A GENERAL MATHEMATICAL MODEL WITH TWO VARIABLES OF A FILTRATION PROCESS APPLLIED TO MEMBRANES

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IMA Preprint Series # 2205

(May 2008)

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A General Mathematical Model with two Variables of a Filtration Process Applied to Membranes

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Abstract

In this work a new Mathematical model in two variables is presented for the permeate flux, flow and fluid velocity of a tubular synthetic membrane. This model allows to analyze with more details the behavior of the flow and flux along the membrane and the time. Although the one outlines this fact from the basic principles of the fluids mechanics, it is interesting to see that you arrives to the experimental expressions for the flux described in the magazines of the topic. They are also expressions for the variations of the concentration and the active area of the membrane. Of the differential equations and of their exact solution they come off four types of flux observed experimentally

Key words, ultrafiltration, mathematical model, permeate flux.

1. Introduction

In the last years the technology of membranes appeared as an emergent tool with a high impact in the innovation of productive processes. At the present time, the developed operations that are industrially set up are inverse osmosis (IO), ultrafiltration (UF) and microfiltration (MF). Nowadays, ultrafiltration is used in a wide variety of fields, from chemical industry to medical applications [1]. As it is known, the biological products are usually in diluted suspensions and blended with other substances, for instance, metabolites and microorganisms. Many of these biological products are unstable at high temperatures and sensitive to pressure and shear stress.

In these cases, the tangential flow MF or UF have been used in concentration operations, division and purification of high added value products, such as antibiotics, enzymes, albumin and other bio-products [2-4]. In addition, the membrane technology offers great comparative advantages such as small energy consumption, operation flexibility, quick paying-off and operation at room temperature or close to it.

However, the use of UF membranes in a large-scale industrial processes is still limited due to the problem of the boundary polarization layer growth. The consequent contamination or fouling of the membrane makes that the flux declines as the work cycle lengthens, since the biological products are spread blocking the filtration channels of the membrane [5-6]. For this reason, the process requires a great effort to investigate new strategies of membranes cleaning with tangential flows.

In our opinion, to improve the membrane cleaning it is necessary a better understanding of the fouling during the process and a design of a more appropriate mathematical model that explains in detail the variations of the permeate flux. In this work, we study a general model with two independent variables, based on the continuity equation for a multicomponent incompressible fluid that flows inside a filtration membrane.

This mathematical model shows the behavior of the fluid velocity inside the membrane, the time and spatial variation of the flow, the concentration profiles, the permeate flux and the variation of the active membrane area during the fouling process. Although this model is very general, it describes, in a simple way, 3 particular cases. The models are described in the specialized bibliography of the topic.

2. Differential equation for the flow, flux and velocity inside the membrane.

It is clear that when we analyze in detail the filtration problem in a membrane two different phenomena can be observed in relationship with the variation of the permeate flux. One of them, a phenomenon studied for many cases, is the total variation of the permeate flux with time, which is modeled by the function $J_P(t)$ taking into account the experimental determination of this flux. It is worth remembering that the value of the permeate flux in an arbitrary time interval is the sum of the permeate contribution of all the differentials length of the membrane in that time interval, which does not mean *that* the permeate flux along the membrane is the same for all the differentials length. In other words, the value of the curve $J_p(t)$ is the integral sum of all the differentials contributions of the differentials length, but each of them according to the distribution $J(x)$ along the membrane. In a model of two variables this can be mathematically written as follows,

$$J_P(t) = \int_0^L J(x,t)dx \tag{1}$$

where, L denotes the length of the membrane, or the sum of the differentials length along the membranes, if they are connected to each other. It occurs that the concentration of the fluid that circulates inside of the membrane increases causing the saturation of the membrane in a different way, especially if the membrane is long or there is a series of several connected membranes. Besides this, if the permeation is large the flow inside the membrane decays along the membrane as well as its velocity, which provokes that the shears become smaller generating more favorable conditions for the saturation of the membrane.

In short membranes this phenomenon can be disregarded depending on the type of membrane, the solution to concentrate and if it can be assumed that the permeate fluid through the membrane is constant everywhere.

Figure 1 shows a balance of the total flow of an element of length Δx, in a membrane with arbitrary but constant area (generally circular). In the Figure it can be observed that the output flow will be smaller than the input flow due to part of it is filtrated through the membrane.

In the mentioned balance any restriction to the permeate flux was imposed. Therefore, the analysis is general and valid for any mathematical function that models the flux. In the Figure it can be seen that if the membrane area remains constant the velocity of the fluid inside the membrane will change according to the permeate flux, not only with the time but also with the length of the membrane. Due to this behavior, it is proposed a function for the permeate flux that depends on the variables x and t as can be seen in Figure 1.

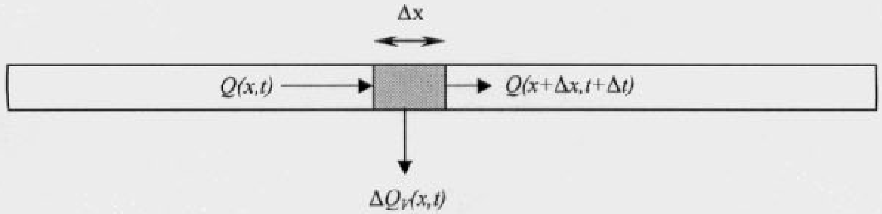

Figure 1

Flows balance in a membrane differential area

Because of the principle of mass conservation in the dripping of an incompressible fluid we have,

$$Q(x,t) - \Delta Q_V(x,t) = Q(x+\Delta x, t+\Delta t) \tag{2}$$

where

$Q(x,t)$ is the flow entering into the elemental volume.

$\Delta Q_V(x,t)$ is the permeate flow leaving the differential area of the membrane

$Q(x+\Delta x, t+\Delta t)$ is the variable flow leaving the cross-area of the membrane.

Taking the Taylor expansion of these expressions up to the linear term in (x,t) we obtain,

$$Q(x,t) - \Delta Q_V(x,t) = Q(x,t) + \frac{\partial}{\partial x}Q(x,t)\Delta x + \frac{\partial}{\partial t}Q(x,t)\Delta t \tag{3}$$

then simplifying,

$$-\Delta Q_V(x,t) = \frac{\partial}{\partial x}Q(x,t)\Delta x + \frac{\partial}{\partial t}Q(x,t)\Delta t \quad . \tag{4}$$

Dividing by Δx expression (4),

$$-\frac{\Delta Q_V(x,t)}{\Delta x} = \frac{\partial}{\partial x}Q(x,t) + \frac{\Delta t}{\Delta x}\frac{\partial}{\partial t}Q(x,t) \tag{5}$$

taking the limit it can be rewritten as it follows,

$$-\frac{\partial Q_V(x,t)}{\partial x} = \frac{\partial}{\partial x}Q(x,t) + \frac{\delta t}{\delta x}\frac{\partial}{\partial t}Q(x,t) \tag{6}$$

Repeating the process but dividing by Δt and approaching the limit to zero, the expression remains,

$$-\frac{\partial}{\partial t}Q_V(x,t) = \frac{\partial}{\partial x}Q(x,t)\frac{\delta x}{\delta t} + \frac{\partial}{\partial t}Q(x,t) \tag{7}$$

Considering that $\frac{\delta x}{\delta t}$ has velocity dimension, and beside that it, calibrate the units of the differential equations (6) and (7), we will denote the calibration function or characteristic velocity by $v(x,t)$ and define it as,

$$v = \frac{\delta x}{\delta t} = v(x,t) \tag{8}$$

Thus it is possible to rewrite equations (6) and (7) as follows,

$$-\frac{\partial}{\partial x} Q_V(x,t) = \frac{\partial}{\partial x} Q(x,t) + \frac{1}{v(x,t)} \frac{\partial}{\partial t} Q(x,t) \tag{9}$$

$$-\frac{\partial}{\partial t} Q_V(x,t) = v(x,t) \frac{\partial}{\partial x} Q(x,t) + \frac{\partial}{\partial t} Q(x,t) \tag{10}$$

Multiplying by $v(x,t)$ the equation (9) and solving the system it is obtained,

$$v(x,t) \frac{\partial}{\partial x} Q_V(x,t) = \frac{\partial}{\partial t} Q_V(x,t) \tag{11}$$

from which expression the following nonlinear partial differential equation for $Q_V(x,t)$ is obtained,

$$\frac{\partial}{\partial x} Q_V(x,t) - \frac{1}{v(x,t)} \frac{\partial}{\partial t} Q_V(x,t) = 0 \tag{12}$$

If the permeate flow is referred to the total active area of the membrane A_0, the differential equation is expressed in terms of the permeate flux $J(x,t)$, defined as $Q_V(x,t) = J(x,t)A_0$,

$$\frac{\partial}{\partial x} J(x,t) - \frac{1}{v(x,t)} \frac{\partial}{\partial t} J(x,t) = 0 \tag{13}$$

which is the searched general differential equation. Since this equation does not have any mathematical restriction it represents a general model, which relates the calibration function or the characteristic velocity with the permeate flux, at any position along all the membrane and in an arbitrary time t. If a simpler model is analyzed for an instant in time t, i.e. as if a instantaneous photograph of what it is occurring in the membrane in two cross-sectional consecutive areas, separated by a Δx, were taking it could be observed the following situation (Figure 2),

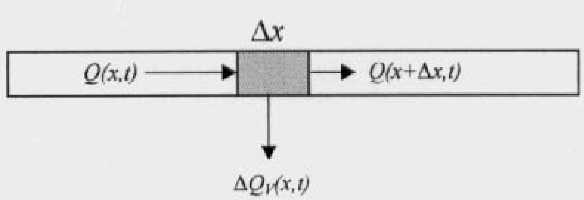

Figure 2

Simplified model of the flows balance in an instant time t

In this simplified model the balance of flows is,

$$Q(x,t) - \Delta Q_V(x,t) = Q(x,t) + \frac{\partial}{\partial x} Q(x,t)\Delta x \tag{14}$$

then simplifying it becomes,

$$-\Delta Q_V(x,t) = \frac{\partial}{\partial x} Q(x,t)\Delta x \tag{15}$$

to know the variation of the permeate flow is enough if the expression is divided by Δx, approaching the limit to zero, so,

$$-\frac{\partial Q_V(x,t)}{\partial x} = \frac{\partial}{\partial x} Q(x,t) \tag{16}$$

this equation expresses that the permeate flow variation due to a differential variation of length δx in the membrane is an exact measurement of the flow variation in this length interval. If the permeate increases the internal flow decreases, this is the reason of the negative sign of expression (16).

On the contrary, to determine the permeate flux variation in a time interval Δt, it is possible to divided expression (15) by Δt and taking the limit tending to zero, obtaining in this way,

$$-\frac{\partial Q_V(x,t)}{\partial t} = \frac{\partial}{\partial x} Q(x,t)\frac{\delta x}{\delta t} \tag{17}$$

as the aim is to determinate the permeate flux variation $J(t)$ in a differential time interval δt from a variation of the internal flow during a shift in the length of the membrane δx, the shift δx of a plane area of a fluid which circulates inside the membrane would be related with the time interval during which is desired to measure the change of the permeate flow. This relationship is the linear velocity of the fluid.

As a consequence, independently of the flow variation $Q(x,t)$ in the interval Δx, this must be achieved in the interval Δt during which the variation of the permeate flow is measured.

If the fluid velocity were zero, the following Δx would never be reached, so there would not be a variation of the internal flow. Moreover, independently of the length of the interval Δt, the permeate flux would be zero in the following differential Δx. If $v(x,t)$ is the fluid velocity/speed inside the membrane, it could also be written,

$$v(x,t) = \frac{1}{S} Q(x,t) \tag{18}$$

Being S a constant measured in m^2 that represents a normal area of the membrane.

In the same way in which a system in a time instant t was modeled, it is possible to model in an area x of the membrane in an interval Δt, since the model is based on independent variables. In this case, the model is represented in Figure 3.

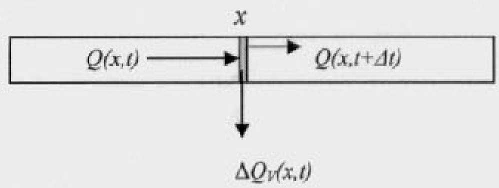

Figure 3
Flows balance in a particular area of the membrane, at two different times

$$Q(x,t) - \Delta Q_V(x,t) = Q(x,t) + \frac{\partial}{\partial t} Q(x,t)\Delta t \tag{19}$$

Simplifying expression (19) and following the same procedure than in (16) and (17) the following relationships are obtained,

$$-\frac{\partial Q_V(x,t)}{\partial t} = \frac{\partial}{\partial t} Q(x,t) \tag{20}$$

and,

$$-\frac{\partial Q_V(x,t)}{\partial x} = \frac{\partial}{\partial t} Q(x,t) \frac{\delta t}{\delta x} \tag{21}$$

Replacing (8) in equation (21) it yields to,

$$-v(x,t) \frac{\partial Q_V(x,t)}{\partial x} = \frac{\partial}{\partial t} Q(x,t) \tag{22}$$

equaling (20) and (22) or (17) and (16), equation (11) is obtained,

$$\frac{\partial}{\partial x} Q_V(x,t) - \frac{1}{v(x,t)} \frac{\partial}{\partial t} Q_V(x,t) = 0 \tag{23}$$

and a similar equation for $Q(x,t)$.

Rewriting equations (9) and (10) using the fluid velocity,

$$-\frac{\partial}{\partial x} Q_V(x,t) = \frac{\partial}{\partial x} Q(x,t) + \frac{1}{v(x,t)} \frac{\partial}{\partial t} Q(x,t) \tag{24}$$

$$-\frac{\partial}{\partial t} Q_V(x,t) = v(x,t) \frac{\partial}{\partial x} Q(x,t) + \frac{\partial}{\partial t} Q(x,t) \tag{25}$$

if now equations (24) and (25) are differentiated in a crossed form and equation (18) is used, differential equations that relate the fluid velocity with the permeate flow are obtained. After differentiating, the following equations are obtained,

$$-S\frac{\partial^2}{\partial x\partial t}Q_V(x,t)=S\frac{\partial^2}{\partial x\partial t}v(x,t)+S\frac{\partial}{\partial t}\left[\frac{1}{v(x,t)}\frac{\partial}{\partial t}v(x,t)\right] \tag{26}$$

$$-S\frac{\partial^2}{\partial t\partial x}Q_V(x,t)=S\frac{\partial}{\partial x}\left[v(x,t)\frac{\partial}{\partial x}v(x,t)\right]+S\frac{\partial^2}{\partial t\partial x}v(x,t) \tag{27}$$

Resolving these equations with equal first terms the following expression is fulfilled,

$$\frac{\partial}{\partial x}\left[v(x,t)\frac{\partial}{\partial x}v(x,t)\right]=\frac{\partial}{\partial t}\left[\frac{1}{v(x,t)}\frac{\partial}{\partial t}v(x,t)\right] \tag{28}$$

which is a differential equation only in $v(x,t)$, then,

$$\frac{\partial}{\partial x}\left[v(x,t)\frac{\partial}{\partial x}v(x,t)\right]=\frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\left(\frac{1}{2}v^2(x,t)\right)\right]=\frac{1}{2}\frac{\partial^2}{\partial x^2}v^2(x,t) \tag{29}$$

$$\frac{\partial}{\partial t}\left[\frac{1}{v(x,t)}\frac{\partial}{\partial t}v(x,t)\right]=\frac{\partial}{\partial t}\left[\frac{\partial}{\partial t}\ln v(x,t)\right]=\frac{\partial^2}{\partial t^2}\ln v(x,t) \tag{30}$$

that can be rewritten as follows,

$$\frac{1}{2}\frac{\partial^2}{\partial x^2}v^2(x,t)-\frac{\partial^2}{\partial t^2}\ln v(x,t)=0 \tag{31}$$

which is the differential equation for the fluid velocity inside the membrane or, in this case, it is also for the calibration function where any consideration about the variation form of the permeate functions has been made.

An equation based on the flow that circulates inside the membrane could be written as,

$$\frac{\partial}{\partial x}\left[\frac{1}{S}Q(x,t)\frac{\partial}{\partial x}\frac{1}{S}Q(x,t)\right]=\frac{\partial}{\partial t}\left[\frac{S}{Q(x,t)}\frac{\partial}{\partial t}\frac{1}{S}Q(x,t)\right] \tag{32}$$

after simplifying,

$$\frac{1}{S^2}\frac{\partial}{\partial x}\left[Q(x,t)\frac{\partial}{\partial x}Q(x,t)\right]=\frac{\partial}{\partial t}\left[\frac{1}{Q(x,t)}\frac{\partial}{\partial t}Q(x,t)\right] \tag{33}$$

which is transformed in the following differential equation,

$$\frac{1}{2S^2}\frac{\partial^2}{\partial x^2}\left[Q^2(x,t)\right]-\frac{\partial^2}{\partial t^2}\left[\ln Q(x,t)\right]=0 \tag{34}$$

being this the differential equation for the flow that circulates inside the membrane. With the aim of obtaining a solution for this equation, first a solution of (31) can be calculated to find the fluid velocity $v(x,t)$ and then multiplied it by S. All of the differential equations are dimensionally correct.

3. A general solution of the differential equation for the characteristic velocity *v(x,t)* and for the flow *Q(x,t)*.

To integrate the differential equation (34) it could be possible to begin with the general integration of the velocity equation (31) and multiply its solution by S. The relationship below is the differential equation for the fluid velocity,

$$\frac{1}{2}\frac{\partial^2}{\partial x^2}v^2(x,t)=\frac{\partial^2}{\partial t^2}\ln v(x,t) \tag{35}$$

and the general integration is going to be made by the separate variables method.
Defining a new variable function of (x,t) in the following way,

$$u(x,t)=\exp[T(t)]\exp[X(x)]=e^{T(t)}e^{X(x)}=\exp[[T(t)]+[X(x)]] \tag{36}$$

with the purpose of integrating the following change of variables is made,

$$v^2(x,t)=u(x,t) \tag{37}$$

so it is possible to write,

$$v(x,t)=\sqrt{u(x,t)}=\sqrt{e^{T(t)}e^{X(x)}}=e^{\frac{1}{2}T(t)}e^{\frac{1}{2}X(x)}=\frac{1}{2}\exp[[T(t)]+[X(x)]] \tag{38}$$

replacing the variables change in the logarithmic equation leads to,

$$\begin{gathered}2\ln v(x,t)=\ln u(x,t)\\ \ln v(x,t)=\frac{1}{2}\ln u(x,t)\end{gathered} \tag{39}$$

replacing in the differential equation,

$$\frac{1}{2}\frac{\partial^2}{\partial x^2}u(x,t)-\frac{1}{2}\frac{\partial^2}{\partial t^2}\ln u(x,t)=0 \tag{40}$$

or,

$$\frac{\partial^2}{\partial x^2}u(x,t)-\frac{\partial^2}{\partial t^2}\ln u(x,t)=0 \tag{41}$$

Calculating the corresponded derivatives to the proposed function $u(x,t)=T(t)X(x)$ it is found that,

$$\frac{\partial}{\partial x}u(x,t)=X'(x)e^{T(t)}e^{X(x)}=X'(x)u(x,t) \tag{42}$$

$$\frac{\partial^2}{\partial x^2}u(x,t)=X''(x)u(x,t)+[X'(x)]^2u(x,t) \tag{43}$$

and for the derivatives respect to t we have $\ln u(x,t)=\ln T(t)+\ln X(x)$, then,

$$\frac{\partial^2}{\partial t^2}\ln u(x,t)=\frac{\partial^2}{\partial t^2}[T(t)+X(x)]=\frac{\partial^2}{\partial t^2}[T(t)]=T''(t) \tag{44}$$

replacing the derivatives in the corresponded differential equation,

$$\frac{\partial^2}{\partial x^2}u(x,t)-\frac{\partial^2}{\partial t^2}\ln u(x,t)=0 \tag{45}$$

$$\left\{X''(x)+[X'(x)]^2\right\}e^{X(x)}e^{T(t)}=T''(t) \tag{46}$$

which is a differential equation of separated variables, and if could be written as follows,

$$\left\{X''(x)+[X'(x)]^2\right\}e^{X(x)}=T''(t)e^{-T(t)} \tag{47}$$

The above relationship leads to two independent differential equations,

$$\left\{X''(x)+[X'(x)]^2\right\}e^{X(x)}=k \tag{48}$$

$$T''(t)e^{-T(t)}=k \tag{49}$$

the equation in function of the variable t could be integrated in the following way,

$$T''(t)e^{-T(t)}=k \tag{50}$$

$$T''(t)=ke^{T(t)} \tag{51}$$

multiplying by $T'(t)$ both terms of the equality,

$$T''(t)T'(t) = ke^{T(t)}T'(t) \tag{52}$$

being possible to rewrite this relationship as,

$$T''(t)T'(t) = \frac{d}{dt}\left[\frac{1}{2}T'(t)T'(t)\right] = ke^{T(t)}T'(t) = \frac{d}{dt}\left[ke^{T(t)}\right] \tag{53}$$

integrating once over t leads to,

$$\frac{1}{2}\left[T'(t)\right]^2 = ke^{T(t)} \tag{54}$$

$$T'(t) = \sqrt{2k}e^{\frac{1}{2}T(t)} \tag{55}$$

changing the variables in the following way,

$$\left[T(t)\right] = 2\ln g(t) = \ln g(t)^2 \tag{56}$$

Differentiating respect to t,

$$\left[T'(t)\right] = 2\frac{g'(t)}{g(t)} \tag{57}$$

replacing in the equation $T'(t) = \sqrt{2k}e^{\frac{1}{2}T(t)}$, the expression is transformed in,

$$T'(t) = 2\frac{g'(t)}{g(t)} = \sqrt{2k}e^{\frac{1}{2}(2\ln g(t))} = \sqrt{2k}e^{\ln g(t)} \tag{58}$$

so (57) is going to be written as,

$$T'(t) = 2\frac{g'(t)}{g(t)} = \sqrt{2k}e^{\frac{1}{2}(2\ln g(t))} = \sqrt{2k}e^{\ln g(t)} = \sqrt{2k}g(t) \tag{59}$$

being transformed in,

$$
\begin{aligned}
2\frac{g'(t)}{g(t)} &= \sqrt{2k}\,g(t) \\
-\frac{g'(t)}{[g(t)]^2} &= -\frac{\sqrt{2k}}{2} = -\sqrt{\frac{k}{2}}
\end{aligned}
\tag{60}
$$

but the first term is the derive of $\frac{1}{g(t)}$, as a consequence it could be denoted that,

$$
-\frac{g'(t)}{[g(t)]^2} = \frac{d}{dt}\left[\frac{1}{g(t)}\right] = -\sqrt{\frac{k}{2}} \tag{61}
$$

in which function $g(t)$ is now,

$$
\frac{1}{g(t)} = -\sqrt{\frac{k}{2}}t + C' \tag{62}
$$

reordering the equation,

$$
g(t) = -\frac{1}{\sqrt{\frac{k}{2}}t + C'} \tag{63}
$$

and replacing this unto (56) it is found that,

$$
[T(t)] = \ln\left[-\frac{1}{\sqrt{\frac{k}{2}}t + C'}\right]^2 = \ln\frac{1}{\left(\sqrt{\frac{k}{2}}t + C'\right)^2} \tag{64}
$$

or,

$$
[T(t)] = \ln\frac{1}{\left(\sqrt{\frac{k}{2}}t + C''\right)^2} \tag{65}
$$

$e^{T(t)}$ can be written in a simple way as,

$$e^{T(t)} = \frac{1}{\left(\sqrt{\frac{k}{2}}t + C'\right)^2} \tag{66}$$

Integrating the expression over x,

$$\begin{aligned} &\left\{X''(x) + [X'(x)]^2\right\} e^{X(x)} = k \\ &X''(x) + [X'(x)]^2 = ke^{-X(x)} \end{aligned} \tag{67}$$

Choosing the following variables transformation,

$$\begin{aligned} &X(x) = \ln u(x) \\ &X'(x) = \frac{u'(x)}{u(x)} \\ &X''(x) = \frac{u(x)u''(x) - u'(x)u'(x)}{[u(x)]^2} \end{aligned} \tag{68}$$

replacing in the differential equation in (67), we obtained,

$$\frac{u(x)u''(x) - u'(x)u'(x)}{[u(x)]^2} + \frac{u'(x)u'(x)}{[u(x)]^2} = ke^{-\ln u(x)} = k\frac{1}{u(x)} \tag{69}$$

which is transformed in,

$$u''(x) = k \tag{70}$$

being its integration trivial,

$$u(x) = \frac{k}{2}x^2 + bx + c \tag{71}$$

considering the change of variables $X(x) = \ln u(x)$, therefore,

$$X(x) = \ln u(x) = \ln\left(\frac{k}{2}x^2 + bx + c\right) \tag{72}$$

and,

$$e^{X(x)} = \frac{k}{2}x^2 + bx + c \tag{73}$$

the previous expression can also be written as,

$$e^{X(x)} = \left(\sqrt{\frac{k}{2}} x + c' \right)^2 \tag{74}$$

which is also a solution of the differential equation $u''(x) = k$, rebuilding the function for the velocity,

$$v(x,t) = \sqrt{u(x,t)} = \sqrt{e^{T(t)} e^{X(x)}} = \left(\frac{\left(\sqrt{\frac{k}{2}} x + c' \right)^2}{\left(\sqrt{\frac{k}{2}} t + C' \right)^2} \right)^{\frac{1}{2}} \tag{75}$$

or,

$$v(x,t) = \frac{\sqrt{\frac{k}{2}} x + c'}{\sqrt{\frac{k}{2}} t + C'} \tag{76}$$

Multiplying all the terms in the numerator and denominator by the relationship $\bar{k} / \sqrt{k/2}$ and renaming the constants of equation (76) it is possible to write,

$$v(x,t) = \frac{\bar{k}x + c_1}{\bar{k}t + c_2} \tag{77}$$

the exact solution of the differential equation (31). Bearing in mind when the mentioned equation was established, only the functions developments using the series of Taylor until the linear term were taking in consideration, and without considering the second order differentials. As a consequence, this solution is restricted to the linealazed model.

Therefore, the solution of the differential equation (34) for the flow would be the same of (77), multiplied by the constant S.

$$Q(x,t) = S \frac{\bar{k}x + c_1}{\bar{k}t + c_2} \tag{78}$$

where S is the area of the membrane and a constant of the system.

Another different method to find this characteristic velocity $v(x,t)$ would be calculating it from equation (23) or from (13) and replacing the required functions for experimental approximations or supposing good approximation functions to the actual data of the permeate flux.

From the equation (13) is easily to calculate the calibration function or the characteristic velocity $v(x,t)$, reaching that,

$$v(x,t) = \frac{\frac{\partial}{\partial t}J(x,t)}{\frac{\partial}{\partial x}J(x,t)} = \frac{-\frac{\partial}{\partial t}Q_V(x,t)}{-\frac{\partial}{\partial x}Q_V(x,t)} \tag{79}$$

i.e. $v(x,t)$ represents a function that relates the rate of change of the flow permeate by the membrane with the time, referred to the spatial rate of change of the same flow In this equation the negative signs were introduced by convenience, without modifying the expression of it.

To simplify the solutions of the differential equations the functions $Q_V(x,t)$ and $J(x,t)$ are going to be considerer as functions of separate variables, i.e.,

$$Q_V(x,t) = Q_V(x)Q_V(t) \tag{80}$$

$$J(x,t) = J(x)J(t) \tag{81}$$

As a result, equation (79) could be written as,

$$v(x,t) = \frac{\left(\frac{\partial}{\partial t}J(t)\right)J(x)}{\left(\frac{\partial}{\partial x}J(x)\right)J(t)} = \frac{-\left(\frac{\partial}{\partial t}Q_V(t)\right)Q_V(x)}{-\left(\frac{\partial}{\partial x}Q_V(x)\right)Q_V(t)} \tag{82}$$

From the models developed in the literature and taking same considerations of modeling, it is possible to determine a way (not the only one), to this function of the characteristic velocity. This means that this velocity must be built in a way that represents the filtration models and afterwards it fits to the actual solutions.

For example, for the permeate flow during a certain time $Q_V(t)$ a model similar to the exponential one could be chosen, as the ones developed at the beginning of this paper, or their linear approximation,

$$Q_V(t) = Q_{SS} + Ke^{-kt} \qquad k > 0 \tag{83}$$

Consequently, the numerator of equation (82) could be expressed as,

$$-\frac{\partial}{\partial t}[Q_V(t)] = Kke^{-kt} \qquad k > 0 \tag{84}$$

As it was described with the calculus in the first part of the paper, the fitted constant k is in general small, since the decrease of the fluxes with the time is generally slow. As a consequence, a linear approximation of this function would be enough in several real cases, and it can be denoted as,

$$-\frac{\partial}{\partial t}[Q_V(t)] = Kke^{-kt} = Kk\frac{1}{e^{kt}} \cong \frac{Kk}{1+kt} \tag{85}$$

this function is excellent to model the variations of the flux with the time, since making k a little bigger it is possible to get a faster decrease of the permeate flux with the time, and making k very small it is possible to get an almost constant value of the function. Furthermore, a wide spectrum of intermediate variations not only with k but also with the constants at $t = 0$ is obtained.

For the denominator of the calibration function the one is chosen based on a similar reasoning, but in this case a function in the variable x is needed. Due to the velocity of the feed flow decreases along the membrane because of the permeate flux through it, the hydraulic tangential shears will be higher at the beginning of the membrane and lower at the end of it. This difference in the internal flow velocity would increase if the length of the membrane is higher, or if there are several membranes connected in series. In this way one could think that the membrane fouling would be less at the beginning than at the end of it. On the other hand, the solution concentrates as the permeate comes out the membrane, which indicates that the feed concentration increases and, as a consequence; the fouling will be even higher. According to these considerations, we have to model a decreasing permeate flux through the membrane. In fact, if the membrane were extremely long, at the end of it the concentration would be very high and there would not be almost liquid to permeate. Because of it, it could be proposed a model of variation similar to the one of the flow with the time, which has to be adequately parameterized in each case. In this way it is possible to write,

$$Q_V(x) = Q'_{SS} + K'e^{-k'x} \qquad k' > 0 \tag{86}$$

The derivative and the approximation until the linear term are written as,

$$-\frac{\partial}{\partial x}[Q_V(x)] = K'k'e^{-k'x} = K'k'\frac{1}{e^{k'x}} \cong \frac{K'k'}{1+k'x} \tag{87}$$

Replacing functions (85) and (87) calculated in (82) leads to the following expression,

$$v(x,t) = \frac{-\left(\frac{\partial}{\partial t}Q_V(t)\right)Q_V(x)}{-\left(\frac{\partial}{\partial x}Q_V(x)\right)Q_V(t)} = \frac{Kke^{k'x}\left(Q'_{SS} + K'e^{-k'x}\right)}{K'k'e^{kt}\left(Q_{SS} + Ke^{-kt}\right)} = \frac{KkQ'_{SS}e^{k'x} + KK'k}{K'k'e^{kt}Q_{SS} + KK'k'} \tag{88}$$

Rewriting the constants it is possible to simplify the above expression as,

$$v(x,t) = \frac{k_1 e^{k'x} + k_2}{k_3 e^{kt} + k_4} \tag{89}$$

if both the numerator and denominator are developed as Taylor series functions, until the linear term, following the same procedure than in the previous model, the following expression is reached,

$$v(x,t)=\frac{k_1(1+k'x)+k_2}{k_3(1+kt)+k_4}=\overline{K}\frac{\overline{k}x+c_1}{\overline{k}t+c_2} \tag{90}$$

taking $\overline{K}=1$ makes this equal to the determined by the exact and analytical resolution of the differential equations. This model proposed for the calibration function or characteristic velocity is developed as in the previous case until the linear term of the exponentials.

4. An analytical and exact solution of the permeate flux *J(x,t)*.

Because of the solution for the flow velocity or the calibration function inside of the membrane $v(x,t)$ was already obtained, it is possible to integrate in an analytical and exact manner the differential equation for a permeate flux $J(x,t)$,

$$\frac{\partial}{\partial x}J(x,t)-\frac{1}{v(x,t)}\frac{\partial}{\partial t}J(x,t)=0 \tag{91}$$

replacing the solution by the function $v(x,t)$ in the following equation leads to,

$$\frac{\partial}{\partial x}J(x,t)-\left(\frac{\overline{k}t+c_2}{\overline{k}x+c_1}\right)\frac{\partial}{\partial t}J(x,t)=0 \tag{92}$$

that can be written as,

$$\left(\overline{k}x+c_1\right)\frac{\partial}{\partial x}J(x,t)=\left(\overline{k}t+c_2\right)\frac{\partial}{\partial t}J(x,t) \tag{93}$$

if the permeate flux is expressed as $J(x,t)=J(x)J(t)$ and the corresponding derivative of (93) are determined,

$$\begin{aligned}\frac{\partial}{\partial x}J(x,t)&=J'(x)J(t)\\ \frac{\partial}{\partial t}J(x,t)&=J'(t)J(x)\end{aligned} \tag{94}$$

introducing these derivatives in (93) a differential equation of separated variables is obtained,

$$\left(\overline{k}x+c_1\right)J'(x)J(t)=\left(kt+c_2\right)J'(t)J(x) \tag{95}$$

$$\left(\bar{k}x + c_1\right)\frac{J'(x)}{J(x)} = -k$$
$$\left(\bar{k}t + c_2\right)\frac{J'(t)}{J(t)} = -k \tag{96}$$

from which it is possible to obtain two independent differential equations and equal them to an arbitrary constant,

$$\frac{J'(x)}{J(x)} = -k\frac{1}{\bar{k}x + c_1}$$
$$\frac{J'(t)}{J(t)} = -k\frac{1}{\bar{k}t + c_2} \tag{97}$$

easy to integrate and with the following solutions,

$$J(x) = \frac{1}{\left(\bar{k}x + c_1\right)^{\frac{k}{\bar{k}}}}$$
$$J(t) = \frac{1}{\left(\bar{k}x + c_2\right)^{\frac{k}{\bar{k}}}} \tag{98}$$

making $\frac{k}{\bar{k}} = n$ and regrouping the constants a general solution for the permeate flux is obtained,

$$J(x,t) = J(x)J(t) = \frac{1}{\left(\bar{k}x + c_1\right)^n}\frac{1}{\left(\bar{k}x + c_2\right)^n} \tag{99}$$

which takes the following form,

$$J(x,t) = J(x)J(t) = \frac{1}{\left(c_1\left(\frac{1}{c_1}\bar{k}x + 1\right)\right)^n}\frac{1}{\left(c_2\left(\frac{1}{c_2}\bar{k}x + 1\right)\right)^n} \tag{100}$$

or

$$J(x,t) = \frac{\frac{1}{c_1^{\,n}}}{(1+\beta x)^n}\frac{\frac{1}{c_2^{\,n}}}{(1+\lambda t)^n} = \frac{J_{(0,0)}}{(1+\beta x)^n(1+\lambda t)^n} \tag{101}$$

where,

$$\beta = \frac{1}{c_1}\bar{k}, \quad \lambda = \frac{1}{c_2}\bar{k}, \quad J_{(0,0)} = \frac{1}{c_1^{\ n}}\frac{1}{c_2^{\ n}} \tag{102}$$

in the point $(x,t) = (0,0)$ the permeate flux is named $J_{(0,0)}$.

$$J(x,t) = \frac{J_{(0,0)}}{(1+\beta x)^n (1+\lambda t)^n} \tag{103}$$

expression of the analytical and exact solution of the differential equation of the permeate flux (91).

Figure 4 shows a typical distribution of the permeate flux $J(x,t)$. In addition a membrane with lenght L has been drawn in the x axis of the reference system.

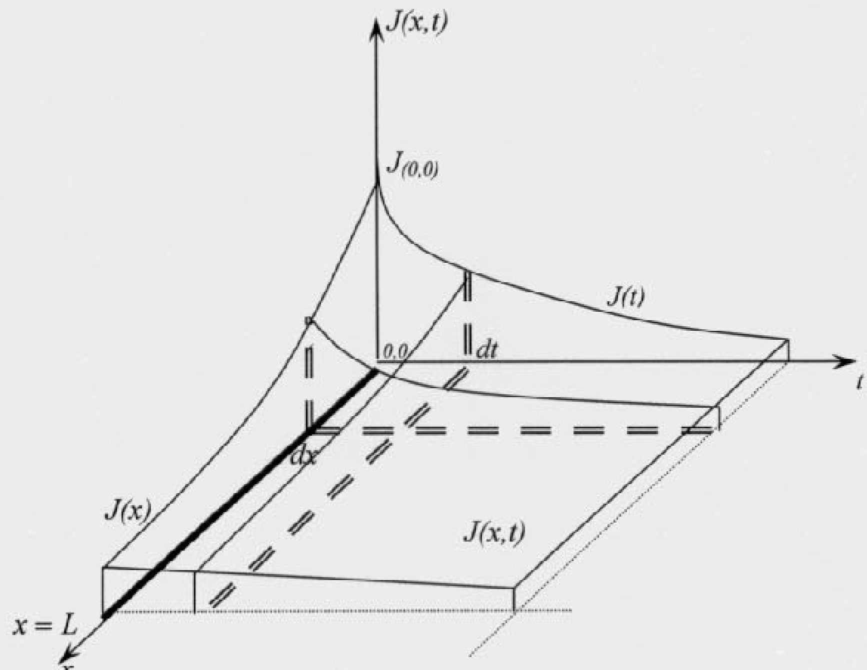

Figure 4
Bidimensional distribution of the permeate flux

5. Interpretation of some flux models.

Having the general solution for the permeate flux,

$$J(x,t) = \frac{J_{(0,0)}}{(1+\beta x)^n (1+\lambda t)^n} \tag{104}$$

it is possible to calculate the flux in function of the time integrating the variable x over all the membrane length, in a graphic of $J_p(t)$, like of the Figure 5. The y-axis represents the permeate flux in all of the membrane at an arbitrary time t. Then,

$$J_p(t) = \int_0^L J(x,t)dx = \int_0^L \frac{J_{(0,0)}}{(1+\beta x)^n (1+\lambda t)^n} dx \qquad (105)$$

this relationship explains that the measured permeate flux in a certain instant t is the sum of all the contributions of the permeate differential fluxes through the membrane for all the dx of length. For this reason the equation is integrated between 0 and L being L the total length of the membrane or the sum of all of the lengths of the membranes connected in series.

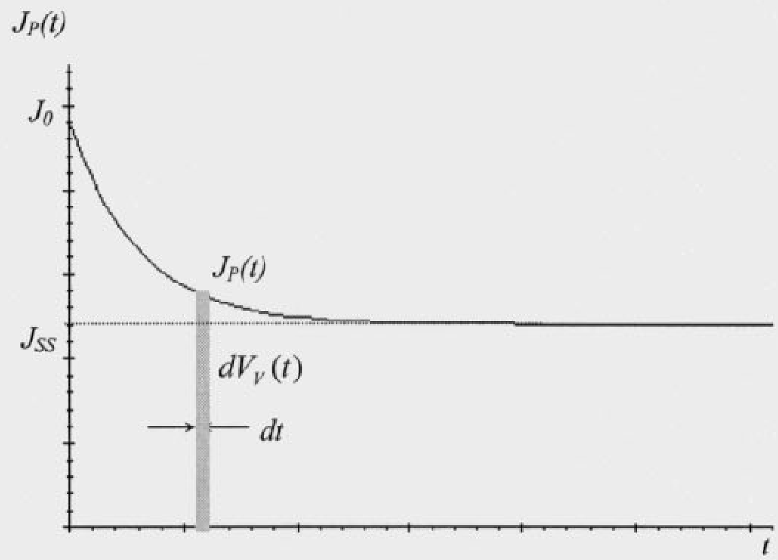

Figure 5

Definition of parameters in a typical curve of permeate flux in an UF experiment with BSA.

The most representative cases of the flux models are for $n = 1$, $n = 2$ and $n = 0.5$.

Case $n = 1$ (Pore blocking model)

The integral equation of the flux for this model is,

$$J_p(t) = \int_0^L J(x,t)dx = \int_0^L \frac{J_{(0,0)}}{(1+\beta x)(1+\lambda t)} dx \qquad (106)$$

which integral is,

$$J_p(t) = \frac{\ln(1+\beta L)}{\beta} \frac{J_{(0,0)}}{(1+\lambda t)} = \frac{J_0}{(1+\lambda t)} \qquad (107)$$

a model of flux for Pore blocking model, (Huisman I. H. et al. 2004). The value for the permeate flux for $t = 0$ is formed for an expression that consider not only the constant L, which represents the membrane length, but also the parameter β, which characterized the fouling profile through the membrane.

In this model the constant λ is a parameter that describes the fouling characteristics associated to its process, and it is given by the following expression,

$$\lambda = \frac{\sigma P_{tm}}{\mu R_m} \tag{108}$$

where $\sigma\left[m^{-1}\right]$ is a parameter that characterized the potential fouling of the feed solution, R_m is the resistance of the membrane, P_{tm} is the effective transmembrane pressure and μ is the solution viscosity.

Case $n = 2$ (Internal pore plugging model)

In this case equation (105) could be written as,

$$J_p(t) = \int_0^L J(x,t)dx = \int_0^L \frac{J_{(0,0)}}{(1+\beta x)^2(1+\lambda t)^2}dx \tag{109}$$

which integral is,

$$J_p(t) = \frac{J_{(0,0)}L}{(1+\beta L)(1+\lambda t)^2} = \frac{J_{(0)}}{(1+\lambda t)^2} \tag{110}$$

This expression corresponds to the internal pore plugging model, (Huisman I. H. et al. 2004). In this model the parameter λ is equal to,

$$\lambda = \frac{\beta J_0}{e\varepsilon} \tag{111}$$

where, in this case, β is an dimensionless parameter, which characterized the potential fouling of the solution, e is the width of the membrane and ε is the initial anular fraction of the membrane, being $\varepsilon = N\pi r_0^2$, where N is the number of pores per membrane area unity and r_0 is the initial ratio of the pore before fouling.

Case $n = 0.5$ Cake filtration model

In this case equation (105) remains as,

$$J_P(t) = \int_0^L \frac{J_{(0,0)}}{\sqrt{(1+\beta x)}\sqrt{(1+\lambda t)}} dx \tag{112}$$

integrating leads to the following result,

$$J_P(t) = \frac{2J_{(0,0)}(\sqrt{(1+\beta L)}-1)}{\sqrt{(1+\lambda t)}} = \frac{J_{(0)}}{\sqrt{(1+\lambda t)}} \tag{113}$$

which is the cake filtration model, (Huisman I. H. et al. 2004).

In this model the constant λ is a parameter, which describes the characteristic of the fouling associated to the process, denoted as,

$$\lambda = \frac{2\alpha P_{tm} A_0}{\mu R_m^2} \tag{114}$$

where $\alpha \left[m^{-4}\right]$ is a parameter that characterized the potential of fouling of the feed solution, R_m is the resistance of the membrane, P_{tm} is the effective transmembrane pressure, μ is the solution viscosity and A_0 is the total active area of the membrane. Figure 6 demonstrates an arbitrary graphic for the surface pore blocking model with the aim to showing the permeate flux determination in two dimensions.

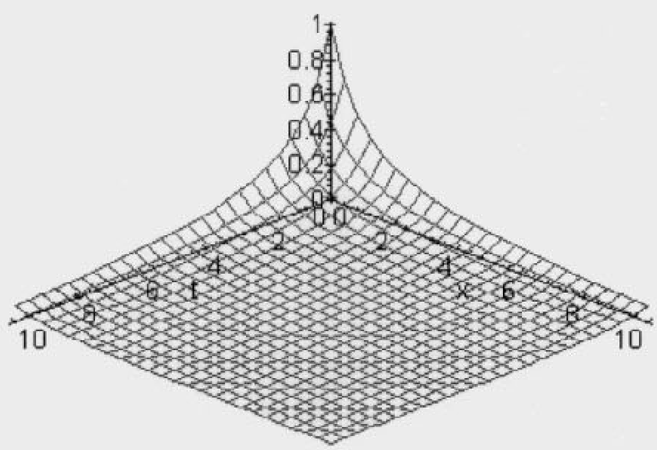

Figure 6
Theoretical model of a permeate flux

This graphic illustrates the permeate flux decay through the membrane and in function of the time, as it is expressed in (105) for $n = 1$. Figure 7 shows the profiles of the permeate flux in function of the time for four values of x =*Cte*.

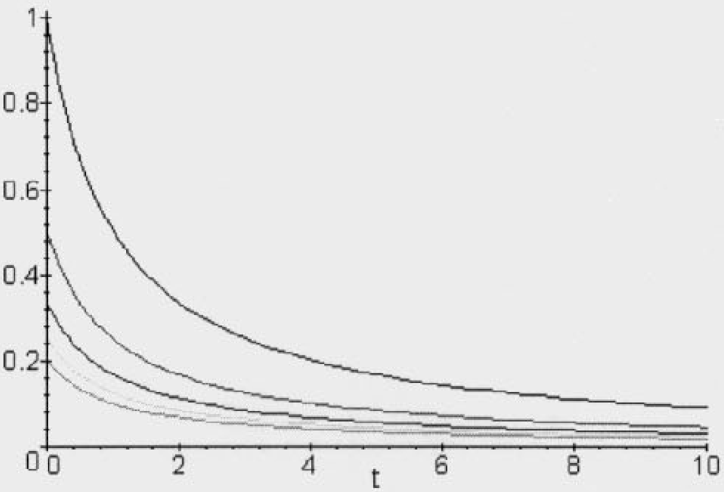

Figure 7
Profiles of the two-dimensions model for the permeate flux at $x = C$

To determine the constants it has to be consider that,

$$J(x,t) = \frac{J_{(0,0)}}{(1+\beta x)^n (1+\lambda t)^n} \tag{115}$$

$J(0,0) = J_{(0,0)}$ at $(x,t) = (0,0)$, this value is unique, being the permeate flux in the initial instant in the at the beginning of the membrane, i.e. in $x = 0$. But this is not J_0, which value is,

$$J(0) = J_0 = J_{(0,0)} \ln(1+\beta L)^{\frac{1}{\beta}} \tag{116}$$

the value would be higher as the length of the membrane L increases, and lower if the parameter β increases. The derivative of $J(t)$ in function of the time is,

$$\frac{d}{dt}[J(t)] = \frac{d}{dt}\left[\frac{\ln(1+\beta L)}{\beta}\frac{J_{(0,0)}}{(1+\lambda t)}\right] = \frac{d}{dt}\left[\frac{J_0}{(1+\lambda t)}\right] = -\frac{\lambda}{\beta}\ln(1+\beta L)\frac{J_{(0,0)}}{(1+\lambda t)^2} \tag{117}$$

evaluating at $t = 0$,

$$\frac{d}{dt}[J(t)]_{t=0} = -J_{(0,0)}\frac{\lambda}{\beta}\ln(1+\beta L) \tag{118}$$

expression that represents the slope of the permeate flux at $t = 0$.

Integration of the case for a general model of filtration

For an arbitrary model the equation for the flux would be written in function of an arbitrary exponent n by the general expression for a flux in two variables,

$$J_P(t) = \int_0^L J(x,t)dx = \int_0^L \frac{J_{(0,0)}}{(1+\beta x)^n (1+\lambda t)^n} dx \tag{119}$$

integrating,

$$J_P(t) = \frac{(1+\beta L)^{1-n} - 1}{\beta(1-n)} \frac{J_{(0,0)}}{(1+\lambda t)^n} = \frac{J_0}{(1+\lambda t)^n} \qquad n \neq 1 \tag{120}$$

in this equation is necessary to save the indetermination for $n = 1$, which is the case of the surface pore blocking model.

Among the representative model of membranes found in the literature, there are some that take into consideration some special cases in the characteristic velocity, for instance, the case in which the permeate is disregarded compared with the flow that circulates through the membrane, so it is possible to consider,

$$v(x,t) \cong v_0 \tag{121}$$

For this case the differential equation for the permeate flow becomes,

$$\frac{\partial}{\partial x} Q_V(x,t) - \frac{1}{v_0} \frac{\partial}{\partial t} Q_V(x,t) = 0 \tag{122}$$

which solution is,

$$Q_V(x,t) = Q_{V0} e^{-k(x+v_0 t)} \tag{123}$$

replacing the differential equation leads to,

$$\frac{\partial}{\partial x} Q_V(x,t) = \frac{\partial}{\partial x} Q_{V0} e^{-k(x+v_0 t)} = -k Q_{V0} e^{-k(x+v_0 t)} = -k Q_V$$

$$\frac{\partial}{\partial t} Q_V(x,t) = \frac{\partial}{\partial t} Q_{V0} e^{-k(x+v_0 t)} = -k v_0 Q_{V0} e^{-k(x+v_0 t)} = -k v_0 Q_V$$

then, the solution is verifying by,

$$-k v_0 Q_V + k v_0 Q_V = 0$$

at $x = t = 0$

$$Q_V(0,0) = Q_{V0} \tag{124}$$

expression that represents the initial caudal at $t = 0$ at the inlet of the membrane. If it is divided the permeate caudal by the active area of the membrane, we reaches to the membrane flux equation,

$$J_V(x,t) = J_{V0}e^{-k(x+v_0t)} \tag{125}$$

This is the final expression of this model, with constant flow velocity. If there is a stationary flow in the membrane, after an arbitrary time t, the equation becomes,

$$Q_V(x,t) = Q_{SS} + Q_{V0}e^{-k(x+v_0t)} \tag{126}$$

or, in terms of the flux,

$$J_V(x,t) = J_{SS} + J_{V0}e^{-k(x+v_0t)} \tag{127}$$

which also is in agreement with the corresponded differential equation.

Other interesting case of this model is that it considers the situation in which the permeate flux has the following form,

$$J(t) = J_0 t^{-b} \tag{128}$$

as it is presented by (Cheyran, Munir 1998). Taking into account the characteristic velocity equation and rewriting it,

$$v(x,t) = \overline{K}\frac{\overline{k}x + c_1}{\overline{k}t + c_2} = c_1\left(\frac{k_1 x + \overline{K}}{\overline{k}t + c_2}\right) \tag{129}$$

Choosing $\mathrm{k}_1 \approx 0$ and $\mathrm{c}_2 = 0$, $v(x,t)$, the expression above takes the following form,

$$v(x,t) = \frac{\overline{K}c_1}{\overline{k}t} = \frac{K}{t} \tag{130}$$

introducing this relationship in the equation of the permeate flow (12),

$$\frac{\partial}{\partial x}Q_V(x,t) - \frac{1}{v(x,t)}\frac{\partial}{\partial t}Q_V(x,t) = 0 \tag{131}$$

it gives the following differential equation,

$$\frac{\partial}{\partial x}Q_V(x,t)-\left(\frac{t}{K}\right)\frac{\partial}{\partial t}Q_V(x,t)=0 \tag{132}$$

after separating the variables,

$$\frac{Q_V'(t)}{Q_V(t)}=-\frac{b}{t}$$
$$\frac{Q_V'(x)}{Q_V(x)}=-\frac{b}{K} \tag{133}$$

where $b>0$ is an arbitrary constant. The solutions for these equations are trivial,

$$\ln Q_V(t)=-b\ln t+c'=-b\ln t+\ln c''=\ln\left(c''t^{-b}\right)$$
$$\ln Q_V(x)=-\frac{b}{K}x+c''' \tag{134}$$

being transform in,

$$Q_V(x,t)=Q_V(x)Q_V(t)=\left(c''t^{-b}\right)\left(e^{-\frac{b}{K}x}e^{c'''}\right)=ct^{-b}e^{-\frac{b}{K}x} \tag{135}$$

The above expression could be transform in a equation for the permeate flux if it is divided by the initial active area of the membrane, leading to,

$$J_V(x,t)=\frac{1}{A_0}ct^{-b}e^{-\frac{b}{K}x} \tag{136}$$

to see the variation of the permeate flux function with t a integration over the x variable in all the membrane length, then,

$$J(t)=\int_0^L J_V(x,t)dx=\int_0^L\frac{1}{A_0}ct^{-b}e^{-\frac{b}{K}x}dx=\frac{Kc}{bA_0}\left(1-e^{-\frac{b}{K}L}\right)t^{-b}=J_0t^{-b} \tag{137}$$

if we develop the constant expression in terms of growing potentials of L, it is possible that choosing a constant K big enough, the permeate flux model can be written as,

$$\frac{Kc}{bA_0}\left(1-e^{-\frac{b}{K}L}\right)=\frac{cL}{A_0}-\frac{cbL^2}{2A_0K}+\frac{cb^2L^3}{6A_0K^2}-\frac{cb^3L^4}{24A_0K^3}+\ldots \tag{138}$$

expression that when it is K included becomes,

$$\frac{Kc}{bA_0}\left(1-e^{-\frac{b}{K}L}\right) \cong \frac{cL}{A_0} \tag{139}$$

so the permeate flux can be expressed as,

$$J(t) = \frac{Kc}{bA_0}\left(1-e^{-\frac{b}{K}L}\right)t^{-b} = \frac{cL}{A_0}t^{-b} = J_0 t^{-b} \tag{140}$$

6. Differential equation for the concentration and its analytical and exact solution.

Given a solution formed by a solvent and a solute, its concentration can be defined as,

$$C(x,t) = \frac{dm}{dV(x,t)} \tag{141}$$

the flow of the solution is defined as follows,

$$Q(x,t) = \frac{dV(x,t)}{dt} \tag{142}$$

a simple equation for the mass flow of the solute could be,

$$Q(x,t)C(x,t) = \frac{dm}{dV(x,t)}\frac{dV(x,t)}{dt} = \frac{dm}{dt} \tag{143}$$

as the mass flow must be constant if there is not accumulation or elimination of the solute inside the membrane, therefore,

$$Q(x,t)C(x,t) = Q(x+\Delta x, t+\Delta t)C(x+\Delta x, t+\Delta t) \tag{144}$$

if this differential equation is developed in series of Taylor,

$$Q(x,t)C(x,t) = \left[Q(x,t) + \frac{\partial}{\partial x}Q(x,t)\Delta x + \frac{\partial}{\partial t}Q(x,t)\Delta t\right]\left[C(x,t) + \frac{\partial}{\partial x}C(x,t)\Delta x + \frac{\partial}{\partial t}C(x,t)\Delta t\right] \tag{145}$$

taking into account until the linear term, disregarding the second order differentials, it is obtained,

$$Q(x,t)C(x,t) = \left[Q(x,t) + \frac{\partial}{\partial x}Q(x,t)\Delta x + \frac{\partial}{\partial t}Q(x,t)\Delta t\right]\left[C(x,t) + \frac{\partial}{\partial x}C(x,t)\Delta x + \frac{\partial}{\partial t}C(x,t)\Delta t\right] \tag{146}$$

denoting,

$$dQ(x,t) = \frac{\partial}{\partial x} Q(x,t)\Delta x + \frac{\partial}{\partial t} Q(x,t)\Delta t \tag{147}$$

$$dC(x,t) = \frac{\partial}{\partial x} C(x,t)\Delta x + \frac{\partial}{\partial t} C(x,t)\Delta t \tag{148}$$

and disregarding the differential terms of second order,

$$Q(x,t)C(x,t) = Q(x,t)C(x,t) + Q(x,t)dC(x,t) + dQ(x,t)C(x,t) \tag{149}$$

reordering the expression,

$$Q(x,t)dC(x,t) + dQ(x,t)C(x,t) = 0 \tag{150}$$

this equation is the exact differential of,

$$d[Q(x,t)C(x,t)] = Q(x,t)dC(x,t) + dQ(x,t)C(x,t) \tag{151}$$

so, it is possible to write,

$$Q(x,t)C(x,t) = k^* \tag{152}$$

replacing $Q(x,t)$ for its value in the following equation it leads to,

$$\frac{k^*}{C(x,t)} dC(x,t) + dQ(x,t)C(x,t) = 0 \tag{153}$$

or, written in an other form,

$$dQ(x,t) = -\frac{k^*}{C^2(x,t)} dC(x,t) \tag{154}$$

$dQ(x,t)$ is the permeate flow in a differential cross-area of the membrane and in this way, proportional to the permeate flux, i.e. $dQ(x,t) = A_0 dJ(x,t)$,

$$A_0 dJ(x,t) = -\frac{k^*}{C^2(x,t)} dC(x,t) = d\left(\frac{k^*}{C(x,t)}\right) \tag{155}$$

deducing that,

$$A_0 J(x,t) = \frac{k^*}{C(x,t)} \tag{156}$$

or,

$$C(x,t) = \frac{k^*}{A_0}\frac{1}{J(x,t)} = k^{**}\frac{1}{J(x,t)} \tag{157}$$

The above equation shows that the concentration is proportional to the inverse function of the permeate flux. Introducing (104) in the previous relationship it is obtained that,

$$C(x,t) = k^{**}\frac{(1+\beta x)^n(1+\lambda t)^n}{J_{(0,0)}} = \bar{k}(1+\beta x)^n(1+\lambda t)^n \tag{158}$$

the concentration profiles in function of time and space also depend on the exponent n, in the same form that for the permeate flux. In the above equation, if x and t increases, the feed concentration depends on the flow model and the length of the membrane. Thus, for a membrane with length L, the concentration profile in function of time will be,

$$C(t) = \int_0^L \bar{k}(1+\beta x)^n(1+\lambda t)^n dx = \bar{k}\frac{\left((1+\lambda t)^n\left((1+\beta L)^{n+1}-1\right)\right)}{\beta(n+1)} \tag{159}$$

which is valid for $n=1, n=2$ and $n=0.5$.

7. Differential equation for the variation of the active area of the membrane.

To analyze the decrease of the membrane active area during the filtration process, it is assumed that the fouling is blocking a fraction of the available pores of the membrane. As the immediate effect of the fouling is the reduction of the permeate flow, it could be consider that the decrease of the area is proportional to the instant value of the permeate flow, if the transmembrane pressure is maintained constant. Expressing this idea in a mathematically form,

$$dA(x,t) = -\sigma dQ(x,t) \tag{160}$$

Where the differential terms are,

$$\begin{aligned} dQ(x,t) &= \frac{\partial}{\partial x}Q(x,t)\Delta x + \frac{\partial}{\partial t}Q(x,t)\Delta t \\ dA(x,t) &= \frac{\partial}{\partial x}A(x,t)\Delta x + \frac{\partial}{\partial t}A(x,t)\Delta t \end{aligned} \tag{161}$$

introducing (161) in equation (160),

$$\frac{\partial}{\partial x}A(x,t)\Delta x+\frac{\partial}{\partial t}A(x,t)\Delta t=-\sigma\left[\frac{\partial}{\partial x}Q(x,t)\Delta x+\frac{\partial}{\partial t}Q(x,t)\Delta t\right] \tag{162}$$

which can be written as a function of the flow velocity as,

$$\frac{\partial}{\partial x}A(x,t)+\frac{1}{v(x,t)}\frac{\partial}{\partial t}A(x,t)=-\sigma S\left[\frac{\partial}{\partial x}v(x,t)+\frac{1}{v(x,t)}\frac{\partial}{\partial t}v(x,t)\right] \tag{163}$$

If the general expression of velocity is introduced,

$$v(x,t)=\frac{\bar{k}x+c_1}{\bar{k}t+c_2} \tag{164}$$

the second term of (163) becomes zero, then, the differential equation for the membrane active area is,

$$\frac{\partial}{\partial x}A(x,t)+\frac{1}{v(x,t)}\frac{\partial}{\partial t}A(x,t)=0 \tag{165}$$

introducing the velocity in (165) and making $A(x,t)=A(x)A(t)$,

$$A'(x)A(t)+\frac{\bar{k}t+c_2}{\bar{k}x+c_1}A'(t)A(x)=0 \tag{166}$$

which becomes in the following equations with separated variables,

$$\frac{A'(x)}{A(x)}\left[\bar{k}x+c_1\right]=k \tag{167}$$

$$\frac{A'(t)}{A(t)}\left[\bar{k}t+c_2\right]=-k \tag{168}$$

their solutions are simple,

$$A(x)=\left[\bar{k}x+c_1\right]^{\frac{k}{\bar{k}}} \tag{169}$$

$$A(t)=\left[\bar{k}t+c_2\right]^{-\frac{k}{\bar{k}}} \tag{170}$$

denoting $\dfrac{k}{\bar{k}}=n$ equation (171) is the corresponded solution,

$$A(x,t) = \left[\frac{\bar{k}x + c_1}{\bar{k}t + c_2} \right]^n \tag{171}$$

If we integrated over the x variable, for all the length of the membrane L, the variation of the active membrane area with the time is obtained,

$$A(t) = \int_0^L A(x,t)dx = \int_0^L \left[\frac{\bar{k}x + c_1}{\bar{k}t + c_2} \right]^n dx \tag{172}$$

and the integral is,

$$A(t) = \frac{\left(\bar{k}L + c_1\right)^n (\bar{k}L + c_1) - c_1^{n+1}}{(n+1)\bar{k}\left(\bar{k}t + c_2\right)^n} \tag{173}$$

with a general form,

$$A(t) = \frac{K(L,n)}{\left(1 + \gamma t\right)^n} \tag{174}$$

meaning that the active area characterized by the exponent n, as in the case of the different types of flows and length of it. This more general result is in agreement with the theoretical result found in [4] and verified experimentally. In both cases a decrease of the flow or the permeate flow with the same form of the reduction of the active area of the membrane is observed.

References

1. Cheyran Munir (1998) la ecuación (6.3 a) de la página 243 del "Ultrafiltration and microfiltration Handbook" de, Technomic Publishing Co. 1998.

2. Jaffrin M.Y., L.H. Ding, Ch. Couvreur, P. Khari, Effect of ethanol on ultrafiltration of bovine albumin solutions with organic membranes. J. Membr. Sci., 124 (1997) 233-241.

3. Huisman, I.H., P. Prádanos and A. Hernández, The effect of protein-protein and protein-membrane interactions on membrane fouling in ultrafiltration, J. Membr. Sci., 179 (2000) 79.

4. Tarazaga C. C., M.E. Campderrós, A. Pérez Padilla, Physical cleaning by means of electric field in the ultrafiltration of a biological solution. J. Membr. Sci., 278 (2006) 219-224.

5. Tarazaga C. C., M.E. Campderrós, A. Pérez Padilla, Physical cleaning by means of electric field in the ultrafiltration of a biological solution. J. Membr. Sci., 278 (2006) 219-224.

6. Visvanathan C., R. Ben Aim, Application of an electric field for the reduction of particle and colloidal membrane fouling in crossflow microfiltration, Sep.Sci and Technol.,24 (1989) 383.398.

7. Yu-Hsiang Weng, Lin Han Chaung-Hsieh, Hsin-Hua Lee, Kung-Cheh Li and C.P. Huang, Removal of arsenic and humic substances (HSs) by electro-ultrafiltration (EUF) J.Hazar.Mat.,122 (2005) 171-176.

8. Zumbusch, W. Kulcke and G. Brunner, Use of alternating electrical fields as anti-fouling strategy in ultrafiltration of biological suspensions - Introduction of a new experimental procedure for crossflow filtration, J. Membr. Sci., 142 (1998) 75-86.