Infinite Products

A thesis submitted to the faculty of the Graduate School of the University of Minnesota by Doris L. Brown, in partial fulfillment of the requirements for the degree of Master of Arts, May 25, 1912.

Infinite Products

Part I

A treatment of infinite products without the aid of infinite series.

Definition of an Infinite Product

Let $f_0, f_1, f_2, \cdots$ be any set of numbers none of which is zero. Form the infinite product

$$f_0 \cdot f_1 \cdot f_2 \cdot f_3 \cdots\cdots$$

and denote the product of the first n factors by p_n. (i.e. $p_n = f_0 \cdot f_1 \cdot f_2 \cdots f_{n-1}$)

Allow n to increase without limit. If p_n approaches a limit P, different from zero, (i.e. if $\lim\limits_{n=\infty} p_n = P$ and $P \neq 0$) the infinite product is said to converge, or to be convergent.

If $\lim_{n \doteq \infty} p_n = \infty$ or 0, or does not exist at all, the infinite product is said to diverge, or to be divergent.

Example: $2 \cdot \frac{1}{\sqrt{2}} \cdot \sqrt[3]{2} \cdot \frac{1}{\sqrt[4]{2}} \cdots$ is a convergent product because $\lim_{n \doteq \infty} p_n = \lim_{n \doteq \infty} 2^{1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots \frac{1}{n}}$ $= 2^{\log 2}$ since the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}\cdots$ is convergent and equal to $\log 2$.

Example: $2 \cdot \sqrt{2} \cdot \sqrt[3]{2} \cdot \sqrt[4]{2} \cdots$ is a divergent infinite product, because $\lim_{n \doteq \infty} p_n =$ $\lim_{n \doteq \infty} 2^{(1+\frac{1}{2}+\frac{1}{3}\cdots+\frac{1}{n})} = \infty$ because the series $1+\frac{1}{2}+\frac{1}{3}+\cdots$ is divergent.

Example: $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdots$ is a divergent infinite product, because $\lim_{n \doteq \infty} p_n = \lim_{n \doteq \infty} \frac{1}{2^n} = 0$

Example: $1 \cdot (-1) \cdot 1 \cdot (-1) \cdots$ is a divergent product, because $\lim_{n \doteq \infty} p_n$ does not exist. $p_n = +1$ when n is of the form $4k$ or $4k+1$ and $p_n = -1$ when n is of the form $4k+2$ or $4k+3$.

Example: $2 \cdot \frac{1}{2} \cdot 2 \cdot \frac{1}{2} \cdots$ The product of n terms is 2 or 1 according as the number of factors is odd or even. It is called an oscillating product.

Definition of a limit.

An infinite product has just been defined as the limit of the product of n factors as n increases without limit. It will be advisable, before proceeding further, to state clearly what is meant by saying that a variable approaches a limit. The definition ordinarily given is about like this: "when a variable approaches a constant in such a way that the difference between the two may become and remain less than any assigned positive quantity, however small, the constant is called the limit of the variable." This is not quite definite enough for the purposes of this paper. The limits that we are going to consider are of this kind;

$$\lim_{x=a} f(x) = A$$ where x is either a continuous variable or a variable integer.

We say that the limit is equal to A when and only when it is possible to find a number δ, such that $|f(x) - A| < \varepsilon$[1] when $|x-a| < \delta$[2], where ε is a preassigned quantity which may be as small as we please. (The 1 and 2 over the ε and δ are meant to indicate that the ε is chosen first and the δ is found later.) A special case is

$\lim_{x \doteq \infty} f(x) = A$, which means that for any assigned ε, it is possible to find a number M such that $|f(x) - A| < \overset{1}{\varepsilon}$ when $x > M$

Example: $\lim_{n \doteq \infty} \sqrt[n]{A} = 1$

$$\text{or} \quad \left|\sqrt[n]{A} - 1\right| < \overset{1}{\varepsilon} \qquad \text{when} \quad n > \overset{2}{M}$$

$$\therefore \quad \sqrt[n]{A} < \varepsilon + 1$$

$$A < (1+\varepsilon)^n$$

$$\log A < n \log(1+\varepsilon)$$

$$\frac{\log A}{\log(1+\varepsilon)} < n$$

Therefore n must be chosen greater than the quantity $\dfrac{\log A}{\log(1+\varepsilon)}$.

The following <u>fundamental theorems concerning limits</u> will be needed from time to time. See Osgood's Calculus, page 16.

I. The limit of a constant times a variable is equal to the product of the constant into the limit of the variable.

$$\lim (cx) = c \lim x$$

II The limit of the sum of n variables is equal to the sum of the limits of these variables, n being any fixed positive number.

$$\lim (x_1 + x_2 + \cdots x_n) = \lim x_1 + \lim x_2 + \cdots \lim x_n$$

III The limit of the product of n variables is equal to the product of their limits, n being any fixed positive integer.

$$\lim (x_1 \cdot x_2 \cdots x_n) = \lim x_1 \cdot \lim x_2 \cdots \lim x_n$$

IV. The limit of the quotient of two variables is equal to the quotient of their limits, provided that the limit of the divisor is not zero.

$$\lim \frac{x}{y} = \frac{\lim x}{\lim y}, \quad \text{if } \lim y \neq 0$$

<u>A Fundamental Principle concerning the existence of a limit</u>.

If s_n is a variable that 1) always increases when n increases;

$$s_{n'} > s_n \qquad n' > n \;;$$

but that 2) always remains less than some definite fixed number A;

$$s_n < A$$

for all values of n, then s_n approaches a limit U:

$$\lim_{n=\infty} s_n = U$$

This limit, U, is not greater than A;

$$U \leqq A$$

This fundamental theorem will be used frequently.

See - Introduction to Infinite Series - Osgood. page 4.

Infinite Products, all of whose factors are greater than one

Test for convergence. Direct comparison.
(1) Let $f_0 \cdot f_1 \cdot f_2 \cdots$ be an infinite product, each of whose factors is greater than 1, the convergence of which it is desired to test.

If a product (2) $a_0 \cdot a_1 \cdot a_2 \cdots$ whose factors are greater than 1, already known to be convergent, can be found whose factors are never less than the factors of the product (1) to be tested, then (1) is a convergent product.

For let $p_n = f_0 \cdot f_1 \cdot f_2 \cdots f_{n-1}$

$$A_n = a_0 \cdot a_1 \cdot a_2 \cdots a_{n-1}$$

and $\lim\limits_{n \doteq \infty} A_n = U, \quad U \neq 0$;

then since $A_n < U$

and $p_n \leqq A_n$, by hypothesis

$$p_n < U, \quad U \neq 0$$

Therefore p_n approaches the limit P^*, $(P \neq 0$

* See Fundamental Principle – page 6

and $P \leqq U$) as n increases indefinitely, and the infinite product (1) is convergent

<u>Infinite products, whose factors are both greater than 1 and less than 1</u>

<u>Theorem</u>

1) Let the factors of an infinite product be alternately greater than 1 and less than 1 and written in the form

$$f_0 \cdot \frac{1}{f_1} \cdot f_2 \cdot \frac{1}{f_3} \; ---- \qquad \text{where } f_0 > 1 \, ;$$

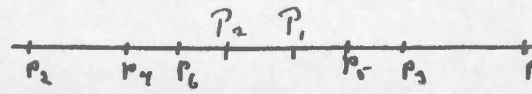
2) let each f be $\leqq$ its predecessor
   (i.e. $f_{n+1} \leqq f_n$) ;

3) and let $\lim\limits_{n \doteq \infty} f_n = 1$

Then the infinite product is convergent.

<u>Proof</u>

Let $\quad p_n = f_0 \cdot \frac{1}{f_1} \cdot f_2 \cdot \frac{1}{f_3} \; --- \quad$ to n factors



Plot p_1, p_2, p_3 --- on a straight line. Then it is seen that $p_1\, p_3\, p_5 \cdots$ always move to the left (because $p_3 = p_1 \cdot \frac{1}{f_1} \cdot f_2 \leqq p_1$ since $f_2 \leqq f_1$; also $p_5 = p_3 \cdot \frac{1}{f_3} \cdot f_4 \leqq p_3$ since $f_4 \leqq f_3$; etc) but they never advance so fas as p_2, for p_2 represents the smallest possible product, since $f_{n+1} \leqq f_n$. Therefore they approach a limit P_1

(Fundamental Principle, page 6)

$$\therefore \lim_{m \doteq \infty} p_{2m+1} = P_1$$

Similarly the points $p_2, p_4, p_6 \cdots$ always move to the right, (since $p_4 = p_2 \cdot f_2 \cdot \frac{1}{f_3} \geqq p_2$, for $f_2 \geqq f_3$ etc) but they never advance so far as p_1 for p_1 is the greatest possible product since $f_{n+1} \leqq f_n$. Therefore they approach a limit P_2

(Principle, page 6)

$$\therefore \lim_{m \doteq \infty} p_{2m} = P_2$$

But since

$$p_{2m+1} = f_{2m} \cdot p_{2m}$$

$$\lim p_{2m+1} = \lim p_{2m} \cdot \lim f_{2m}$$

(Theorem III, page 5)

and since $\lim_{n \doteq \infty} f_n = 1$ (by hypothesis)

we have $\lim p_{2m+1} = \lim p_{2m}$

or $P_1 = P_2$

Therefore p_n approaches a limit $P \neq 0$ and the product is convergent.

The same is true if the first factor f_0 is less than 1 as may be proved by considering the product obtained by leaving off the factor f_0. The new product will begin with a factor greater than 1 and will be convergent by the theorem just proved. The original product will be the new product multiplied by f_0.

Example: The alternating product

$$2 \cdot \frac{1}{\sqrt{2}} \cdot \sqrt[3]{2} \cdot \frac{1}{\sqrt{2}} \cdots\cdots$$

is convergent because $f_{n+1} < f_n$ and $\lim_{n \doteq \infty} f_n = 1$

Example: The alternating product

$$* \ \frac{\pi}{2} = \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdots}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdots}$$

is convergent because $f_{n+1} < f_n$ and $\lim_{n \doteq \infty} f_n = 1$

* Note – This product for $\frac{\pi}{2}$ is due to Wallis – See page 54

Note. In an alternating series, where $|u_{n+1}| \leqq u_n$, the condition for convergence is that $\lim_{n \doteq \infty} u_n = 0$. This limit is approached from the + and − sides alternately.

It has just been shown that in an alternating infinite product, where $f_{n+1} \leqq f_n$, the condition for convergence is that the $\lim_{n \doteq \infty} f_n = 1$. This limit is approached from the two sides of 1, alternately.

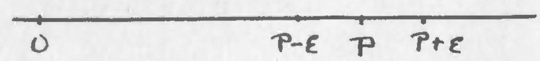
A necessary condition for Convergence.

Let $f_0 \cdot f_1 \cdot f_2 \cdots$ be any convergent infinite product of factors greater than 1 and less than 1.

Then $\lim f_n = 1$

Proof By hypothesis

$\lim_{n \doteq \infty} p_n = P$ exists and $P \neq 0$



Plot the number P, on a line. Mark off the points $P-\varepsilon$ and $P+\varepsilon$, where ε is any positive number $< P$. Since $\lim_{n \doteq \infty} p_n = P$, it is possible to find an M such that when $n > M$, the points p_n lie within the interval $P-\varepsilon$ and $P+\varepsilon$.

$$\text{Then} \quad |p_{n+1} - p_n| < 2\varepsilon \qquad n > M;$$

$$\text{and} \quad |p_n| > |P| - \varepsilon$$

$$\text{then} \quad \left|\frac{p_{n+1}}{p_n} - 1\right| < \frac{2\varepsilon}{p_n} < \frac{2\varepsilon}{|P|-\varepsilon}$$

$\dfrac{2\varepsilon}{|P|-\varepsilon}$ approaches zero as ε approaches zero

$$\therefore \lim_{n \doteq \infty} \frac{p_{n+1}}{p_n} = 1$$

$$\text{But} \quad \frac{p_{n+1}}{p_n} = f_n$$

$$\therefore \lim_{n \doteq \infty} f_n = 1$$

Note. This condition is necessary but not sufficient.

Example: $2 \cdot \sqrt{2} \cdot \sqrt[3]{2} \cdot \sqrt[4]{2} \cdots\cdots$

$\lim_{n \doteq \infty} f_n = 1$ but the series is divergent.

(Proof, page 2)

Remark. It is frequently convenient when studying the convergence of infinite products, to reject a few factors at the beginning and to consider the new product only.

That the convergence of this product is necessary and sufficient for the convergence of the original product is evident, since

$$p_n = f_0 \cdot f_1 \cdots f_{m-1} \cdot f_m \cdot f_{m+1} \cdots f_{n-1}$$
$$= f \cdot p_{n-m} \qquad \text{where } f = f_0 \cdot f_1 \cdots f_{m-1}$$

Here f is a constant and therefore p_n will converge to a limit if p_{n-m} does, and conversely, p_{n-m} will converge if p_n does.

(Theorem I, page 5)

Consequently, it will not be necessary to consider negative factors, for sooner or later all factors become positive since $\lim_{n \to \infty} f_n = 1$ and therefore, we may discard a finite number of factors at the beginning including all that are negative.

Just as in series, a series with all negative terms may be handled by the theory of series having all positive terms; so in infinite products, a product whose factors are all less than 1 may be handled by the theory of products having factors all greater than 1; i.e. by replacing each factor by its reciprocal. If an infinite product has a finite limit, also the product of reciprocals will have a finite limit, namely the reciprocal of the other limit.

Therefore we shall consider products all of whose factors are greater than 1, and products some of whose factors are greater than 1 and some less than 1.

<u>General case of convergence</u> – when factors are both greater than 1 and less than 1.

Let $f_0 \cdot f_1 \cdot f_2 \cdot f_3 \cdots$ be any infinite product and let

$g_0 \cdot g_1 \cdot g_2 \cdots$ denote the product of all factors which are greater than 1, and

$\frac{1}{e_0} \cdot \frac{1}{e_1} \cdot \frac{1}{e_2} \cdots$ the product of the factors which are less than 1.

For example, if the f product is

$$2 \cdot \frac{1}{\sqrt{2}} \cdot \sqrt[3]{2} \cdot \frac{1}{\sqrt[4]{2}} \cdots$$

the g-product is

$$2 \cdot \sqrt[3]{2} \cdot \sqrt[5]{2} \cdots$$

and the $\frac{1}{e}$-product is

$$\frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt[4]{2}} \cdot \frac{1}{\sqrt[6]{2}} \cdots$$

Let

$$g_m = g_0 \cdot g_1 \cdot g_2 \cdots g_{m-1}$$

$$\frac{1}{e_p} = \frac{1}{e_0} \cdot \frac{1}{e_1} \cdot \frac{1}{e_2} \cdots \frac{1}{e_{p-1}} \qquad \text{where } n = p + m$$

$$p_n = f_0 \cdot f_1 \cdot f_2 \cdots f_{n-1}$$

where m is the number of factors greater than 1, among the first n factors, and g_m their product, p is the number of factors less than 1 and $\frac{1}{e_p}$ their product.

Then whatever value n may have

$$p_n = g_m \cdot \frac{1}{e_p}$$

When n increases without limit, either m or p, or both m and p increases without limit, since $n = m + p$. In case either m or p remains finite, the product may be readily handled by discarding these factors at the beginning and considering the remaining product of factors all greater than 1 or all less than 1, as the case may be

When m and p both increase without limit, cases arise.

Case I. When both g_m and $\frac{1}{e_p}$ approach limits -

Let $\lim_{m \doteq \infty} g_m = G \qquad G \neq 0$

and $\lim_{p \doteq \infty} \frac{1}{e_p} = \frac{1}{E} \qquad \frac{1}{E} \neq 0$

Then the p_n product will also converge.
(Theorem III, page 5)

$$\lim_{n \doteq \infty} p_n = P \qquad P \neq 0$$

and $P = G \cdot \frac{1}{E}$

Case II When at least one of the products g_m or e_p approaches no limit. The p-product may converge or may diverge.

If either the g_m or e_p product diverges while the other converges, the p-product will diverge. If both the g_m and e_p products diverge, the p-product may be convergent or may be divergent.

Example:

$2 \cdot \frac{1}{\sqrt{2}} \cdot \sqrt[3]{2} \cdot \frac{1}{\sqrt[4]{2}} \cdots$ is a convergent product (Proof, page 2)

But $2 \cdot \sqrt[3]{2} \cdot \sqrt[5]{2} \cdots$ is a divergent product (Proof, page 2)

and $\frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt[4]{2}} \cdot \frac{1}{\sqrt[6]{2}} \cdots$ is a divergent product,

because $\lim\limits_{n \doteq \infty} p_n = \lim\limits_{n \doteq \infty} 2^{-\frac{1}{2}(1+\frac{1}{2}+\frac{1}{3}+\cdots\frac{1}{n})} = 2^{-\infty} = 0$

Example:

$2 \cdot \frac{1}{\sqrt[3]{2}} \cdot \sqrt[3]{2} \cdot \frac{1}{\sqrt[9]{2}} \cdot \sqrt[5]{2} \cdots$ is divergent

because $2 \cdot \sqrt[3]{2} \cdot \sqrt[6]{2} \cdots$ is divergent (Proof, page)

and $\frac{1}{\sqrt[3]{2}} \cdot \frac{1}{\sqrt[9]{2}} \cdot \frac{1}{\sqrt[27]{2}} \cdots$ is convergent.

This product converges because $\lim\limits_{n \doteq \infty} p_n = \lim\limits_{n \doteq \infty} 2^{-(\frac{1}{3}+\frac{1}{3^2}+\cdots\frac{1}{3^n})}$ $= 2^{-\frac{1}{2}} = \frac{1}{2}\sqrt{2}$

Now form the product of factors greater than 1, by replacing each factor less than 1 by its reciprocal. Hereafter, this product will be called the <u>conjugate</u> product. A product whose terms are all greater than 1, is its own conjugate product.

Let $f_0' f_1' f_2' \cdots$ be this product.

Then if $p'_n = f'_0 \cdot f'_1 \cdot f'_2 \cdots f'_{n-1}$ it is clear that

$$p'_n = g_m \cdot e_p \qquad \text{where } n = m + p$$

since the g-factors have remained unchanged and the $\frac{1}{e}$-factors have been replaced by their reciprocals.

<u>Theorem</u>. If the conjugate product converges, the original product will also converge.

<u>Proof</u>.

$$\lim_{n \doteq \infty} p'_n = P' \qquad \text{by hypothesis}$$

also $p'_n = g_m \cdot e_p$

No matter how many factors of the g and e-products be taken, the result cannot exceed the limit P'. But every factor of the g and e products is greater than 1 and therefore the product of $m+p$ terms increases as m and p increase. By the principle of page 6, each of these g and e products converges, and therefore the g and $\frac{1}{e}$ products converge. Consequently we have Case I of page 17 and the original p-product converges.

The converse theorem is not true.

Introduction and Removal of Parentheses.

Theorem Given the infinite product

(1) $p = f_0 \cdot f_1 \cdot f_2 \cdots$

Form a new product by enclosing each pair of factors in a parenthesis.

(2) $P = (f_0 \cdot f_1) \cdot (f_2 \cdot f_3) \cdot (f_4 \cdot f_5) \cdots$

If product (1) is convergent toward a limit A, the product (2) will converge to the same limit A.

Proof

Let $p_n = f_0 \cdot f_1 \cdot f_2 \cdots f_{n-1}$

and $P_N = (f_0 \cdot f_1) \cdot (f_2 \cdot f_3) \cdot (f_4 \cdot f_5) \cdots (f_{p-1} \cdot f_p)$

(N = the number of parentheses $= (\frac{p \pm 1}{2})$]

It is seen that $P_N = p_{n'}$ if n' is properly chosen i.e. if the last term of P_N is $(f_{p-1} \cdot f_p)$, the

the last term of $p_{n'}$ is f_p. Therefore P_N and $p_{n'}$ approach the same limit for as n' and N become infinite P_N takes on no values that are not taken on by $p_{n'}$. $p_{n'}$ however, takes on no values that are not taken on by P_N.

This argument holds for parentheses enclosing any finite number of factors. The number of factors need not be the same for all parentheses.

We have just seen that it is allowable to insert parentheses in a convergent product. The question now arises as to whether it is always allowable to remove parentheses.

Example: $(2 \cdot \frac{1}{2}) \cdot (2 \cdot \frac{1}{2}) \cdots\cdots$ is a convergent product whose limit is 1

$2 \cdot \frac{1}{2} \cdot 2 \cdot \frac{1}{2} \cdots\cdots$ is an oscillating product

Example: $* \ \frac{\pi}{2} = \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdots}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdots\cdots} \qquad = 2 \cdot \left(\frac{1}{3} \cdot 2\right) \cdot \left(\frac{1}{3} \cdot 4\right)\left(\frac{1}{5} \cdot 4\right) \cdots$

* See page 10

Removing parentheses we have

$$2 \cdot \frac{1}{3} \cdot 2 \cdot \frac{1}{3} \cdot 4 \cdot \frac{1}{5} \cdot 4 \cdots \quad \text{which is divergent}$$

$$\text{since } \lim_{n \doteq \infty} f_n \neq 1$$

For this reason we may not omit the 1 in the denominator and rewrite

$$\frac{\pi}{2} = \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdots}{3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdots}$$

If this were allowable we should have

$$\frac{\pi}{2} = \left(\frac{2}{3}\right)^2 \cdot \left(\frac{4}{5}\right)^2 \cdots \quad = \left(\frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdots\right)^2$$

Evidently parentheses may not always be removed without destroying the convergence. It follows from the preceding theorem that if the product after the removal remains convergent, it converges to the same limit as before and that therefore the removal is allowable.

<u>Theorem</u> When a product P, containing parentheses is convergent, and when the factors of the product after the removal of parentheses approach one as a limit, and

when the number of factors in every parenthesis is finite, then it is permissable to remove parentheses.

<u>Hypothesis</u> 1) $\lim_{n=\infty} f_n = 1$, where f_n is a factor after the removal of parentheses.

2) The maximum number of factors in any parenthesis is K.

<u>Proof</u>. Let $P = P_0 \cdot P_1 \cdot P_2 \cdots$ be the convergent product, of which each factor P_i is a parenthesis. After the removal of the parentheses let the product be

$$P' = f_0 \cdot f_1 \cdot f_2 \cdot f_3 \cdots\cdots$$

Let P_n' denote the product of n factors of the second product.

$$\text{i.e. } P_n' = f_0 \cdot f_1 \cdot f_2 \cdots f_{n-1}$$

It may happen that, if the parentheses are put back in, the factor f_{n-1} will be the last factor in a parenthesis. Then P_n' will equal P_N, where N is less than or equal to n. But it is more likely that f_{n-1} will be somewhere in the middle of a parenthesis. Therefore we can choose N so that P_N will include all the factors of P'_n and some more too.

i.e. $P_N = P'_n \cdot (f_n \cdots f_{n'-1})$, where $0 \leqq (n'-n) < K$

By hypothesis P_N approaches a limit. P'_n will approach the same limit if we can show that $(f_n \cdots f_{n'-1})$ approaches a limit 1.

$$\lim (f_n \cdot f_{n+1} \cdots f_{n'-1}) = \lim f_n \lim f_{n+1} \cdots \lim f_{n'-1}$$

(Theorem III, page 5)

But $\lim f_i = 1$ by hypothesis

$(i = n, n+1, \cdots n'-1)$

$$\therefore \lim (f_n \cdot f_{n+1} \cdots f_{n'-1}) = 1$$

$$\therefore \lim P_N = \lim P'_n$$

The Commutative Law

The value of an ordinary product is independent of the order of the factors (e.g., $abc = bca$). In an infinite product, this may or may not be so.

For example;

(1) $\quad 2 \cdot \frac{1}{\sqrt{2}} \cdot \sqrt[3]{2} \cdot \frac{1}{\sqrt[4]{2}} \cdot \sqrt[5]{2} \cdots\cdots$ is convergent.

(It is an alternating product, see page 10)

(2) $\quad 2 \cdot \sqrt[3]{2} \cdot \frac{1}{\sqrt{2}} \cdot \sqrt[5]{2} \; \sqrt[7]{2} \cdot \frac{1}{\sqrt[4]{2}} \cdots\cdots$ is convergent also, but to a different limit, because

$$(1) = 2^{1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots} = 2^{\alpha} \quad \text{where } \alpha < \tfrac{5}{6}$$

$$(2) = 2^{1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\cdots} = 2^{\beta} \quad \text{where } \beta > \tfrac{5}{6}$$

<u>Theorem</u> In a convergent product of factors greater than 1, the factors may be rearranged at pleasure without altering the value of the product.

<u>Proof</u>.

Let $p_n = f_0 \cdot f_1 \cdot f_2 \cdots\cdots f_{n-1}$,

and $\lim\limits_{n \doteq \infty} p_n = P, \quad P \neq 0$

After rearrangement let the product of n' terms be

$$p'_{n'} = f'_0 \cdot f'_1 \cdot f'_2 \cdots f'_{n'-1}$$

Then $p'_{n'}$ always increases as n' increases because each factor is greater than 1, but no matter how large n' may be taken and then held fast, n can be taken so large that p_n will include all the factors of $p'_{n'}$ and more too.

$$\therefore \quad p'_{n'} < p_n < P$$

Therefore no matter how large n' be taken, $\qquad p'_{n'} < P$;

hence $p'_{n'}$ approaches a limit $P' \leqq P$.

(Fundamental Principle, page 6)

We may now turn things about and regard the f-product as generated by the rearrangement of factors of the f'-product, and the same reasoning shows that

$$P \leqq P'$$

$$\therefore \quad P = P'$$

<u>Theorem</u> In a convergent product of factors both greater than 1 and less than 1, whose conjugate product is convergent, the factors may be rearranged at pleasure without altering the value of the product.

(1) $3 \cdot \frac{1}{\sqrt[3^2]{3}} \cdot \sqrt[5^2]{3} \cdot \frac{1}{\sqrt[7^2]{3}} \cdots$ is convergent

because it is an alternating product and $\lim_{n \doteq \infty} f_n = 1$

The conjugate product

$$3 \cdot \sqrt[3^2]{3} \cdot \sqrt[5^2]{3} \cdot \sqrt[7^2]{3} \cdots\cdots$$ is also convergent

because $\lim p_n = \lim_{n \doteq \infty} 3^{1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \frac{1}{(2n-1)^2}}$

and $1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$ is a convergent series because $\lim n\left(1 - \frac{u_{n+1}}{u_n}\right) = 2$. Therefore the factors of (1) may be rearranged at pleasure.

<u>Proof.</u>

Let $p_n = \frac{1}{e_p} \cdot g_m$ where $n = m + p$,

g_m is the product of factors greater than 1 and $\frac{1}{e_p}$ is the product of factors less than 1.

Let P_n' be the product of n terms of the conjugate product.

Then $P_n' = g_m \cdot e_p$

But P'_n converges by hypothesis, therefore g_m and e_p converge and p_n converges.

Let (Theorem, page 19)

$$\lim p_n = P,$$

$$\lim g_m = G \quad \text{and} \quad \lim e_p = E.$$

Then $\lim \frac{1}{e_p} = \frac{1}{E}$

and $P = G \cdot \frac{1}{E}$

Let $f_0' \cdot f_1' f_2' f_3' \cdots$ be the product after any rearrangement of factors and let

$$p'_{n'} = g'_{m'} \cdot \frac{1}{e'_{p'}}.$$

But $\lim g'_{m'} = \lim g_m = G$

and $\lim \frac{1}{e'_{p'}} = \lim \frac{1}{e_p} = \frac{1}{E}$

(Theorem, page 25)

$$\therefore \lim p'_{n'} = G \cdot \frac{1}{E} = P$$

__Note__. An infinite product whose conjugate product is convergent is said to be __absolutely convergent__ because its factors can be rearranged at pleasure, without altering the value of the product. A product whose conjugate product is not convergent, is called a <u>conditionally convergent</u> product.

<u>Theorem</u> It is possible to re-arrange the factors of any conditionally convergent product so that the product will converge to any preassigned limit

<u>Proof</u>

Let $f_0 \cdot f_1 \cdot f_2 \, f_3 \cdots$ be a conditionally convergent product and

$g_0 \cdot g_1 \cdot g_2 \cdots$ the product of factors greater than 1

$\frac{1}{e_0} \cdot \frac{1}{e_1} \cdot \frac{1}{e_2} \cdots$ the product of factors less than 1

Then each of these products is divergent.

(See page 17)

Let P be the preassigned limit. Rewrite the product as follows; take enough g-factors to make the product just exceed P; then take enough $\frac{1}{e}$-factors to make it just less than P. Then take enough g-factors to make it just exceed P, etc. (This can be done because the g and $\frac{1}{e}$ products are divergent) Since $\lim_{n \doteq \infty} g_n = 1$ and $\lim_{n \doteq \infty} \frac{1}{e_n} = 1$, the amount by which the product of terms in the re-arranged product differs from P becomes less and less and approaches the limit 0. Therefore the product converges to the limit P.

Theorem

If $P = f_0 \cdot f_1 \cdot f_2 \cdots$

$G = g_0 \cdot g_1 \cdot g_2 \cdots\cdots$ are two convergent products, they may be multiplied together, factor by factor

$$P \cdot G = f_0 \cdot g_0 \cdot f_1 \cdot g_1 \cdot f_2 \cdot g_2 \cdots\cdots$$

Proof

Let $p_n = f_0 \cdot f_1 \cdots\cdots f_{n-1}$

$g_n = g_0 \cdot g_1 \cdots\cdots g_{n-1}$

Then $p_n \cdot g_n = (f_0 \cdot g_0) \cdot (f_1 \cdot g_1) \cdots\cdots (f_{n-1} \cdot g_{n-1})$

As $n \doteq \infty$, $p_n \cdot g_n$ converges to $P \cdot G$

(Theorem III, page 5)

hence $P \cdot G = (f_0 \cdot g_0)(f_1 \cdot g_1)(f_2 \cdot g_2) \cdots\cdots$

The parentheses may now be dropped.

(Theorem, page 22)

$$\therefore P \cdot G = f_0 \cdot g_0 \cdot f_1 \cdot g_1 \cdot f_2 \cdot g_2 \cdots\cdots$$

Double Products of factors greater than 1

The following is a double product.

$$\begin{array}{l} f_0^{(0)} \cdot f_1^{(0)} \cdot f_2^{(0)} \cdot f_3^{(0)} \cdots\cdots \\ f_0^{(1)} \cdot f_1^{(1)} \cdot f_2^{(1)} \cdot f_3^{(1)} \cdots\cdots \\ f_0^{(2)} \cdot f_1^{(2)} \cdot f_2^{(2)} \cdot f_3^{(2)} \cdots\cdots \\ \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots \end{array}$$

Definition A double product of factors greater than 1 is said to be convergent, if every simple product that can be formed from it, each factor of the double product appearing once and only once in the simple product, is a convergent product.

This definition of convergence is unsatisfactory until we prove the following theorem.

Theorem

Let $p = f_0 \cdot f_1 \cdot f_2 \cdots\cdots$ be any simple product of the double product and

$p' = f_0' \cdot f_1' \cdot f_2' \cdots\cdots$ be any other simple

simple product of the same double product. If p converges to the value P, p' converges to the same value.

Proof

Let $p_n = f_0 \cdot f_1 \cdots f_{n-1}$

and $p'_{n'} = f'_0 f'_1 \cdots f'_{n'-1}$

$p'_{n'} \leqq P$, $P \neq 0$ since it is possible to take n so large that p_n will include all the factors of $p'_{n'}$ and more too. Therefore $p'_{n'}$ approaches a limit P' and p' is convergent.

(Fundamental Principle, page 6)

$$P' \leqq P \quad , \quad P \neq 0$$

Now start with the convergent product p' and in the same way prove that $P \leqq P'$.

$$\therefore \quad P = P'$$

Theorem If P is a convergent double product of factors greater than 1, then

1) each row of P is a convergent product

2) the product $f^{(0)} \cdot f^{(1)} \cdot f^{(2)} \cdots$ is a convergent product where $f^{(k)}$ is the value of the product formed by the kth row of P.

3) the product $f^{(0)} \cdot f^{(1)} \cdot f^{(2)} \cdots$ converges to the same value P, to which the double product converges.

<u>Proof</u> 1)

$f_0^{(0)} \cdot f_1^{(0)} \cdot f_2^{(0)} \cdots f_{n-1}^{(0)} < P$ and it increases as n increases. Therefore it converges.

(Fundamental Principle, page 6)

Call its value $f^{(0)}$. Similarly every other row converges.

2) Let $P_K = f^{(0)} \cdot f^{(1)} \cdot f^{(2)} \cdots f^{(K-1)}$

where $f^{(0)} = f_0^{(0)} \cdot f_1^{(0)} \cdot f_2^{(0)} \cdots$ (the first row of the double product)

$f^{(1)} = f_0^{(1)} \cdot f_1^{(1)} \cdot f_2^{(1)} \cdots$ (" second " " " " ")

$\cdots\cdots\cdots$

Then $P_K = \left(f_0^{(0)} \cdot f_1^{(0)} f_2^{(0)} \cdots\right) \cdot \left(f_0^{(1)} f_1^{(1)} f_2^{(1)} \cdots\right) \cdots \left(f_0^{(K-1)} f_1^{(K-1)} \cdots\right)$
where each parenthesis is a row of the double product. But P_K does not exhaust the double product, no matter how large k is taken.

$\therefore P_K < P$ where P is the value of the double product. Also P_K increases as k increases. Therefore the product $f^{(0)} f^{(1)} f^{(2)} \cdots$ converges to the value $P' \leqq P$.

(Fundamental Principle, page 6)

3) To prove $P' = P$

Let $p = f_0 \cdot f_1 \cdot f_2 \cdots$ be any simple product of P and let

$$p_n = f_0 \cdot f_1 \cdots f_{n-1}$$

$\lim_{n \doteq \infty} p_n = P$ where P is the value of the double product. Then n can be chosen so that

$p_n > P - \overset{2}{\varepsilon}$ when $n > \overset{2}{m}$ and ε is any preassigned small number. Choose n large enough and hold it fast.

In the double product there is a last row in which a factor of this p_n appears; call it the l-th row; and there is a last column, call it the m-th column. Consider the part of the product contained within this rectangle R. The terms of p_n all lie within this rectangle.

Let $\rho = (f_0^{(0)} \cdot f_1^{(0)} \cdot f_2^{(0)} \cdots f_{m-1}^{(0)})(f_0^{(1)} \cdot f_1^{(1)} \cdot f_2^{(1)} \cdots f_{m-1}^{(1)}) \cdots$
$(f_0^{(l)} \cdot f_1^{(l)} \cdot f_2^{(l)} \cdots f_{m-1}^{(l)})$

This product, ρ, is the product of all factors in the rectangle R.

$$\therefore \rho \geqq p_n \quad \text{and} \quad \therefore \rho > P - \varepsilon$$

But if k be chosen equal to l

$$P_k > \rho \qquad \therefore P_K > P - \varepsilon$$

$$\therefore \lim_{k \doteq \infty} P_k = P \quad \text{or } P' = P$$

Converse Theorem

If P is a double product of factors greater than 1, and if each row of P is a convergent product, and if the product of the value of the rows (viz: $f^{(0)} \cdot f^{(1)} \cdot f^{(2)} \cdots$) converges to a value P, — then the double product converges to the same value.

Proof

Let $p = f_0 \cdot f_1 \cdot f_2 \cdots$ be any simple product of P and let

$$p_n = f_0 \cdot f_1 \cdot f_2 \cdots f_{n-1}$$

Choose n arbitrarily and hold it fast. Then box in the factors of p_n by the rectangle R, as in the preceding theorem.

Then, if $P_l = f^{(0)} \cdot f^{(1)} \cdots f^{(l-1)}$

$$p_n \leqq P_l < P$$

$\therefore$ $p_n < P$ and the double product converges to $P' \leqq P$

(Fundamental Principle, page 6)

To prove $P = P'$

Choose l so that $P_l > P - \varepsilon$ and then hold it fast. Now choose m so that the product of the factors in the first row to the left of the m-th column differs from $f^{(1)}$ by less than δ_ε. Do the same for each of the l rows and then select m large enough to be used for all at once. Then the product of terms in the rectangle R will differ from P_l by less than ε.

If g is the product of all factors within R

$$g > P_l - \varepsilon$$

and $\therefore$ $$g > P_l - \varepsilon > P - 2\varepsilon$$

$$\therefore \lim g = P$$

$$\text{or } \lim p_n = P, \quad \text{or } P' = P$$

The results of the last two theorems may be summarized as follows:

The necessary and sufficient condition for the convergence of a double product of factors greater than 1, is that

1) each row of the double product shall converge

2) the product of the values of the rows shall converge.

Double Products of factors greater than 1 and less than 1.

A double product of factors greater than 1 and less than 1, is said to be convergent if all of its simple products are convergent.

Theorem If a double product is convergent, all of its simple products converge to the same value.

Proof All of its simple products are convergent by hypothesis.

Let $p = f_0 \cdot f_1 \cdot f_2 \cdots\cdots$ be any one of its simple products. Then p can not be conditionally convergent, for if it were, it would be possible to rearrange its terms so as to make it a divergent product, and that would be contrary to the hypothesis that all the simple products are convergent. Therefore p is absolutely convergent and therefore unlimitedly commutative. Therefore all of its simple products converge to the same value.

<u>Definition</u> The value of a double product is the value of any one of its simple products.

<u>Theorem</u> The necessary and sufficient condition for the convergence of a double product, is that the conjugate product converge.

<u>Proof</u>. Let P be a double product and P_1 be the conjugate product.

1) If P converges, every one of its simple products converges <u>absolutely</u>, and therefore every simple product of P_1 converges. Therefore P_1 is a convergent double product and the condition stated is <u>necessary</u>.

2) If P_1 converges, every one of its simple products converges and therefore every simple product of P is absolutely convergent. Therefore P is a convergent double product, and the condition stated is <u>sufficient</u>.

<u>Note</u> A conditionally convergent double product is impossible.

<u>Theorem</u> Let P be any convergent double product.

1) Each row is a convergent product

2) The product $f^{(0)} \cdot f^{(1)} \cdot f^{(2)} \cdots$ is convergent, where $f^{(i)}$ is the value of the i-th row.

3) The product $f^{(0)} \cdot f^{(1)} \cdot f^{(2)} \cdots$ converges to the same value P, to which the double product converges.

<u>Proof</u> 1) Let P_1 be the conjugate double product of P. It is convergent. (Theorem, page 39)
Each row of P_1 is a convergent product.
(Theorem, page 33)
Therefore each row of P is absolutely convergent.

2) To prove $f^{(0)} \cdot f^{(1)} \cdot f^{(2)} \cdots$ convergent, where $f^{(i)} = f_0^{(i)} f_1^{(i)} f_2^{(i)} \cdots$
Consider the corresponding product
$$F^{(0)} \cdot F^{(1)} \cdot F^{(2)} \cdots$$
of the conjugate double product, where $F^{(i)} = F_0^{(i)} \cdot F_1^{(i)} \cdot F_2^{(i)} \cdots$
This product is convergent
(Theorem, page 33)
Therefore $f^{(0)} \cdot f^{(1)} \cdot f^{(2)} \cdots$ is absolutely convergent.

3) The value P of the double product is equal to the

value P' of the product $f^{(0)} \cdot f^{(1)} \cdot f^{(2)} \cdots$

Proof

Let $P_k = f^{(0)} \cdot f^{(1)} \cdots f^{(k-1)}$

Choose k so large that

$|P' - P_k| < \epsilon$ where ϵ is any chosen positive quantity. Then hold k fast. Now consider the rectangle R, of factors contained in the 1st l columns and the 1st k rows of P.

Choose l so that the product of the factors of any row of the rectangle differs from the product of that row of the double product by less than $\sqrt[k]{\epsilon}$. Let g denote the product of the factors of the rectangle.

Then $|g - P_k| < \epsilon$

but since $|P' - P_k| < \epsilon$

$$|P' - g| < 2\epsilon$$

This inequality will remain true if we increase k and l. Choose k and l so large that

$$|P - g| < \epsilon$$

Then $|P' - P| < 3\epsilon$

$$\therefore P' = P$$

Part II

A treatment of infinite products with the aid of infinite series.

Every infinite product $f_0 f_1 f_2 \cdots$ may be written in the form

$$(1+\alpha_0)(1+\alpha_1)(1+\alpha_2)\cdots\cdots$$

where $\lim_{n \doteq \infty} \alpha_n = 0$ and $\alpha_n \neq -1$

Theorem The necessary and sufficient condition for the convergence of the infinite product

$$(1+\alpha_0)(1+\alpha_1)(1+\alpha_2)\cdots$$

is the convergence of the series $\alpha_0+\alpha_1+\alpha_2+\cdots$

Case I When $\alpha_i \geqq 0$ $\qquad i = 0, 1, 2, 3 \cdots$

Proof 1) Necessary condition.

The infinite product converges, by hypothesis

$$\begin{aligned} p_n = 1 & \\ & + \alpha_0 + \alpha_1 + \alpha_2 + \cdots \alpha_{n-1} \\ & + \alpha_0\alpha_1 + \alpha_0\alpha_2 + \cdots \alpha_1\alpha_2 + \cdots \quad \alpha_{n-2}\alpha_{n-1} \\ & + \alpha_0\alpha_1\alpha_2 + \alpha_0\alpha_1\alpha_3 + \cdots\cdots \\ & \cdots\cdots \\ & + \alpha_0\alpha_1\alpha_2\cdots\alpha_{n-1} \end{aligned}$$

$p_n > \alpha_0 + \alpha_1 + \alpha_2 + \cdots \alpha_{n-1}$ since all the terms are greater than or equal to zero. Therefore if p_n and s_n, $(s_n = \alpha_0 + \alpha_1 + \alpha_2 + \cdots \alpha_{n-1})$ were plotted on a line, the points p_n would move to the right more quickly than s_n. That is, s_n is a variable which increases as n increases but always remains less than p_n. Therefore s_n approaches a limit and

$\alpha_0 + \alpha_1 + \alpha_2 + \cdots$ is a convergent series.

(Principle, page 6)

2) Sufficient condition.

$\alpha_0 + \alpha_1 + \alpha_2 + \cdots$ is convergent, by hypothesis

$$\therefore \lim_{n \doteq \infty} s_n = A$$

$$\alpha_0 + \alpha_1 + \alpha_2 + \cdots \alpha_{n-1} \leqq A$$

Squaring both sides

$$\alpha_0^2 + \alpha_1^2 + \cdots + 2\alpha_0\alpha_1 + 2\alpha_0\alpha_2 + \cdots + 2\alpha_1\alpha_2 + \cdots + 2\alpha_{n-2}\alpha_{n-1} \leqq A^2$$

Since all the terms are positive

$$\alpha_0\alpha_1 + \alpha_0\alpha_2 + \cdots + \alpha_1\alpha_2 + \cdots + \alpha_{n-2}\alpha_{n-1} \leqq \frac{A^2}{2}$$

Similarly

$$\alpha_0\alpha_1\alpha_2 + \alpha_0\alpha_1\alpha_3 + \cdots + \alpha_1\alpha_2\alpha_3 + \cdots \alpha_{n-3}\alpha_{n-2}\alpha_{n-1} \leqq \frac{A^3}{3!}$$

etc.

Adding these inequalities,

$$p_n \leqq 1 + A + \frac{A^2}{2} + \frac{A^3}{3!} + \frac{A^4}{4!} + \text{------}$$

This series is convergent for all values of A, and therefore p_n approaches a limit as n increases, for p_n increases but always remains less than or equal to the value of the above series.

Case II   where $\alpha_i \leqq 0$

Proof   1) Necessary condition

The infinite product converges, by hypothesis.

Then $\lim\limits_{n \doteq \infty} \alpha_n = 0$ since $\lim\limits_{n \doteq \infty} f_n = 1$

and it is possible to find a positive integer m such that

$$|\alpha_n| < 1 \qquad \text{when } n > m$$

$\therefore (1 + \alpha_n) = (1 - |\alpha_n|)$ which is a positive quantity when $n > m$.

Consider the product

$$\frac{1}{1+\alpha_m} \cdot \frac{1}{1+\alpha_{m+1}} \cdot \frac{1}{1+\alpha_{m+2}} \text{ ----} \qquad (A)$$

Each denominator is greater than zero, since $|\alpha_n| < 1$ if $n > m$

Therefore each term may be written in the form

$$\frac{1}{1-|\alpha_n|} = 1 - 1 + \frac{1}{1-|\alpha_n|} = 1 + \frac{\alpha_n}{1-|\alpha_n|} \qquad \text{where } n > m$$

(A) then becomes

$$\left[1+\frac{|\alpha_m|}{1-|\alpha_m|}\right]\left[1+\frac{|\alpha_{m+1}|}{1-|\alpha_{m+1}|}\right]\left[1+\frac{|\alpha_{m+2}|}{1-|\alpha_{m+2}|}\right]\cdots\cdots \qquad (B)$$

By case I, the necessary and sufficient condition for the convergence of (B) and therefore of (A), is the convergence of the series

$$\frac{|\alpha_m|}{1-|\alpha_m|}+\frac{|\alpha_{m+1}|}{1-|\alpha_{m+1}|}+\frac{|\alpha_{m+2}|}{1-|\alpha_{m+2}|}+\cdots\cdots \qquad (C)$$

The necessary and sufficient condition for the convergence of (C) is the convergence of the series

$$|\alpha_m|+|\alpha_{m+1}|+|\alpha_{m+2}|+\cdots\cdots$$

for $\displaystyle\lim_{n \doteq \infty} \frac{|\alpha_n|}{\frac{|\alpha_n|}{1-|\alpha_n|}} = 1$ and therefore both series converge or diverge together.

Therefore the necessary and sufficient condition for the convergence of A is the convergence of the series

$$|\alpha_m|+|\alpha_{m+1}|+|\alpha_{m+2}|+\cdots\cdots$$

But (A) is the necessary condition for the convergence of the given infinite product and therefore the necessary condition for the convergence of the given infinite product is the convergence of the series $\alpha_0+\alpha_1+\alpha_2+\cdots\cdots$

2) Sufficient condition.

By hypothesis, the infinite series

$$\alpha_0+\alpha_1+\alpha_2+\cdots \quad \text{converges}$$

$$\therefore \lim_{n\doteq\infty} \alpha_n = 0$$

$$\therefore |\alpha_n| < 1 \quad \text{when } n > m$$

Then since

$$\lim \frac{|\alpha n|}{\frac{|\alpha n|}{1-|\alpha n|}} = 1 \quad \text{the series (C) converges.}$$

Therefore the product (B) converges, also (A), and consequently the given product.

<u>Case III.</u> where $\alpha_i > 0$ and $\alpha_j < 0$

The sufficient condition for the convergence of the product, is that the series $\alpha_0+\alpha_1+\alpha_2+\alpha_3+\cdots$ converge absolutely.

<u>Proof.</u> The factors of the product will be of the forms $(1+\alpha_i)$ and $(1-\alpha_j)$.

Let $p'_{n'}$ equal the product of n' factors greater than 1 and $p''_{n''}$, the product of n'' factors less than 1.

Then $p_n = p'_{n'} \cdot p''_{n''}$ where $n = n'+n''$

By hypothesis $|\alpha_0|+|\alpha_1|+|\alpha_2|+\cdots$ converges.

Therefore the series

$$v_0 + v_1 + v_2 + \cdots \qquad \text{where } v_i > 0$$

and $w_0 + w_1 + w_2 + \cdots$ converge where $w_i < 0$

By cases I and II, $p'_{n'}$ and $p''_{n''}$ converge and approach the limits p' and p''.

Therefore the infinite product converges

$$p = p' \cdot p'' \qquad \text{(Theorem III, page 5)}$$

Tests for Convergence

Since the convergence of the infinite product depends upon the convergence of the series $\alpha_0 + \alpha_1 + \alpha_2 + \cdots$, the following well known tests for convergence may be used.

1. Comparison test
2. Test-ratio.

$$\lim_{n \doteq \alpha} \frac{a_{n+1}}{a_n} = T < 1, \text{ convergent}$$

$$\text{"} \quad \frac{a_{n+1}}{a_n} = T > 1, \text{ divergent}$$

$$\frac{a_{n+1}}{a_n} = T = 1, \text{ no test}$$

3. Test-ratio

$$n\left(1-\frac{a_{n+1}}{a_n}\right) = \sigma > 1, \text{ convergent}$$
$$= \sigma < 1, \text{ divergent}$$
$$= \sigma = 1, \text{ no test.}$$

Example:

Test the convergence of the product

$$\frac{3}{2}\cdot\frac{7}{6}\cdot\frac{13}{12}\cdot\frac{21}{20}\cdots\cdots$$

or $\left(1+\frac{1}{2}\right)\cdot\left(1+\frac{1}{6}\right)\cdot\left(1+\frac{1}{12}\right)\cdot\left(1+\frac{1}{20}\right)\cdots\cdots$

The series $\frac{1}{2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots$ is convergent, by comparison with the geometric series

$$\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots$$

Therefore the product converges.

Example:

Test the convergence of the product

$$\frac{3}{2}\cdot\frac{6}{4}\cdot\frac{11}{8}\cdot\frac{20}{16}\cdots\cdots$$

or $\left(1+\frac{1}{2}\right)\left(1+\frac{2}{2^2}\right)\left(1+\frac{3}{2^3}\right)\left(1+\frac{4}{2^4}\right)\cdots\cdots$

The series $\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+\cdots\cdots$ converges; for

$$\frac{a_{n+1}}{a_n} = \frac{n+1}{2^{n+1}}\cdot\frac{2^n}{n} = \frac{n+1}{2n} = \frac{1}{2}+\frac{1}{2n}$$

$$\therefore \lim_{n=\infty}\frac{a_{n+1}}{a_n} = \frac{1}{2}$$

Therefore the product $\frac{3}{2}\cdot\frac{6}{4}\cdot\frac{11}{8}\cdot\frac{20}{16}\cdots$ converges.

Theorem. The necessary and sufficient condition for the convergence of the infinite product,

$$(1+\alpha_0)(1+\alpha_1)(1+\alpha_2)----$$

where $\alpha_i \neq -1$ and $\alpha_i > 0$

is the convergence of the infinite series

$$\log(1+\alpha_0)+\log(1+\alpha_1)+\log(1+\alpha_2)+----$$

Proof

Let $p_n = (1+\alpha_0)(1+\alpha_1)---- (1+\alpha_{n-1})$

Taking logarithms

$$\log p_n = \log(1+\alpha_0)+\log(1+\alpha_1)+---\log(1+\alpha_{n-1})$$

1) Necessary condition.

Hypothesis. The infinite product converges. i.e. $\lim_{n \doteq \infty} p_n$ exists and $\lim p_n \neq 0$.

Let $\lim_{n \doteq \infty} p_n = P$

Since the logarithmic curve $y = \log x$, is single valued and continuous and always increases as x increases, $\log p_n$ approaches a limit when $n \doteq \infty$.

But $\log p_n$ is the sum of the first n terms of the series $\log(1+\alpha_0)+\log(1+\alpha_1)+-----$, therefore the series approaches a limit as n increases and is therefore convergent.

2) Sufficient condition

Hypothesis - The series

$\log(1+\alpha_0) + \log(1+\alpha_1) + \cdots$ converges

Therefore if $p_n = (1+\alpha_0)(1+\alpha_1) + \cdots (1+\alpha_{n-1})$,

$\lim_{n \doteq \infty} \log p_n$ exists.

Let $\lim_{n \doteq \infty} \log p_n = P$, then from the character of the curve $y = \log x$, p_n approaches a limit when $n \doteq \infty$. Therefore the infinite product converges.

<u>Theorem</u> A necessary condition for the convergence of the infinite product

$$f_0 \cdot f_1 \cdot f_2 \cdots$$

is that $\lim_{n \doteq \infty} f_n = 1$

<u>Proof</u> by means of the logarithmic series.

Take the logarithm of the product and get $\log f_0 + \log f_1 + \log f_2 + \cdots$

The necessary condition for the convergence of this series is

$$\lim_{n \doteq \infty} \log f_n = 0$$

But if $\lim \log f_n = 0$, $f_n = 1$

Therefore the necessary condition for the convergence of the infinite product is that $\lim_{n \doteq \infty} f_n = 1$

This proof is simpler than the one given in Part I, without the use of series.

Absolute Convergence

Theorem The necessary and sufficient condition for the absolute convergence of a product, is the absolute convergence of the series

$$\log(1+\alpha_0) + \log(1+\alpha_1) + ---$$

Proof. The necessary and sufficient condition for the interchange of terms in the series is the absolute convergence of the series. Any rearrangement of terms in the series produces a corresponding rearrangement in the product, therefore the necessary and sufficient condition for the absolute convergence of the product is the absolute convergence of the series

$$\log(1+\alpha_0) + \log(1+\alpha_1) + \log(1+\alpha_2) - - - - -$$

<u>Theorem</u> The necessary and sufficient condition for the absolute convergence of the infinite product $(1+\alpha_0)(1+\alpha_1)(1+\alpha_2) - - - -$, is the absolute convergence of the series,

$$\alpha_0 + \alpha_1 + \alpha_2 + - - - - -$$

<u>Proof</u> If we have given two series

$$u_0 + u_1 + u_2 + - - - \qquad (A)$$

$$v_0 + v_1 + v_2 + - - - \qquad (B)$$

of which (B) is absolutely convergent, then if the $\lim\limits_{n \doteq \infty} \dfrac{u_n}{v_n}$ exists, the series (A) is absolutely convergent.

Applying this theorem to the series

$$\alpha_0 + \alpha_1 + \alpha_2 + - - - \qquad (C)$$

$$\log(1+\alpha_0) + \log(1+\alpha_1) + \cdots \qquad (D)$$

we get $\quad \lim \dfrac{\alpha_n}{\log(1+\alpha_n)} = 1$

since $\lim\limits_{x \doteq 0} \dfrac{x}{\log(1+x)} = 1$ and since $\lim\limits_{n \doteq \infty} \alpha_n = 0$

Therefore the necessary and sufficient condition for the absolute convergence of the series (C) is the absolute convergence of the series (D). But by the preceding theorem the necessary and sufficient condition for the absolute convergence of the infinite product, $(1+\alpha_0)(1+\alpha_1)(1+\alpha_2)----$ is the absolute convergence of the series,

$$\log(1+\alpha_0)+\log(1+\alpha_1)+ ------$$

Consequently the necessary and sufficient condition for the absolute convergence of the infinite product,

$$(1+\alpha_0)(1+\alpha_1)(1+\alpha_2)----$$

is the absolute convergence of the series,

$$\alpha_0+\alpha_1+\alpha_2+ ----$$

Algebraic Transformation of Infinite Products

Let $A = (1+\alpha_0)(1+\alpha_1)(1+\alpha_2) ---$

$\quad B = (1+\beta_0)(1+\beta_1)(1+\beta_2) ----$

be two convergent infinite products

Then $AB = (1+\alpha_0)(1+\beta_0)(1+\alpha_1)(1+\beta_1) -----$

Proof

$$\log A = \log(1+\alpha_0) + \log(1+\alpha_1) + \cdots$$

$$\log B = \log(1+\beta_0) + \log(1+\beta_1) + \cdots$$

are two convergent series. They may be added term by term.

$$\therefore \log A + \log B = \log(1+\alpha_0) + \log(1+\beta_0) + \log(1+\alpha_1) + \cdots$$

$$\text{or } \log AB = \log(1+\alpha_0)(1+\beta_0)(1+\alpha_1)(1+\beta_1)\cdots$$

$$\therefore AB = (1+\alpha_0)(1+\beta_0)(1+\alpha_1)(1+\beta_1)$$

Special Infinite Products.

Among the few infinite products which have been developed from time to time, there is one of special interest, known as Wallis's formula.

$$\frac{\pi}{2} = \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdots}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdots}$$

(See Part I, page 10)

Wallis arrived at this result in a rather peculiar manner, while studying the quadrature of the circle.

The same expression may be obtained more readily by means of definite integrals.

$$\int_0^1 \frac{x^{2n}\,dx}{\sqrt{1-x^2}} = \frac{1\cdot 3\cdot 5 \cdots (2n-1)}{2\cdot 4\cdot 6 \cdots 2n}\cdot\frac{\pi}{2}$$

$$\int_0^1 \frac{x^{2n+1}\,dx}{\sqrt{1-x^2}} = \frac{2\cdot 4\cdot 6 \cdots 2n}{3\cdot 5\cdot 7 \cdots (2n+1)}$$

But $$\int_0^1 \frac{x^{2n-1}\,dx}{\sqrt{1-x^2}} > \int_0^1 \frac{x^{2n}\,dx}{\sqrt{1-x^2}} > \int_0^1 \frac{x^{2n+1}}{\sqrt{1-x^2}}dx$$

since x is never greater than 1

$$\therefore \frac{2\cdot 4\cdot 6 \cdots (2n-2)}{3\cdot 5\cdot 7 \cdots (2n-1)} > \frac{1\cdot 3\cdot 5 \cdots (2n-1)}{2\cdot 4\cdot 6 \cdots 2n}\cdot\frac{\pi}{2} > \frac{2\cdot 4\cdot 6 \cdots 2n}{3\cdot 5\cdot 7 \cdots (2n+1)}$$

$$\therefore \frac{2\cdot 2\cdot 4\cdot 4\cdot 6 \cdots (2n-2)2n}{1\cdot 3\cdot 3\cdot 5\cdot 5 \cdots (2n-1)(2n-1)} > \frac{\pi}{2} > \frac{2\cdot 2\cdot 4\cdot 4 \cdots 2n\cdot 2n}{1\cdot 3\cdot 3\cdot 5 \cdots (2n-1)(2n+1)}$$

These two products differ by only one factor which approaches the limit 1, as n becomes infinite. Consequently

$$\frac{\pi}{2} = \frac{2\cdot2\cdot4\cdot4\cdot6\cdot6\cdots}{1\cdot3\cdot3\cdot5\cdot5\cdot7\cdots}$$

While working at the same problem of squaring the circle, Vieta found that the ratio of the area of the square of diagonal 2, to the area of the circumscribing circle, or the value of $\frac{2}{\pi}$, is the infinite product

$$\sqrt{\tfrac{1}{2}}\cdot\sqrt{\tfrac{1}{2}+\tfrac{1}{2}\sqrt{\tfrac{1}{2}}}\cdot\sqrt{\tfrac{1}{2}+\tfrac{1}{2}\sqrt{\tfrac{1}{2}+\tfrac{1}{2}\sqrt{\tfrac{1}{2}}}}\cdots\cdots.$$