.8) that there exists an integer K such that for all $n > K$.

$$\|x(t_n) - x_\infty\|_{\mathbb{R}^N} < \frac{\sigma}{4}, \quad 0 < F(t_n) - f(x_\infty) < \varepsilon. \tag{2.9}$$

Define

$$\bar{t}_n = \sup\{t > t_n \big| \, \|x(s) - x_\infty\|_{\mathbb{R}^N} < \sigma, \ \forall\, s \in [t_n, t].\}$$

In what follows, we recall the simple argument introduced in [16], which provides a convenient way to apply the Łojasiewicz inequality (also for Łojasiewcz–Simon inequality).

Proposition 2.1. *There exists $n_0 \geq K$, such that $\bar{t}_{n_0} = +\infty$.*

Proof. The proof follows from the contradiction argument in [16]. Suppose $\forall\, n \geq K$, $t_n < \bar{t}_n < +\infty$. We can apply Theorem 2.1 on interval $[t_n, \bar{t}_n]$. As a consequence, the length of the trajectory $x(t)$ between $[t_n, \bar{t}_n]$ is

$$\int_{t_n}^{\bar{t}_n} \|x_t(\tau)\|_{\mathbb{R}^N} d\tau = \int_{t_n}^{\bar{t}_n} \frac{1}{\|\nabla f(x(\tau))\|_{\mathbb{R}^N}} (-\frac{d}{d\tau} F(\tau)) d\tau$$

$$
\begin{aligned}
&\le \int_{t_n}^{\bar{t}_n} \frac{1}{|F(\tau)-f(x_\infty)|^{1-\theta}}(-\frac{d}{d\tau}F(\tau))d\tau \\
&= \frac{1}{\theta}\Big[(F(t_n)-f(x_\infty))^\theta-(F(\bar{t}_n)-f(x_\infty))^\theta\Big] \\
&< \frac{1}{\theta}(F(t_n)-f(x_\infty))^\theta < \frac{1}{\theta}\varepsilon^\theta < \frac{\sigma}{4}. \qquad (2.10)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\|x(\bar{t}_n)-x_\infty\|_{\mathbb{R}^N} &\le \|x(\bar{t}_n)-x(t_n)\|_{\mathbb{R}^N}+\|x(t_n)-x_\infty\|_{\mathbb{R}^N} \\
&< \int_{t_n}^{\bar{t}_n}\|x_t(\tau)\|_{\mathbb{R}^N}d\tau+\frac{\sigma}{4}<\frac{\sigma}{2}, \qquad (2.11)
\end{aligned}
$$

which is a contradiction to the definition of $\bar{t}_n$. □

Since $\|x(t)-x_\infty\|_{\mathbb{R}^N}<\sigma$, $\forall\, t\ge t_{n_0}$, we infer that

$$\int_0^{+\infty}\|x_t(\tau)\|_{\mathbb{R}^N}d\tau<+\infty.$$

Consequently,

$$\|x(t_1)-x(t_2)\|_{\mathbb{R}^N}\le\int_{t_1}^{t_2}\|x_t(\tau)\|_{\mathbb{R}^N}d\tau\to 0, \qquad \text{as } t_1,\ t_2\to 0.$$

Hence, $x(t)$ is uniformly convergent in $\mathbb{R}^N$. Combined with (2.5), one arrives at our goal (2.7).

Finally, let us study the convergence rate of $x(t)$ to x_∞. We only have to consider Case 2, since Case 1 is trivial. We know from Theorem 2.1 that $\forall\, t\ge t_{n_0}$,

$$-\frac{d}{dt}[F(t)-f(x_\infty)]^\theta = \theta\|x_t\|^2_{\mathbb{R}^N}[F(t)-f(x_\infty)]^{\theta-1}\ge\theta\|F(t)-f(x_\infty)\|^{1-\theta}_{\mathbb{R}^N}, \qquad (2.12)$$

which indicates (cf. [12, 41])

$$F(t)-f(x_\infty)\le C(1+t)^{-\frac{1}{1-2\theta}}, \qquad \forall\, t\ge t_{n_0}. \qquad (2.13)$$

As a result,

$$\int_t^\infty\|x_t(\tau)\|_{\mathbb{R}^N}d\tau\le-\frac{1}{\theta}\int_t^\infty\frac{d}{d\tau}[F(t)-f(x_\infty)]^\theta d\tau\le C(1+t)^{-\frac{\theta}{1-2\theta}}, \qquad \forall\, t\ge t_{n_0}. \qquad (2.14)$$

After choosing proper constant C, we can get

$$\|x(t)-x_\infty\|_{\mathbb{R}^N}\le C(1+t)^{-\frac{\theta}{1-2\theta}}, \qquad \forall\, t\ge 0. \qquad (2.15)$$

3 Preliminaries

Recall the well-established functional setting for periodic boundary value problems (cf. for instance, [35, Chapter 2]):

$$
\begin{aligned}
H^m_p(Q) &= \{v\in H^m(\mathbb{R}^n)\mid v(x+e_i)=v(x)\}, \\
\dot{H}^m_p(Q) &= H^m_p(Q)\cap\left\{v:\ \int_Q v(x)dx=0\right\},
\end{aligned}
$$

$$\begin{aligned} H &= \{v \in L^2_p(Q),\ \nabla \cdot v = 0\}, \quad \text{where } L^2_p(Q) = H^0_p(Q), \\ V &= \{v \in \dot{H}^1_p(Q),\ \nabla \cdot v = 0\}, \\ V' &= \text{the dual of } V. \end{aligned}$$

For simplicity, we denote the inner product on $L^2_p(Q)$ as well as H by $(\cdot,\cdot)$ and the associated norm by $\|\cdot\|$. We shall denote by C the genetic constants depending on $\lambda, \gamma, \nu, Q, f$ and the initial data. Special dependence will be pointed out explicitly in the text if necessary. Since the parameters λ and γ do not play important roles in the proof, we set $\lambda = \gamma = 1$ for the sake of simplicity. Following [35], one can define mapping S

$$Su = -\Delta u, \quad \forall\, u \in D(S),$$

with

$$D(S) = \{u \in H, Su \in H\} = \dot{H}^2_p \cap H.$$

The operator S can be seen as an unbounded positive linear self-adjoint operator on H. If $D(S)$ is endowed with the norm induced by $\dot{H}^0_p(Q)$, then S becomes an isomorphism from $D(S)$ onto H. We refer to [35] for more detailed properties of operator S.

We recall the interior elliptic estimate, which states that for any $U_1 \subset\subset U_2$ there is a constant $C > 0$ depending only on U_1 and U_2 such that $\|d\|_{H^2(U_1)} \le C(\|\Delta d\|_{L^2(U_2)} + \|d\|_{L^2(U_2)})$. In our case, we can choose Q' to be the union of Q and its neighborhood copies. Then we have

$$\|d\|_{H^2(Q)} \le C(\|\Delta d\|_{L^2(Q')} + \|d\|_{L^2(Q')}) = 9C(\|\Delta d\|_{L^2(Q)} + \|d\|_{L^2(Q)}). \tag{3.1}$$

The following embedding inequalities will be frequently used in the subsequent proofs:

Lemma 3.1. *(cf. [35]) If $n = 2$, we have*

$$\|u\|_{L^\infty(Q)} \le c\|u\|^{\frac{1}{2}} \|u\|^{\frac{1}{2}}_{H^2(Q)}, \quad \forall\, u \in H^2_p(Q),$$

If $n = 3$, then

$$\|u\|_{L^\infty(Q)} \le c\|u\|^{\frac{1}{4}} \|u\|^{\frac{3}{4}}_{H^2(Q)}, \quad \forall\, u \in H^2_p(Q).$$

Here, we note that $\|u\|_{H^2(Q)}$ can be estimated by $\|\Delta u\|$ and $\|u\|$ in sprit of the elliptic estimate (3.1).

The global existence of weak/classical solutions to system (1.1)–(1.5) has been proven in [34, Theorem 1.1]. More precisely, we have

Proposition 3.1. *Assume that $(v_0, d_0) \in V \times H^2_p(Q)$. Then, if either $n = 2$ or $n = 3$ with the large viscosity assumption $\nu \ge C(\lambda, \gamma, v_0, d_0)$ (see (5.24)), problem (1.1)–(1.5) admits a global solution such that*

$$v \in L^\infty(0,\infty; V), \quad d \in L^\infty(0,\infty; H^2), \tag{3.2}$$

and

$$v \in L^2(0,\infty; D(S)), \quad d \in L^2(0,T; H^3), \tag{3.3}$$

where $T > 0$ is arbitrary. Moreover, $v, d \in C^\infty(Q)$ for all $t \in (0,T)$.

The proof for Proposition 3.1 relies on a modified Galerkin method introduced in [23]. After generating a sequence of approximate solutions, one can use the Ladyzhenskaya method (cf. [18, 36]) to get high-order energy estimates, which enable us to pass to the limit. Furthermore, a weak solution together with high-order derivative estimates implies a strong solution, i.e. $v \in L^2(0,T;D(S))$ and $d \in L^2(0,T;H^3(Q))$. Finally, a bootstrap argument based on Serrin's result [32] (cf. also [17]) and Sobolev embedding theorems leads to the existence of classical solutions.

The Lyapunov functional for problem (1.1)–(1.5) is

$$\mathcal{E}(t) = \frac{1}{2}\|v(t)\|^2 + \frac{\lambda}{2}\|\nabla d(t)\|^2 + \lambda \int_Q F(d(t))dx. \tag{3.4}$$

As mentioned before, our system has the following *basic energy law*

$$\frac{d}{dt}\mathcal{E}(t) = -\nu\|\nabla v(t)\|^2 - \lambda\gamma\|\Delta d(t) - f(d(t))\|^2, \quad t \geq 0. \tag{3.5}$$

First, we shall show a continuous dependence result on initial data, from which the uniqueness of regular solutions to problem (1.1)–(1.5) follows.

Lemma 3.2. *Suppose that (v_i, d_i) $(i = 1, 2)$ are global solutions to problem (1.1)–(1.5) corresponding to initial data $(v_{0i}, d_{0i}) \in V \times H^2_p(Q)$ $(i = 1, 2)$. Moreover, we assume that for any $T > 0$, the following estimate holds*

$$\|v_i(t)\|_{H^1} + \|d_i(t)\|_{H^2} \leq M, \quad \forall\, t \in [0,T]. \tag{3.6}$$

Then for any $t \in [0,T]$, we have

$$\begin{aligned} & \|(v_1 - v_2)(t)\|^2 + \|(d_1 - d_2)(t)\|^2_{H^1} + \int_0^t (\nu\|\nabla(v_1 - v_2)(\tau)\|^2 + \|\Delta(d_1 - d_2)(\tau)\|^2)d\tau \\ \leq\ & 2e^{Ct}(\|v_{01} - v_{02}\|^2 + \|d_{01} - d_{02}\|^2_{H^1}), \end{aligned} \tag{3.7}$$

where C is a constant depending on M but not on t.

Proof. Denote

$$\bar{v} = v_1 - v_2, \quad \bar{d} = d_1 - d_2. \tag{3.8}$$

Since (v_i, d_i) are solutions to problem (1.1)–(1.5), we have

$$v_{1t} + v_1 \cdot \nabla v_1 - \nu\Delta v_1 + \nabla P_1 = -\nabla \cdot [\nabla d_1 \odot \nabla d_1 + (\Delta d_1 - f(d_1)) \otimes d_1], \tag{3.9}$$

$$\nabla \cdot v_1 = 0, \tag{3.10}$$

$$d_{1t} + v_1 \cdot \nabla d_1 - d_1 \cdot \nabla v_1 = \Delta d_1 - f(d_1), \tag{3.11}$$

and

$$v_{2t} + v_2 \cdot \nabla v_2 - \nu\Delta v_2 + \nabla P_2 = -\nabla \cdot [\nabla d_2 \odot \nabla d_2 + (\Delta d_2 - f(d_2)) \otimes d_2], \tag{3.12}$$

$$\nabla \cdot v_2 = 0, \tag{3.13}$$

$$d_{2t} + v_2 \cdot \nabla d_2 - d_2 \cdot \nabla v_2 = \Delta d_2 - f(d_2). \tag{3.14}$$

Multiplying $v_1 - v_2$ with the subtraction of (3.12) from (3.9), and multiplying $(d_1 - d_2) - (\Delta d_1 - \Delta d_2)$ with the subtraction of (3.14) from (3.11), we add these two resultants together. By integration by parts, we infer from the periodic boundary conditions that

$$\begin{aligned}
& \frac{1}{2}\frac{d}{dt}\left(\|\bar{v}\|^2 + \|\bar{d}\|^2 + \|\nabla\bar{d}\|^2\right) + \nu\|\nabla\bar{v}\|^2 + \|\nabla\bar{d}\|^2 + \|\Delta\bar{d}\|^2 \\
= \ & -(v_2\cdot\nabla\bar{v},\bar{v}) - (\bar{v}\cdot\nabla v_1,\bar{v}) - (\Delta d_2\cdot\nabla\bar{d},\bar{v}) + (\Delta d_2\otimes\bar{d},\nabla\bar{v}) \\
& -((f(d_1)-f(d_2))\otimes d_1,\nabla\bar{v}) - (f(d_2)\otimes\bar{d},\nabla\bar{v}) + (f(d_1)-f(d_2),\Delta\bar{d}) \\
& +(v_2\cdot\nabla\bar{d},\Delta\bar{d}) - (\bar{d}\cdot\nabla v_2,\Delta\bar{d}) - (f(d_1)-f(d_2),\bar{d}) - (\bar{v}\cdot\nabla d_1,\bar{d}) \\
& -(v_2\cdot\nabla\bar{d},\bar{d}) + (d_1\cdot\nabla\bar{v},\bar{d}) + (\bar{d}\cdot\nabla v_2,\bar{d}). \qquad (3.15)
\end{aligned}$$

From assumption (3.6), $\|v\|_{H^1}$ and $\|d\|_{H^2}$ are uniformly bounded in $[0,T]$. Hence, by using the Sobolev embedding theorems, we can estimate the righthand side term by term (the calculation presented here is for the three dimensional case and it is also valid for two dimensional case).

$$\begin{aligned}
& |(v_2\cdot\nabla\bar{v},\bar{v})| + |(\bar{v}\cdot\nabla v_1,\bar{v})| \\
\le \ & \|v_2\|_{L^6}\|\nabla\bar{v}\|\|\bar{v}\|_{L^3} + \|\bar{v}\|_{L^4}^2\|\nabla v_1\| \\
\le \ & C\|\nabla\bar{v}\|(\|\nabla\bar{v}\|^{\frac{1}{2}}\|\bar{v}\|^{\frac{1}{2}} + \|\bar{v}\|) + C(\|\nabla\bar{v}\|^{\frac{3}{4}}\|\bar{v}\|^{\frac{1}{4}} + \|\bar{v}\|)^2 \\
\le \ & \varepsilon\|\nabla\bar{v}\|^2 + C\|\bar{v}\|^2. \qquad (3.16)
\end{aligned}$$

$$\begin{aligned}
& |(\Delta d_2\cdot\nabla\bar{d},\bar{v})| + |(\Delta d_2\otimes\bar{d},\nabla\bar{v})| \\
\le \ & \|\Delta d_2\|\|\nabla\bar{d}\|_{L^3}\|\bar{v}\|_{L^6} + \|\Delta d_2\|\|\bar{d}\|_{L^\infty}\|\nabla\bar{v}\| \\
\le \ & C(\|\Delta\bar{d}\|^{\frac{1}{2}}\|\nabla\bar{d}\|^{\frac{1}{2}} + \|\nabla\bar{d}\|)(\|\nabla\bar{v}\| + \|\bar{v}\|) + C(\|\Delta\bar{d}\|^{\frac{3}{4}}\|\bar{d}\|^{\frac{1}{4}} + \|\bar{d}\|)\|\nabla\bar{v}\| \\
\le \ & \varepsilon(\|\Delta\bar{d}\|^2 + \|\nabla\bar{v}\|^2) + C(\|\bar{d}\|_{H^1}^2 + \|\bar{v}\|^2). \qquad (3.17)
\end{aligned}$$

$$\begin{aligned}
& |((f(d_1)-f(d_2))\otimes d_1,\nabla\bar{v})| + |(f(d_2)\otimes\bar{d},\nabla\bar{v})| + |(f(d_1)-f(d_2),\Delta\bar{d})| \\
& +|(f(d_1)-f(d_2),\bar{d})| \\
\le \ & (\|f(d_1)-f(d_2)\|\|d_1\|_{L^\infty} + \|f(d_2)\|_{L^\infty}\|\bar{d}\|)\|\nabla\bar{v}\| + \|f(d_1)-f(d_2)\|(\|\Delta\bar{d}\| + \|\bar{d}\|) \\
\le \ & C(\|f'(\xi)\|_{L^\infty} + 1)\|\bar{d}\|\|\nabla\bar{v}\| + C\|f'(\xi)\|_{L^\infty}\|\bar{d}\|(\|\Delta\bar{d}\| + \|\bar{d}\|) \\
\le \ & \varepsilon(\|\nabla\bar{v}\|^2 + \|\Delta\bar{d}\|^2) + C\|\bar{d}\|^2, \qquad (3.18)
\end{aligned}$$

where $\xi = ad_1 + (1-a)d_2$ with $a \in [0,1]$.

$$\begin{aligned}
& |(v_2\cdot\nabla\bar{d},\Delta\bar{d})| + |(\bar{d}\cdot\nabla v_2,\Delta\bar{d})| \\
\le \ & \|v_2\|_{L^6}\|\nabla\bar{d}\|_{L^3}\|\Delta\bar{d}\| + \|\nabla v_2\|\|\bar{d}\|_{L^\infty}\|\Delta\bar{d}\| \\
\le \ & C(\|\Delta\bar{d}\|^{\frac{1}{2}}\|\nabla\bar{d}\|^{\frac{1}{2}} + \|\nabla\bar{d}\|)\|\Delta\bar{d}\| + C(\|\Delta\bar{d}\|^{\frac{3}{4}}\|\bar{d}\|^{\frac{1}{4}} + \|\bar{d}\|)\|\Delta\bar{d}\| \\
\le \ & \varepsilon\|\Delta\bar{d}\|^2 + C\|\bar{d}\|_{H^1}^2. \qquad (3.19)
\end{aligned}$$

$$\begin{aligned}
& |(\bar{v}\cdot\nabla d_1,\bar{d})| + |(v_2\cdot\nabla\bar{d},\bar{d})| + |(d_1\cdot\nabla\bar{v},\bar{d})| + |(\bar{d}\cdot\nabla v_2,\bar{d})| \\
\le \ & \|\nabla d_1\|_{L^3}\|\bar{v}\|\|\bar{d}\|_{L^6} + \|v_2\|_{L^6}\|\nabla\bar{d}\|\|\bar{d}\|_{L^3} + \|d_1\|_{L^\infty}\|\nabla\bar{v}\|\|\bar{d}\| + \|\nabla v_2\|\|\bar{d}\|_{L^4}^2 \\
\le \ & \varepsilon\|\nabla\bar{v}\|^2 + C(\|v\|^2 + \|\bar{d}\|_{H^1}^2). \qquad (3.20)
\end{aligned}$$

Choosing ε small enough in the above estimates, we infer from (3.15) that

$$\frac{d}{dt}(\|\bar{v}\|^2 + \|\bar{d}\|_{H^1}^2) + \nu\|\nabla\bar{v}\|^2 + \|\Delta\bar{d}\|^2 \le C(\|\bar{v}\|^2 + \|\bar{d}\|_{H^1}^2), \tag{3.21}$$

where C is a constant depending on $\|v_i\|_{H^1}$ and $\|d_i\|_{H^2}$ but not on t. By Gronwall's inequality, we can see that for any $t \in [0,T]$,

$$\|\bar{v}(t)\|^2 + \|\bar{d}(t)\|_{H^1}^2 + \int_0^t (\nu\|\nabla\bar{v}(\tau)\|^2 + \|\Delta\bar{d}(\tau)\|^2)d\tau \le 2e^{Ct}(\|\bar{v}(0)\|^2 + \|\bar{d}(0)\|_{H^1}^2). \tag{3.22}$$

The proof is complete. $\square$

Remark 3.1. *Since the global classical solution (v,d) to problem* (1.1)–(1.5) *obtained in Proposition 3.1 is uniformly bounded in $V \times H^2$ (cf. [34], see also Lemma 4.2 and Lemma 5.1 below), it immediately follows from Lemma 3.2 that this global solution is unique.*

Next, we look at the following elliptic periodic boundary value problem

$$\begin{cases} -\Delta d + f(d) = 0, & x \in Q, \\ d(x+e_i) = d(x), & x \in \partial Q. \end{cases} \tag{3.23}$$

Define

$$E(d) := \frac{1}{2}\|\nabla d\|^2 + \int_Q F(d)dx. \tag{3.24}$$

It is not difficult to see that the solution to (3.23) is a critical point of $E(d)$ and conversely the critical point of $E(d)$ is a solution to (3.23).

In order to apply the Łojasiewicz–Simon approach to prove the convergence to equilibrium, we shall introduce a suitable Łojasiewicz–Simon type inequality that is related to our problem. In particular, we have

Lemma 3.3. [Łojasiewicz–Simon inequality] *Let ψ be a critical point of $E(d)$. Then there exist constants $\theta \in (0, \frac{1}{2})$ and $\beta > 0$ depending on ψ such that for any $d \in H_p^1(Q)$ satisfying $\|d-\psi\|_{H_p^1(Q)} < \beta$, it holds*

$$\|-\Delta d + f(d)\|_{(H_p^1(Q))'} \ge |E(d) - E(\psi)|^{1-\theta}, \tag{3.25}$$

where $(H_p^1(Q))'$ is the dual space of $H_p^1(Q)$.

Remark 3.2. *Lemma 3.3 can be viewed as an extended version of Simon's result [33] for scalar functions using the L^2-norm. For the proof of this result, we refer to [14, Chapter 2, Theorem 5.2].*

The following lemma turns out to useful in the study of long-time behavior of solutions to evolution equations.

Lemma 3.4. [41, Lemma 6.2.1] *Let T be given with $0 < T \le +\infty$. Suppose that $y(t)$ and $h(t)$ are nonnegative continuous functions defined on $[0,T]$, which satisfy the following conditions:*

$$\frac{dy(t)}{dt} \le c_1 y(t)^2 + c_2 + h(t), \quad \int_0^T y(t)dt \le c_3, \quad \int_0^T h(t)dt \le c_4,$$

where c_i $(i = 1, 2, 3, 4)$ are given nonnegative constants. Then for any $r \in (0, T)$, the following estimate holds:

$$y(t+r) \le \Big(\frac{c_3}{r} + c_2 r + c_4\Big) e^{c_1 c_3}, \quad \forall\, t \in [0, T-r].$$

Furthermore, if $T = +\infty$, then

$$\lim_{y\to+\infty} y(t) = 0.$$

4 Long-time Behavior in Two Dimensional Case

In this section, we prove the convergence of global solutions to single steady states as time tends to infinity in the two dimensional case. In 2-D case, an important property for the global solution to problem (1.1)–(1.5) is the following high-order energy inequality, which played a crucial role in the proof of global existence result in [34].

Denote

$$A(t) = \|\nabla v(t)\|^2 + \lambda \|\Delta d(t) - f(d(t))\|^2. \tag{4.1}$$

We recall that it has been assumed that $\lambda = \gamma = 1$. Besides, since viscosity ν does not play a crucial role in the 2-D case, we also set $\nu = 1$ in this section for the sake of simplicity. Then we have (cf. [34, (45)])

Lemma 4.1. *In two dimensional case, the following inequality holds for the classical solution (v, d) to problem (1.1)–(1.5):*

$$\frac{d}{dt}A(t) + \frac{1}{2}(\|\Delta v(t)\|^2 + \|\nabla(\Delta d(t) - f(d(t)))\|^2) \le C(A^2(t) + 1), \quad \forall\, t \ge 0, \tag{4.2}$$

where C is a constant depending on $f, Q, \|v_0\|, \|d_0\|_{H^1(Q)}$.

4.1 Convergence to Equilibrium

Based on the high-order energy inequality (4.2), we are able to show the decay property of the velocity field v.

Lemma 4.2. *For any $t \ge 0$, the following uniform estimate holds*

$$\|v(t)\|_{H^1} + \|d(t)\|_{H^2} \le C, \tag{4.3}$$

where C is a constant depending on $f, Q, \|v_0\|_{H^1}, \|d_0\|_{H^2}$. Furthermore,

$$\lim_{t\to+\infty} (\|v(t)\|_{H^1} + \| - \Delta d(t) + f(d(t))\|) = 0. \tag{4.4}$$

Proof. It follows from the basic energy law (3.4) that

$$\mathcal{E}(t) + \int_0^t A(\tau) d\tau \le \mathcal{E}(0) < \infty, \quad \forall\, t \ge 0. \tag{4.5}$$

(4.5) implies the uniform estimate

$$\|v(t)\| + \|d(t)\|_{H^1} \le C, \quad \forall\, t \ge 0. \tag{4.6}$$

Concerning the uniform bound (4.3), we take $r = 1$ in Lemma 3.4 to get

$$\|\nabla v(t)\| + \| - \Delta d(t) + f(d(t))\| \le C, \quad \forall\, t \ge 1, \tag{4.7}$$

where C does not depend on t. On the other hand, for any $t \in [0,1]$, it follows from (4.2) and the fact $\int_0^1 A(t)dt \le C$ that

$$\sup_{0\le t\le 1} A(t) \le e^{\int_0^1 A(t)dt} A(0) + C \le C. \tag{4.8}$$

Besides, from the continuous embedding $H^1 \hookrightarrow L^p$ $(1 \le p < \infty)$ and (4.6) we have

$$\|\Delta d\| \le \| - \Delta d + f(d)\| + \|f(d)\| \le \| - \Delta d + f(d)\| + C\left(1 + \|d\|^3_{L^6}\right) \le C. \tag{4.9}$$

Therefore, (4.3) follows from (4.7)–(4.9). Furthermore, (4.5) together with Lemma 4.1 and Lemma 3.4 yields that

$$\lim_{t\to+\infty} (\|\nabla v(t)\| + \|\Delta d(t) - f(d(t))\|) = 0. \tag{4.10}$$

By the Poincaré inequality for $v \in V$, we conclude (4.4). The proof is complete. $\square$

Let $\mathcal{S}$ be the set

$$\mathcal{S} = \{(0,u) \mid -\Delta u + f(u) = 0, \text{ in } Q,\ u(x + e_i) = u(x) \text{ on } \partial Q\}.$$

The ω-limit set of $(v_0, d_0) \in V \times H^2_p(Q) \subset L^2_p(Q) \times H^1_p(Q)$ is defined as follows:

$$\begin{aligned}\omega(v_0, d_0) \quad = \quad & \{(v_\infty(x), d_\infty(x)) \mid \text{ there exists } \{t_n\} \nearrow \infty \text{ such that} \\ & (v(t_n), d(t_n)) \to (v_\infty, d_\infty) \text{ in } L^2(Q) \times H^1(Q),\ \text{as } t_n \to +\infty\}.\end{aligned}$$

We infer from Lemma 4.2 that

Proposition 4.1. $\omega(v_0, d_0)$ *is a nonempty bounded subset in* $H^1_p(Q) \times H^2_p(Q)$*, which is compact in* $L^2_p(Q)\times H^1_p(Q)$*. Besides, all asymptotic limiting points* (v_∞, d_∞) *of problem* (1.1)–(1.5) *belong to* $\mathcal{S}$*. In other words,* $\omega(v_0, d_0) \subset \mathcal{S}$*.*

In what follows, we prove the convergence for director field d. For any initial data $(v_0, d_0) \in V \times H^2_p(Q)$, it follows from Lemma 4.2 that $\|d\|_{H^2}$ is uniformly bounded. Proposition 4.1 implies that there is an increasing unbounded sequence $\{t_n\}_{n\in\mathbb{N}}$ and a function $d_\infty \in H^2_p(Q)$ such that

$$\lim_{t_n\to+\infty} \|d(t_n) - d_\infty\|_{H^1} = 0. \tag{4.11}$$

Moreover, d_∞ satisfies the equation

$$-\Delta d_\infty + f(d_\infty) = 0, \quad x \in \Omega, \quad d_\infty(x + e_i) = d_\infty(x) \text{ on } \partial Q. \tag{4.12}$$

We prove the convergence result following a simple argument first introduced in [16] (see also Section 2), in which the key observation is that after a certain time t_0, $d(t)$ will fall into a certain small neighborhood of d_∞ and stay there forever.

From the basic energy law (3.4), we can see that $\mathcal{E}(t)$ is decreasing on $[0,\infty)$, and it has a finite limit as time goes to infinity because it is nonnegative. Therefore, it follows from (4.4) and (4.11) that

$$\lim_{t_n\to+\infty} \mathcal{E}(t_n) = E(d_\infty). \tag{4.13}$$

On the other hand, we can infer from (3.4) that $\mathcal{E}(t) \geq E(d_\infty)$, for all $t > 0$, and the equal sign holds if and only if, for all $t > 0$, $v = 0$ and d solves problem (4.12).

As for the ODE system in Section 2, we now consider all possibilities.

Case 1. If there is a $t_0 > 0$ such that $\mathcal{E}(t_0) = E(d_\infty)$, then for all $t > t_0$, we deduce from (3.4) that

$$\|\nabla v\| \equiv 0, \quad \| - \Delta d + f(d)\| \equiv 0. \tag{4.14}$$

It follows from (1.3), (4.14) and (4.20) that for $t \geq t_0$, $\|d_t\| = 0$. Namely, d is independent of time for all $t \geq t_0$. Due to (4.11), we conclude that $d(t) \equiv d_\infty$ for $t \geq t_0$.

Case 2. For all $t > 0$, we suppose that $\mathcal{E}(t) > E(d_\infty)$. First we assume that the following claim holds true.

Proposition 4.2. *There is a $t_0 > 0$ such that for all $t \geq t_0$, $\|d(t) - d_\infty\|_{H^1} < \beta$. Namely, for all $t \geq t_0$, $d(t)$ satisfies the condition in Lemma 3.3.*

In this case, it follows from Lemma 3.3 that

$$|E(d) - E(d_\infty)|^{1-\theta} \leq \| - \Delta d + f(d)\|_{(H^1_p)'} \leq \| - \Delta d + f(d)\|, \quad \forall\, t \geq t_0. \tag{4.15}$$

The fact $\theta \in (0, \frac{1}{2})$ implies that $0 < 1 - \theta < 1$, $2(1 - \theta) > 1$. Consequently,

$$\|v\|^{2(1-\theta)} = \|v\|^{2(1-\theta)-1}\|v\| \leq C\|v\|.$$

Then we infer from the basic inequality

$$(a + b)^{1-\theta} \leq a^{1-\theta} + b^{1-\theta}, \qquad \forall\, a, b \geq 0$$

that

$$\begin{aligned}
(\mathcal{E}(t) - E(d_\infty))^{1-\theta} &\leq \left(\frac{1}{2}\|v\|^2 + |E(d) - E(d_\infty)|\right)^{1-\theta} \\
&\leq \left(\frac{1}{2}\|v\|^2 + \| - \Delta d + f(d)\|^{\frac{1}{1-\theta}}\right)^{1-\theta} \\
&\leq \left(\frac{1}{2}\right)^{1-\theta}\|v\|^{2(1-\theta)} + \| - \Delta d + f(d)\| \\
&\leq C\|v\| + \| - \Delta d + f(d)\|.
\end{aligned} \tag{4.16}$$

Therefore, a direct calculation yields that

$$\begin{aligned}
-\frac{d}{dt}(\mathcal{E}(t) - E(d_\infty))^\theta &= -\theta(\mathcal{E}(t) - E(d_\infty))^{\theta-1}\frac{d}{dt}\mathcal{E}(t) \\
&\geq \frac{C\theta(\|\nabla v\| + \| - \Delta d + f(d)\|)^2}{C\|v\| + \| - \Delta d + f(d)\|} \\
&\geq C_1(\|\nabla v\| + \| - \Delta d + f(d)\|), \quad \forall\, t \geq t_0,
\end{aligned} \tag{4.17}$$

where C_1 is a constant depending on v_0, d_0, Q and θ.
Integrating from t_0 to t, we get

$$(\mathcal{E}(t) - E(d_\infty))^\theta + C_1\int_{t_0}^t (\|\nabla v(\tau)\| + \| - \Delta d(\tau) + f(d(\tau))\|)d\tau$$

$$\leq \quad (\mathcal{E}(t_0) - E(d_\infty))^\theta < \infty, \quad \forall\, t \geq t_0. \tag{4.18}$$

Since $\mathcal{E}(t) - E(d_\infty) \geq 0$, we conclude that

$$\int_{t_0}^{\infty} (\|\nabla v(\tau)\| + \| - \Delta d(\tau) + f(d(\tau))\|) d\tau < \infty. \tag{4.19}$$

On the other hand, it follows from equation (1.3), (4.3) and Sobolev embedding theorems that

$$\begin{aligned} \|d_t\| &\leq \|v \cdot \nabla d\| + \|d \cdot \nabla v\| + \| - \Delta d + f(d)\| \\ &\leq \|v\|_{L^4} \|\nabla d\|_{L^4} + \|d\|_{L^\infty} \|\nabla v\| + \| - \Delta d + f(d)\| \\ &\leq C\|\nabla v\| + \| - \Delta d + f(d)\|. \end{aligned} \tag{4.20}$$

Hence,

$$\int_{t_0}^{\infty} \|d_t(\tau)\| d\tau < +\infty, \tag{4.21}$$

which easily implies that as $t \to +\infty$, $d(t)$ converges in $L^2(Q)$. This and (4.11) indicate that

$$\lim_{t \to +\infty} \|d(t) - d_\infty\| = 0. \tag{4.22}$$

Since $d(t)$ is uniformly bounded in $H^2(Q)$ (cf. (4.3)), by standard interpolation inequality we have

$$\lim_{t \to +\infty} \|d(t) - d_\infty\|_{H^1} = 0. \tag{4.23}$$

On the other hand, the uniform bound of d in $H^2(Q)$ implies the weak convergence

$$d(t) \rightharpoonup d_\infty, \quad \text{in } H^2(Q). \tag{4.24}$$

However, the decay property of the quantity $A(t)$ (cf. Lemma 4.2) could tell us more. Namely, we could get strong convergence of d in H^2. To see this, we keep in mind that

$$\begin{aligned} \|\Delta d - \Delta d_\infty\| &\leq \|\Delta d - \Delta d_\infty - f(d) + f(d_\infty)\| + \|f(d) - f(d_\infty)\| \\ &\leq \|\Delta d - f(d)\| + \|f'(\xi)\|_{L^4} \|d - d_\infty\|_{L^4} \\ &\leq \|\Delta d - f(d)\| + C\|d - d_\infty\|_{H^1}. \end{aligned} \tag{4.25}$$

The above estimate together with (4.4) and (4.23) yields

$$\lim_{t \to +\infty} \|d(t) - d_\infty\|_{H^2} = 0. \tag{4.26}$$

To finish the proof, it remains to show that Proposition 4.2 always holds true for the global solution $d(t)$ to system (1.1)–(1.5). Define

$$\bar{t}_n = \sup\{\, t > t_n |\ \|d(s) - d_\infty\|_{H^1} < \beta,\ \forall\, s \in [t_n, t]\}. \tag{4.27}$$

It follows from (4.11) that for any $\varepsilon \in (0, \beta)$, there exists an integer N such that when $n \geq N$,

$$\|d(t_n) - d_\infty\|_{H^1} < \varepsilon, \tag{4.28}$$

$$\frac{1}{C_1} (\mathcal{E}(t_n) - E(d_\infty))^\theta < \varepsilon. \tag{4.29}$$

On the other hand, we know that the orbit of the solution d is continuous in H^1. It follows from (4.3) that $d \in L^\infty(0, +\infty; H^2)$. As a consequence, $d \in L^2(t, t+1; H^2)$ for any $t \geq 0$. The basic energy law and (4.20) imply $d_t \in L^2(t, t+1; L^2)$. Thus, for any $t \geq 0$, it holds $d \in C([t, t+1]; H^1)$. The continuity of the orbit of d in H^1 and (4.28) yield that

$$\bar{t}_n > t_n, \quad \text{for all} \;\; n \geq N. \tag{4.30}$$

Then there are two possibilities:

(i). If there exists $n_0 \geq N$ such that $\bar{t}_{n_0} = +\infty$, then from the previous discussions in Case 1 and Case 2, the theorem is proved.

(ii) Otherwise, for all $n \geq N$, we have $t_n < \bar{t}_n < +\infty$, and for all $t \in [t_n, \bar{t}_n]$, $E(d_\infty) < \mathcal{E}(t)$. Then from (4.18) with t_0 being replaced by t_n, and t being replaced by $\bar{t}_n$, we obtain from (4.29) that

$$\int_{t_n}^{\bar{t}_n} (\|\nabla v(\tau)\| + \| - \Delta d(\tau) + f(d(\tau))\|) d\tau < \varepsilon. \tag{4.31}$$

Thus, it follows that (cf. (4.20))

$$\begin{aligned} \|d(\bar{t}_n) - d_\infty\| &\leq \|d(t_n) - d_\infty\| + \int_{t_n}^{\bar{t}_n} \|d_t(\tau)\| d\tau \\ &\leq \|d(t_n) - d_\infty\| + C \int_{t_n}^{\bar{t}_n} (\|\nabla v(\tau)\| + \| - \Delta d(\tau) + f(d(\tau))\|) d\tau \\ &< C\varepsilon, \end{aligned} \tag{4.32}$$

which implies that $\lim_{n\to+\infty} \|d(\bar{t}_n) - d_\infty\| = 0$. Since $d(t)$ is relatively compact in H^1, there exists a subsequence of $\{d(\bar{t}_n)\}$, still denoted by $\{d(\bar{t}_n)\}$ converging to d_∞ in H^1, i.e., when n is sufficiently large,

$$\|d(\bar{t}_n) - d_\infty\|_{H^1} < \beta,$$

which contradicts the definition of $\bar{t}_n$ that $\|d(\bar{t}_n) - d_\infty\|_{H^1} = \beta$.

Summing up, we have considered all the possible cases and the conclusion (1.6) is proved. □

4.2 Convergence Rate

In this part, we shall prove the estimate for convergence rate (1.8). This can be achieved by several steps. Notice that compared to small molecule system (cf. [39]), because of the differences lying in the higher order energy inequality, more work on the estimate is needed here.

Step 1. As has been shown in the literature (cf. [12, 41]), an estimate on the convergence rate in certain lower–order norm could be obtained directly from the Łojasiewicz–Simon approach. From Lemma 3.3 and (4.17), we have

$$\frac{d}{dt}(\mathcal{E}(t) - E(d_\infty)) + C_1(\mathcal{E}(t) - E(d_\infty))^{2(1-\theta)} \leq 0, \quad \forall\, t \geq t_0, \tag{4.33}$$

which implies

$$\mathcal{E}(t) - E(d_\infty) \leq C(1+t)^{-\frac{1}{1-2\theta}}, \quad \forall\, t \geq t_0. \tag{4.34}$$

Integrating (4.17) on (t, ∞), where $t \geq t_0$, it follows from (4.20) that

$$\int_t^\infty \|d_t(\tau)\| d\tau \leq \int_t^\infty (C\|\nabla v(\tau)\| + \| - \Delta d(\tau) + f(d(\tau))\|) d\tau \leq C(1+t)^{-\frac{\theta}{1-2\theta}}. \tag{4.35}$$

By adjusting the constant C properly, we obtain

$$\|d(t) - d_\infty\| \le C(1+t)^{-\frac{\theta}{1-2\theta}}, \quad t \ge 0. \tag{4.36}$$

Step 2. In Step 1, we only obtain the convergence rate of d (in L^2). Unlike for the temperature variable in some phase–field systems (cf. [38] and references cited therein), although we have got some decay information for the velocity field v such that

$$\int_t^\infty \|\nabla v(\tau)\| d\tau \le C(1+t)^{-\frac{\theta}{1-2\theta}}, \tag{4.37}$$

we are not able to prove convergence rate of v directly. This is because now v satisfies a Navier–Stokes equation which is much more complicated than the heat equation for the temperature variable in phase–field systems. As a result, one cannot easily obtain relation between $\|\nabla v\|$ and v_t (in certain possible norm) from the equation (1.1) itself. However, it is possible to achieve our goal by using the idea in [38] where we use higher–order energy estimates and construct proper differential inequalities (cf. also [8, 37, 39]). Besides, the convergence rate of d in higher order norm can be proved simultaneously.

The steady state corresponding to problem (1.1)–(1.5) formally satisfies the following system:

$$\begin{aligned} v_\infty \cdot \nabla v_\infty - \nu \Delta v_\infty + \nabla P_\infty &= -\nabla \cdot [\nabla d_\infty \odot \nabla d_\infty + (\Delta d_\infty - f(d_\infty)) \otimes d_\infty], && (4.38)\\ \nabla \cdot v_\infty &= 0, && (4.39)\\ v_\infty \cdot \nabla d_\infty - d_\infty \cdot \nabla v_\infty &= \Delta d_\infty - f(d_\infty), && (4.40) \end{aligned}$$

with periodic boundary conditions. Lemma 4.2 implies that all limiting points of system (1.1)–(1.5) have the form $(0, d_\infty) \in \mathcal{S}$. As a result, system (4.38)–(4.40) can be reduced to

$$\begin{aligned} \nabla P_\infty + \nabla \left(\frac{|\nabla d_\infty|^2}{2} \right) &= -\nabla d_\infty \cdot \Delta d_\infty, && (4.41)\\ -\Delta d_\infty + f(d_\infty) &= 0. && (4.42) \end{aligned}$$

In (4.41), we use the fact that

$$\nabla \cdot (\nabla d_\infty \odot \nabla d_\infty) = \nabla \left(\frac{|\nabla d_\infty|^2}{2} \right) + \nabla d_\infty \cdot \Delta d_\infty.$$

As in [39], subtracting the stationary problem (4.41)–(4.42) from the evolution problem (1.1)–(1.3), we obtain that

$$\begin{aligned} & v_t + v \cdot \nabla v - \nu \Delta v + \nabla (P - P_\infty) + \nabla \left(\left(\frac{|\nabla d|^2}{2} \right) - \left(\frac{|\nabla d_\infty|^2}{2} \right) \right) \\ = \; & -\nabla \cdot [(\Delta d - f(d)) \otimes d] - \nabla d \cdot \Delta d + \nabla d_\infty \cdot \Delta d_\infty, && (4.43)\\ & \nabla \cdot v = 0, && (4.44)\\ & d_t + v \cdot \nabla d - d \cdot \nabla v = \Delta (d - d_\infty) - f(d) + f(d_\infty). && (4.45) \end{aligned}$$

Multiplying (4.43) by v and (4.45) by $-\Delta d + f(d) = -\Delta(d - d_\infty) + f(d) - f(d_\infty)$, respectively, integrating over Q, and adding the results together, we have

$$\frac{d}{dt} \left(\frac{1}{2} \|v\|^2 + \frac{1}{2} \|\nabla d - \nabla d_\infty\|^2 + \int_Q [F(d) - F(d_\infty) - f(d_\infty)(d - d_\infty)] dx \right)$$

$$
\begin{aligned}
& +\nu\|\nabla v\|^2+\|\Delta d-f(d)\|^2 \\
= \ & (v,\nabla d_\infty\cdot\Delta d_\infty)=(v,\nabla d_\infty\cdot(\Delta d_\infty-f(d_\infty)))+(v,\nabla F(d_\infty)) \\
= \ & 0.
\end{aligned}
\tag{4.46}
$$

Multiplying (4.45) by $d-d_\infty$ and integrating in Ω, we obtain

$$
\begin{aligned}
& \frac{1}{2}\frac{d}{dt}\|d-d_\infty\|^2+\|\nabla(d-d_\infty)\|^2 \\
= \ & -(v\cdot\nabla d, d-d_\infty)+(d\cdot\nabla v, d-d_\infty)-(f(d)-f(d_\infty), d-d_\infty):=I_1.
\end{aligned}
\tag{4.47}
$$

The right hand side can be estimated as follows

$$
\begin{aligned}
|I_1| \ \leq \ & \|v\|_{L^4}\|\nabla d\|_{L^4}\|d-d_\infty\|+\|\nabla v\|\|d\|_{L^\infty}\|d-d_\infty\|+\|f'(\xi)\|_{L^\infty}\|d-d_\infty\|^2 \\
\leq \ & C\|\nabla v\|\|d-d_\infty\|+C(\|\nabla(d-d_\infty)\|^{\frac{1}{2}}\|d-d_\infty\|^{\frac{1}{2}}+\|d-d_\infty\|)^2 \\
\leq \ & \varepsilon_1\|\nabla v\|^2+\frac{1}{2}\|\nabla(d-d_\infty)\|^2+C\|d-d_\infty\|^2.
\end{aligned}
\tag{4.48}
$$

Multiplying (4.47) by $\alpha>0$ and adding the resultant to (4.46), using (4.48), we get

$$
\begin{aligned}
& \frac{d}{dt}\left(\frac{1}{2}\|v\|^2+\frac{1}{2}\|\nabla d-\nabla d_\infty\|^2+\frac{\alpha}{2}\|d-d_\infty\|^2+\int_Q(F(d)-F(d_\infty))dx\right. \\
& \left.-\int_Q f(d_\infty)(d-d_\infty)dx\right)+(\nu-\alpha\varepsilon_1)\|\nabla v\|^2+\|\Delta d-f(d)\|^2+\frac{\alpha}{2}\|\nabla(d-d_\infty)\|^2 \\
\leq \ & C\alpha\|d-d_\infty\|^2.
\end{aligned}
\tag{4.49}
$$

On the other hand, by the Taylor's expansion

$$
F(d)=F(d_\infty)+f(d_\infty)(d-d_\infty)+f'(\xi)(d-d_\infty)^2,
\tag{4.50}
$$

where $\xi=ad+(1-a)d_\infty$ with $a\in[0,1]$.
Then we deduce that

$$
\begin{aligned}
& \left|\int_Q[F(d)-F(d_\infty)dx-f(d_\infty)(d-d_\infty)]dx\right|=\left|\int_Q f'(\xi)(d-d_\infty)^2dx\right| \\
\leq \ & \|f'(\xi)\|_{L^\infty}\|d-d_\infty\|^2\leq C_2\|d-d_\infty\|^2.
\end{aligned}
\tag{4.51}
$$

Let us define now, for $t\geq 0$,

$$
\begin{aligned}
y(t) \ = \ & \frac{1}{2}\|v(t)\|^2+\frac{1}{2}\|\nabla d(t)-\nabla d_\infty\|^2+\frac{\alpha}{2}\|d(t)-d_\infty\|^2+\int_Q(F(d(t))dx-F(d_\infty))dx \\
& -\int_Q f(d_\infty)(d(t)-d_\infty)dx.
\end{aligned}
\tag{4.52}
$$

In (4.49) and (4.52), we choose

$$
\alpha\geq 1+2C_2>0, \quad \varepsilon_1=\frac{\nu}{2\alpha}.
\tag{4.53}
$$

As a result,

$$
y(t)+C_2\|d-d_\infty\|^2\geq\frac{1}{2}(\|v\|^2+\|d-d_\infty\|_{H^1}^2).
\tag{4.54}
$$

Furthermore, we infer from (4.54) that for certain constants $C_3, C_4 > 0$,

$$\frac{d}{dt}y(t) + C_3 y(t) \le C_4\|d - d_\infty\|^2 \le C(1+t)^{-\frac{2\theta}{1-2\theta}}. \tag{4.55}$$

By Gronwall's inequality, we have (cf. [37,38])

$$y(t) \le C(1+t)^{-\frac{2\theta}{1-2\theta}}, \quad \forall\, t \ge 0, \tag{4.56}$$

which together with (4.54) implies that

$$\|v(t)\| + \|d(t) - d_\infty\|_{H^1} \le C(1+t)^{-\frac{\theta}{1-2\theta}}, \quad \forall\, t \ge 0. \tag{4.57}$$

Step 3. In the last step, we shall prove the convergence rate in the same space where the initial data stay. In Section 3.1, it has been shown that, once we could obtain the uniform bound of d in H^2, we are able to obtain strong convergence of d in H^2 instead of weak convergence. By reinvestigating the higher-order energy estimate for the subtracted system (4.43)–(4.45) (cf. Lemma 4.1), we can obtain a further result, which provides the same rate estimate of (v, d) in $H^1 \times H^2$ as (4.57).

Taking the time derivative of $A(t)$, we obtain (cf. also [34, (43)])

$$\begin{aligned}
& \frac{1}{2}\frac{d}{dt}A(t) + (\nu\|\Delta v\|^2 + \|\nabla(\Delta d - f(d)\|^2) \\
= \; & (\Delta v, v\cdot\nabla v) + (\Delta v, \nabla\cdot(\nabla d \odot \nabla d)) + (\Delta v, \nabla\cdot((\Delta d - f(d)) \otimes d)) \\
& + (\nabla(\Delta d - f(d)), \nabla(v\cdot\nabla d)) - (\nabla(\Delta d - f(d)), \nabla(d\cdot\nabla v)) \\
& + (\Delta d - f(d), f'(d)d_t) \\
= \; & I_2 + \dots + I_7.
\end{aligned} \tag{4.58}$$

Noticing that we have got uniform bounds for $\|v\|_{H^1}$ and $\|d\|_{H^2}$ before (see Lemma 4.2), in what follows we estimate I_i $(i = 2, \dots, 7)$ term by term.

$$\begin{aligned}
|I_2| &= |(\Delta v, v\cdot\nabla v)| \le \|\Delta v\|\|v\|_{L^\infty}\|\nabla v\| C\|\Delta v\|^{\frac{3}{2}}\|\nabla v\|^{\frac{3}{2}} \\
&\le \; \le C\|\Delta v\|^{\frac{3}{2}}\|\nabla v\|^{\frac{1}{2}} \le \varepsilon\|\Delta v\|^2 + C_\varepsilon\|\nabla v\|^2.
\end{aligned} \tag{4.59}$$

$$\begin{aligned}
|I_3| &= |(\Delta v, \nabla\cdot(\nabla d \odot \nabla d))| = |(\Delta v, \nabla\cdot(\nabla d \odot \nabla d)) - (\Delta v, \nabla P_\infty)| \\
&= |(\Delta v, \nabla d\cdot\Delta d) - (\Delta v, \nabla d_\infty \cdot \Delta d_\infty)| \\
&\le |(\Delta v, \nabla d\cdot\Delta d) - (\Delta v, \nabla d\cdot\Delta d_\infty)| + |(\Delta v, \nabla d\cdot\Delta d_\infty) - (\Delta v, \nabla d_\infty\cdot\Delta d_\infty)| \\
&=: I_{3a} + I_{3b}.
\end{aligned} \tag{4.60}$$

Since

$$\begin{aligned}
\|\nabla\Delta d\| &\le \|\nabla(\Delta d - f(d))\| + \|f'(d)\nabla d\| \le \|\nabla(\Delta d - f(d))\| + \|f'(d)\|_{L^4}\|\nabla d\|_{L^4} \\
&\le \|\nabla(\Delta d - f(d))\| + C\|\Delta d\|^{\frac{1}{2}}\|\nabla d\|^{\frac{1}{2}} \le \|\nabla(\Delta d - f(d))\| + C,
\end{aligned} \tag{4.61}$$

we have

$$I_{3a} = |(\Delta v, \nabla d\cdot\Delta d) - (\Delta v, \nabla d\cdot\Delta d_\infty)| \le \|\Delta v\|\|\nabla d\|_{L^\infty}\|\Delta d - \Delta d_\infty\|$$

$$
\begin{aligned}
&\le C\|\Delta v\|\|\Delta d-\Delta d_\infty\|(\|\nabla d\|^{\frac12}\|\nabla\Delta d\|^{\frac12}+\|\nabla d\|)\\
&\le C\|\Delta v\|(\|\Delta d-f(d)\|+\|f(d)-f(d_\infty)\|)(\|\nabla(\Delta d-f(d))\|+C)^{\frac12}\\
&\le \varepsilon\|\Delta v\|^2+C_\varepsilon(\|\Delta d-f(d)\|+\|f(d)-f(d_\infty)\|)^2(\|\nabla(\Delta d-f(d))\|+C)\\
&\le \varepsilon\|\Delta v\|^2+\varepsilon\|\nabla(\Delta d-f(d))\|^2\\
&\quad +C_\varepsilon[1+(\|\Delta d-f(d)\|+\|f(d)-f(d_\infty)\|)^2](\|\Delta d-f(d)\|+\|f(d)-f(d_\infty)\|)^2\\
&\le \varepsilon\|\Delta v\|^2+\varepsilon\|\nabla(\Delta d-f(d))\|^2+C_\varepsilon(\|\Delta d-f(d)\|+\|f(d)-f(d_\infty)\|)^2\\
&\le \varepsilon\|\Delta v\|^2+\varepsilon\|\nabla(\Delta d-f(d))\|^2+C_\varepsilon\|\Delta d-f(d)\|^2+C_\varepsilon\|d-d_\infty\|^2, \qquad (4.62)
\end{aligned}
$$

$$
\begin{aligned}
I_{3b} &= |(\Delta v,\nabla d\cdot\Delta d_\infty)-(\Delta v,\nabla d_\infty\cdot\Delta d_\infty)|\le\|\Delta v\|\|\nabla(d-d_\infty)\|_{L^\infty}\|\Delta d_\infty\|\\
&\le C\|\Delta v\|(\|\nabla(d-d_\infty)\|^{\frac12}\|\nabla\Delta(d-d_\infty)\|^{\frac12}+\|\nabla(d-d_\infty)\|)\\
&\le \varepsilon\|\Delta v\|^2+C_\varepsilon\|\nabla(d-d_\infty)\|\|\nabla\Delta(d-d_\infty)\|+C_\varepsilon\|\nabla(d-d_\infty)\|^2\\
&\le \varepsilon\|\Delta v\|^2+C_\varepsilon\|\nabla(d-d_\infty)\|\|\nabla(\Delta d-f(d))\|\\
&\quad +C_\varepsilon\|\nabla(d-d_\infty)\|\|\nabla(f(d)-f(d_\infty))\|+C_\varepsilon\|\nabla(d-d_\infty)\|^2\\
&\le \varepsilon\|\Delta v\|^2+\varepsilon\|\nabla(\Delta d-f(d))\|^2+C_\varepsilon\|\nabla(d-d_\infty)\|^2\\
&\quad +C_\varepsilon\|\nabla(d-d_\infty)\|(\|f'(d)\nabla(d-d_\infty)\|+\|(f'(d)-f'(d_\infty))\nabla d_\infty\|)\\
&\le \varepsilon\|\Delta v\|^2+\varepsilon\|\nabla(\Delta d-f(d))\|^2+C_\varepsilon\|\nabla(d-d_\infty)\|^2\\
&\quad +C_\varepsilon\|\nabla(d-d_\infty)\|(\|f'(d)\|_{L^\infty}\|\nabla(d-d_\infty)\|+\|f'(d)-f'(d_\infty)\|_{L^4}\|\nabla d_\infty\|_{L^4})\\
&\le \varepsilon\|\Delta v\|^2+\varepsilon\|\nabla(\Delta d-f(d))\|^2+C_\varepsilon\|\nabla(d-d_\infty)\|^2\\
&\quad +C_\varepsilon\|\nabla(d-d_\infty)\|\|f''(\xi)\|_{L^\infty}\|d-d_\infty\|_{L^4}\\
&\le \varepsilon\|\Delta v\|^2+\varepsilon\|\nabla(\Delta d-f(d))\|^2+C_\varepsilon\|\nabla(d-d_\infty)\|^2+C_\varepsilon\|\nabla(d-d_\infty)\|^{\frac32}\|d-d_\infty\|^{\frac12}\\
&\le \varepsilon\|\Delta v\|^2+\varepsilon\|\nabla(\Delta d-f(d))\|^2+C_\varepsilon\|\nabla(d-d_\infty)\|^2+C_\varepsilon\|d-d_\infty\|^2. \qquad (4.63)
\end{aligned}
$$

Next,

$$
\begin{aligned}
I_4+I_6 &= (\Delta v,\nabla\cdot((\Delta d-f(d))\otimes d))-(\nabla(\Delta d-f(d)),\nabla(d\cdot\nabla v))\\
&= -(d\cdot\nabla\Delta v,\Delta d-f(d))+(\Delta d-f(d),\Delta(d\cdot\nabla v))\\
&= -(d\cdot\nabla\Delta v,\Delta d-f(d))+\int_Q(\Delta d_i-f_i)\Delta(d_j\nabla_jv_i)\,dx\\
&= -(d\cdot\nabla\Delta v,\Delta d-f(d))\\
&\quad +\int_Q(\Delta d_i-f_i)(\Delta d_j\nabla_jv_i+2\nabla_kd_j\nabla_k\nabla_jv_i+d_j\Delta\nabla_jv_i)\,dx\\
&= -(d\cdot\nabla\Delta v,\Delta d-f(d))+\int_Q(\Delta d_i-f_i)(d_j\nabla_j\Delta v_i)\,dx+(\Delta d-f,\Delta d\cdot\nabla v)\\
&\quad +2\int_Q(\Delta d_i-f_i)(\nabla_kd_j\nabla_k\nabla_jv_i)\,dx\\
&= (\Delta d-f(d),\Delta d\cdot\nabla v)+2\int_Q(\Delta d_i-f_i)(\nabla_kd_j\nabla_k\nabla_jv_i)\,dx\\
&= (\Delta d-f(d),\Delta d\cdot\nabla v)+2(\Delta d-f(d),(\nabla d\cdot\nabla)\cdot\nabla v)\\
&=: \tilde I_4+\tilde I_6. \qquad (4.64)
\end{aligned}
$$

$$
|\tilde I_4| \le \|\Delta d-f(d)\|_{L^4}\|\Delta d\|\|\nabla v\|_{L^4}
$$

$$
\begin{aligned}
&\le C(\|\nabla(\Delta d - f(d))\|^{\frac12}\|\Delta d - f(d)\|^{\frac12} + \|\Delta d - f(d)\|)(\|\Delta v\|^{\frac12}\|\nabla v\|^{\frac12} + \|\nabla v\|)\\
&\le \varepsilon(\|\nabla(\Delta d - f(d))\|^2 + \|\Delta v\|^2) + C_\varepsilon(\|\Delta d - f(d)\|^2 + \|\nabla v\|^2).\\
|\tilde I_6| &\le C\|\Delta d - f(d)\|_{L^4}\|\nabla d\|_{L^4}\|D^2 v\|\\
&\le C(\|\nabla(\Delta d - f(d))\|^{\frac12}\|\Delta d - f(d)\|^{\frac12} + \|\Delta d - f(d)\|)\|\Delta v\|\\
&\le \varepsilon(\|\nabla(\Delta d - f(d))\|^2 + \|\Delta v\|^2) + C_\varepsilon\|\Delta d - f(d)\|^2. \qquad (4.65)
\end{aligned}
$$

As a result,

$$
\begin{aligned}
|I_4 + I_6| &= |\tilde I_4 + \tilde I_6| \le |\tilde I_4| + |\tilde I_6|\\
&\le 2\varepsilon(\|\nabla(\Delta d - f(d))\|^2 + \|\Delta v\|^2) + C_\varepsilon(\|\Delta d - f(d)\|^2 + \|\nabla v\|^2). \qquad (4.66)
\end{aligned}
$$

Besides,

$$
\begin{aligned}
|I_5| &= |(\nabla(\Delta d - f(d)), \nabla(v\cdot\nabla d))|\\
&\le \|\nabla(\Delta d - f(d))\|(\|\nabla v\|\|\nabla d\|_{L^\infty} + \|v\|_{L^\infty}\|D^2 d\|_{L^2})\\
&\le \frac{\varepsilon}{2}\|\nabla(\Delta d - f(d))\|^2 + C_\varepsilon(\|\nabla\Delta d\|\|\nabla d\|\|\nabla v\|^2 + C_\varepsilon\|\Delta v\|\|v\|\\
&\le \frac{\varepsilon}{2}\|\nabla(\Delta d - f(d))\|^2 + C_\varepsilon\|\nabla(\Delta d - f(d))\|\|\nabla v\|^2 + C_\varepsilon\|\nabla v\|^2\\
&\quad +\varepsilon\|\Delta v\|^2 + C_\varepsilon\|v\|^2\\
&\le \varepsilon(\|\nabla(\Delta d - f(d))\|^2 + \|\Delta v\|^2) + C_\varepsilon\|\nabla v\|^2 + C_\varepsilon\|v\|^2. \qquad (4.67)
\end{aligned}
$$

$$
\begin{aligned}
I_7 &= (\Delta d - f(d), f'(d)d_t)\\
&= -(\Delta d - f(d), f'(d)v\cdot\nabla d) + (\Delta d - f(d), f'(d)d\cdot\nabla v)\\
&\quad +(\Delta d - f(d), f'(d)(\Delta d - f(d)))\\
&=: I_{7a} + I_{7b} + I_{7c}. \qquad (4.68)
\end{aligned}
$$

$$
\begin{aligned}
|I_{7a}| &\le \|\Delta d - f(d)\|\|f'(d)\|_{L^\infty}\|v\|_{L^4}\|\nabla d\|_{L^4} \le C\|\Delta d - f(d)\|^2 + C\|\nabla v\|^2,\\
|I_{7b}| &\le \|\Delta d - f(d)\|\|f'(d)\|_{L^\infty}\|d\|_{L^\infty}\|\nabla v\| \le C\|\Delta d - f(d)\|^2 + C\|\nabla v\|^2,\\
|I_{7c}| &\le \|\Delta d - f(d)\|^2\|f'(d)\|_{L^\infty} \le C\|\Delta d - f(d)\|^2.
\end{aligned}
$$

Therefore,

$$
|I_7| \le |I_{7a}| + |I_{7b}| + |I_{7c}| \le C\|\Delta d - f(d)\|^2 + C\|\nabla v\|^2. \qquad (4.69)
$$

Summing up, we obtain that

$$
\begin{aligned}
&\frac12\frac{d}{dt}A(t) + (\nu - 6\varepsilon)\|\Delta v\|^2 + (1 - 5\varepsilon)\|\nabla(\Delta d - f(d)\|^2\\
\le\ & C_5(\|\nabla v\|^2 + \|\Delta d - f(d)\|^2) + C_6(\|v\|^2 + \|d - d_\infty\|^2_{H^1}), \qquad (4.70)
\end{aligned}
$$

where C_5, C_6 are positive constants depending on $\varepsilon, \|\nabla v\|, \|d\|_{H^2}, \|d_\infty\|_{H^2}$. Taking

$$
\varepsilon \in \left(0, \frac{1}{12}\min\{1,\nu\}\right),
$$

we have

$$
\frac12\frac{d}{dt}A(t) + \frac{\nu}{2}\|\Delta v\|^2 + \frac12\|\nabla(\Delta d - f(d)\|^2 \le C_5 A(t) + C_6(\|v\|^2 + \|d - d_\infty\|^2_{H^1}). \qquad (4.71)
$$

Recalling (4.54), multiplying (4.71) by a small positive constant α_1 and adding the resultant to (4.49), we obtain that

$$\begin{aligned}
&\frac{d}{dt}\left(y(t)+\frac{\alpha_1}{2}A(t)\right)+\frac{\alpha_1\nu}{2}\|\Delta v\|^2+\frac{\alpha_1}{2}\|\nabla(\Delta d-f(d)\|^2\\
&+(\nu-\alpha\varepsilon_1-C_5\alpha_1)\|\nabla v\|^2+(1-C_5\alpha_1)\|\Delta d-f(d)\|^2+\frac{\alpha}{2}\|\nabla(d-d_\infty)\|^2\\
\leq\ & C\alpha\|d-d_\infty\|^2+C_6\alpha_1(\|v\|^2+\|d-d_\infty\|^2_{H^1}). \qquad (4.72)
\end{aligned}$$

Constants α,ε_1 are chosen as in (4.53). Then we can take

$$\alpha_1\in\left(0,\min\left\{\frac{1}{2C_5},\frac{\nu}{4C_5}\right\}\right). \qquad (4.73)$$

Using the estimates (4.56), (4.57), we infer from (4.72) that

$$\begin{aligned}
\frac{d}{dt}\left(y(t)+\frac{\alpha_1}{2}A(t)\right)+C_7\left(y(t)+\frac{\alpha_1}{2}A(t)\right) &\leq C_7y(t)+C_8(\|v\|^2+\|d-d_\infty\|^2_{H^1})\\
&\leq C_9(1+t)^{-\frac{2\theta}{1-2\theta}},\quad \forall\, t\geq 0. \qquad (4.74)
\end{aligned}$$

Again using Gronwall's inequality, we have

$$y(t)+\frac{\alpha_1}{2}A(t)\leq C_{10}(1+t)^{-\frac{2\theta}{1-2\theta}},\quad \forall\, t\geq 0, \qquad (4.75)$$

which together with (4.56) implies that (cf. (4.54))

$$\begin{aligned}
0\leq A(t) &\leq -\frac{2}{\alpha_1}y(t)+\frac{2C_{10}}{\alpha_1}(1+t)^{-\frac{2\theta}{1-2\theta}}\leq\frac{2C_2}{\alpha_1}\|d-d_\infty\|^2+\frac{2C_{10}}{\alpha_1}(1+t)^{-\frac{2\theta}{1-2\theta}}\\
&\leq C_{11}(1+t)^{-\frac{2\theta}{1-2\theta}},\quad \forall\, t\geq 0. \qquad (4.76)
\end{aligned}$$

The above estimate yields

$$\|\nabla v(t)\|+\|\Delta d(t)-f(d(t))\|\leq C(1+t)^{-\frac{\theta}{1-2\theta}},\quad \forall\, t\geq 0. \qquad (4.77)$$

Recalling (4.25), it follows from (4.77) that

$$\|\Delta d(t)-\Delta d_\infty\|\leq C(1+t)^{-\frac{\theta}{1-2\theta}},\quad \forall\, t\geq 0. \qquad (4.78)$$

Summing up, we can deduce the required estimate (1.8) from (4.57), (4.77) and (4.78).

The proof of Theorem 1.1 is complete. □

5 Results in Three Dimensional Case

In this section we prove the corresponding results in 3-D case, namely, Theorem 1.2 and Theorem 1.3. When the space dimension is three, we can still set $\lambda=\gamma=1$ for the sake of simplicity (cf. [23, 34]). However, to prove Theorem 1.2, the (constant) viscosity ν plays an essential role, which cannot be neglected. The largeness of ν is required to guarantee the existence of the global solution.

The following property is useful to understand the asymptotic behavior of the solutions to problem (1.1)–(1.5).

Theorem 5.1. *For any $R > 0$, whenever*

$$\|\nabla v\|^2(0) + \|\Delta d - f(d)\|^2(0) \le R,$$

there exists a constant $\varepsilon_0 \in (0,1)$, depending on ν, f, Q and R, such that either

(1) problem (1.1)–(1.5) has a unique global classical solution (v,d) with uniform estimate

$$\|v(t)\|_{H^1(Q)} + \|d(t)\|_{H^2(Q)} \le C, \quad \forall\, t \ge 0, \tag{5.1}$$

or

(2) there is a $T_* \in (0,\infty)$ such that

$$\mathcal{E}(T_*) \le \mathcal{E}(0) - \varepsilon_0,$$

where

$$\mathcal{E}(t) = \frac{1}{2}\|v(t)\|^2 + \frac{1}{2}\|\nabla d(t)\|^2 + \int_Q F(d(t))\,dx.$$

Proof. Suppose that (v,d) is a solution of problem (1.1)–(1.5). First, we can see from the basic energy law that

$$\mathcal{E}(t) \le \mathcal{E}(0), \quad \forall\, t \ge 0, \tag{5.2}$$

which implies

$$\|v(t)\| + \|d(t)\|_{H^1} \le C, \quad \forall\, t \ge 0. \tag{5.3}$$

By a direct calculation, we have (cf. (4.58), (4.64))

$$\begin{aligned}
& \frac{1}{2}\frac{d}{dt}A(t) + (\nu\|\Delta v\|^2 + \|\nabla(\Delta d - f(d)\|^2) \\
= \; & (\Delta v, v\cdot\nabla v) + (\Delta v, \nabla\cdot(\nabla d \odot \nabla d)) + (\nabla(\Delta d - f(d)), \nabla(v\cdot\nabla d)) \\
& + (\Delta d - f(d), \Delta d\cdot\nabla v) + 2(\Delta d - f(d), (\nabla d\cdot\nabla)\cdot\nabla v) \\
& + (\Delta d - f(d), f'(d)d_t).
\end{aligned} \tag{5.4}$$

We remark that the above (formal) calculation is valid for the classical solutions, but it can be justified by a proper approximating procedure (cf. [23,34]). We now estimate the right-hand side term by term. Since we only know the uniform estimate of $\|v\|$ and $\|d\|_{H^1}$, and due to the lack of maximal principle to the d equation, the estimates here are different from those in the previous section and in [23].

$$\begin{aligned}
|(\Delta v, v\cdot\nabla v)| &\le \|\Delta v\|\|v\|_{L^\infty}\|\nabla v\| \le C\|\Delta v\|(\|\Delta v\|^{\frac{3}{4}} + 1)\|\nabla v\| \\
&\le \frac{\nu}{8}\|\Delta v\|^2 + \frac{C}{\nu^7}\|\nabla v\|^8.
\end{aligned} \tag{5.5}$$

Similar to (4.61), in the 3-D case,

$$\begin{aligned}
\|\nabla\Delta d\| &\le \|\nabla(\Delta d - f(d))\| + \|f'(d)\|_{L^3}\|\nabla d\|_{L^6} \le \|\nabla(\Delta d - f(d))\| + C(\|\Delta d\| + 1) \\
&\le \|\nabla(\Delta d - f(d))\| + C\|\Delta d - f(d)\| + C.
\end{aligned} \tag{5.6}$$

As a result,

$$|(\Delta v, \nabla\cdot(\nabla d \odot \nabla d))|$$

$$
\begin{aligned}
&= (\Delta v, \Delta d \nabla d) \le \|\Delta d\|\|\Delta v\|\|\nabla d\|_{L^\infty} \le C\|\Delta d\|\|\Delta v\|(\nabla \Delta d\|^{\frac{3}{4}} + 1) \\
&\le C|\Delta d\|\|\Delta v\|(\|\nabla(\Delta d - f(d))\|^{\frac{3}{4}} + C\|\Delta d - f(d)\|^{\frac{3}{4}} + C) \\
&\le \frac{1}{8}\|\nabla(\Delta d - f(d))\|^2 + \frac{\nu}{8}\|\Delta v\|^2 + C(\|\Delta d - f(d)\|^8 + 1). && (5.7)
\end{aligned}
$$

$$
\begin{aligned}
&|(\nabla(\Delta d - f(d)), \nabla(v \cdot \nabla d))| \\
&\le \|\nabla(\Delta d - f(d))\|(\|\nabla v\|\|\nabla d\|_{L^\infty} + \|v\|_{L^\infty}\|D^2 d\|) \\
&\le C\|\nabla(\Delta d - f(d))\|[\|\nabla v\|(\nabla \Delta d\|^{\frac{3}{4}} + 1) + C(\|\Delta v\|^{\frac{3}{4}} + 1)(\|\Delta d\| + 1)] \\
&\le \frac{1}{8}\|\nabla(\Delta d - f(d))\|^2 + \frac{\nu}{8}\|\Delta v\|^2 + C(\|\nabla v\|^8 + \|\Delta d - f(d)\|^8 + 1). && (5.8)
\end{aligned}
$$

$$
\begin{aligned}
&|(\Delta d - f(d), \Delta d \cdot \nabla v)| \\
&\le \|\Delta d - f(d)\|_{L^3}\|\Delta d\|\|\nabla v\|_{L^6} \\
&\le C(\|\nabla(\Delta d - f(d))\|^{\frac{1}{2}}\|\Delta d - f(d)\|^{\frac{1}{2}} + \|\Delta d - f(d)\|) \\
&\quad \times(\|\Delta d - f(d)\| + 1)(\|\Delta v\| + \|\nabla v\|) \\
&\le \frac{1}{8}\|\nabla(\Delta d - f(d))\|^2 + \frac{\nu}{8}\|\Delta v\|^2 + C(\|\Delta d - f(d)\|^6 + \|\nabla v\|^2 + 1). && (5.9)
\end{aligned}
$$

$$
\begin{aligned}
&2|(\Delta d - f(d), (\nabla d \cdot \nabla) \cdot \nabla v)| \\
&\le C\|\Delta d - f\|_{L^3}\|\nabla d\|_{L^6}\|v\|_{H^2} \\
&\le C(\|\nabla(\Delta d - f(d))\|^{\frac{1}{2}}\|\Delta d - f(d)\|^{\frac{1}{2}} + \|\Delta d - f(d)\|)(\|\Delta d\| + 1)(\|\Delta v\| + 1) \\
&\le \frac{1}{8}\|\nabla(\Delta d - f(d))\|^2 + \frac{\nu}{8}\|\Delta v\|^2 + C\|\Delta d - f(d)\|^6 + C. && (5.10)
\end{aligned}
$$

As before, the last term on the right-hand side can be expressed into three terms

$$
\begin{aligned}
(\Delta d - f(d), f'(d)d_t) &= -(\Delta d - f(d), f'(d)v \cdot \nabla d) + (\Delta d - f(d), f'(d)d \cdot \nabla v) \\
&\quad +(\Delta d - f(d), f'(d)(\Delta d - f(d))). && (5.11)
\end{aligned}
$$

Then we have

$$
\begin{aligned}
|(\Delta d - f(d), f'(d)v \cdot \nabla d)| &\le \|\Delta d - f(d)\|\|f'(d)\|_{L^3}\|\nabla d\|_{L^6}\|v\|_{L^\infty} \\
&\le C\|\Delta d - f(d)\|(\|\Delta d\| + 1)(\|\Delta v\|^{\frac{3}{4}} + 1) \\
&\le \frac{\nu}{8}\|\Delta v\|^2 + C(\|\Delta d - f(d)\|^{\frac{16}{5}} + 1), && (5.12)
\end{aligned}
$$

$$
\begin{aligned}
|(\Delta d - f(d), f'(d)d \cdot \nabla v)| &\le \|\Delta d - f(d)\|\|f'(d)\|_{L^3}\|d\|_{L^\infty}\|\nabla v\|_{L^6} \\
&\le C\|\Delta d - f(d)\|(\|\Delta d\| + 1)(\|\Delta v\| + 1) \\
&\le \frac{\nu}{8}\|\Delta v\|^2 + C(\|\Delta d - f(d)\|^4 + 1), && (5.13)
\end{aligned}
$$

$$
\begin{aligned}
&|(\Delta d - f(d), f'(d)(\Delta d - f(d)))| = \|f'(d)\|_{L^3}\|\Delta d - f(d)\|^2_{L^3} \\
&\le C(\|\nabla(\Delta d - f(d))\|^{\frac{1}{2}}\|\Delta d - f(d)\|^{\frac{1}{2}} + \|\Delta d - f(d)\|)^2 \\
&\le \frac{1}{8}\|\nabla(\Delta d - f(d))\|^2 + C\|\Delta d - f(d)\|^2. && (5.14)
\end{aligned}
$$

Summing up, we can conclude that

$$\frac{d}{dt}A(t) \le C_*(A(t)^4 + 1), \tag{5.15}$$

where C_* is a constant that only depends on ν, f, Q, $\|v_0\|$ and $\|d_0\|_{H^1}$.

If the initial data satisfy

$$A(0) = \|\nabla v_0\|^2 + \|\Delta d_0 - f(d_0)\|^2 \le R,$$

we consider the following initial value problem for a nonlinear ODE:

$$\frac{d}{dt}Y(t) = C_*(Y(t)^4 + 1), \quad Y(0) = A(0) \le R.$$

We denote by $I = [0, T_{max})$ the maximal existence interval of $Y(t)$ such that

$$\lim_{t \to T_{max}^-} Y(t) = \infty.$$

On the other hand, it is easy to see that for any $t \in I$, $0 \le A(t) \le Y(t)$. Consequently, $A(t)$ exists on I. Moreover, T_{max} is determined by $Y(0)$ and C_* such that $T_{max} = T_{max}(Y(0), C_*)$ is increasing when $Y(0) \ge 0$ is decreasing. We take $t_0 = \frac{1}{2}T_{max}(R, C_*) > 0$. Then it follows that $Y(t)$ as well as $A(t)$ is uniformly bounded on $[0, t_0]$. This fact together with the argument in [34] and Lemma 3.2 implies the local existence of a unique (classical) solution of problem (1.1)–(1.5) at least on $[0, t_0]$.

If (2) is not true, we have

$$\mathcal{E}(t) \ge \mathcal{E}(0) - \varepsilon_0, \quad \forall\, t \ge 0.$$

From the basic energy law, we infer that

$$\int_0^\infty \int_Q \left(\nu|\nabla v(t)|^2 + |\Delta d(t) - f(d(t))|^2\right)\, dxdt \le \varepsilon_0, \quad \forall\, t \ge 0.$$

Hence, there exists a $t_* \in [\frac{t_0}{2}, t_0]$ such that

$$\nu\|\nabla v(t_*)\|^2 + \|\Delta d(t_*) - f(d(t_*))\|^2 \le \frac{2\varepsilon_0}{t_0}.$$

Choosing $\varepsilon_0 > 0$ such that

$$\frac{2}{\min\{1, \nu\}} \frac{\varepsilon_0}{t_0} \le R,$$

we have $A(t_*) \le R$. Taking t_* as the initial time, we infer from the above argument that $A(t)$ is uniformly bounded at least on $[0, \frac{3t_0}{2}] \subset [0, t_* + t_0]$. Moreover, the bound only depends on R, C_* but not on the length of existence interval. We can extend the local solution step by step to infinity such that

$$A(t) \le C, \quad \forall\, t \ge 0, \tag{5.16}$$

where C is uniform in time. The proof is complete. □

Remark 5.1. *Theorem 5.1 implies that if the energy $\mathcal{E}$ does not "drop" too fast, problem* (1.1)–(1.5) *admits a global unique classical solution. This assumption can be verified for certain special cases, which are stated in the following corollaries.*

Corollary 5.1. *Let $d^* \in H^2_p(Q)$ be an absolute minimizer of the functional*

$$E(d) = \frac{1}{2}\|\nabla d\|^2 + \int_Q F(d)dx.$$

There exists a constant $\sigma \in (0,1]$ that may depend on ν, f, Q and d_* such that for initial data $(v_0, d_0) \in V \times H^2_p(Q)$ satisfying

$$\|v_0\|_{H^1} + \|d_0 - d_*\|_{H^2} \le \sigma,$$

problem (1.1)–(1.5) *admits a unique global classical solution.*

Proof. Without loss of generality, we assume that $\sigma \le 1$. From assumption

$$||v_0||_{H^1} + ||d_0 - d^*||_{H^2} \le \sigma \le 1,$$

we infer that

$$\begin{aligned} \|v_0\|^2_{H^1} + \|\Delta d_0 - f(d_0)\|^2 &\le \|v_0\|^2_{H^1} + 2\|\Delta d_0 - \Delta d^*\|^2 + 2\|f(d_0) - f(d^*)\|^2 \\ &\le K_1(\|v_0\|_{H^1} + \|d_0 - d^*\|_{H^2})^2 \le K_1. \end{aligned} \tag{5.17}$$

In addition, since d^* is the absolute minimizer of $E(d)$, we have

$$\begin{aligned} \mathcal{E}(0) - \mathcal{E}(t) &\le \mathcal{E}(0) - E(d(t)) \le \mathcal{E}(0) - E(d^*) \\ &\le \frac{1}{2}\|v_0\|^2 + \frac{1}{2}\left(\|\nabla d_0\|^2 - \|\nabla d_*\|^2\right) + \int_Q F(d_0) - F(d_*)dx \\ &\le \frac{1}{2}\sigma^2 + C\sigma \le K_2\sigma. \end{aligned}$$

Here K_1 and K_2 are positive constants that only depend on d_*, ν, f (not on σ).

Take $R = K_1$, $\varepsilon_0 = K_2\sigma$, respectively. We can apply Theorem 5.1 by choosing

$$\sigma = \min\left\{1,\ \frac{K_1}{4K_2}T_{max}(K_1, C_*)\min\{1, \nu\}\right\}. \tag{5.18}$$

The proof is complete. ☐

Corollary 5.1 implies that if the initial velocity v_0 is small in H^1 and initial director d_0 is properly close to the absolute minimizer d_* of functional $E(d)$ in H^2, problem (1.1)–(1.5) admits a unique global classical solution. However, from the proof of Theorem 5.1 we can somewhat relax the "smallness" requirement from $H^1 \times H^2$ to $L^2 \times H^1$.

Corollary 5.2. *Let $d^* \in H^2_p(Q)$ be an absolute minimizer of the functional*

$$E(d) = \frac{1}{2}\|\nabla d\|^2 + \int_Q F(d)dx.$$

For any initial data $(v_0, d_0) \in V \times H^2_p(Q)$, there exists a constant $\sigma \in (0,1]$, which depends on ν, f, Q, d_*, $\|v_0\|_{H^1}$ and $\|d_0\|_{H^2}$ such that if

$$\|v_0\| + \|d_0 - d_*\|_{H^1} \le \sigma,$$

problem (1.1)–(1.5) *admits a unique global classical solution.*

Proof. Without loss of generality, we assume that $\sigma \le 1$. Set

$$K_1 := \|\nabla v_0\|^2 + \|\Delta d_0 - f(d_0)\|^2 < \infty. \tag{5.19}$$

Moreover, we have

$$\mathcal{E}(0) - \mathcal{E}(t) \le K_2\sigma,$$

where K_2 is a positive constant that only depend on d_*, ν, f (not on σ). As in the proof of Corollary 5.1, we take $R = K_1$, $\varepsilon_0 = K_2\sigma$ and choose

$$\sigma = \min\left\{1,\ \frac{K_1}{4K_2} T_{max}(K_1, C_*) \min\{1, \nu\}\right\}. \tag{5.20}$$

The conclusion follows from Theorem 5.1. Here, we note that now σ depends on the norm of $\|v_0\|_{H^1}$ and $\|d_0\|_{H^2}$ while in Corollary 5.1, σ does not. □

In what follows, we proceed to prove the conclusions in Theorem 1.2 and Theorem 1.3. First, we have the following result for both cases.

Proposition 5.1. *In three dimensional case, for the unique classical solution (v, d) obtained in Proposition 3.1, Corollary 5.1 and Corollary 5.2, it holds*

$$\lim_{t\to+\infty} A(t) = 0. \tag{5.21}$$

Proof. (1) For the large viscosity case (cf. Proposition 3.1), after refining the argument in [34], we indeed have the following differential inequality (the detailed calculation is left to interested readers):

Lemma 5.1. *We consider 3-D case. Set $\tilde{A}(t) = A(t) + 1$. For arbitrary $\nu_0 > 0$, if $\nu \ge \nu_0 > 0$, then the following inequality holds for the classical solution (v, d) to problem* (1.1)–(1.5)*:*

$$\frac{d}{dt}\tilde{A}(t) \le -\left(\nu - M_1\nu^{\frac{1}{2}}\tilde{A}(t)\right)\|\Delta v\|^2 - \left(1 - \frac{M_1}{\nu^{\frac{1}{4}}}\tilde{A}(t)\right)\|\nabla(\Delta d - f(d))\|^2 + M_2\tilde{A}(t), \tag{5.22}$$

where M_1, M_2 are constants depending on $f, |Q|, \|v_0\|, \|d_0\|_{H^1}$, M_2 may also depend on ν_0.

Based on Lemma 5.1, we can show the uniform estimate for $A(t)$ by using the idea in [23]. It follows from the basic energy law that

$$\int_t^{t+1} \tilde{A}(\tau)d\tau \le \int_t^{t+1} A(\tau)d\tau + 1 \le \tilde{M}, \quad \forall\, t \ge 0, \tag{5.23}$$

where $\tilde{M} > 0$ is a constant depending only on $\|v_0\|, \|d_0\|_{H^1}$. Take ν large enough satisfying

$$\nu^{\frac{1}{4}} \ge M_1(\tilde{A}(0) + M_2\tilde{M} + 4\tilde{M}) + 1. \tag{5.24}$$

Then by (5.22), there must be some $T_0 > 0$ such that

$$\nu - M_1\nu^{\frac{1}{2}}\tilde{A}(t) \ge 0, \quad 1 - \frac{M_1\tilde{A}(t)}{\nu^{\frac{1}{4}}} \ge 0,$$

for all $t \in [0, T_0]$. Moreover, on $[0, T_0]$,

$$\frac{d}{dt}\tilde{A}(t) \leq M_2\tilde{A}(t). \tag{5.25}$$

Denote $T_* = \sup T_0$. First we show that $T_* \geq 1$ by a contradiction argument.
If $T_* < 1$, then

$$\tilde{A}(T_*) \leq \tilde{A}(0) + M_2 \int_0^1 \tilde{A}(t)dt \leq \tilde{A}(0) + M_2\tilde{M}.$$

On the other hand, from the definition of T_*, we have

$$\nu < \max\{M_1^2\tilde{A}^2(T_*), M_1^4\tilde{A}^4(T_*)\},$$

which contradicts (5.24).

Next, if $T_* < +\infty$, (5.23) implies that there is a $t_1 \in [T_* - \frac{1}{2}, T_*]$ such that

$$\tilde{A}(t_1) \leq 4\tilde{M}. \tag{5.26}$$

As a result,

$$\tilde{A}(T_*) \leq 4\tilde{M} + M_2 \int_{t_1}^{T_*} \tilde{A}(t)dt \leq 4\tilde{M} + M_2\tilde{M}. \tag{5.27}$$

From the definition of T_*, we have

$$\nu < \max\{M_1^2\tilde{A}^2(T_*), M_1^4\tilde{A}^4(T_*)\},$$

which together with (5.27) also yields a contradiction with (5.24). Hence, we have the uniform estimate

$$\tilde{A}(t) \leq \frac{\nu^{\frac{1}{2}}}{M_1}, \quad \forall\, t \geq 0, \tag{5.28}$$

which implies that

$$A(t) \leq C, \quad \forall\, t \geq 0, \tag{5.29}$$

where C is a constant depending on $f, |Q|, \|v_0\|_{H^1}, \|d_0\|_{H^2}$. Thus, we infer from (5.22) that

$$\frac{d}{dt}A(t) = \frac{d}{dt}\tilde{A}(t) \leq M_2\tilde{A}(t) \leq C.$$

Due to the fact that $A(t) \in L^1(0, \infty)$ (cf. (4.5)), we can conclude that (5.21) holds.

(2) Now we consider the near equilibrium case. When the assumptions in Theorem 5.1 (or Corollary 5.1 / Corollary 5.2) are satisfied, we have (5.16) holds for all $t \geq 0$. Thus (5.15) implies

$$\frac{d}{dt}A(t) \leq C_*(A^4(t) + 1) \leq C, \quad \forall\, t \geq 0.$$

By the same argument as in (1), we obtain (5.21). The proof is complete. □

After previous preparations, we can proceed to prove the results in Theorem 1.2 and Theorem 1.3.

Proof of Theorem 1.2 and Theorem 1.3. Based on Proposition 5.1, for both large viscosity case and near equilibrium case, one can argue exactly as in Section 3.1 to conclude that

$$\lim_{t \to +\infty} (\|v\|_{H^1} + \|d - d_\infty\|_{H^2}) = 0. \tag{5.30}$$

Notice that we now have uniform bounds for $\|v\|_{H^1}$ and $\|d\|_{H^2}$. Then we are able to show the estimate on convergence rate (1.8) for both cases. To this end, we can check the argument for 2-D case step by step. By applying corresponding Sobolev embedding theorems in 3-D, we can see that all calculations in Section 4.2 are valid with minor modifications. Hence, the details are omitted here. We complete the proofs for Theorem 1.2 and Theorem 1.3. □

6 Remark on the Flows with Non-vanishing Average Velocity

We briefly discuss the flows with non-vanishing average velocity. Due to the periodic boundary condition (1.4), by integration of (1.1) over Q, we get

$$\frac{d}{dt}\left(\frac{1}{|Q|}\int_Q v(t)dx\right) = 0, \tag{6.1}$$

which implies

$$m_v := \frac{1}{|Q|}\int_Q v(t)dx \equiv \frac{1}{|Q|}\int_Q v_0 dx, \quad \forall\, t \geq 0, \tag{6.2}$$

where $|Q|$ is the measure of Q.

Our main results (Theorems 1.1–1.3) in this paper are valid for the flow with vanishing average velocity (see the definition of function space V), namely, $m_v = 0$. In that case, we can apply the Poincaré inequality to $v \in V$ such that $\|v\| \leq C\|\nabla v\|$. This enables us to show that under the dissipations of system (1.1)–(1.5), the velocity of the flows will tends to zero and the director d will converge to a steady state.

When the non-vanishing average flow v is considered, as for the single Navier–Stokes equation (cf. [35]), we set

$$v = \tilde{v} + m_v. \tag{6.3}$$

The we transform problem (1.1)–(1.5) into the following system for variables $\tilde{v}$ and d:

$$\begin{aligned} \tilde{v}_t + \tilde{v}\cdot\nabla\tilde{v} - \nu\Delta\tilde{v} + m_v\cdot\nabla\tilde{v} + \nabla P &= -\lambda\nabla\cdot[\nabla d \odot \nabla d + (\Delta d - f(d)) \otimes d], && (6.4)\\ \nabla\cdot\tilde{v} &= 0, && (6.5)\\ d_t + \tilde{v}\cdot\nabla d + m_v\cdot\nabla d - d\cdot\nabla\tilde{v} &= \gamma(\Delta d - f(d)), && (6.6) \end{aligned}$$

subject to the corresponding periodic boundary conditions and initial conditions

$$\begin{aligned} &\tilde{v}(x+e_i) = \tilde{v}(x), \quad d(x+e_i) = d(x), \qquad \text{for } x \in \partial Q, && (6.7)\\ &\tilde{v}|_{t=0} = \tilde{v}_0(x) = v_0(x) - m_v, \quad \text{with } \nabla\cdot v_0 = 0, \quad d|_{t=0} = d_0(x), \quad \text{for } x \in Q. && (6.8) \end{aligned}$$

Introduce the new energy functional

$$\tilde{\mathcal{E}}(t) = \frac{1}{2}\|\tilde{v}\|^2 + \frac{\lambda}{2}\|\nabla d\|^2 + \lambda\int_Q F(d)dx. \tag{6.9}$$

It is not difficult to check that system (6.4)–(6.8) still enjoys the *basic energy law*

$$\frac{d}{dt}\tilde{\mathcal{E}}(t) = -\nu\|\nabla\tilde{v}\|^2 - \lambda\gamma\|\Delta d - f(d)\|^2, \quad t \geq 0. \tag{6.10}$$

By a similar argument, we can still prove the global existence of classical solution $(\tilde{v}, d)$ to problem (6.4)–(6.8) under the same assumptions as in Proposition 3.1, Theorem 5.1, Corollary 5.1 and Corollary 5.2. Moreover, we can prove the same higher-order energy inequalities like Lemma 4.1, Lemma 5.1 and (5.15) for $(\tilde{v}, d)$.

As far as the long-time behavior of the global solution is concerned, following a similar arguments in previous sections, we can conclude that

$$\lim_{t\to+\infty}(\|\tilde{v}(t)\|_{H^1} + \|\Delta d(t) - f(d(t))\|) = 0. \tag{6.11}$$

Recalling (6.3), we infer from (6.11) that

$$\lim_{t\to+\infty}\|v(t) - m_v\|_{H^1} = 0. \tag{6.12}$$

However, in general we cannot conclude similar results on the convergence of d like in Theorems 1.1–1.3. (6.11) implies that the 'limit' of d, which is denoted by $\hat{d}$, will satisfy $\Delta\hat{d} - f(\hat{d}) = 0$ with corresponding periodic boundary condition. Let us look at the 'limiting' case such that $v = \hat{v} = m_v$ and $d = \hat{d}$. It follows from (1.5) that

$$\frac{D}{Dt}\hat{d} = \hat{d}_t + \hat{v}\cdot\nabla\hat{d} = 0. \tag{6.13}$$

Consequently, $\hat{d}$ is purely transported and it ($i.e.$, $\hat{d}(x(X,t),t)$) remains unchanged when the molecule moves through a flow field with velocity m_v. However, the local rate of change $\hat{d}_t$ may not be zero, since the convective rate of change may not vanish. Hence, in the Eulerian coordinates, or in Q, $\hat{d}(x,t)$ may change in time. As a result, there might be no steady state for the director. Obviously, this is different from the situation in the previous sections, where all the three rates of change are vanishing in the limiting case. We can look at a simple example. In the case of periodic boundary condition, let $\hat{v} = (1,0)$ and $\hat{d}(x,0) = \hat{d}_0(x)$ for $x \in Q$. We can see that in the Eulerian coordinates, $\hat{d}(x,t)$ $(x \in Q)$ is a periodic function in time such that for $t \geq 0$, $\hat{d}(x,t) = \hat{d}(x,t+1)$ with $T = 1$ being the period.

7 Future Work

As mentioned in Section 1, the current system (1.1)–(1.5) is a special case of the general Ericksen–Leslie system which describes the flow of liquid crystal materials (cf. [2, 21, 25]):

$$\begin{aligned} v_t + v\cdot\nabla v + \nabla p &= -\nabla(\nabla d \odot \nabla d) + \nabla\cdot\sigma, && (7.1)\\ \nabla\cdot v &= 0, && (7.2)\\ -\lambda_1(d_t + v\cdot\nabla d - \omega d) - \lambda_2 A d &= \Delta d - f(d), && (7.3)\end{aligned}$$

with proper boundary conditions and initial conditions. Here

$$\begin{aligned} \sigma_{ij} &= \mu_1 d_k d_p A_{kp} d_i d_j + \mu_2 d_j N_i + \mu_3 d_i N_j + \mu_4 A_{ij} + \mu_5 d_j d_k A_{ki} + \mu_6 d_i d_k A_{kj}, && (7.4)\\ N &= d_t + v\cdot\nabla d - \omega d. && (7.5)\end{aligned}$$

The coefficients satisfy the the following relations

$$\mu_1 \geq 0,\ \mu_4 > 0,\ \mu_5 + \mu_6 \geq 0. \tag{7.6}$$

$$\lambda_1 = \mu_2 - \mu_3 < 0, \quad \lambda_2 = \mu_5 - \mu_6 = -(\mu_2 + \mu_3). \tag{7.7}$$

(7.7) is achieved from the hydrodynamic point of view in order to guarantee the entropy condition, that is, the second law of thermodynamics (cf. [21]). The relation $\lambda_2 = -(\mu_2 + \mu_3)$ is called Parodi's condition, which is derived from the Onsager reciprocal relation.

Remark 7.1. *If we set* $\mu_1 = \mu_3 = \mu_6 = 0$, $\mu_2 = -1$, $\mu_5 = 1$, *then* $\lambda_1 = -1$, $\lambda_2 = 1$. *In this case, it is consistent with our system* (1.1)–(1.3).

Note that compared to the one considered in our paper, due to so many stress terms in (7.1), the system (7.1)–(7.3) is much more complicated. However, if we impose the range of λ_2 as

$$|\lambda_2| \le \sqrt{-\lambda_1}\sqrt{\mu_5 + \mu_6},$$

using the relation (7.7), we find that the system (7.1)–(7.3) (with proper boundary conditions) enjoys the following basic energy law: (cf. [40])

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left(\|v\|^2 + \|\nabla d\|^2 + 2\int_Q F(d)\,dx\right) &= -\mu_1\|d^T A d\|^2 - \frac{\mu_4}{2}\|\nabla v\|^2 + \frac{1}{\lambda_1}\|\Delta d - f(d)\|^2 \\ &\quad - \left(\mu_5 + \mu_6 + \frac{(\lambda_2)^2}{\lambda_1}\right)\|Ad\|^2. \end{aligned} \tag{7.8}$$

Thanks to the basic energy law (7.8), one can recover the general Ericksen–Leslie system by energetic variational approaches. Furthermore, through different types of energetic methods, say, Onsager's maximal dissipation law and principle of virtual work, we are able to distinguish the dissipative part and conservative part among all stress terms.

Our forthcoming goal is to study the wellposedness as well as long-time dynamics of the system (7.1)–(7.3) with proper boundary conditions (for instance, the periodic b.c.) under proper conditions. It has been shown in the literature that the coefficients play important roles in the mathematical study of system (7.1)–(7.3). For instance, in [25], the existence of classical solutions and the asymptotic stability of the solutions were discussed under additional assumption $\lambda_2 = 0$, which is crucial in the analysis therein that certain maximum principle for d only holds under this condition. However, the physical meaning of conditions like $\lambda_2 = 0$ seems to be unclear (cf. [25]). We intend to investigate the existence of global (classical) solutions to system (7.1)–(7.3) under more physical relevant conditions, for instance, the viscosity μ_4 is assume to be sufficiently large (cf. [40]). Furthermore, the numerical simulation for the general system is an issue worth thinking.

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