PROPER AND PERFECT COOPERATIVE EQUILIBRIUM FOR CLASSICAL AND RATIONAL GAMES

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PROPER AND PERFECT COOPERATIVE EQUILIBRIUM FOR CLASSICAL AND RATIONAL GAMES

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Abstract

Many years ago Selten in [15] refined the concept of Nash equilibria for n-person mixed extension of finite strategic games. Afterwards, in an important piece of work, Myerson in [13] refined Selten's concept of perfect equilibrium into the concept of proper equilibrium. In [13] he presented an existence proof of proper points. On the other hand Aumann in [1] introduced and proved the concept of correlated equilibria and Hart and Schmeidler [5] proved in an elegant way its existence using an elementary proof.

Finally Marchi in [9] has introduced the concept of cooperative equilibrium and its existence.

In this paper we will introduce and study the existence of perfect and proper cooperative equilibrium points in the sense of Selten and Myerson for classical and rational games as introduced by Marchi in [10].

1- Introduction

The subject of equilibrium points in games of strategy is an old subject. Many authors claim the introduction of important concepts of equilibrium in what seems some kind of competition. This is due to Cournot, whose contribution considered only the two player case. Wald in mathematical economy has developed an important concept of equilibrium. However the landmark for n-person strategic games in normal form was introduced by Nash in [14], where he has proved and introduced a very deep concept of equilibrium where each person obtains its maximum by playing its corresponding component assuming that all the remaining players abide by it.

Several attempts to generalize and approach equilibrium points have been studied by different authors. In the decade of the 60's namely Wn Wen-Tsün and Jiang Jia-He [16] and Marchi [8].

It is in the 70's that there is a breakthrough with the work of Selten [15], where he refines the concept of Nash equilibrium points into the perfect equilibrium points. In this way he obtains a non-empty subset of Nash equilibrium points, which have more stability property, related with perturbations of possible errors of the players. Moreover Myerson in [13] has refined the concept of Selten of perfect equilibrium points which are a proper subset of the perfect. He has introduced a deep and very elegant existence theorem for them. There are other several refinements in the literature, which might be seen in Van Damme [4].

In the aspect of cooperative theory Marchi has introduced and extended in [9] the cooperative equilibrium points from the paper [10].

These points in a conceptual aspect might describe the intercommunication and "cooperative actions" among the players. These are of the kind of point due to Aumann [1], the correlated one. However even though they posses similar conceptual properties the mathematical relationship has not yet been studied. This would be an interesting subject of research.
Following our argument here in this paper we are going to extend the cooperative equilibrium in the sense of perfectness and properness.

2- Perfect and Proper g-cooperative equilibrium points for general games

Consider an n-person game in normal form

$$\Gamma = (S_1, \dots S_n, V_1, \dots V_n)$$

where each S_1 is a non empty finite set, and each V_1 is a real-valued-function defined in the domain $S_1 x S_2 x \dots x S_n$. The set S_i is the set of pure strategies available to player i. Each V_i is the utility function for player i, so that

$$V_i(S_1 \dots S_n)$$

would be the payoff to player i (measured in some von Neumann-Morgenstern utility scale) and $(S_1, \dots, S_n)$ is the combination of strategies chosen by the players.
For any finite set M, let $\Delta(M)$ be the set of all probability distribution over M, thus the set of all mixed strategies over S_i is

$$\Delta(S_i) = \left\{ \sigma_i \in R^{|S_i|} \;\middle|\; \sum_{s_i \in S_i} \sigma_i(s_i) = 1 \quad \sigma_i(s'_i) \geq 0 \;\forall s'_i \in S_i \right\}$$

On the other hand consider the cooperative communicated functions

$$q^i : \Delta\left(\mathop{X}_{i=1}^{n} S_i\right) \longrightarrow \Delta\left(\mathop{X}_{j \neq i} S_j\right)$$

It is straightforward to extend the utility functions to mixed strategies for correlated actions.

Let $z \in \Delta\left(\mathop{X}_{i=1}^{n} S_i\right)$ then $\pi_i(z) \in \Delta(S_i)$

let the marginal mixed strategy for player i, that is to say

$$\pi_i(z)(s_i) = \sum_{s_1 \dots s_{i-1}, s_{i+1} \dots s_n} z(s_1, \dots s_n) = \sum_{s-i} z(s_i, s-i)$$

Then we have

$$\nu_i(\pi_i(z), q^i(z)) = \sum_{s_i} \pi_i(z)(s_i) \sum_{s-i} q^i(z)(s-i)\; V_i(s_i, s-i).$$

We remind the reader that a q-cooperative point for a game $\ni$ with communication functions q^i is a point ε) (X S_i) such that

$$V_i(\pi_i(\bar{z}),\ q^i(\bar{z})) \geq V_i(\sigma_i, q^i(\bar{z})) \qquad \forall_i\ \forall \sigma_i \in \Delta(S_i)$$

The interpretation is clear from an intuitive point of view.
Now we introduce the corresponding functions

$$V_j(s_j^*)\ /\ q^i(z)) = \sum_{s_i} \sum_{s_{-i}} V_j(s_j, s_{-j})\ q^j(z)(s_{-j})\ \sigma_j^*(s_j)$$

where

$$\sigma_j^*(s_j) = \begin{cases} 1 & \text{if } s_j = s_j^* \\ 0 & \text{otherwise} \end{cases}$$

We have the following
<u>Proposition 1:</u> z is q-cooperative equilibrium point if and if

$$V_j(s_j \mid q^j(\bar{z})) < V_j(s_j' \mid q^j(\bar{z})) \text{ implies } \pi_j(\bar{z})(s_j) = 0 \text{ for all } j,\ s_j, s_j' \in S_j.$$

<u>Proof</u>: In order to see this result consider the equality

$$V_i(\sigma_i,\ q^i(z)) - V_i(\sigma_i',\ q^i(z))$$

$$= \sum_{s_i} \sum_{s_i'} \sigma_i(s_i)\sigma_i'(s_i')\ [V_i(s_i | q^i(z)) - V_i(s_i' | q^i(z))]$$

Then the proposition follows easily from the definition of q-cooperative equilibrium (q.e.d.).
Now we are going to introduce Selten's approach for our points. We say that a point z is ε-perfect q-cooperative equilibrium point if $\varepsilon>0$ an

$$z \in \Delta^0(\mathop{X}_{i=1}^{n} S_i) = \left\{ z \in \Delta(\mathop{X}_{i=1}^{n} S_i) : z(s_i, s_{-i}) > 0\ \forall s_i s_{-i} \right\}$$

such that if

$$V_i(s_i, q^i(z)) < V_i(s_i', q^i(z))$$

implies

$\pi_i(z)(s_i) \leq \varepsilon$ for each i, s_i, $s'_i \in S_i$.
In the same way we now follow the ideas of Myerson. We say that a point

$$z \in \Delta^0(\mathop{X}_{i=1}^{n} S_i)$$

is an ε-proper q-cooperative equilibrium point if

$$V_i(s_i | q^i(z)) < V_i(s_i' | q^i(z)) \text{ implies } \pi_i(z)(s_i) \leq \varepsilon \pi_i(z)(s_i')$$

for $0<\varepsilon< 1$, and each i, s_i, $s'_i \in S_i$.

Having the idea of ε-perfect q-cooperative point, now we say that a point

$$\bar{z} \in \Delta(\mathop{X}_{i=1}^{n} S_i)$$

is an perfect q-cooperative point if there exist some sequences

$$\{\varepsilon_k\}_{k=0}^{\infty} \text{ and } \{z^k\}_{k=0}^{\infty} \text{ such that}$$

$$\text{each } \varepsilon_k > 0 \text{ and } \lim_{k \to \infty} \varepsilon_k = 0$$

each z^k is an ε_k-proper q-cooperative equilibrium and

$$\lim_{k \to \infty} z^k(s) = z(s) \qquad \forall s \in \mathop{X}_{i=1}^{n} S_i$$

In a similar way for proper q-equilibrium points. We have the following result

<u>Theorem 2</u>: For an n-person game $\ni$ with communication function q^i, there always exist a proper q-cooperative equilibrium point.

<u>Proof</u>: We show first that there exists an ε- proper q- cooperative equilibrium for any $0<\varepsilon<1$.

Let $m= \max |S_i|$. Given ε, let $\delta = \frac{1}{m\ n} \varepsilon^{m.n}$ and let

$$\Delta^*(\mathop{X}_{i=1}^{n} S_i) = \{ z \in \Delta(\mathop{X}_{i=1}^{n} S_i) ; (z(s) \geq \delta \quad \forall s \in \mathop{X}_{i=1}^{n} S_i \}$$

We observe that $\Delta^*(\mathop{X}_{i=1}^{n} S_i)$ is non-empty compact and convex subset of $\Delta^0(\mathop{X}_{i=1}^{n} S_i)$.

Now define the point to set map

$$F : \Delta^*(\mathop{X}_{i=1}^{n} S_i) \Rightarrow \Delta^*(\mathop{X}_{i=1}^{n} S_i)$$

by

$$F(z) = \{ \bar{z} \in \Delta^*(\mathop{X}_{i=1}^{n} S_i) \mid \text{ if } V_i(s_i \mid q^i(z)) < V_i(s_i' \mid q^i(z))$$

$$\text{implies } \pi_i(\bar{z})(s_i) \leq \varepsilon\, \pi_i(\bar{z})(s_i') \ \forall i \quad \forall s_i \quad \forall s_i' \in S_i \}$$

For any z the points in F (z) satisfy a finite collection of linear inequalities, so F (z) is a convex closed set.

Now we check that it is non-empty. Let $\rho_i(s_i)$ be the number of pure strategies $s'_i \in S_i$ such that

$$V_i(s_i \mid q^i(z)) < V_i(s'_i \mid q^i(z))$$

then letting

$$\tilde{z}(s_i, \ldots s_n) = \varepsilon^{\rho_i(s_i)} \Big/ \sum_{s'_1 \in S_1} \varepsilon^{\rho_1(s'_1)} \ldots \varepsilon^{\rho_n(s_n)} \Big/ \sum_{s'_n \in S_n} \varepsilon^{\rho_n(s'_n)}$$

we observe that $\tilde{z}(s_i \ldots s_n) \geq \frac{\varepsilon^{nm}}{nm}$ and

$$\pi_i(\tilde{z})(s_i) = \varepsilon^{\rho_i(s_i)} \Big/ \sum_{s_i \in S_i} \varepsilon^{\rho i(s'_i)} \leq \frac{\varepsilon\, \varepsilon^{\rho i(s'_i)}}{\sum_{s'_i \in S_i} \varepsilon^{\rho i(s'_i)}} = \varepsilon\, \pi_i(\bar{z})(s'_i)$$

so

$$\tilde{z} \in F(z).$$

Then F satisfies all the conditions of Kakutani Fixed Point theorem, so there exist some $z(\varepsilon) \in \Delta^*(\overset{n}{\underset{i=1}{X}} S_i)$ such that

$$z(\varepsilon) \in F(z(\varepsilon))$$

This is clearly a ε-proper q-cooperative equilibrium point.

In this way for any $0 < \varepsilon < 1$ there exists an ε-proper q-equilibrium point. By the compactness of the set $\check{\Delta}(\overset{n}{\underset{i=1}{X}} S_i)$ there must be a convergent subsequence and a proper q-cooperative equilibrium point

$$z = \lim_{\varepsilon \to 0} z(\varepsilon) . \qquad \text{(q.e.d.)}$$

As a trivial consequence of this theorem we have the following result.

<u>Corollary 3</u>: For an n-person game $\ni$ with communication functions q^i, there always exists a perfect q-cooperative equilibrium point.

3- Proper and Perfect q.p-cooperative equilibrium points for rational games.

In this section we are going to study cooperative equilibria for rational games and its natural rational mixed extension. These games appear for the first time in the paper of von Neumann where he generalized the minimax theorem. Marchi has extended it to equilibrium point for rational games. These games appear to be very important in expanding economy problems, generalizing the work of Los [6], Moeschlin [11] and Morgenstern and Thompson [12]. For example see Cesco and Marchi [3]. By the way there is a theorem regarding generalized rational n-person games, which is very useful in this subject of expanding economies. Let us quote Boyce and Marchi [2].

Now, since the rational games are not so much well-known in the literature we will introduce them accordingly. Let

$$\Gamma = \left\{ S_i \ \dots \ S_n \ ; \quad \frac{M_i}{N_i} \ , \ \dots , \frac{M_n}{N_n} \right\}$$

were the payoff functions

$$M_i, N_i \ ; \ \overset{n}{\underset{i=1}{X}} S_i \quad \longrightarrow \quad R$$

are arbitrary functions. Without loss of generality we assume that $M(s_i \ \dots \ s_n) > 0$

$$\forall \ (s_i \ \dots \ s_n) \in \overset{n}{\underset{i=1}{X}} S_i .$$

Then introducing the corresponding communication functions

$$q^i, \ p^i \ : \ \Delta \left(\overset{n}{\underset{i=1}{X}} S_i \right) \longrightarrow \Delta \left(\underset{j \neq i}{X} S_j \right)$$

we can extend in a natural way its rational mixed extension

$$\bar{\Gamma} = \left\{ \Delta (X \ S_i) \ ; \ \frac{M_1}{N_1} \ , \ \dots , \ \frac{M_n}{N_n} \right\}$$

where the new expectations are naturally computed from the previous in the following way

$$\frac{M_i(\sigma_i, q^i(z))}{M_i(\sigma_i, p^i(z)} \qquad z \in \Delta \left(\overset{n}{\underset{i=1}{X}} S_i \right) \qquad \sigma_i \in \Delta(S_i)$$

Then we say that a point $\overline{z}$ is a p.q.-cooperative equilibrium in $\ni$ if

$$\frac{M_i(\pi_i(\bar{z}), q^i(\bar{z}))}{N_i(\pi_i(\bar{z}), p^i(\bar{z}))} \geq \frac{M_i(\sigma_i, q^i(\bar{z}))}{N_i(\sigma_i, p^i(\bar{z}))} \qquad \forall_i \ \forall \sigma_i \in \Delta(S_i)$$

The intuitive meaning is clear from the context and the definition. In order to prove we need the following lemma which might be seen in Marchi [9].

<u>Lemma 4:</u> Let $\emptyset$; $\Sigma \times \Sigma \to R$ be a real continuous function from a non-empty convex and compact set Σ in an euclidean space such that

$$\emptyset\ (.,z)$$

is concave in $\overline{z}$ $\in \Sigma$ for each z , then there exists a point

$$\emptyset(\bar{z},\bar{z}) \geq \emptyset\ (z,\bar{z}) \qquad \forall z \in \Sigma$$

Now using this strong result we are going to prove the following existence result.

<u>Theorem 5:</u> The rational mixed extension of a rational finite n-person game with communication functions q, p has always a q, p-cooperative equilibrium point.
<u>Proof:</u> Consider the function

$$\varphi : \Delta(\mathop{X}_{i=1}^{n} S_i) \times \Delta(\mathop{X}_{i=1}^{n} S_i) \rightarrow R$$

defined by

$$\varphi_i(z,w) = M_i(\pi_i(z), q^i(w))\ M_i(\pi_i(w), p^i(w)) - M_i(\pi_i(w), q^i(w))\, M_i(\pi_i(z), p^i(w))$$

and

$$\varphi(z,w) = \sum_{i=1}^{n} \varphi_i(z,w)$$

It is clear from the definition of π_i and the payoff functions that φ is linear in the first variable for any fixed $w \in \Delta(\mathop{X}_{i=1}^{n} S_i)$. Then by Lemma 4 there exists a $\bar{z} \in \Delta(\mathop{X}_{i=1}^{n} S_i)$ such that

$$\varphi(\bar{z}, \bar{z}) \geq \varphi(z, \bar{z}) \qquad \forall z \in \Delta(\mathop{X}_{i=1}^{n} S_i)$$

Now consider for given $\sigma_1, \ldots, \sigma_n$ the point

$$\bar{z}(\sigma_1, \ldots, \sigma_n) \in \Delta(\overset{n}{\underset{i=1}{X}} S_i)$$

defined by

$$z(\sigma_1, \ldots, \sigma_n)(s_1, \ldots, s_n) = \sigma_1(s_1) \ldots \sigma_n(s_n).$$

Then we have that $\varphi(\bar{z}, \bar{z}) = 0$ and

$$0 \geq \varphi(z(\sigma_1, \ldots, \sigma_n), \bar{z}) = \sum_{i=1} \varphi_i(z(\sigma_1, \ldots, \sigma_n), \bar{z})$$

On the other have it holds $\pi_i(z(\sigma_1, \ldots, \sigma_n)) = \sigma_i \ \forall_i$.

Now if we choose σ_i arbitrary and $\sigma_j = \pi_j(\hat{z})$ for $j \neq i$.

Then $\varphi_j(z(\sigma_1, \ldots, \sigma_n), \bar{z}) = 0$ for $j \neq i$ and

$$\begin{aligned} 0 \geq \varphi(z(\sigma_1, \ldots, \sigma_n), \bar{z}) &= \varphi_i(z(\sigma_i, \ldots, \sigma_n), \bar{z}) \\ &= M_i(\sigma_i, q^i(\bar{z}))M_i(\pi_i(\bar{z}), p^i(\bar{z})) \\ &\quad - M_i(\pi_i(\bar{z}), q^i(\bar{z}))\, M_i(\sigma_i, p^i(\bar{z})) \end{aligned}$$

If this is done for each i = 1,...,n we have that the point z is an q, p- cooperative equilibrium point of the rational mixed extension (q.e.d.).

Now let us introduce the notation

$$\bar{M}_i(s_i/q^i(z)) = M_i(\bar{\sigma}_i/q^i(z))$$

where

$$\bar{\sigma}_i(s_i') = \begin{cases} 1 & s_i'=s_i \\ 0 & \text{otherwise} \end{cases}$$

Similarly for N_i.
Now we have for rational mixed extension game a similar result as Proposition 1 for common games.

Proposition 6: z is a q,p-cooperative equilibrium point for the rational mixed extension if and only if

$$\bar{M}_i(s_i/q^i(\bar{z}))\ M_i(s_i'/p^i(\bar{z})) - \bar{M}_i(s_i'/q^i(\bar{z}))\ \bar{M}_i(s_i/p^i(\bar{z})) > 0$$

Implies $\pi_i\bar{z}(s_i') = 0 \quad \forall_i\ \forall s_i,\ s_i' \in S_i.$

Proof: The result follows easily from the equality

$$M_i(\sigma_i, q^i(z))\ N_i(\sigma_i',\ p^i(z)) - M_i(\sigma_i', q^i(z))\ N_i(\sigma_i, p^i(z))$$

$$= \sum_{s_i \in S_i} \sum_{s_i' \in S_i} \sigma_i(s_i)\sigma_i'(s_i')\ [\bar{M}_i(s_i/q^i(z))\ \bar{N}_i(s_i'/p^i(z))$$

$$- \bar{M}_i(s_i'/q^i(z))\ \bar{N}_i(s_i, p^i(z))]$$

$$= \sum_{s_i \in S_i} \sum_{s_i \in S_i} \sigma_i(s_i')\ \sigma_i'(s_i')\ \Omega_i(s_i,\ s_i',\ z).\ \text{(q.e.d.)}$$

Now we define an ε- perfect q,p-cooperative equilibrium point z if

$\bar{z} \in \Delta^0(\mathop{X}_{i=1}^{n} S_i)$ and

$\Omega_i(s_i, s_i', \bar{z}) < 0$ implies $\pi_i(z)(s_i') \leq \varepsilon \quad i, \forall s_i, s_i' \in S_i.$

for any given $\varepsilon > 0$.

Similarly an ε -proper q,p-cooperative equilibrium point z if

$\bar{z} \in \Delta^0(\mathop{X}_{i=1}^{n} S_i)$ and

$$\Omega_i(s_i, s_i', \bar{z}) < 0 \ \text{ implies } \pi_i(z)(s_i') \leq \varepsilon \pi_i\ z(s_i) \quad \forall i, s_i, s_i'.$$

It is clear that a ε -proper is ε -perfect.

An in common games we define accordingly the perfect and proper q,p equilibrium points. It is transparent that a proper point is perfect and a perfect is equilibrium.

We now state our final result

Theorem 7: The rational mixed extension of a finite rational game with communication functions p^i, q^i, has always a proper q,p-cooperative equilibrium.

Proof: We just sketch the proof since it is similar to that of theorem 2. For $0 < \varepsilon < 1$ let $\delta = \frac{1}{m\,n} \varepsilon^{mn}$ where m=max (s_i). Consider the multivalued function

$$F : \Delta^*(\overset{n}{\underset{i=1}{X}} S_i) \Rightarrow \Delta^*(\overset{n}{\underset{i=1}{X}} S_i)$$

defined by

$$\bar{F}(z) = \left\{ \bar{z} \in \Delta^*(\overset{n}{\underset{i=1}{X}} S_i) \;\middle|\; \text{if } \Omega_i(s_i, s_i', z) < 0 \; \pi_i(z)(s_i') \leq \varepsilon \pi_i(z)(s_i) \quad \forall i \; \forall s_i \; \forall s_i' \right\}$$

Apply Kakutani's theorem and take a convergent subsequence $\bar{z}(\varepsilon) \longrightarrow \bar{z}$. $\bar{z}$ is a proper q,p equilibrium point (q.e.d.)

In order to finish the discussion let us say that then result for rational games might be not only interesting by themselves but also for possible application to the subject of expanding economies as mentioned above.

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