TITLE : STUDIES IN RE-ARRANGEMENT OF TERMS OF INFINITE SERIES

A Thesis submitted to the Faculty of the Graduate School of the University of Minnesota by

KATHERINE MELVINA DOWNEY

in partial fulfillment of the requirements for the degree of Master of Arts

June 12 - 1913 .

O U T L I N E :

Real progress in the study of infinite series was delayed until the nineteenth century, for it was not until then that any clear distinction between convergent and divergent series was recognized. Until that time we find the majority of mathematicians using all series as though they were convergent. In the early part of the nineteenth century, Cauchy and Bolzano were among the first to make a study of sequences and apply this knowledge to the subject of infinite series.

According to Reiff, we may divide the history of infinite series into three periods: the period of Newton and Leibnitz, the period of Euler and his contemporaries, and the period beginning with Gauss and extending through the work on uniform convergence . The study of infinite series during the first period seems to have arisen from the necessity of integrating fractional and irrational expressions by means of integrating series equivalent to these expressions . Newton and Leibnitz gave various methods for developing a function into a series . Yet the notion of convergence was extremely faulty, for we find mathematicians sanctioning such views as the following : "Since the series $1-1+1-1 \cdots\cdots$ is equal to 1 onehalf the time , and to zero the other half of the time , $1-1+1-1 \cdots\cdots = \frac{1}{2}$

But , if (1-1) is taken as a term , the series becomes $1-1+1-1+1-1 \cdots\cdots = 0$.

Consequently, we find Guido Grando concluding that 0 = 1/2 and that from these results he has a proof of the creation of the world out of nothing . Both Newton and Leibnitz believed in the correctness of the following :

$$\ldots \ldots \frac{1}{n^2} + \frac{1}{n} + 1 + n + n^2 + \cdots \ldots = 0.$$

Even during the second period , there was a noticeable laxity in regard to the distinction between convergence and divergence ; but considerable attention was given to the form of the series . At this time Taylor and Mac Laurin came forward with their work , while La Grange was the first one to point out the power of Taylor's theorem . Euler gave more effort to obtaining the sum of a given series ; to him we owe the production of the theory of definite integrals . Apparently the only onesin this period who realized the true nature of a divergent series were Nicholas Bernoulli and D' Alembert .

At the beginning of the third period Gauss developed a general criterion for the convergence of infinite series . In addition to this , Cauchy deduced several special tests , and investigated the limit of a product of convergent series . Later considerable work in finite series was carried on by Abel , Voss , and Pringsheim . The study of general criteria was continued by Du Bois Reymond , De Morgan , Dini of Paris and Kumner . The convergence of Fourier series was first investigated by Drichlet , but later work was done by G. Cantor and Du Bois Raymond . During the middle of the nineteenth century , we have the investigation by Seidel and Stokes concerning uniform convergence ; this threw light on Weier -

strass' work in regard to the integration of series .

In conclusion of this historical outline, it may be added that no attention was given to the re-arrangement of the terms of an infinite series until a paper was published by Dirichlet in 1837 . This paper was the first recognition of the difference between absolutely convergent and conditionally convergent series when subjected to a re-arrangement of terms .

Before proceeding to the convergence or divergence of special series , we find it necessary to define convergent series and gave a survey of the tests for convergence . We cannot properly define convergent series until we define the convergent sequence . Let us think of an infinite sequence of quantities $a_1, a_2, a_3, \cdots\cdots a_n, \cdots$, where each term has a certain , definite , preassigned order . If a_n approaches a definite finite limit as n approaches infinity , the sequence is called convergent . The following general tests stated by Cauchy can be shown to be a necessary and sufficient condition for the convergence of a sequence ; corresponding to any preassigned number ε , a positive integer $n > m$ (agiven quantity) exists such that $| a_{n+p} - a_n | < \varepsilon$ for every positive integer p . Any sequence which is not convergent is considered divergent .

An infinite series is defined as the sum of the terms of an infinite sequence . If there exists an infinite sequence $a_1, a_2, a_3, \cdots\cdots\cdots, a_n, \cdots\cdots$, the series $a_1 + a_2 + a_3 + a_4 + \cdots\cdots + a_n + \cdots\cdots$ will be convergent provided the sequence $S_1, S_2, S_3, \cdots S_n \cdots$

is convergent . Where $S_1 = a_1$, $S_2 = a_1 + a_2$,

$S_3 = a_1 + a_2 + a_3, \ldots\ldots\ldots\ldots, S_n = a_1 + a_2 + \ldots + a_n + \ldots\ldots$

Then , if s equals the limit of the sequence

$$S_1, S_2, S_3, S_4, \ldots\ldots\ldots\ldots, S_n, \ldots\ldots\ldots,$$

$$S = \lim_{n \to \infty} S_n = \lim (a_1 + a_2 + a_3 + a_4 + \ldots + a_n \ldots).$$

Then , if a definite , finite limit s exists for the sequence $S_1, S_2, S_3, S_4, \ldots\ldots\ldots\ldots S_n, \ldots\ldots\ldots,$ the series $a_1 + a_2 + a_3 + \ldots\ldots\ldots + a_n + \ldots\ldots\ldots\ldots$ is convergent .

Cauchy's general test for the convergence of a sequence gives the following when applied to a series: corresponding to any preassigned positive number ε , an integer n should exist such that the sum of any number of terms beginning with a_{m+1} is less than ε , or such that the absolute value of $a_{m+1} + a_{m+2} + \ldots\ldots\ldots + a_{m+p}$ is less than ε, for the sequence $S_1, S_2, S_3, \ldots\ldots, S_n, \ldots$ is then convergent . The above condition may be written

$$|S_{m+p} - S_m| < \varepsilon . \quad n > m.$$

While this is the most general test for convergence , it is often difficult to obtain the limit of the sum of the series . For this reason , special tests have been worked out . The special tests for series of positive terms have been divided into several classes . The first class includes those tests which depend upon only one term of the series in question ; the second class , those tests which depend upon two terms of the given series ; the third class, those tests involving three or more terms of the given series .

In a series of positive terms the sum is continually increasing . Consequently , if the sum of the series becomes less than some finite quantity as the number of terms approaches infinity , the series is convergent . The most general test of the first kind is the test based upon the comparison of the given series with a series known to be convergent or divergent .

Given : a convergent series $c_1 + c_2 + c_3 + \ldots + c_n + \ldots$ whose sum approaches C , and a divergent series

$d_1 + d_2 + d_3 + d_4 + \ldots + d_n + \ldots$

To show that a series $a_1 + a_2 + a_3 + \ldots + a_n + \ldots$ is convergent if $\lim_{n \to \infty} \frac{a_n}{c_n} \leq K$ (K is not infinite) and divergent if $\lim. \frac{a_n}{d_n} \geq H$ ($H > 0$).

If $\lim_{n \to \infty} \frac{a_n}{c_n} < K$, we know that

$$a_m < K(c_m),$$
$$a_{m+1} < K(c_{m+1}),$$
$$\cdots\cdots\cdots\cdots$$
$$a_{m+p} < K(c_{m+p}).$$

Then , adding these inequalities , we have

$$(a_m + a_{m+1} + a_{m+2} + \cdots\cdots\cdots + a_{m+p}) < K(c_m + c_{m+1} + \cdots \; c_{m+p})$$

Since the series $\sum_{n=1}^{n=\infty} c_n$ has the limit C , we know that

$$c_m + c_{m+1} + c_{m+2} + \cdots\cdots + c_{m+p} < C \text{ (a finite quantity).}$$

If K is finite , $K(c_m + c_{m+1} + \cdots\cdots\cdots\cdots + c_{m+p})$ is finite and $a_m + a_{m+1} + a_{m+2} + \cdots\cdots\cdots\cdots + a_{m+p}$ is less than a finite quantity . Then ~~comma~~ , since the sum of all the terms after the mth. is less than a finite quantity , we know that $\sum_{n=1}^{n=\infty} a_n$ is a convergent series when $\lim_{n \to \infty} \frac{a_n}{c_n} < K$. (K is not infinite).

Suppose , however , that $\lim_{n \to \infty} \frac{a_n}{d_n} > H$, $H > 0$.

Then
$$a_{m+1} > H(d_{m+1}),$$
$$a_{m+2} > H(d_{m+2}),$$
$$a_{m+3} > H(d_{m+3}).$$
$$\cdots\cdots\cdots\cdots$$
$$a_{m+p} > H(d_{m+p}).$$

Adding these inequalities , we have

$$a_{m+1} + a_{m+2} + a_{m+3} + \cdots\cdots\cdots\cdots > H(d_{m+1} + d_{m+2} + \cdots\cdots)$$

Since $\sum_{n=1}^{n=\infty} d_n$ is a divergent series , we know that

$H\,(d_{m+1} + d_{m+2} + \ldots\ldots\ldots\ldots + d_{m+p} + \ldots\ldots$ is greater than any finite quantity if H is not 0 . Then , if the sum of all the terms after the mth. in the series $\sum_{n=1}^{n=\infty} a_n$ is infinite , the series $\sum_{n=1}^{n=\infty} a_n$ is divergent .

When we compare a given series $\sum_{n=1}^{n=\infty} a_n$ with the geometrical progression $a + ar + ar^2 + \ldots + ar^{n-1} \ldots\ldots$, we obtain Cauchy's test of the first kind , which may be stated as follows : Let the limit $\lim_{n \to \infty} \sqrt[n]{a_n} = K$. If K is less than 1 , the given series is convergent ; if K is greater than 1 , the given series is divergent . When K equals 1 , the given series may be convergent or divergent . When the logarithmic series is used for comparison with a given series , we obtain tests which are frequently convenient. The logarithmic series $\frac{1}{2(\lg.2)^p} + \frac{1}{3(\lg.3)^p} + \ldots\ldots + \frac{1}{n(\lg.n)^p} + \ldots + \frac{1}{(n+k)(\lg.(n+k))^p} + \ldots\ldots$ is divergent when $p \leq 1$, and is convergent when $p > 1$.

Tests of the second kind have been given us by D' Alembert , Raabe , and Kummer . If the ratio of two consecutive terms of a given series is compared with the ratio of two corresponding terms of a known series , we obtain D'Alembert's test : "A positive series $u_1 + u_2 + u_3 + u_4 + \ldots\ldots + u_n + \ldots$ is convergent if the limit $\frac{u_{n+1}}{u_n}$ is less than 1 for values of n greater than a finite quantity m . The series $\sum_{n=1}^{n=\infty} u_n$ is divergent if the lim. $\frac{u_{n+1}}{u_n}$ is greater than 1 where n > m . " If the lim. $\frac{u_{n+1}}{u_n}$ equals 1 for values of n > m , the series maybe convergent or divergent . In that case Raabe's test often works effectively . " If the $\lim_{n \to \infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) > 1$, $\sum_{n=1}^{n=\infty} u_n$ converges. If this limit is < 1, $\sum_{n=1}^{n=\infty} u_n$ diverges". If the test of Raabe gives 1 , we must seek other tests . This test maybe proved directly from Kummer's test , which is : " A positive series with a general term U_n is convergent if , for every value of n greater than m (a given finite quantity) , The

$$\lim_{n \to \infty} (u_n V_n - u_{n+1} V_{n+1}) > K\, u_{n+1},$$ or if $$\lim_{n \to \infty} (u_n V_n - u_{n+1} V_{n+1}) < 0$$ where $\sum_{n=1}^{n=\infty} \frac{1}{V_n}$ is a divergent series and K is any positive quantity > 0.

A positive series $\sum_{n=1}^{n=\infty} u_n$ is divergent if the

$$\lim_{n \to \infty} (u_n V_n - u_{n+1} V_{n+1}) < 0,$$ where $\sum_{n=1}^{n=\infty} \frac{1}{V_n}$ represents a divergent series .

A well known test of the third kind is Cauchy's condensation test . "If f(n) be positive for all values of n and constantly decrease as n increases , $\sum_{n=1}^{n=\infty} f(n)$ is convergent or divergent according as the series $\sum_{n=1}^{n=\infty} a^n f(a)^n$ is convergent or divergent where a is any positive integer not less than 2 ."

The special tests given above have all been given for series containing only positive terms . If we have a series with alternating signs , it may be shown to be convergent if the limit of the nth. term as n approaches infinity is zero , and if each term is numerically less than the preceding . The convergence of an alternating series will be absolute if the series formed from the absolute values of the terms of the given series is convergent. It can easily be shown that any series is convergent if the series formed of the absolute values of its terms is convergent .

Having reviewed some of the most common tests for convergence , we are now able to observe the effect of certain rearrangements of terms upon the convergence or divergence of special series . First we shall take the specific series given by Dirichlet , which can both be shown to be convergent : $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} \cdots\cdots$

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \cdots\cdots$$

I (a) Given the series $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots\cdots - \frac{1}{\sqrt{2k}} + \frac{1}{\sqrt{2k+1}} \cdots\cdots$

Since $\lim_{n \to \infty} \frac{1}{\sqrt{n}} = 0$, we see that the nth. term approaches zero. Since $\frac{1}{\sqrt{2k}} > \frac{1}{\sqrt{2k+1}} > \frac{1}{\sqrt{2k+2}}$, each term is numerically greater than the preceding . Therefore this series is convergent .

I. (b) Given the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{2k} + \frac{1}{2k+1} \cdots$

Since $\lim_{n \to \infty} \frac{1}{n} = 0$, the nth. term approaches zero . Since $\frac{1}{2k} > \frac{1}{2k+1} > \frac{1}{2k+2} > \ldots$, each term is numerically greater than the preceding , and this series is also convergent .

Re-arranging the terms of I (a) so that two positive terms are followed by a negative , we have II(a)

$$1 + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{7}} - \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{9}} + \frac{1}{\sqrt{11}} - \frac{1}{\sqrt{6}} \cdots\cdots$$

Since a convergent series remains convergent however its terms maybe enclosed in parentheses , we can write this series $\left(1 + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{5}} + \frac{1}{\sqrt{7}} - \frac{1}{\sqrt{4}}\right) + \left(\frac{1}{\sqrt{9}} + \frac{1}{\sqrt{11}} - \frac{1}{\sqrt{6}}\right)$

$+ \cdots\cdots + \left(\frac{1}{\sqrt{4n-3}} + \frac{1}{\sqrt{4n-1}} - \frac{1}{\sqrt{2n}}\right) \cdots\cdots$

We have here a series of positive terms, for $\frac{1}{\sqrt{4n-3}} + \frac{1}{\sqrt{4n-1}} > \frac{1}{\sqrt{2n}}$, since n must always be a positive integer . Let us test the series for divergence by comparison with the series

$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots\cdots + \frac{1}{n} + \cdots\cdots$

Then $\frac{a_n}{d_n}$ becomes $\frac{a_n}{\frac{1}{n}}$ or $n \cdot a_n$.

If the re-arranged series is divergent , $\lim_{n \to \infty} n \cdot a_n > h$ where $h > 0$.

$$\lim_{n\to\infty} n \cdot a_n = \lim_{n\to\infty} n\left(\frac{1}{\sqrt{4n-3}} + \frac{1}{\sqrt{4n-1}} - \frac{1}{\sqrt{2n}}\right)$$

$$\doteq \lim_{n\to\infty}\left(\frac{n}{\sqrt{4n-3}} + \frac{n}{\sqrt{4n-1}} - \frac{1}{\sqrt{2n}}\right).$$

$$= \lim_{n\to\infty}\left(\frac{n}{\sqrt{4n-3}} + \frac{n(\sqrt{2n} - \sqrt{4n-1})}{\sqrt{\frac{2}{n}(4n-1)}}\right)$$

$$= \lim_{n\to\infty}\left(\frac{1}{\sqrt{\frac{4}{n} - \frac{3}{n^2}}} + \frac{\sqrt{2n} - \sqrt{4n-1}}{\sqrt{\frac{2}{n}(4n-1)}}\right)$$

$$\doteq \lim_{n\to\infty}\left(\frac{1}{\sqrt{\frac{4}{n} - \frac{3}{n^2}}} + \frac{\sqrt{2} - \sqrt{4-\frac{1}{n}}}{\sqrt{\frac{2}{n}\left(4-\frac{1}{n}\right)}}\right)$$

$$= \infty + \infty \text{ or } \infty.$$

The series II (a) is a divergent series since $\lim_{n \to \infty} n . a_n$ is greater than any finite quantity h .

Re-arranging the terms of I (b) so that two positive terms are followed by a negative , we have II(b)

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \ldots\ldots, \text{or}$$

$$\left(1 + \frac{1}{3}\right) - \frac{1}{2} + \left(\frac{1}{5} + \frac{1}{7}\right) - \frac{1}{4} + \left(\frac{1}{9} + \frac{1}{11}\right) - \frac{1}{6} \ldots\ldots$$

$$\ldots\ldots\ldots - \frac{1}{n} + \frac{1}{2n+1} + \frac{1}{2n+3} - \frac{1}{n+2} \ldots\ldots,$$

an alternating series . It is clear that as n approaches infinity , the nth. term approaches zero . It remains to show each term less than the preceding .

$$\frac{1}{2n+1} + \frac{1}{2n+3} = \frac{4n+4}{4n^2+8n+3} .$$

$$= \frac{1}{n+1 - \frac{1}{4n+4}} .$$

n is $< \left(n+1 - \frac{1}{4n+4}\right)$, for $\frac{1}{4n+4}$ must always be a proper fraction. Then $\frac{1}{n} > \left(\frac{1}{2n+1} + \frac{1}{2n+3}\right)$.

$$\frac{1}{n+1-\frac{1}{2n+2}} > \frac{1}{n+2}$$ since the denominator , (n+1) minus a proper fraction , is surely less than the denominator n+2 .

$$\therefore \frac{1}{n} > \left(\frac{1}{2n+1} + \frac{1}{2n+3}\right) > \frac{1}{n+2} .$$

Then the series II (b) is convergent since the nth. term approaches zero as $n \to \infty$, and each term of the series is less than its preceding term .

We find then that , when the two conditionally convergent series , I(a) and I(b) , are subjected to the re-arrangement specified above , the de-ranged series II(a) is divergent , while the series II(b) is convergent.

Our given series I(a) and I(b) may be represented by the series III(c)

$$1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \frac{1}{5^p} \cdots\cdots - \frac{1}{(2k)^p} \cdots\cdots$$

The deranged series II(a) and II(b) may be represented by the series III(d)

$$1 + \frac{1}{3^p} - \frac{1}{2^p} + \frac{1}{5^p} + \frac{1}{7^p} - \frac{1}{4^p} \cdots\cdots\cdots ,$$

Let us investigate the deranged series III(d) to determine what values of p make the series convergent and what values make the series divergent . Series II(b) is what series

III(d) becomes when p equals 1 . Then we know that III(d) is convergent when p equals 1 .

First let p be > 1 . III(d) maybe written as the series

$$\left(1 + \frac{1}{3^p} - \frac{1}{2^p}\right) + \left(\frac{1}{5^p} + \frac{1}{7^p} - \frac{1}{4^p}\right) \cdots\cdots\cdots\cdots$$

$$\cdots\cdots + \left(\frac{1}{(4n-3)^p} + \frac{1}{(4n-1)^p} - \frac{1}{(2n)^p}\right) + \cdots\cdots\cdots\cdots$$

All the terms except the first of the series $\sum_{n=1}^{n=\infty}\left(\frac{1}{(4n-3)^p} + \frac{1}{(4n-1)^p} - \frac{1}{(2n)^p}\right)$ are negative . Consequently , let us test the convergence of the series

$$\sum_{n=1}^{n=\infty}\left(-\frac{1}{(4n-3)^p} - \frac{1}{(4n-1)^p} + \frac{1}{(2n)^p}\right),$$

since we know that a series is convergent if the series of its absolute values is convergent . $\sum_{n=1}^{n=\infty}\frac{1}{n^p}$ is known to be a convergent series when $p > 1$; then the series above is convergent if $\lim_{n\to\infty}\frac{a_n}{\frac{1}{n^p}} < h$ where $h \neq \infty$ and $p > 1$.

$$\lim_{n\to\infty}\frac{a_n}{\frac{1}{n^p}} = \lim_{n\to\infty} n^p\left[-\frac{1}{(4n-3)^p} - \frac{1}{(4n-1)^p} + \frac{1}{(2n)^p}\right]$$

$$= \lim_{n\to\infty}\left[-\frac{n^p}{(4n-3)^p} - \frac{n^p}{(4n-1)^p} + \frac{n^p}{(2n)^p}\right].$$

Since p may be regarded as having the same value in the numerator and denominator ,

$$\lim_{n \to \infty} \frac{a_n}{\frac{1}{n^p}} = \lim_{n \to \infty} \left[\frac{-1}{(4-\frac{3}{n})^p} - \frac{1}{(4-\frac{1}{n})^p} + \frac{1}{2^p} \right].$$

$$= -\frac{1}{4^p} - \frac{1}{4^p} + \frac{1}{2^p}.$$

$$= -\frac{2}{4^p} + \frac{1}{2^p}.$$

$$= \frac{1}{2^p} - \frac{1}{2^{2p-1}}.$$

Since in our given comparison series $\sum_{n=1}^{n=\infty} \frac{1}{n^p}$, p is greater than 1 , we know that $\lim_{n \to \infty} \frac{a_n}{c_n}$ is a positive proper fraction , for $\frac{1}{2^p}$ must be $> \frac{1}{2^{2p-1}}$ for values of $p > 1$. Then the series III(d) is convergent when $p \geq 1$.

Let the series III(d) be tested to determine what values of p make it divergent . $\sum_{n=1}^{n=\infty} \frac{1}{n}$ is known to be a divergent series ; then a series $\sum_{n=1}^{n=\infty} a_n$ is diver - gent when $\lim_{n \to \infty} \frac{a_n}{\frac{1}{n}} > h$ where $h > 0$.

$$\lim_{n \to \infty} \frac{a_n}{\frac{1}{n}} = \lim_{n \to \infty} n \left[\frac{1}{(4n-3)^p} + \frac{1}{(4n-1)^p} - \frac{1}{(2n)^p} \right].$$

$$= \lim_{n \to \infty} \left[\frac{n}{(4n-3)^p} + \frac{n[(2n)^p - (4n-1)^p]}{[2n(4n-1)]^p} \right]$$

$$= \lim_{n \to \infty} \left[\frac{1}{\left(\frac{4}{n^{p-1}} - \frac{3}{n^p}\right)^p} + \frac{2^p - (4-\frac{1}{n})^p}{\left(\frac{8}{n^{p-1}} - \frac{2}{n^p}\right)^p} \right]$$

$$\lim_{n\to\infty} \frac{a_n}{\frac{1}{n^p}} = \lim_{n\to\infty} \left[\frac{1}{\left(\frac{4}{n^{\frac{1}{p}-1}} - \frac{3}{n^{\frac{1}{p}}}\right)^p} + \frac{2^p - \left(4 - \frac{1}{n}\right)^p}{\left(\frac{8}{n^{\frac{1}{p}-1}} - \frac{2}{n^{\frac{1}{p}}}\right)^p} \right].$$

For all values of p less than 1 and greater than 0 , $n^{\frac{1}{p}-1}$ is a quantity with a positive exponent . $\therefore \frac{4}{n^{\frac{1}{p}-1}}$ and $\frac{8}{n^{\frac{1}{p}-1}}$ are each equal to 0 as n approaches infinity . For values of $p < 1$ and > 0

$$\lim_{n\to\infty} \frac{a_n}{\frac{1}{n}} = \frac{1}{0-0} + \frac{2^p - 4^p}{0}$$
$$= \infty + \infty .$$

Then series III(d) is divergent for all values of p less than 1 but greater than 0 . It maybe noted that the series

$$1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \frac{1}{5^p} \cdots\cdots\cdots$$

is absolutely convergent for all values of $p > 1$ and conditionally convergent for all values of p less than or equal to 1 but > 0 . The re-arranged series

$$1 + \frac{1}{3^p} - \frac{1}{2^p} + \frac{1}{5^p} + \frac{1}{7^p} - \frac{1}{4^p} \cdots\cdots\cdots$$

is convergent for values of $p \geq 1$ and divergent for $0 < p < 1$.

Let the terms of the series I(a)

$$1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} \cdots\cdots\cdots$$

and the series I(b)

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\cdots\cdots$$

be re-arranged so that we have a positive quantity followed by a negative , two positives followed by a negative , three positives followed by a negative , and so on . We shall see that each series with this re-arrangement be - comes divergent .

IV(a) which is derived from I(a) may be written

$$\left(1-\frac{1}{\sqrt{2}}\right)+\left(\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{5}}-\frac{1}{\sqrt{4}}\right)+\left(\frac{1}{\sqrt{7}}+\frac{1}{\sqrt{9}}+\frac{1}{\sqrt{11}}-\frac{1}{\sqrt{6}}\right)$$

$$+\ \cdots\cdots\ +\left(\frac{1}{\sqrt{n^2-n+1}}+\frac{1}{\sqrt{1+n(n-1)+2}}+\cdots+\frac{1}{\sqrt{1+n(n-1)+2(n-1)}}-\frac{1}{\sqrt{2n}}\right)+\cdots$$

The nth. term may be simplified to the form

$$\frac{1}{\sqrt{n^2-n+1}}+\frac{1}{\sqrt{n^2-n+3}}+\frac{1}{\sqrt{n^2-n+5}}+\cdots+\frac{1}{\sqrt{n^2+n-1}}-\frac{1}{\sqrt{2n}}.$$

IV(a) as grouped above is a series of positive terms . This can be seen by noticing the nth. term , which is positive if $\frac{n}{\sqrt{n^2+n-1}}>\frac{1}{\sqrt{2n}}$, or $\frac{1}{\sqrt{1+\frac{1}{n}-\frac{1}{n^2}}}>\frac{1}{\sqrt{2n}}$. Since n is always a positive integer, $\frac{1}{n}$ and $\frac{1}{n^2}$ are proper fractions and $\frac{1}{n}$ is never less than $\frac{1}{n^2}$. Then $\sqrt{1+\frac{1}{n}-\frac{1}{n^2}}$ is less than $\sqrt{2n}$. Consequently, $\frac{1}{\sqrt{1+\frac{1}{n}-\frac{1}{n^2}}}>\frac{1}{\sqrt{2n}}$ and IV(a) is a series of positive terms . Let this series be compared with the divergent series $\sum_{n=1}^{n=\infty}\frac{1}{n}$.

$$\lim_{n\to\infty} n.\,a_n = n\left[\frac{1}{\sqrt{n^2-n+1}}+\frac{1}{\sqrt{n^2-n+3}}+\cdots+\frac{1}{\sqrt{n^2+n-1}}-\frac{1}{\sqrt{2n}}\right].$$

The limit $\lim\limits_{n\to\infty} n.\,a_n$ will be greater than h $(h>0)$

If $\lim\limits_{n\to\infty} n.\,a'_n>h$ where $a'_n=\frac{n}{\sqrt{n^2+n+1}}-\frac{n}{\sqrt{2n}}$.

$$\lim_{n \to \infty} n \cdot a'_n = \lim_{n \to \infty} \left(\frac{n^2}{\sqrt{n^2+n+1}} - \frac{n}{\sqrt{2n}} \right)$$

$$= \lim_{n \to \infty} \left(\frac{n}{\sqrt{1+\frac{1}{n}+\frac{1}{n^2}}} - \frac{\sqrt{n}}{\sqrt{2}} \right)$$

$$= \lim_{n \to \infty} \left(\frac{n\sqrt{2} - \sqrt{n+1+\frac{1}{n}}}{\sqrt{2}\sqrt{1+\frac{1}{n}+\frac{1}{n^2}}} \right)$$

$$= \lim_{n \to \infty} \left(\frac{\sqrt{2} - \sqrt{\frac{1}{n}+\frac{1}{n^2}+\frac{1}{n^3}}}{\frac{\sqrt{2}}{n}\sqrt{1+\frac{1}{n}+\frac{1}{n^2}}} \right)$$

$$= \frac{\sqrt{2} - 0}{0} = \infty.$$

Then the series IV(a) is divergent .

IV(b) is the series

$$1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{5} - \frac{1}{4} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} \cdots\cdots$$

This series may be grouped so that we have for IV(b)

$$\left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{5} - \frac{1}{4}\right) + \cdots\cdots$$

$$+ \left(\frac{1}{1+n(n-1)} + \frac{1}{1+n(n-1)+2} + \cdots\cdots + \frac{1}{1+n(n-1)+2(n-1)} - \frac{1}{2n}\right) \cdots\cdots$$

The nth. term when simplified becomes

$$\frac{1}{n^2-n+1} + \frac{1}{n^2-n+3} + \cdots\cdots\cdots + \frac{1}{n^2+n-1} - \frac{1}{2n}.$$

The series IV(b) as grouped above is a series of positive terms . The nth. term maybe seen to be positive by a method similar to that used for IV(a) .

Let the series IV(b) be compared with the divergent series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots\cdots + \frac{1}{n} + \cdots$$

$$\lim_{n\to\infty} n.\,a_n = \lim_{n\to\infty} n\left(\frac{1}{n^2-n+1} + \frac{1}{n^2-n+3} + \cdots + \frac{1}{n^2+n-1} - \frac{1}{2n}\right)$$

But IV(b) will be divergent if

$$\lim_{n\to\infty} n\left(\frac{n}{n^2+n+1} - \frac{1}{2n}\right) > H, \text{ a quantity } > 0.$$

$$\lim_{n\to\infty} n\left(\frac{n}{n^2+n+1} - \frac{1}{2n}\right) = \lim_{n\to\infty}\left(\frac{n^2}{n^2+n+1} - \frac{1}{2}\right).$$

$$= \lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}+\frac{1}{n^2}} - \frac{1}{2}\right).$$

$$= 1 - \frac{1}{2} \text{ or } \frac{1}{2}.$$

Therefore the series IV(b) is divergent .

From the re-arrangements of the terms of I(a) and I(b) we see that the terms of conditionally convergent series maybe re-arranged to produce either convergent or divergent series . But the re-arrangement of the terms of absolutely convergent series gave series which were absolutely convergent . These results will be found to agree with the general laws for re-arrangement which will be stated in another section of this paper .

It has been noted that a certain re-arrangement of terms , when applied to a conditionally convergent series , may produce a series which is divergent ; and , when applied to another conditionally convergent series , it may produce a convergent series . Let us study a type of re-arrangement which , when applied to any convergent series , produces a convergent series . As we shall see , these transformed series are often more valuable for purposes of computation than the original series. Let the series A, $a_1 + a_2 + a_3 + a_4 + \cdots\cdots + a_n + \cdots$ be a convergent series . A study of the two series , (a) and (b) , is given below .

(a) $\frac{a_1}{2} + \frac{a_1 + a_2}{2} + \frac{a_2 + a_3}{2} + \cdots\cdots + \frac{a_{n-1} + a_n}{2} \cdots$
is a series in which the order of each term is changed by only half a term from its order in series A .

(b) $\frac{2a_1 + a_2}{3} + \frac{a_1 + a_2 + a_3}{3} + \frac{a_2 + a_3 + a_4}{3} \cdots + \cdots + \frac{a_{n-1} + a_n + a_{n+1}}{3} \cdots\cdots$

The sum of the series A , (a) and (b) may be shown to approach the same limit in the following way :

Let $S_n = a_1 + a_2 + a_3 + \cdots\cdots\cdots + a_n$.

$T_n = \frac{a_1}{2} + \frac{a_1 + a_2}{2} + \cdots\cdots\cdots + \frac{a_{n-1} + a_n}{2}$.

Let $U_n = \frac{2a_1+a_2}{3} + \frac{a_1+a_2+a_3}{3} + a_2 + a_3 + \frac{a_{n-1}+a_n+a_{n+1}}{3}$

(1). $T_n = \frac{a_1}{2} + \frac{a_1}{2} + \frac{a_2}{2} + \frac{a_2}{2} + \frac{a_3}{2} + \cdots + \frac{a_{n-1}}{2} + \frac{a_{n-1}}{2} + \frac{a_n}{2}.$

$T_n = a_1 + a_2 + a_3 + \cdots + a_{n-1} + \frac{a_n}{2}.$

Then $S_n = T_n + \frac{a_n}{2}.$

But a_n approaches 0 as n approaches infinity .

Then $\lim_{n\to\infty} S_n = \lim_{n\to\infty} T_n.$

Therefore , A and (a) have the same limit .

(2) $U_n = \frac{2a_1+a_2}{3} + \frac{a_2}{3} + \frac{a_2}{3} + \frac{a_3}{3} + \frac{a_3}{3} + \frac{a_3}{3} + \frac{a_3}{3} + \cdots$

$\cdots + \frac{a_{n-1}}{3} + \frac{a_{n-1}}{3} + \frac{a_{n-1}}{3} + \frac{2a_n}{3} + \frac{a_{n+1}}{3}$

$= a_1 + a_2 + a_3 + \cdots + a_{n-1} + \frac{2a_n}{3} + \frac{a_{n+1}}{3}.$

Then $S_n = U_n + \frac{a_n}{3} - \frac{a_{n+1}}{3}.$

$\frac{a_n}{3}$ and $\frac{a_{n+1}}{3}$ each approach 0 as n approaches infinity. Then $\lim_{n\to\infty} S_n = \lim_{n\to\infty} U_n$. Consequently A , (b) and (a) all have the same limit .

It may be noticed that the nth. term of the T_n series differs from the nth. term of the S_n series by the amount $\frac{a_n}{2}$, while the nth. term of the U_n series differs from the nth. term of the S_n series by the amount

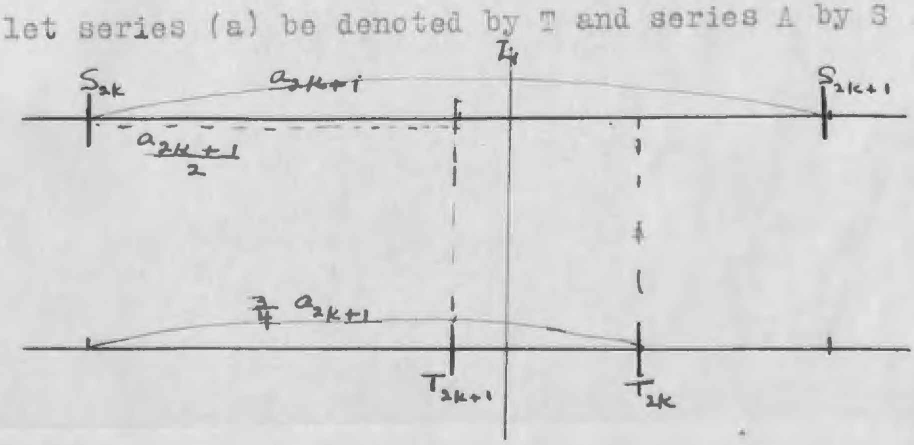
$\frac{a_n}{3} - \frac{a_{n+1}}{3}$. The series (a) and (b) are of practical value for computation when their individual terms approach 0 more rapidly than the individual terms of the series A. This , we can show , will depend upon the signs of the terms of series A . Series (a) and (b) are more effective than A when A is an alternating series .

If the given series is an alternating convergent series , fewer terms of series (a) than of A are required to obtain the sum of the given series within the same degree of error whenever $a_{2k} < \frac{3}{2} a_{2k+1}$.

Series A becomes $a_1 - a_2 + a_3 - a_4 \ldots\ldots\ldots\ldots$

Series (a) becomes $\frac{a_1}{2} + \frac{a_1 - a_2}{2} - \frac{a_2 - a_3}{2} + \cdots + \frac{a_{2k-1} - a_{2k}}{2} - \cdots$

A sufficient condition for the conclusions given above maybe obtained by the aid of a geometrical representation. Here let series (a) be denoted by T and series A by S .



Since $S_n = T_n + \frac{a_n}{2}$,

$$S_{2k+1} = T_{2k+1} + \frac{a_{2k+1}}{2} \text{ or}$$

$$T_{2k+1} = S_{2k+1} - \frac{a_{2k+1}}{2}.$$

Since $S_{2k} = T_{2k} - \frac{a_{2k}}{2}$,

$$T_{2k} = S_{2k} + \frac{a_{2k}}{2}.$$

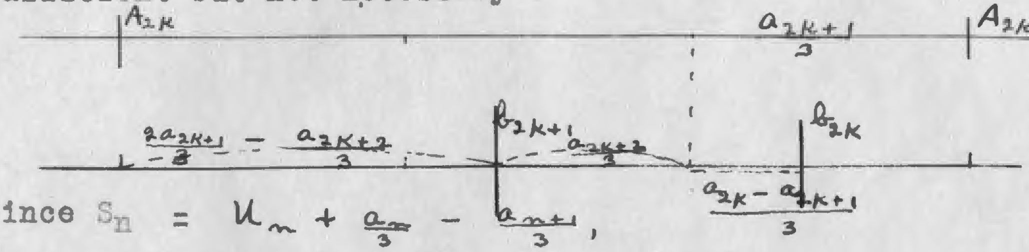
By means of the theorem : "If S denoted the sum of alternating series , $a_1 - a_2 + a_3 - a_4 + a_5 - a_6 \dots\dots\dots$ the sums, a_1, $a_1 - a_2$, $a_1 - a_2 + a_3$, $\dots\dots\dots$, are alternately greater than and less than the true sum S ," we know the limit L must be between S_{2k} and S_{2k+1}, and it must also lie between T_{2k} and T_{2k+1}. Then if ~~T~~ T_{2k} lies to the left of S_{2k+1}, the limit L will be nearer T_{2k} than S_{2k}. To make T_{2k} fall to the left of S_{2k+1}, $\frac{a_{2k}}{2}$ must be less than a_{2k+1}. This is a sufficient condition that T_{2k} is nearer L than S_{2k} is. But to have T_{2k+1} nearer L than S_{2k+1}, $\frac{a_{2k}}{2}$ must be $\leq \frac{3}{4} a_{2k+1}$ Then a sufficient condition that fewer terms of T than of S are required to obtain the sum of the given series within the same degree of error is : $\frac{a_{2k}}{2} \leq \frac{3}{4} a_{2k+1}$ or $a_{2k} \leq \frac{3}{2} a_{2k+1}$.

Let A be the series $a_1 - a_2 + a_3 - a_4 \ldots\ldots$

and b the series

$$\frac{2a_1 - a_2}{3} + \frac{a_1 - a_2 + a_3}{3} - \frac{a_2 - a_3 + a_4}{3} \ldots\ldots\ldots$$

Fewer terms of b than of A give the sum of the given series within the same degree of error whenever $2a_{2k}$ is $<$ $(3a_{2k+1} - a_{2k+2})$. This condition, as we shall see is sufficient but not necessary.



Since $S_n = U_n + \frac{a_n}{3} - \frac{a_{n+1}}{3}$,

$$A_{2k} = b_{2k} - \frac{a_{2k}}{3} - \frac{a_{2k+1}}{3}.$$

$$b_{2k} = A_{2k} + \frac{a_{2k}}{3} + \frac{a_{2k+1}}{3}.$$

b_{2k+1} will always fall to the right of $A_{2k} + \frac{2a_{2k+1}}{3}$ and $A_{2k} + \frac{2a_{2k+1}}{3}$

Since $A_{2k+1} = b_{2k+1} + \frac{a_{2k+1}}{3} + \frac{a_{2k+2}}{3}$,

$$b_{2k+1} = A_{2k+1} - \frac{a_{2k+1}}{3} - \frac{a_{2k+2}}{3}.$$

b_{2k+1} will always fall between $A_{2k} + \frac{a_{2k+1}}{3}$ and $A_{2k} + \frac{2a_{2k+1}}{3}$

The limit L falls nearer b_{2k+1} than A_{2k+1} whenever

$$\frac{2a_{2k+1} - a_{2k+2} + a_{2k}}{3} > \frac{a_{2k+2}}{3} + \frac{a_{2k+1}}{3} + \frac{a_{2k}}{3} = a_{2k} < (3a_{2k+1} - 2a_{2k+2}).$$

The limit L falls nearer b_{2k+1} than A_{2k+1} whenever

$$\frac{a_{2k+2} - a_{2k+1} + a_{2k}}{3} \leq \frac{1}{2} \frac{a_{2k+2} + a_{2k+1}}{3}$$

or $2a_{2k} < (3a_{2k+1} - a_{2k+2})$

If this condition exists , we can see that the condition $a_{2k} < (3a_{2k\ 1} - 2a_{2k\ 2})$ must exist .

Let the preceding statements be verified means of the conditionally convergent series A'

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot - \frac{1}{n} + \frac{1}{n+1} \cdot \cdot \cdot \cdot$$

The general series $\frac{a_1}{2} + \frac{a_1 + a_2}{2} + \frac{a_2 + a_3}{2} + \cdot \cdot \cdot \cdot \cdot \cdot \cdot$

becomes a', $\frac{1}{2} + \frac{1}{4} - \frac{1}{12} + \frac{1}{24} - \frac{1}{40} + \frac{1}{60} - \frac{1}{84} + \frac{1}{128}$

$$- \frac{1}{144} + \frac{1}{180} - \frac{1}{242} + \frac{1}{264} - \frac{1}{312} \cdot \cdot \cdot \cdot$$

Since we know that the sum of a finite number of terms of an alternating convergent series differs from the limit of the sum of the series by an amount less than the first term omitted , the error is less than 1/10 if we take for the sum of our series the sum of the first two terms since the first term omitted is less than 1/10 . The value of a is computed correct within the quantity 1/100 if we take the sum of the first seven terms since 1/128 is the 1st term omitted ; the value of a' is computed correct within the quantity 1/1000 if we take the sum of the first 22 terms since the 23rd term is $\frac{1}{2(506)}$.

The general series b, $\frac{2a_1 + a_2}{3} + \frac{a_1 + a_2 + a_3}{3} + \frac{a_2 + a_3 + a_4}{3} + \ldots\ldots$ when applied to A' becomes b'

$$\frac{1}{2} + \frac{5}{18} - \frac{15}{36} + \frac{17}{180} - \frac{13}{180} + \frac{37}{630} \ldots\ldots\ldots\ldots$$

The value of b' is computed correct within the quantity 1/10 if we take the sum of the first three terms , for the fourth term is $\frac{17}{180}$. The value of b' is computed correct within 1/100 if we take the sum of the first thirtythree terms . The thirtythird term is $\frac{-\frac{1}{32} + \frac{1}{33} - \frac{1}{34}}{3}$ or .0101 (numerically >.01)

The 34th. term is $\frac{\frac{1}{33} - \frac{1}{34} + \frac{1}{35}}{3}$ or .0091 (numerically <.01)

The value of b' may be computed correct within the quantity 1/1000 if we take the sum of the first 333 terms . For the 333rd. term is $\frac{-\frac{1}{332} + \frac{1}{333} - \frac{1}{334}}{3}$ or .001007, which is numerically >.001 . The 334th. term is

$$\frac{1}{333} - \frac{1}{334} + \frac{1}{335} \text{ or } .000992 \text{ (numerically } < .001\text{)}.$$

The value of series A' may be computed with an error less than 1/10 if we take the sum of the first nine terms , with an error less than 1/100 if we take the sum of the first 99 terms , with an error less than 1/1000 if we take the sum of the first 999 terms . So we see that the sum of the given series can be computed within the same degree of accuracy by means of the fewest number of of accuracy by means of the fewest number of terms if we use a' , but a fewer number of terms is required when we use b' than when we use A' .

We have seen that series a , b and A all approach the same limit . Looking at the particular series a' , b' and A' we can see that the limits lie in the same region .

S	for	series	A'	lies	between	.745 and .645
S	"	"	a'	"	"	.6943 and .693
S	"	"	b'	"	"	.706 and .693

Let us also verify the general statements :

(1) that $a_{2k} < \frac{3}{2} a_{2k+1}$ is a sufficient condition for the case when series a' is more useful than A' in obtaining an approximate value of our given series

(2) that $a_{2k} < 3a_{2k+1} - 2a_{2k+2}$ is sufficient to make T_{2k} nearer U_{2k} than S_{2k}.

Let $|a_{2k}| = a_n = \frac{1}{n}$

$$a_{2k+1} = a_{n+1} = \frac{1}{n+1}$$

If $a_{2k} \leq \frac{3}{2}\, a_{2k+1}$

$$\frac{1}{n} \leq \frac{3}{2(n+1)}.$$

$2(n+1) \leq 3n$ for all values of $n > 1$.

(2). If $a_{2k} \leq |3a_{2k+1} - 2a_{2k+2}|$,

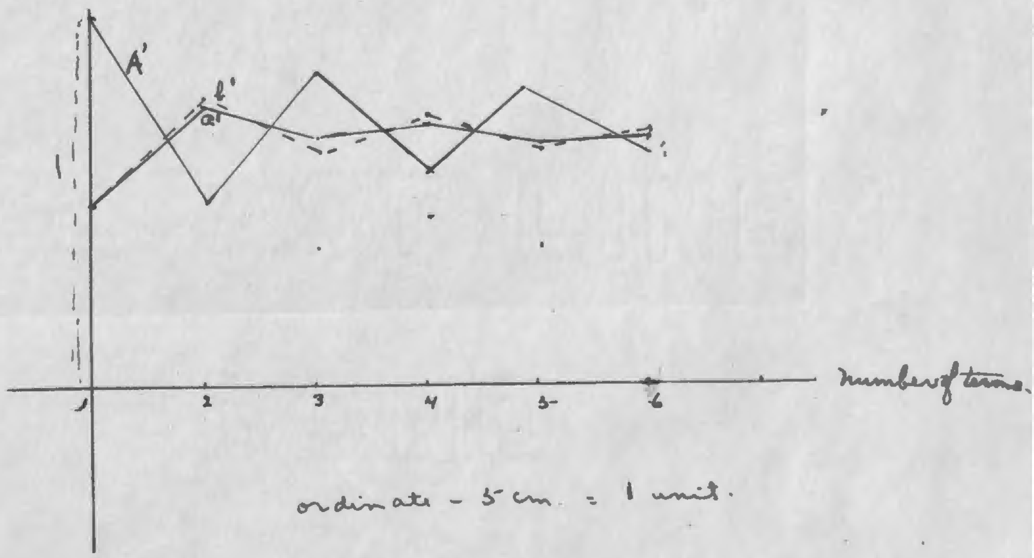
$$\frac{1}{n} \leq \frac{3}{n+1} + \frac{2}{n+2}$$

$(n+1)(n+2) < (4n^2 + 4n)$ or

$n^2 + 3n + 2 < 4n^2 + 4n$, for

$n^2 + 3n < 4n^2 + 4n - 2$ for all of $n \geq 2$.

It is interesting to note that while series A and b are conditionally convergent , series a is absolutely convergent . The graphs given below where the sums are plotted against the number of terms bring out the difference of the sums of the three series as the number of terms is increased .



ordinate - 5 cm. = 1 unit.

Let us test series a' for absolute convergence.

It may be written : $\frac{1}{2} + \frac{1-\frac{1}{2}}{2} + \frac{-\frac{1}{2}+\frac{1}{3}}{2} + \frac{\frac{1}{3}-\frac{1}{4}}{2} \dots\dots$

or $\frac{1}{2} + \frac{1-\frac{1}{2}}{2} - \frac{\frac{1}{2}-\frac{1}{3}}{2} + \frac{\frac{1}{3}-\frac{1}{4}}{2} \dots\dots\dots\dots$

The absolute value of the nth term is $\frac{1}{2(n-1)} - \frac{1}{2n}$ or $\frac{1}{2n(n-1)}$.

The " " " " n+1th " " $\frac{1}{2n} - \frac{1}{2(n+1)}$ " $\frac{1}{2n(n+1)}$.

Applying D'Alembert's test we have

$$\lim_{n \to \infty} \frac{\frac{1}{2n(n+1)}}{\frac{1}{2n(n-1)}} = 1$$

Applying Raabe's test we have

$$\lim_{n \to \infty} n\left(\frac{\frac{1}{2n(n-1)}}{\frac{1}{2n(n+1)}} - 1\right) = \lim_{n \to \infty} n\left(\frac{n+1}{n-1} - 1\right)$$

$$= \lim_{n \to \infty} \frac{2n}{n-1}$$

$$= \lim_{n \to \infty} \frac{2}{1-\frac{1}{n}} \text{ or } 2.$$

Therefore series a' is absolutely convergent .

Furthermore , it may be shown that if the series A' $a_1 - a_2 + a_3 - a_4 + a_5 \dots\dots\dots\dots$ is a convergent series , the series a'

$$\frac{a_1}{2} + \frac{a_1 - a_2}{2} - \frac{a_2 - a_3}{2} + \frac{a_3 - a_4}{2} - \frac{a_4 - a_5}{2} \dots\dots\dots$$

is absolutely convergent . The series of absolute values of ~~small~~ a' is written:

$$\frac{a_1}{2} + \frac{a_1 - a_2}{2} + \frac{a_2 - a_3}{2} + \text{-------------}$$

or we have

$$\frac{a_1}{2} + \frac{a_1}{2} - \frac{a_2}{2} + \frac{a_2}{2} - \frac{a_3}{2} + \frac{a_3}{2} \cdots\cdots - \frac{a_{n-1}}{2} + \frac{a_{n-1}}{2} - \frac{a_n}{2} \cdots$$

a telescopic series , which has for the sum of n terms the quantity , $a_1 - \frac{a_n}{2}$. The limit of this series is the $\lim_{n \to \infty}$ of $a_1 - \frac{a_n}{2}$, which is a_1 , for the nth term approaches 0 as n approaches infinity .

But series b, $\frac{2a_1 - a_2}{3} + \frac{a_1 - a_2 + a_3}{3} - \frac{a_2 - a_3 + a_4}{3} \cdots$ can never be absolutely convergent unless A, $a_1 - a_2 + a_3 \cdots$ .... is absolutely convergent . For the series of absolute values of b is

$$\frac{2a_1 - a_2}{3} + \frac{a_1 - a_2 + a_3}{3} + \frac{a_2 - a_3 + a_4}{3} + \frac{a_3 - a_4 + a_5}{3}$$

$$+ \frac{a_4 - a_5 + a_6}{3} + \frac{a_5 - a_6 + a_7}{3} + \frac{a_6 - a_7 + a_8}{3} + \frac{a_7 - a_8 + a_9}{3}$$

$$+ \frac{a_8 - a_9 + a_{10}}{3} + \frac{a_9 - a_{10} + a_{11}}{3} + \cdots\cdots$$

or $$\frac{2a_1}{3} - \frac{a_2}{3} + \frac{a_1}{3} + \left(-\frac{a_2}{3} + \frac{a_3}{2}\right) + \left(\frac{a_2}{3} - \frac{a_3}{3}\right) + \frac{a_4}{3} + \frac{a_3}{3}$$

$$+ \left(-\frac{a_4}{3} + \frac{a_5}{3}\right) + \left(\frac{a_4}{3} - \frac{a_5}{3}\right) + \frac{a_6}{3} + \frac{a_5}{3} + \left(-\frac{a_6}{3} + \frac{a_7}{3}\right)$$

$$+ \left(\frac{a_6}{3} - \frac{a_7}{3}\right) + \frac{a_8}{7} + \frac{a_7}{3} \cdots\cdots$$

or $$\frac{2a_1}{3} - \frac{a_2}{3} + \frac{a_1}{3} + \frac{a_4}{3} + \frac{a_3}{3} + \frac{a_6}{3} + \frac{a_5}{3} + \cdots$$

or $$a_1 - \frac{a_2}{3} + \frac{1}{3}\left(a_4 + a_3 + a_6 + a_5 \cdots\cdots\right),$$

a series which is convergent only when the series $a_1 - a_2 + a_3 - a_4 \ldots$ is absolutely convergent.

We find that series b', $\frac{2-\frac{1}{2}}{3} + \frac{1-\frac{1}{2}+\frac{1}{3}}{3} - \frac{\frac{1}{2}-\frac{1}{3}+\frac{1}{4}}{3} \ldots\ldots$ verifies these conclusions . Since it is derived from a series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots\ldots\ldots\ldots\ldots\ldots$ which is conditionally convergent , the series of its absolute values should be divergent . Let us test this series $\frac{2-\frac{1}{2}}{3} + \frac{1-\frac{1}{2}+\frac{1}{3}}{3} + \frac{\frac{1}{2}-\frac{1}{3}+\frac{1}{4}}{3} + \ldots\ldots\ldots$ for divergence by comparison with the divergent series $\sum_{n=1}^{n=\infty} \frac{1}{n}$.

$$\lim_{n\to\infty} \frac{a_n}{d_n} = \lim_{n\to\infty} \frac{\frac{1}{3(n-1)} - \frac{1}{3n} + \frac{1}{3(n+1)}}{\frac{1}{n}}.$$

$$= \lim_{n\to\infty} \frac{n(n^2+2n+2)}{3n(n-1)(n+1)}.$$

$$= \lim_{n\to\infty} \frac{n^2+2n+2}{3(n^2-1)}.$$

$$= \lim_{n\to\infty} \frac{1+\frac{2}{n}+\frac{2}{n^2}}{3\left(1-\frac{1}{n^2}\right)} = \frac{1}{3}, > \ell > 0.$$

Therefore series b is conditionally convergent as has been stated .

Having studied some special re-arrangements of terms of infinite series , let us look at some more general considerations . That such a study of the laws in regard to re-arrangement of terms of infinite series is necessary , may be seen by the many errors in the work of prominent mathematicians who disregarded these laws . Let us examine carefully some typical examples taken from the first volume of the "Solutions of Cambridge Problems" published by I . M . F. Wright in 1825 .

Given : $\frac{1}{1} + \frac{1}{2^2} + \frac{1}{3^2} \cdots\cdots\cdots\cdots = \frac{\pi^2}{6}$.

To find the sum of the series

$$\frac{1}{1^2 . 2 . 3} + \frac{1}{2^2 . 3 . 4} + \cdots\cdots\cdots + \frac{1}{n^2(n+1)(n+2)} + \cdots\cdots,$$

which is seen to be absolutely convergent . The nth term, is

$$\frac{1}{n^2(n+1)(n+2)} = \frac{1}{2n^2} - \frac{3}{4n} + \frac{3}{4(n+1)} + \frac{1}{4(n+1)(n+2)}.$$

Then the author writes

$$" S = \frac{1}{2}\left(\frac{1}{1} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\cdots \infty\right) - \frac{3}{4}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots\cdots \infty\right) + \frac{3}{4}\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} \cdots\cdots \infty\right) + \frac{1}{4}\left(\frac{1}{2.3} + \frac{1}{3.4} + \cdots\cdots \infty\right).$$

$$= \frac{1}{2} \cdot \frac{\pi^2}{6} - \frac{3}{4} + \frac{1}{4}\left(\frac{1}{2.3} + \frac{1}{3.4} + \cdots\cdots\right).$$

Let $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots\cdots\cdots \infty = S$.

Then $\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots\cdots\cdots \infty = S - \frac{1}{2}$.

$\therefore \frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5} + \cdots\cdots \infty = \frac{1}{2}$, by subtraction. Hence then, we finally obtain,

$$S = \frac{\pi^2}{12} - \frac{3}{4} + \frac{1}{8} = \frac{\pi^2}{12} - \frac{5}{8}."$$

Mr. Wright's procedure consisted first in writing for each term of the given series the algebraic sum of four fractions . This gives the series C

$$\left(\frac{1}{2} - \frac{3}{4} + \frac{3}{8} + \frac{1}{4.2.3}\right) + \left(\frac{1}{2.2^2} - \frac{3}{8} + \frac{1}{4} + \frac{1}{4.3.4}\right)$$

$$+\left(\frac{1}{2.3^2} - \frac{1}{4} + \frac{3}{16} + \frac{1}{4.4.5}\right) + \cdots\cdots\cdots$$

$$+\left(\frac{1}{2n^2} - \frac{3}{4n} + \frac{3}{4(n+1)} + \frac{1}{4(n+1)(n+2)}\right) + \cdots\cdots$$

which is equivalent to the given series provided the grouping given here is retained . If we change the grouping so as to obtain an alternating series , we have

$$\frac{1}{2} - \frac{3}{4} + \frac{13}{24} - \frac{3}{8} + \frac{47}{144} - \frac{1}{4} + \frac{25}{120} - \frac{3}{16} \cdots$$

which is not absolutely convergent, since our series may be written $\frac{1}{2} - \frac{3}{4}(1) + \frac{13}{24} - \frac{3}{4}\left(\frac{1}{2}\right) + \frac{47}{144} - \frac{3}{4}\left(\frac{1}{3}\right) + \frac{25}{120} - \frac{3}{4}\left(\frac{1}{4}\right) \cdots\cdots$

To test for absolute convergence make the negative terms positive . Then we have a positive quantity preceding each term of a divergent series $\frac{3}{4}(1+\frac{1}{2}+\frac{1}{3}+\cdots\cdots)$ and our series is necessarily divergent . Since the series above is not absolutely convergent , the author has no right to re-arrange its terms into the four infinite series , for , as we shall see later , it is only an absolutely convergent series which may be subjected to this re-arrangement and retain the same limit . He has re-arranged the terms of C so as to produce four series , the first of which is formed by the first terms of the successive groups of C, the second of which is formed by the second terms of the groups of C, and so on . In addition to this error due to his disregard for the laws of re-arrangement , we find that the author has assumed

$$-\frac{3}{4}\left(1+\frac{1}{2}+\frac{1}{3}\cdots\cdots\infty-\frac{1}{2}-\frac{1}{3}-\frac{1}{4}\cdots\cdots\infty\right)$$

is equal to - 3/4 . The correct result is $\infty-\infty$, which is indeterminate . He has committed the same sort of error when he subtracts from $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots\infty=S$ the series $\frac{1}{3}+\frac{1}{4}+\frac{1}{5}\cdots\cdots\infty=S-\frac{1}{2}$, and writes 1/2 for the result , since in so doing he has assumed S to be a finite quantity .

Let us look at another example , where we have a series which is known to be conditionally convergent . The author writes "$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \cdots\cdots$", (which is correct.).

$$\text{"}\log 2 = \frac{1}{1.2} + \frac{1}{3} - \frac{1}{4.5} - \frac{1}{6} + \frac{1}{7.8} + \frac{1}{9} \cdots$$

$$= \left(\frac{1}{1.2} - \frac{1}{4.5} + \frac{1}{7.8} \cdots\cdots \text{etc}\right) + \frac{1}{3}\left(\log \frac{1}{2}\right.$$

$$\frac{2}{3}\log 2 = + \frac{1}{132} - \frac{1}{446} + \frac{1}{608} - - - - - - - - \Big)\text{."}$$

In this last statement he makes the mistake of giving the terms of this conditionally convergent series an infinite displacement . He finishes by saying

$$\text{"}\log 2 = \frac{1}{1.2} - \frac{1}{4.5} + - - - - - - \text{etc} + \frac{1}{3}\log 2 .$$

$$\frac{2}{3}\log 2 = \frac{1}{1.2} - \frac{1}{4.5} + \frac{1}{7.8} - \frac{1}{10.11} \cdots\cdots$$

If the statement preceding the last two had been obtained correctly , this conclusion would be justifiable .

Still another example of this same type emphasizes the laxity in regard to re-arrangement of terms of conditionally convergent series . Mr. Wright says :

$$"\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} \text{-----------}$$

$$= 1 - \frac{1}{3} + \frac{2}{5} - \frac{1}{5} - \frac{1}{7} + \frac{2}{7} - \frac{1}{9} - \frac{1}{11} + \frac{2}{13} \text{--------}$$

$$= \left(1 + \frac{2}{5} + \frac{2}{9} + \frac{2}{13} \cdots\cdots\right) - \frac{1}{3} - \frac{1}{5} - \frac{1}{7} - \frac{1}{9}$$

$$- \frac{1}{11} \text{-----------}."$$

In the statement above we see that he has re-arranged the terms of the conditionally convergent series and has again given some of them an infinite displacement . He repeats the error when he writes

$$"\frac{\pi}{4} = 1 + \frac{2}{5} - \frac{1}{3} + \frac{2}{9} - \frac{1}{5} + \frac{2}{13} - \frac{1}{7} + \text{------}$$

$$= 1 + \frac{1}{3.5} + \frac{1}{5.9} + \text{----------} = \infty$$ (by reducing to common denominators)."

So we might sight many examples of these errors occurring in articles written in the first part of the 19th century . These serve to emphasize the fact that no extensive work in the summation of series can be properly carried on without a careful consideration of the general laws for re-arrangement of the terms of infinite series .

I.

Re-arrangement of Terms of Absolutely Convergent Series .

Let the series A, $a_1 + a_2 + a_3 + \cdots\cdots$ be an absolutely convergent series . Let B represent a series which contains all the terms of A in an order where the position of each term differs from its position in A by a finite number of terms . Then series B equals series A and series B is also absolutely convergent . Since the proof of this theorem is found in any text-book on infinite series , it seems unnecessary to give it here .

The terms of any series may be so re-arranged that some of them receive an infinite displacement , viz. the deranged series may consist of a finite or an infinite number of infinite series . Let us illustrate these re-arrangements to make clear the two theorems which will follow . Suppose the given series to be

(1) $a_1 + a_2 + a_3 + a_4 + \cdots\cdots + a_n \cdots$

Writing first all the terms with odd subscripts , then all those with even subscripts , we have series (2)

$a_1 + a_3 + a_5 + a_7 + \cdots\cdots + a_{2n-1} + a_2 + a_4 + a_6 + a_8 + \cdots\cdots + a_{2n}$, which consists of .

the 2 infinite series,

$$a_1 + a_3 + a_5 + a_7 + \cdots\cdots$$

and $a_2 + a_4 + a_6 + a_8 + \cdots\cdots$.

Series (3) is the result of a re-arrangement which gives an infinite number of infinite series.

(3) $B_1 + B_2 + B_3 + B_4 + \cdots$

where
$$B_1 = a_1 + a_{11} + a_{21} \cdots + a_{f(10)+1} + \cdots$$
$$B_2 = a_2 + a_4 + a_6 + \cdots + a_{f(2)} + \cdots$$
$$B_3 = a_3 + a_9 + a_{15} + \cdots + a_{f(3)} + \cdots$$
$$B_4 = a_5 + a_{25} + a_{35} + \cdots + a_{f(5)} + \cdots$$
$$B_5 = a_7 + a_{49} + a_{77} + \cdots + a_{f(7)} + \cdots$$

In B_1 occurs every term whose index is a number of which the last figure is 1 . In B_2 occurs every term whose index is a multiple of 2 . B_3 contains each term not in a preceding series whose index is a multiple of 3 . B_4 contains each term not in a preceding series whose index is a multiple of 5 . Each remaining series has terms whose subscripts are multiples of a given prime number which is not used as a subscript in any of the preceding series . No term is to be repeated .

Theorem : If the term of an absolutely convergent series be re-arranged into a finite number of infinite series , the sum of the resulting series will be the same as the sum of the original series .

Given : the absolutely convergent series

$$a_1 + a_2 + a_3 + \cdots + a_n + a_{n+1} \cdots = A.$$

Let the terms of A be so - re-arranged that we have a finite number k of infinite series which we may designate by S', S'', S''', $\cdots\cdots S^k$.

First we may show that each of the series S', S'', S''', S^{IV}, $\cdots\cdots$, S^k is absolutely convergent .

Let us consider the series S'. Every term in $|S'|$ is found in $|A|$. No matter how large we choose n' for $|S'_m|$, we can find an m for $|A_m|$ such that $|S'_m| < |A_m| < |A|$, where $|A_m| = |a_1| + |a_2| + |a_3| + \cdots + |a_m|$.

Then the series $|S'_m|$ has a limit , which is less than $|A|$.

Likewise it may be shown that $|S''|$, $|S'''|$, $\cdots\cdots |S^k|$ each have limits $< A$. Therefore each of the k series is absolutely convergent .

Let $S'_m = a'_1 + a'_2 + a'_3 \cdots\cdots a'_m$

Let $S_n'' = a_1'' + a_2'' + a_3'' + \cdots\cdots + a_n''$.

$S_n''' = a_1''' + a_2''' + a_3''' + \cdots\cdots + a_n'''$.

$- - - - - - - - - - - - - - - -$

$S_n^k = a_1^k + a_2^k + \cdots\cdots + a_n^k$.

All the terms of the k series given above constitute all the terms of the series A . Since we have a finite number of absolutely convergent series we may add them term by term and obtain:

$$S_n' + S_n'' + S_n''' + \cdots\cdots S_n^k$$
$$= a_1' + a_1'' + a_1''' + \cdots a_1^k + a_2' + a_2'' + a_2''' \ldots + a_2^k$$
$$+ \cdots\cdots\cdots + a_n^k.$$

We have now one series every term of which occupies a definite place in a given series . That this series is absolutely convergent can be shown by means of the following theorem : "If any two absolutely convergent series be added term by term , the resulting series is absolutely convergent and hence its terms can be re-arranged at pleasure provided the order of each term is changed by only a finite number of terms . " Consequently, we can re-arrange the terms in the series

$$a_1' + a_1'' + \cdots\cdots a_1^k + a_2' + a_2'' + \cdots\cdots + a_2^k$$

$$+ a_3' + a_3'' + \cdots\cdots + a_3^k + \cdots\cdots\cdots a_m^k$$

so that we obtain the given series A_n whose limit is A.

Therefore, $\lim_{n \to \infty} (S_n' + S_n'' + S_n''' + \cdots\cdots S_n^k) = \lim_{n \to \infty} A_n$

$$S' + S'' + S''' + \cdots\cdots S^k = A.$$

Theorem : If the terms of an absolutely convergent series be re-arrnaged into an infinite number of infinite series, the sum of the resulting series remains the same as the sum of the original series .

Given : An absolutely convergent series

$$A = a_1 + a_2 + a_3 + \cdots + a_n + \cdots\cdots$$

Let the terms of A be re-arranged so that we have an infinite number 1 , 2 , 3, - - - - - - -,k , of infinite series.

$$S', \ S'', \ S''', \ S^{IV}, \cdots\cdots\cdots, S^k.$$

To show that $S' + S'' + S''' + \cdots\cdots + S^k = A$.

In the same way as in the preceding theorem,it can be shown that $S', \ S'', \ S''', \cdots\cdots\cdots\cdots S^k$ are absolutely convergent series . Let

$$S' + S'' + S''' + \cdots\cdots + S^l + S^{l+1} + \cdots + S^k = S.$$

In the given series A we can find a subscript m so large that $(|a_m| + |a_{m+1}| + |a_{m+2}| + \cdots + |\ \ |) < \varepsilon$.
Then in the series $S_m' + S_m'' + S_m''' + \cdots + S_m^k = S_m$.
we can always find a subscript m so large that

$$|S_m^m + S_m^{n+1} + S_m^{n+2} + \cdots| < |a_m + a_{m+1} + \cdots\cdots|.$$

$$\text{Let } |S_m^m + S_m^{n+1} + \cdots\cdots| = B$$

$$\text{"} \ |a_m + a_{m+1} + \cdots\cdots| = C.$$

If $n \geq m$, we know $B < C$ for C contains all terms of B but B does not contain all terms of C, for some of the terms of C are found in each of the series preceding B. Then since $(A - |a_1 + a_2 + \cdots + a_m|) < \varepsilon$ we see that $(A - |S' + S'' + \cdots S^n|) < \varepsilon$, and the series $S' + S'' + S''' + \cdots + S^k$ is an absolutely convergent series whose sum approaches the limit A.

II.

Re-arrangement of Terms of Conditionally Convergent Series.

A series which is convergent but not absolutely convergent is called a conditionally convergent series. We find that re-arrangement of the terms of this type of series may change the limit of the sum of the series . By means of re-arrangement not only may the sum of the terms of the series be made to approach any preassigned quantity , as was shown by Riemann , but it may be made to oscillate between any two limits whatever , even between the limits ∞ and $-\infty$. Let us note how the sum of the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots\ldots\ldots$ is changed by a re-arrangement such that we have two positive terms are followed by a negative term . The deranged series becomes

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \cdots$$

$$\cdots + \frac{1}{4n-1} + \frac{1}{4n-3} - \frac{1}{2n} \cdots\cdots\cdots$$

The sum of this series is the limit of the expression

$$\sum_{n=1}^{n=\infty}\left(\frac{1}{4n-1} + \frac{1}{4n-3} - \frac{1}{2n}\right)$$

This is shown by Pierpont in article 111 of volume 2 of his "Theory of Functions of Real Variables" to be 3/2 the sum of the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \cdots\cdots\cdots$

Let the terms of the given series be re-arranged so that we have a positive term followed by two negatives. The deranged series becomes $\left(1 - \frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{6} - \frac{1}{8}\right)$ $+ \left(\frac{1}{5} - \frac{1}{10} - \frac{1}{12}\right) + ---- + \left(\frac{1}{2n-1} - \frac{1}{4n-2} - \frac{1}{4n}\right) + \cdots$

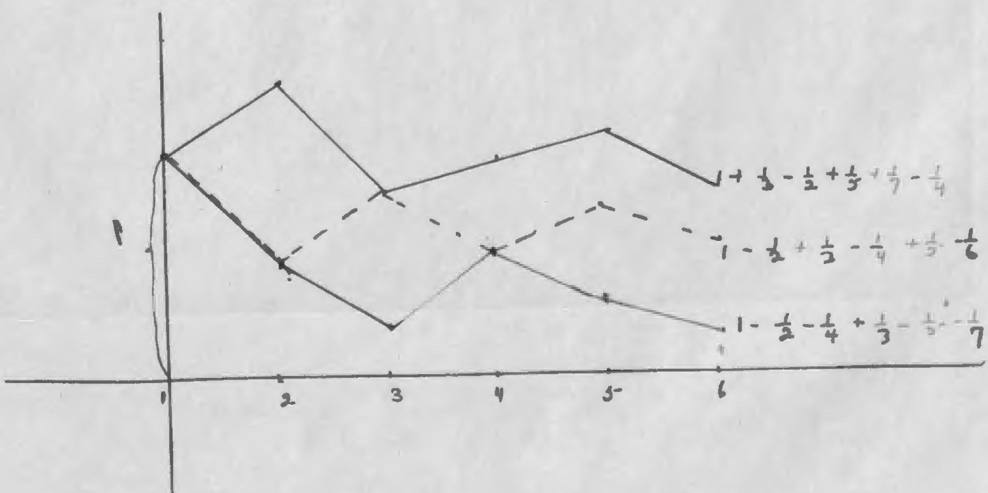
The sum of this series is the limit of the expression

$$\sum_{n \to \infty}\left(\frac{1}{2n-1} - \frac{1}{4n-2} - \frac{1}{4n}\right) \text{ or } \frac{1}{2}\sum_{n=1}^{n=\infty}\left(\frac{1}{2n-1} - \frac{1}{2n}\right).$$

The series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} - - - - -$ may be regarded as the limit of $\sum_{n=1}^{n=\infty}\left(\frac{1}{2n-1} - \frac{1}{2n}\right)$. Therefore , the limit of the deranged series is $\frac{1}{2}$ the lim of $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \cdots$

Thus we see that the series $1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \cdots$ and the series $1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \cdots\cdots$ each have sums different from the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \cdots\cdots$

. The differnece in the sums is emphasized by the following graphs .



ordinate 3 cm. = 1 unit.

Having studied the effect upon the sum of a conditionally convergent series due to some special re-arrangement , let us produce such a re-arrangement which will make the given series oscillate between any two limits as 1/2 and 1/4 . This may be done by adding positive terms , taken in the order in which they occur in the given series until the sum exceeds 1/2 and then adding enough negative terms until the total sum is just less than 1/4 . So if we continue these operations we obtain a series which oscillates between the limits 1/2 and 1/4 . Given the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - - - - - - - - - - - - -$$

The series

$$1 + \left(-\frac{1}{2} - \frac{1}{4} - \frac{1}{6}\right) + \left(\frac{1}{3} + \frac{1}{5}\right) + \left(-\frac{1}{8} - \frac{1}{10} - \frac{1}{12} - \frac{1}{14}\right) + \left(\frac{1}{7} + \frac{1}{9} + \frac{1}{11}\right) - \frac{1}{16} - \frac{1}{18} - \frac{1}{20} - \frac{1}{22} - \frac{1}{24} - \frac{1}{26} - \frac{1}{28} - \frac{1}{30} - \frac{1}{32} + \left(\frac{1}{13} + \frac{1}{15} + \frac{1}{17} + \frac{1}{19}\right) - - - - - - - -$$

oscillates between the limits 1/2 and 1/4 . The sequence of the successive sums is

$$1, \frac{1}{12}, .6166, .2376, .5824, .222, \ldots$$

In the same way we may make the given series oscillate between any other two values as 1 and 3/4 .

We then have

$$\left(1+\frac{1}{3}\right)+\left(-\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{7}+\frac{1}{9}\right)-\frac{1}{6}-\frac{1}{8}+\frac{1}{11}+\frac{1}{13}$$

$$+\frac{1}{15}+\frac{1}{17}-\frac{1}{10}-\frac{1}{12}-\frac{1}{14}+\text{- - - - - - - -}.$$

The sequence of the successive sums is

1.333 , .5833 , 1.0372 , .7456 , 1.0388, .7141, - - - -

The given series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}$ - - - - - - may be made to oscillate between any number of limits . For example , suppose the series be made to oscillate between the values 1.5 , 2 , and 4 by the following method: Add enough positive terms until the sum is just less than 4 , add enough negative terms until the sum is just greater than 1.5 , add enough positive terms until the sum is just less than 2 , add enough negative terms until the sum is just greater than 1.5 . By repeatedly performing the above operation we obtain the required series .

The given series may even be made to oscillate between the limits ∞ and $-\infty$. For example , we may add positive and negative terms so that the successive sums are 1 more , -2 or less , 3 or more , -4 or less , 5 or more , and so on . We have then an alternating series each of whose terms is numerically larger than the corresponding terms of $1 - 2 + 3 - 4 + 5 - 6 \dots\dots\dots$ which alternatesbetween the limits ∞ and $-\infty$ as the number of terms is indefinitely increased .

So in conclusion we may now make the general statement : A conditionally convergent series may by means of re-arrangement of its terms be made to oscillate be - tween any finite limit or even infinite limits .

Let A', $a_1 - a_2 + a_3 - a_4 \dots\dots$ be a given conditionally convergent series . The successive sums of this series may be made to oscillate between any finite limits A and B $(A > B)$ by repeatedly adding e - nough positive terms until the sum is just greater than A and then enough negative terms until the sum is just less than B . This process may be continued indefinitely since there are an infinite number of terms . Further - more , as a great number of terms of the given series is used , the successive sums become nearer A and B, since each term of the given series is numerically smaller than the preceding .

By means of a given alternating series which oscillates between the limits ∞ and $-\infty$, we can make any conditionally convergent series oscillate between ∞ and $-\infty$. For we may repeatedly add enough positive terms and enough negative terms so that each term of the deranged series is numerically $>$ the corresponding term of the series which is known to oscillate between ∞ and $-\infty$ as the number of terms is indefinitely increased.

B I B L I O G R A P H Y :

While the nature of this paper is such that few references have been consulted , there has been access to those given below . A working knowledge of infinite series has been obtained from the lectures given by Dr. George N. Bauer .

Theory of Infinite Series — Bromwich

Mathematical Analysis Vol. I. — Goursat-Hedrick

Theory of Functions of Real Variables Vol. II. — Pierpont

Uber den Gebrauch divergenter Reihen in der Zeit von 1750-1860 — H. Burkhardt

Allgemeine Theorie der Divergenz und Convergenz von Reihen mit positiven Gliedern — A. Pringsheim

Geschichte der unendlichen Reihen — Reiff

History of Mathematics — Cajori

Solution of Cambridge Problems — I.M.F.Wright

Mathematische Annalen

Crelle's Journal Vol. I. - XII.