W.Mc.

REPORT

of

COMMITTEE ON EXAMINATION

This is to certify that we the undersigned, as a Committee of the Graduate School, have given Karl John Holzinger final oral examination for the degree of Master of Arts. We recommend that the degree of Master of Arts be conferred upon the candidate.

Minneapolis, Minnesota

June 2 1917

W. H. Bussey
Chairman

Royal R. Shumway

John T. Tate

REPORT

of

Committee on Thesis

The undersigned, acting as a Committee of the Graduate School, have read the accompanying thesis submitted by Karl John Holzinger for the degree of Master of Arts. They approve it as a thesis meeting the requirements of the Graduate School of the University of Minnesota, and recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Arts.

W. H. Bussey (acting for A. L. Underhill)
Chairman

Burt L. Newkirk

John T. Tate

191

A DISCUSSION OF THE INTRINSIC EQUATIONS FOR

PLANE CURVES, CURVES IN SPACE, AND SURFACES

A Thesis submitted to the

Faculty of the Graduate School of the

UNIVERSITY OF MINNESOTA

by

Karl J. Holzinger

In partial fulfillment of the requirements

for the degree of

Master of Arts

June

1917

Part One.

Plane Curves.

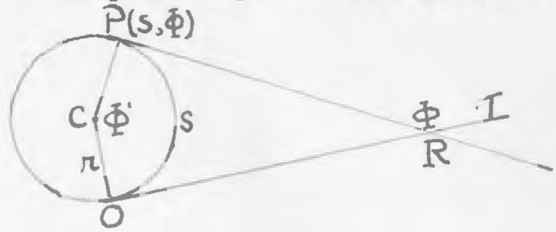
The Derivation of Intrinsic Equations for Plane Curves.

A plane curve is ordinarily expressed by any one of three systems of coordinates. The ones usually adopted are Cartesian, polar, or intrinsic. In the last system the coordinates are: s and Φ . Here s represents the distance of any point, P, from a chosen fixed point on the curve, the distance being measured along the curve. Φ represents the angle made by the tangent at P, with some arbitrarily fixed tangent to the curve. It is evident that s and Φ , thus chosen, are independent of all fixed points or lines of reference other than the fixed point and fixed tangent of the curve itself,- hence the name intrinsic equation is employed for the form,

(1) $$S = f(\Phi).$$

It is possible to build up the intrinsic equation of a curve from physical or geometrical conditions without reference to any other system of coordinates.

A simple example of this is the circle;

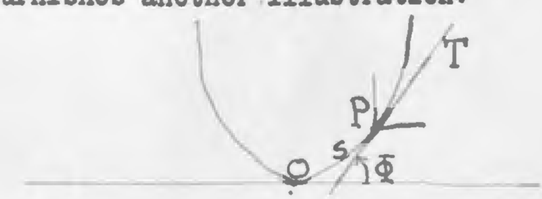


Take O for the fixed point, and the tangent at O as the tangent of reference. Let $P(s,\Phi)$ be any point on the circle, and the arc OP be designated by s. The angle Φ , or TRP, is also equal to the central angle Φ' , or OCP. Since the arc is equal to the radius times the angle measured in radians, we may write immediately,

$$s = r\Phi,$$

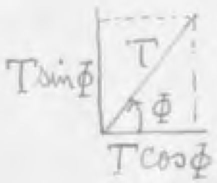
which is the intrinsic equation of the circle.

The derivation of the catenary from physical condititions furnishes another illustration:



This is the familiar curve made by a heavy flexible string supported at both ends. The only force exerted by one portion of the string on an adjacent portion is the pull along the string i.e. the tension which has different values at different points along the string, and is therefore a function of the coordinates of a point on the curve. The tension at any point, P, has to support the weight of the portion of the string below that point, and react against a certain amount of side pull, due to the fact that string would hang vertically if its ends were not forcibly held apart. As in the previous example take O as the fixed point. This will then be the point from which s is measured and at which the fixed tangent is drawn. Then $s = OP$ and $W = ms$, since the weight of the string is proportional to its length. The weight is balanced by the vertical component of the tension, T, giving the relation

$$T \sin \Phi = ms .$$



As there is no external horizontal force acting, the horizontal component of the tension is a constant i.e.

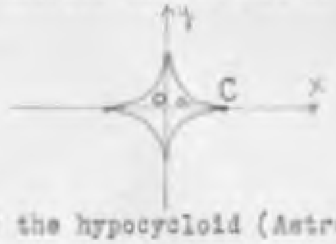
$$T \cos \Phi = C.$$

If the first of these equations is divided by the second, we get the simple relation

$$s = b \tan \Phi$$

in which b has the value $\frac{c}{m}$. This is the desired intrinsic equation.

When neither physical nor geometrical conditions furnish a ready means of obtaining the equation of a curve in intrinsic coordinates, it may be obtained from the equations already given in either Cartesian or polar coordinates. Two examples will illustrate this method of derivation:



The equation of the hypocycloid (Astroid) in Cartesian coordinates is

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}.$$

If, in the above figure, we choose C as the fixed point from which to measure s, and also as the point at which the fixed tangent is drawn (here the x axis), we have from the formula

$$s = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx,$$

an expression for the length of arc,

$$s = \frac{3}{2} a^{\frac{1}{3}} \left(a^{\frac{2}{3}} - x^{\frac{2}{3}}\right)$$

from c to any arbitrary point (x,y) on the curve. Also, the slope of the tangent is given in the form,

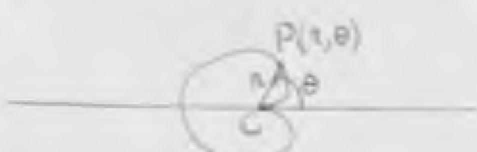
$$\tan \Phi = -\left(\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}\right)$$

If we now eliminate x and y from these equations, we get the expression,

$$2s = 3a \sin^2 \Phi,$$

which is the required intrinsic equation.

The intrinsic equation of the Cardioid in polar coordinates may be obtained in a similar manner.



The equation of the Cardioid in polar coordinates is,

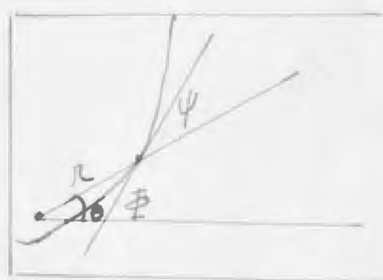
$$r = a(1 - \cos\theta).$$

For the point from which s is measured, choose the polar origin C, and the tangent at that point i.e. the polar axis, as the tangent of reference. If $P(r,\theta)$ is any point on the curve, the length of arc s is given by the formula,

$$s = \int_0^\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = 4a\left(1 - \cos\frac{\theta}{2}\right).$$

Since,

$$\Phi = \psi + \theta$$



and,

$$\tan\psi = r\frac{d\theta}{dr}$$

we have,

$$\Phi = \theta + \tan^{-1} r\frac{d\theta}{dr} = \theta + \tan^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{3}{2}\theta.$$

By means of the relation above the desired equation between s and Φ is obtained in the form,

$$s = 4a\left(1 - \cos\frac{\Phi}{3}\right).$$

In the last two of these examples it was found necessary to perform an integration to obtain the expression for the length of arc. The general method employed in both was to obtain expressions for s and Φ in terms of the original variables, and then eliminate the latter. This method is therefore valid whenever the formula for s can be conveniently integrated. It could not be used in the case of the ellipse.

The intrinsic equations thus obtained have a distinct value in determining curvature. The curvature at a point, P, is defined by the expression,

$$K = \frac{d\Phi}{ds}$$

and the radius of curvature as the reciprocal,

$$R = \frac{ds}{d\Phi}$$

The radius of curvature of a curve given in intrinsic coordinates may thus be ordinarily obtained by a single differentiation with respect to the tangent angle.

For the Astroid above

$$R = \frac{3}{2} a \sin(2\Phi)$$

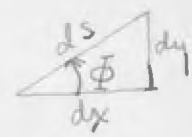
and for the cardioid,

$$R = \frac{4}{3} a \sin\left(\frac{\Phi}{3}\right).$$

Compared with the rather cumbersome formulae used in obtaining the curvature for equations in Cartesian or polar coordinates, the above method has the distinct advantage of simplicity, especially in case the intrinsic equation of the curve is known in the form s as an explicit function of Φ .

The problem just dealt with has been to find the intrinsic equation of a curve under certain conditions, or from equations in other coordinates. The inverse problem is to pass from the intrinsic equation to one in Cartesian (or polar) coordinates. By making use of the following relations:

$$\tan \Phi = \frac{dy}{dx}, \quad \cos \Phi = \frac{dx}{ds},$$



$$\sin \Phi = \frac{dy}{ds}, \quad \sqrt{dx^2 + dy^2} = ds,$$

it is sometimes possible to reverse the method used in the above problems and obtain, for example, the desired equations in Cartesian coordinates from the given intrinsic equation. The method, however, is not general, and greater difficulties in integration arise than in the direct problem.

The catenary will illustrate the method. The intrinsic equation is,

$$s = a \tan \Phi .$$

From the relations just above we may write

$$s^2 = a^2 \left(\frac{ds^2}{dx^2} - 1 \right),$$

or,

$$\frac{a\,ds}{(a^2 + s^2)^{\frac{1}{2}}} = dx .$$

After integrating, squaring, and making use of the relation,

$$s = a \frac{dy}{dx},$$

we finally have the expression,

$$dy = \frac{1}{2} \left(e^{\frac{x}{a}} - e^{-\frac{x}{a}} \right) dx,$$

which, after integration, gives rise to the equation,

$$y = \frac{a}{2} \left(e^{\frac{x}{a}} - e^{-\frac{x}{a}} \right).$$

This is the equation of the catenary in Cartesian coordinates, where the former fixed point, O, has become the point (O,a) and the former fixed tangent has as its equation

$$y - a = 0.$$

The desirability of a general method for studying these problems is evident. Such a method is developed by Scheffer* in his discussion of the natural (intrinsic) equation. In order to get a clear understanding of this topic, it is first necessary to understand the origin and development of the differential invariants upon which this method is based.

*Scheffer's Anwendung der Differential and Integral Rechnung auf Geometrie. Vol. I P. 84.

Differential Invariants.*

Scheffer designates by intrinsic properties of a curve, those which remain unchanged when the curve is subjected to any movement in the plane and which are essential for its shape and measurement. An analytical expression for these properties is sought. This can be done by determining certain magnitudes, which are well-defined when a curve is given, and which remain unchanged when the curve is moved in the plane. These magnitudes are called the differential invariants of a plane curve. We have here to consider the quantities which

*Scheffer Vol. I P. 74

depend upon a curve point P(x,y) and its vicinity. The curve may be represented in the vicinity of the point P(x,y) in the form,

$$\Delta y = f'(x)\frac{\Delta x}{1!} + f''(x)\frac{\Delta x^2}{2!} + f'''(x)\frac{\Delta x^3}{3!} + \cdots$$

The differential invariants defined above thus may include the quantities,

$$x,\ y,\ y',\ y'',\ y''' \cdots\cdots$$

Any function of these magnitudes is in general, not a differential invariant, as for example

$$\phi = y - xy'$$

which is the y - intercept of the tangent. If the curve is moved with respect to its axes, the value of this function changes, as is evident geometrically.

The expression for the curvature at any point has the form,

$$K = \frac{y''}{\pm\sqrt{(1+y'^2)^3}}$$

Due to the ambiguity in sign we will find it more convenient to use the square of the curvature, i.e.

$$K^2 = \frac{y''^2}{(1+y'^2)^3}$$

This function, as is evident geometrically, is a differential invariant. This will also be shown analytically.

The equations of transformation for the movement of a point $\bar{P}(\bar{x},\bar{y})$ over into a point $P(x,y)$ are,

(2) $$\begin{cases} x = (\bar{x}-a)\cos\alpha + (\bar{y}-b)\sin\alpha, \\ y = -(\bar{x}-a)\sin\alpha + (\bar{y}-b)\cos\alpha. \end{cases}$$

and inversely,

(3) $$\begin{cases} \bar{x} = x\cos\alpha - y\sin\alpha + a, \\ \bar{y} = x\sin\alpha + y\cos\alpha + b, \end{cases}$$

By these relations the derivatives y', y'', y''', ... are easily expressed in terms of the derivatives, $\bar{y}'$, $\bar{y}''$, $\bar{y}'''$, ... In particular,

(4) $$\begin{cases} y' = \dfrac{dy}{dx} = \dfrac{\bar{y}' - tg\,\alpha}{\bar{y}'\,tg\,\alpha + 1}, \\[2ex] y'' = \dfrac{dy'}{dx} = \dfrac{\bar{y}''}{(\bar{y}'\sin\alpha + \cos\alpha)^3} \end{cases}$$

If next, the function $\bar{K}^2$ is formed by substituting the last two expressions in the formula for K^2, we get,

$$\bar{K}^2 = \frac{\bar{y}''^2}{(1+\bar{y}'^2)^3},$$

which is identical in form with the original expression for the curvature, and consequently K^2 is a differential invariant.

If $J(x, y, y', y'', \ldots.)$ is a differential invariant, then the relation,

(5) $$J(x, y, y', y'' \ldots) = \bar{J}(\bar{x}, \bar{y}, \bar{y}', \bar{y}'' \ldots),$$

must hold true for all movements in the plane. It is clear from equations (2) and (3), that for a translation (5) will only hold true in case the first two arguments are omitted, viz. x and y. All differential invariants, then, will have to be functions of the derivatives alone. Since every movement in the plane consists of a translation and a rotation, we next consider the analytical conditions imposed by the latter motion upon (5). A rotation about the origin is expressed by the equations,

$$\begin{cases} x = \bar{x}\cos\alpha - \bar{y}\sin\alpha, \\ y = \bar{x}\sin\alpha + \bar{y}\cos\alpha. \end{cases}$$

For all such movements the function J must remain unchanged. To show this directly we would have to compute all functions of the form (4), and these soon become very complicated. An artifice is therefore employed.

We will first show that if U and V are two functions of y', y'', y''', ... which change only by an additive constant for all movements in the plane, i.e.

(6)
$$\begin{aligned} U(y', y'', y''' \cdots) &= U(\bar{y}', \bar{y}'', \bar{y}''' \cdots) + C_1 \equiv \bar{U} + C_1, \\ V(y', y'', y''' \cdots) &= V(\bar{y}', \bar{y}'', \bar{y}''' \cdots) + C_2 \equiv \bar{V} + C_2 \end{aligned}$$

then $\dfrac{dU}{dV}$ is a differential invariant. This is readily shown by taking the differentials in (6) and forming the quotients,

$$\frac{dU}{dV} = \frac{d\bar{U}}{d\bar{V}}.$$

To apply this method we may take for U some known differential invariant J, and find another function V which changes only by an additive constant. The constant C for J is equal to zero. For V we may take the tangent angle Φ, since for all movements in the plane it differs from its original value by an additive constant only, as is geometrically evident, and is seen analytically as follows:

Since $tg\,\Phi = y'$ and $tg\,\bar{\Phi} = \bar{y}'$ the first of equations (4) gives

$$tg\,\Phi = \frac{tg\,\bar{\Phi} - tg\,\alpha}{tg\,\bar{\Phi}\,tg\,\alpha + 1} = tg(\bar{\Phi}-\alpha)$$

so that,

$$\bar{\Phi} = \phi + \alpha + n\pi \quad (n,\ \text{integer}).$$

We therefore conclude that whenever J is a differential invariant $\dfrac{dJ}{d\Phi}$ is also a differential invariant.

<u>If J is chosen equal to K^2, the following system of differential invariants appears by repeated application of the above principle:</u>

(7) $$\underline{K^2,\ \frac{d(K^2)}{d\phi},\ \frac{d^2(K^2)}{d\phi^2},\ \text{- - - -}}$$

These are said to be invariants of the second, third, fourth.... order, since the highest derivatives contained in each are of the second, third, and fourth order etc. respectively.

In case the equations of the curve are given in the parameter form,

$$x = \phi(t),$$
$$y = \psi(t).$$

then, K^2 and Φ are given by the values,

$$K^2 = \frac{(\phi'\psi'' - \psi'\phi'')^2}{(\phi'^2 + \psi'^2)^3},$$

and,

$$\Phi = \operatorname{arc\,tg} \frac{\psi'}{\phi'}.$$

For any function J, we therefore have,

$$\frac{dJ}{d\Phi} = \frac{dJ}{dt} \cdot \frac{dt}{d\Phi} = \frac{\phi'^2 + \psi'^2}{\phi'\psi'' - \psi'\phi''} \cdot \frac{dJ}{d\Phi}.$$

If now K^2 is substituted in place of J in the last expression, we have,

(8) $$\frac{d(K^2)}{d\Phi} = 2\left[\frac{\phi'\psi''' - \psi'\phi'''}{(\phi'^2 - \psi'^2)^2} - 3\,\frac{(\phi'\psi'' - \psi'\phi'')(\phi'\phi'' + \psi'\psi'')}{(\phi'^2 + \psi'^2)^3}\right]$$

which is a differential invariant of third order.

The Natural Equation.

The question next arises as to when two curves in the plane are congruent to one another i.e. can be brought into coincidence by a rigid motion. From the expression for the curvature in parameter notation we may in general obtain t as a function of the curvature. The substitution of this value in equation (8) of the last section will give an expression in the form

(9) $$\frac{d(K^2)}{d\Phi} = \omega(K^2),$$

the right side being a known function of K^2. A corresponding expression will appear for the second curve. If now the two curves <u>are</u> congruent, they will have the same value of K^2 and $\frac{d(K^2)}{d\Phi}$ at homologous points, since these are differential invariants. Consequently, for both curves, the same relation (9) exists between the two invariants.

The converse problem is to show that all curves for which the relation (9) holds, are congruent. We choose from that region of K^2 in which $\omega(K^2)$ is a single-valued analytic function, a value K_0^2 which is not equal to zero, and for which also the function does not disappear. Next, choose as the origin that point on the curve for which $K^2 = K_0^2$, and the positive tangent at that point as the positive x-axis. The particular choice of axes does not affect (9) since only differential invariants appear there. Since $\omega(K_0^2) \neq 0$, there is for K^2 a vicinity of K_0^2 in which $\omega(K^2)$ will not disappear, so that the integral of (9), which has the form,

$$\int_{K_0^2}^{K^2} \frac{d(K^2)}{\omega(K^2)} = \Phi$$

becomes a single-valued analytic function of K^2 which for $K^2 = K_0^2$ takes the value zero or has the form

$$\Omega(K^2) - \Omega(K_0^2) = \Phi .$$

Inversely, this equation defines K^2 as a single-valued analytical function of Φ, which for $\Phi = 0$ takes the value K_0^2. If both plus and minus signs are taken into account, there result two functions K of Φ differing only in sign. To find the equation of the curve in rectangular coordinates, we make use of the equations*,

$$(10) \quad \begin{cases} x = \int_0^{\Phi} \frac{\cos\Phi}{K} d\Phi , \\ y = \int_0^{\Phi} \frac{\sin\Phi}{K} d\Phi , \end{cases}$$

where K^2 is a funtion of Φ satisfying (9) and which for $\Phi = 0$ takes the value K_0^2. The above values K_0 and $-K_0$ are substituted in equations (10) in place of K. Thus will appear two curves which satisfy the condition (9), and which are congruent with one another, for one goes into the other upon substituting $K = -K$ or what is the same thing, $x = -x$ and $y = -y$. By a rotation about the origin the two curves may be brought into coincidence. The necessary and sufficient conditions are thus established that two plane curves shall be congruent i. e. the relation (9) must hold for both. In view of this property

*Scheffer Vol. I Page 52

the relation (9) is called by Scheffer the natural equation.

It will be noticed that the intrinsic equation (1) involves the arc, expressed as an explicit function of the tangent angle. In the form (9) adopted by Scheffer, we have the differential invariant of the third order equal to a function of the differential invariant of the second order. The advantage of the form (9) adopted by Scheffer lies in the fact that it has been shown to be a necessary and sufficient condition for the congruence of two curves, and that it is always possible theoretically to pass to the equations of the curve in other coordinates.

As an illustration of the above theory, we next assume the function $\omega(K^2)$ of (9) to be known, and proceed to find the equations of the curve in Cartesian coordinates from the intrinsic equation (9) thus given. An easy example is where ω is proportional to K^2 itself i.e. (9) has the form,

$$\frac{d(K^2)}{d\Phi} = -2aK^2,$$

or,

$$\frac{d(K^2)}{K^2} = -2a\,d\Phi.$$

After integration and proper choice of axes, we obtain,

$$K^2 = e^{-2a\Phi},$$

and,

$$K = e^{-a\Phi}.$$

Now from equations (10) we have,

$$\begin{cases} x = \int_0^{\Phi} \cos\Phi \cdot e^{a\Phi}\, d\Phi, \\ y = \int_0^{\Phi} \sin\Phi\, e^{a\Phi}\, d\Phi. \end{cases}$$

By integration we find,

(11)
$$x = e^{a\Phi}\left[\frac{a\cos\Phi + \sin\Phi}{a^2+1}\right],$$
$$y = e^{a\Phi}\left[\frac{a\sin\Phi - \cos\Phi}{a^2+1}\right]$$

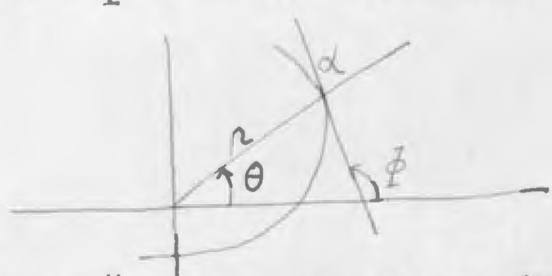
which are the equations of the curve with respect to the parameter Φ . To recognize the form of this curve more clearly, we will obtain its equation in polar coordinates. If equations (11) are squared and added, we have,

(12)
$$x^2 + y^2 = \frac{e^{2a\Phi}}{(a^2+1)}$$

We next make the substitution,

$$x^2 + y^2 = r^2$$

and eliminate Φ in the following manner:



Since $\tan\theta = \frac{y}{x}$ we have from equations (11),

$$\tan\theta = \frac{a\sin\Phi - \cos\Phi}{a\cos\Phi + \sin\Phi} = \frac{a\,tg\,\Phi - 1}{a + tg\,\Phi}.$$

Setting $\tan\alpha = \frac{1}{a}$ (α is a constant angle) the last equation gives,

$$\tan\theta = \tan(\Phi - \alpha)$$

and therefore,

$$\phi = \theta + \alpha.$$

Making this substitution in (12) we find, after simplification

$$r = \frac{e^{a(\theta+\alpha)}}{\sqrt{a^2+1}} = \frac{e^{a\alpha}}{\sqrt{a^2+1}} \cdot e^{a\theta} = b \cdot e^{a\theta},$$

which is a logarithmic spiral in polar coordinates.

In the light of the above theory we can also study the problem suggested on page 7 of passing from the equation in intrinsic coordinates to one in Cartesian coordinates. These may be found by obtaining a value for K and substituting it in the equations (10). If the intrinsic equation is given in the form (1) the value for K is found very simply as was shown. If however, the natural equation is given in the form (9), we have to perform an integration to find the value of K. Both methods will now be illustrated.

An example is found in the astroid. The intrinsic equation in the form (1) was found to be,

$$2S = 3a \sin^2 \Phi,$$

and the curvature is given by the expression,

$$K = \frac{2}{3a} \csc(2\Phi) = \frac{1}{3a \sin\Phi \cos\Phi}.$$

The substitution of this last value in equations (10) gives,

$$x = 3a \int_0^{\Phi} \sin\Phi \cos^2\Phi \, d\Phi,$$

$$y = 3a \int_0^{\Phi} \sin^2\Phi \cos\Phi \, d\Phi,$$

and upon integration we find,

$$x = -a\cos^3\Phi \ ; \ y = a\sin^3\Phi,$$

and from these equations we may write,

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$$

which is the familiar equation of the astroid.

We next set up the natural equation of the astroid (illustrative of Scheffer's method) and from it obtain a value for K. This example will, of course, have no practical significance here but will serve to illustrate the above theory and show the method in case the natural equation is given in the form (9). The value for the curvature just found above may be put in the form,

$$K = C_1\left(\csc[2\Phi]\right) \qquad , \left[C_1 = \frac{2}{3a}\right]$$

Squaring, differentiating with respect to Φ , and simplifying, we get the natural equation of the astroid in Scheffer's form,

$$\frac{d(K^2)}{d\Phi} = -4K^2 \cot(2\Phi).$$

On separating the variables and integrating we find,

$$K^2 = C\left[\sin(2\Phi)\right]^{-2}$$

By choosing the axes so that when $\Phi = 0$, $K = 1$, C will equal one i.e. choose the positive x-axis parallel to the positive tangent at the point where R equals one. We may now write

$$K = \pm\left[\sin(2\Phi)\right]^{-1} = \pm\left[2\sin\Phi\cos\Phi\right]^{-1}.$$

From the equations (10) we have,

$$X = \frac{1}{2}\int_0^{\Phi} \cos^2\Phi \sin\Phi \, d\Phi,$$

$$y = \frac{1}{2}\int_0^{\Phi} \sin^2\Phi \cos\Phi \, d\Phi.$$

By integration we obtain the parametric equations in the form

$$X = -\frac{\cos^3\Phi}{6},$$

$$y = +\frac{\sin^3\Phi}{6},$$

and from these equations we again obtain the equation of the astroid in the form,

$$X^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}},$$

where $$a^{\frac{2}{3}} = 6^{-\frac{2}{3}}.$$

Part Two.

Curves in Space.

The Intrinsic Equations for Curves in Space.

Analogous to the method of developing the theory of intrinsic equations for plane curves, Scheffer* takes up the treatment of the curve in space. Corresponding to the analytical quantities ** by which a plane curve is expressed in the neighborhood of a point, these analytical quantities for a curve in space include the coordinates in parameter form i.e.

(1) $$x = \phi(t), \quad y = \psi(t), \quad z = \theta(t),$$

and their successive derivatives with respect to some variable, here taken as the length of arc s.

The equations of transformation for the movement of a point, P (x,y,z), over into a point, $\bar{P}(\bar{x},\bar{y},\bar{z})$ are,

(2) $$\begin{cases} \bar{x} = \alpha_1 x + \alpha_2 y + \alpha_3 z + a, \\ \bar{y} = \beta_1 x + \beta_2 y + \beta_3 z + b, \\ \bar{z} = \gamma_1 x + \gamma_2 y + \gamma_3 z + c, \end{cases}$$

and the derivatives of the coordinates may be expressed by the formulae,

(3) $$\begin{cases} \bar{x}^{(l)} = \alpha_1 x^{(l)} + \alpha_2 y^{(l)} + \alpha_3 z^{(l)}, \\ \bar{y}^{(l)} = \beta_1 x^{(l)} + \beta_2 y^{(l)} + \beta_3 z^{(l)}, \\ \bar{z}^{(l)} = \gamma_1 x^{(l)} + \gamma_2 y^{(l)} + \gamma_3 z^{(l)}. \end{cases}$$

*Scheffer Vol. I P. 278. **Scheffer Vol. I. P. 269.

If $J(x,y,z,\ x',y',z',\ x'',\ y'',\ z'' \ldots\ldots\ldots)$ is a differential invariant, then, as in the plane, the relation

(4) $$J(x,y,z,\ x',y',z',\ x''\ldots)\qquad \bar{J}(\bar{x},\bar{y},\bar{z},\ \bar{x}',\bar{y}',\bar{z}',\ \bar{x}''\ldots)$$

must hold true for all movements in space. From equations (2) it is easily seen that for a <u>translation</u>, the relation (4) will only hold true in case the function J is free from the first three arguments i.e. the three coordinates x,y,z.

We have now only left to consider the effect of the <u>rotation</u> about a fixed point upon the function

$$J(x',y',z',\ x''\ ,y''\ ,z''\ \ldots\ldots),$$

and in order to do this we have to resort to an artifice. From equations (2) it appears that for two points P_1 and P_2 for one position of the curve, and the two corresponding points $\bar{P}_1$ and $\bar{P}_2$, for another position of the curve, the correspondence between the coordinates

$$x_1,\ y_1,\ z_1, \quad \text{and} \quad \bar{x}_1,\ \bar{y}_1,\ \bar{z}_1,$$

is the same as that between the coordinates

$$x_2,\ y_2,\ z_2, \quad \text{and} \quad \bar{x}_2,\ \bar{y}_2,\ \bar{z}_2.$$

It also appears in equations (3) that this same correspondence holds between the derivatives

$$x',\ y',\ z', \quad \text{and} \quad \bar{x}',\ \bar{y}',\ \bar{z}',$$

and in general between,

$$x^{(l)},\ y^{(l)},\ z^{(l)}, \quad \text{and} \quad \bar{x}^{(l)},\ \bar{y}^{(l)},\ \bar{z}^{(l)}.$$

In place, then, of the differential invariant of the second order,

$$J(x',\ y',\ z',\ \ x'',\ y'',\ z''),$$

we may now consider the function

$$J(x_1,\ y_1,\ z_1,\ \ x_2,\ y_2,\ z_2),$$

and demand that it shall remain unchanged when the points P_1 and P_2 are brought into new positions $\bar{P}_1$ and $\bar{P}_2$ by rigid motions of the curve about the origin O. Geometrically this means that the triangle OP_1P_2 must be congruent to the triangle $O\bar{P}_1\bar{P}_2$, and this will only be true in case the three relations

$$\overline{OP}_1^2 = \overline{O\bar{P}}_1^2, \qquad \overline{OP}_2^2 = \overline{O\bar{P}}_2^2, \text{ and } \quad \overline{P_1P}_2^2 = \overline{\bar{P}_1\bar{P}}_2^2$$

or,

$$x_1^2+y_1^2+z_1^2, \qquad x_2^2+y_2^2+z_2^2, \text{ and } \quad x_1x_2+y_1y_2+z_1z_2$$

remain unchanged for any rotation about the origin. By the correspondence noted above, we may now shift back to derivatives in place of subscripts , and obtain the following expressions,

$$x'^2+y'^2+z'^2, \qquad x''^2+y''^2+z''^2, \text{ and } \quad x'x''+y'y''+z'z''$$

which are the required differential invariants of the first and second order. The first of these expressions is equal to 1,*

*Scheffer Vol. I P. 466 (Tables).

and the last is equal to zero, while the remaining one may be written in the form

(5) $$1/r^2 = x''^2 + y''^2 + z''^2,$$

which is the expression for the square of the curvature (that this expression should appear as a differential invariant of second order was to be expected from geometrical considerations). From the formula for the torsion,

(6) $$\frac{1}{S} = -\frac{1}{x''^2 + y''^2 + z''^2}\begin{vmatrix} x' & x'' & x''' \\ y' & y'' & y''' \\ z' & z'' & z''' \end{vmatrix},$$

it can be easily shown that it is also a differential invariant as is again evident geometrically. The derivatives of (5) and (6) will also be differential invariants, so that we may sum up that the important theorem:

<u>For a curve in space, the quantities,</u>

(7) $$\underline{K^2, K'^2, KK'', K'K''', KK^{IV}, K'K^{V} \ldots\ldots}$$
$$\underline{T, T'^2, T'', T'T''', T^{IV} \ldots\ldots}$$

<u>will remain unchanged for all movements in space</u>, where K and T are the curvature and torsion respectively, and the differentiation is with respect to s.

The Intrinsic Equations Obtained.

By a method paralleling that of the chapter on plane curves, Schaffer shows that any two curves in space are congruent when and only when the square of the curvature (5) and the torsion (6) are the same for corresponding points on the two curves. We may now by an argument analogous to that used in the plane case conclude:

The natural equations of a curve in space are given by the expressions,

(8) $$\left(\frac{d\frac{1}{r}}{ds}\right)^2 = U\left(\frac{1}{r^2}\right), \quad \frac{1}{S} = V\left(\frac{1}{r^2}\right).$$

or,

(9) $$\frac{1}{r^2} = F(s), \quad \frac{1}{S} = G(s).$$

The Determination of all Curves with given Intrinsic Equations.

It remains to investigate all curves which are given by the intrinsic equation in the form (9). The method, in brief, is, having given the curvature and torsion as functions of s, to compute the nine functions $\alpha, \beta, \gamma, l, m, n, \lambda, \mu, \nu,$ from the Frenet formulas*, and next show that one of these sets is sufficient to determine the curve. The Cartesian coordinates will then have the form

(10) $$x = \int (\alpha)\, ds \quad , y = \int (\beta)\, ds \quad , z = \int (\gamma)\, ds.$$

It appears at once from the formulas referred to above, that the equations

(11) $$\frac{du}{ds} = \frac{v}{r} \quad , \quad \frac{dv}{ds} = -\frac{u}{r} - \frac{w}{S} \quad , \quad \frac{dw}{ds} = \frac{v}{S} \, ,$$

have three sets of solutions:

$$\alpha, l, \lambda ; \quad \beta, m, \mu ; \quad \gamma, n, \nu ;$$

These three sets are the direction cosines of three mutually perpendicular lines. Hence the quantities u, v, w must satisfy the relation,

(12) $$u^2 + v^2 + w^2 = 1$$

*Scheffer Vol. I. P. 467 (Tables III C).

The relation (12) may be factored in the form

$$(u+iv)(u-iv) = (1+w)(1-w),$$

and we may introduce with Darboux two functions ξ and η, which are defined by the relations,

(13) $$\xi = \frac{u+iv}{1-w}, \quad \eta = -\frac{u+iv}{1+w}.$$

We find after differentiation and simplification of equations (13) that we have two equations,

$$\frac{d\xi}{ds} = \frac{i}{2\rho} - \frac{i}{r}\xi - \frac{i}{2\rho}\xi^2,$$

and,

$$\frac{d\eta}{ds} = \frac{i}{2\rho} - \frac{i}{r}\eta - \frac{i}{2\rho}\eta^2.$$

Thus ξ and η appear as solutions of the single equation

(14) $$\frac{d\theta}{ds} = \frac{i}{2\rho} - \frac{i}{r}\theta - \frac{i}{2\rho}\theta^2.$$

On solving equations (13) for u, v, and w we find,

$$u = \frac{1-\xi\eta}{\xi-\eta}, \quad v = i\frac{1+\xi\eta}{\xi-\eta}, \quad w = \frac{\xi+\eta}{\xi-\eta}.$$

Our problem reduces then to the integration of equation (14) to determine ξ and η which may then be substituted in the last equations given above for u, v, and w.

The equation (14) may be written in the more general form

(15) $$\frac{d\theta}{ds} = L + 2M\theta + N\theta^2,$$

where L, M, N are functions of s, and is known as the Riccati differential equation.

From the theory of differential equations we may write the general integral of equation (15) in the form,

(16) $$\theta = \frac{aP+Q}{aR+S},$$

where a is a constant and P, Q, R, S are functions of s. We now consider

(17) $$\xi_i = \frac{a_iP+Q}{a_iR+S}, \quad \eta_i = \frac{b_iP+Q}{b_iR+S} \quad (i = 1, 2, 3)$$

as six particular integrals of the equation (16). One of the sets of solutions of (11) will have the form

(18) $$\alpha = \frac{1-\xi_1\eta_1}{\xi_1-\eta_1}, \quad l = i\,\frac{1+\xi_1\eta_1}{\xi_1-\eta_1}, \quad \lambda = \frac{\xi_1+\eta_1}{\xi_1-\eta_1},$$

and similar expressions will appear for the other two sets of direction cosines given at the bottom of page 37.

It can be shown that these three sets of solutions are equivalent.

If the values of equations (17) are substituted in equations (18) we obtain the desired expressions for α, β, γ *.
We may then make use of equations (10) and write the equations of the curve whose radii of first and second curvature are ρ and σ respectively, in the form

(19)
$$x = \int \frac{(P^2 - R^2) - (Q^2 - S^2)}{2(PS - QR)}\, ds,$$
$$y = i\int \frac{(P^2 - R^2) + (Q^2 - S^2)}{2(PS - QR)}\, ds,$$
$$z = \int \frac{RS - PQ}{PS - QR}\, ds.$$

where P, Q, R, S are determined from equations (16).

Problems.

I. A simple example will illustrate the above theory in its relation to that of the preceding part. If we assume the torsion to be zero, the Riccati equation (14) will have the form

$$\frac{d\theta}{ds} = -\frac{i\theta}{\rho}.$$

or,

$$\frac{d\theta}{\theta} = -\frac{i\,ds}{\rho}.$$

Integrating this expression we find

$$\theta = C e^{-i\int\frac{ds}{\rho}}$$

Since the general form of the solution of the Riccati equation is

$$\theta = \frac{aP+Q}{aR+S}, \tag{16}$$

we may here write,

$$P = e^{-i\int\frac{ds}{\rho}}, \quad Q = 0, \quad R = 0, \quad S = 1$$

Substituting these last values in the third equation of (19) we have

$$z = \int_0^s 0 \cdot ds,$$

or,

$$z = 0,$$

which shows, as already known that when the torsion is zero, the curve is a plane curve, whose equations may be found in the first two equations (19).

II, As further illustration of the above theory, we will next take an example* where the functions $F(s)$ and $\varphi(s)$ of equations (9) are constants (real) i.e.

$$\rho = a \ , \quad r = b \ , \quad \frac{\rho}{r} = \frac{a}{b} = c \ (c, \text{ real}).$$

If the expression $r = \frac{s}{c}$ is substituted in equations (14) we obtain the Riccati equation

$$\frac{d\theta}{ds} = \frac{i}{2s}\left(1 - 2c\theta - \theta^2\right)$$

Two particular integrals will then be roots of the equation

$$\theta^2 + 2c\theta - 1 = 0$$

and may be written in the form

$$\theta_1 = -c - \sqrt{c^2+1} \ ; \quad \theta_2 = -c + \sqrt{c^2+1} \ ; \quad \theta_1\theta_2 = -1.$$

From these values we obtain the general solution (16) in the form

$$\theta = \frac{a e^{it}\cdot\theta_2 - \theta_1}{a e^{it} - 1}$$

where we have put

$$t = \frac{\theta_2 - \theta_1}{2}\int\frac{ds}{s} = \frac{\sqrt{c^2+1}}{c}\int\frac{ds}{r}.$$

From this general solution we may now write,

$$P = e^{it}\theta_2 \ , \quad R = e^{it};$$
$$Q = -\theta_1 \ , \quad S = -1.$$

By choosing properly the constants a_i and b_i of equation (17) i.e. so that in pairs they have the [illegible] [illegible] −1, we may compute the values for α, β, γ of (18) and by substituting these last values in equations (10) obtain the Cartesian coordinate of the curve. This will be equivalent to substituting

*Differential Geometry - Eisenhart pp 29-30.

the values for P, Q, R, S in equations (19). The equations will be

(20)
$$x = \frac{c}{\sqrt{c^2+1}}\int \cos t \, ds,$$
$$y = \frac{c}{\sqrt{c^2+1}}\int \sin t \, ds,$$
$$z = \frac{s}{\sqrt{c^2+1}}.$$

From the last of these equations we see that the direction cosine which the tangent makes with the Z - axis, is a constant i.e. the tangent to the curve forms with the XY- plane a constant angle. This a well known property of a helix.

Equations (20) may now be written in the form

$$x = g\int \cos\left(\frac{g}{b}s\right) ds,$$
$$y = g\int \sin\left(\frac{g}{b}s\right) ds,$$
$$z = \frac{bs}{ad},$$

where

$$g = \frac{\sqrt{a^2+b^2}}{a}$$

Integrating these equations we find

$$x = b\sin\left(\frac{g}{b}s\right),$$
$$y = -b\cos\left(\frac{g}{b}s\right),$$
$$z = \frac{bs}{ad}.$$

and from the first two of the above equations we may write

$$x^2+y^2 = b^2$$

This shows that the helix is wound on a right circular cylinder of radius, b.

In case C has imaginary values, the resulting curve in the form (19) will be imaginary, so that for real curves we must choose real values of C.

III. As a special case of the above example we next choose $\rho = r = \sqrt{2}\cdot s$, hence C=I. Equations (20) will then have the form

$$x = \frac{1}{\sqrt{2}}\int \cos(\log s)\,ds,$$

$$y = \frac{1}{\sqrt{2}}\int \sin(\log s)\,ds,$$

$$z = s,$$

since,

$$t = \sqrt{2}\int \frac{ds}{\sqrt{2}\,s} = \log s.$$

These equations may be integrated by parts and we find,

$$x = \frac{s}{2\sqrt{2}}\left[\sin\log s + \cos\log s\right],$$

$$y = \frac{s}{2\sqrt{2}}\left[\sin\log s - \cos\log s\right].$$

From these two equations we may write

$$x^2 + y^2 = \frac{s^2}{4} = \frac{z^2}{4}$$

which shows that the helix in this case is wound on a circular cone.

IV. A second special case is found in the example where the intrinsic equation of the curve is written in the form

$$\rho = r = \frac{s^2+4}{\sqrt{2}} \qquad (C=1)$$

The value of t is found to be

$$t = 2\int \frac{ds}{s^2+4} = \tan^{-1}\frac{s}{2}.$$

Equations (20) will then have the form

$$x = \frac{1}{\sqrt{2}}\int \cos\left(\tan^{-1}\frac{s}{2}\right)ds,$$

$$y = \frac{1}{\sqrt{2}}\int \sin\left(\tan^{-1}\frac{s}{2}\right)ds,$$

which integrate at once into

$$x = \sqrt{2}\log\left(\frac{s \pm \sqrt{s^2+4}}{2}\right),$$

$$y = \sqrt{\frac{s^2+4}{2}}$$

Eliminating s from these last two equations we obtain the expression

$$x = a\log\left(\frac{y \pm \sqrt{y^2-a^2}}{a}\right),$$

where

$$a = \sqrt{2}$$

This curve is a catenary*. In this example, therefore, the helix is wound on a cylinder whose cross-section in a plane parallel to the XY plane is a catenary.

*Kiepert and Stegemann P 337-338.

Part Three.

Surfaces.

The Derivatives of the Rectangular Coordinates for the Equations of a Surface and the Three Fundamental Equations.

If the parametric equations of a surface are given in the form

(1) $$x = \phi(u,v), \; y = \psi(u,v), \; z = \theta(u,v),$$

Scheffer* shows that the partial derivatives of second order, with respect to u and v, of the rectangular coordinates x,y,z may be expressed by means of the partial derivatives of first order of x,y,z, the fundamental quantities E,F,G,L,M,N, and their partial derivatives of first order. The equations have the form

(2) $$\begin{aligned} x_{uu} &= L\underline{X} + \frac{1}{2D^2}(E_u G + E_v F - 2F_u F)x_u + \frac{1}{2D^2}(-E_u F - E_v E + 2F_u E)x_v, \\ x_{uv} &= M\underline{X} + \frac{1}{2D^2}(E_v G - G_u F)x_u + \frac{1}{2D^2}(-E_v F + G_u E)x_v, \\ x_{vv} &= N\underline{X} + \frac{1}{2D^2}(-G_v F - G_u G + 2F_v G)x_u + \frac{1}{2D^2}(G_v E + G_u F - 2F_v F)x_v, \end{aligned}$$

and similar expressions appear for y and z by cyclic interchange. We may also write

(3) $$\begin{aligned} \underline{X}_u &= \frac{1}{D^2}(FM - GL)x_u + \frac{1}{D^2}(FL - EM)x_v, \\ \underline{X}_v &= \frac{1}{D^2}(FN - GM)x_u + \frac{1}{D^2}(FM - EN)x_v, \end{aligned}$$

and similar forms for Y and Z where X,Y,Z are the direction cosines of the normal to the surface.

*Scheffer Vol. II P 332, Satz 1.

Scheffer* shows that if E,F,G,L,M,N are such functions of u and v that satisfy three fundamental equations, and for which $EG - F^2 \neq 0$, then there exists at least one surface which has E,F,G,L,M,N as its fundamental quantities of first and second order. The three fundamental equations are

$$\frac{LN - M^2}{D^2} = \frac{1}{2D^2}\left(2F_{uv} - E_{vv} - G_{uu}\right) + \frac{E}{4D^4}\left(G_u^2 + E_v G_v - 2G_v F_u\right) + \frac{G}{4D^4}\left(E_v^2 + E_u G_u - 2E_u F_v\right) + \frac{F}{4D^4}\left(E_u G_v - E_v G_u - 2F_u G_u - 2F_v E_v + 4F_u F_v\right) \tag{4}$$

$$L_v - M_u = \frac{E_v G - G_u F}{2D^2} L - \frac{E_u G - G_u E + 2(E_v - F_u)F}{2D^2} M - \frac{-E_u F - E_v E + 2F_u E}{2D^2} N, \tag{5}$$

$$N_u - M_v = \frac{G_u E - E_v F}{2D^2} N - \frac{G_v E - E_v G + 2(G_u - F_v)F}{2D^2} M - \frac{-G_v F - G_u G + 2F_v G}{2D^2} L. \tag{6}$$

The first of these equations is called the Gauss equation, and the last two Mainardi-Codazzi equations, because they were first discovered by these men.

* Scheffer Vol II, p 350, Satz 4

Differential Invariants for Surfaces.

The development of the differential invariants for a surface is similar to that for a plane curve and space curve. We consider the equations of the surface given in the form

(1) $$x = \phi(u,v), \quad y = \psi(u,v), \quad z = \omega(u,v).$$

As in Part Two the equations of transformation of a point $P(x,y,z)$ over into a point $\bar{P}(\bar{x},\bar{y},\bar{z})$ will be

(7) $$\begin{aligned} \bar{x} &= \alpha_1 x + \alpha_2 y + \alpha_3 z + a, \\ \bar{y} &= \beta_1 x + \beta_2 y + \beta_3 z + b, \\ \bar{z} &= \gamma_1 x + \gamma_2 y + \gamma_3 z + c. \end{aligned}$$

If these equations, in particular, are to represent a translation, then, as in the preceding sections, any differential invariant must be free of the first three arguments x,y,z. Hence the invariant function J will have the form

$$J(x_u, y_u, z_u, x_v, y_v, z_v, x_{uu}, y_{uu} \cdots)$$

The six fundamental quantities are known to be differential invariants*. We may also apply the theorem** according to which the nth order partial derivatives of the rectangular coordinates are expressed in terms of the first order partial derivatives of the coordinates, and also the six fundamental quantities and their

* Scheffer p 16 Satz 3; p 123 Satz 7.
** Scheffer p 322

partial derivatives. In view of this we may now write the function J in the form

$$J(x_u, x_v, y_u, y_v, z_u, z_v; E, F, G, E_u \cdots; L, M, N, L_u \cdots)$$

It can be further shown that a differential invariant involving the first order partial derivatives of the coordinates is expressible solely as a function of E, F, and G. Finally it is possible to show that for all movements in space a function of the coordinates x, y, z and their higher partial derivatives with respect to u and v, will then and only then remain unchanged when it is a function of the six fundamental quantities and their partial derivatives. The function J may now be written

$$J(E, F, G, L, M, N, E_u \cdots L_v \cdots).$$

The Determination of the Surfaces with Given Fundamental Quantities.

Analagous to equations (11) page 27, equations (2) and (3) of this section will be used to determine the rectangular coordinates of the surface with given fundamental quantities. The method, in brief, will be to determine the quantities

$$x_u, x_v, y_u, y_v, z_u, z_v, X, Y, Z.$$

After this is accomplished we make use of the total differentials of x, y, z in the form

$$dx = x_u\,du + x_v\,dv \text{ etc.}$$

and may then write the equations of the surface in the form

(8)
$$x = \int x_u\,du + x_v\,dv,$$
$$y = \int y_u\,du + y_v\,dv,$$
$$z = \int z_u\,du + z_v\,dv,$$

which are the required Cartesian coordinates of the surface.

In the computation of x_u, x_v etc. we employ two auxiliary functions (as in the previous section) defined by the relations

(9)
$$\xi = \frac{(F+iD)x_u - E x_v}{D\sqrt{E}\,(1-X)},$$
$$\eta = -\frac{(F+iD)x_u - E x_v}{D\sqrt{E}\,(1+X)},$$

where it is assumed that E does not equal zero. The case $E = 0$ and $G \neq 0$ is similarly treated, while if $E = G = 0$, we have

$$\xi = \frac{2x_v}{\sqrt{2F}}\cdot\frac{1}{1-X}, \quad \eta = -\frac{2x_v}{\sqrt{2F}}\cdot\frac{1}{1+X}.$$

From equations (9) we find

(10)
$$x_u = i\sqrt{E}\,\frac{1+\xi\eta}{\xi-\eta},$$
$$x_v = \frac{-D}{\sqrt{E}}\cdot\frac{1-\xi\eta}{\xi-\eta} + i\frac{F}{\sqrt{E}}\,\frac{1+\xi\eta}{\xi-\eta},$$

and

$$X = \frac{\xi+\eta}{\xi-\eta}.$$

In order to determine the ξ and η it can also be shown that they will satisfy two differential equations of the form

$$
\begin{aligned}
\sigma_u &= A + B\sigma + C\sigma^2, \\
\sigma_v &= \bar{A} + \bar{B}\sigma + \bar{C}\sigma^2,
\end{aligned} \tag{11}
$$

where

$$
\begin{aligned}
A &= \frac{EM-(F+iD)L}{2D\sqrt{E}}, \quad \bar{A} = \frac{EN-(F+iD)M}{2D\sqrt{E}}, \\
B &= i\,\frac{E_uF+E_vE-2F_uE}{2DE}, \quad \bar{B} = i\,\frac{E_vF-G_uE}{2DE}, \\
C &= \frac{EM-(F-iD)L}{2D\sqrt{E}}, \quad \bar{C} = \frac{EN-(F-iD)M}{2D\sqrt{E}}.
\end{aligned} \tag{12}
$$

As solutions of equations (11) we may now write from equations (9) the three pairs of solutions

$$
\begin{aligned}
\xi_1 &= \frac{(F+iD)x_u-Ex_v}{D\sqrt{E}\,(1-X)}, \quad \eta_1 = -\frac{(F+iD)x_u-Ex_v}{D\sqrt{E}\,(1+X)}, \\
\xi_2 &= \frac{(F+iD)y_u-Ey_v}{D\sqrt{E}\,(1-Y)}, \quad \eta_2 = -\frac{(F+iD)y_u-Ey_v}{D\sqrt{E}\,(1+Y)}, \\
\xi_3 &= \frac{(F+iD)z_u-Ez_v}{D\sqrt{E}\,(1-Z)}, \quad \eta_3 = -\frac{(F+iD)z_u-Ez_v}{D\sqrt{E}\,(1+Z)}.
\end{aligned}
$$

The insertion of these values in equations (10) will then give the desired first order derivatives of the coordinates in the form

$$
\begin{aligned}
x_u &= i\sqrt{E}\,\frac{1+\xi_1\eta_1}{\xi_1-\eta_1}, \quad x_v = -\frac{D}{\sqrt{E}}\,\frac{1-\xi_1\eta_1}{\xi_1-\eta_1} + i\,\frac{F}{\sqrt{E}}\,\frac{1+\xi_1\eta_1}{\xi_1-\eta_1}, \\
y_u &= i\sqrt{E}\,\frac{1+\xi_2\eta_2}{\xi_2-\eta_2}, \quad y_v = -\frac{D}{\sqrt{E}}\,\frac{1-\xi_2\eta_2}{\xi_2-\eta_2} + i\,\frac{F}{\sqrt{E}}\,\frac{1+\xi_2\eta_2}{\xi_2-\eta_2}, \\
z_u &= i\sqrt{E}\,\frac{1+\xi_3\eta_3}{\xi_3-\eta_3}, \quad z_v = -\frac{D}{\sqrt{E}}\,\frac{1-\xi_3\eta_3}{\xi_3-\eta_3} + i\,\frac{F}{\sqrt{E}}\,\frac{1+\xi_3\eta_3}{\xi_3-\eta_3}.
\end{aligned} \tag{13}
$$

and also

$$X = \frac{\xi_1 + \eta_1}{\xi_1 - \eta_1}, \quad Y = \frac{\xi_2 + \eta_2}{\xi_2 - \eta_2}, \quad Z = \frac{\xi_3 + \eta_3}{\xi_3 - \eta_3}.$$

In order that the value of these first derivatives shall actually give the six fundamental quantities, it is necessary that $\xi_1, \eta_1, \xi_2, \eta_2, \xi_3, \eta_3$ shall in pairs form a harmonic ratio. Next we will show how the values of ξ, η of equations (11) may be computed. These equations are in fact two partial differential equations and they may be combined into a single total differential equation in the form

(14) $$d\sigma = (A + B\sigma + C\sigma^2)\,du + (\bar{A} + \bar{B}\sigma + \bar{C}\sigma^2)\,dv,$$

where A,B,C etc. have the values as given by equations (12).

In order to determine the solution of equations (13), the problem may be reduced to that of solving an ordinary differential equation of the Riccati type. This is brought about by making use of a substitution

(15) $$\frac{v - v_0}{u - u_0} = a$$

where u_0, and v_0 are particular values of u and v, and a is an arbitrary constant. The Riccati equation will appear as

(16) $$\frac{d\tau}{du} = (A + a\bar{A}) + (B + a\bar{B})\tau + (C + a\bar{C})\tau^2.$$

The solution T of equation (16) will contain the arbitrary constant a of the substitution (15). The elimination of this constant from the solution of (16) and the substitution (15) will then give the desired solution of the total differential equation (14) in the form

(17) $$\sigma(u,v) = \frac{P(u,v)C + Q(u,v)}{R(u,v)C + S(u,v)},$$

where C is the constant of integration and P,Q,R,S are functions of u and v.

Next the six functions

$$\xi_i = \frac{a_i P + Q}{a_i R + S}, \quad \eta_i = \frac{b_i P + Q}{b_i R + S} \quad (i = 1,2,3)$$

appear as three pairs of solutions of equations (14). In order that any two pairs $\xi_i\,\eta_i$ shall form a harmonic ratio, as is necessary, the following choice of the six constants is made:

$$a_1 = 0, \; b_1 = \infty, \; a_2 = 1, \; b_2 = -1, \; a_3 = -i, \; b_3 = +i.$$

Hence the values of ξ_i and η_i will be given as follows

$$\xi_1 = \frac{Q}{S}, \quad \xi_2 = \frac{P+Q}{R+S}, \quad \xi_3 = \frac{P+iQ}{R+iS},$$

$$\eta_1 = \frac{P}{R}, \quad \eta_2 = \frac{P-Q}{R-S}, \quad \eta_3 = \frac{P-iQ}{R-iS}.$$

If these values of ξ_i and η_i are inserted in equations (13) the following values for x_u and y_u etc. are obtained,

$$x_u = i\sqrt{E}\cdot\frac{RS+PQ}{RQ-SP}, \qquad x_v = -\frac{D}{\sqrt{E}}\cdot\frac{(RS-PQ)}{(RQ-SP)} + i\frac{F}{\sqrt{E}}\frac{RS+PQ}{RQ-SP},$$

$$y_u = i\sqrt{E}\,\frac{R^2-S^2+P^2-Q^2}{2(RQ-SP)}, \quad y_v = -\frac{D}{\sqrt{E}}\frac{R^2-S^2-P^2+Q^2}{2(RQ-SP)} + i\frac{F}{\sqrt{E}}\frac{R^2-S^2+P^2-Q^2}{2(RQ-SP)}$$

If these last values are inserted in equations (8) the rectangular coordinates of the surface are obtained in the following form:

$$x = \int \frac{i(RS+PQ)(E\,du+F\,dv)-(RS-PQ)D\,dv}{E(RQ-SP)},$$

(18)

$$y = \int \frac{i(R^2-S^2+P^2-Q^2)(E\,du+F\,dv)-(R^2-S^2-P^2+Q^2)D\,dv}{2E(RQ-SP)},$$

$$z = \int \frac{(R^2+S^2+P^2+Q^2)(E\,du+F\,dv)+i(R^2+S^2-P^2-Q^2)D\,dv}{2E(RQ-SP)}.$$

where P, Q, R, S are determined from (17).

Illustrative Problems.

I. By way of illustrating the above theory we will first assume that the quantities E,F,G,L,M,N are known functions of u and v, and from these derive the rectangular coordinates of the surface.

We will consider the case,

$$E = r^2, \qquad L = -r,$$
$$F = 0, \qquad M = 0,$$
$$G = 1, \qquad N = 0,$$

where r is a constant. From equations (12) we obtain the values for A,B,C, A,B,C as

(12')
$$A = \frac{i}{2}, \qquad \bar{A} = 0,$$
$$B = 0, \qquad \bar{B} = 0,$$
$$C = -\frac{i}{2}, \qquad \bar{C} = 0.$$

Equation (14) will then have the form

(14')
$$d\sigma = \frac{i}{2}\left[1 - \sigma^2\right] du,$$

which is seen to be immediately in the form of an ordinary Riccati equation, so we do not need to employ the transformation (15). Separating the variables in equation (14') we have

$$\int \frac{d\sigma}{1-\sigma^2} = \frac{i}{2}\int du.$$

On integration we find

$$\log \frac{1+\sigma}{1-\sigma} = iu + C,$$

or,

$$\frac{1+\sigma}{1-\sigma} = Ce^{iu},$$

from which we obtain σ in the form

(17') $$\sigma = \frac{Ce^{iu} - 1}{Ce^{iu} + 1}.$$

Referring to equations (17) we see that

$$P = e^{iu}, \qquad R = e^{iu},$$
$$Q = -1, \qquad S = +1.$$

The substitution of these values in equations (18) gives immediately

(18') $$\begin{aligned} x &= \int \frac{dv}{a}, \\ y &= \int -\frac{(e^{iu} - e^{-iu})}{2i} du = -\int \sin u \, du, \\ z &= \int \frac{e^{iu} + e^{-iu}}{2} du = \int \cos u \, du. \end{aligned}$$

The integration of these last equations gives

$$x = \frac{v}{a},$$

$$y = \cos u,$$

$$z = \sin u,$$

which are the familiar equations for a circular cylinder whose axis is perpendicular to the y-z plane.

In case E, F, G, L, M, N are given as functions of u and v, in general, the Riccati equations of the form (16) are very difficult and often impossible to solve in a finite form. In practice, therefore, very few problems may be solved by the above method. An alternative method may sometimes be employed however, by studying the parameter lines on the surface when the fundamental quantities of the surface are given. In order to do this we discuss the geometry of the u and v curves on the surface.

II. We will consider the case where E, F, G, L, M, N are given as functions of the parameter u only. Also, by choosing the parameter lines as the lines of curvature on the surface it follows that

$$F = M = 0$$

We can still further simplify the form of the six fundamental quantities by arranging the u curves among themselves in such a way that $E = 1$. As a result the fundamental quantities appear in the form

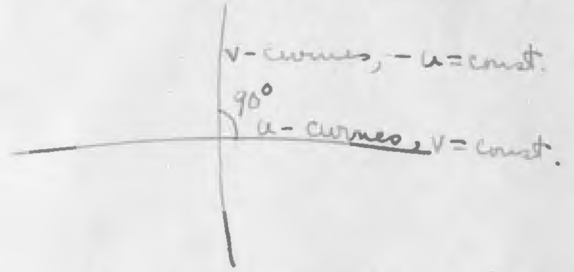
$$E = 1, \quad F = 0, \quad G = G(u), \quad L = L(u), \quad M = 0, \quad N = N(u).$$

The derivatives of the rectangular coordinates* will now appear in the simple form

(2′)
$$x_{uu} = L X,$$
$$x_{uv} = \frac{g_u}{2g} x_v = \frac{\partial \sqrt{g}}{\partial u} x_v,$$
$$x_{vv} = N X - \frac{g_u}{2} x_u.$$

*Scheffer Vol. II. p 564. Tables. Also cf. equations (2).

The Discussion of the u-curves i.e. v constant.



The element of arc for a surface is given by the formula

$$ds = \sqrt{E\,du^2 + 2F\,du\,dv + G\,dv^2}$$

For the curves v = constant, dv = o, so that we have

$$ds = du$$

We next compute the direction cosines of the tangent and principal normal to the u-curves. These are given by the relations

$$\alpha = \frac{dx}{ds} = x_u ,$$
$$\beta = \frac{dy}{ds} = y_u ,$$
$$\gamma = \frac{dz}{ds} = z_u .$$

Similarly, the direction cosines of the principal normal are

$$l = \lambda\frac{d\alpha}{ds} = \lambda x_{uu} = \lambda L_1 X ,$$
$$m = \lambda\frac{d\beta}{ds} = \lambda y_{uu} = \lambda L_1 Y ,$$
$$n = \lambda\frac{d\gamma}{ds} = \lambda z_{uu} = \lambda L_1 Z .$$

From these last equations we may at once conclude that the principal normals to the u-curves are parallel to the normals of the surface i.e. coincident at the same point.

The curvature is given by the formula

$$\frac{1}{\rho^2} = \sum \left(\frac{d\alpha}{ds}\right)^2 \qquad = L^2(u),$$

so that we see that the curvature is variable along the u-curves. The torsion is found from the relation

$$\frac{1}{T} = -\rho^2 \begin{vmatrix} x' & y' & z' \\ x'' & y'' & z'' \\ x''' & y''' & z''' \end{vmatrix},$$

where

$$x' = x_u,$$
$$x'' = x_{uu} = L X,$$
$$x''' = L_u X + L X_u \quad \text{etc.}$$

Therefore we have

$$\frac{1}{T} = -\frac{L_u}{L} \begin{vmatrix} x_u & y_u & z_u \\ X & Y & Z \\ X & Y & Z \end{vmatrix} + \frac{1}{R_1} \begin{vmatrix} x_u & y_u & z_u \\ X & Y & Z \\ x_u & y_u & z_u \end{vmatrix} = 0,$$

which shows that the u-curves are plane curves.

The Discussion of the v-curves.

For the curves $u = \text{constant}$

$$ds = \sqrt{G}\, dv$$

so that the direction cosines of the tangent may be written

$$\alpha = \frac{1}{\sqrt{g}} x_v,$$
$$\beta = \frac{1}{\sqrt{g}} y_v,$$
$$\gamma = \frac{1}{\sqrt{g}} z_v.$$

The direction cosines of the principal normal are found by the formulae

$$l = \lambda \frac{d\alpha}{ds} = \lambda\left(\frac{\partial \alpha}{\partial u}\frac{du}{ds} + \frac{\partial \alpha}{\partial v}\frac{dv}{ds}\right) \text{etc.}$$

Since $\frac{du}{ds} = 0$ and $\frac{\partial \alpha}{\partial v} = \frac{1}{\sqrt{g}} x_{vv}$ we have, using ($2'$),

$$l = \frac{\lambda}{g} x_{vv} = \lambda N X,$$

$$m = \frac{\lambda}{g} y_{vv} = \lambda N Y,$$

$$n = \frac{\lambda}{g} z_{vv} = \lambda N Z,$$

We may therefore conclude that the principal normals to the v-curves and those of the surface are coincident.

The curvature and torsion of the v-curves may be shown to have the values

$$\frac{1}{\rho^2} = N^2(u),$$

and,

$$\frac{1}{\tau} = 0,$$

respectively. Since the curvature is constant for any particular value of u and the torsion is zero, the v-curves will appear as circles on the surface.

It is to be noted that the u and v curves, which are orthogonal, are so situated that the principal normals coincide with the normals to the surface, while the binormal of one is the tangent to the other. Furthermore the v-curves are circles and the u-curves are plane curves with the curvature varying at each point. We may therefore conclude that the surfaces in question, which are characterized by having their six fundamental quantities functions of a single parameter only, are surfaces of revolution.