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Abstract. We establish a new type of local asymptotic formula for the Green's function of a parabolic operator with non-constant coefficients. Our procedure leads to a construction of approximate solutions to parabolic equations which are accurate to arbitrary prescribed order in time.

Contents

1. Introduction

We establish a new type of local estimate for the Green's function of a parabolic operator with non-constant coefficients that *do not depend on time*.

We consider the following second-order differential operator L acting on functions in $\mathbb{R}^N$,

$$Lu(x) := \sum_{i,j=1}^{N} a_{ij}(x)\partial_i\partial_j u(x) + \sum_{k=1}^{N} b_k(x)\partial_k u(x) + c(x)u(x), \tag{1.1}$$

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where $x = (x_1, ..., x_N) \in \mathbb{R}^N$, $\partial_k := \frac{\partial}{\partial x_k}$, and the coefficients a_{ij}, b_i, and c and all their derivatives are assumed to be bounded. (We write $a_{ij}, b_j, c \in \mathcal{C}_b^\infty\left(\mathbb{R}^N\right)$). The approach in this paper can be extended to the case of certain operators with unbounded coefficients that appear in applications, such as $\partial_t - (ax^2\partial_x^2 + bx\partial_x + c)$ acting on $\mathbb{R}_t \times \mathbb{R}_x$. This is part of a work in progress[9], but see also below for a more detailed discussion of this issue.

We further assume that L is *uniformly strongly elliptic*, namely that there exists a constant $\gamma > 0$ such that

$$\sum_{ij} a_{ij}(x)\xi_i\xi_j \geq \gamma\|\xi\|^2, \quad \|\xi\|^2 := \sum_{i=1}^N \xi_i^2, \tag{1.2}$$

for all $(\xi, x) \in \mathbb{R}^N \times \mathbb{R}^N$. We define the matrix $A := [a_{ij}]$, which we take to be symmetric. In view of the applications we are interested in, we take u and the coefficients of L to be real-valued.

We study the initial value problem (IVP) for the parabolic operator $\partial_t - L$:

$$\begin{cases} \partial_t u(t,x) - Lu(t,x) = 0 & \text{in } (0,\infty) \times \mathbb{R}^N \\ u(0,x) = f(x), \quad f \in \mathcal{C}^\infty(\mathbb{R}^n) & \text{on } \{0\} \times \mathbb{R}^N, \end{cases} \tag{1.3}$$

where $f \in \mathcal{C}^\infty(\mathbb{R}^N)$, $u \in \mathcal{C}^\infty([0,\infty) \times \mathbb{R}^N)$. We can also replace $\mathbb{R}^n$ with a manifold of bounded geometry [9].

Then it is known that there exists a function $\mathcal{G}^L \in \mathcal{C}^\infty((0,\infty) \times \mathbb{R}^N \times \mathbb{R}^N)$ such that

$$u(t,x) = \int_{\mathbb{R}^N} \mathcal{G}^L(t,x,y) f(y) dy, \quad t > 0, \tag{1.4}$$

is a solution of the above equation, and it is unique if f (and hence u) satisfy certain growth conditions, specified later (see e.g. [10], page 237). We will often write $\mathcal{G}^L(t,x,y) = \mathcal{G}_t^L(x,y)$. In case we have uniqueness, we shall also use the notation $u(t) = e^{tL}f$. The operator e^{tL} is then called the *solution operator* of the problem (1.3), and its kernel $\mathcal{G}_t^L$ the *Green's function*, or *fundamental solution* of L, or *conditional probability density* in applications to probability.

For L with constant coefficients and for a few other cases, one can explicitly compute the kernel $\mathcal{G}^L$. In general, it is not known how to provide explicit formulas for $\mathcal{G}^L$, though there exists well-known asymptotic formulas, which gives good approximation to the Green's function for t small and x close to y. For example, interpreting the operator L as a Laplace-Beltrami operator on a manifold plus lower order terms, lead to heat kernel expansions of them form

$$\begin{aligned}\mathcal{G}_t(x,y) &= \frac{1}{(2\pi t)^{N/2}} e^{-\frac{d(x,y)^2}{2t}+W(x,y)} \\ &\quad \cdot \sqrt{\frac{d(x,y)^{N-1}}{\Psi(x,y)}} \left(1+\mathcal{G}^{(1)}(x,y)t+\cdots \mathcal{G}^{(n)}(x,y)t^n\right)\end{aligned} \tag{1.5}$$

where $d(x,y)$ is the geodesic distance between x and y. (Among the vast literature we refer to [25], to [14] for a pseudo-differential perspective, and to [17, Chapter 5] for a more precise description). However, one difficulty in the practical implementation of this approach is that except again in special cases, there is no closed form solution to the geodesic equations used in defining $d(x,y)$ and thus must be computed numerically. Another approach uses the stationary phase approximation which gives:

$$\mathcal{G}^L(t,x,y) \sim \sum_{j\geq 0} p_j\left(x, t^{-1/2}(x-y)\right) e^{-(x-y)^T A(x)^{-1}\cdot(x-y)/4t}, \tag{1.6}$$

where $p_j(x,z)$ is a polynomial of degree j in z, and $A(x) := [a_{ij}(x)]$. (We follow here Taylor [28, Chapter 7, Section 13].) It is well-known that even functions of the Laplacian, in particular the heat semigroup, can be studied using the solution operator of the corresponding wave equation [23, 8].

The main aim of this paper is to construct an expansion of the form (1.6) where the polynomials p_j are *explicitly, algorithmically computable.* Our approach is more elementary than found in the literature and rely on an iterative time-ordered formula for the solution operator e^{tL}, equation (2.8), a scaling argument, and a suitable Taylor's expansion of the coefficients of L, equation (3.7). We also give global error estimates on $\mathbb{R}^N$ in both weighted and regular Sobolev spaces. In particular, our approximation is valid uniformly in x and y, provided t is small enough. Since the iterative formula is obtained via repeated applications of Duhamel's principle, we could also treat certain classes of semilinear equations following Kato's method, which allows to take rougher data as well (see [19] in the context of the Navier-Stokes equations).

Our main result is the following theorem. Below, $W_a^{m,p}$ is an exponentially weighted Sobolev space defined in (2.1). When $a = 0$, we recover estimates in the usual Sobolev spaces.

Theorem 1.1. *Let $f \in W_a^{m,p}(\mathbb{R}^N)$, $m \in \mathbb{R}$, $a \in \mathbb{R}$, $1 < p < \infty$. For any $t \in [0,T]$, $0 < T < \infty$, and $n \in \mathbb{Z}_+$, we have*

$$e^{tL} f(x) = \int_{\mathbb{R}^N} \mathcal{G}_t^{[n]}(x,y) f(y) dy + t^{(n+1)/2} \mathbb{E}_{t,n} f(x),$$

where

(1.7)

$$\mathcal{G}_t^{[n]}(x,y) = t^{-N/2} \sum_{\ell=0}^{n} t^{\ell}/2 \left[\mathcal{P}^{\ell}(x, x_0, \partial_x) \left(\frac{e^{\frac{(x-w)^T A(x_0)^{-1}(x-w)}{4}}}{\sqrt{(4\pi)^N \det(A(x_0))}} \right) \right] \Bigg|_{\substack{x_0 = x \\ w = x + (y-x)/\sqrt{t}}},$$

with $\mathcal{P}^{\ell}$ differential operators of order 2ℓ, and

$$\|\mathbb{E}_{t,n} f\|_{W_a^{m,p}} \leq C(T, m, n, p, a) \|f\|_{W_a^{m,p}}. \tag{1.8}$$

where C is independent of $t \in [0,T]$.

A localization procedure as in [21] will allow us to pass from operators on $\mathbb{R}^N$ to operators on manifolds M of bounded geometry (again based on [8]). More precisely, our results will extend to operators of the form $L = \sum_{ij}^N a_{ij} \partial_i \partial_j + \sum_{ij}^N b_i \partial_i + c$, defined on a subset Ω of $\mathbb{R}^N$ such that the metric $g = \sum_{ij}^N a^{ij} dx_i dx_j$ is complete of bounded geometry on Ω and the coefficients of our differential operator are bounded in *normal coordinates*. Here the matrix $[a^{ij}]$ is the inverse of the matrix A, following the usual conventions. Such metrics arise naturally when resolving boundary singularities (see [24] for a complete treatment of heat calculus on manifolds with boundary). We refer to [1, 18, 31] for recent papers dealing with partial differential equations on manifolds with metrics of this form. In particular, we can deal with certain operators having polynomial coefficients such as those arising in Probability and its applications, for example in the Black-Scholes option pricing equation [4]

$$Lu(x) = x^2 \partial_x^2 u(x) + r(x \partial_x u(x) - u(x)).$$

Our results also apply to differential operators arising in stochastic volatility models (c.f. [2, 7, 12, 13, 16, 22]). On the other hand, our results apply to operators of the form $x^{2\beta} \partial_x^2$, $0 < \beta < 1$ only *locally*. A good framework for obtaining differential operators with unbounded coefficients that satisfy our assumptions is that of Lie manifolds [1]. This will be discussed in detail in [9].

Fully explicit calculations and concrete, practical applications will be given in the second part of this paper [9].

We conclude this introduction with an outline of the paper. In Section 2, we define the weighted and regular Sobolev spaces of initial data for the parabolic equation and introduce the class of operators L under study. We also briefly discuss mapping properties of the semigroup generated by L and use them to justify the time-ordered perturbation expansion of e^{tL}. In Section 3, we exploit local in space and time dilations of the Green's function together with a certain Taylor expansion of the operator L to rewrite the perturbation expansion as a formal power series in $s = \sqrt{t}$. In Section 4, we employ commutator estimates to derive computable formula for each term in the expansion. finally, in Section 5, we rigorously justify our expansion by means of pseudodifferential calculus.

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2. Preliminaries

We consider the class of second-order differential operators L of the form (1.1) with coefficients uniformly bounded in $C_b^\infty(\mathbb{R}^N)$. This class forms a convex set,which we denote by $\mathbb{L}$, in the algebra of differential operators $\mathrm{Diff}(\mathbb{R}^N)$. We also define the subset of all operators in $\mathbb{L}$, satisfying the ellipticity estimate (1.2) for a given ellipticity constant γ, and we denote it by $\mathbb{L}_\gamma$. This definition can be extended to operators on manifolds of bounded geometry M (see [8, 21, 27]). For example, when $M = \mathbb{R}^N$ with the Euclidean metric, the class $\mathbb{L}$ considered in [21] recovers the class $\mathbb{L}$ considered in this paper.

In what follows, we denote the inner product on $\mathbb{R}^N$ by $(u, v) = \int_{\mathbb{R}^N} u(x)v(x)dx$. We also recall the definition of and some basic facts about L^p-based Sobolev spaces $W^{r,p}(\mathbb{R}^N)$. For $1 \le p \le \infty$, $r \in \mathbb{R}$:

$$W^{r,p}(\mathbb{R}^N) := \{u : \mathbb{R}^N \to \mathbb{C} \ \langle \xi \rangle^r \hat{u} \in L^p(\mathbb{R}^N)\}$$

$$W^{r,p}(\mathbb{R}^N) := \{u : \mathbb{R}^N \to \mathbb{C},\ (1 - \Delta)^{r/2} u \in L^p(\mathbb{R}^N)\},$$

where $\langle \xi \rangle := (1+|\xi|^2)^{1/2}$ and $\hat{u}$ is the Fourier Transform of u. If $r \in \mathbb{Z}_+$,

$$W^{r,p}(\mathbb{R}^N) = \{u : \mathbb{R}^N \to \mathbb{C},\ \partial_x^\alpha u \in L^p(\mathbb{R}^N),\ |\alpha| \le r\}.$$

Since we dimension N is fixed throughout the paper, we will usually write $W^{r,p}$ for $W^{r,p}(\mathbb{R}^N)$. When $1 < p < \infty$, the dual of $W^{r,p}$ is the Sobolev space $W^{-r,p'}$ with $1/p + 1/p' = 1$.

We are interested in considering the IVP (1.3) in the largest-possible space of initial data f where uniqueness holds. We therefore introduce exponentially weighted Sobolev spaces. Given a fixed point $x_0 \in \mathbb{R}^N$, we set $\langle x \rangle_{x_0} = (1 + |x - x_0|^2)^{1/2}$ and define $W_a^{m,p}(\mathbb{R}^N)$ for $m \in \mathbb{Z}_+$,

$a \in \mathbb{R}$, $1 < p < \infty$, by

$$\text{(2.1)} \quad W_a^{m,p}(\mathbb{R}^N) := e^{-a\langle x\rangle_{x_0}} W^{m,p}(\mathbb{R}^N)$$
$$= \{u : \mathbb{R}^N \to \mathbb{C},\ \partial_x^\alpha\big(e^{a\langle x\rangle_{x_0}} u(\cdot)\big) \in L^p(\mathbb{R}^N),\ |\alpha| \le m\},$$

with norm $\|u\|^p_{W_a^{m,p}} := \|e^{a\langle x\rangle_{x_0}} u\|^p_{W^{m,p}} = \sum_{|\alpha|\le m} \|\partial_\xi^\alpha\big(e^{a\langle x\rangle_{x_0}} u(x)\big)\|^p_{L^p}$. The spaces $W_a^{m,p}$ do not depend on the choice of the point x_0. We observe that $W_0^{m,p} = W^{m,p}$. The spaces $W_a^{m,p}$, for $m \in \mathbb{Z}_-$, can be defined by duality as $W_a^{m,p} = (W_{-a}^{-m,p'})^*$.

When $p = 2$, as customary we will write $W^{s,2} = H^s$, $W_a^{s,2} = H_a^s$.

We begin with recalling some properties of L and the associated solution operator e^{tL} to the IVP (1.3).

2.1. **Mapping properties.** Given a Banach X, we shall denote by $\mathcal{C}([0,\infty), X)$ the space of continuous functions $u : [0,\infty) \to X$. We assume that $X \subset L^1_{\rm loc}(\mathbb{R}^N)$, and that L is an unbounded operator on X with domain $\mathcal{D}(L)$. By a *classical solution* of (1.3) we mean a function

$$u \in \cap\mathcal{C}([0,\infty), X) \cap \mathcal{C}^1((0,\infty), X) \cap \mathcal{C}((0,\infty), \mathcal{D}(L)),$$

such that $\partial_t u = Lu$ in X for all $t > 0$. In particular, $u(0) = f$ must belong to the closure of $\mathcal{D}(L)$ in X. In the case of interest here, if $X = L^p$, then $\mathcal{D}(L) = W^{2,p}$, which is dense in L^p.

A family of (bounded) linear operators $U(t)$ on X, $t \ge 0$, will be called a $\mathcal{C}^0$ or *strongly continuous* semigroups of operators if $U(t)$ forms a semigroup in t and $U(t)u - u \to 0$ in X as $t \to 0_+$. $U(t)$ will be called an *analytic* semigroup if $U(t)$ is an analytic function of t in a sector $\Delta \subset \mathbb{C}$ that contains the nonnegative real axis with continuity at 0 [26]. In particular, an analytic semigroup is a $\mathcal{C}^0$ semigroup.

If $L \in \mathbb{L}_\gamma$, then the solution operator e^{tL} of (1.3) forms an analytic semigroup in $L^p(\mathbb{R}^N)$ for any $1 < p < \infty$ (see e.g. [20, Theorem 3.1.3, page 73]). This result follows from spectral properties of L on Sobolev spaces, and in particular from the useful lemma below, the proof of which is standard.

Lemma 2.1. *Let $L \in \mathbb{L}_\gamma$. There exist a constant $C > 0$, uniform on $\mathbb{L}_\gamma$ such that*

$$C(u,u) \le -(Lu,u) \le \gamma(\nabla u, \nabla u) + C(u,u).$$

For any $m \in \mathbb{Z}_+$, $1 < p < \infty$, the norm $|||v|||_{2m} := \|u\|_{L^p} + \|L^m u\|_{L^p}$ is equivalent to the norm $\|\cdot\|_{W^{2m,p}}$ on $W^{2m,p}(\mathbb{R}^N)$.

The fact that e^{tL} is an analytic semigroup on L^p has several consequences for the solvability of the problem (1.3) and mapping properties of e^{tL}. For the sake of clarity and completeness, we include here a sketch of the proofs. Our proofs also serve the purpose to justify the perturbative expansion discussed in Section 2.2. Further details can be found in [20, 26]. Below $p \in (1, \infty)$ is fixed, and the constants appearing in the estimates depend on p, but not on $L \in \mathbb{L}_\gamma$.

Proposition 2.2. *There exist constants $C, C_r > 0$ and $\omega \in \mathbb{R}$ such that, for any $L \in \mathbb{L}_\gamma$:*

(i) The problem (1.3) *has a unique classical solution $u(t) = e^{tL} f$, $t \in [0, \infty)$, for each $f \in L^p$. This solution satisfies $\|e^{tL} f\|_{L^p} \leq Ce^{\omega t}\|f\|_{L^p}$.*

(ii) For each $r \in \mathbb{R}$, $\|e^{tL} f\|_{W^{r,p}} \leq C_{r,T}\|f\|_{W^{r,p}}$, $t \in (0, T]$, $0 < T < \infty$.

Proof. Lemma 2.1 (i) gives that e^{tL} is a C^0 semigroup by the Hille-Yosida theorem. Consequently, we have $\|e^{tL} f\|_{L^2} \leq Ce^{\omega t}\|f\|_{L^2}$ for some constants $C > 0$ and $\omega \in \mathbb{R}$ independent of $L \in \mathbb{L}_\gamma$, by standard properties of C^0 semigroups in Banach spaces. To prove (ii), we notice that (i) implies, for $t \in [0, T]$,

$$\begin{aligned}\|e^{tL} f\|_{W^{2m,p}} &\leq C|||e^{tL} f|||_{2m,p} = C\big(\|e^{tL} f\|_{L^p} + \|L^m e^{tL} f\|_{L^p}\big) \\ &= C\big(\|e^{tL} f\|_{L^p} + \|e^{tL} L^m f\|_{L^p}\big) \leq C\big(\|f\|_{L^p} + \|L^m f\|_{L^p}\big) = C|||f|||_{2m} \\ &\leq C\|f\|_{W^{2m,p}},\end{aligned}$$

with C depending on m and T. Thought L may not be self-adjoint, the adjoint L^* is an operator of the same type, so the estimate above holds for L^*. We can then extend it to $W^{-2m,p}$, $m \in \mathbb{Z}_+$, by duality and to any $W^{r,p}$ by interpolation (see e.g. [5]). This completes the proof. $\square$

We discuss smoothing properties of e^{tL}, it is convenient to first assume $L^* = L$, that is that L is self-adjoint. We will remove this assumption afterwards.

Corollary 2.3. *There exist constants $C_{r,s} > 0$ such that, for any $L \in \mathbb{L}_\gamma$ with $L = L^*$:*

(i) $\|e^{tL} f\|_{W^{r,p}(\mathbb{R}^N)} \leq C_{r,s} t^{(s-r)/2}\|f\|_{W^{s,p}(\mathbb{R}^N)}$, $r \geq s$ real.

(ii) There exists a $\mathcal{G}^L_t(x, y) \in \mathcal{C}^\infty((0, \infty) \times \mathbb{R}^N \times \mathbb{R}^N)$ such that

$$u(t, x) = \int_{\mathbb{R}^N} \mathcal{G}^L_t(x, y) f(y) dy. \tag{2.2}$$

Proof. The part (i) can be proved using resolvent estimates and a scaling-in-time argument (see e.g. [21] in the context of manifolds with bounded geometry). Part (ii) folllows from the Schwartz kernel theorem (see for example [28, Chapter 7]), since from (i) e^{tL} maps compactly supported distributions in $\mathcal{E}'(\mathbb{R}^N)$ to smooth functions in $C^\infty(\mathbb{R}^N)$. In fact, if we denote by $<,>$ the duality pairing between C^∞ and $\mathcal{E}'$, we explicitly have:

$$\mathcal{G}_t^L(x,y) = < \delta_x, e^{tL}\delta_y >, \tag{2.3}$$

where δ_x, δ_y represent Dirac delta's at x and y respectively. □

We now proceed to eliminate the assumption that $L^* = L$ in the above result. First, let us notice that if $L, L_0 \in \mathbb{L}_\gamma$, $V = L - L_0$. Then (1.3) becomes

$$\begin{cases} \partial_t u - L_0 u = Vu & \text{in } (0,\infty) \times \mathbb{R}^N \\ u = f \in \mathcal{C}^\infty(\mathbb{R}^N) & \text{on } \{0\} \times \mathbb{R}^N. \end{cases} \tag{2.4}$$

It is well-know that applying Duhamel's formula, gives a Volterra integral equation of the first kind for u. If $L = L_0^*$, the solution of the integral equation is a classical solution of (2.4). in fact, it is enough that e^{tL_0} generates an analytic semigroup (see [26, Theorem 2.4, page 107]).

Lemma 2.4. *Assume that $L, L_0 \in \mathbb{L}_\gamma$, and that $L_0 = L_0^*$. Then, the solution $u(t)$ to the problem (2.4) is given by:*

$$\begin{aligned} u(t) := e^{tL} f &= e^{tL_0} f + \int_0^t e^{(1-\tau)L_0}(L - L_0) e^{\tau L} f d\tau \\ &= e^{tL_0} f + \int_0^t e^{(1-\tau)L_0} V e^{\tau L} f ds, \end{aligned} \tag{2.5}$$

for any initial data $f \in W^{s,p}$, $s \in \mathbb{R}$, $1 < p < \infty$.

Proof. We fix $f \in W^{s,p}$. By integrating in time in (2.4), u formally satisfies the following Volterra integral equation for $t \in [0,T]$:

$$u(t) = e^{tL_0} f + \int_0^t e^{(1-\tau)L_0} V u(\tau)\, d\tau = \Phi_T(u),$$

that is, u is formally a fixed point of the affine map Φ_T. We take $0 < T \leq 1$, and let B_M be a ball of radius M about the origin in $C([0,T], W^{s,p}(\mathbb{R}^N))$. By Proposition 2.2, part (ii, since $V \in \mathbb{L}$, $\|Ve^{\tau L} v\|_{W^{s-2,p}} \leq C(s,p)M$, so that by Corollary 2.3, part (i), $\|e^{(1-\tau)L_0} V e^{\tau L} v\|_{W^{s,p}} \leq C(s,p)\tau^{-1/2} M$. Hence, Φ is a contraction from B_M to B_M if $M > (1 - CT^{1/2})^{-1} e^\omega \|f\|_{W^{s,p}}$ and T is small enough. Therefore, there exists a unique fixed point u in B_M. Since the equation is linear and M does not depend on T, as long as $T \in (0, T_0]$, T_0 small enough, bootstrapping leads to the existence and uniqueness of u in $C([0,\infty), W^{s,p})$.

Therefore, when $f \in L^p$ u must agree with the unique classical solution to (2.4), which exists by Proposition 2.2, since $e^{t\,L_0}$ is an analytic semigroup, so $e^{tL_0}v$ is smooth in x and differentiable in t, for $t > 0$. $\square$

We now extend Corollary 2.3 to non self-adjoint operators L and to the exponentially weighted spaces $W_a^{s,p}$.

Proposition 2.5. *Let $r \geq s$ and $a \in \mathbb{R}$ in a bounded set. There exists a constant $C > 0$ such that for any $L \in \mathbb{L}_\gamma$ and $t \in (0, 1]$, we have*

$$\|u(t)\|_{W_a^{r,p}} \leq Ct^{(s-r)/2}\|f\|_{W_a^{s,p}(\mathbb{R}^N)}.$$

The constant C above is independent of r, s, and a, as long as they belong to a bounded set. We also recall that $W_a^{s,p}$ is independent of the choice of the point x_0 (c.f. (2.1)), so that C is uniform in $x_0 \in \mathbb{R}^N$.

Proof. We first observe that proving the result for L and $a \neq 0$ is the same as proving the result for $L_1 := e^{a\langle\xi\rangle}Le^{-a\langle\xi\rangle}$ and $a = 0$. Since for bounded a we can assume that $L_1 \in \mathbb{L}_\gamma$ (by increasing the defining constants of $\mathbb{L}_\gamma$, if necessary), we may assume that $a = 0$.

Let us assume also that $s \leq r < s + 1$. Let $L_0 := (L + L^*)/2$ and $V = L - L_0$. Then $V \in \mathbb{L}$. By Lemma 2.4,

$$e^{tL} = e^{tL_0} + \int_0^t e^{(t-\tau)L_0}Ve^{\tau L}d\tau.$$

Hence, the norm $\|e^{tL}\|_{W^{s,p}\to W^{r,p}}$ of e^{tL} as linear map $W^{s,p} \to W^{r,p}$ is bounded by

$$\|e^{tL_0}\|_{W^{s,p}\to W^{r,p}}+\int_0^t \|e^{(t-\tau)L_0}\|_{W^{s-1,p}\to W^{r,p}}\|V\|_{W^{s,p}\to W^{s-1,p}}\|e^{\tau L_1}\|_{W^{s,p}\to W^{s,p}}d\tau$$

$$\leq C\Big(t^{(s-r)/2}+\int_0^t (t-\tau)^{(s-1-r)/2}d\tau\Big) = C\big(t^{(s-r)/2}+t^{(s-r+1)/2}\big) \leq Ct^{(r-s)/2},$$

where we have used in the last inequality that $0 < t \leq 1$, and where $C > 0$ is a generic constant, different at each appearance. The general case follows from this one by writing $e^{tL} = \big(e^{tL/m}\big)^m$, for $m \geq r - s$ and then writing $\|e^{tL}\|_{W^{s,p}\to W^{r,p}} \leq C(t/m)^{m(s-r)/(2m)} = C't^{(s-r)/2}$. $\square$

In particular, we obtain the existence of the Green's function $\mathcal{G}_t^L(x, y)$ for any $L \in \mathbb{L}_\gamma$, defined again via formula (2.3). We shall also write $\mathcal{G}_t^L(x, y) = e^{tL}(x, y)$. For constant coefficient operators, the Green's function can be determined explicitly.

Remark 2.6. If L (1.1) is a constant coefficient operator

$$L^0 = \sum_{i,j=1}^{n} a_{ij}^0 \partial_i \partial_j + \sum_{k=1}^{n} b_k^0 \partial_k + c^0 \tag{2.6}$$

and $A^0 := (a_{ij}^0)$ is the matrix of highest order coefficients, assumed to satisfy $a_{ij}^0 = a_{ji}^0$, we have the explicit formula

$$\mathcal{G}_t^{L^0}(x,y) = e^{tL^0}(x,y) = \frac{e^{c^0 t}}{\sqrt{(4\pi t)^n \det(A^0)}} e^{\frac{(x+b^0t-y)^t (A^0)^{-1} (x+b^0t-y)}{4t}}. \tag{2.7}$$

2.2. **Perturbative expansion.** The purpose of this section is to obtain a time-ordered perturbative expansion of e^{tL}, $L \in \mathbb{L}_\gamma$, in terms of e^{tL_0} for a fixed element $L_0 \in \mathbb{L}_\gamma$. Later, L_0 will be obtained by freezing the highest-order coefficients of L at a given point x_0 and dropping the lower-order terms. This expansion is well-known. Here, we concentrate on obtaining global error estimates in weighted Sobolev spaces.

For any two normed spaces X and Y, we denote by $\mathcal{B}(X,Y)$ the normed space of continuous operators $X \to Y$. For each $k \in \mathbb{Z}_+$, we also denote by

$$\begin{aligned}\Sigma_k &:= \{\tau = (\tau_0, \tau_1, \dots, \tau_k) \in \mathbb{R}^{k+1},\ \tau_j \geq 0, \sum \tau_j = 1\} \\ &\simeq \{\sigma = (\sigma_1, \dots, \sigma_k) \in \mathbb{R}^k,\ 1 \geq \sigma_1 \geq \sigma_2 \geq \dots \sigma_{k-1} \geq \sigma_k \geq 0\}\end{aligned}$$

the *standard unit simplex* of dimension k. The bijection above is given by $\sigma_j = \tau_j + \tau_{j+1} + \ldots + \tau_k$. Using this bijection, for any operator-valued function f of $\mathbb{R}^N$ we can write

$$\int_{\Sigma_k} f(\tau) d\tau = \int_0^1 \int_0^{\sigma_1} \dots \int_0^{\sigma_{k-1}} f(1-\sigma_1, \sigma_1-\sigma_2, \dots, \sigma_{k-1}-\sigma_k, \sigma_k) d\sigma_k \dots d\sigma_1$$

Throughout, operator-valued integrals are taken in the sense of Bochner (see e.g. [33]). We begin with a preliminary lemma.

Lemma 2.7. *Let $L, L_0 \in \mathbb{L}_\gamma$ and let V_j be such that $\langle x\rangle^{-b_j} V \in \mathbb{L}$, $j = 1, \dots, k$, $k \in \mathbb{Z}_+$. Then*

$$\Phi(\tau) = e^{\tau_0 L_0} V_1 e^{\tau_1 L_0} \dots e^{\tau_{k-1} L_0} V_k e^{\tau_k L_0}, \qquad \tau \in \Sigma_k$$

and

$$\Phi(\tau) = e^{\tau_0 L_0} V_1 e^{\tau_1 L_0} \dots e^{\tau_{k-1} L_0} V_k e^{\tau_k L}, \qquad \tau \in \Sigma_k$$

define functions $\Phi \in L^1(\Sigma_k, \mathcal{B}(W_a^{s,p}(\mathbb{R}^N), W_{a-k|b|}^{r,p}(\mathbb{R}^N)))$ for any $a \in \mathbb{R}$, $r \geq s$, $1 < p < \infty$, and any multi index $b = (b_1, \dots, b_k) \in \mathbb{Z}_+^k$.

Above we use the standard multi index notation $|b| = \sum_{j=1}^k b_j$.

Proof. First, we observe that by definition the sets $\mathcal{V}_j := \{\tau_j > \frac{1}{2(k+1)}\}$, $j = 1, \ldots, k$, cover σ_k. On each set $\mathcal{V}_j$, we use uniform in time boundedness in $W^{s,p}_a$ or $W^{r,p}_a$, which follows from Proposition 2.5 with $r = s$, for each operator $e^{\tau_i L_0}$, $i \neq j$ (and same for $e^{\tau_i L}$ if $i = k$), while we employ the mapping properties in Proposition 2.5 with $r \geq s$ for $e^{\tau_j L_0}$ (or $e^{\tau_j L}$ if $j = k$). If $V_j \in \mathbb{L}$, we conclude that

$$\Phi(\tau) : W^{s,p}_a \to W^{r,p}_a, \qquad \tau \in \mathcal{V}_j,$$

with operator norm $\sim C\tau_j^{(s-r-2k)/2} \in L^1(\mathcal{V}_j)$, since each V_j is second order and they appear k times. If $\langle x\rangle^{-b_j} V_j \in \mathbb{L}$, we have instead:

$$\Phi(\tau) : W^{s,p}_a \to W^{r,p}_{a-k|b|}, \qquad \tau \in \mathcal{V}_j,$$

with operator norm $\sim C\tau_j^{(s-r-2k)/2} \in L^1(\mathcal{V}_j)$, since we loose a factor of b_j in the weight when V_j is applied. □

By iterating Duhamel's formula in Lemma 2.4, we obtain a time-ordered expansion of e^{tL}.

Proposition 2.8. *Let $d \in \mathbb{Z}_+$. Then, for each $L, L_0 \in \mathbb{L}_\gamma$,*

$$\begin{aligned} e^{tL} &= e^{tL_0} + t\int_{\Sigma_1} e^{t\tau_0 L_0} V e^{t\tau_1 L_0} d\tau \\ &+ t^2 \int_{\Sigma_2} e^{t\tau_0 L_0} V e^{t\tau_1 L_0} V e^{t\tau_2 L_0} d\tau + \cdots + \\ &+ t^d \int_{\Sigma_p} e^{t\tau_0 L_0} V e^{t\tau_1 L_0} \ldots e^{t\tau_{d-1} L_0} V e^{t\tau_d L_0} d\tau \\ &+ t^{d+1} \int_{\Sigma_{d+1}} e^{t\tau_0 L_0} V e^{t\tau_1 L_0} \ldots e^{t\tau_d L_0} V e^{\tau_{d+1} L} d\tau, \end{aligned} \tag{2.8}$$

where $V = L - L_0$, and each integral is a well-defined Bochner integral.

The positive integer d will be called the *iteration level* of the approximation. Later in the paper, V will be replaced by a Taylor approximation of L, so that V will have polynomial coefficients in x, so we have included this case in the lemma above.

Proof. Recall that Lemma 2.4 gives

$$e^{tL} - e^{tL_0} = \int_0^t e^{(1-\zeta)L_0} V e^{\zeta L)} d\zeta = \int_0^1 e^{t(1-\tau)(L_0)} V e^{t\tau L)} t d\tau.$$

with the substitution $\zeta = t\tau$. This is in fact our result for $p = 1$.

The result for any p then follows by induction using the above formula for $\xi = t\sigma_p$. Explicitly, for $t = 1$:

$$
\begin{aligned}
e^L &= e^{L_0} + \int_{\Sigma_1} e^{(1-\sigma_1)L_0} V e^{\sigma_1 L_0} d\sigma + \int_{\Sigma_2} e^{(1-\sigma_1)L_0} V e^{(\sigma_1-\sigma_2)L_0} V e^{\sigma_2 L_0} d\sigma \\
&\quad + \cdots + \int_{\Sigma_{d-1}} e^{(1-\sigma_1)L_0} V \ldots e^{(\sigma_{d-2}-\sigma_{n-1})L_0} V e^{\sigma_{d-1} L} d\sigma \\
&= e^{L_0} + \int_{\Sigma_1} e^{(1-\sigma_1)L_0} V e^{\sigma_1 L_0} d\sigma + \int_{\Sigma_2} e^{(1-\sigma_1)L_0} V e^{(\sigma_1-\sigma_2)L_0} V e^{\sigma_2 L_0} d\sigma \\
&\quad + \cdots + \int_{\Sigma_{d-1}} e^{(1-\sigma_1)L_0} V \ldots e^{(\sigma_{d-2}-\sigma_{n-1})L_0} V e^{\sigma_{d-1} L_0} d\sigma + \ldots \\
&\quad + \int_{\Sigma_{d-1}} \int_0^{\sigma_{d-1}} e^{(1-\sigma_1)L_0} V \ldots e^{(\sigma_{d-2}-\sigma_{d-1})L_0} V e^{(\sigma_{d-1}-\sigma_d)L_0} V e^{\sigma_d L} d\sigma d\sigma_n \\
&= e^{L_0} + \int_{\Sigma_1} e^{(1-\sigma_1)L_0} V e^{\sigma_1 L_0} d\sigma + \int_{\Sigma_2} e^{(1-\sigma_1)L_0} V e^{(\sigma_1-\sigma_2)L_0} V e^{\sigma_2 L_0} d\sigma \\
&\quad + \cdots + \int_{\Sigma_d} e^{(1-\sigma_1)L_0} V e^{(\sigma_1-\sigma_2)L_0} \ldots e^{(\sigma_{d-1}-\sigma_d)L_0} V e^{\sigma_d L} d\sigma,
\end{aligned}
$$

where each integral is well defined as a Bochner integral by the Lemma. □

3. Local dilations and perturbative expansions

In this section, we tackle the task of deriving an algorithmically computable approximation to e^{tL}. We exploit the perturbative expansion (2.8) with L_0 the operator obtained by freezing the highest-order coefficents of L at a given, but arbitrary, point $x_0 \in \mathbb{R}^N$, and dropping the lower-order terms (see (3.10a) below). Then, we approximate $L - L_0$ by an appropriate Taylor expansion, so that each of the terms in (2.8) except the last one can be explicitly computed using commutators, as discussed in Section 4.

First, using a suitable rescaling in space and time, we replace the problem of determining an asymptotic expansion of the kernel $\mathcal{G}_t^L(x, y) = e^{tL}(x, y)$ of e^{tL} by the problem of determining an asymptotic expansion of the kernel $\mathcal{G}_1^{L^{s,x_0}}(x, y) = e^{L^{s,x_0}}(x, y)$ of $e^{L^{s,x_0}}$ for a suitable family of operators L^{s,x_0} parameterized by $s = \sqrt{t}$, and by the point $x_0 \in \mathbb{R}^N$. The point x_0 is fixed throughout this section, but it will be allowed to vary later on and eventually we will set $x_0 = x$. The family L^{s,x_0} has limit precisely L_0. Since we will let x_0 vary later, we write $L_0 = L_0^{x_0}$.

For any $s > 0$, we consider the action on functions of dilating x by s about x_0 and t by s^2 about 0. If $f : \mathbb{R}^N \to \mathbb{R}$, $u : [0, \infty) \times \mathbb{R}^N \to \mathbb{R}$,

we then set
(3.1)

$$f^{s,x_0}(x) := f(x_0 + s(x - x_0)), \qquad u^{s,x_0}(t,x) := u(s^2 t, x_0 + s(x - x_0)),$$

and,

$$L^{s,x_0} := \sum_{i,j=1}^{N} a_{ij}^{s,x_0}(x)\partial_i\partial_j + s\sum_{i=1}^{N} b_i^{s,x_0}(x)\partial_i + s^2 c^{s,x_0}(x). \tag{3.2}$$

Then

$$L^{s,x_0}u^{s,x_0} = s^2(Lu)^{s,x_0}, \quad (\partial_t - L^{s,x_0})u^{s,x_0} = s^2[(\partial_t - L)u]^{s,x_0} \tag{3.3}$$

In particular, we have the following simple lemma, which we record for further reference.

Lemma 3.1. *If u solves* (2.4), *then u^{s,x_0} solves*

$$\begin{cases} \partial_t u^{s,x_0} - L^{s,x_0}u^{s,x_0} = 0 & \textit{in } (0,\infty)\times\mathbb{R}^N \\ u^{s,x_0} = f^{s,x_0} \in \mathcal{C}_c^\infty(\mathbb{R}^N) & \textit{on } \{0\}\times\mathbb{R}^N. \end{cases} \tag{3.4}$$

3.1. **Dilations and Green's functions.** We want to study the IVP (3.4) and the Green's function of its associated solution operator $e^{tL^{s,x_0}}$. We can reduce to study the special case $x_0 = 0$.

The definition of the Green's function and Lemma 3.1 then gives

$$\begin{aligned} u^{s,0}(t,x) &= \int_{\mathbb{R}^N} \mathcal{G}_t^{L^{s,0}}(x,y) f^{s,x_0}(y)dy = \int_{\mathbb{R}^N} \mathcal{G}_t^{L^{s,0}}(x,y) f(sy)dy \\ &= s^{-N}\int_{\mathbb{R}^N} \mathcal{G}_t^{L^{s,0}}(x,\frac{y}{s}) f(y)dy. \end{aligned} \tag{3.5}$$

On the other hand,

$$u^{s,0}(t,x) = u(s^2t, sx) = \int_{\mathbb{R}^N} \mathcal{G}_{s^2t}^{L}(sx,y) f(y)dy,$$

which implies

$$\mathcal{G}_t^{L^{s,0}}(x,\frac{y}{s}) = s^N\mathcal{G}_{s^2t}^{L}(sx,y) \;\Leftrightarrow\; \mathcal{G}_t^{L^{s,0}}(x,y) = s^N\mathcal{G}_{s^2t}^{L}(sx,sy).$$

In other words

$$\mathcal{G}_t^{L}(x,y) = s^{-N}\mathcal{G}_{s^{-2}t}^{L^{s,0}}(s^{-1}x, s^{-1}y)$$

If we now translate to $x_0 \neq 0$ and choose $s = \sqrt{t}$, we obtain the desired correspondence between $\mathcal{G}_t^L$ and $\mathcal{G}_1^{L^{s,x_0}}$, which we also record for further reference.

Lemma 3.2. *Assume $L \in \mathbb{L}$ and let x_0 be a fixed, but arbitrary, point in $\mathbb{R}^N$. Then, for any $s > 0$,*

$$\begin{aligned}\mathcal{G}_t^L(x,y) &= s^{-N}\mathcal{G}_1^{L^{s,x_0}}(x_0 + s^{-1}(x - x_0), x_0 + s^{-1}(y - x_0)) \\ &= t^{-\frac{N}{2}}\mathcal{G}_1^{L^{\sqrt{t}}}(x_0 + t^{-\frac{1}{2}}(x - x_0), x_0 + t^{-\frac{1}{2}}(y - x_0)), \ \textit{if } s = t^{-\frac{1}{2}}.\end{aligned}$$

3.2. **Perturbative expansion of $e^{L^{s,x_0}}$.** Since Lemma 3.2 gives us an immediate procedure for obtaining the Green function $\mathcal{G}_t^L(x,y)$ of $\partial_t - L$ from the Green's function $\mathcal{G}_t^{L^{s,x_0}}(x,y)$ of $\partial_t - L^{s,x_0}$, we now concentrate on obtaining a perturbative expansion for the latter.

We write $V_1^{s,x_0} := L^{s,x_0} - L_0^{x_0}$. Then, V_1^{s,x_0} takes the role of V in the perturbative expansion (2.8) for the operator $e^{L^{s,x_0}}$, that is:

$$\begin{aligned}e^{L^{s,x_0}} = e^{L_0^{x_0}} + \int_{\Sigma_1} e^{\tau_0 L_0^{x_0}} V_1^{s,x_0} e^{\tau_1 L_0^{x_0}} d\tau W_1^{s,x_0} e^{\tau_2 L_0^{x_0}} d\tau + \cdots + \\ \int_{\Sigma_d} e^{\tau_0 L_0^{x_0}} V_1^{s,x_0} e^{\tau_1 L_0^{x_0}} \ldots e^{\tau_{d-1} L_0^{x_0}} V_1^{s,x_0} e^{\tau_d L_0^{x_0}} d\tau + \\ \int_{\Sigma_{d+1}} e^{\tau_0 L_0^{x_0}} V_1^{s,x_0} e^{\tau_1 L_0^{x_0}} \ldots e^{\tau_d L_0^{x_0}} V_1^{s,x_0} e^{\tau_{d+1} L^{s,x_0}} d\tau.\end{aligned} \tag{3.6}$$

In a sense to be made precise below, $V_1^s = \mathcal{O}(s)$. Consequently, if we let the iteration level $d \to \infty$ in (2.8) we obtain a formal power series in s. We will rigorously show in Section 5 using the exponentially weighted Sobolev spaces $W_a^{s,p}$ that (2.8) gives rises to an asymptotically convergent series in s as $s \to 0$ and will derive global error bounds in $W^{s,p}$ and $W_a^{s,p}$ for the partial sums.

Let $n \in \mathbb{Z}_+$ be a fixed integer and consider the Taylor expansion of the operator L^{s,x_0} up to order n in s around $s = 0$,

$$L^{s,x_0} = \sum_{m=0}^{n} s^m L_m^{x_0} + V_{n+1}^{s,x_0} \tag{3.7}$$

were V_{n+1}^{s,x_0} is the remainder term in the expansion:

$$V_{n+1}^{s,x_0} = s^{n+1} L_{n+1}^{s,x_0}.$$

The operators $L_m^{x_0}$, $1 \leq m \leq n$, are given by

$$L_m^{x_0} := \frac{1}{m!}\left(\frac{d^m}{ds^m} L^{s,x_0}\right)\Big|_{s=0}, \tag{3.8}$$

and are independent of s, while

$$L_{n+1}^{s,x_0} := \frac{1}{(n+1)!}\left(\frac{d^{n+1}}{d\theta^{n+1}} L^{\theta,x_0}\right)\Big|_{\theta=\alpha s}, \tag{3.9}$$

for some $0 < \alpha < 1$, and hence it still depends on s.

Remark 3.3. From the form of L^{s,x_0} in equation (3.2) it follows that the operators $L^{x_0}_m$, L^{s,x_0}_{n+1} have coefficients that are *polynomials in* $(x-x_0)$ *of degree at most* m. The coefficients of the polynomials themselves are bounded functions of x_0. More precisely, the coefficients of the second order derivative terms are of degree at most m in $(x-x_0)$, while the coefficients of the first order derivatives term are of degree at most $m-1$ in $(x-x_0)$, and the coefficients of the zero order derivative term is of degree at most $m-2$ in $(x-x_0)$.

The first few terms of the Taylor expansions are explicitly:

$$L^{x_0}_0 = \sum_{i,j=1}^{N} a_{ij}(x_0)\partial_i\partial_j, \tag{3.10a}$$

$$L^{x_0}_1 = \sum_{i,j=1}^{N} \left((x-x_0)\cdot\nabla a_{ij}(x_0)\right)\partial_i\partial_j + \sum_{i=1}^{N} b_i(x_0)\partial_i, \tag{3.10b}$$

$$\begin{aligned} L^{x_0}_2 = \sum_{i,j=1}^{N} \frac{1}{2}\big((x-x_0)^T\ \nabla^2 a_{ij}(x_0)\ (x-x_0)\big)\,\partial_i\partial_j + \\ + \sum_{i=1}^{N}((x-x_0)\cdot\nabla b_i(x_0))\partial_i + c(x_0). \end{aligned} \tag{3.10c}$$

Since $L^{x_0}_0$ has coefficients that are constant in x, from formula (2.7) we obtain

$$e^{tL^{x_0}_0} = \frac{1}{\sqrt{(4\pi t)^N \det A^0}} e^{\frac{(x-y)^t (A^0)^{-1}(x-y)}{4t}}, \tag{3.11}$$

where $A^0 := A(x_0)$.

Furthermore $V^s_1 := L^{s,x_0} - L^{x_0}_0$ can be written as $V^s_1 := \sum_{m=0}^{n} s^m L^{x_0}_m + V^{s,x_0}_{n+1}$, which gives a more precise form to (3.6) as a formal power series in s. To describe each term of this power series more explicitly and to formulate the main results in this section we need to introduce some notation.

Definition 3.4. *For any positive fixed integers k,ℓ we shall denote by $\mathfrak{A}_{k,\ell}$ the set of multi-indexes $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in \mathbb{Z}^k_+$, such that $|\alpha| := \sum \alpha_j = \ell$. Furthermore, we define $\mathfrak{A}_\ell := \bigcup_{k=1}^{\ell} \mathfrak{A}_{k,\ell}$.*

We note that, since $\alpha_i \geq 1$, the set $\mathfrak{A}_{k,\ell}$ is empty if $\ell < k$.

Proposition 3.5. *The set $\mathfrak{A}_\ell$ contains $2^{\ell-1}$ elements.*

Proof. For any given, $1 \le k \le \ell$, the number of elements in the set $\mathfrak{A}_{k,\ell}$ is the number of sequences $\{\alpha_1, \alpha_2, ...\alpha_k\}$ of size k which add up to ℓ and is, therefore, given by $\binom{\ell-1}{\ell-k}$. Therefore the number of elements in $\mathfrak{A}_\ell$ is given by $\sum_{k=1}^{\ell} \binom{\ell-1}{\ell-k} = \sum_{k=0}^{\ell-1} \binom{\ell-1}{k} = 2^{\ell-1}$. □

We shall need the following lemmas on the behavior of each operator in the Taylor expansion of L^{s,x_0}. We recall that $\langle x \rangle_{x_0} = (1+|x-x_0|^2)^{1/2}$. The first is an immediate consequence of Remark 3.3.

Lemma 3.6. *The family*

$$\{\langle x \rangle_{x_0}^{-j} L_j^{x_0},\ j = 0, \dots, n,\ \langle x \rangle_{x_0}^{-n-1} L_{n+1}^{s,x_0};\ s \in (0,1],\ x_0 \in \mathbb{R}^N\}$$

defines a bounded subset of $\mathbb{L}$.

Lemma 3.7. *For each given* $\epsilon > 0$, *the family*

$$\{e^{-\epsilon \langle x \rangle_{x_0}} L_j^{x_0},\ j = 0, \dots, n,\ e^{-\epsilon \langle x \rangle_{x_0}} L_{n+1}^{s,x_0};\ s \in (0,1],\ x_0 \in \mathbb{R}^N\}$$

is a bounded subset of $\mathbb{L}$.

Proof. The lemma follows from Lemma lemma.poly.growth and the simple observation that $\langle x \rangle_{x_0}^{j} e^{-\epsilon \langle x \rangle_{x_0}} \le C$, with C independent of x_0. □

We are now in the position to describe the expansion 3.6 more explicitly. We recall that d is the iteration level of the approximation and n is the order of the Taylor expansion. Up to this point, n and p are unrelated. Below, *we choose* $n \ge d$. We also recall that $\mathfrak{A}_{k,\ell} \equiv \emptyset$, if $\ell < k$. This condition will be understood.

Definition 3.8. *Assume* $1 \le k \le d+1$, $1 \le \ell \le (n+1)^2$. *For each multi-index* $\alpha \in \mathfrak{A}_{k,\ell}$, *we let*

$$\Lambda_{\alpha,x_0}^{k,\ell} := \int_{\Sigma_k} e^{\tau_0 L_0^{x_0}} L_{\alpha_1}^{x_0} e^{\tau_1 L_0^{x_0}} L_{\alpha_2}^{x_0} \cdots L_{\alpha_k}^{x_0} e^{\tau_k L_0^{x_0}} ds, \tag{3.12}$$

if $1 \le \ell \le (n+1), 1 \le k \le d$, *and*

$$\Lambda_{\alpha,s,x_0}^{d+1,\ell} := \int_{\Sigma_k} e^{\tau_0 L_0^{x_0}} L_{\alpha_1}^{x_0} e^{\tau_1 L_0^{x_0}} L_{\alpha_2}^{x_0} \cdots L_{\alpha_k}^{x_0} e^{\tau_k L^{s,x_0}} ds, \tag{3.13}$$

if $1 \le \ell \le (n+1), k = d+1$. *Then, we set*

$$\Lambda_{x_0}^{\ell,k} := \sum_{\alpha \in \mathfrak{A}_{\ell,k}} \Lambda_{\alpha,x_0}^{k,l}, \qquad \text{if } 1 \le \ell \le (n+1), 1 \le k \le d, \tag{3.14}$$

and

$$\Lambda_{s,x_0}^{d+1,\ell} := \sum_{\alpha \in \mathfrak{A}_{d+1,\ell}} \Lambda_{\alpha,s,x_0}^{d+1,\ell}, \qquad \text{if } 1 \le \ell \le (n+1), k = d+1. \tag{3.15}$$

If $\ell \geq (n+1)$ in (3.12) or (3.13), some or all of the operators $L^{x_0}_{\alpha_j}$ will correspond in fact to L^{s,x_0}_{n+1}. We distinguish this case by writing $\Lambda^{k,\ell}_{\alpha,s,x_0}$ and $\Lambda^{k,\ell}_{s,x_0}$ even when $k < d+1$.

In what follows, when no confusion may arise, we will drop the explicit dependence on x_0. However, in Section 5, x_0 will be allowed to vary and we will reinstate the full notation. We also observe that each $\Lambda^{k,l}_{x_0}$ or $\lambda^{k,\ell}_{s,x_0}$ is well defined as a Bochner integral by Lemma 2.7.

With this definition, the perturbative expansion (3.6) can be written as follows:

$$
\begin{aligned}
e^{L^{s,x_0}} &= e^{L_0^{x_0}} + \sum_{\ell=1}^{n}\sum_{k=1}^{d} s^\ell \Lambda^{k,\ell}_{x_0} + \\
&\quad + \sum_{\ell=n+1}^{(n+1)^2}\sum_{k=1}^{d} s^\ell \Lambda^{k,\ell}_{s,x_0} + \sum_{\ell=d+1}^{(n+1)^2} s^\ell \Lambda^{d+1,\ell}_{s,x_0} \\
&= e^{L_0^{x_0}} + \sum_{\ell=1}^{n}\sum_{k=1}^{d} s^\ell \Lambda^{k,\ell}_{x_0} + \\
&\quad + s^{d+1}\left(\sum_{\ell=n+1}^{(n+1)^2}\sum_{k=1}^{d} s^{(\ell-d-1)} \Lambda^{k,\ell}_{s,x_0} + \sum_{\ell=d+1}^{(n+1)^2} s^{(\ell-d-1)} \Lambda^{d+1,\ell}_{s,x_0}\right) \\
&= e^{L_0^{x_0}} + \sum_{\ell=1}^{n}\sum_{k=1}^{d} s^\ell \Lambda^{k,\ell}_{x_0} + s^{d+1}\mathbb{E}^{s,x_0}_{d,n},
\end{aligned}
\tag{3.16}
$$

where $\mathbb{E}^{s,x_0}_{d,n}$ represents the error in the approximation. We see from this formula that in order to have a consistent expansion, at least formally, *we must choose $n = d$, as we will do from now on.* Then, the approximation $e^{L^{s,x_0}} \sim e^{L_0^{x_0}} + \sum_{\ell=1}^{n}\sum_{k=1}^{\ell} s^\ell \Lambda^{k,\ell}_{x_0}$ is "accurate" to order s^n. We will rigorously justify this assertion in Section 5, where we will prove that:

$$
e^{L^{s,x_0}} - e^{L_0^{x_0}} - \sum_{\ell=1}^{n}\sum_{k=1}^{\ell} s^\ell \Lambda^{k,\ell}_{x_0} = o(s^n)
$$

in $\mathcal{B}(W^{s,p}(\mathbb{R}^N); W^{s,p}(\mathbb{R}^N))$. (We recall that we can always assume $k \leq \ell$.)

4. Commutator calculations

The purpose of this section is to give an explicitly computable representation of the perturbative expansion (3.16) as

$$
e^{L^{s,x_0}} \sim \mathbb{P}_n(s,x,\partial) e^{L_0^{x_0}}
\tag{4.1}
$$

where $\mathbb{P}_n(s, x, \partial)$ is a differential operator with polynomial coefficients in $(x-x_0)$ and s. Both the order of the operator as well as the degree of the polynomial coefficients depend on the order of the Taylor expansion n, which also equals the iteration level. We give an explicit characterization of $\mathbb{P}_n$ and an iterative procedure to calculate it in Theorem 4.7. The main idea is to show that each $\Lambda_{\alpha}^{k,\ell}$ in (3.12) can be written as a differential operator $\mathcal{P}_{\alpha}^{k,\ell}$ acting on $e^{L_0^{x_0}}$, and thus using (3.15) show that the perturbative expansion (3.16) can be rewritten in this form as well.

Definition 4.1 (Spaces of Differentiations)**.** *For any nonnegative integers a, b we denote by $\mathcal{D}(a, b)$ the vector space of all differentiations of degree at most a and order at most b. We extend this definition to negative indices by defining $\mathcal{D}(a, b) = \{0\}$ if either a or b is negative. By degree of A we mean the highest power of the polynomials appearing as coefficients in A.*

Definition 4.2 (Adjoint Representation)**.** *For any two differentiations $A_1 \in \mathcal{D}(a_1, b_1)$ and $A_2 \in \mathcal{D}(a_2, b_2)$ we define $\mathrm{ad}_{A_1}(A_2)$ by*

$$\mathrm{ad}_{A_1}(A_2) := [A_1, A_2] = A_1A_2 - A_2A_1 \tag{4.2}$$

and for any integer $j \geq 1$ we define $\mathrm{ad}_{A_1}^j(A_2)$ recursively by

$$\mathrm{ad}_{A_1}^j(A_2) := \mathrm{ad}_{A_1}(\mathrm{ad}_{A_1}^{j-1}(A_2)) \tag{4.3}$$

Proposition 4.3. *Suppose $A_1 \in \mathcal{D}(a_1, b_1)$ and $A_2 \in \mathcal{D}(a_2, b_2)$. Then for any integer $k \geq 1$,*

$$\mathrm{ad}_{A_1}^k(A_2) \in \mathcal{D}(k(a_1 - 1) + a_2, k(b_1 - 1) + b_2). \tag{4.4}$$

Proof. We first notice that

$$\mathrm{ad}_{A_1}(A_2) \in \mathcal{D}(a_1 - 1 + a_2, b_1 - 1 + b_2). \tag{4.5}$$

Next, from (4.3) we have

$$\mathrm{ad}_{A_1}^k(A_2) = \mathrm{ad}_{A_1}(\mathrm{ad}_{A_1}(\mathrm{ad}_{A_1}(\mathrm{ad}_{A_1}(\dots)))) \tag{4.6}$$

so that an application of (4.5) k times yields the result. □

Lemma 4.4. *Let m, k be fixed integers ≥ 1. Let $L_0^{x_0} \in \mathcal{D}(0, 2)$ be the constant coefficient operator given by* (3.10a) *and $L_m^{x_0} \in \mathcal{D}(m, 2)$ be the operator given by* (3.8)*. Then,*

$$\mathrm{ad}_{L_0^{x_0}}^k(L_m^{x_0}) \in \mathcal{D}(m - k, m + 2). \tag{4.7}$$

In particular,

$$\mathrm{ad}^k_{L_0^{x_0}}(L_m^{x_0}) = 0 \quad \forall k > m. \tag{4.8}$$

Proof. Applying Lemma 4.4 we see that $\mathrm{ad}^k_{L_0^{x_0}}(L_m^{x_0}) \in \mathcal{D}(m-k, m+2)$. If $k > m$, then by definition $\mathcal{D}(m-k, m+2) = \{0\}$ and we obtain (4.8). $\square$

Lemma 4.5. *Let $L_0^{x_0} \in \mathcal{D}(0,2)$ be defined as in* (3.10a)*, and $L_m^{x_0} \in \mathcal{D}(m,2)$ be defined as in* (3.8)*. Then for any $\theta \in (0,1)$,*

$$e^{\theta L_0^{x_0}} L_m^{x_0} = P_m(\theta, x, x_0, \partial) e^{\theta L_0^{x_0}} \tag{4.9}$$

where $P_m \in \mathcal{D}(m+2, m+2)$ is given by

$$P_m(\theta, x, x_0, \partial) := L_m^{x_0} + \sum_{i=1}^m \frac{\theta^i}{i!} \mathrm{ad}^i_{L_0^{x_0}}(L_m^{x_0}). \tag{4.10}$$

Proof. Recall the Baker-Campbell-Hausdorf formula (see e.g. [3, 6, 15])

$$[e^A, B] = \left([A,B] + \frac{1}{2!}[A,[A,B]] + \frac{1}{3!}[A,[A,[A,B]]] + \dots \right) e^A. \tag{4.11}$$

In general this formula is an infinite series. Setting $A = \theta L_0^{x_0}$, $B = L_m^{x_0}$ we have

$$\begin{aligned}
&e^{\theta L_0^{x_0}} L_m^{x_0} \\
&= \left(L_m^{x_0} + \theta[L_0^{x_0}, L_m^{x_0}] + \frac{\theta^2}{2!}[L_0^{x_0},[L_0^{x_0}, L_m^{x_0}]] + \frac{\theta^3}{3!}[L_0^{x_0},[L_0^{x_0},[L_0^{x_0}, L_1]]] + \cdots \right) e^{\theta L_0^{x_0}} \\
&= \left(L_m^{x_0} + \sum_{i=i}^m \frac{\theta^i}{i!} \mathrm{ad}^i_{L_0^{x_0}}(L_m^{x_0}) \right) e^{\theta L_0^{x_0}} = P_m(\theta, x, x_0, \partial) e^{\theta L_0^{x_0}}
\end{aligned}$$

where the finiteness of the sum is due to Lemma 4.4. The indicated properties of $P_m(\theta, x, x_0, \partial) e^{\theta L_0^{x_0}}$ also follow directly from Lemma 4.4. $\square$

Lemma 4.6. *For a given multi-index $\alpha \in \mathfrak{A}_{k,\ell}$ let*

$$\mathcal{P}_\alpha^{k,\ell}(x, x_0, \partial) := \int_{\Sigma_k} \prod_{i=1}^k P_{\alpha_i}^{k,\ell}(1-\sigma_i, x, x_0, \partial) d\sigma \tag{4.12}$$

where $P_{\alpha_i}^{k,\ell}$ is defined in (4.10) *with $m = \alpha_j$, for some $j = 1, \dots, k$. Then*

$$\Lambda_\alpha^{k,\ell} = \mathcal{P}_\alpha^{k,\ell}(x, x_0, \partial) e^{L_0^{x_0}} \tag{4.13}$$

where $\mathcal{P}^{k,\ell}_{\alpha}$ is a differential operator of order $2k$ and degree $\leq \ell = |\alpha| := \sum_{i=1}^{k} \alpha_i$.

Proof. The proof is a calculation based on the repeated application of Lemma 4.5 on $\Lambda^{k,\ell}_{\alpha}$. Fix $\alpha \in \mathfrak{A}_{k,\ell}$. Then,

(4.14)

$$\begin{aligned}
\Lambda^{k,\ell}_{\alpha} &= \int_{\Sigma_k} e^{(1-\sigma_1)L_0^{x_0}} L_{\alpha_1} e^{(\sigma_1-\sigma_2)L_0^{x_0}} L_{\alpha_2} e^{(\sigma_2-\sigma_3)L_0^{x_0}} L_{\alpha_3} \cdots L_{\alpha_k} e^{\sigma_k L_0^{x_0}} d\sigma \\
&= \int_{\Sigma_k} P^{k,\ell}_{\alpha_1}(1-\sigma_1, x, x_0, \partial) e^{(1-\sigma_2)L_0^{x_0}} L_{\alpha_2} \cdots L_{\alpha_k} e^{\sigma_k L_0^{x_0}} d\sigma \\
&= \int_{\Sigma_k} P^{k,\ell}_{\alpha_1}(1-\sigma_1, x, x_0, \partial) P_{\alpha_2}(1-\sigma_2, x, x_0, \partial) e^{(1-\sigma_3)} \cdots L_{\alpha_k} e^{\sigma_k L_0^{x_0}} d\sigma \\
&= \int_{\Sigma_k} P^{k,\ell}_{\alpha_1}(1-\sigma_1, x, x_0, \partial) P^{k,\ell}_{\alpha_2}(1-\sigma_2, x, x_0, \partial) \cdots P^{k,\ell}_{\alpha_k}(1-\sigma_k, x, x_0, \partial) e^{L_0^{x_0}} d\sigma \\
&= \int_{\Sigma_k} \prod_{i=1}^{k} P^{k,\ell}_{\alpha_i}(1-\sigma_i, x, x_0, \partial) e^{L_0^{x_0}} d\sigma \\
&= \left(\int_{\Sigma_k} \prod_{i=1}^{k} P^{k,\ell}_{\alpha_i}(1-\sigma_i, x, x_0, \partial) d\sigma \right) e^{L_0^{x_0}}
\end{aligned}$$

□

Finally, we set

$$\begin{aligned}
\mathcal{P}^{\ell,k}(x, x_0, \partial) &:= \sum_{\alpha \in \mathfrak{A}_{k,\ell}} \mathcal{P}^{k,\ell}_{\alpha}(x, x_0, \partial) \\
&= \sum_{\alpha \in \mathfrak{A}_{k,\ell}} \int_{\sigma_k} \prod_{i=1}^{k} P^{k,\ell}_{\alpha_i}(1-\sigma_i) d\sigma,
\end{aligned} \tag{4.15}$$

so that we have

$$\begin{aligned}
\Lambda^{k,\ell} &= \sum_{\alpha \in \mathfrak{A}_{k,\ell}} \Lambda^{k,\ell}_{\alpha} \\
&= \sum_{\alpha \in \mathfrak{A}_{\ell}} \mathcal{P}_{\alpha}(x, x_0, \partial) e^{L_0^{x_0}} \\
&= \sum_{\alpha \in \mathfrak{A}_{\ell}} \int_{\sigma_k} \prod_{i=1}^{k} P^{k,\ell}_{\alpha_i}(1-\sigma_i, x, x_0, \partial) d\sigma e^{L_0^{x_0}} \\
&= \mathcal{P}^{k,\ell}(x, x_0, \partial) e^{L_0^{x_0}}
\end{aligned} \tag{4.16}$$

We now state the main result of this section. Below, we set $\mathcal{P}^{0,\ell} = 1$ for any ℓ.

Theorem 4.7. *The perturbative expansion* (3.16) *of* $e^{L^{s,x_0}}$ *can be written in the form*

$$e^{L^{s,x_0}} = \mathbb{P}_n(x, x_0, s, \partial)e^{L_0^{x_0}} + s^{n+1}\mathbb{E}_n^{s,x_0} \tag{4.17}$$

where

$$\mathbb{P}_n(s, x, x_0, \partial) = \sum_{\ell=1}^{n} s^\ell \sum_{k=0}^{\ell} \mathcal{P}^{k,\ell}(x, x_0, \partial), \tag{4.18}$$

is a differential operator of order $2n$ *and degree* n *in its polynomial coefficients.*

Proof. Starting with (3.16), we have

$$\begin{aligned} e^{L^{s,x_0}} &= e^{L_0^{x_0}} + \sum_{\ell=1}^{n}\sum_{k=1}^{\ell} s^\ell \Lambda^{k,\ell} + s^{n+1}\mathbb{E}_{p,n}^{s,x_0} \\ &= e^{L_0^{x_0}} + \sum_{\ell=1}^{n}\sum_{k=1}^{\ell} \mathcal{P}^{k,\ell} e^{L_0^{x_0}} + s^{n+1}\mathbb{E}_n^{s,x_0} \\ &= \mathbb{P}_n(s, x, x_0, \partial)e^{L_0^{x_0}} + s^{n+1}\mathbb{E}_n^{s,x_0}. \end{aligned} \tag{4.19}$$

Since each $\mathcal{P}^{k,\ell}$ is a differential operator of order $2k$ and degree $\leq \ell = |\alpha|$, with $\alpha \in \mathfrak{A}_{k,\ell}$, $\mathbb{P}_n(s, x, x_0, \partial)$ is a differential operator of order $2n$ and degree n. $\square$

5. Error estimates

In this final section, we prove all the bounds necessary to justify the validity of the asymptotic expansion (4.17). Then, (4.17) will give our main result Theorem 1.1 by selecting $x_0 = x$, $s = t^1/2$ and, for example, $n = p$, and using the correspondence between e^{tL} and $e^{L^{\sqrt{t},x}}$ given in Lemma 3.2.

We shall, therefore, no longer assume that the dilation center $x_0 \in \mathbb{R}^N$ is fixed. Hence, we will restore the notational dependence on x_0. To emphasize further that x_0 is allowed to vary arbitrarily, we set $z := x_0$. We are going to estimate the main term $\mathbb{P}_{d,n}(x, z, s, \partial)e^{L_0^{x_0}}$ and the reminder term $s^{d+1}\mathbb{E}_{d,n}^{s,z}$ separately.

We begin by observing that Lemma 2.7 and formulas (3.14) and (3.15) give:

$$\begin{aligned} \Lambda_z^{k,\ell} &\in \mathcal{B}(W_a^{q,p}, W^{r,a-\ell}), \\ \Lambda_{s,z}^{k,\ell} &\in \mathcal{B}(W_a^{q,p}, W^{r,a-\ell}), \end{aligned} \tag{5.1}$$

for all $1 < p < \infty$, $r \geq q$, $a \in \mathbb{R}$.

In order to show that (4.17) is an asymptotic series in s in $\mathcal{B}(W^{r,p}, W^{r,p})$, we need to establish *uniform* bounds in s for the operator norm of $\Lambda_z^{k,\ell}$ and $\Lambda_{s,z}^{k,\ell}$. By (5.1) and the Schwartz Kernel Theorem, both $\Lambda_z^{k,\ell}$ and $\Lambda_{s,z}^{k,\ell}$ are integral operators, the kernel of which we denote respectively by $K_z^{k,\ell}$ and $K_{s,z}^{k,\ell}$.

Then, by Lemma 3.2, we can write, setting $s = t^{1/2}$, $z = x$:

$$\begin{aligned}
\mathcal{G}_t^L(x,y) &= t^{-N/2}\Big(\mathcal{G}_1^{L_0^x}(x, x+(y-x)/\sqrt{t}) + \\
&\quad + \sum_{\ell=1}^{n} t^{\ell/2} \sum_{k=1}^{\ell} K_x^{k,\ell}(x, x+(y-x)/\sqrt{t}) + \\
&\quad + t^{(n+1)/2} E_n^{s,x}(x, x+(y-x)/\sqrt{t})\Big) \\
&=: \mathcal{G}_t^{[n]}(x,y) + t^{(n+1)/2} E_{t,n}(x,y).
\end{aligned} \tag{5.2}$$

with $E_n^{s,z}$ the Schwartz kernel of the operator $\mathbb{E}_n^{s,z}$.

We will show below that each $K_x^{k,\ell}(x, x+(y-x)/\sqrt{t})$ is the kernel of a pseudodifferential operator, and use symbol calculus to derive the desired uniform bounds. We refer to [29] for all relevant properties of pseudodifferential operators. Below, we follow the usual convention and set $D = \frac{1}{i}\partial$, $(i = \sqrt{-1})$.

Theorem 5.1. *Let $1 \leq \ell, k \leq n$. Then, $t^{-N/2} K_x^{k,\ell}(x, x + t^{-1/2}(y-x))$ is the kernel of a pseudodifferential operator with symbol $a^{k,\ell}(x, t\xi) \in S^{-\infty}$, such that the family $\{a^{k,\ell}(x,t\xi)\}_{t\in(0,1]}$ is uniformly bounded in t in the Hörmander class $S^0_{1,0}$. Consequently, for any $1 < p < \infty$, any $r \in \mathbb{R}$,*

$$\|a^{k,\ell}(x, tD)\|_{W^{r,p}\to W^{r,p}} \leq C_{r,p},$$

for a constant $C_{r,p}$ independent of ℓ and $t \in (0,1]$.

Proof. We split the proof in different parts. First we study the operator $\Lambda_z^{k,\ell} = \mathcal{P}^{k,\ell} e^{L_0^z}$ as an operator in x depending on a parameter z. We use $\sigma(P_z)(x,\xi)$ to denote the symbol of a pseudo-differential operator $P_z(x,D)$. By abuse of notation, we set $\sigma(\mathcal{P}_z^{k,\ell})(x,\xi) = \sigma(\mathcal{P}_z^{k,\ell}(x, \partial/i))$.

Let ϕ be a standard, smooth cut-off function centered at the origin and let $\phi_z(x) = \phi(x-z)$. We set $\Lambda_{z,\phi}^{k,\ell} = M_{\phi_z}\Lambda_z^{k,\ell} = \phi_z\,\mathcal{P}^{k,\ell} e^{L_0^z}$, where M_{ϕ_z} is the multiplication operator by ϕ_z. We observe that $\sigma(\Lambda_{x,\phi}^{k,\ell})(x,\xi) = \sigma(\Lambda_x^{k,\ell})(x,\xi)$. We also note that $\sigma(M_{\phi_z}\,\mathcal{P}_z^{k,\ell}) = \phi_z\,\sigma(\mathcal{P}_z^{k,\ell})$, since the symbol of ϕ_z does not depend on ξ. Standard estimates (see e.g. [29]) shows that for each $z \in \mathbb{R}^N$ fixed, $\sigma(e^{L_0^z}) \in S^{-\infty}$ and

$\sigma(M_{\phi_z}\,\mathcal{P}^{k,\ell}) \in S^{2k}_{1,0}$ with seminorms uniformly bounded in z. Differentiating in z will similarly give uniform bounds for the symbol. In particular, for each triple of multiindices α, β, γ, and $N \in \mathbb{Z}_+$, there exist constants $C > 0$ depending on L, α, β, γ, N, but not on x, z, ξ such that

$$|\partial^\alpha \partial_x^\beta\, \partial_z^\gamma\, \sigma(e^{L_0^z})(x,z,\xi)| \leq C\,(1+\xi^2)^{-N/2}, \tag{5.3}$$

$$|\partial_\xi^\alpha\, \partial_x^\beta\, \partial_z^\gamma\, \sigma(M_{\phi_z}\mathcal{P}^{k,\ell})(x,z,\xi)| \leq C\,(1+\xi^2)^{(2k-\alpha)/2}, \tag{5.4}$$

The first estimates follow directly from (3.11) and the fact $L \in \mathbb{L}$, *i.e.,* its coefficients along with all their derivatives are uniformly bounded on $\mathbb{R}^N$. For the second estimate, we use also that $\mathcal{P}_z^{k,\ell}$ s a differential operator of order $2k$ with coefficients that are polynomial in $(x-z)$ and bounded functions in z with all their derivatives, depending only on L, that ϕ and all its derivatives are bounded on $\mathbb{R}^N$ and that $(x-z)$ is bounded on the support of ϕ.

From symbol calculus, it follows that the composition $\Lambda^{k,\ell}_{z,\phi} = \phi_z\,\mathcal{P}^{k,\ell}\,e^{L_0^z}$ is a family of pseudodifferential operators in $S^{-\infty}$ depending on the parameter z, the symbol of which satisfies estimates similar to (5.3). In particular, $\sigma(\Lambda^{k,\ell}_{x,\phi})(x,\xi) = \sigma(\Lambda^{k,\ell}_x)(x,x,\xi) \in S^{-\infty}$.

We next consider $K^{k,\ell}_x(x,,y)$, the Schartz kernel of the operator $\Lambda^{k,\ell}_x$, and form the dilated kernel $K^{k,\ell}_{t,x}(x,y) = (t^{-N/2}K^{k,\ell}_x(x, x+t^{-1/2}(y-x))$, $t \in (0,1]$. For each $0 < t \leq 1$ fixed, $K^{k,\ell}_{t,x}(x,y)$ is clearly the kernel of a pseudo-differential operator with symbol in $S^{-\infty}$. We denote its symbol by $a^{k,\ell}_t(x,\xi)$, so that $a^{k,\ell}_t(x,D)$ is a smoothing operator. The family of symbols $\{a^{k,\ell}_t\}_{t\in(0,1]}$ is not uniformly bounded in $S^{-\infty}$, but it is uniformly bounded in $S^0_{0,1}$, which will be sufficiently to establish the necessary mapping properties. To prove the uniform bound in $S^0_{1,0}$, it is enough to show that $a^{k,\ell}_t$ has the form

$$a^{k,\ell}_t(x,\xi) = a^{k,\ell}(x, t^{1/2}\xi), \qquad a^{k,\ell} \in S^0_{1,0}. \tag{5.5}$$

This observation follows immediately from the definition of the symbol class $S^0_{1,0}$. To this end, we recall the connection between the kernel K and the symbol a of a pseudo-differential operator:

$$K(x,y) = \int_{\mathbb{R}^N} a(x,\xi)\, e^{-i\,(y-x)\cdot\xi}\, d\xi.$$

If we set $\sigma(\Lambda_x^{k,\ell})(x,\xi) = \sigma(\Lambda_{x,\phi}^{k,\ell})(x,\xi) = a_\ell(x,\xi)$, then

$$\begin{aligned} K_{t,x}^{k,\ell}(x,y) &= (t^{-N/2} K_x^{k,\ell}(x, x + t^{-1/2}(y-x)) \\ &= t^{-N/2} \int_{\mathbb{R}^N} a_\ell(x,\xi)\, e^{-i\, t^{-1/2}(y-x)\cdot\xi}\, d\xi \\ &= \int_{\mathbb{R}^N} a_\ell(x, t^{1/2}\xi') e^{-i\,(y-x)\cdot\xi'}\, d\xi', \end{aligned}$$

where the last equality follows from the change of variables $\xi = t^{1/2}\xi'$. This establishes (5.5) with $a^{k,\ell}(x,\xi) = \sigma(\Lambda_{x,\phi}^{k,\ell})(x,\xi)$, which is a symbol is $S^0_{1,0}$ by (5.3)-(5.4). Finally, we exploit known mapping properties of pseudo-differential operators to conclude the proof , *i.e.,* the property that any pseudo-differential operator with symbol in $S^0_{1,0}$ is a bounded linear operator on any Sobolev space $W^{r,p}$, $1 \le r < \infty$, $s \in \mathbb{R}$ (we refer again to [29]). □

From (5.2), we immediately obtain the desired estimate on the principal part of the asymptotic expansion.

Corollary 5.2. *For each $1 < p < \infty$, $r \in \mathbb{R}$, and any $f \in W^{r,p}$*

$$\int_{\mathbb{R}^N} \mathcal{G}_t^{[n]}(x,y) f(y)\, dy,$$

is uniformly bounded in t in $W^{r,p}$.

We now move to the error term $E_{t,n}$ in (5.2). Below we study the mapping properties for the operator $\mathbb{E}_{t,n}$ with kernel $E_{t,n}$, which is not immediately in the form of a pseudodifferential operator. Consequently, we are not able to derive as sharp bounds as in Theorem 5.1 above. Nevertheless, the bounds we derive are sufficient to show the error term is negligible for $t \to 0^+$ in any Sobolev space, provided n is chosen large enough.

We use the representation (5.2) for the kernel $K_{s,z}^{k,\ell}$ of the operator $\Lambda_{s,z}^{k,\ell}$, (5.1), and the Riesz lemma to obtain the needed mapping properties for $\mathbb{E}_{t,n}$.

We begin by observing that

$$\partial_x^\alpha \partial_z^\beta \partial_y^\gamma K_{\ell,z}(x,y) = \partial_x^\alpha\big(\delta_x(\partial_z^\beta \Lambda_{s,z}^{k,\ell} \partial_y^\gamma \delta_y)\big) := \langle \partial_x^\alpha \delta_x, \partial_z^\beta \Lambda_{s,z}^{k,\ell} \partial_y^\gamma \delta_y\rangle. \tag{5.6}$$

Since all the coefficients (and their derivatives) of L are bounded, the derivative $\partial_z^\beta \Lambda_{s,z}^{k,\ell}$ will satisfy the same mapping properties as $\Lambda_{s,z}^{k,\ell}$. For each multiindex α, $\partial^\alpha \delta_y \in H^{-q}(\mathbb{R}^N)$ for $q > N/2 + |\alpha|$ and has norm independent of y. Also, since δ_y has compact support, $\delta_y \in H_a^{-q}$ for all $a \in \mathbb{R}$ with

$$\|\partial^\alpha \delta_y\|_{H_a^{-q}} := \|e^{a<y-z>} \partial^\alpha \delta_y\|_{H^{-q}} \le C_{q,a} e^{a\langle y-z\rangle},$$

where we recall $\langle x\rangle = (1+|x|^2)^{1/2}$. (We recall also that the space H_a^{-q} is independent of the choice of dilation center $z = x_0$.)

In turn, for each multiindex α, β, γ, if $q > N/2 + \max(|\alpha|, |\gamma|)$,

$$\begin{aligned}\left|\partial_x^\alpha \partial_z^\beta \partial_y^\gamma K_{s,z}^{k,\ell}(x,y)\right| &\le C\left|\langle \partial^\alpha \delta_x\,, \partial_z^\beta \Lambda_{s,z}^{k,\ell}\, \partial^\gamma \delta_y\rangle\right| \\ &\le C\|\partial^\alpha \delta_x\|_{H^{-q}_{|\beta|+\ell-a}} \|\partial_z^\beta \Lambda_{s,z}^{k,\ell}\|_{H^{-q}_{-a} \to H^q_{a-\ell-|\beta|}} \|\partial^\gamma \delta_y\|_{H^{-q}_{-a}} \\ &\le Ce^{(|\beta|+\ell-a)\langle x-z\rangle - a\langle y-z\rangle}.\end{aligned}$$

We now make the following substitutions in view of (5.2):

$$z = x, \qquad y \to x + (y-x)/\sqrt{t}, \qquad s = \sqrt{t},$$

and use the chain rule to conclude that, if λ is a multiindex such that $|\lambda| \le r$, $r \in \mathbb{Z}_+$, for any $a \in \mathbb{R}$:

$$\tag{5.7a} \begin{aligned}&\left|\partial_x^\lambda K_{\sqrt{t},x}^{k,\ell}(x, x + t^{-1/2}(y-x))\right| \\ &\le C \sum_{|\alpha|+|\beta|+|\gamma|=|\lambda|} t^{-|\gamma|/2} \left|\partial_x^\alpha \partial_z^\beta \partial_w^\gamma K_{\sqrt{t},z}^{k,\ell}(x,w)\right|_{z=x, w=x+t^{-1/2}(y-x)} \\ &\le Ct^{-r/2} e^{-a\langle y-x\rangle/\sqrt{t}},\end{aligned}$$

$$\tag{5.7b} \begin{aligned}&\left|\partial_y^\lambda K_{\sqrt{t},x}^{k,\ell}(x, x + t^{-1/2}(y-x))\right| \\ &\le Ct^{-|\lambda|/2} \left|\partial_w^\gamma K_{\sqrt{t},z}^{k,\ell}(x,w)\right|_{z=x, w=x+t^{-1/2}(y-x)} \\ &\le C_{a,r} t^{-r/2} e^{-a\langle y-x\rangle/\sqrt{t}},\end{aligned}$$

This estimate together with Riesz Lemma gives a first mapping property. Let $\mathbb{E}_{t,n}$ be the operator with kernel $E_{t,n}$ defined in (5.2).

Lemma 5.3. *For any $r \in \mathbb{Z}_+$ and $1 < p < \infty$,*

$$\|\mathbb{E}_{t,n}\|_{L^p \to W^{r,p}} \le Ct^{-r/2}, \qquad t \in (0,1].$$

.

Proof. By Riesz' Lemma, it is enough to check that

$$\tag{5.8} \int_{\mathbb{R}^N} |\partial_x^\lambda E_{t,n}(x,y)|\, dx \le Ct^{-r/2}$$

and

$$\tag{5.9} \int_{\mathbb{R}^N} |\partial_x^\lambda E_{t,n}(x,y)|\, dx \le Ct^{-r/2}$$

with a constant C independent of x and y, and for any multinidex λ with $|\lambda| \le r$. But these estimates follow directly from equation (5.7) since

$$\partial_x^\lambda E_{t,n}(x,y) = t^{-N/2} \sum_{\ell=n+1}^{(n+1)^2} t^{(\ell-n-1)/2} \partial_x^\lambda K^{k,\ell}_{\sqrt{t},x}(x, x+t^{-1/2}(y-x))$$

and the factor $t^{-N/2}$ is absorbed in the exponential, by a change of variables. $\square$

From the Lemma, using that $W^{r,p} \subset L^p$, for any $r \geq 0$, $1 < p < \infty$, we obtain immediately that

$$\|\mathbb{E}_{t,n}\|_{W^{r,p} \to W^{r,p}} \leq C t^{-r/2}, \qquad t \in (0,T].$$

This estimate is not strong enough to yields Theorem 1.1, but we observe that if we continue the expansion of $e^{tL^{s,x}}$ in (4.17) to order $n+r+1$, then we can write $\mathbb{E}_{t,n}$ as a sum of terms of the type $\sqrt{t}^{(\ell-n)} \Lambda_x^{k,\ell}$ with $n < \ell < n+r$, which are uniformly bounded in $t \in [0,T]$ on any Sobolev space, plus a remainder term of the type $\sqrt{t}^r \mathbb{E}_{t,n+r}$ for which we have the mapping properties as above, so that

$$\|\sqrt{t}^r \mathbb{E}_{t,n+r}\|_{W^{r,p} \to W^{r,p}} \leq C(T,r,p,n), \qquad t \in (0,T].$$

We therefore conclude that in fact $\mathbb{E}_{t,n}$ is uniformly bounded in $t \in [0,T]$ on any Sobolev space.

Corollary 5.4. *For any $r \in \mathbb{Z}_+$ and $1 < p < \infty$, there exists a constant $C = C(T,r,p,L) > 0$ such that*

$$\|\mathbb{E}_{t,n}\|_{W^{r,p} \to W^{r,p}} \leq C(T,n,r.p), \qquad t \in (0,T].$$

.

Finally, this corollary and Corollary 5.2 give our main result Theorem 1.1.

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