THE UNIVERSITY OF MINNESOTA

GRADUATE SCHOOL

Report

of

Committee on Examination

901

This is to certify that we the undersigned, as a committee of the Graduate School, have given Helen Whitaker final oral examination for the degree of

Master of Science

We recommend that the degree of

Master of Science

be conferred upon the candidate.

Wm. O. Beal Chairman

R. W. Brink

John T. Tate

Date June 2, 1922

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THE UNIVERSITY OF MINNESOTA

GRADUATE SCHOOL

Report

of

Committee on Thesis

The undersigned, acting as a Committee of the Graduate School, have read the accompanying thesis submitted by Helen Whitaker for the degree of Master of Science. They approve it as a thesis meeting the requirements of the Graduate School of the University of Minnesota, and recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Science.

Wm O. Beal
Chairman

R. W. Brink

John T. Tate

Date May 16, 1922

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A Comparison of Methods of Computing the orbit of an asteroid from three places, based on calculations of the orbit of Eros.

A Thesis

submitted by

Helen Whitaker

in partial fulfillment

of the requirements for the Master's Degree.

May 6, 1922.

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I

Aside from the first graphical approximations made by Newton and the straight line assumptions of Euler and Lambert, the theory of orbit computations has developed from the work of Laplace and Gauss. The method of Laplace determines the elements of the orbit from the six quantities represented by the rectangular coordinates of the body and the velocity components at the time of the second observation, obtaining these quantities from geometrical and dynamic considerations; while that of Gauss determines the rectangular coordinates of the first and third observations, from the projections on the three fundamental planes of the triangles formed by the sun and the three positions of the body, and computes the elements directly from these six quantities. Recent modifications have added to the ease and speed of computation and have improved the results, yet these two methods are fundamental to all theories. For this reason, a comparison of methods of computing orbits from three places should be based on a study of the theories of Laplace and Gauss, and should include as typical of later methods and improvements, a consideration of some one more modern theory.

Accepting this assumption, the method of Laplace and the method of Gauss are the first two to be included in this paper. The computation follows the exposition of these methods, as found in F. R. Moulton's "An Introduction to Celestial Mechanics."* (A) It is a study based upon the geometric conditions of the triangle formed by the earth, sun and object, and the dynamic conditions of motion of the object around the sun in accordance with the law

*Second Revised Edition 1914. Pages 191-257.

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of gravitation, omitting the assumption of Laplace, but substituting his solution for the velocity component of the second observation in place of the coordinates of the first and third observations of Gauss. Dr. Moulton attempts "to clarify the problem mathematically and to show precisely where the results in orbit calculation become poorly determined, to define the extent of the indetermination and to make use of the fact that the final results are only partially determinate to abbreviate the computation."*

The purpose of this paper is to study the claims made by Dr. Moulton in critisism of each method. The conclusions are based on actual computations of the orbit of the asteroid Eros, using for each method the same fundamental data, the same logarithmic tables (except in one case, explained below) and the same methods of computing and recording. The identical elements in the three theories, the number of operations, the time required for computing, the results obtained, and the conditions under which the computing was done, have been taken into consideration in estimating the ease of computation. The results show the feasibility of the short cuts suggested in the third method.

In order to make the computation a purely mathematical test of formulae and to eliminate errors of observation, the dates of the three observations - 1900, October 26.0, October 31.5, November 7.0 - have been chosen arbitrarily, near the time of perihelion passages, and at intervals of 5.5 days and 6.5 days respectively, so that all series used will converge rapidly. The positions were computed from the elements given by E. Millosevick

* Memoir, Page 1.

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in the Astronomische Nachtrichten,* by the formulae given in Moulton's Celestial Mechanics.** The elements given in this Ephemeris were also used as a standard with which to compare the final result of computation of elements by the three methods.

The rectangular coordinates of the sun and the log radius vector of the earth were taken from the Sonnen Ephemeris in the Astronomische Jahrbuch fur 1900. Other astronomical constants used were taken from the Nautical Almanac for the current year of 1922.

In all computations, Bruhn's New Manual of Logarithms (seven place) have been used except where the author suggests four or five place tables in the third method, since it does not affect the accuracy of the result. Here a set of five place tables compiled by David A. Rothrock and a four place, one page, table in H. Black's Laboratory Manual in Physics, were used.

*No. 3609, Vol. 151, pages 137-138. Lull orbita di (433) Eros in base all osservazioni degli anni 1898-1899.

**Chapter V.

(A) The method decided upon to illustrate modern developments is that presented by F. R. Moulton in his "Memoir on the Theory of Determining Orbits", in the Astronomical Journal, 1914, Nos. 661, 662, 663. All references to this paper are given according to its own page number, to which 102 must be added to get the Journal page number.

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II

Any comparison of methods of orbit computation cannot attempt a final solution of the problem, or name any one method as superior in every detail, for every type of problem. Ever since the time of Gauss and of Laplace, such claims have been made by the loyal supporters of each method, influenced in their judgement, in all probability, by their familiarity with the formulae that they advised. History seems to show that when either method is taken as a basis, improvements can be made which add to the ease of computation or manipulation of formula, or which give a closer agreement between the observations and the derived elements.

It hardly seems advisable therefore, to attempt an absolute comparison between the methods of Gauss and of Laplace. It is worth while, however, to study the inherent weaknesses of each method, as well as the advantages, and to see how far these weaknesses are eliminated by later methods, in this case by Moulton's method as presented in his Memoir.

The old methods show the following disadvantages:

(1) Fundamental to any method is the small indeterminations due to the nature of the problem and the inaccuracies in the observations. For instance, if the determinant of the three observations is small (i. e. the observations close together) the solution is poorly determined and these uncertainties are evident in the elements. If the orbit is such that the eccentricity is small, perihelion is poorly determined, and if the inclination is small, node is subject to large errors. Uncertainties of this type cannot be eliminated by more accurate formulae, for they are inherent to the problem. "If desired, small corrections can be

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determined so that all of the equations will be satisfied. But to make the adjustments is justifiable neither theoretically or practically, because the accuracy is only apparent, and even if it were not so, it would not be required by observers. The place for the fine adjustments is in the definitive determination of the orbit based on all valuable observational data." *

(2) Theoretically the method of Laplace is less accurate than that of Gauss, as it introduces in its development the motion of the observer. Logically the motion of the observer at the time of of observation has nothing to do with the orbit of the observed body, and should not enter into the problem.

(3) In correcting for the change in position due to the time required for light to come to the observer from the observed body and in making a second approximation of the elements by either method, practically all the determinants amd the auxillary quantities must be recomputed.

Dr. Moulton in his Memoir attempts to improve these methods as follows;

(1) He makes no claim for eliminating the fundamental inaccuracies, but determines their field.

(2) By combining the geometric conditions of Gauss method with the dynamic conditions of the motion of the object, the defect in theory of the Laplace method is eliminated.

(3) By grouping into determinants the quantities not changed by successive approximations much of the recomputing is eliminated.

(4) By the use of four or five place tables in all but the final solution for X Y Z, and X' Y' Z', the work of computing is lessened. The number of places that must be used depends on the

*Moulton's Memoir. Page 16.

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determinant of the positions of the observed body and should be that of the smallest row or column of the determinant.

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III

A study of the development of the theory of both the "Introduction to Celestial Mechanics" and the "Memoir on the Theory of Determining Orbits" discloses two types of errors, which affect the formulas necessary for computation. For the first group of typographical errors, it is sufficient to list the formula as printed, and the error with the reference to the page where the correct formula may be found. All errors in theory, however, have been proved by a complete derivation of the formulae involved.

In the Memoir, the following typographical errors are noted:

(1) On page 19, formula (43), we find

$$K_6 = \begin{vmatrix} X_1 + X_3 - 2X_2 & \lambda_1 & \lambda_2 - \lambda_1 \\ Y_1 + Y_3 - 2Y_2 & \mu_1 & \mu_2 - \mu_1 \\ Z_1 + Z_3 - 2Z_2 & \nu_1 & \nu_2 - \nu_1 \end{vmatrix}$$

This is miscopied from page 13, where the formula reads correctly:

$$K_6 = -\begin{vmatrix} X_1 + X_3 - 2X_2 & \lambda_1 & \lambda_2 - \lambda_1 \\ Y_1 + Y_3 - 2Y_2 & \mu_1 & \mu_2 - \mu_1 \\ Z_1 + Z_3 - 2Z_2 & \nu_1 & \nu_2 - \nu_1 \end{vmatrix}$$

(2) Page 20, formula (42) we find

$$G_3 = \frac{-2(\epsilon^2/\tau^2) - 1/3\ \upsilon\tau\epsilon Q + \upsilon\epsilon(\tau P - 1/2\ \epsilon Q)}{1 - \epsilon/\tau + 1/2\ \upsilon\tau(\tau P - 1/3\ \epsilon Q)}$$

This is miscopied from page 13, where the formula reads correctly:

$$G_3 = \frac{-2(\epsilon^2/\tau^2) - 1/3\ \upsilon\tau\epsilon Q + \upsilon\epsilon(\tau P - 1/3\,\epsilon Q)}{1 - \epsilon/\tau + 1/2\ \upsilon\tau(\tau P - 1/3\,\epsilon Q)}$$

In the text we find the following typographical errors:

(1) On page 228 following formula (71), we find;

$$x = x_0 + x_0'\tau + 1/2\ x_0''\tau^2 + 1/6\ x_0'''\tau^3 + 1/24\ x_0^{iv}\tau^4 + 1/120\ x_0^{v}\tau^5.$$

$$y = y_0 + y_0'\tau + 1/2\ y_0''\tau^2 + 1/6\ y_0'''\tau^3 + 1/24\ y_0^{iv}\tau^4 + 1/120\ y_0^{v}\tau^5$$

$$z_0 = z_0 + z_0'\tau + 1/2\ z_0''\tau^2 + 1/6\ z_0'''\tau^3 \pm 1/24\ z_0^{iv}\tau^4 + 1/120\ z_0^{v}\tau^5$$

By symetry it is evident that the last equation should read:

$$z = z_0 + z_0'\tau + 1/2\ z_0''\tau^2 + 1/6\ z_0'''\tau^3 + 1/24\ z_0^{iv}\tau^4 + 1/120\ z_0^{v}\tau^5$$

(2) On page 255, formula (89), we find:

$$\begin{vmatrix}\lambda_1 & \lambda_3\\ \mu_1 & \mu_3\end{vmatrix}\rho_3 = \frac{(2,3)}{(1,2)}\begin{vmatrix}\lambda_1 & x_1\\ \mu_1 & y_1\end{vmatrix} - \frac{(1,3)}{(1,2)}\begin{vmatrix}\lambda_1 & x_2\\ \mu_1 & y_2\end{vmatrix} + \begin{vmatrix}\lambda_1 & x_3\\ \mu_1 & y_3\end{vmatrix} + \rho_2\frac{(1,3)}{(2,3)}\begin{vmatrix}\lambda_1 & \lambda_2\\ \mu_1 & \mu_2\end{vmatrix}$$

This is miscopied from page 236, where the formula reads correctly:

$$\begin{vmatrix}\lambda_1 & \lambda_3\\ \mu_1 & \mu_3\end{vmatrix}\rho_3 = \frac{(2,3)}{(1,2)}\begin{vmatrix}\lambda_1 & x_1\\ \mu_1 & y_1\end{vmatrix} - \frac{(1,3)}{(1,2)}\begin{vmatrix}\lambda_1 & x_2\\ \mu_1 & y_2\end{vmatrix} + \begin{vmatrix}\lambda_1 & x_3\\ \mu_1 & y_3\end{vmatrix} + \rho_2\frac{(1,3)}{(1,2)}\begin{vmatrix}\lambda_1 & \lambda_2\\ \mu_1 & \mu_2\end{vmatrix}$$

The errors in theory, all found in the "Introduction to Celestial Mechanics" include the formulae given below with the corrections proved necessary by the derivations which follow.

(1) Page 221, formula (62).

$$\Delta\phi = \frac{\delta\phi\cdot\epsilon}{4\epsilon_1 - \epsilon_2}$$

This should read

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$$\Delta \phi = \frac{\delta \phi \cdot \epsilon}{\epsilon_2 - 4\epsilon_1}$$

(2) Page 234, formula (87),

$$\Delta_2 \rho_2 = K + \frac{\tau^2}{4r_2^3} PK_1 + \frac{\tau \epsilon}{12r_2^3} QK_2$$

and on page 235,

$$\Delta_2 \rho_2 = K + \frac{\tau^2 K_1}{4r_2^3}$$

This should read

$$\Delta_2 \rho_2 = K + \frac{\tau^2}{4r_2^3} PK_1 + \frac{\tau \epsilon}{12r_2^3} QK_2 + \frac{\epsilon}{2\tau} K_2$$

and

$$\Delta_2 \rho_2 = K + \frac{\tau^2 K_1}{4r_2^3} + \frac{\epsilon K_2}{2\tau}$$

(3) Page 255, formula (47),

$$N \sin m = R_2 \sin \psi_2$$

$$N \cos m = R_2 \cos \psi_2 - \frac{K}{\Delta_2}$$

$$M = \frac{4 \Delta_2 R^3 \sin^3 \psi}{K_1} > 0$$

This should read

$$N \sin m = R_2 \sin \psi_2$$

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$$N \cos m = R_2 \cos \psi_2 - \frac{K}{\Delta_2} - \frac{\epsilon K_2}{2\tau \Delta_2}$$

$$M = \frac{4 N \Delta_2 R^3 \sin^3 \psi}{\tau^2 K_1} > 0$$

(4) Page 254, formula (88),

Formula for K_2 must therefore be included with K and K_1. This should read

$$K = -1/2 \begin{vmatrix} \lambda_1 & X_1 + X_3 - 2X_2 & \lambda_3 \\ \mu_1 & Y_1 - Y_3 - 2Y_2 & \mu_3 \\ \nu_1 & Z_1 - Z_3 - 2Z_2 & \nu_3 \end{vmatrix}$$

$$K_1 = - \begin{vmatrix} \lambda_1 & X_1 + X_3 & \lambda_3 \\ \mu_1 & Y_1 - Y_3 & \mu_3 \\ \nu_1 & Z_1 - Z_3 & \nu_3 \end{vmatrix}$$

$$K_2 = + \begin{vmatrix} \lambda_1 & X_3 - X_1 & \lambda_3 \\ \mu_1 & Y_3 - Y_1 & \mu_3 \\ \nu_1 & Z_3 - Z_1 & \nu_3 \end{vmatrix}$$

(1) Page 221, formula (62),

$$4 \log \sin \phi_0 - \log M - \log \sin (\phi_0 + m) = \epsilon$$

Giving ϕ_0 the increment $\Delta \phi_0$ such that $\epsilon = 0$

$$4 \log \sin (\phi_0 + \Delta\phi_0) - \log M - \log \sin (\phi_0 + \Delta\phi_0 + m) = 0$$

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log sin (ϕ_0 + $\Delta\phi_0$) differs from log sin ϕ_0 by a correction obtained by dividing the tabular difference by the unit of the table and multiplying by the correction of the angle, $\Delta\phi_0$.

Let tabular difference at sin $\phi_0 = \epsilon_1$; at sin(ϕ_0 + m) = ϵ_2 then

$$4 \log \sin\phi + \frac{4\epsilon_1}{\delta\phi}\Delta\phi - \log M - \log\sin(\phi + m) - \frac{\epsilon_2}{\delta\phi}\Delta\phi = 0$$

but

$$4 \log\sin\phi - \log M - \log\sin(\phi + m) = \epsilon$$

substituting

$$\epsilon + \frac{4\epsilon_1}{\delta\phi}\Delta\phi - \frac{\epsilon_2}{\delta\phi}\Delta\phi = 0$$

Solving for $\Delta\phi$,

$$\Delta\phi = \frac{\epsilon\delta\phi}{\epsilon_2 - 4\epsilon_1}, \text{ book gives } \frac{\epsilon\delta\phi}{4\epsilon_1 - \epsilon_2} = \Delta\phi$$

(2) Page 234, formula (87),

On making use of equations (86), equation (83) becomes:

$$(83)\ \Delta_2 \ \rho_2 = -\frac{(2,\ 3)}{(1,\ 3)} D^{(1)} + D^{(2)} - \frac{(1,\ 2)}{(1,\ 3)} D^{(3)}$$

where

$$D_1 = \begin{vmatrix} \lambda_1 & X_1 & \lambda_3 \\ \mu_1 & Y_1 & \mu_3 \\ \nu_1 & Z_1 & \nu_3 \end{vmatrix}$$

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$$D_2 = \begin{vmatrix} \lambda_1 & X_2 & \lambda_3 \\ \mu_1 & Y_2 & \mu_3 \\ \nu_1 & Z_2 & \nu_3 \end{vmatrix}$$

$$D_3 = \begin{vmatrix} \lambda_1 & X_3 & \lambda_3 \\ \mu_1 & Y_3 & \mu_3 \\ \nu_1 & Z_3 & \nu_3 \end{vmatrix}$$

(86)

$$\frac{(2,\ 3)}{(1,\ 3)} = 1/2 + \frac{\epsilon}{2\tau} + 1/4\ u\tau^2\ P + \frac{\tau\epsilon}{12}\ uQ$$

$$\frac{(1,\ 2)}{(1,\ 3)} = 1/2 - \frac{\epsilon}{2\tau} + 1/4\ u\tau^2\ P - \frac{\tau\epsilon}{12}\ uQ$$

Substituting,

$$\Delta_2\ \rho_2 = -\ 1/2\ D^{(1)} - \frac{\epsilon}{2\tau}\ D^{(1)} - 1/4\ u\tau^2\ PD^{(1)} - \frac{\tau\epsilon}{12}\ uQD^{(1)} + D^{(2)} - 1/2\ D^{(3)} - 1/4\ u\tau^2\ PD^{(3)} + \frac{\epsilon}{2\tau}\ D^{(3)} + \frac{\tau\epsilon u}{12}\ QD^{(3)}$$

Combining and taking $u = 1/r^3$

$$\Delta_2\ \rho_2 = -\ 1/2\ D^{(1)} + D^{(2)} - 1/2\ D^{(3)} - \frac{\tau^2}{4r^3}\ PD^{(1)} - \frac{\tau^2}{4r^3}\ PD^{(3)} - \frac{\tau\epsilon}{12r^3}\ QD^{(1)} + \frac{\tau\epsilon}{12r^3}\ QD^{(3)} - \frac{\epsilon}{2\tau}\ D^{(1)} - \frac{\epsilon}{2\tau}\ D^{(3)}$$

Taking $K = -\ 1/2\ D^{(1)} + D^{(2)} - 1/2\ D^{(3)}$

$K_1 = -\ D^{(1)} - D^{(2)}$

$K_2 = -\ D^{(1)} + D^{(2)}$

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(87)

$$\Delta_2 \rho_2 = K + \frac{\tau^2}{4r^3} PK_1 + \frac{\tau\epsilon}{12r^3} QK_2 + \frac{\epsilon}{2\tau} K_2$$

"Since the left member of (87) is of the third order, the right member also must be of the third order."(1)* "K_2 is of the second order."(2)* τ is of the first order. ϵ is of the second order."(3)*

$$\frac{\epsilon}{2\tau} K_2$$

$$\frac{\text{second}}{\text{first}} \quad \text{second}$$

Term is of the third order and must be included.

Introduction to Celestial Mechanics. (1) Page 235. (2)* Page 235. (3)* Page 234.

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(3) Page 255, formula (47). This refers back to formulae on pages 235 and 236.

$$\Delta_2 \rho = K + \frac{\tau^2 K_1}{4r^3} + \frac{\epsilon}{2\tau} K_2$$

Substituting

$$\rho = R \frac{\sin(\psi + \phi)}{\sin \phi}$$

$$r = R \frac{\sin \psi}{\sin \phi}$$

and dividing by Δ_2

$$R \frac{\sin(\psi + \phi)}{\sin \phi} = \frac{K}{\Delta_2} + \frac{\psi^2 K_1 \sin^3 \phi}{4 \Delta_2 R^3 \sin^3 \psi} + \frac{\epsilon}{2\tau\Delta_2} K_2$$

expanding and multiplying by $\sin \phi$,

$$R \sin\psi \cos \phi - R \cos\psi \sin\phi = \frac{K \sin\phi}{\Delta_2} + \frac{\tau^2 K_1 \sin^4 \phi}{4 \Delta_2 R^3 \sin^3 \psi} + \frac{\epsilon \sin \phi K_2}{2\tau \Delta_2}$$

$$R \sin\psi \cos\phi + \left[R \cos\psi \sin\phi - \left(\frac{K}{\Delta_2} + \frac{\epsilon K_2}{2\tau\Delta_2}\right) \sin\phi \right]$$

$$= \frac{\tau^2 K_1}{4 \Delta_2 R^3 \sin^3 \psi} \sin^4 \phi$$

$$R \sin\psi \cos\phi + \sin\phi \left[R \cos\phi - \left(\frac{K}{\Delta_2} + \frac{\epsilon K_2}{2\tau \Delta_2}\right) \right]$$

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then

$$\frac{N \sin m \cos\phi + N \cos m \sin\phi}{N} = \frac{1}{\dfrac{N\, 4\, \Delta_2\, R^3 \sin^3\psi}{\tau^2 K_1}} \sin^4\phi$$

Where

$$N \sin m = R \sin\psi$$

and

$$N \cos m = R \cos\psi - \frac{K}{\Delta_2} - \frac{\epsilon K_2}{2\tau\, \Delta_2}$$

Then

$$\sin(m + \phi) = \frac{1}{M} \sin^4\phi$$

where

$$M = \frac{4\, N\, \Delta_2\, R^3 \sin^3\psi}{\tau^2 K_1}$$

The theoretical errors here noted are evidently checked by the results of the computation which show

(1)

ϕ_0	30
$\phi_0 + m$	$6^\circ\ 40'43''$
Sin ϕ_0	9. 69897
M	9. 76489
Sin $(\phi + m)$	9. 06558
M $\sin(\phi_0 + m)$	883 047
Sin $^4 \phi_0$	879 588
ϵ	3459
$4\ \epsilon_1$	7.2
ϵ_2	0.4
$4\ \epsilon_1 - \epsilon_2$	6.8
$\Delta\phi$	$- 1^0\ 24'\ 47''$
ϕ	$28^0\ 35'\ 13''$

(2)

$$R_2 \cos\psi = - .7656\ 2550$$

$$- \frac{K}{\Delta_2} = -2.0502\ 5660$$

$$- \frac{\epsilon K_2}{2\tau \Delta_2} = + 1.3519\ 6293$$

Evidently both terms are of the same order.

(3) Using the corrected formula shows M

Gauss -------- 9.766 5915

Laplace ------ 9.765 9158

Moulton ------ 9.76489

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IV.

The following are the formulae necessary for orbit computations by each of the three methods:

Method of Gauss,

$$(64)\quad \Delta_2 = \begin{vmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \\ \nu_1 & \nu_2 & \nu_3 \end{vmatrix}$$

$$(88)\quad \left\{ \begin{aligned} K &= -\,1/2 \begin{vmatrix} \lambda_1 & X_1 + X_3 - 2X_2 & \lambda_3 \\ \mu_1 & Y_1 + Y_3 - 2Y_2 & \mu_3 \\ \nu_1 & Z_1 + Z_3 - 2Z_2 & \nu_3 \end{vmatrix} \\ K_1 &= -\begin{vmatrix} \lambda_1 & X_1 + X_3 & \lambda_3 \\ \mu_1 & Y_1 + Y_3 & \mu_3 \\ \nu_1 & Z_1 + Z_3 & \nu_3 \end{vmatrix} \\ K_2 &= +\begin{vmatrix} \lambda_1 & X_3 - X_1 & \lambda_3 \\ \mu_1 & Y_3 - Y_1 & \mu_3 \\ \nu_1 & Z_3 - Z_1 & \nu_3 \end{vmatrix} \end{aligned} \right.$$

$$(46)\quad R_2 \cos\psi_2 = X\lambda + Y\mu + Z\nu$$

$$N \sin m = R_2 \sin\psi_2$$

$$(47)\quad N \cos m = R_2 \cos\psi_2 - \frac{K}{\Delta_2} - \frac{\epsilon K_2}{2\tau\Delta_2}$$

$$M = \frac{4\, N \Delta_2\, R^3 \sin^3\psi}{\tau^2 K_1}$$

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(48) $\sin^4\phi = M \sin(\phi + m)$

(46)
$$r_2 = \frac{R_2 \sin\psi}{\sin\phi}$$
$$\rho_2 = \frac{R_2 \sin(\phi + \psi)}{\sin\phi}$$

(89)
$$\begin{vmatrix} \lambda_1 & \lambda_3 \\ \mu_1 & \mu_3 \end{vmatrix} \rho_1 = \begin{vmatrix} X_1 & \lambda_3 \\ Y_1 & \mu_3 \end{vmatrix} - \frac{(1, 3)}{(2, 3)} \begin{vmatrix} X_2 & \lambda_3 \\ Y_2 & \mu_3 \end{vmatrix} + \frac{(1, 2)}{(2, 3)} \begin{vmatrix} X_3 & \lambda_3 \\ Y_3 & \mu_3 \end{vmatrix} - \rho_2 \frac{(1, 3)}{(2, 3)} \begin{vmatrix} \lambda_2 & \lambda_3 \\ \mu_2 & \mu_3 \end{vmatrix}.$$

$$\begin{vmatrix} \lambda_1 & \lambda_3 \\ \mu_1 & \mu_3 \end{vmatrix} \rho_3 = \frac{(2, 3)}{(1, 2)} \begin{vmatrix} \lambda_1 & X_1 \\ \mu_1 & Y_1 \end{vmatrix} - \frac{(1, 3)}{(1, 2)} \begin{vmatrix} \lambda_1 & X_2 \\ \mu_1 & Y_2 \end{vmatrix} + \begin{vmatrix} \lambda_1 & X_3 \\ \mu_1 & Y_3 \end{vmatrix} + \rho_2 \frac{(1, 3)}{(1, 2)} \begin{vmatrix} \lambda_1 & \lambda_2 \\ \mu_1 & \mu_2 \end{vmatrix}$$

(85)
$$k(t_j - t_2) = \tau_j$$
$$2\tau = \tau_3 - \tau_1$$
$$2\epsilon = \tau_3 + \tau_1$$

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$$
(86)\left\{
\begin{aligned}
\frac{(1,\ 3)}{(2,\ 3)} &= \frac{1}{\frac{1}{2} + \frac{\epsilon}{2\tau} + \frac{\tau^2}{4r_2^3}} \\[2ex]
\frac{(1,\ 2)}{(2,\ 3)} &= \frac{1 - \frac{\epsilon}{\tau} + \frac{\tau^2}{2\ r_2^3}}{1 + \frac{\epsilon}{\tau} + \frac{\tau^2}{2\ r_2^3}} \\[2ex]
\frac{(2,\ 3)}{(1,\ 2)} &= \frac{1 + \frac{\epsilon}{\tau} + \frac{\tau^2}{2\ r_2^3}}{1 - \frac{\epsilon}{\tau} + \frac{\tau^2}{2\ r_2^3}} \\[2ex]
\frac{(1,\ 3)}{(1,\ 2)} &= \frac{1}{\frac{1}{2} - \frac{\epsilon}{2\tau} + \frac{\tau^2}{4\ r_2^3}}
\end{aligned}
\right.
$$

$$
(8)\qquad
\begin{aligned}
x &= \rho\lambda - X \\
y &= \rho\mu - Y \\
z &= \rho\nu - Z
\end{aligned}
$$

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Method of Laplace,

(26) $k(t_j - t_2) = \tau_j$

(67) $P = -\tau_1 \tau_3(\tau_3 - \tau_1) = -k\tau_1 \tau_3(t_3 - t_1)$

(64, 65) $D = \frac{2}{P}\begin{vmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \\ \nu_1 & \nu_2 & \nu_3 \end{vmatrix}$

(67) $D_1 = -\frac{\tau_3^2}{P}\begin{vmatrix} \lambda_1 & \lambda_2 & X \\ \mu_1 & \mu_2 & Y \\ \nu_1 & \nu_2 & Z \end{vmatrix} - \frac{\tau_1^2}{P}\begin{vmatrix} \lambda_2 & \lambda_3 & X \\ \mu_2 & \mu_3 & Y \\ \nu_2 & \nu_3 & Z \end{vmatrix}$

$$D_2 = \frac{2\tau_3}{P}\begin{vmatrix} \lambda_1 & \lambda_2 & X \\ \mu_1 & \mu_2 & Y \\ \nu_1 & \nu_2 & Z \end{vmatrix} + \frac{2\tau_1}{P}\begin{vmatrix} \lambda_2 & \lambda_3 & X \\ \mu_2 & \mu_3 & Y \\ \nu_2 & \nu_3 & Z \end{vmatrix}$$

(46) $R\cos\psi = X\lambda + Y\mu + Z\nu$

$N\sin m = R\sin\psi$

(47) $N\cos m = R\cos\psi - \frac{D_1}{DR^3}$

$M = -\frac{N R^3 D}{D_1}\sin^3\psi > 0$

(48) $\sin^4\phi = M\sin(\phi + m)$

(46)
$r = R\,\frac{\sin\psi}{\sin\phi}$

$\rho = R\,\frac{\sin(\psi + \phi)}{\sin\phi}$

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(44) $\rho' = \frac{D_2}{2D}\left[\frac{1}{R^3} - \frac{1}{r^3}\right]$

(8) $x = \rho\lambda - X$ etc.

$y = \rho\mu - Y$

$z = \rho\nu - Z$

$$\lambda' = \frac{-\tau_3\lambda_1}{\tau_1(\tau_1 - \tau_3)} - \frac{(\tau_3 + \tau_1)\lambda_2}{-\tau^3(-\tau_1)} \bullet \frac{\tau_1\lambda_3}{(\tau_3 - \tau_1)\tau_3} \quad \ldots\ldots$$

(32) $$\mu' = \frac{-\tau_3\mu_1}{\tau_1(\tau_1 - \tau_3)} - \frac{(\tau_3 + \tau_1)\mu_2}{-\tau^3(-\tau_1)} - \frac{\tau_1\mu_3}{(\tau_3 - \tau_1)\tau_3} \quad \ldots\ldots$$

$$\nu' = \frac{-\tau_3\nu_1}{\tau_1(\tau_1 - \tau_3)} - \frac{(\tau_3 + \tau_1)\nu_2}{-\tau^3(-\tau_1)} - \frac{\tau_1\nu_3}{(\tau_3 - \tau_1)\tau_3}$$

$$X' = \frac{t_2 - t_3}{(t_1 - t_2)(t_1 - t_3)} X_1 + \frac{2t_2 - t_3 - t_1}{(t_2 - t_1)(t_2 - t_3)} X_2$$

$$+ \frac{t_2 - t_1}{(t_3 - t_1)(t_3 - t_2)} X_3 \quad \ldots\ldots$$

$$Y' = \frac{(t_2 - t_3)}{(t_1 - t_2)(t_1 - t_3)} Y_1 + \frac{2t_2 - t_3 - t_1}{(t_2 - t_1)(t_2 - t_3)} Y_2$$

$$+ \frac{t_2 - t_1}{(t \;\; - t \;\;)(t_3 - t_2)} Y_3 \ldots\ldots$$

$$Z' = \frac{t_2 - t_3}{(t_1 - t_2)(t_1 - t_3)} Z_1 + \frac{2t_2 - t_3 - t_1}{(t_2 - t_1)(t_2 - t_3)} Z_2$$

$$+ \frac{t_2 - t_1}{(t_3 - t_1)(t_3 - t_2)} Z_3$$

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$$
\begin{aligned}
x' &= \rho'\lambda + \rho\lambda' - X' \\
y' &= \rho'\mu + \rho\mu' - Y' \qquad (8)\\
z' &= \rho'\nu + \rho\nu' - Z'
\end{aligned}
$$

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$$(20)\qquad \Delta_2 = \begin{vmatrix} \lambda_1 + \lambda_3 - 2\lambda_2 & \lambda_2 & \lambda_3 - \lambda_2 \\ \mu_1 + \mu_3 - 2\mu_2 & \mu_2 & \mu_3 - \mu_2 \\ \nu_1 + \nu_3 - 2\nu_2 & \nu_2 & \nu_3 - \nu_2 \end{vmatrix}$$

$$(39)\left\{\begin{aligned} K &= \begin{vmatrix} \lambda_1 & X_1 + X_3 - 2X_2 & \lambda_3 - \lambda_1 \\ \mu_1 & Y_1 + Y_3 - 2Y_2 & \mu_3 - \mu_1 \\ \nu_1 & Z_1 + Z_3 - 2Z_2 & \nu_3 - \nu_1 \end{vmatrix} \\ K_1 &= \begin{vmatrix} \lambda_1 & X_1 - X_3 & \lambda_3 - \lambda_1 \\ \mu_1 & Y_1 - Y_3 & \mu_3 - \mu_1 \\ \nu_1 & Z_1 - Z_3 & \nu_3 - \nu_1 \end{vmatrix} \\ K_2 &= \begin{vmatrix} \lambda_1 & X_1 + X_3 & \lambda_3 - \lambda_1 \\ \mu_1 & Y_1 + Y_3 & \mu_3 - \mu_1 \\ \nu_1 & Z_1 + Z_3 & \nu_3 - \nu_1 \end{vmatrix} \end{aligned}\right.$$

$$(43a)\left\{\begin{aligned} K_3 &= -\begin{vmatrix} X_1 + X_3 - 2X_2 & \lambda_2 & \lambda_3 - \lambda_2 \\ Y_1 + Y_3 - 2Y_2 & \mu_2 & \mu_3 - \mu_2 \\ Z_1 + Z_3 - 2Z_2 & \nu_2 & \nu_3 - \nu_2 \end{vmatrix} \\ K_4 &= \begin{vmatrix} X_3 - X_2 & \lambda_2 & \lambda_3 - \lambda_2 \\ Y_3 - Y_2 & \mu_2 & \mu_3 - \mu_2 \\ Z_3 - Z_2 & \nu_2 & \nu_3 - \nu_2 \end{vmatrix} \\ K_5 &= \begin{vmatrix} X_2 & \lambda_2 & \lambda_3 - \lambda_2 \\ Y_2 & \mu_2 & \mu_3 - \mu_2 \\ Z_2 & \nu_2 & \nu_3 - \nu_2 \end{vmatrix} \end{aligned}\right.$$

$$k(t_j - t_2) = \tau_j$$

(34)
$$2\epsilon = \tau_1 + \tau_3$$
$$2\tau = \tau_3 - \tau_1$$

(43b)
$$K_6 = -\begin{vmatrix} X_1 + X_3 - 2X_2 & \lambda_1 & \lambda_2 - \lambda_1 \\ Y_1 + Y_3 - 2Y_2 & \mu_1 & \mu_2 - \mu_1 \\ Z_1 + Z_3 - 2Z_2 & \nu_1 & \nu_2 - \nu_1 \end{vmatrix}$$

$$K_7 = \begin{vmatrix} X_2 - X_1 & \lambda_1 & \lambda_2 - \lambda_1 \\ Y_2 - Y_1 & \mu_1 & \mu_2 - \mu_1 \\ Z_2 - Z_1 & \nu_1 & \nu_2 - \nu_1 \end{vmatrix}$$

$$K_8 = \begin{vmatrix} X_2 & \lambda_1 & \lambda_2 - \lambda_1 \\ Y_2 & \mu_1 & \mu_2 - \mu_1 \\ Z_2 & \nu_1 & \nu_2 - \nu_1 \end{vmatrix}$$

(45) $$R_2 \cos\psi = -\lambda_2 X_2 - \mu_2 Y_2 - \nu_2 Z_2$$

(46)
$$N \sin m = R_2 \sin\psi$$
$$N \cos m = R_2 \cos\psi - \frac{1}{2\Delta_2}\left(K + \frac{\epsilon}{\tau} K_1\right)$$

(44) $$\Phi = \frac{\tau^2}{4} PK_2 + \frac{\tau\epsilon}{12} QK_1$$

(46) $$M = \frac{N \Delta_2 R_2^3 \sin^3\psi}{\Phi} > 0$$

(47) $$\sin^4\phi = M \sin(\phi + m)$$

(45)
$$r_2 = R_2 \frac{\sin\psi}{\sin\phi}$$
$$\rho_2 = R_2 \frac{\sin(\psi + \phi)}{\sin\phi}$$

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$$F_1 = \frac{2\,\frac{\epsilon^2}{\tau^2} + u\tau\,(1 - \frac{\epsilon}{\tau})\,(\tau P + \frac{1}{3}\,\epsilon\,Q)}{1 + \frac{\epsilon}{\tau} + \frac{1}{2}\,u\tau\,(\tau P + \frac{1}{3}\,\epsilon\,Q)}$$

$$F_3 = \frac{2\,\frac{\epsilon^2}{\tau^2} - u\tau\,(1 + \frac{\epsilon}{\tau})\,(\tau P - \frac{1}{3}\,\epsilon\,Q)}{1 - \frac{\epsilon}{\tau} + \frac{1}{2}\,u\tau\,(\tau P - \frac{1}{3}\,\epsilon\,Q)}$$

(42)

$$G_1 = \frac{-\,2\,\frac{\epsilon^2}{\tau^2} + \frac{1}{3}\,u\tau\epsilon\,Q - u\epsilon\,(\tau P + \frac{1}{3}\,\epsilon\,Q)}{1 + \frac{\epsilon}{\tau} + \frac{1}{2}\,u\tau\,(\tau P + \frac{1}{3}\,\epsilon\,Q)}$$

$$G_3 = \frac{-\,2\,\frac{\epsilon^2}{\tau^2} - \frac{1}{3}\,u\tau\epsilon\,Q + u\epsilon\,(\tau P - \frac{1}{3}\,\epsilon\,Q)}{1 - \frac{\epsilon}{\tau} + \frac{1}{2}\,u\tau\,(\tau P + \frac{1}{3}\,\epsilon\,Q)}$$

$$u = \frac{1}{r^3}$$

$$\Delta_2\rho_1 = K_3 + 2\,\frac{\epsilon}{\tau}\,K_4 + F_1\,K_5 + G_1\,(K_4 + K_5)$$

(43)

$$\Delta_2\rho_3 = K_6 + 2\,\frac{\epsilon}{\tau}\,K_7 + F_3\,K_8 + G_3\,(K_8 - K_7)$$

$$x_2 = \lambda_2\rho_2 + X_2$$

(48) $$y_2 = \mu_2\rho_2 + Y_2$$

$$z_2 = \nu_2\rho_2 + Z_2$$

$$(g_3 - g_1)\,x' = X_3 - X_1 - (f_3 - f_1)x_2 + \lambda_1(\rho_3 - \rho_1) + (\lambda_3 - \lambda_1)\,\rho_3$$

(49) $$(g_3 - g_1)\,y' = Y_3 - Y_1 - (f_3 - f_1)y_2 + \mu_1(\rho_3 - \rho_1) + (\mu_3 - \mu_1)\,\rho_3$$

$$(g_3 - g_1)\,z' = Z_3 - Z_1 - (f_3 - f_1)z_2 + \nu_1(\rho_3 - \rho_1) + (\nu_3 - \nu_1)\,\rho_3$$

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$$f_3 - f_1 = 2\tau\Big[- u\epsilon - 1/2\, up\tau^2 - 3/2\, up\epsilon^2 - 1/6\, u (u - 15p^2 + 3q)\tau^2\epsilon - 1/8\, up(u - 7p^2 + 3q)\tau^4 \ldots\Big]$$

$$g_3 - g_1 = 2\tau\Big[1 - 1/6\, u\tau^2 - 1/2\, u\epsilon^2 + up\tau^2\epsilon + 1/120\, u(u - 45p^2 + 9q)\ {}^4 \ldots\ldots\Big]$$

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The formulae listed above are those necessary for determining the coordinates of the body from the earth or the coordinates and the velocity components, as the method may demand; and do not include time aberration, successive approximations, or a solution for the elements.

The correction for the time aberration for the interval τ_j;

$$\Delta\tau_j = - \frac{\rho + \rho' \ \tau_j}{V} \quad \text{or} \quad \frac{\rho_j}{V}$$

where V is the velocity of light, or for the time t_j

$$\Delta t_j = - 498.65 \ \rho_j$$

All formulae involving t or τ in any term must be recomputed with these new values for t or τ.

The elements can be determined from the formulae of the type found in Moulton's "Introduction to Celestial Mechanics", chapter V, which will not be listed here.

The successive approximations suggested by Dr. Moulton depend on the determination of the auxiliary quantities P, Q, p, and q from equations:

$$\text{(14)} \quad r_2^2 p = x_2 x_2' + y_2 y_2' + z_2 z_2'$$

$$r_2^2 q = x_2'^2 + y_2'^2 + z_2'^2 - \frac{1}{r_2}$$

$$P = 1 - \frac{\epsilon^2}{\tau^2} - 2p\epsilon + 1/12\ (7\ u - 15p^2 + 3q)\tau^2 \ldots\ldots\ldots$$

$$\text{(36)} \quad Q = 1 - \frac{\epsilon^2}{\tau^2} + 3/2\ p\ \frac{\tau^2}{\epsilon} - 3p\epsilon + 1/60\ (37u - 765p^2 + 153q)\tau^2$$

$$+ 1/4\ p\ (3u + 14p^2 - 6q)\ \frac{\tau^4}{\epsilon} \quad \ldots\ldots\ldots$$

and their substitutions, where they appear in the formulae above.

The computation of the above formulae for the orbit of Eros show the results:

Preparation of Data.

Elements of the orbit.

M	304	23	59.7
π	121	9	22.0
ω	177	38	41.6
Ω	303	30	40.4
i	10	49	38.9
ϕ	12	52	48.2
$\log a$	0.163 8027		
μ	2015.1274		

Computed positions 1900.

Oct. 26.0	α_1	36	31	38.6
	δ_1	52	41	47.3
Oct. 31.5	α_2	34	06	50.7
	δ_2	53	41	2.1
Nov. 7.0	α_3	30	52	52.3
	δ_3	54	16	13.7

Direction cosines of object.

λ_1	9.687 5244
μ_1	9.557 1673
ν_1	9.900 6054
λ_2	9.690 4870
μ_2	9.521 3383
ν_2	9.906 2068
λ_3	9.700 0475
μ_3	9.476 5541
ν_3	9.909 4398

Geo-centric coordinates of the sun.

X_1	- .837 3335
Y_1	- .490 8087
Z_1	- .212 9228
X_2	- .781 0493
Y_2	- .561 3207
Z_2	- .243 5094
X_3	- .705 3982
Y_3	- .637 9420
Z_3	- .276 7516

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log R = 9.996 5893

log k = 8.235 5814

ϵ = 7.934 5540

τ = 9.013 7323

τ_1 = 8.975 9441n

τ_3 = 9.048 4948

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Method of Gauss.

Determinants:

Δ_2	6.439 3169
2K	7.052 1551n
K_1	8.850 0548
K_2	7.950 4900n

Correction for Time Aberration:

ϵ	7.934 5742
τ	9.013 7365

Distances:

r_2	0.125 0017	r_2	0.124 9633
ρ_2	9.611 8658	ρ_2	9.611 7236
ρ_1	9.635 4760	ρ_1	9.635 3446
ρ_3	9.586 4773	ρ_3	9.586 3240

Coordinates of the body:

x_1	0.020 2154	x_2	9.991 9317	x_3	9.952 6476
y_1	9.810 6282	y_2	9.843 3396	y_3	9.877 0943
z_1	9.745 4192	z_2	9.758 2049	z_3	9.770 7857

x'	9.861 0297n
y'	9.716 4793
z'	9.215 4371

Elements of the orbit:

Ω	$304^\circ\ 10'\ 17''$
$\Delta\Omega$	$39'\ 37''$
i	$10^\circ\ 48'\ 58''$
Δi	$31''$

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Method of Laplace.

Determinants:		Correction for time aberration	
D	9.401 1928	τ_1	8.976 1466
D_1	9.240 0514	τ_3	9.048 3540
D_2	9.253 5738n		
Distances:			
r_2	0.129 1945	r_2	0.129 1914
ρ_2	9.627 1598	ρ_2	9.627 1127
ρ'	9.339 6515n	ρ'	9.342 8108n
Velocity components:			
λ'	8.803 9343	X'	- 0.632 3374
μ'	9.472 7920n	Y'	+ 0.717 7688
ν'	8.921 6189	Z'	+ 0.311 3748
Coordinates and velocity components of the body:			
x	9.995 1205	x'	9.853 2879n
y	9.846 3813	y'	9.714 9707
z	9.767 1213	z'	9.228 0953
Elements of the orbit:			
☊	305° 41' 25"	Δ☊	2° 21'
i	11° 4' 10"	Δ i	14'.5

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Method of Moulton.

Determinants:

	7 place	5 place
Δ_2	+ .000 275	+ .000 275
K	+ .001 127	+ .001 127
K_1	- .008 922	- .008 920
K_2	- .070 803	- .070 80
K_3	+ .000 563	+ .000 563
K_4	- .002 338	- .002 340
K_5	+ .020 555	+ .020 56
K_6	+ .000 567	+ .000 567
K_7	- .002 118	- .002 218
K_8	+ .015 641	+ .015 635

Distances:

r_2 0.12 417 ρ_2 9.60 877

Correction for time aberration:

ϵ	7.93 457	K	.001 128
τ	9.01 374	K_2	- .070 805
		K_8	.000 5674

X_1	.837 3453	X_2	.781 0621	X_3	.705 4119
Y_1	.561 3062	Y_2	.490 7923	Y_3	.637 9299
Z_1	.243 5031	Z_2	.212 9157	Z_3	.276 7462

r_2	0.124492
ρ_2	9.61 155
ρ_1	9.63 329
ρ_3	9.58 843

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Coordinates and velocity components of the object:

x	9.99 190	x'	9.85 187n
y	9.84 330	y'	9.72 075
z	9.75 810	z'	9.25 123

Elements of the orbit:

i	$10^{\circ}\ 45'\ 37''$	Δi	$4'$
Ω	$306^{\circ}\ 37'\ 45''$	$\Delta\Omega$	$3^{\circ}\ 7'$

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Second Approximations.

Constants:

p	9.11 041n
q	8.55 408
P	9.99 903
Q	9.87 953

Distances:

r_2	0.12 547
ρ_2	9.61 359
ρ_1	9.63 497
ρ_3	9.59 005

Coordinates and velocity components:

x	9.99 232	x'	9.85 060n
y	9.84 370	y'	9.72 151
z	9.75 927	z'	9.25 293

Elements of the orbit:

Ω	$306^\circ\ 39'\ 34.6''$	$\Delta\Omega$	$3^\circ\ 9'$
i	$10^\circ\ 48'\ 06''$	Δi	$1'\ 33''$

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Third Approximation.

Constants:

p	9.10 367n
q	8.54 839
P	9.99 904
Q	9.88 165

Distances:

r_2	0.12 547
ρ_2	9.61 358
ρ_1	9.63 516
ρ_3	9.58 993

Coordinates:and velocity components:

x	9.99 232	x'	9.85 112n
y	9.84 370	y'	9.72 110
z	9.75 927	z'	9.25 007

Elements of the orbit:

i	$10^{\circ}\ 48'\ 05''$	Δi	$1'\ 34''$
Ω	$306^{\circ}\ 24'\ 51''$	$\Delta\Omega$	$2^{\circ}\ 54'$

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V

The relative ease of computation by the three methods can be determined by; (a) a study of the relative difficulty of mathematical operations, (b) a count of the number of operations of each type involved in the above formulae, and (c) a totaling of the count (b) by means of (a).

The study under (a), of the relative difficulty of mathematical operations, was carried out by a speed test on operations used; getting logarithms and anti-logarithms with four, five and seven place tables, additions and subtractions of four, five and seven digit numbers; and involution and evolution by easy multiplications or divisions as they occur in the use of logarithms. The results of this speed test were checked in part by student computers and found to agree in ratio with the results given below.

The results of nine problems of each type show,

Time per problem per column:

Addition --------------------	.9 seconds.
Subtraction ----------------	1.0 "
Powers ---------------------	.8 "
Root -----------------------	.9 "

This shows that the arithmetical problems may be considered in a group, and in the ratio of the number of places; 4 : 5:: 7 seconds.

Time per problem:

	Logarithms.		Anti-logarithms.	
	Not int.	Int.	Not int.	Int.
4 place	3.3	9.3	6.4	15.0
5 place	8.2	17.4	12.3	20.3
7 place	11.2	29.0	16.4	32.0

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This shows the ratio of four place : five : seven place =

9 : 17 : 29 3 : 8 : 11	for logarithms
15 : 20 : 32 6 : 12 : 16	for anti-logarithms

= approx. 1 : 2 : 3.

Therefore in terms of 7 place = 1; 4 place weight = 1/3
5 place = 2/3.

It also shows in every case a difference in time of logarithm and anti-logarithm in ratio of

$$\frac{17}{20} = \frac{26}{28} = \frac{29}{32} = \text{approx. } \frac{10}{11}$$

Therefore in terms of logarithm = 1, anti-logarithm weight = 1.1

Comparing the time of arithmetical problems with the time of logarithms, the ratio is

$$\frac{4}{17} = \frac{5}{26} = \frac{7}{29} = \text{approx. } 1/4$$

Therefore in terms of lagarithm = 1, anti-logarithm weight = .25

In each case, weights have been expressed in simple arithmetical ratios to facilitate their use. This is allowable as it is within the accuracy of the method used to determine the weights.

Weights to be applied

Logarithmic tables,	7 place	-------- 1
	5 place	-------- 2/3
	4 place	-------- 1/3
Operations,	7 place	---------1
	5 place	---------1/4
	4 place	---------11/10

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The count to be made for (b) therefore need involve only these separate headings: arithmetical operations, logarithms, and anti-logarithms, noting in each case the number of places under consideration. The rule has been adopted of considering the number of places in the first group the same as the logarithmic table being used, as in general this is true and the weight of this group is small enough to make minor irregularities negligible. Since in only three or four cases interpolation was unnecessary, no account has been taken of these cases.

In the three methods, the following identical formulae and parts of formulae are to be found;

$$\lambda = \cos\delta\cos\alpha$$

$$\mu = \cos\delta\sin\alpha$$

$$\nu = \sin\delta$$

For three positions

$$R_2 \cos\psi = \pm X\lambda \pm Y\mu \pm Z\nu$$

$$N \sin m = R \sin\psi$$

$$N \cos m = R \cos\psi -$$

$$k(t_j - t_2) = \tau_j$$

$$M = N \Delta_2 R^3 \sin^3\psi$$

$$\sin^4\phi = M \sin(\phi + m)$$

$$r = R\frac{\sin\psi}{\sin\phi}$$

$$\rho = R \frac{\sin(\phi + \psi)}{\sin\phi}$$

$$u = \frac{1}{r^3}$$

The determinant $\begin{vmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \\ \nu_1 & \nu_2 & \nu_3 \end{vmatrix}$ also appears in all three methods, but as it can be computed more advantageously when grouped with other determinants than when computed alone, it is not included here. The formula for determining x, y, and z has been omitted as one method requires its application three times and the other two methods but once.

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Table 1.

		Identical parts.	Gauss.	Laplace	Moulton.
	4				77
Arith.	5	32 = 26	104	161	232 = 60
	7	6			95
	4				43
Logs.	5	24 = 9	32	29	87 = 17
	7	15			37
	4				48
Anti-logs.	5	7 = 7	40	52	98 = 16
	7				34

Table 2.

		Gauss.	Laplace.	Moulton.
	4			77
Arith.	5	136	193	86
	7			101
	4			43
Logs.	5	58	53	26
	7			42
	4			48
Anti-log.	5	47	59	23
	7			34

Table 3.

	Weights.	Gauss.	Laplace.	Moulton.		
				4	5	7
Arith.	1/4	34	48	19	22	25
Log.	1	58	53	42	26	42
Anti-log.	1 + 1/10	52	65	53	25	37
Totals weighted for operations.		184	166	114	73	104

Table 4.

	Weights.	Gauss.	Laplace.	Moulton.	
				As computed	As possible to compute.
4 Place.	1/3			38	62
5 Place.	1/2			48	
7 Place.	1	184	166	104	104
Total weighted for tables.		184	166	190	166

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Table 1 shows the number of arithmetical operations, logarithms, and antilogarithms, in the identical formulae, and in the methods of Gauss, Laplace and Moulton, respectively. Where different tables had been used, the fact is noted.

On table 2, the count on identical formulae is added to the other counts, to show the total operations necessary for each method.

On table 3, the results of section (a), have been applied as weights to the different operations and in table 4, to the different logarithmic tables and a grand total for each method arrived at, thus. For the method of Moulton, the first total is as the computation was carried out in this paper, the second is when 4-place tables substitute for 5-place where they were used.

The totals given in table 3 suggest the Laplace method shorter than Gauss, and the Moulton method as carried out in this paper is the same as Gauss, but capable of being made as brief as the Laplace method in the first approximation. This conclusion is of course merely suggestive, being inaccurate to the extent that the weights applied to table 2 have been determined for only a few individuals. While they are applicable for this particular problem, they do not admit of generalization without further tests.

A similar count for the work in correcting for time aberration shows,

Table 2.

	Gauss.	Laplace.	Moulton.
Arith.	68	142	153
Logs.	20	26	59
Anti-logs.	23	32	66

Table 3.

	Gauss.	Laplace	Moulton	7 place	5 place
Arith.	17	37	38 =	19	+ 19
Logs.	20	26	59 =	27	+ 32
Anti-logs.	25	35	73 =	37	+ 36
	62	98		83	87

Moulton weighted for log. tables 83
58
141

weighted for a 4-place table 83
58
112

This shows that the process by Moulton's method is the longest, even using 4-place tables.

However, this method is also the most accurate, as neither of the other two methods take into consideration the changes in X Y Z due to aberration. With this factor included the work of computing by the first two methods would be considerably larger.

A brief summary of the work involved in successive approximations on the third method gives:

Arith.	149	weighted	37
Log.	22	"	22
Anti-log.	56	"	62
			121 total.
			81 weighted for 5-place.

The result is weighted to a five place table as practically all can be so computed, and the large part can be done by four place tables.

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A comparison with the time of the first approximation shows that more than half of the work of recomputing is saved.

The conclusions arrived at above, may be checked roughly by comparing the actual time spent in the computation of each method. In fact, if the computations had been carried out, under the same general conditions, this would be theoretically, valuable data. However, the order in which the methods were taken up is an important factor in invalidating these results, for the later computations are better arranged, contain fewer arithmetical errors, required fewer recomputations to check errors, and include formulae or quantities that had already been computed for the first method; as for instance, the time aberration constant, interpolation for log radius vector of the earth, etc.

Methods are listed in order of computation for the first approximation only,

	Dates of computation.	Interval between successive computations.	Time per computation.	total.
Moulton	Nov. 10 to 16	1 day	4 1/3 hrs.	23
Laplace	Jan. 4 to 16	2 days.	3 1/2 hrs.	18
Gauss	Jan. 25 to Mar. 9.	9 days.	3 hrs.	20

When the total times are as close as this, the interpretation must depend upon balancing regularity in computing and superiority of the Moulton method against the added skill of later computations, the fewer errors, and the quantities carried over from the Moulton method, and seems to show a slight preference for the method of Moulton.

For other approximations on Moulton's method, we compare:

First approximation plus elements 23 hrs.

Second approximation plus elements 12 hrs.

Third approximation plus elements 14 hrs.

This agrees fairly well, considering conditions, with the theoritical ratio 1 : 2 obtained above.

The comparison of four place and seven place tables can be checked by the fact that the computation of the ten determinants at the beginning of Moulton's method required 5 1/2 hours by seven place tables and 2 hours by four place tables. The theoretical value determined was 1 : 3.

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From the computations we learn therefore, that the results by the three methods vary but little and are corrected but slowly by successive approximations; that the elements of the definitive orbit shows a greater variation, representing evidently a more accurate determination. These results prove the truth of the statement that there are fundamental inaccuracies which cannot be eliminated, and that a nice selection of data to satisfy the observations belongs to a study of all observations rather than a method of three observations.

The claim that the problem has been cleared mathematically, has not been included in the scope of this paper, except to note that the fundamental erroneous assumption of Laplace has not been introduced.

The third claim for the Moulton method is that by grouping the quantities which are not changed in successive approximations, the labor can be reduced. For the correction for time aberration, the Moulton method has been shown longer, as three of the determinants must be recomputed, but more accurate, because the coordinates of the sun are also corrected. For successive approximations the labor has been shown to be less than half that of the first approximation. No determinants needed to be recomputed.

The use of four and five place tables in place of seven has been shown to save from one third to two thirds the time, both by a speed check on the operations and by the time taken in a double computation of one part of the method. By applying these results to the operations involved in the solution the first approximation

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of the method of Moulton was found to be as short, if not shorter, than the methods of Laplace and of Gauss on which it is based. The total time spent in computing by the three methods, allowing for the conditions under which the computing was done, agrees with the result.

By a comparison of results it is evident that the use of four or five place tables does not affect the accuracy of the solution.

Therefore it seems that the methods of Gauss and of Laplace show differences in accuracy and in shortness, difficult to measure quantitatively, and include certain discrepancies whcih have been greatly improved by the method of Moulton, which shortens the labor, eliminates inconsistent assumptions, and points out the fundamental indeterminations of the problem.

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