REPORT

of

Committee on Thesis

The undersigned, acting as a Committee of the Graduate School, have read the accompanying thesis submitted by George C. Priester for the degree of Master of Science They approve it as a thesis meeting the requirements of the Graduate School of the University of Minnesota, and recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Science

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THEORY OF COLUMNS

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George C. Priester

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THEORY OF COLUMNS

INTRODUCTION

This thesis properly falls into the three following divisions:

1. A review, comparison and discussion of the most important formulae that have been proposed for an axially loaded column.
2. An investigation of the general principles and conditions of the state of stress in a column.
3. The determination of a simple algebraic equation which gives the breaking strength of a column for a wide range of its slenderness ratio. This equation should be so flexible that it is possible to adapt it to the experimental data of various types of columns.

SHORT RESUMÉ OF COLUMN FORMULAE

In engineering practice the general equation of flexure is $E\,I\,\frac{d^2y}{dx^2} = M$ where E is the modulus of elasticity of the material, I the moment of inertia of a cross section of the column and M the bending moment.

<u>Euler, 1757.</u>

For an axially loaded column Euler proposed the formula $E\,I\,\frac{d^2y}{dx^2} = -Py$ to represent the condition where the axis of the column is the x- axis, a line in the bending plane perpendicular to the axis is the y- axis, and P is the axial load. By quadratures the equation reduces to $x = \left(\frac{E\,I}{P}\right)^{\frac{1}{2}} \arcsin\left(\frac{y}{f}\right)$, where f is the maximum deflection under the load P. Eliminating x and y this reduces to $P = K\pi^2\frac{EI}{l^2}$ or $P/a = \frac{K\pi^2E}{\left(\frac{l}{r}\right)^2}$ where K is a constant depending on the end conditions, l is the length of the column, r is the least radius of gyration of the section, and a is the cross-section area.

Hodgkinson, 1840.

Hodgkinson in experimenting on comparatively long round cast iron columns found results that agreed approximately with those obtained from Euler's formula. However, he proposed the formula $P = Q \frac{d^{\alpha}}{l^{\beta}}$ where Q, α and β are empirical constants depending upon the material. P represents the breaking load, d the diameter of the column and l its length.

Rankine - Gordon 1860.

Rankine decided that columsn of ordinary length fail by combined flexure and compression and therefore writes $S = S_1 + \frac{P}{a}$ where S is the maximum stress in any fiber, S_1 the stress due to flexure, and $\frac{P}{a}$ the stress due to direct compression. To use this equation he writes it as follows: $S = \frac{P}{a}\left[1+\phi\left(\frac{l}{r}\right)^2\right]$ where ϕ is an empirical constant varying with the end conditions of the column and the kind of material. This form is often called Gordon's formula but he probably first proposed it as follows: - $S = \frac{P}{a}\left[1+\alpha\left(\frac{l}{d}\right)^2\right]$ where d is the diameter of the column and α an empirical constant.

Ritter, 1873.

Ritter proposed the formula $\frac{P}{a} = \dfrac{S}{1+\frac{S_e}{\mu E}\left(\frac{l}{r}\right)^2}$ which is similar to Rankine's, where S_e is the elastic limit of the material and μ a function of π depending on the end conditions.

Crehore, 1879.

Crehore derived the formula $\frac{P}{a} = \dfrac{S}{1+\frac{S}{c\pi^2 E}\left(\frac{l}{r}\right)^2}$ which is similar to Ritter's except S replaces S_e, and c is a constant depending only on the end conditions.

Johnson, T. H., 1886.

T. H. Johnson proposed the straight line formula $S = K - m\,l/r$ where K and m are empirical constants which are to be determined from experimental data on the various materials and conditions under which columns are used.

Merriman - Ritter 1894.

The formula derived by Merriman and Ritter in 1894 is practically the same as the one given by Ritter in 1873.

$$\frac{P}{a} = \frac{S}{1 + \frac{nS_e}{4\pi^2 E}\left(\frac{l}{r}\right)^2}$$

n being a constant depending on end conditions.

Tetmajer (about 1895)

From a great many experiments Tetmajer found that Euler's equation gave too large values for the breaking load of iron columns whose slenderness ratio $(\frac{l}{r})$ was less than 150, and also, that the maximum stress for values of l/r less than 150 was nearly a straight line. He writes the equation $S = S' - C\left(\frac{l}{r}\right)$ where S' is the maximum compressive strength of a short block and C have such value that the line becomes tangent to Euler's curve.

Cain-Bresse, 1897.

Cain derived the formula for an "ideal" column $P = \frac{\pi^2 EI}{l l_1}$ from an analysis given by Bresse where Bresse found an expression for the deflection of a column under a given axial load. In Cain's formula l is the original length of the column and l_1 the length under the load P.

Johnson, J. B.

J. B. Johnson proposed the parabolic formula

$$\frac{P}{a} = S\left[1 - \frac{mS}{16\pi^2 E}\left(\frac{l}{r}\right)^2\right]$$

where m is a coefficient depending on end conditions.

GENERAL PRINCIPLES

The reason that no satisfactory formula for the determination of the strength of a column has been found is that the stresses in it are generally a combination of flexure and compression. With a small load the stress is generally direct compression, but due to local defects in the material, an increase in load causes the column to bend and thereby produces a combination of flexure and direct compression. The relation between these two quantities has never been satisfactorily determined from a mathematical basis, or, in other words, their physical relation has not lent itself to a mathematical expression.

Experience has shown that the ultimate unit load P/a for long columns is always less than the unit compressive strength for short prisms. On short prisms the total stress is uniformly distributed over the entire area, while on long columns this condition is no longer maintained. Bending under an appreciable load is always experienced in columns whenever the slenderness ratio is greater than 30 and this bending increases with an increase of this ratio, the other quantities remaining constant. The unequal distribution of stress over a given section of a column may be assumed to be due to this bending. Experiments verify this assumption wherever the elastic limit of the material has not been exceeded.

When an increasing axial load P is applied to a straight column the stress is at first evenly distributed over the entire cross-section, but as the load increases the column becomes bent due to the physical defects in the column and the conditions under which the load is applied. This bending leaves the load P no longer axial to every cross-section of the column. The eccentricity of the load produces a moment which is resisted by a non-uniformly distributed stress. Therefore the maximum unit stress in a column may be expressed as follows: $S = \frac{P}{a} + S_1$ where S_1 is the stress due to bending. The evaluation of S_1 for any given load P has not been satisfactorily determined for every condition.

In applying a load to a column the distribution of stress before bending is represented in A of figure 1. After bending, by figures B, C, D, and

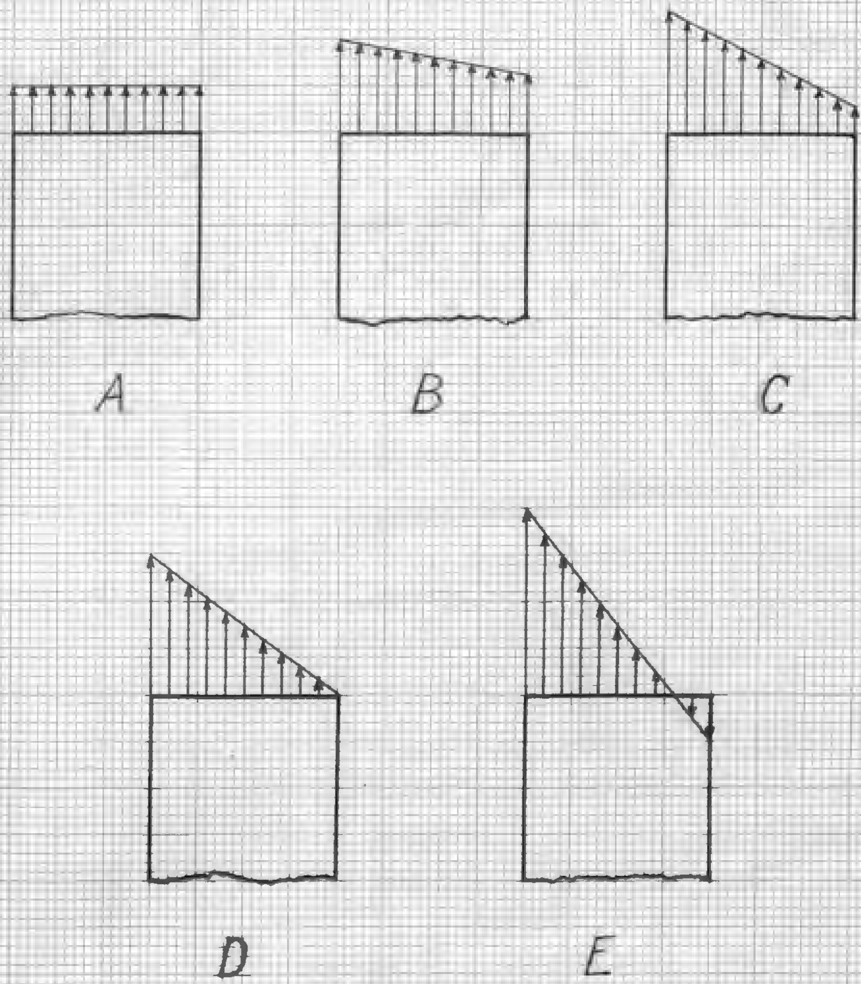


Figure 1.

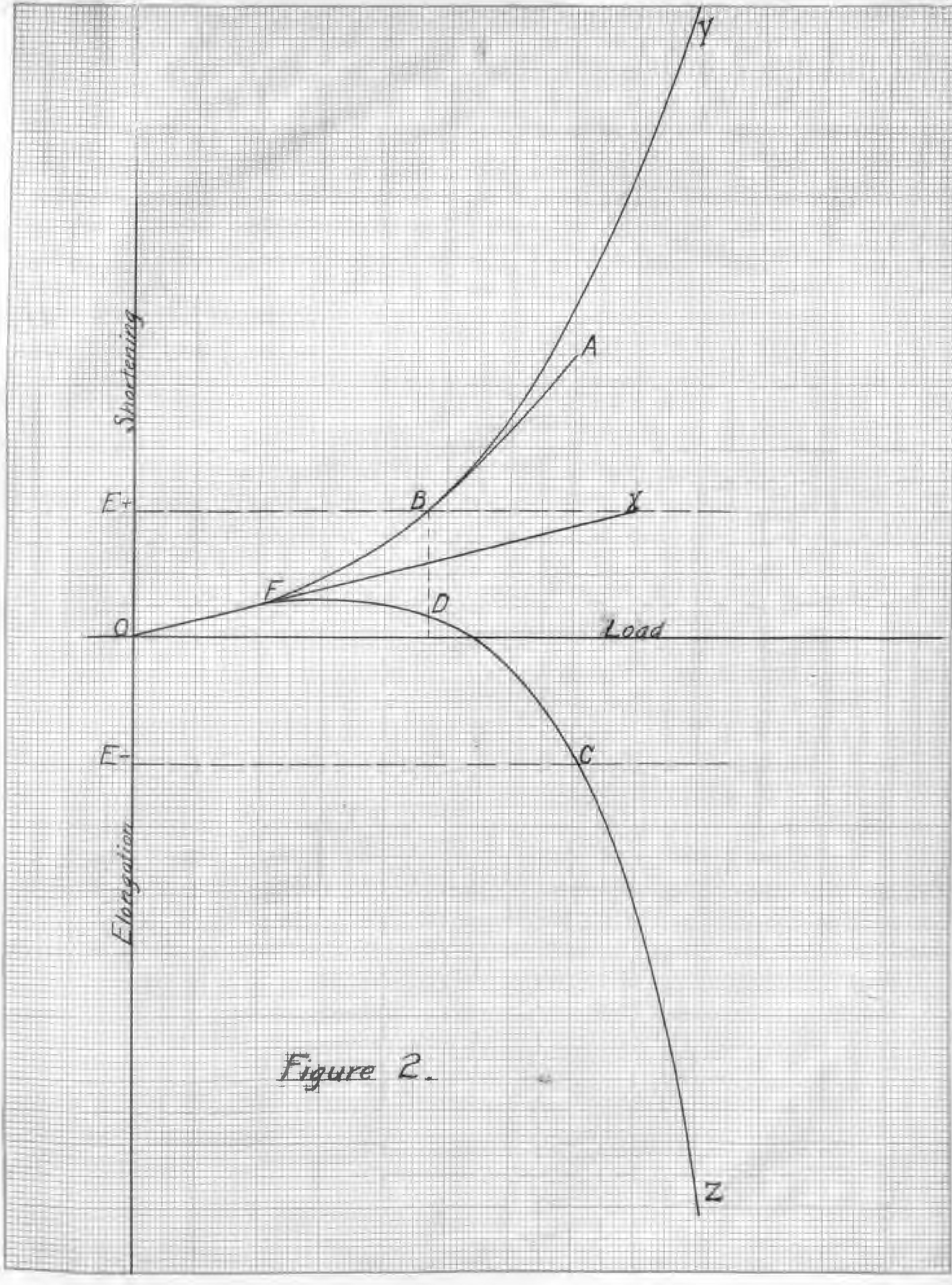


Figure 2.

E in the order of the amount of load applied. In B, with an increase in load over that applied in A, the inner fiber of the column has an increase in compressive stress over the average. The stress in the outer fiber has increased over that shown in A but it is less than the average stress in B. In C, D, and E, average stress continues to increase beyond the average in each successive increase of load. However, the stress in the outer fiber in C begins to decrease and in D there is no compressive stress in the outer fiber. With a further increase in load the outer fiber goes into tension as shown in E.

In Figure 2 the shortening and elongation of the inner and outer fibers are plotted against the load P. e + is the ordinate where the elastic limit is reached in compression, and e- the ordinate where the elastic limit is reached in tension. OX is the direction of the line representing the relation between the average shortening and the load P. This line terminates at X since the ordinate of the elastic limit is reached at that point. However, before going that far, the load P represented by the abscissa of F is where the column begins to bend and here the inner fiber begins to shorten beyond the average and the outer fiber to lengthen. The relation between the inner fiber and the load P might be the curve O F A but at B the elastic limit is reached and the shortening is no longer in the same proportion as before, but larger, and so its true relation would be represented by the curve O F B Y. For the outer fiber the relation might be given by the curve O F D C Z. The curve for the outer fiber has no abrupt change in direction at D which has the same abscissa as B but must continue along this curve to C where the elastic limit in elongation is passed. Here it suffers a discontinuity and then continues on to Z.

To determine the state of stress at anysection an expression for the elastic curve must be found. The equation of the elastic curve is generally determined from the radius of curvature which is

$$R = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

or assuming the length of the curve to be equal to x by $\frac{1}{R} = \frac{d^2y}{dx^2}$

If for R we substitute E I/M we have $\frac{d^2y}{dx^2} = \frac{M}{EI}$ which is our equation for an elastic curve. This equation lends itself immediately to the calculation of stress due to flexure in beams where the load is applied transversely, since an expression for the moment M is readily found. However, there seems to be no rational basis for the use of this expression for the elastic curve in this form, when applied to columns where the load is axial and the moment is indeterminate.

AGREEMENT AND LIMITATIONS OF THE FORMULAE.

Euler's formula $P = \frac{K\pi^2 EI}{l^2}$ when used to find the breaking load of a given column does not make use of the required unit stress to rupture the material. It represents that load at which sufficient flexure is produced so that the column will fail by simple bending. Euler's formula is therefore a criterion for determining the load which will cause a column to fail by lateral flexure. Experiments verify these assumptions where the slenderness ratio of a column with round ends is greater than 150 and for square and fixed ends greater than 200 and 250 respectively. When the slenderness ratio is less than those given above, the breaking load is found to be less than that given by Euler's formula. This leads one to believe that slender columns fail by simple bending, while shorter columns fail by a combination of flexure and direct compressive stress. While Euler's formula is derived from a rational basis when taking into account only the flexure of a bent column, the use of this formula must be limited to the region in which his assumptions are the governing factors.

The formula proposed by Hodgkinson is empirical and applicable only to the type of columns which were used in his experiments. While his formula is unlike that of Euler and the results obtained from his experiments upon which it is based satisfy both within practical limits, it is evident that the use of this formula must be limited.

The Rankine-Gordon formula is based on the condition that the maximum unit stress is the sum of the compressive stress P/a and the flexure stress due to bending. The flexure stress is given by the relation $S_1 = \frac{Mc}{I}$ where M is the bending moment due to the load P. If f is the lateral deflection of the column we have $S_1 = \frac{P\,f\,c}{I}$ or $S_1 = \frac{P\,f\,c}{a\,r^2}$. Then with the assumption that f varies directly as $\frac{l^2}{c}$ as is the case in the theory of beams the equation

$S = \frac{P}{a}\left[1 + \phi\left(\frac{l}{r}\right)^2\right]$ is written.

There is no valid reason why the above assumption should be used since the deflection of a beam and a column may not be analogous. Also the introduction of the factor ϕ which is an empirical constant depending upon the material and end conditions causes the formula to lose its rationality and to become empirical and suitable only for those columns for which ϕ has been actually determined. The values of ϕ have been determined for various conditions, and in the range of l/r from 20 to 150 this formula may readily be used when these facts are at hand and the conditions under which they have been determined fit the problem.

In Ritter's formula which is similar to Rankine's ϕ is expressed in terms of the elastic limit of the material, the modulus of elasticity and a function of π. This formula gives values within practical limits when l/r is from 20 to 150 for wrought iron and structural steel but for brittle material and wood, the values are unsatisfactory.

Crehore derived practically the same formula except thatthe unit stress of the elastic limit is replaced by the maximum unit stress of the material under a breaking load. This formula has about the same objections as the Ritter formula.

The Merriman-Ritter formula presents the same objections since it is no different from the two preceding formulae except in the method of expressing the coefficient of $(\frac{1}{r})^2$.

T. H. Johnson found that the maximum values of P/a for a given material was practically a straight line when plotted as a function of 1/r. Since this formula is purely empirical a formula for every different condition must be evolved. Also the range of 1/r is limited in each case because the same material under the same conditions does not follow this formula. The values he proposed for the constants give a range of 1/r from 20 to about 120.

Tetmajer's formula is empirical and seems to fit conditions within practical limits wherever its constants have been determined for particular materials and conditions.

The last two mentioned formulae are probably the foundation of the straight line formula in use by nearly all of our present designers of columns.

While Cain's formula has a rational basis it covers no wider range than Euler's.

J. B. Johnson's parabolic formula is an empirical formula which begins for short prisms with a value of P/a = 3 and approaches tangency to Euler's curve at about one half of the value of P/a. It is therefore limited in range but within this range it gives satisfactory results.

THE MATHEMATICAL THEORY

The complete mathematical determination of a state of stress in an elastic body is dependent upon four conditions for which we must find a mathematical interpretation.

(1) The equations of equilibrium may be written as follows:

$$\frac{\delta X_x}{\delta x} + \frac{\delta X_y}{\delta y} + \frac{\delta Z_x}{\delta z} + \rho X = 0$$

$$\frac{\delta X_y}{\delta x} + \frac{\delta Y_y}{\delta y} + \frac{\delta Y_z}{\delta z} + \rho Y = 0$$

$$\frac{\delta Z_x}{\delta x} + \frac{\delta Y_z}{\delta y} + \frac{\delta Z_z}{\delta z} + \rho Z = 0$$

(2) The boundary conditions as follows:

$$X_x \cos(x,\nu) + X_y \cos(y,\nu) + Z_x \cos(z,\nu) = X_\nu .$$
$$X_y \cos(x,\nu) + Y_y \cos(y,\nu) + Y_z \cos(z,\nu) = Y_\nu .$$
$$Z_x \cos(x,\nu) + Y_z \cos(y,\nu) + Z_z \cos(z,\nu) = Z_\nu .$$

where ν is the direction of the outward drawn normal to the bounding surface and X_ν is the x- component of the applied surface traction.

(3) The strain components in terms of the stress components as follows:

$$E e_{xx} = X_x - \sigma Y_y - \sigma Z_z . \qquad e_{yz} = \frac{Y_z}{\mu} .$$
$$E e_{yy} = Y_y - \sigma Z_z - \sigma X_z . \qquad e_{zx} = \frac{Z_x}{\mu} .$$
$$E e_{zz} = Z_z - \sigma X_x - \sigma Y_y . \qquad e_{xy} = \frac{X_y}{\mu} .$$

or the stress components in terms of the strain components as follows:

$$X_x = \lambda(e_{xx} + e_{yy} + e_{zz}) + 2\mu e_{xx} . \qquad Y_z = \mu e_{yz} .$$
$$Y_y = \lambda(e_{xx} + e_{yy} + e_{zz}) + 2\mu e_{yy} . \qquad Z_x = \mu e_{zx} .$$
$$Z_z = \lambda(e_{xx} + e_{yy} + e_{zz}) + 2\mu e_{zz} . \qquad X_y = \mu e_{xy} .$$

(4) the relation between the strain components and displacement as follows:

$$e_{xx} = \frac{\delta u}{\delta x} . \qquad e_{yy} = \frac{\delta v}{\delta y} . \qquad e_{zz} = \frac{\delta w}{\delta z} .$$

$$e_{yz} = \frac{\delta w}{\delta y} + \frac{\delta v}{\delta z} . \quad e_{zx} = \frac{\delta u}{\delta z} + \frac{\delta w}{\delta x} . \quad e_{xy} = \frac{\delta v}{\delta x} + \frac{\delta u}{\delta y} .$$

It is evident that to fit the conditions of stress in an axially loaded column to the above conditions is necessarily very complicated and if any method of shortening the work is attempted by assumptions that the relative displacements of parts of the strained body are small care must be taken that this condition is fulfilled in the final result.

In the problem of the elastica as applied to a thin rod subject only to a bending stress we have for the equations of equilibrium $T = -R \cos \theta$ $N = -R \sin \theta$ $\frac{dG'}{dS} + N = 0$. Letting $G' = -B \frac{d\theta}{dS}$ we have the equation $B \frac{d^2\theta}{dS}$ we have the equation $B \frac{d^2\theta}{dS^2} + R \sin \theta = 0$. Here we find that the assumptions are not complete for the conditions of stress in the ordinary column and therefore of no value for practical considerations.

The above notation is taken from Love's Mathematical Theory of Elasticity and for any further developments along these lines this treatise is given as a reference. The objection to the general treatment of the problem is the laborious method necessary to develop the expression for the stress and when derived the expression is in such condition as to be inapplicable to a practical problem.

A practical formula must be simple and expressed in such terms that the required stress can be calculated by the methods of computation familiar to the engineer. A formula which covers an exceedingly wide range of values of l/r and at the same time can be expressed in a simple practical form is the formula which is the most desirable.

A SIMPLE FORMULA FOR A WIDE RANGE OF L/R.

From the many experiments performed upon columns the maximum stress at which a column will fail with a range of l/r gives an idea of the form of an equation which should fit their relation when plotted on coordinate paper. In the following pages an attempt to find such an equation is given. No attempt has been made to interpret the physical nature of these constants and the formula proposed in its present state is purely empirical.

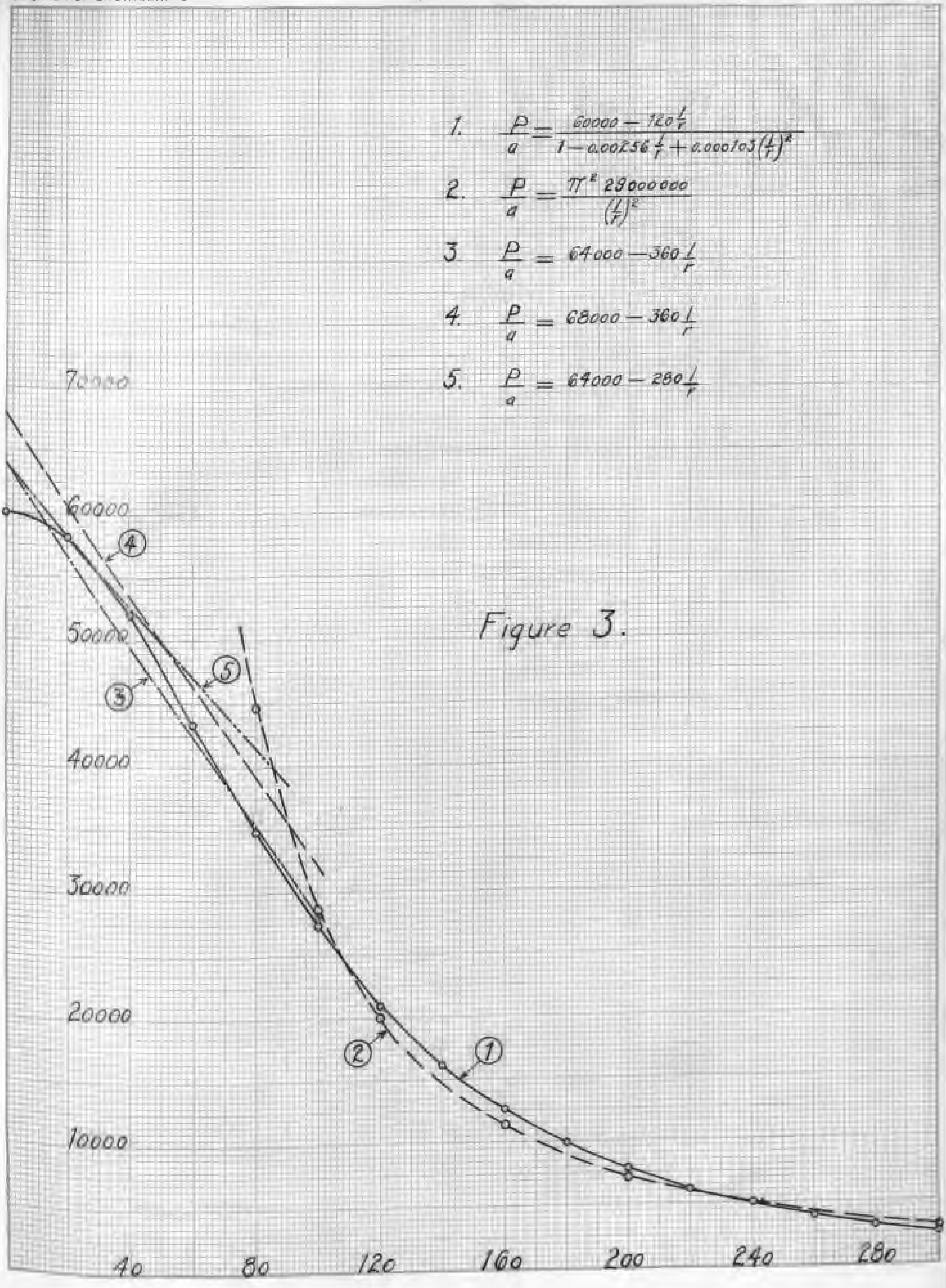
A short block of structural steel which will fail with a compressive stress of 60000 #/sq. in.and in which the modulus of elasticity is about 29,000,000 #/sq. in. is taken as a model. Any other rational assumptions could be used and similar results obtained.

Three straight line formulae proposed by designers of columns to fit a range of l/r from 30 to 120 are

(a) $P/a = 17000 - 90\ l/r$

(b) $P/a = 16000 - 70\ l/r$

(c) $P/a = 16000 - 90\ l/r$



1. $\frac{P}{a} = \frac{60000 - 120\frac{l}{r}}{1 - 0.00256\frac{l}{r} + 0.000103\left(\frac{l}{r}\right)^2}$
2. $\frac{P}{a} = \frac{\pi^2\, 28000000}{\left(\frac{l}{r}\right)^2}$
3. $\frac{P}{a} = 64000 - 360\frac{l}{r}$
4. $\frac{P}{a} = 68000 - 360\frac{l}{r}$
5. $\frac{P}{a} = 64000 - 280\frac{l}{r}$
70000
60000
50000
40000
30000
20000
10000
40
80
120
160
200
240
280
1
2
3
4
5

Figure 3.

P/a is the working stress for a given l/r. The breaking stress is at about four times these values. The formulae then for the breaking stress would be

(a) $P/a = 68000 - 360\ l/r$

(b) $P/a = 64000 - 280\ l/r$

(c) $P/a = 64000 - 360\ l/r$

These equations intersect Euler's equation for values of l/r between 80 and 120. Since the straight line formula is known to represent the maximum stress for l/r from 20 to 120 and Euler's formula for l/r greater than 120, it is desirable to find an expression which will approximate these values within practical limits and have a range of l/r say from 0 to at least 300.

Equation of various types are suggested when a study is made of the above condition and a few are given.

(1) $y = a + bx + cx^2 + dx^3$

(2) $y = a + \frac{b}{N + x} + \frac{c}{(N + x)^2} + \frac{d}{(N + x)^3}$

(3) $y = \frac{a + bx}{e^{cx}}$

(4) $y = \frac{a + bx}{1 + cx^2}$

(5) $y = \frac{a + bx}{1 + cx + dx^2}$

where y = P/a and x = l/r.

Equation (1) is a cubic equation which can be made to fit the conditions only over a limited range of values and could not be classed as satisfactory. It also contains x to the third power and cannot be solved readily for x. Equations of type (1) can be made to follow this curve very closely providing higher powers of x are introduced but this is not desirable.

Equation (2) while different from (1) might be said to satisfy to the same degree and also to possess the same objections.

Equation (3) which is less flexible than the former equations, is unsatisfactory because of the difficulty of solution for both y and x and very difficult to determine for a wide range of conditions.

Equation (4) can be made to satisfy the conditions except at certain values of l/r. If however we introduce the 1st power of x in the denominator this equation now becomes equation (5) which will satisfy within practical calculation the given conditions.

By choosing (a) as equal to the crushing strength of a short prism and selecting the proper values of l/r and finding the corresponding values of P/a from the straight line and Euler's formula for the given material, the coefficients b, c, and d may be found.

To illustrate this take the figures for a piece of medium steel, where E is 29,000,000 #/sq. in. and the crushing strength of a short prism is 60,000 #/sq. in. For x (l/r) take 40, 120, and 240, and y(P/a) take 52000, 21000 and 5000 respectively, solving for b, c and d we have b = - 120 c = - 0.00256 d = 0.000103.

With the above values of the constants this curve is plotted in Figure 3 in comparison with the three straight line and Euler's formulae.

It is evident that for a short column say with l/r from 0 to 40 that there is very little difference in its crushing strength, that from 40 to 120 there is a gradual decrease in the value and beyond 120 the values follow Euler's curves very closely. The equation suggested conforms very closely to this evidence. It is also possible to find l/r in terms of P/a since the equation is of the second degree in l/r.

The equation is applicable to any material with the various end conditions of columns. By finding a number of values of P/a from different values of l/r and determining the constants of the equation from selected values of these, there results an equation which will be satisfactory for a wide range of values of l/r.

It is beyond the scope of this paper to give any physical interpretation of the constants in the equation farther than that the value of a should be the ultimate unit compressive stress of the material for which the formula is used. In order to make a satisfactory interpretation for them it is necessary to investigate more fully the exact state of stress in a column under very wide and varying conditions. With the modern instruments designed for the measurement of fiber stress this investigation is not a remote possibility.

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