A Proposed Algebra Problem-Analysis Model

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Dedication

Everything here is dedicated to my mom and dad for always supporting me in whatever crazy life adventures I take, even when those adventures move me thousands of miles away, and to Katie and Henry, for being so totally awesome.

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Abstract

The National Mathematics Advisory Panel (2008) states that algebra is a gateway to high school graduation and college success. While existing research emphasizes the importance of quality algebra instruction, the current body of research on algebra problem-analysis for struggling secondary students is small. This paper proposes a problem-solving model to help support those students struggling with algebra. The model integrates the recommendations from math policy boards and research. It is composed of five core sections, each section focusing on a specific critical component of school algebra. The study examines the relationship between the five skills within the model to an established measure of algebra, as well as the validity of the measures being used to assess the different skill areas The results indicate that there is a significant relationship between the five sections of the model and algebra proficiency, and that the model is able to identify non-proficiency students with a high degree of accuracy.

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CHAPTER 1: INTRODUCTION

Introduction

Mathematics is a fundamental skill that is required to successfully maneuver through adult life. The expectations regarding the type of mathematics skills that students are expected to possess by the time they leave school are changing, and are very different from just a few decades ago: students are expected to not only be able to computer numbers but reason and think mathematically (National Research Council, 2001). The different areas of math vary, but they are all important. Recently, much focus has been shifted towards algebra, and requiring students to have some level of proficiency prior to graduation (National Mathematics Advisory Panel (2008).

Statement of the Problem

The National Mathematics Advisory Panel (2008) states that algebra is a gateway to later achievement, and that there is a strong correlation between the completion of algebra II and the likelihood of college success and college graduation. Students who complete algebra II are also more likely to have higher college GPAs, graduate from college, and higher earnings in later life (Gaertner, Kim, DesJardins, & McClarty, 2014).

Algebra proficiency is a requirement for employment in growing fields like engineering, medicine, science, programming, and technology (Zeldin, Britner, & Pajares, 2008; Steen, 1999; Usiskin, 1995). There is a call from these and other professional communities for more students who are prepared to enter a career in the STEM field (NCTM, 2014). For many students, the choice to pursue the types of college programs leading to STEM careers happens as early as middle school, and those choices are often a reflection of a student's learning experiences with science and math content, and not on their ability alone (Crisp, Nora, & Taggart, 2009; Hazari, Tai, & Sadler, 2007; Wai et al., 2010).

There is a call for algebra proficiency at the high school level as well. At the present time a number of states require that students take and pass at least one algebra course while in high school (American Diploma Project Network, 2009), making algebra a gateway to not just college but to high school graduation (Stein, Kaufman, Sherman, & Hillen, 2011). To meet this demand, schools are requiring that students take algebra in eighth grade or earlier.

Earlier exposure to content can be beneficial to some students (Rickles, 2013), but for others it may prove to be harmful (Stein, Kaufman, Sherman, & Hillen, 2011). Requiring students to take higher-level math courses without adequate preparation can result in higher failure rates and lower grades in both their current class and future classes (Allensworth, Nomi, Montgomery, & Lee, 2009; Domina, 2014; Clotfelter, 2012). Putting struggling students in algebra classes with higher performing students might impact the performance of both higher and lower performing students (Nomi, 2012).

The educational and professional systems in the United States are designed in a way that require all students to possess a working knowledge of algebra, but current academic trends indicate that this goal is still a ways off. On the National Assessment of Educational Progress, an assessment meant to represent the average U.S. student achievement in different academic areas, the average score for a 12th grade student on the mathematics test's algebra scale was 155, which falls significantly below the expected proficiency score of 176 and indicates only basic understanding of algebra and its applications and procedures (National Center for Education Statistics [NCES], 2014). There are many potential causes of the gaps in achievement outlined above, but what remains clear is that a large number of students are struggling with algebra.

All students require quality core instruction in the classroom, but approximately 10-15% of students will require more intensive academic support and intervention (Averill & Rinaldi, 2011). Thus, even with solid core instruction, these 10-15% of students will fall behind their same-age peers in acquisition of mathematics skills, ultimately impacting their test performance, graduation rates, college choices, careers, and future success. Existing research emphasizes that early levels of low mathematics achievement can predict negative outcomes for students (Casillas et al., 2012). Research indicates that while interventions for high school students can help increase overall academic achievement, it is a struggle to raise the algebra achievement for a growing number of students (Anderson, Yilmaz, & Wasburn-Moses, 2004; Scruggs, Mastropieri, Berkeley, & Graetz, 2010).

Current algebra intervention methods are reflective of traditional methods of algebra instruction; the interventions are oriented towards the student's current curriculum and classwide learning goals and focus on strategies to teach new and difficult concepts, but do not incorporate an evidence-based problem-solving framework of skill acquisition and learning. Before identifying the problems with the interventions focusing primarily on current curriculum, it should be noted that, at the present time, there is not a comprehensive review of curriculum-based versus problem-solving-based algebra interventions, so a comparison of the effectiveness between the two is not possible. With that in mind, this paper posits that there are three major problems with focusing solely on algebra curriculum teaching techniques.

First, failing to identify and teach basic pre-requisite skills to mastery can prevent a student from mastering advanced skills requiring high levels of proficiency with said pre-requisite skills. For example, proficiency in arithmetic problem solving with both whole and rational numbers is a major component for solving algebraic equations (NMAP, 2008). Logically, we can assume that if a student lacks proficiency with whole and rational numbers, they are much less likely to learn to solve algebraic equations proficiently.

Second, focusing only on general instruction can impact the pacing of the class, and students who take more basic algebra classes, or classes extended over two years, are much less likely than their peers to take additional science and math classes that can help them become college ready (Paul, 2005; Stein, Kaufman, Sherman, & Hillen, 2011), and students who take richer math and science classes in high school are much more likely to take additional classes in college and enter a career in the STEM field when compared to those students who did not, irrespective of individual student ability (Crisp, Nora, & Taggart, 2009; Hazari, Tai, & Sadler, 2007; Wai et al., 2010). It stands to reason that when a student lacks a core understanding of the material, and if those core misunderstandings are not the target of intervention, the more errors they will make and the longer it will take to learn new content (Panasuk, 2010). In theory, targeting these deficits will help lay the foundation for more successful and faster-paced learning experiences.

Third, approximately two-thirds of schools across the country are adopting a three-tiered model of instructional supports where each successive tier increases the level of support and targeted instruction for students failing to respond to the previous tier, but only approximately half are targeting math and approximately one-third are in place in high schools (Spectrum K-12 School Solutions, 2010). The current algebra intervention and instruction literature provides general classroom practices that fall under tier 1, and instructional delivery methods that can be implemented in tiers 2 and 3. However, none of the existing research provides guidance as to how to assess algebra difficulties and intervene. Given how frequently students struggle with math general proficiency (Hanushek, Peterson, & Woessmann, 2010) and algebra (NCES, 2014), and the need for guidance regarding a multi-tiered system of support at the high school level, the field needs an evidence-based model that can apply at all three tiers, not just the first.

Fourth, there is a need for an algebra-specific diagnostic and problem-analysis model because, at this time, none exist. Problem analysis is the systematic assessment and evaluation of a problem to find the potential causes as well as what is enabling the problem to continue to exist with a goal of isolating and intervening with one or more specific skills (Christ & Arañas, 2014). Analyses to identify potential causes of academic problems could include multiple environmental, educational, and instructional components as well as specific student skill deficits (Christ, Burns, & Ysseldyke, 2005). The process of problem analysis involves the explicit and systematic testing of hypotheses and clearly defined variables that are related, alterable, and relevant to an intervention (Christ, 2008). There has been limited research in problem analysis for mathematics and even less so for algebra. Below, I will discuss a proposed model for problem analysis with algebra.

Study Purpose

The need for an algebra research is paramount because achievement data indicates our students continue struggling with algebra, but there is not a well-developed problem-analysis model for intervening with struggling students. Direct instructional activities and guidelines exist for a number of the content areas within algebra (Foegen, 2008; NMAP, 2008), but academic interventions are more successful if they directly address the student deficit (Burns, VanDerHeyden, & Boice, 2008). As of now there are no clear guidelines about how to assess for specific skill and knowledge-based deficits, and then select appropriate, evidence-based interventions.

Identification of these specific deficits should be guided by the use of valid assessments that target each area (Thompson, 2004). At this time, however, there are no standardized mathematics assessments that target the proposed areas in the model. Consequently, there are currently no standardized algebra assessments that can help guide researchers in crafting a problem-solving model. If there is going to be an evidence-based problem-solving model leading to effective algebra interventions, appropriate assessments should exist.

The proposed study will attempt to evaluate the validity of using a five-part problem-solving model to identify core deficits for students struggling with algebra. The five core sections of the problem-solving model support and inform each other, and are all required to establish a basis for algebra proficiency. Being able to systematically assess and evaluate the specific knowledge and skill deficits that are leading a student to struggle will lead to increasingly effective instructional modifications and interventions, and when interventions are occurring, these analyses will allow for a more thorough connection between the intervention and instructional components being taught.

The rationale behind this study is to propose a model that utilizes an evidence-based problem-analysis model to identify specific skill and knowledge deficits related to algebra, and to develop an assessment that effectively assesses a student's performance with each of the target areas.

The following research questions will guide this research:

1. What is the relationship between each of the five sections within the problem-analysis model to an established measure of algebra?

2. To what extent do assessment data support the proposed five-factor structure?

3. To what extent can the five sections and each subsection within the problem-analysis model accurately identify the level of a student's difficulty with algebra as measured by a criterion?

Organization of the Dissertation

The following literature review will define the type of algebra that is focused on in this paper. In that review, important components and skills required for successful algebra learning will be identified and defined. Additionally, methods of assessing algebra skills and providing support will be discussed. The methods section will then describe the participants, procedures, measures, and analyses being used. The results sections will analyze the results, and the discussion section will discuss their meaning, importance, and potential impact.

CHAPTER 2: LITERATURE REVIEW

The first section of the literature review will discuss the current model of mathematics proficiency, so as to define what this paper will refer to when discussing the subject. The second section of the review will define what the literature deems necessary to be covered under the title of school algebra. This will serve to contain the content being discussed in the paper, as the term algebra covers a wide range of topics. The third section of the review will identify the skills required to be proficient with algebra. The fourth section will discuss certain skills more in depth. The last two sections will lay the foundation for the dissertation's research.

Current Model of Mathematics Proficiency

The term proficiency refers to what someone knows how to do, can do, and ultimately wants to do (Schoenfeld, 2007). When it comes to academics, specifically math, we expect that a student who demonstrates proficiency to be knowledgeable, flexible, and resourceful with a given skill or set of skills (Foegen, Olson, & Impecoven-Lind, 2008; Goertz & Duffy, 2003). The National Research Council (NRC, 2001) identifies mathematics proficiency as possessing five different yet interwoven "strands," conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition.

Conceptual Understanding

Conceptual understanding is recognizing and understanding the core underlying ideas of a subject such as the relationships and reasons that underlie the mathematics problems in a certain area (Byrnes & Wasik, 1991; Hiebert & Lefevre, 1986). It is knowledge that can be generalized to a specific area and underlying core principals, and does not necessarily refer to a specific set of problems. Thus, conceptual knowledge can be implicit or explicit and applied flexibly (Rittle-Johnson, Siegler, & Alibali, 2001; Schneider & Stern, 2010). Building on this, Crooks and Alibali (2014) identified six separate yet related types of conceptual knowledge; relationships, rules and facts, procedures, categories, symbols, and structures.

Researchers measure conceptual understanding several ways including having students complete conceptual-based mathematics items (Helwig, Anderson, & Tindal, 2002) or assessing underlying mathematics principles such as inversion and commutative property (Canobi et al., 1998; Geary, 2006). However, many items used to assess conceptual understanding have been criticized for actually measuring application rather than underlying concepts (Zaslofsky & Burns, 2014), and students may be able to employ and continuously use strategies related to mathematics principles without a developed conceptual understanding (Canobi, 2009).

Conceptual understanding involves describing mathematical ideas and problem solving processes beyond identifying the individual steps to solving a problem (Bartell et al., 2013). Having strong conceptual understanding allows for more flexible problem solving, recognition of whether a problem is correct or incorrect, and identifying whether an achieved solution is reasonable (Baroody, Feil, & Johnson; Rittle-Johnson, Siegler, & Alibali, 2001; Booth et al., 2013; Crooks & Alibali, 2014). The focus on description and explanation helps facilitate a deeper understanding of the material that can lead to longer lasting and transferable knowledge (Krebs, 2003; Crooks & Alibali, 2014; Renkl et al., 1998).

Solely relying on procedural recall can lead to errors and leave students lacking a solid base on which to build new knowledge (NRP, 2008). The possession of a deep understanding of mathematical content that provides flexible application and problem solving is a critical uniting thread for successfully understanding algebra (Woodbury, 2000).

Procedural Fluency

Procedural fluency refers to a person's knowledge of rules, symbols, and sequence of steps required to solve mathematics problems (Zamarian, Lopez-Rolon, & Delazer, 2007), and his or her ability to carry out mathematical operations accurately, efficiently, and appropriately (Baroody, Feil, & Johnson, 2007; NRC, 2008). Skills in this area are often exhibited when students quickly retrieving correct answers and proficiently completing algorithms to compute mathematical operations (Geary, 1993; Rittle-Johnson et al., 2001). Procedural fluency helps facilitate comparing magnitudes, estimation, calculation skills, and mental problem solving (Byrnes & Wasik, 1997).

Although it is unclear whether conceptual understanding or procedural fluency develops first, or in what order they develop (Rittle-Johnson & Siegler, 1998), conceptual understanding may provide the basis for procedural fluency. For example, students with conceptual understanding should be able to apply the certain concepts of understanding to solving familiar problems, even if they do not have procedural fluency regarding a certain topic (Greeno, Riley, & Gelman, 1984). Moreover, the strategies used by a student to solve a mathematics problem can be based on the procedures associated with the problem or an invented approach based on strong conceptual understanding of the problem being addressed (Star & Rittle-Johnson, 2011; Xin et al., 2011). Having the support of conceptual understanding reduces errors in problem solving, increases retention, and allows application and extension of knowledge (Baroody, Feil, & Johnson, 2007).

Star (2005) attempts to reconceptualize procedural fluency as being either shallow or deep; shallow understanding leads to the errors commonly associated with procedural fluency, but deep understanding allows for problem solving flexibility through choosing of appropriate procedures. The ability to efficiently utilize multiple problem solving procedures reduces the likelihood of error and increases problem-solving effectiveness (Star, 2005; Krebs, 2003; Baroody, Feil, & Johnson, 2007). Being too practiced with one type of problem-solving procedure can lead students to overly on what may be an inappropriate procedure, but if a student has learned multiple algebra problems-solving methods, they will often use the more efficient method (Newton, Star, & Lynch, 2010).

Strategic Competence

Strategic competence is defined as the ability to formulate, represent, and solve mathematical problems and is often referred to as problem formulation and problem solving (NRC, 2001; Hiebert & Grouws, 2007). Similar to the conceptualization of procedural fluency by

Star (2005), strategic competence helps facilitate efficient and strategic use of problem-solving methods. It is influenced by previous knowledge and structure of the task (Ross et al., 2011; Staats & Batteen, 2009; Townsend, Lannin, & Barker, 2009). Students draw on the conceptual understanding of the problem and their fluency with procedures to aid in flexible strategy use (Van Dooren, 2002). Other factors that impact strategic competence are social influences and visual images generated by the student to solve the problem (Lannin, Barker, and Townsend, 2006).

Students who display strategic competence analyze tasks, monitor performance, and shift between different problem-solving methods (Özdemir & Pape, 2012). That student will figure out how to solve a problem by knowing strategies to analyze and solve problems, identifying potential appropriate strategies before addressing the problem, and shifting between these different strategies based on their evaluation of the effectiveness (Özdemir & Pape, 2012). When presented with algebra problems, the student should also be able to understand the key features of the problem, formulate the problem appropriately using numbers or symbols, and ignore irrelevant information (NRC, 2001). This requires an understanding of relationships and how to flexibly apply solutions to both routine and non-routine problems (NRC, 2001). Finally, a student should be able to develop multiple representations of a given mathematics concept when presented with a problem (Panasuk, 2010).

Adaptive Reasoning

Adaptive reasoning is defined as the "capacity for logical thought, reflection, explanation, and justification" (NRC, 2001 [p. 5]). Adaptive reasoning is the ability to reflect on the choices made while problem-solving, provide justifiable reasons for the decisions, construct arguments and counter examples, and make logical conclusions based on the information (NRC, 2001; Kaasila, Pehkonen, & Hellinen, 2010; Kilpatrick, 2001). Developing reasoning may take the form of informal methods, understanding through persona experiences, and visualization of problems (Kasmer & Kim, 2012). It assists with justifying mathematical decisions, and reasoning with mathematics facts through inductive and deductive reasoning using patterns, analogies, and metaphors (NRC, 2001). Adaptive reasoning is closely related with strategic competence; strategic competence helps a student formulate a problem-solving plan while adaptive reasoning helps the student understand if that plan is effective (NRC, 2001).

Having prior knowledge, a positive disposition and possessing a strong conceptual understanding of algebra is crucial to reasoning successfully because without them, reasoning strategies will likely not be implemented appropriately (Kasmer, & Kim, 2012; Nathan & Koellner, 2007). Other components of adaptive reasoning include synthetizing and integrating information; analyzing; evaluating; generalizing; connecting; predicting; and justifying (Mullis et al., 2001). Reasoning is a major struggle for students moving from arithmetic to algebra (Walkington, Petrosino, & Sherman, 2013). Students begin experiencing success when they are able to make explicit connections, especially through big ideas (Kasmer, & Kim, 2012).

Productive Disposition

Productive disposition refers to a person's tendency to perceive mathematics as valuable, believe that they are a capable learner of math, and understand that there is a value to mathematics beyond the classroom (NRC, 2001). Students with productive dispositions are resilient, persevere when presented unfamiliar or challenging problems, and have a drive to solve problems because they want to know the answer, not just because they are being required to complete the task (Krebs, 2003).

Productive disposition can be broken down into to two main components: finding value in the task, and finding value in oneself as a learner (Gilbert, 2014). Students who find value on the task show higher levels of engagement, which can contribute significantly to academic progress (Christenson, Reschly, & Wylie, 2012; Reyes et al., 2012). Finding value often means relating content to real-world examples, and these sorts of applications of content can increase achievement and strengthen conceptual understanding, strategic application, and adaptive reasoning (Friedlander & Arcavi, 2012; Phelan et al., 2011; Newton, Star, & Lynch, 2010).

Students who perceive themselves as successful mathematics learners have higher rates of achievement (Pietsch, Walker, & Chapman, 2003), and students who feel that they are successful at mathematics also show higher rates of achievement (Fast et al., 2010; Galla et al., 2014; Lewis et al., 2012).

Students with higher rates of productive disposition are more engaged in class, and students who are engaged are more likely to learn than their peers who have less productive dispositions (Christenson, Reschly, & Wylie, 2012; Gilbert, 2014). Among its many benefits, fostering high levels of student engagement assists with algebra skill acquisition and understanding of material (Ross & Willson, 2012), finding value within the class, decreasing mathematics anxiety (Martin et al., 2012; Martin et al., 2014), and increasing levels of academic resilience (Finn & Rock, 1997, Moller et al., 2014; Nichols & White, 2001).

Synthesis

Theoretical and applied mathematics research strongly support that acknowledging and addressing mathematics proficiency is crucial across all areas of mathematics, from basic calculation to algebra to calculus and beyond, as proficiency moves someone beyond rote memorization of content to promote generalization and flexibility (NRC, 2001; Rittle-Johnson & Siegler, 1998). Conceptual understanding (Crooks and Alibali, 2014; Woodbury, 2000) and procedural fluency (Newton, Star, & Lynch, 2010; Star, 2005) provide the thorough knowledge base that supports and facilitates learning of both strategic competence (Van Dooren, 2002) and adaptive reasoning (Kaasila, Pehkonen, & Hellinen, 2010; Kilpatrick, 2001), and when all are tied together with a productive disposition (Gilbert, 2014; Christenson, Reschly, & Wylie, 2012; Reyes et al., 2012), all are necessary and working in concert to support algebra proficiency (NRC, 2001).

While the importance of strategic competence and adaptive reasoning to algebra proficiency is acknowledged throughout the literature, there is a dearth of experimental research on supporting the growth of struggling algebra students in these areas through intervention

(NMAP, 2008). In addition, the conceptual understanding and procedural fluency intervention research literature focuses primarily on elementary and early middle school aged students and calculation procedures, neglecting the needs of later middle school and high school students. The importance of mathematics proficiency across all areas of the subject is undeniable (NRC, 2001), but more experimental research needs to be conducted to identify ways of developing that proficiency in struggling middle and high school learners.

Definition of School Algebra

Much like the definition of mathematics proficiency, the definition of algebra is multifaceted and involves many dynamic components, ultimately promoting many different ideas as to what algebra actually is. To some algebra is the study and memorization of mathematical formulas and procedures, while to others algebra is the process of abstraction, generalization, and reasoning (Kortering, deBettencourt, & Braziel, 2005). In some respect both of these views can be considered correct, as over the past two centuries the concept of what algebra is and how it should be taught has changed dramatically (Kilpatrick & Izsák, 2008).

The National Council of Teachers of Mathematics (NMAP, 2008) defines algebra as a way of thinking and a set of concepts and skills that enable students to generalize, model, and analyze mathematical situations, and which provides a systematic way to investigate relationships, helping to describe, organize, and understand the world. This definition encompasses a wide range content, so to provide more specifics we will focus on the type of algebra instruction that exists within public schools and is labeled as school algebra (NRC, 2001). The NMAP (2008) identified the core components of school algebra as those that should be learned throughout high school, and which is most commonly split between Algebra I and Algebra II but may also be interwoven though a number of other courses like geometry, trigonometry, and statistics. The NMAP (2008) identified five major topics of school algebra. Each will be briefly described below.

Symbols and Expressions

The first major topic of school algebra is that of symbols and expressions. Symbols are items or signs that represent something else (i.e. a value, a process), and they are used to make statements about things. Symbols include basic expressions like + to mean addition and – to mean subtraction as well as more complex symbols like Σ, representing sigma and indicating a summation of all the values within a given series (Van De Walle, Karp, & Bay-Williams, 2013). An expression refers to a mathematic phrase made up of a finite number of symbols that can include constants, variables, operations, functions, or a number of other symbols (Kaufman & Schwitters, 2004). The National Research Council (2008) identifies three different skills under the heading of symbols and expressions that students should know: polynomial expressions, rational expressions, and arithmetic and finite geometric sequences. Polynomial expressions include real numbers and variables; more than one term; can only include addition, subtraction, and multiplication; and include exponents. Rational expressions (or fractional expressions) occur when one polynomial is divided by another polynomial. Finally, in an arithmetic sequence a number is found by adding a constant from one term to the next, in a geometric sequence each number is found by multiplying a constant to the previous term (NRC, 2008).

Instruction with symbols and expressions, or using variables and equivalence, are two of the biggest challenges for students transitioning from arithmetic to algebra (NMAP, 2008). Algebra instruction research has identified the following instructional techniques as effective in teaching these concepts: modeling thinking using multiple representations; justifying reasoning behind solutions; critiquing others; use of visual tools; comparing different equations to see if they are similar; cooperative learning; graphic organizers; and modeling, feedback, and guided practice (Knuth, Alibali, Hattikudur, McNeil, & Stephens, 2008; Gain & Sheffield, 2015; Haas, 2005; Maccini, McNaughton, & Ruhl, 1999; Rakes, Valentine, McGatha, & Ronau, 2010; Lannin et al., 2008).

Linear Equations

The second topic, linear equations, involves one or two variables, the numbers and knowns in the equation represent constant values and not zero, the unknowns are variables, and when graphed the number falls on a straight line (Kaufman & Schwitters, 2004). One simple example might look like $Ax + By = C$. Solving linear equations using multiple strategies has long been considered a critical base skill for students in algebra (Wolfram, 2014). According to the NRC (2008), a student should be able to able to use linear equations to solve problems and graph linear equations, linear inequalities (linear equations using signs of inequality (ex. less than, greater than, not equal to)), and multiple linear functions.

As students begin working with linear equations in seventh grade, and they often serve as one of the major entrances into algebra for students, the research is filled with instructional methods for teaching linear equations (CCSS, 2011). The focus on variables and equivalence is a major component of linear equations (Pirie & Martin, 1997). Two standard approaches for teaching linear equations are one, having students change the equation on both sides until an answer is provided, and two, using inverse operations to change values and signs on either side of the equation (Pirie & Martin, 1997). One way to support student learning is use arithmetic sequences and tables to organize and chart the different values of an equation (Matsuura & Harless, 2012). Another is to use concrete examples by having students construct different items using outcomes from linear equations (Hedin, 2007).

Quadratic Equations and Polynomials

Third are quadratic equations and polynomials. Quadratic equations are equations that contain one variable with an exponent of two (ex. $X^2 = 25$), and under this category the NRC (2008) identifies four different critical skills. First, a student should be able to factor, or multiply, quadratic polynomials (polynomials with an exponent of two) with integer coefficients (a number of symbol multiplied by a variable or unknown) (NRC, 2008). Second, a student should be able to complete the square in quadratic equations. This means rearranging the equation into squared parts. Third, a student should understand the quadratic formula ($ax^2 + bx + c = 0$) and be able to factor general quadratic polynomials, or breaking them down so they can be solved. Finally, a student should be able to use the quadratic equation to solve problems (Kaufman & Schwitters, 2004).

With regards to polynomials, a student should be able to factor polynomials (ex. $X^3 – 3x^2 – 2x + 6 = (x-3)(x^2 – 2)$ as well as find the roots of a polynomial (NRC, 2008). The root of a polynomial indicated that the solution of a polynomial equation equals zero. Second, a student should be able to understand and use complex numbers. A complex number is any number that can be expressed by a + bi, where a and b are real numbers and i is an imaginary unit. An imaginary number is labeled as so because, when squared, it gives a negative result, which does not happen. Imaginary numbers allow us to find the square roots of negative numbers (Kaufman & Schwitters, 2004; Wolfram, 2014). Third, a student should understand the Fundamental Theorem of Algebra, which states that every polynomial equation having complex coefficients and degree greater than or equal to one has at least one complex root (NRC, 2008). The degree of a term refers to the sum of the variable's exponents in the problem, and a complex root happens when a quadratic factor is not able to be reduced over real numbers. Fourth, a student should be able to understand binomial coefficients and Pascal's triangle. Binomial coefficients are generally written $\frac{n}{k}$ and refer to the number of combinations of *k* items that can be taken from *n* items, and is read as "n choose k". Pascal's Triangle arranges the binomial coefficients in a way that allows you to see the probability to any combination. It is a triangular pattern where the outside of the triangle is one, and every internal number is the sum of the two numbers above it (Wolfram, 2014). The last topic area is Mathematical induction and the binomial theorem. Mathematical induction is a two-step process of proving mathematical statements. First, the statement is proven true for the first number. Second, it is proven that if the statement is proven true for one number, it is proven true for the next number. The binomial theorem is a method of calculating the power of a binomial without having to multiply the problem (Wolfram, 2014).

Students begin using quadratic equations and polynomials in high school (CCSS, 2011). This will build on previously acquired knowledge of variables, equivalence, linear equations, and other skills, and as with those skills, it is crucial that students develop conceptual understanding of quadratic equations (NRC, 2001). A major concept that students need to understand is that binomial multiplication is an extension of the distributive property, because if they do not, they will likely be unable to multiply trinomials successfully (Kalder, 2012).

Once a student gains conceptual knowledge, they are ready to gain procedural fluency. One common procedural method for multiplying quadratic equations is FOIL, which stands for: firsts, outsides, insides, and lasts (Kalder, 2012). Another is a method found by O'Neil (2006) to be effective in helping struggling students with quadratic equations is more organized and mechanical than FOIL. Using tables, students write the first set of numbers on the y-axis, and the second set on the x-axis, and the multiply each set of numbers in the corresponding cell. The numbers within the cells are then summed diagonally, and the answers provide the value for the answer of the equation (O'Neil, 2006).

Functions

The fourth school algebra content area is that of functions. A function is a relationship between two different sets of numbers, and the relationship between the two sets of numbers is associated with an object, often associated with the symbol f (Van De Walle et al., 2013). There are many different kinds of functions that a student should use to solve problems and graph: linear, quadratic, polynomials, nonlinear, exponential, logarithmic, and trigonometric (Kaufman & Schwitters, 2004; Wolfram, 2014). Linear functions (i.e. $f(x) = ax + b$) grow in a consistent and linear manner and have the values representing real numbers. A quadratic function is written $f(x) = ax^2 + bx + c$, where $a \neq 0$, and whose graph is a parabola. Polynomial functions are functions incorporating integer powers that are not negative, like $f(x) = a_n x^2 + a_{n-1} x^{2-1} + \ldots + a_1 x + a_0$. A non-linear function is a function that, when graphed, is not linear (i.e. $ax^2 + b$). Exponential functions are written $f(x) = b^x$, where b is greater than zero and not equal to 1. Logarithmic functions are written $f(x) = log_b x$, where b is greater than zero and not equal to 1, and x is greater than 0. Logarithmic functions are the opposite of exponential functions. Finally, trigonometric functions (a.k.a. circular functions) are the functions of an angle that help relate an angle of a triangle to the length of its sides. These functions are made up of cosecant, cosine, cotangent, secant, sine, and tangent (Kaufman & Schwitters, 2004; Wolfram, 2014).

A simple method commonly used by elementary and early middle school students for learning about functions include using “function machines” which have an input, some sort of change to the number, and an output; recording those values in a table; and repeating with different inputs (Steketee & Scher, 2011). This is called the rules of assignment perspective, and while this can act as a good entry point, it can also lead to misconceptions and errors stemming from students not recognizing the continuous relationship between values, the continuous growth (Steketee & Scher, 2011; Weber, 2014). The function as covariation perspective allows for a varying relationship between two quantities, and one effective way to help develop this perspective is to have students graph multiple quantities and see the variation (Weber, 2014).

Combinatorics

The final content area suggested by the NMAP (2008) is combinatorics and finite probability. Combinatorics is a field of study that involves ordering and listing, combining, and rearranging the mathematical properties and relations of certain things (Wolfram, 2014). The NMAP (2008) further indicates that a student should be able to apply the binomial thermo and Pascal’s Triangle to develop combinations and permutations.

When learning combinatorics, students can become easily overwhelmed with symbols and algorithms (Tarlow, 2008). One way to help introduce students to this somewhat advanced concept, is to use the concrete-representation-abstract method and familiar situations. For example, Tarlow (2008) introduced combinatorics to students using the problem of different pizza toppings. To help conceptualize the subject matter, students first wrote out graphics with all the possible choices, and then with instruction and guided practice, worked through developing and understanding equations that represented the problem (Tarlow, 2008).

Synthesis

The research on effective teaching methods for algebra is vast, dense and strong. There were three major themes across the research. First, conceptual understanding of the content is crucial, and it is critical that the student understand what is being taught before procedural fluency is attempted. Second, it is important a student have an understanding of previously taught content that will be utilized in the new content. Third, effective instructional techniques such as direct instruction, using multiple representations, and facilitating content discussions remain effective across all areas of algebra.

Skills Required for School Algebra

The National Research Council (2001) reported that proficiency with whole numbers and rational numbers are prerequisites to advanced mathematical proficiency. Computation of single and multi-digit problems, estimation, mental arithmetic, and an ability to solve word problems all fall under proficiency with whole numbers. Proficiency with rational numbers means understanding factions and decimals, solving problems with the numbers; use of integers and proportional reasoning; and being able to applying them in multiple contexts.

The National Mathematics Advisory Panel (2008) identified certain skill areas in which a student should be proficient before they can succeed in algebra. The first area is whole numbers. Under this heading, NMAP (2008) recommends that a student should understand basic operations; commutative, associative, and distributive properties; applying operations to problem solving; estimation and magnitude; and being fluent with algorithms. The second area that a student should have fluency with is working with fractions. These skills should include comparing and representing rational numbers; performing calculations with rational numbers; and converting and understanding the relationships between different rational numbers. NMAP (2008) identifies having fluency with aspects of geometry and measurement as crucial to establishing proficiency. This includes understanding linear functions; analyzing properties of shapes; and solving for length, angels, and areas.

The Southern Regional Education Board (2003), a non-profit organization providing educational guidance to 16 southern states, identified a set of content-specific skills that are requirements for future algebra proficiency. These necessary skills include: using, comparing, and ordering a variety of number forms including fractions, decimals, integers, percepts, and percents; computing (addition, subtraction, multiplication, and division) with these numbers forms; identifying the greatest common factor, least common multiple, and prime factorization; writing and using ratios, rates, and proportions; drawing and using a variety of geometric shapes; measure length with appropriate tools and formulas; understand and use the Pythagorean Theorem to solve problems; gather, organize, display, and interpret data; use and understand probabilities; write, simplify, and solve algebraic equations using addition, subtracting, multiplication, and division; create, analyze, and generalize a variety of patterns; and understand and represent algebraic functions both through formulas and graphically.

Bush and Karp (2013) identified four broad categories for specific algebra skills including, (a) ratios and the proportional relationships, (b) the number system, (c) expressions and equations, and (d) functions. They expanded on the concept of number systems to include more specific skills for algebra including fractions, decimals and percentages, integers, exponents, order of operations, properties of numbers, and comparing and ordering. Expressions and equations include equality, variables, algebraic expressions, and algebraic equations.

Finally, the Common Core State Standards [CCSS] (2010) along with a number of research findings emphasize the importance of generalizing algebra skills (Ellis, 2011; Lannin, 2005), integrating them with understanding of a symbol system (Christou & Vosniadou, 2012; Kaput, 1989), and exploring with them as generalizations and functions (Hackbarth & Wilsman, 2008). Of critical connections they include: numbers, place values, basic facts, and computation (Carraher et al., 2006; Lee et al., 2011); proportional reasoning (Booth, Netwon, & Twiss-Garrity, 2014); measurement (Soares, Blanton, & Kaput, 2006); geometry (Charbonneau, 1996); and data analysis (CCSS, 2010).

Foundational Skills

When learning new skills, building on more foundational skills while integrating new information with previously learned information helps reinforces concepts and increases the likelihood that a person will be able to generalize and apply the new knowledge (Woolfolk, 2008). If a solid foundation does not exist, then it is likely someone will not develop proficiency with those advanced skills (Kraiger, Ford, & Salas, 1993).

Comparing and ordering. Having solid number sense with whole numbers is critical to practically all areas of mathematics (Zaslavsky, 2001; Hallet, Nunes, Bryant, & Thorpe, 2012; NRC, 2001; Siegler et al., 2012). Having a conceptual understanding of fractions, decimals, and percentages as actual numbers is a critical first step to proficiency with advanced mathematics (NRC, 2001; NMAP, 2008; Wu, 2001). Understanding the magnitude of rational numbers, especially fractions, and how they compare to one another is predictive of future mathematics achievement, regardless of the level of other mathematics skills including computation (Bailey, Hoard, Nugent, & Geary, 2012; Booth & Newton, 2012; Siegler, Thompson, & Schneider, 2011).

Ordering rational numbers requires an understanding of magnitude where a student has to compare the sizes of at least two different numbers (Booth, Newton, & Twiss-Garrity, 2014). The ability to recognize different quantities and compare is required for developing equivalence, equation solving, and proportional reasoning (Cetin and Ertekin, 2011). Moreover, recognizing and comparing different quantities assists in understanding and creating algebraic expressions as well as applying and generalizing algorithms (Brown & Quinn, 2006; Brown & Quinn, 2007).

Understanding, comparing and ordering integers is typically taught during kindergarten and first grade, learning fractions as number is taught during third grade, and fraction equivalence, ordering, and comparing rational numbers is taught beginning in grade four (CCSS, 2010). Research has found, however, that having understanding of fractions through recognizing distributions and sharing of whole numbers can begin as early as kindergarten (Cwikla, 2014). Early introduction of rational numbers may help support future fraction proficiency, which is a critical skills that's required for a student's transition from arithmetic to algebra (NMAP, 2008). With the knowledge that struggles understanding fractions is one of the biggest challenges for students in algebra, the research strongly supports the idea that having a strong conceptual understanding and knowledge of procedural fluency with fractions facilitates development of strategic competence and adaptive reasoning, all of which are critical for almost all levels of algebra (NRC, 2001).

When developing conceptual understanding and procedural fluency with the magnitude, comparing, and ordering of rational numbers, the research has found certain instructional components to be effective in helping improve student outcomes. These include presenting material using a concrete-representational-abstract progression of content; having students verbalize their problem-solving process and reasoning behind their answers; and using meaningful and familiar examples and scenarios (Moone & de Groot, 2007; Ortiz, 2006; Bray & Abreu-Sanches, 2010). Additional methods include having students evaluate and understand the reason for their errors; utilizing estimation and mental imagery to compare and order rational numbers; using a number line; and estimating magnitudes across different representations simultaneously (Bray & Abreu-Sanches, 2010; Shaughnessy, 2011; Whitin & Whitin, 2012).

The research on number sense and understanding fractions is similar to the proficiency literature in that it rarely addresses the needs of struggling secondary students through targeted interventions. Even the literature on fraction proficiency, a skill that has been widely acknowledged to be a sort of gatekeeper for algebra success (NMAP, 2008), has a very small literature base for secondary students. Future intervention research needs to target these skill gaps in struggling secondary students in the algebra classroom.

Calculation. Solving problems using whole numbers and rational numbers forms the basis of practically all advanced mathematics, including school algebra (Geary, 1994; Linchevski & Livneh, 1999; NMAP, 2008). Single-digit problem solving proficiency is critical to solving multi-digit problems, and a student must be proficient with these basic functions before he or she is able to generalize, adapt, reason, and strategically utilize the skills in more advanced ways (Humberstone & Reeve, 2008; Kinach, 2014; Linchevski & Livneh, 1999). Knowledge of fractions and calculations, especially division, is predictive of success in higher grades (Siegler et al., 2012).

Successful computation with both integers and rational numbers form the basis for almost all aspects of algebra, including linear equations, quadratic equations, and functions (NRP, 2008). Having both a strong conceptual understanding and procedural fluency with calculations helps promote generalization, efficiency problem solving, and strategic use of procedures, all of which are critical to algebra proficiency (NRC, 2001). Supported by strong number sense and sense of rational numbers, calculation forms the basis for almost all aspects of algebra.

When delivering computation instruction and interventions, it is important to target the specific mathematic proficiency needs of the student, whether it be conceptual understanding or procedural fluency, because targeting the correct area will increase the effectiveness of the intervention (Burns et al., 2015). It is also important to ensure certain prerequisite skills have been acquired, as number sense and the ability to compare and order numbers, both integers and rational, is critical to computation success (Johanning, 2011; Cengiz & Rathouz, 2011).

Intervention and instruction for both integer and rational number computation should encourage generalization and flexibility with computation (O'Loughlin, 2007). Students need time to think about and discuss the content, understand the big ideas, and learn about the connections between concepts and different areas of learning (Cengiz & Rathouz, 2011). The research is flush with techniques focusing on computation conceptual understanding and procedural fluency. Some methods include breaking down problems with different methods such as problem strings and open number lines (O'Loughlin, 2007); encouraging estimation to evaluate the plausibility and appropriateness of results; and using visual models and diagrams across different problem representations (Johanning, 2011).

Word problems. The NCTM (2001) identifies solving word problems as an important component to mathematics proficiency. Proficiency of word problems with using arithmetic skills, and the requirements of breaking down the problem into workable components, provides a basis for solving advanced problems in algebra (Brenner et al., 1997; Mayer, 1982). Students' performance on word problems predicts their understanding of variable and the appropriate use of the equals sign, two components to algebraic thinking which will be discussed in the next section (Fuchs et al., 2012).

Proficiency with word problems requires a student to conceptualize the problem being presented, identifying the important information, disregard irrelevant information, and plan out a problem-solving strategy (NRC, 2001; Xin, Wiles, & Lin, 2008). Of particular importance with solving word problems is the understanding of what symbols to use (Clement, 1982). A strong conceptual understanding of integers, rational numbers, and calculation strategies are typically necessary for developing proficiency with word problems (Xin, Wiles, & Lin, 2008). If a student struggles word problems using integers and rational numbers, they are more than likely to struggle when solving algebraic word problems (NRP, 2008).

Solving word problems begins as early as kindergarten and runs parallel to computation expectations (CCSS, 2011). Students need to have proficiency with number sense and calculation before they can be successful with word problems (Englard, 2010). When encountering early word problems, students will often develop their own methods of solving problems, but as the difficulty of content increases, students will often rely on superficial, inefficient, and ultimately incorrect methods of problem solving if they are not taught problem-solving strategies (Englard, 2010). Developing student proficiency with word problems shares many of the same instructional characteristics as number sense and computation, such as having students represent their knowledge in multiple ways and explaining their reasoning (Whitin & Whitin, 2008). Instructional methods that have been successful in facilitating growth include having students write out their own story problems, drawing or modeling relationships within the problems, and identifying similarities and differences between problems (Drake & Barlow, 2007; England, 2010; Whitin & Whitin, 2008).

Core Skills: Algebraic Thinking

Algebraic thinking helps evolve one's understanding and generalization of basic skills, informs new facts being learned, and guides problem-solving with novel and abstract problems (Banchoff, 2008; Kinach, 2014; Johanning, 2004). It's important to note that algebraic thinking does not develop independently, and proficiency is more likely acquired if the thought components are introduced early on in one's mathematic career alongside topics like basic arithmetic (Warren & Cooper, 2008; Cai & Moyer, 2008; Ferrucci, Kaur, Carter, & Yeap, 2008).

Equals sign and variables. A major component of the transition from the straight-forward thinking about arithmetic to the flexible and abstract thinking of algebra requires the student to evolve their ideas of what the symbols of arithmetic mean and to develop a meaningful use of symbols (Herscovics & Lunchevski, 1994; NMAP, 2008). Perhaps the most common misconception of the struggling algebra student involves the equals sign (Kieran, 2008; McNeil & Alibali, 2005). As a product of learning basic operations, many students perceive the sign to mean "the answer is," when in reality, and of special importance in algebra, the symbol means "is equivalent to" (Kieran, 2008).

Understanding the concept of the equals sign is critical to algebraic understanding (Pillay, Wilss, & Boulton-Lewis, 1998). Misconceptions of the equals sign are formed early on (McNeil & Alibali, 2005), and a student's understanding of an equals sign, whether correct or incorrect, remains stable across elementary and middle schools (Knuth, Stephens, McNeil, & Alibali, 2006). There is a strong correlation between a student's understanding of the equals sign and equation solving, and weak understanding of the equals sign can impede conceptual understanding development of the most core algebra topics, such as linear equations (NRC, 2001). A fundamental characteristic of algebra is the flexible use of numbers, and the idea of solving for equivalence on both sides of an equation rather than for an answer on side is a crucial component of that (NRP, 2008).

Another struggle students have is with understanding variables. According to Küchemann (1978), there is a hierarchy of variables: first, letters are directly evaluated; second, letters can be ignored and the solution can be found by working around the variable; third, the letter serves as an object; fourth, the letter is a specific unknown; fifth, the letter serves as a generalized number; and sixth, the letter serves as a variable. Students struggle with representing unknown quantities, and understanding that variables can be a range of quantities, and that when one value changes another does as well (Kuchemann, 1978; McNeil et al., 2010; Stacey & Macgregor, 1997; Lucariello, Tine, & Ganley, 2014; Philipp, 1992). In regards to numbers and symbols, many struggling students erroneously believe that algebraic symbols are static, the numbers are literal, each number can only have one value, and they have a sign bias which means that unless there is a negative sign, the student assumes the variable is positive (Christou & Vosniadou, 2012).

Students typically begin creating numerical expressions in fifth grade, are introduced to variables and simple algebraic expressions in sixth, and equivalent expressions in seventh grade (CCSS, 2011). The instruction and intervention literature targeting these areas typically use middle or high school students as the subjects, which is a much more appropriate population when discussing algebra proficiency. However, the studies often do not observe the impact of the intervention on overall algebra proficiency. Research has found the following instructional strategies are effective in increasing students' knowledge of equalities and variables: modeling thinking using multiple representations, justifying reasoning behind solutions, critiquing others, use of visual tools like equality strings, and comparing different equations to see if they are similar (Gain & Sheffield, 2015; Knuth, Alibali, Hattikudur, McNeil, & Stephens, 2008).

Relationship Between Arithmetic and Algebra

Despite both being a fundamental part of mathematics and appearing very much the same, the processes behind arithmetic and algebra are very different (Herscovics & Linchevski, 1994). Arithmetic thinking is concrete and focuses on using procedures and operations to gain an answer to a problem, while algebraic thinking utilizes generalizations and reasoning to search out relations among values and variables (Stacey & MacGregor, 2000). The transition from arithmetic to algebra is one of the biggest hurdles for many students, as it requires less of a reliance on specific operational problem solving to more of a relational understanding between problems, and a shift from concrete operations to more abstract representations and strategies (Cai & Moyer, 2008; Herscovics & Linchevski, 1994).

The generalization of strategies is a key component of school algebra and algebraic reasoning (Kinach, 2014). Generalization requires a flexible knowledge of numbers, operations, and symbols of arithmetic (Becker & Rivera, 2005; Humbersome & Reeve, 2008). Developing these generalizations requires knowledge of the number systems and basic arithmetic (Kaput Carraher, & Blanton, 2008). An early form of this generalization occurs when a student is required to solve for missing values in simple equations (CCSS, 2010). Students who fail to develop this generalization ability are at an increase risk of incorrectly applying different algebraic ideas to problems and struggling to learn new ideas (Christou & Vosiadou, 2012).

The transition from arithmetic to algebra is complicated process with many moving parts like understanding equivalence and unknown quantities. The majority of research in the area of this transition focuses on those individual components rather then the transition as a whole. As stated earlier, much of the research on equivalence and variables occurs with middle school students, but there is research occurring with elementary school students in hopes of facilitating the transition sooner. Instructional methods to approaching this goal include: teaching students to think of numbers as numbers and not just objects; change language so instead of finding “what a problem equals”, find “what makes the statement true”; construct open number sentences;

introduce symbols as variables prior to letters, as students may be confused by letters being a range of values; and use tables and graphs to show the changing nature of different values (Earnest & Balti, 2008; Rivera, 2006).

Patterns. The ability to identify patterns and the relations is a critical component of algebra proficiency (Kaput, Carraher, & Blanton, 2008; Kieran 2004; Cai & Moyer, 2008; Warren & Cooper; SREB, 2003), and the requirements of stating, verifying, and justifying patterns may help bridge the transition from arithmetic to algebra (Rivera & Becker, 2009). Learning about and using patterns typically begins with pictorial patterns and moves to arithmetic progressions before moving to geometric patterns and functions (Bush & Karp, 2013). The step-by-step sequential change where patterns are observed, continued, and described are three critical components to the patterning process (Lee & Freiman, 2006).

In algebra, patterning is important when solving complicated problems like linear equations, where the solution to the equations depend on a systematic change in some sort of variables or answer (Stacey, 1989). Arithmetic and especially geometric patterns help a student understand functional relationships, observe growth, and think about unknowns, equivalence, and ranges of values (Lee & Freiman, 2006; Markworth, 2012). Patterns expose students to using inputs, outputs, and measuring rate of change, and pattern recognition and the systematic increase or decrease of numbers is a primary component of functions (Walkowiak, 2014). They also help students analyze describe extend and generalize numbers (Zazkis & Liljedahl, 2002).

Students begin explaining simple arithmetic patterns starting in third grade, explaining patterns with shapes in fourth grade, begin describing patterns in sixth, and begin identifying patterns in data and using functions in eighth grade (CCSS, 2011). There is a plethora of instructional research on patterns, but little intervention research. One possible reason for this discrepancy is that while recognizing and generalizing patterns is a critical component of algebra, teaching patterns as a specific "topic" is not particularly effective in increasing student growth (Lee & Freiman, 2006). Instructions regarding patterns can be started as early as kindergarten and be integrated throughout elementary and middle school mathematics (Lee & Freiman, 2006; Friel & Markworth, 2009).

The first step to pattern instruction is helping students recognize a pattern (Lee & Freiman, 2006). This can prove challenging in that not all students who recognize patterns are able to generalize them, and sometimes students search for patterns where there are none (Lee & Freiman, 2006; Rivera & Becker, 2009). One method to help students recognize patterns is to use a concrete-representational-abstract approach using two-and-three dimensional geometric figure counting (Beigie, 2011; Rivera & Becker, 2009). The second step with patterns involves generalizing, which is the construction and justification of formulas that represent the pattern (Rivera & Becker, 2009). One method is to help develop this skill is to have students compare and evaluate different growing patterns, ask which one is correct, and have them justify their answer (Lee & Freiman, 2006).

Proportional reasoning. Proportional reasoning, which uses ratios to compare different quantities, is considered a critical component of algebraic thinking (Bright, Joyner, & Wallis, 2003; NRC, 2001; NMAP, 2008; SREB, 2003; Bush & Karp, 2013; CCSS, 2010; Fujimura, 2001). Types of proportional reasoning problems include missing numbers; comparing numbers; and qualitative predicting and comparison (Özgün-Koca & Altay, 2009), and it is made up of three different components: ratios; ratios being equal; and finding the relevant information while disregarding the irrelevant (Özgün-Koca & Altay, 2009).

Proportional reasoning should be taught early in a student's mathematics career (Tjoe & Torre, 2014). It serves to unite many of the concepts taught throughout elementary and middle school by assisting with the transition from arithmetic to algebra (Jitendra et al., 2013; NCTM, 2000; Lesh, Post, & Behr, 1988). It also requires many of the skills critical to algebra, including flexible use of the equals sign, generalization of numbers, number transformations, and using different types of variables. (Lesh, Post, & Behr, 1988). When students struggle with proportional reasoning, it is often because of struggles with the multiplicative properties; not having a conceptual understanding of ratios; misunderstanding variables; and the relationships between the values (Özgün-Koca & Altay, 2009; Singh, 2000; Lawton, 2003).

Proportional reasoning is typically first addressed in sixth grade when students begin using ratio concepts to reason and solve problems, and continues through grade seven and eight when the problems become more complex, and in grade nine students begin using proportional reasoning in the real of functions (CCSS, 2011). Similar to the other areas addressed through this literature review, much of the research on teaching proportional reasoning happens at an instruction level.

Creating pictorial representations of the problem can help build conceptual understanding of proportional reasoning and minimize the interference of errors such as using additive reasoning (Ruchti & Bennett, 2013). An additional strategy that has been effective in helping develop conceptual understanding of proportional reasoning are the schema-based approach that follows the following steps: Priming the structure of the problem, visual representations, explicit teaching of problem solving methods, and procedural flexibility (Jitendra, Star, Dupuis, & Rodriguez, 2013). Strategies for teaching proportional reasoning include; using familiar contexts; facilitating class discussions; using the concrete-representational-abstract approach; and avoid solely focusing on finding missing values (Banker, 2012; Fielding-Wells, Dole, & Makar, 2014).

Core Skills: Factual Knowledge

According to the NMAP (2008), Curriculums should focus on simultaneously developing a student's conceptual understanding, procedural fluency, and problem solving skills. Students struggling with algebra also benefit from a focus on understanding the core concepts behind algebra instruction, having the prerequisite knowledge required for each task, explicitly teaching the components of the content including vocabulary, and engaging the students in verbal dialogues regarding the content (Witzel, Smith, & Brownell, 2001). Supporting these findings, Impecoven-Lind & Foegen (2010) identified the three core areas where students struggle with when learning algebra: cognitive processes (attention, memory, language, meta-cognition), content foundations (declarative, procedural, conceptual), and algebra concepts.

Problem solving. Without a doubt, direct instruction of algebra content encompasses the biggest portion of the algebra instruction. The Common Core State Standards (2010) identifies the following areas as what should be targeted in Algebra I: Using exponents and rational exponents; rational and irrational numbers; reasoning quantitatively; interpreting and writing linear, exponential, and quadratic expressions; performing arithmetic operations on polynomials; creating equations that describe numbers and relationships; understanding and reasoning with equations; solving equations and inequalities with one variable; graphing and solving equations; understanding the concept of a function; interpreting and analyzing functions; and constructing and comparing linear, quadratic, and exponential models.

The Minnesota state standards (2007) are similar to the Common Core. Minnesota requires that algebra students be able to use equations and inequalities involving linear expressions to represent problems; solve equations and inequalities using symbols and graphs; generate equivalent numerical and algebraic expressions; use algebraic properties to evaluate expressions; Understand the differences between linear and non-linear functions; use tables; descriptions; and graphs to represent linear functions; and apply functions in mathematical and real-world situations.

There is a dense literature on instructional techniques for teaching students algebra, and a somewhat smaller literature on algebra interventions. In three separate meta-analyses on the characteristics of effective algebra instruction for struggling learners, the authors had to rely on fewer, older studies with smaller sample sizes (Haas, 2005; Maccini, McNaughton, & Ruhl, 1999; Rakes, Valentine, McGatha, & Ronau, 2010). However, their conclusions were similar and reflective of the recommendations provided by NRC (2001) and NMAP (2008). When teaching struggling learners, instruction should include: a primary focus on conceptual understanding, cooperative learning, multiple representations, assessment strategies, graphic organizers, spiraling strategies, manipulatives, and modeling, feedback, and guided practice (Haas, 2005; Maccini, McNaughton, & Ruhl, 1999; Rakes, Valentine, McGatha, & Ronau, 2010).

Vocabulary. Developing the language of mathematics is critical for all students to help develop mathematics proficiency (Adoniou & Qing, 2014). Limited mathematics vocabulary can have an impact on a student's ability to understand a problem, express answers or their thought process, and remember and recall new, specific information (Panasuk, 2011). However, being exposed to vocabulary can help increase a student's understanding and retention of academic material (Little & Box, 2011). For struggling learners, language acquisition proves a major barrier to algebra proficiency (Rakes, McGatha, & Ronau, 2010). The language of procedures and concepts needs to clearly defined and linked to past examples, especially for increasingly abstract concepts, as clarifying and supporting students' awareness of the language of symbols, syntax, and ambiguity is linked to language and algebra proficiency (MacGregor & Price, 2002).

Students encounter mathematics vocabulary as soon as they begin learning about mathematics, and new words have to be learned with every new subject. Traditional vocabulary instruction outside the mathematics classroom often occurs through incidental learning and direct instruction using definitions and contextual information (Dunston & Tyminski, 2013). These strategies, while sometimes effective, can leave a student wanting, as students are not often exposed to this type of vocabulary outside the classroom. While sometimes neglected, focusing on vocabulary is critical as it can help enhance a student's conceptual understanding (Rubenstein, 2007; Dunston & Tyminski, 2013). Instructional strategies to help teach struggling students vocabulary include: clearly stating meanings and distinguishing between multiple uses; teach roots and origins; pre-teach content; cooperative learning; writing journals; personal glossaries; and visual cues (Livers & Bay-Williams, 2014; Bruun, Diaz, & Dykes, 2015; Rubenstein, 2007). One graphic organizer that has been found to be effective is the Frayer Model, where students chart words using four sections of the organizer: essential characteristics, nonessential characteristics, examples, and non-examples (Dunston & Tyminski, 2013).

Conceptual understanding. Having a strong conceptual understanding of different topics in mathematics is a critical component of mathematics proficiency across all areas (NCTM, 2001), but perhaps its no more important than in the study of algebra. As it has been repeated throughout this paper, conceptual understanding is critical to problem-solving, flexible application of strategies, generalization of skills, reasoning, and learning new material (Byrnes & Wasik, 1991; Hiebert & Lefevre, 1986; Rittle-Johnson, Siegler, & Alibali, 2001; Schneider & Stern, 2010; Rakes, Valentine, McGatha, & Ronau, 2010). The specific importance of conceptual understanding and word problems is described above, and this section will go more into detail about the construct of conceptual understanding.

Panasuk (2011) theorized conceptual understanding as having three phases. The first phase, phase 0, indicates that the student does not recognize the same problem concept presented in different ways. The second phase, phase 1, indicates the student might observe some similarities between the two representations, but does not fully recognize the relationship. Phase 2, the final phase, indicates that the student can recognize the relationship and translate that relationship between forms (Panasuk, 2011).

It is critical that conceptual understanding be taught directly when presenting new material, as instruction in conceptual understanding was effective for students who did not demonstrate understanding of the underlying mathematical construct (Burns, 2011, Burns et al., 2015). Assessing conceptual understanding within different areas of algebra often takes the form of providing multiple types of problem-solving opportunities and having students explain their answers and/or certain concepts (Lucariello, Tine, & Ganley, 2014). Observing a student's choices between correct and incorrect answers can indicate levels of algebraic conceptual understanding (Booth, Lange, Koedinger, & Newton, 2013).

Reinforcing Value in Practice: Authentic Application

Certain theories of learning emphasize the importance of utilizing and applying of newly learned information across multiple settings to help improve maintenance and generalization of the content, which are two important components of skill proficiency (Harris & Pressley, 1991; Woolfolk, 2008). Algebra is no different: meaningful, real-world application of concepts helps facilitate conceptual understanding, strategic application, and adaptive reasoning (Friedlander & Arcavi, 2012; Phelan et al., 2011; Newton, Star, & Lynch, 2010).

It is important that struggling students find algebra to be meaningful and have value. Those who find value in what they're learning are more engaged, show greater academic growth, and show higher levels of academic engagement (Christenson, Reschly, & Wylie, 2012). In contrast, a student who finds less value with a subject will likely be less engaged, show less growth, and lower levels of engagement (Reyes et al., 2012).

By high school, the perceptions of students struggling with mathematics, both of themselves and mathematics as a whole, are more likely to be negative, thereby counter-productive, ultimately resulting in high levels of disengagement (Christenson, Reschly, & Wylie, 2012). Disengagement has a negative impact their learning and academic performance, which in turn can strengthen a student's perceptions, ultimately forcing the student into a negative cycle of performance and expectations. If we want algebra instruction and intervention to be effective, it is imperative for these students to develop a productive disposition (NRC, 2001).

Having a productive mindset is crucial to finding and maintaining success with advanced mathematics (Fast et al., 2010; NRC, 2001). Within the current model, mindset addresses a student's self-efficacy and self-concept regarding mathematics learning. Self-efficacy refers to a person's belief as to whether or not they are able to perform certain tasks and be successful (Bandura, 1977), while self-concept refers to a person's perceptions of himself or herself as formed by their individual experiences (March & Craven, 1997). Students who have higher levels of self-efficacy regarding their mathematics skills show stronger growth rates and higher rates of achievement compared to students with lower rates of self-efficacy (Fast et al., 2010; Galla et al., 2014; Lewis et al., 2012). In addition, higher rates of achievement are found in students who integrate the idea of being a successful mathematics learner into their self-concept (Pietsch, Walker, & Chapman, 2003).

The research on the importance of having a productive mindset, seeing oneself as an able mathematics learner, and finding value on the material, is strong (NRC, 2001). The studies cover a wide range of students, ages, and subjects. However, the majority of the research is predictive in nature, leaving little to no research on directly linking improvements in productive mindset to improvements in achievement. Bresó, Schaufeli, and Salanova (2011) found that increasing a student's self-efficacy decreased anxiety and raised academic performance levels, but their sample size was small and their outcome measure was a final exam score. Future research needs to address whether improving one's productive mindset can help boost levels of achievement in struggling algebra learners.

The Uniting Thread: Engagement

The importance of engagement in the classroom, especially for struggling students, cannot be understated, which is why engagement acts as an umbrella for the other four sections of the intervention model. Not only do engaged students show greater levels of academic growth than those who are not engaged (Christenson, Reschly, & Wylie, 2012; Reyes et al., 2012), they show greater levels of class attendance, class participation, and work completion (Christenson et al., 2008), are less likely to drop out of school, and more likely to enroll in post-secondary education (Finn, 1993). In addition, engagement can act as a crucial mediator between self-efficacy, self-concept, and achievement (Galla et al., 2014).

Fostering high levels of student engagement with algebra is critically important, not only because it assists with skill acquisition and understanding of material (Ross & Willson, 2012), but also because struggling students are potentially entering algebra class with dangerously low levels of engagement. While motivation is found to be a mild to moderate predictor of initial mathematics achievement, it is a strong predictor of mathematics achievement growth (Murayama et al., 2013). There is often a decline in mathematics engagement from elementary to middle to high school (Marks, 2000), and this drop is significant starting in middle school as the content increases in difficulty (Martin et al., 2014). Helping increase student engagement can combat this drop, and within mathematics, specially addressing engagement has been found to have a positive effect on self-efficacy, the perceived value of the content, class achievement, enjoyment with the material, decreasing mathematics anxiety (Martin et al., 2012; Martin et al., 2014), and increasing levels of academic resilience, even when controlling for various learner characteristics (Finn & Rock, 1997, Moller et al., 2014; Nichols & White, 2001).

One model of student engagement, which is utilized in the Check & Connect engagement intervention program, presents a multidimensional model of engagement and identifies four subtypes of engagement: academic, behavioral, cognitive, and affective (Christenson, Stout, & Pohl, 2012). Academic engagement can be observed through a student's completion of assignments, time on task, grades, or the amount of time spent on schoolwork at home. Behavioral engagement is seen through things like attendance, active participation in class, and participation in after school activities. Cognitive engagement is reflective of a student's motivations to learn, perceived value of school and the content, perception of themselves as an effective learner, and use of learning strategies. Finally, affective learning is reflective of a student's emotional connection and sense of belonging within the school (Christenson, Stout, & Pohl, 2012). Intervening with engagement using this four-factor model of engagement has repeatedly been found to help increase secondary student achievement (Lehr, Sinclair, & Christenson, 2004; Maynard, Kjellstrand, & Thompson, 2014). With the large sample sizes, eclectic populations, and experimental research designs, the research strongly supports both engagement predicting academic achievement and increases of engagement helping support increases in achievement (Christenson, Reschly, & Wylie, 2012; Reyes et al., 2012; Galla et al., 2014; Finn, 1993).

Synthesis

Educators have known for quite some time what characteristics constitute effective traditional algebra instruction with struggling students, yet the current body of research on algebra interventions for secondary low achieving students and students with learning disabilities is small (NCTM, 2000). A major challenge is that research indicates that while interventions for high school students can indeed help increase overall academic achievement, it is a struggle to raise the algebra achievement (Anderson, Yilmaz, & Wasburn-Moses, 2004; Scruggs, Mastropieri, Berkeley, & Graetz, 2010).

The mathematics literature identifies the areas discussed above as critical to algebra proficiency (NRC, 2001), but the majority of research studies have looked at each skill area independently; if interventions are administered, they target the specific area without consideration of the overall impact on algebra performance. In addition, the diagnostic and progress monitoring measures used in most of the studies informal or not standardized, and the samples from the studies are typically elementary or early middle school students. The research literature regarding instructional and intervention supports for older, struggling algebra learners needs to expand if we are going to begin supporting the needs of this growing population.

Conclusion

The current model of what it means to have proficiency in mathematics encompasses the areas of conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition. These five domains work together to support mathematics proficiency, and algebra success. However, to be successful with school algebra, a student should also possess certain skills so they can understand the content and perform the tasks being required of them. These skills fall into different categories; basic skills, algebraic thinking, content knowledge, authentic application, and engagement. As these areas are critical to success in algebra, it is also critical that they be addressed when a student is struggling with the content.

Research has identified different skills required for algebra proficiency, and there is a vast array of literature on instructional techniques for teaching students these skills. However, what does not exist is a systematic way to isolate and assess these skills. Skill isolation is a critical component to both assessment and intervention; appropriate sills need to be assessed in a valid and reliable way, and then interventions should directly target the areas of need (Burns, VanDerHeyden, & Boice, 2008). At the present time there is no adequate measure suited to this purpose, nor is there a model of learning on which the measure can be based. This research will attempt o fill that gap by presenting a problem-solving model for students struggling with algebra.

CHAPTER 3: METHOD

Purpose and Research Questions

The need for an algebra intervention research is paramount because achievement data indicate that our students continue to struggle with algebra, but there is not a well-developed problem-analysis model for intervening with struggling students. Direct instructional activities and guidelines exist for a number of the content areas within algebra (Foegen, 2008; National Mathematics Advisory Panel, 2008), and academic interventions are more successful if they directly address the student deficit (Burns, VanDerHeyden, & Zaslofsky, 2014). As of now there are no clear guidelines about how to assess for specific skill and knowledge-based deficits, and then select appropriate, evidence-based interventions.

Identification of these specific deficits should be guided by the use of valid assessments that target each area (Thompson, 2004). At this time, however, there are no standardized mathematics assessments that target the proposed areas in the model. Consequently, there are currently no standardized algebra assessments that can help guide researchers in crafting a problem-solving model. If there is going to be an evidence-based problem-solving model leading to effective algebra interventions, appropriate assessments should exist.

The study evaluated the validity of using a five-part problem-solving model to identify core deficits for students struggling with algebra. The five core skills of the problem-solving model support and inform each other, and are all required to establish a basis for algebra proficiency. Being able to systematically assess and evaluate the specific knowledge and skill deficits that are leading a student to struggle will lead to increasingly effective instructional modifications and interventions, and when interventions are occurring, these analyses will allow for a more thorough connection between the intervention and instructional components being taught.

The following research questions guided the study:

1. What is the relationship between each of the five skills within the problem-analysis model to an established measure of algebra?

2. To what extent do assessment data support the proposed five-factor structure?

3. To what extent can the five skills and each subskill within the problem-analysis model accurately identify the level of a student's difficulty with algebra as measured by a criterion?

Problem Analysis Model

The topic of algebra is made up of a wide range of topics and skills (Newton, Star, & Lynch, 2010), and the proposed model (Figure 1) identifies five core areas required to be successful in school algebra: basic skills, factual knowledge, algebraic thinking, authentic application, and engagement. Using this model, it is believed that that assessment can be methodically conducted to target a student's specific difficulties with understanding school algebra.

Basic skills. Basic skills within algebra involve a student's ability to use whole numbers and rational numbers to solve a variety of problems. Students must be proficient with the ordering, calculating, and solving word problems with whole numbers and rational numbers before he or she is able to generalize, adapt, reason, and strategically utilize the skills in more advanced ways (Humberstone & Reeve, 2008; Kinach, 2014; Linchevski & Livneh, 1999).

Algebraic thinking. Algebraic thinking is a student's ability to solve problem using the types of thinking patterns required to be successful in algebra (Kieran, 2004). These are transitioning from arithmetic to algebra, understanding the equals sign and variables, proportional reasoning, and identifying patterns and relations.

Factual knowledge. Reflecting the findings from Impecoven-Lind and Foegen (2010) alongside the content required by the CCSS (2010), the factual knowledge skill section of the model has three core subskills, each one representing an aspect of the types of information that should be presented and taught during an intervention: specific content procedures, conceptual understanding, and language fundamentals.

Authentic application. The authentic application skills section of the model addresses a student's mindset regarding himself or herself as an effective mathematics learner. Having a productive mindset is crucial to finding and maintaining success with advanced mathematics (Fast et al., 2010; NRC, 2001), and within the current model, mindset addresses a student's self-efficacy and self-concept regarding mathematics learning. Self-efficacy refers to a person's belief as to whether or not they are able to perform certain tasks and be successful (Bandura, 1977), while self-concept refers to a person's perceptions of himself or herself as formed by their individual experiences (March & Craven, 1997).

Engagement. The final skills section of the model, engagement, assesses four different components of the engagement construct: behavioral, cognitive, and affective (Christenson, Stout, & Pohl, 2012; Wang, Bergin, & Bergin, 2014). Measuring engagement is crucial because fostering high levels of student engagement with algebra is critically important, not only because it assists with skill acquisition and understanding of material (Ross & Willson, 2012), but also because struggling students are potentially entering algebra class with dangerously low levels of engagement.

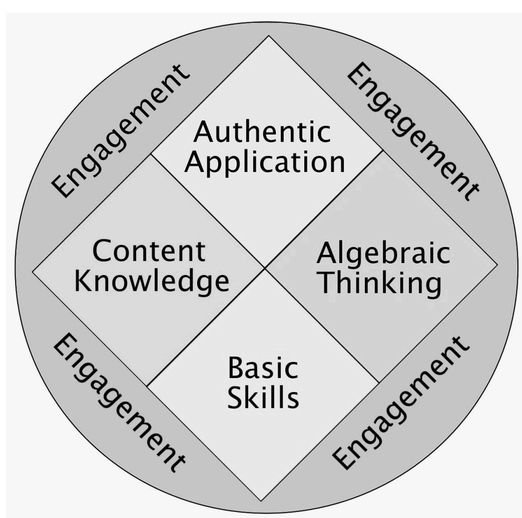

Figure 1. ALGEBRA PROBLEM-ANALYSIS MODEL

Participants

The study included participants from two charter schools located inside an urban midwestern city. The first is a middle school with 225 students, and the second is a high school with 225 students. All of the students voluntarily enrolled in the charter school, and then were randomly selected for enrollment through a lottery system. Together the two schools consist of students from 6th through 12th grade, with approximately 41% receiving free-or-reduced price lunch and 17% receiving special education services. The ages of the participants ranged from 12 to 19 years old.

The student population of the middle school consists of the following ethnicities: 54.1% White, 16.3 % Black, 16.3% Hispanic, 12.8% Asian and Pacific Islander, and .5% Native American. In 2013, 62% of students at the middle school scored within the proficient range for mathematics on the statewide test. The student population of the high school is made up of the following ethnicities: 54.4% White, 19.4 % Black, 15% Hispanic, 10.7% Asian and Pacific Islander, and .5% Native American. The 2013 data indicated that 47.4% of the students at the high school scored in the proficient range on the mathematics state test, and the school reported a graduation rate of 92.6%.

All of the students within the charter school were eligible to participate in the study. The study was discussed with the students during their mathematics classes, and because all students were eligible, passive consent was be used.

A total of 376 students participated in testing, and 327 students were included in the final study. Of those participants, 42% were male and 58% were female. Less than 1% identify as American Indian or Alaskan Native, 11% identify as Asian or Pacific Islander, 16% identify as Hispanic, 23% identify as Black, and 50% identify as White. Of the total number of participants, 10% were receiving special education services and 6% were receiving educational support through a 504 plan. The average age of the participants was 14.1, and the average grade was 8.5. The distribution of students among grades is as follows: grade 6 has 62 students (19%), grade 7 had 59 students (18%), grade 8 had 50 students (15%), grade 9 had 51 students (16%), grade 10 had 40 students (12%), grade 11 had 39 students (12%), and grade 12 had 26 students (8%).

Conceptual Model

The validation process of the model follows a four-step outline. First, the argument should be stated clearly and explicitly. The second step is to evaluate the inferences of the scores by comparing them to relevant and related evidence. The Measure of Academic Progress (MAP; NWEA, 2003) will serve as this criterion measure of a student's level of algebra proficiency.

The third step is to evaluate the assumptions made by the data, and if needed, adjust or modify the measurement procedures. This can be done by focusing on construct validity by identifying a set of axioms that connect the different variables in the study and whose value are estimated by observations. If the variables and their relations are found to be consistent with the theory, the validity of the theory and the measurement scales is supported. This was accomplished through factor analysis. According to Thompson (2004), factor analysis is used to develop theories about the existence of different constructs, and summarize the relationships that can be combined into a set of factors which can be used in future analyses. Confirmatory factor analysis (CFA) will be used for the validation process. CFA is a form of factor analysis that's used when the researcher knows the number of factors being considered, has variables that reflect the given factors, and assumes that the factors will be correlated (Thompson, 2004).

The fourth is to apply and restate the argument. The first step in this will be to examine the diagnostic accuracy of each skills section within the model. Future research can further apply the model in direct intervention research.

Theory-based interpretations should meet three different inferences: scoring, generalization, and the scores being an indicator of the construct. The scoring inference will be addressed by ensuring that the scoring criteria for each assessment be clear and objective. It is recognized that students being used represent a convince sample and there will be sampling error. Future studies should further look at the generalizability of the results.

When evaluating the interpretability of scores as an indicator of a construct, there needs to be both analytic and empirical evidence supporting the claims. Supporting the analytic inference, the content of the assessments is believed to reflect each skill being tested with minimal interference from irrelevant variance. In addition, correlations and factor analysis will be run to determine the fit of the scores with the model. Empirical support will be evidenced by the criterion validity with the MAP. Further research should continue looking at the applicability of the model to algebra performance in both short and long-term.

Measures

Criterion. The MAP is a screening tool for reading and mathematics proficiency in grades 2 through 12 and served as the criterion for the study. The scores from MAP assessments are in a Rasch Unit (RIT) scale, and they are reported in equal intervals. Specific proficiency cut scores varies by grade, and percentiles are also provided. The measured areas are number sense and number systems; estimation and computation; algebra; geometry; measurement; statistics and probability; and problem solving, reasoning, and proofs. The research used the most recent student scores from the fall 2014 MAP mathematics test to measure proficiency. Each students RIT score on the algebra proficiency section of the MAP identifies a student's algebra proficiency level. Scores of 181 and over will be considered proficient. Test-retest reliability of the MAP ranges from .79 to .94, while internal consistency ranges from .61 to .92. Concurrent and predictive validity ranges from .37 to .86, with the majority of validity values ranging from .65 to .85 (NWEA, 2009).

Proposed measures. Assessment and validation of the problem-analysis model were conducted using a set of assessments compiled and modified by the researchers. The assessments will address the skills (Basic Skills, Algebraic Thinking, Content, Application, and Engagement) as well as each subskill (ex. Vocabulary, Conceptual Understanding, and Content Knowledge subskill within the larger Content skills section) as outlined in Table 1. Inclusion of test items were based on their agreement with the definition of the construct as defined above. Because of the pilot nature of the assessments being used, there is no specific reliability data available for many of the assessment components being used. A description of the subtests can be found in Table 2.

Basic skills. The Basic Skills skills section consists of six different probes. The probes assess an individual's ability using whole numbers and rational numbers to compare and order different values; compute single and multi-digit addition, subtraction, multiplication, and division problems; and solving word problems.

Algebraic thinking. In the Algebraic Thinking skills section, test items were adapted from available online copies of the Minnesota Comprehensive Assessment (Minnesota Department of Education, YEAR), Trends in International Mathematics and Science Study (Mullin, Martin, Foy, & Arora, 2012, year), and National Assessment of Educational Progress (NCES, 2010). The questions were taken from assessments designed up through grade eight. The Arithmetic to Algebra subskill will require the student to provide the missing number in an equation, and the problems will use all four primary operations with whole numbers. The problems in the Patterns and Relation subskill will have the student identifying the patterns in a series of numbers. In the Generalization subskill, students will be given equations containing unknown variables on either side of an equals sign, and asked to find the value of one variable when given the value of another. The Proportional Reasoning subskill will have the students solving a series of proportional reasoning problems while showing their work.

Content knowledge. All of the multiple-choice questions within the Content Knowledge skills section were directly adopted from the Pearson (2011) All-In-One Algebra 1 Teaching Resources. The items assess critical algebra skills identified in the Common Core Standards, including solving inequalities; functions; exponents and exponential functions; and polynomials and factoring. The language fundamentals subskill assessment will include questions focusing on the definitions and ideas required in entry algebra classes. A pool of algebra terms will be chosen from key terms in Algebra I textbooks. The terms will be presented to the students, and the students will either be given four definitions and have to select the correct one, or be given four examples of the term and have to select the correct one. Finally, the questions within the conceptual understanding subskill will assess a student's knowledge and understanding of the reasons behind specific entry-level algebra problems. The conceptual understanding subskill assessment will present students a picture and present the student with two equations, one correct and one incorrect, and then ask the student to choose which problem is correct. Having students identify correct or incorrect mathematics problems has been found to be an effective method of assessing conceptual understanding of basic calculation (Burns, 2011) and might be used to differentiate conceptual understanding of linear algebra (Booth, Lange, Koedinger, & Newton, 2013).

Authentic application. Within the Authentic Application skills section, mathematics self-concept, self-efficacy, and levels of mathematics anxiety will be assessed using an adapted version of the Mathematics Attitudes Inventory (MAI; Schneck, 2010). The questions assess areas such as confidence regarding solving mathematics problems, belief in one's performance in math, one's interest in math, and mathematics anxiety. The student is given multiple statements in each of the four areas and ranks whether they strongly agree to strongly disagree using a 5-point Likert sale. The MAI is a scale adapted from the Mulhern and Rae mathematics survey (1998), which has internal consistency ranging from .79 to .93.

Engagement. To measure student engagement with the Engagement skills section, the Classroom Engagement Inventory (CEI; (Wang, Bergin, & Bergin, 2014) will be used. This scale will be used to measure classroom-level engagement within the students' mathematics class by assessing behavioral, cognitive, and affective engagement as well as disengagement. There are a total of 24 items. The 7 items assessing cognitive engagement are on a 7-point scale, while the other items are on a 5-point scale. The scale is made to gather engagement data regarding one specific class. The scale has an internal ranging from .82 to .90 (Wang, Bergin, & Bergin, 2014).

Procedure

Pilot testing. Prior to data collection, the assessments were first administered in a pilot phase. This allowed an examination of the appropriateness of the items and administration procedures as well as the appropriate time frame for completion of each skill. The pilot testing occurred in an 11^{th} grade pre-calculus class of 16 students and a 6^{th} grade math class of 14 students.

After the testing was complete, the students' performance was analyzed and necessary changes made to the assessments and administration procedures. Changes to the assessment included: clarification of directions, implementing an appropriate time frame for completion, item arrangement, and item appropriateness. Clarification of directions was assessed through student feedback and observations of student understanding of the task directions. The appropriate time frame for the assessment was assessed by timing the amount of time it takes for the pilot class to complete the assessment and student feedback. The appropriateness of the items was assessed by reviewing the item discrimination and item difficulty for each problem.

Minor changes were made to the assessment after the pilot testing. The directions were clarified as to make the instructions more direct for the students. Items were rearranged in four skill sections of the test to maintain the easy-to-difficult progression of items. Finally, one item from the vocabulary skill section and one item from the content knowledge skill section were eliminated due to poor discrimination ability and new items were developed.

Data collection. Data collection occurred over the course of 1 month. The academic skills portion (Basic Skills, Algebraic Thinking, and Content Knowledge skills) of the assessment took one class period, or approximately 50 minutes, to complete. The survey portion (Authentic Application and Engagement skills) was administered to the class the following day and took approximately 10 minutes to complete. During the data collection phase, the primary investigator administered the assessments to participants at a classwide level. Instructions were given to the class as a whole within their respective classrooms.

The students were informed that they would be taking an assessment for a research project the day before so as not to surprise them the day of the testing. An administrator described the purpose of the test to the students. Students were then distributed the test packets. The instructions clearly described the process of testing to the student. The basic skills section assessments were administered first followed by the algebraic thought skills section and finally the content skills section. After each section was complete, the students were be given instructions to turn the page and the next task was administered. Once students were finished, the administrator collected the tests.

Statistical Analyses

Question 1: Relationship between proposed model and algebra. The data for question four will be analyzed using correlations between each skill, subskill, and the MAP-M algebra RIT scores. The internal consistency of the measures will all be reported.

Question 2: Proposed five factor structure. The data for question three will be analyzed using confirmatory factor analysis to assess theoretical nature of the algebra-learning construct. The factor analysis will help confirm the proposed factor dimensions of the model. Chi-square, Root Mean Square Error of Approximation (RMSEA), Comparative Fit Index (CFI), Tucker-Lewis Index (TLI), and Standard Root Square Mean Residuals (SRMR) will be used to calculate goodness of fit. The reliability of the assessments will be measured using internal consistency.

Question 3: Diagnostic accuracy. The diagnostic accuracy will be assessed by measuring the sensitivity and specificity of each skill against the MAP-Algebra proficiency score of 181. The sensitivity will be measured by identifying students who score below the proficiency MAP-Algebra score, and identifying the students above the proficiency score will measure the specificity. The diagnostic accuracy will be calculated by measuring the agreement versus the disagreement between the cut score and diagnostic criteria (Stage & Jacobsen, 2003).

Standard setting. There are currently no standards or research-based instructional levels for the different skill-based assessments being used in this research. Therefore, prior to comparing test performance to the criterion measure there needs to be an understanding of what is considered an appropriate proficiency score. Proficiency scores for each assessment were identified through the Angoff method because it has strong evidence as an empirical group method of standard setting (Berk, 1986). Teachers at the participating school served as the expert panel because they had familiarity with both the students and the content (Berk, 1986; Koffler, 1980; Kellow & Willson, 2008). The standard setting method involved four primary steps. First, the panel was presented with the concept of the borderline test-taker. Second, they were asked to imagine a group of 100 borderline test-takers. The panel was then instructed to identify the proportion of that group would answer each question correctly. Finally, the proportions for each problem were summed for each panel member, and then averaged across all of the members' scores to identify a proficiency score for the skill.

Table 1

Characteristics of Subskill Assessments

	Number of Questions	Time Limit
Integers Ordering	6	1 minute
Rational Number Ordering	9	2 minutes
Integers Calculation	100	2 minutes
Rational Number Calculation	30	3 minutes
Integer Word Problems	6	3 minutes
Rational Number Word Problems	6	3 minutes
Patterns	9	3 minutes
Arithmetic-to-Algebra	24	3 minutes
Generalization	6	3 minutes
Proportional Reasoning	6	3 minutes
Vocabulary	9	3 minutes
Conceptual Understanding	6	3 minutes
Problem Solving	10	4 minutes
Authentic Application Survey	20	Unlimited
Engagement Survey	24	Unlimited

CHAPTER 4: RESULTS

The study evaluated the validity of using a five-part problem-solving model to identify core deficits for students struggling with algebra. Within the model, the five core sections of the problem-solving model support and inform each other, and are all required to establish a basis for algebra proficiency.

The following research questions guided the study:

1. What is the relationship between each of the five sections within the problem-analysis model to an established measure of algebra?
2. To what extent do assessment data support the proposed five-factor structure?
3. To what extent can the five sections and each subsection within the problem-analysis model accurately identify the level of a student's difficulty with algebra as measured by a criterion?

Participant Demographics

A total of 373 students participated in testing, and 327 students were included in the final study. Of those participants, 42% were male and 58% were female. Less than 1% identify as American Indian or Alaskan Native, 11% identify as Asian or Pacific Islander, 16% identify as Hispanic, 23% identify as Black, and 50% identify as White. Of the total number of participants, 10% were receiving special education services and 6% were receiving educational support through a 504 plan. The average age of the participants was 14.1, and the average grade was 8.5. The distribution of students among grades is as follows: 62 students (19%) in sixth grade, 59 students (18%) in seventh grade, 50 students (15%) in eighth grade, 51 students (16%) in ninth grade, 40 students (12%) in 10th grade, 39 students (12%) in 11th grade, and 26 students (8%) in 12th grade.

Missing Data

Data were assumed to be missing at random. Data from 44 students were discarded prior to running the analyses. Data from 14 students were discarded because they either entered the class after the administration had already started, or had to leave before the administration was complete. The data from 35 students across all grades were discarded because they were absent on one of the testing days and only completed either the assessment or the survey. All of the students who completed both portions of the assessment did so completely. Missing data were not due to student refusal or non-response, and there were no obvious skipped items or sections. There were no notable patterns in the attendance or behavioral data (e.g., suspensions) for the students with missing data. The descriptive data for the subskill assessments can be found in Table 2, and Table 3 for the overall skill assessments.

Table 2

Descriptive Table of Subskill Measures

	Mean	Standard Deviation
Integers Ordering	5.65	0.94
Rational Number Ordering	4.41	2.12
Integers Calculation	40.81	17.19
Rational Number Calculation	8.78	3.71
Integer Word Problems	4.52	1.59
Rational Number Word Problems	2.54	1.38
Patterns	6.84	1.99
Arithmetic-to-Algebra	18.42	5.19
Generalization	3.35	2.06
Proportional Reasoning	2.93	1.79
Vocabulary	6.32	1.99
Conceptual Understanding	3.86	1.65
Problem Solving	5.20	2.34

Table 3

Descriptive Table of Skill Measures

Skill	Mean	Standard Deviation
Basic Skills	66.70	22.94
Algebraic Thinking	31.53	9.45
Content Knowledge	15.38	4.67

Question #1: What is the relationship between each of the five sections within the problem-analysis model to an established measure of algebra?

To answer the first research question, the data were analyzed using Pearson's correlation between each section, subsection, and Measures of Academic Progress mathematics (MAP-M) Rasch Unit (RIT) RIT score. The correlations between the subsections of the basic skills, algebraic thinking, and content sections are in Table 4, Table 5, and Table 6 respectively.

Subsections. The correlations within the subskill range from weak to very strong. Within the basic subskill sections, the weakest correlation was between ordering integers and solving word problems using rational numbers, while the strongest was between ordering rational numbers and solving word problems using integers. The correlations within the basic skills section indicate that there was a moderate to strong correlation between the subskill measures. The correlations within the algebraic thinking section are stronger, and they all fall into the strong range with the strongest being between patterns and arithmetic to algebra, and the weakest being between patterns and proportional reasoning. The subsection correlations within the content section are the weakest, with the conceptual and problem solving sections having a moderate correlation of $r = .38$, and the strongest correlation being $r = .43$ between problem solving and vocabulary. Finally, the correlation between sections range from strong to very strong, with the weakest being between Basic Skills and Content, and the strongest being between basic skills and algebraic thinking.

Table 4

Pearson Correlations Among the Basic Skills Subsections

	Integer			Rational		
	Ordering (Order-I)	Calculation (Calc-I)	Word Problem WP- I	Ordering Order-R	Calculation Calc-R	Word Problem WP-R
Order - I	1.00*	.32*	.31*	.33*	.35*	.22*
Calc - I	.32*	1.00*	.55*	.52*	.57*	.46*
WP – I	.31*	.55*	1.00*	.61*	.50*	.56*
Order – R	.33*	.52*	.61*	1.00*	.51*	.59*
Calc – R	.35*	.57*	.50*	.51*	1.00*	.46*
WP - R	.22*	.46*	.56*	.59*	.46*	1.00*

Note. *$p < .01$

Table 5

Pearson Correlations Among the Algebraic Thinking Subsections

	Patterns	Arithmetic to Algebra	Generalization	Proportional Reasoning
Patterns	1.00*	.65*	.58*	.55*
Arithmetic to Algebra	.65*	1.00*	.62*	.60*
Generalization	.58*	.62*	1.00*	.62*
Proportional Reasoning	.55*	.60*	.62*	1.00*

Note. *$p < .01$

Table 6

Pearson Correlations Among the Content Knowledge Subsections

	Vocabulary	Conceptual	Problem Solving
Vocabulary	1.00*	.40*	.43*
Conceptual	.40*	1.00*	.38*
Problem Solving	.43*	.38*	1.00*

Note. *p < .01

Table 7

Pearson Correlations Among the Sections

	Basic Skills	Algebraic Thinking	Content Knowledge
Basic Skills	1.00*	.76*	.54*
Algebraic Thinking	.76*	1.00*	.62*
Content Knowledge	.54*	.62*	1.00*

Note. *p < .01

MAP-M criterion. Pearson correlations were calculated for the sections and the composite score of the test in relation to the Measures of Academic Progress (MAP) assessment. The correlations for the three sections and MAP test were as follows: Basic Skills r (325) = .71, p < .001; Algebraic Thinking r (325) = .76, p < .001; and Content Knowledge r (325) = .68, p < .001. The correlation between the MAP and the composite score was r (325) = .79, p < .001. The correlation between the Engagement and Authentic Application sections along with the total composite score and the MAP scores can be found in Table 8. The correlations between the

Authentic Application and Engagement sections are generally weaker than those found in the academic skills sections. The correlations within these sections range from weak to very strong. However, the correlation between the total sum score of the assessment and the MAP math test is very strong at .79.

Table 8

Pearson Correlations Among the Authentic Application, Engagement, MAP, and Total Scores

	Positive	Negative	C	A	BC	BEP	D	MAP	Total
Positive	1.00*	-0.78*	0.37*	0.48*	0.38*	0.42*	-0.31*	0.15*	0.24*
Negative	-0.78*	1.00*	-0.33*	-0.50*	-0.35*	-0.41*	0.35*	-0.18*	-0.23*
C	0.37*	-0.33*	1.00*	0.43*	0.51*	0.53*	-0.29*	0.09	0.13
A	0.48*	-0.50*	0.43*	1.00*	0.43*	0.60*	-0.39*	0.06	0.12
BC	0.38*	-0.35*	0.51*	0.43*	1.00*	0.55*	-0.42*	0.24*	0.22*
BEP	0.42*	-0.41*	0.53*	0.60*	0.55*	1.00*	-0.30*	0.12	0.17*
D	-0.31*	0.35*	-0.29*	-0.39*	-0.42*	-0.30*	1.00*	-0.04	-0.08
MAP	0.15*	-0.18*	0.09	0.06	0.24*	0.12	-0.04	1.00*	0.79*
Total	0.24*	-0.23*	0.13	0.12	0.22*	0.17*	-0.08	0.79*	1.00*

Note. Positive = Positive Mindset; Negative = Negative Mindset; C = Cognitive Engagement; A = Affective Engagement; BC = Behavioral Engagement Compliance; BEP = Behavioral Engagement Participation; D = Disengagement; MAP = Measures of Academic Progress Mathematics RIT Score; Total = Total Score for combined Basic Skills, Algebraic Thought, and Content Knowledge sections.
*$p < .01$

Transforming the Scores. The analyses of the assessment were conducted using raw scores gathered from both the subskill and skill sections. However, it is important to consider the impact and importance of each subskill score on the skill scores, as accurate skill section scores are critical for establishing proficiency levels. For example, a student who scored very high on the integer calculation subskill section may have scored very poorly on the integer and rational word problem measures. Compared with a student who scored very high on the word problem measures but less so on the calculation measures, their scores may appear the same but fail to reflect their true overall level of proficiency with the larger skill.

The scores were transformed so they become standardized and are abled to be compared in a more meaningful way. To do this, z-scores were calculated for each skill and subskill section. The z-scores were calculated using the overall mean and standard deviation. The raw scores were also weighted by using the endogenous r^2 values taken from the confirmatory factor analysis results. The endogenous r^2 value for each subskill measure approximates the explanatory power of that subskill on its corresponding skill. The endogenous r^2 values were multiplied by the raw scores for each student and then summed for a score for each skill section.

The correlations between MAP-M scores, z-scores, and weighted scores were compared against the student raw scores, and these values can be found in Table 9. The correlations indicate that standardizing the scores result in a higher correlation for the Basic Skills, Algebraic Thinking, and Total sections scores. This is likely because the Basic Skills and Algebraic Thinking section both have subskill measures that require a student to complete as many simple arithmetic problems as they can in a short amount a time. Each simple arithmetic problem completed held the same amount of weight as another, more complicated word problem answered correctly.

Table 9

Pearson Correlations Between Z-Scores and Weighted Scores and MAP Scores

Skill Section	Z-Score	Weighted Score	Raw Score
Basic Skills	.81*	.81*	.71*
Algebraic Thinking	.77*	.75*	.76*
Content Knowledge	.68*	.68*	.68*
Total	.85*	.82*	.79*

Note. *$p < .01$

An analysis of the percentage of students who scored proficiency in zero, one, two, or three of the skill areas can be found in Table 10. The fewest number of students were only proficient with Basic Skills, while the greatest number of students lacked proficiency with all three skills. The number of students who were proficient with Content Knowledge only was slightly higher than students who were proficient with Algebraic Thinking only. Students were more likely to be proficient with both Algebraic Thinking and Content Knowledge then with either Basic Skills and Algebraic Thinking or Basic Skills and Content Knowledge. Almost 25% of the students were proficient in all three areas

One possible reason for the low levels of proficiency with only Basic Skills is that a student in middle or high school who has mastered basic skills is likely to have also mastered more advanced skills, and a student who struggles with basic skills have fallen behind their peers in both their proficiency level with those skills and in other areas. Proficiency with Content Knowledge may rely more heavily on current levels of Algebraic Thinking than Basic Skills, while Basic Skills appear to be important to the initial development of Algebraic Thinking.

Table 10

Percentage of Students Scoring Proficient Across Skill Sections

Skill	*Percentage*
Basic Skills	1
Algebraic Thinking	7
Content Knowledge	8
Basic Skills & Algebraic Thinking	9
Basic Skills & Content Knowledge	2
Algebraic Thinking & Content Knowledge	21
All Three Sections	25
No Sections	27

Question #2: To what extent do assessment data support the proposed five-factor structure?

To examine the goodness-of-fit of the model, chi-square was calculated ($X^2 = 443.43$, df = 199, p>.001). The value is significant, which within confirmatory factor analysis indicates a poor model fit and suggests that the proposed model does not adequately describe the data. However, the chi-square value is impacted by sample size, with larger sample sizes often yielding a significant figure (Schreiber et al., 2006). Because of the sensitivity of chi-square to sample size, other goodness-of-fit measures were analyzed. The Root Mean Square Error of Approximation (RMSEA) value is .06 with a 90% confidence interval of .05 to .07, with a rule of .06 being an indicator of good fit (Cheung & Rensyold, 2002; Walkey & Welch, 2010). The Comparative Fit Index (CFI) value is .93 with a cut-off of .90. The Tucker-Lewis Index (TLI) is .92, with a cut-off of .90. The standard root square mean residuals (SRMR) value is .05 with a desired cut-off value of below .08 (Cheung & Rensyold, 2002; Walkey & Welch, 2010). All of the values indicate the model is a good fit.

Because the values indicate the model is of good fit, we can review the partially standardized parameter estimates for the model. In the results, the latent factors are reported with a variance of one while the indicators are reported in their original metric. Therefore, a one-unit increase in the factor score would result in a corresponding increase in the indicator (Brown, 2015). These values can be found Table 11 and Table 12. All of the factor loadings were significant ($p<.001$).

Table 11

Five-Factor Parameter Estimates for Authentic Application and Engagement

	Parameter Estimates	Standard Error	95% Confidence Interval
Authentic Application			
Positive Mindset	2.59*	.14	2.32 to 2.86
Positive Application	2.52*	.14	2.25 to 2.79
Negative Mindset	-2.50*	.17	-2.83 to -2.17
Negative Application	-2.60*	.16	-2.91 to -2.29
Engagement			
Cognitive	5.37*	.44	4.51 to 6.23
Affective	3.65*	.26	3.14 to 4.16
Compliance	1.49*	.11	1.27 to 1.71
Participation	3.22*	.21	2.81 to 3.63
Disengagement	-1.47*	.17	-1.8 to -1.14

*p<.001

Table 12

Five-Factor Parameter Estimates for Basic Skills, Algebraic Thinking, and Content

	Parameter Estimates	Standard Error	95% Confidence Interval
Basic Skills			
Ordering Integers	.41*	.05	.31 to .51
Ordering Rational	1.59*	.10	1.39 to 1.79
Calculation Integers	11.95*	.85	10.28 to 13.62
Calculation Rational	2.51*	.18	2.15 to 2.86
Word Problem Integer	1.30*	.07	1.16 to 1.44
Word Problem Rational	.96*	.07	.82 to 1.10
Algebraic Thinking			
Patterns	1.47*	.10	1.27 to 1.67
Arithmetic to Algebra	4.19*	.24	3.72 to 4.66
Generalization	1.61*	.10	1.41 to 1.81
Proportional Reasoning	1.39*	.08	1.23 to 1.55
Content			
Vocabulary	1.26*	.11	1.04 to 1.48
Conceptual Understanding	.88*	.09	.70 to 1.06
Problem Solving	1.70*	.13	1.45 to 1.95

$p<.001$

Question #3: To what extent can the five sections and each subsection within the problem-analysis model accurately identify the level of a student's difficulty with algebra as measured by a criterion?

There are currently no standards or research-based instructional levels for the different skill-based assessments being used in this research. Therefore, prior to comparing test performance to the criterion measure there needed to be an understanding of what is considered an appropriate proficiency score. Proficiency scores for each assessment were identified through the Angoff method because it has strong evidence as an empirical group method of standard setting. Teachers at the participating school served as the expert panel because they had familiarity with both the students and the content (Berk, 1986; Koffler, 1980; Kellow & Willson, 2008). The standard setting method involved four primary steps. First, the panel was presented with the concept of the borderline test-taker. Second, they were asked to imagine a group of 100 borderline test-takers. The panel was then instructed to identify the proportion of that group would answer each question correctly. Finally, the proportions for each problem were summed for each panel member, and then averaged across all of the members' scores to identify a proficiency score for the section.

The diagnostic accuracy was assessed by measuring the sensitivity and specificity of each section against the MAP math proficiency score of 232, and the results can be found in Table 13. The sensitivity was be measured by identifying students who scored below the proficiency MAP-Algebra score, and identifying the students above the proficiency score measured the specificity. The diagnostic accuracy was calculated by measuring the agreement versus the disagreement between the cut score and diagnostic criteria (Stage & Jacobsen, 2003). The sensitivity value for the overall assessment was 76% while the specificity was 92%. While the Authentic Application and Engagement section scores were included in the overall assessment score, there were no sensitivity, specificity, or accuracy values calculated for them. This is because cut-off scores were not identified for those assessments due to a lack of traditional "correct" and "incorrect" answers.

Table 13

Standard Setting, Sensitivity, Specificity, and Diagnostic Accuracy of Sections and Subsections

Section	Cut-off Score	Sensitivity	Specificity	Accuracy
Ordering - Integers	6	.94	.28	.65
Ordering - Rational	5	.73	.87	.79
Calculation - Integers	40	.76	.70	.73
Calculation - Rational	14	.23	.99	.57
Word Problem - Integers	4	.93	.56	.77
Word Problem - Rational	3	.63	.86	.73
Basic Skills Total	72	.64	.89	.75
Patterns	7	.87	.53	.72
Arithmetic to Algebra	18	.90	.63	.78
Generalization	4	.75	.78	.76
Proportional Reasoning	3	.81	.74	.78
Algebraic Thought Total	32	.87	.77	.83
Vocabulary	7	.72	.68	.70
Conceptual Understanding	4	.77	.58	.69
Problem Solving	5	.81	.54	.74
Content Total	16	.74	.77	.75
Test Total	120	.76	.92	.84

CHAPTER 5: DISCUSSION

The purpose of the study was to examine evidence for the validity of a proposed problem-solving model for identifying skill deficits in students struggling with algebra. Three research questions guided the study. First, what is the relationship between each of the five sections within the problem-analysis model to an established measure of algebra? Second, to what extent does assessment data support the proposed five-factor structure? Third, to what extent does the five sections and each subsection within the problem-analysis model accurately identify the level of a student's difficulty with algebra as measured by a criterion?

The problem-analysis model has five skills sections, and within each skill section were different subskills. To review, the skills are Basic Skills, Algebraic Thinking, Content Knowledge, Engagement, and Authentic Application. The Basic Skills skills section contains the following subskills: comparing and ordering, calculation, and word problems. The Algebraic thinking skills section contains the following subskills: patterns, arithmetic to algebra, proportional reasoning, and generalization. The Content Knowledge skills section contains the following subskills: problem solving, vocabulary, and conceptual understanding. The Authentic Application section contains the following subskills: positive mindset and negative mindset. The Engagement skills section contains the following subskills: cognitive engagement, affective engagement, behavioral engagement, behavioral engagement participation, and disengagement.

Relationship Between Model Components

The correlations between the subsections within the Basic Skills, Algebraic Thinking, and Content Knowledge skills sections range from weak to moderate, with the Algebraic Thinking section having the strongest correlations between subsections. The negative correlation between the Positive and Negative subsections on the Authentic Application section was strong, and the correlations between the subsections on the Classroom Engagement Inventory (CEI) were weak to moderate. The correlations between skill sections are larger, with the Algebraic Thinking section sharing the largest correlation with the other two sections. The correlations indicate that Basic Skills has the weakest connection to Content Knowledge, and the strongest to Algebraic Thinking.

The correlations indicate that the subskills are related to one another, and each skill and subskill provides a unique set of information that, based on the evaluation of the five-factor model, can be assumed to give important information about separate mathematics skills critical to algebra proficiency. There is a weak to moderate correlation between the subskill measures within each subskill section.

The correlations among the overall skills are higher than the subskills. Among the sections, Basic Skills is strongly correlated with Algebraic Thinking, indicating that a student's ability to order, calculate, and solve word problems using integers and rational numbers is closely related to their ability to perform tasks that utilize algebraic thinking. These skills seem to have less impact on their ability to solve problems and recall facts using content knowledge. One possible reason for the weaker correlation is that students may not directly use their arithmetic calculation skills in algebra, but they do so when developing skills related to arithmetic thinking.

The correlation between the total academic assessment score has an overall correlation of $r = .79$, indicating a strong correlation with the MAP test. This correlation indicates that the assessment is a good predictor of the MAP performance. The MAP was used as a criterion measure of mathematics proficiency, and using it as a comparison, it can be interpreted that the designed assessment is also a good indicator of overall math proficiency.

There are four steps in the validation process. The first is to clearly and explicitly state the model. The second step is to evaluate the inferences of the scores by comparing them to relevant and related evidence. Evidence for this step is provided by the high correlation with the MAP. The third step is to evaluate the assumptions made by the data. By using confirmatory factor analysis, the model is deemed a good fit for the data. The combined evidence supports the validation of the interpretation of the data provided by the measures.

Consistency with Previous Research

These findings were consistent with culminated literature on the relation between prerequisite skills required for algebra proficiency (NMAP, 2008). Having an understanding of integer magnitude is the basis for calculation and word problems (Zaslavsky, 2001; Hallet, Nunes, Bryant, & Thorpe, 2012; NRC, 2001; Siegler et al., 2012). Understanding fraction magnitude is especially important, as it not only relates to calculation and word problems, but also algebraic thinking and algebraic problem solving (Bailey, Hoard, Nugent, & Geary, 2012; Booth & Newton, 2012; Siegler, Thompson, & Schneider, 2011). Proficiency with word problems and calculation supports flexible application and algebraic thinking (O'Loughlin, 2007; Fuchs et al., 2012; NRC, 2001).

Algebraic thinking serves as the bridge between arithmetic and algebra problem solving (Warren & Cooper, 2008; Cai & Moyer, 2008; Ferrucci, Kaur, Carter, & Yeap, 2008). The understanding of variables and equivalence assists with the flexible use of arithmetical strategies (Stacey & MacGregor, 2000), which can be applied to solve patterns and functions (Kaput, Carraher, & Blanton, 2008; Kieran 2004) and proportional reasoning (Özgün-Koca & Altay, 2009). Understanding generalization and equivalence promotes flexibility and generalization, which are critical to algebra problem solving (Banchoff, 2008; Kinach, 2014; Johanning, 2004).

Directions for Future Research

Future research should focus on targeted interventions for the different related skills and subskill required for proficiency, with the goal of generalization of skills across algebra. There is a substantial body of research on improving the basic arithmetic skills of struggling students through intervention, but there is substantially less literature on delivering interventions for improving students' algebraic thinking and content knowledge.

Support for the Five-Factor Structure

The significant chi-square value indicates that the proposed five-factor model is not a good fit. However, when running CFA, a large sample size can lead to type II errors, and it is recommended that additional measures of good fit be considered. The Comparative Fit Index (CFI), Tucker-Lewis Index (TLI), Root Mean Square Error of Approximation (RMSEA) and Standard Root Square Mean Residuals (SRMR) values indicate that the model is a good fit.

It can be interpreted, based on the findings, that the proposed model can be used effectively when conducting problem analysis with students struggling with algebra. Each skills section of the model provides a unique grouping of information that is relevant to a student's development of algebra proficiency. Basic skills, algebraic thinking, content knowledge, engagement, and authentic application all play an important role in developing algebra proficiency. Evaluating each skills and subskill section could potentially provide instructional and targeted intervention guidance to educational staff, but the implications that these data have for intervention is an area for future research.

Problem analysis involves the explicit and systematic testing of hypotheses and clearly defined variables that are related, alterable, and relevant to an intervention (Christ, 2008). When a student is learning algebra, they should be receiving quality core instruction. If they struggle, they should receive more intensive and targeted instruction in the subject areas. If they continue to struggle despite these supports, the problem analysis process should begin. The proposed five-factor model provides a framework for explicitly and systematically addressing the skills required to be successful in algebra. The assessment can be administered and areas of difficulty can be identified, and then targeted interventions implemented to target those areas of need.

Consistency with Previous Research

The current finding regarding the factor structure of the scale was consistent with research by Bush and Karp (2013), NMAP (2008), NRC (2001), and SREB (2003). The five-factor model supports the importance of basic skills, algebraic thinking, content knowledge, authentic application, and engagement in developing proficiency. The results support research stating the importance of being able to both recall math facts and apply those concepts with flexibility in problem-solving settings (Kortering, deBettencourt, & Braziel, 2005). The model also supports research underlying the importance of the collaborative nature of basic skills and algebraic thinking in algebra proficiency (Rittle-Johnson & Siegler, 1998; Crooks and Alibali, 2014; Newton, Star, & Lynch, 2010; Star, 2005). The past research has overwhelmingly underlined the critical nature of these skills, but they have not been examined in a systematic, analytic fashion.

Directions for Future Research

Subsequent research should continue to expand on identifying and clarifying skills required for algebra proficiency. These skills can be sought out as early as elementary school where intervention and instruction can be delivered to help lay the learning foundation for later algebra success (Steketee & Scher, 2011).Future research should continue evaluating the validity of the model. This might look like repeating the assessment methods with different populations, expanding it, and including information about individuals using demographics. It is important that the model is representative of actual student performance in the classroom. Using the current model as a predictor, and measure, of long-term success will help establish the validity and practicality of this learning model.

Criterion Measure Evaluation

Sensitivity and specificity were calculated for all the sections scores and the total score for the Basic Skills, Algebraic Thought, and Content Knowledge sections. The Engagement and Authentic Application scores were not included in this total score because the scales did not lend themselves to a cut-off score, and were not included in the standard setting process.

The total score has a sensitivity of .76, indicating that it identifies 76% of students that are proficient with algebra, but fails to identify 24% of those who are proficient. The test fares better with specificity, a value of .92. This means that the test has a 92% chance of identifying a student who is not proficient at algebra. This ratio of low sensitivity to high specificity is somewhat desirable, because in practice it better to over-identify struggling students and deliver additional support rather then failing to identify struggling students, as students who are not struggling but are initially identified as doing so could quickly be exited from intervention if they are seen as not to need the support.

Consistency with Previous Research

One goal of problem-analysis is to isolate and intervene with specific skills (Burns & Gibbons, 2013; Christ, 2008; Christ, Burns, & Ysseldyke, 2005), and for this to occur, valid and reliable assessments that target these skills need to exist (NMAP, 2008). Measuring specific skills is a crucial to successful intervention (Ysseldyke, Burns, Scholin, & Parker, 2010). While broader standardized assessments exist for assessment in mathematics, they do not necessarily target specific algebra skills. The path to validation of the problem-analysis model can provide an avenue to develop more targeted assessments.

Two types of commonly used assessments within schools are screening and diagnostic. Screening assessments are used to identify individuals who are, or may be, at risk for academic struggles and who may benefit from more intensive instruction and/or intervention (Kettler et al., 2014). Diagnostic assessments provide insight into the differing abilities of an individual that provides information on how to differentiate instruction and intervention (Kingston, Scheuring, & Kramer, 2013). Both types of assessments are valuable in the educational setting, and the based on its sensitivity and specificity values, the current assessment may be able to be used for both purposes. However, due to the specific nature of the skills and subskills assessed, it may be better served as a diagnostic assessment. Future research should examine the diagnostic applicability of the problem-analysis model with regards to delivering instruction and intervention to struggling students.

General Discussion

The results of the study indicate that the proposed model can be applied in problem-analysis. Below I will discuss the implications for theory and practice.

Implications for Theory

The research supports the current ideas of prerequisite skills and skill hierarchies for more advanced academic areas (Gagné & Driscoll, 1988). The factors indicate that there is a relationship between a student's basic skills, algebraic thought, engagement, authentic application, and content knowledge in relation to their algebra proficiency, supporting the idea that proficiency with certain mathematics skills are required before proficiency with other skills can be gained. While this hierarchy of learning is accepted in other areas like reading, it is not clear in mathematics. The data do not provide a clear, linear model of learning algebra, but they do provide support the idea that proficiency with more fundamental skills is crucial before proficiency can be gained with advanced skills (Again comparing it to a theory would be great).

The research adds to the literature and supports the theory of mathematics proficiency as outlined by the NRC (2001). Research literature has supported the different yet importance of both conceptual understanding and procedural fluency, but at this time there has no research study directly applying that theory of proficiency to learning algebra. Certain subskills assessments assessed a student's conceptual understanding and procedural fluency in certain areas. That data can provide guidance for further evaluation of the theory of mathematics proficiency applied to algebra learning.

Implications for Practice

While evaluating the use of the model in a problem-analysis context, the current research did not directly apply the model in an educational setting. Therefore, it is important that the practice implications be interpreted with caution. In addition, more research needs to be conducted before specific instructional and intervention recommendations can be made through use of the model.

In practice, the model may help in identifying students who are struggling with algebra. The model can also assist practitioners in identifying specific areas where a student may have deficits. However, data gathered through applying the model should not be the only evidence used in making the claims.

The model can also be used by teachers to analyze why a student may be continuing to struggle with algebra content despite having received quality core instruction in addition to more intensive content support. Future research should search for optimal methods for this instruction support and intervention delivery. However, it is important to note that with a focus on required skills, and if a student is lacking certain skills, that that student not be retained or prevented from advancement, but rather concurrently provided intervention support. One avenue for future research is finding the optimal balance between algebra intervention and instruction.

Limitations

There were limitations of the study. First the population used was a convenience sample. While all students were tested within the two schools, those schools only represented a certain sampling of middle and high school students. To enhance the generalizability and validity of the model, it is critical that the model and procedures be replicated and applied across different populations.

Second, the measures used within the assessments were created for the purpose of testing. Preexisting measures with backing of literature were not used, because none were available. The subskill measures used to examine performance in each area were designed specifically for this assessment. Prior to use they were piloted, the items analyzed for specificity and sensitivity. If appropriate measures are identified, the model can be tested with different assessments targeting the same skills, which help support the validity of the model.

Third, the use of confirmatory factor analysis proved an effective way of analyzing the data and confirming the model. However, the use of CFA does not mean that the model is the only model of fit, nor is it the best model. While mining for more positive results is not recommended, future evaluation of theoretically sound learning models should be considered and compared to the present model.

Fourth, the assessments were designed with a limited testing time in mind due to the practical nature of the assessment administration. This limited time can impact a student's ability to complete the assignments even if they know the information. They also may experience some anxiety and/or fatigue. Research should consider utilizing different modes of testing administration to see how it impacts students' performance and the validity of the model.

Conclusions

Current practice is flush with effective, evidence-based instructional methods for teaching algebra, but there remains little for intervening with struggling students beyond addressing deficits with different instructional methods. This research adds to the literature supporting the skills required for algebra. It also adds to the sparse literature on problem-analysis for students struggling with algebra. It provides a systematic way of identifying skill deficits, which can be used to deliver targeted interventions.

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Appendix

Ordering Integers from Least to Greatest

3, 7, 1, 5 ________________________________

17, 21, 5, 22 ________________________________

5, 44, 19, 38 ________________________________

2, -4, 10, 0 ________________________________

82, -15, -9, 11 ________________________________

1, -41, 0, -20 ________________________________

Ordering Fractions and Decimals from Least to Greatest

$\frac{5}{6}$ $\frac{2}{6}$ 1 $\frac{3}{6}$ ________________________________

.2 . 7 .1 .8 ________________________________

.04 . 005 .11 .9 ________________________________

$\frac{3}{5}$ $\frac{3}{7}$ 1 $\frac{3}{4}$ ________________________________

$\frac{3}{5}$ $\frac{1}{3}$ $\frac{2}{7}$ $\frac{4}{3}$ ________________________________

$\frac{1}{6}$ $\frac{4}{8}$ .3 $\frac{1}{9}$ ________________________________

.25 $\frac{3}{7}$ .6 $\frac{7}{8}$ ________________________________

$\frac{7}{15}$ .27 .3 .77 ________________________________

.15 .61 $\frac{6}{16}$ $\frac{3}{11}$ ________________________________

Fraction and Decimal Computation

$\frac{2}{5}+\frac{2}{5}=$	$\begin{array}{r} 1.7 \\ \underline{+\ 0.8} \end{array}$	$\frac{2}{3}-\frac{1}{5}=$	$\frac{3}{4}-\frac{1}{4}=$	$\begin{array}{r} 0.6 \\ \underline{-\ 0.4} \end{array}$
$\frac{7}{9}\times\frac{1}{2}=$	$\begin{array}{r} 0.52 \\ \underline{+\ 0.17} \end{array}$	$\frac{4}{9}\div\frac{1}{2}=$	$\begin{array}{r} 1.09 \\ \underline{-\ 0.85} \end{array}$	$\begin{array}{r} 0.5 \\ \underline{\times\ 0.5} \end{array}$
$\frac{2}{4}+\frac{1}{3}=$	$2.5 \div 0.5 =$	$\frac{5}{11}\times\frac{1}{4}=$	$\frac{3}{7}\div\frac{5}{9}=$	$3 \div .6 =$
$\begin{array}{r} 0.9 \\ \underline{\times\ 0.2} \end{array}$	$\frac{3}{7}+\frac{1}{7}=$	$\frac{5}{6}\times\frac{2}{3}=$	$\begin{array}{r} .27 \\ \underline{\times\ .13} \end{array}$	$\frac{5}{8}-\frac{4}{8}=$
$5.6 \div 0.8 =$	$\frac{3}{5}-\frac{6}{7}=$	$\frac{8}{11}\div\frac{5}{9}=$	$\begin{array}{r} 2.77 \\ \underline{+\ 1.91} \end{array}$	$\begin{array}{r} 3.33 \\ \underline{-\ 1.94} \end{array}$
$\frac{2}{9}\times\frac{4}{13}=$	$3.2 \div 0.06 =$	$\frac{13}{3}+\frac{7}{4}=$	$2\frac{1}{5}+3\frac{1}{6}=$	$2\frac{3}{4}\div\frac{3}{2}=$

Solving Word Problems

There are 61 passengers and 14 workers on a ship. How many people are on the ship altogether?

There were 17 roses in the garden. Melanie cut 8 roses and put them in a vase. How many roses in the garden are left?

Malik has 64 marbles. He puts an equal number of marbles into each of 4 jars. How many marbles are in each jar?

There are 38 students going on a class trip. The students ride in vans. There are 7 students riding in each van. How many vans are needed to take all the students?

Janelle had 100 stickers. She gave 2 stickers each to 10 friends. How many stickers does Janelle have left?

Danielle bought 4 books about animals, 7 books about outer space, and 1 book about trains. Each book cost $5. How much did Danielle spend on the books?

Solving Word Problems with Fractions and Decimals

Duncan first traveled 4.8 km in a car and then he traveled 1.5 km in a bus. How far did Duncan travel?

Tom ate $\frac{2}{7}$ of a cake, and Jane ate $\frac{2}{7}$ of the cake. How much of the cake did they eat altogether?

A workman cut off $\frac{2}{5}$ of a pipe. The piece he cut off was 3 meters long. How many meters long was the original pipe?

Ann and Jenny divide 560 coins between them. If Jenny gets $\frac{3}{8}$ of the coins, how many coins will Ann get?

Nora is running a race that is 24 miles. She is running at a speed of 6 miles per hour. She has completed $\frac{3}{4}$ of the race. How much longer will it take Nora to finish the race?

Riley has 200 stamps. 35% are from Europe, 10% are from Asia, and 20% are from Australia. The rest of the stamps are from North America. How many of Riley's stamps are from North America?

Finding Patterns

1	2	3	4	__________	__________
12	10	8	6	__________	__________
3	7	11	15	__________	__________
6	12	24	48	__________	__________
150	120	90	60	__________	__________
5	1	-3	-7	__________	__________
$\frac{1}{2}$	1	$1\frac{1}{2}$	2	__________	__________
.1	$\frac{1}{5}$	.3	$\frac{2}{5}$	__________	__________
4	9	16	25	__________	__________

Solve for the Missing Value

Find y if $x = 3$ $5 + x = y + 3$	**Find y if $x = 13$** $x - 7 = 20 - y$
Find y if $x = 4$ $\frac{16}{x} \cdot 5 = 2 + 6y$	**Find y if $x = 3$** $2x + 18 = \frac{y}{4}$
Find y if $x = 9$ $5x + 4 = 11y - 6$	**Find y if $x = 6$** $7 \cdot x = \frac{4y}{2}$

Find the Missing Value

9 +□ 11	9 -□ 3	□ + 8 16	□ -4 8
□ +4 7	□ ×4 16	□ +4 19	□ ×5 25
□ ×8 56	□ ×9 54	36 ÷□ 4	□ +8 21
□ -4 11	□ ÷ 3 9	□ +4 11	32 ÷□ 8
□ -9 8	15 -□ 7	□ - 6 4	□ ÷8 64
9 ×□ 27	□ ÷7 49	8 ×□ 72	35 ÷□ 7

Solving Problems with Proportions and Ratios

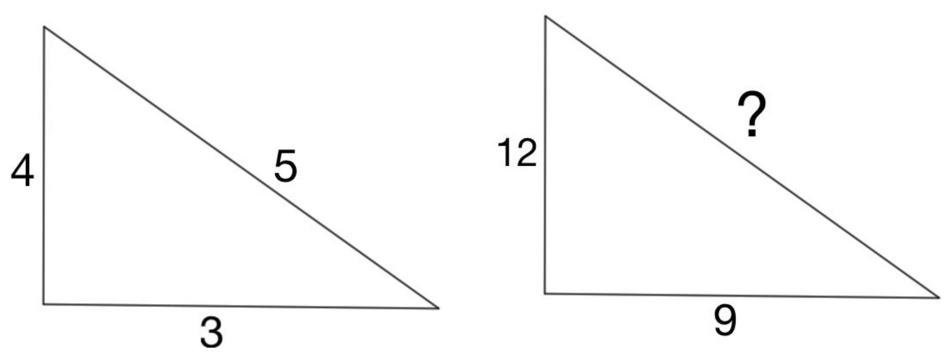
One family paid $75 for 15 hamburgers. What is the cost per hamburger?

If it takes 7 hours to mow 4 lawns, how many lawns can be mowed in 35 hours?

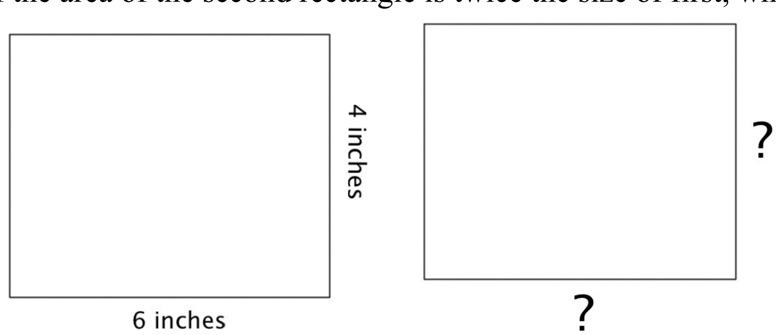
A recipe calls for a milk and sugar in a ratio of 4:2. How much milk will be needed with 1 cup of sugar?

You want to buy a shirt for $27. Which is the better deal: 15% off the price or $3 off the price?

The two triangles are similar. Find the length of the missing side.



If the area of the second rectangle is twice the size of first, what is the length and width?



Vocabulary

The top number in a fraction

a) Numerator
b) Denominator
c) Product

A relation that pairs one input with only one output of the domain

a) Sequence
b) Identity
c) Function

A mathematical sentence that states that two quantities are equal

a) Equation
b) Function
c) Sequence

Replacing each variable with a given number

a) Survey
b) Substitution
c) Equation

The name of the number in $4x^3$

a) Constant
b) Coefficient
c) N^{th} Term

A list of numbers with a common difference

a) Geometric Sequence
b) Non-linear Function
c) Arithmetic Sequence

The ratio of rise over run

a) Slope
b) Translation
c) Isolate

When an algebraic expression has no like terms, negative exponents, or grouping symbols

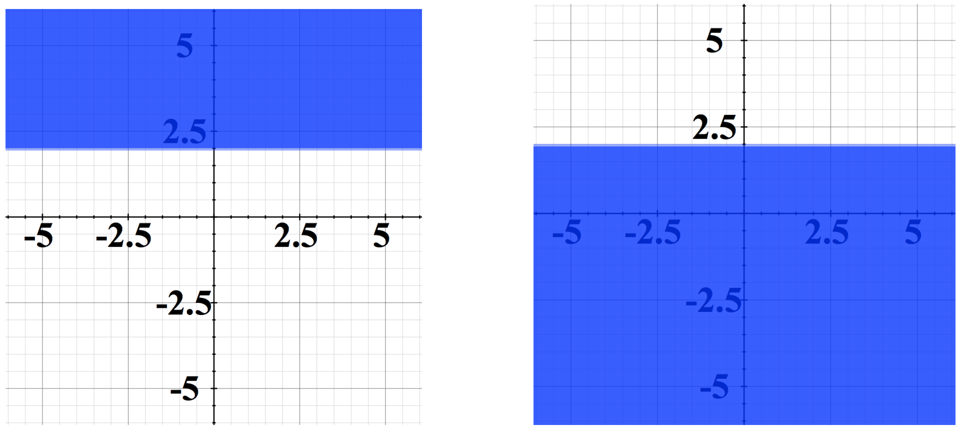
a) Simplest Form
b) Factor
c) Survey

A mathematical problem that contains integers, variables, and operators

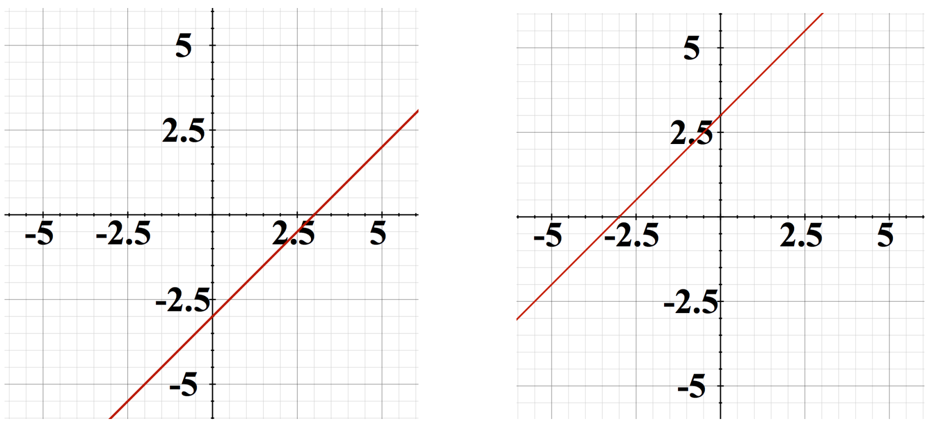
a) Algebraic Expression
b) Arithmetic Expression
c) Geometric Expression

Conceptual Understanding

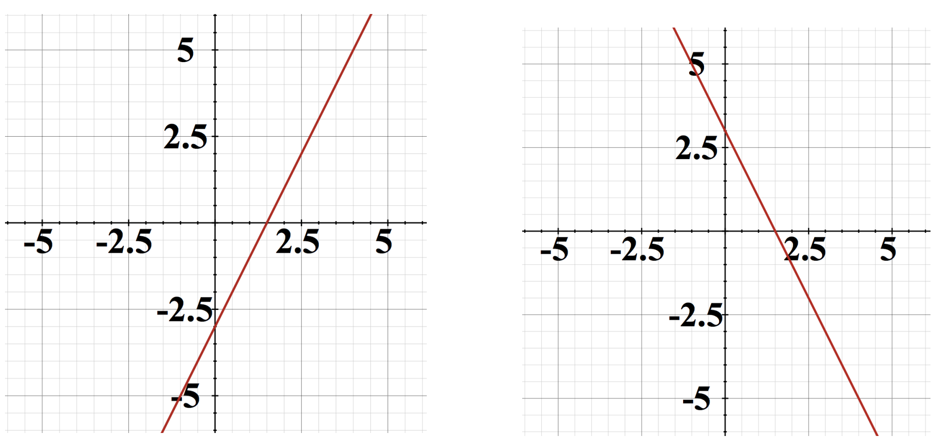
$y < 2$



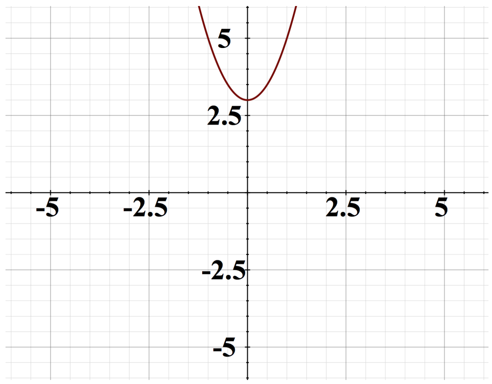
$y = x + 3$

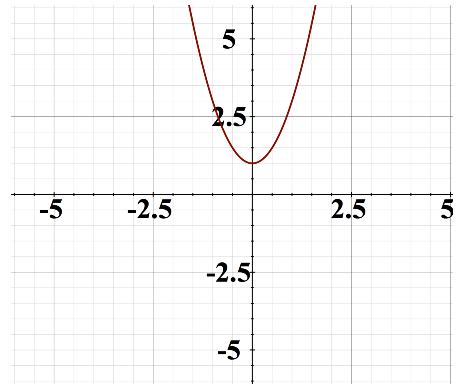


$y = 2x - 3$

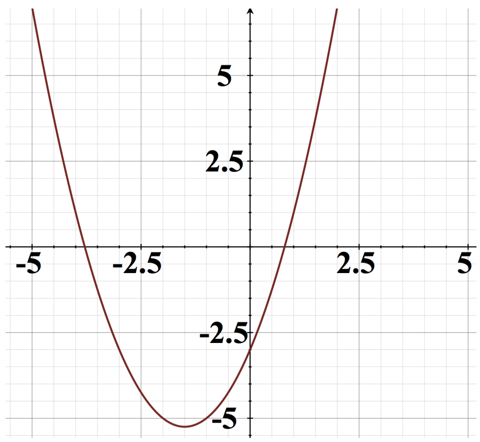


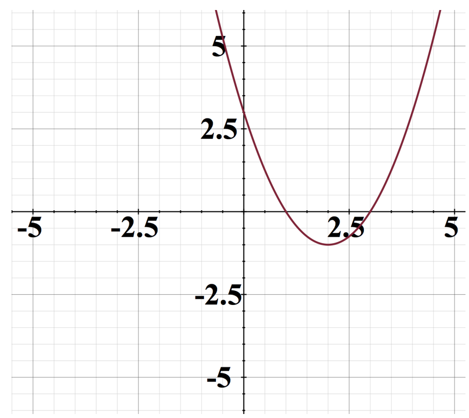
$y = 2x^2 + 1$



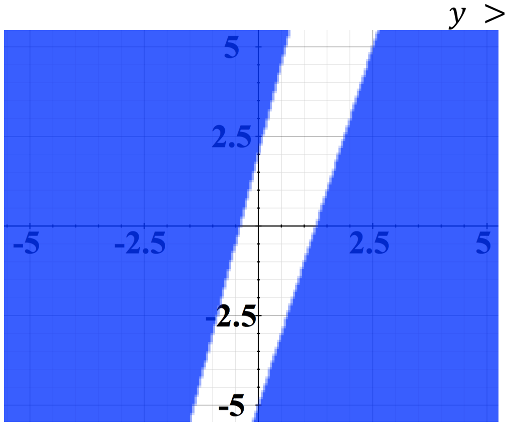


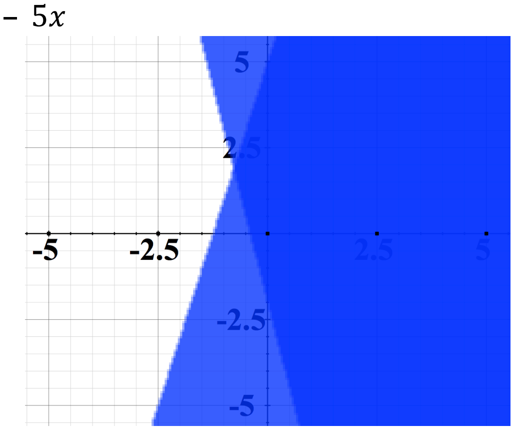
$$y = x^2 + 3x - 3$$





$$y < 4x + 5$$
$$y > -2 - 5x$$





Algebra Skills Review

Solve + $4 \cdot 3x = 36$ 2 3 4	Solve change fraction format $\frac{a}{10} < -2$ $a < -20$ $a > -20$ $a < -5$
Jill earns \$45 per hour. Using p for her pay and h for the hours she works, what function rule represents the situation? $h = 45p$ $p = 45h$ $h = p + 45$	Sarah opens a savings account by depositing \$1000. She deposits an additional \$75 into the account each month. What is an explicit formula that represents the amount of money in the account? $f(n) = 1000 * 75(12)$ $f(n) = 1000 + 75n$ $f(n) = 1000n$
Identify the general rule to the pattern $5, 25, 125, 625, \ldots$ $a_1 = 5;\ a_n = a_{n-1} \cdot 5$ $a_1 = 1;\ a_n = a_{n-1} \cdot 5$ $a_1 = 1;\ a_n = a_{n+5} \cdot 4$	Solve $-17 = -2n + 13 - 8n$ -3 $-2/3$ 3
What is the simplified form? $8b^3c^2 + 4b^3c^2$ $12bc$ $12b^3c^2$ $12b^6c^4$	A student spends no more than 2 hours on his math and English homework. If math takes about twice as long as English, what is the maximum time that the student can spend on English? $1/3$ hour 1 hour $2/3$ hour
What is the simplified form? $8\sqrt{5} + 5\sqrt{5}$ $3\sqrt{5}$ $13\sqrt{5}$ $40\sqrt{5}$	A trip from Ohio to New York is 529 miles. What equation shows the time, t, it takes to go by car if r is the average speed during the trip? $t = \frac{529}{r}$ $t = 529r$ $r = 529t$