A Simple Mathematical Model of Arctic Ocean Ice Extent

A THESIS
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I also would like to thank my husband Shuai for many things, not the least of which is accompanying me, both by keeping me harmonious and supporting me throughout the entire process.

Dedication

This thesis is dedicated to my parents who have given me the opportunity for an education from one of the best institutions – University of Minnesota, and spiritually support me to get through my life abroad in the past 7 years.

This thesis is also dedicated to my Professor Richard McGehee, who guided my path to the Mathematical Climate. Professor McGehee sparks my great curiosity in acknowledging the climate crisis, and inspires me to make a bold prediction by using what I have learned. Without his guidance and persistent help this thesis would not have been possible.

Abstract

The atmospheric concentrations of greenhouse gases, such as carbon dioxide, methane, and nitrous oxide have a significant increase in the past 150 years. Especially, atmospheric concentration of carbon dioxide has increased to unprecedented level by 40% since pre-industrial times[1]. Carbon dioxide forces the Earth's energy budget out of balance by absorbing thermal infrared energy (heat) radiated by the surface, causing the ocean's surface to warm, and melting more and more polar icecaps. Since polar icecaps help to regulate the Earth climate system, the fate of the Arctic icecaps is critical to the future climate.

In the thesis, I construct some simple math models to predict how soon until the Arctic Ocean will be icefree. In the comparison with the comprehensive prediction processes from the IPCC, this thesis considers much fewer components and simpler math models to analyse the trend of the Arctic icecap.

The prediction results in the thesis is approximately year 2035, which is the time of the Arctic Ocean will become icefree in September. This result could be regarded as a possible caution to alert people about the climate crisis.

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Chapter 1

Introduction

Sea ice forms and melts in seawater; this can be seen at the polar icecaps located at the Arctic and Antarctic Poles. Since sea ice can efficiently reflect the Sun's radiance, just like wearing a white shirt to keep people cool when is out in the Sun, sea ice plays a crucial role in the global ocean circulation and regulation of climate. Area, thickness and age of sea ice in the Arctic Sea is the proxy of Earth's temperature.

Since the 1970s, satellites have provided the best method to allow scientists to monitor the inter-annual variations and trends in sea ice cover. Scientists can collect data of Arctic Sea ice during specific time periods to calculate and observe the changes in global temperature and climate.

In recent years, satellite data shows the Arctic sea ice having a shrinking trend; more sea ice is melting into the seawater due to the increasing temperature of the ocean's surface.

Questions - According to the previously mentioned trends in sea ice shrinkage, it is natural to wonder about the main factor causing the ocean's surface temperature to rise, and ask, how soon until the Arctic Sea will be icefree?

There is a lot of convincing evidence showing that some anthropogenic activities cause climate change. Scientists have discovered that the production of electricity using coal and petroleum, and other uses of fossil fuels in transportation and industry, have contributed to the increasing emission of carbon dioxide and other greenhouse gases into the atmosphere. Within the past 200 years, human activity has increased the amount of carbon dioxide in the atmosphere by 40%. These gases absorb heat being radiated

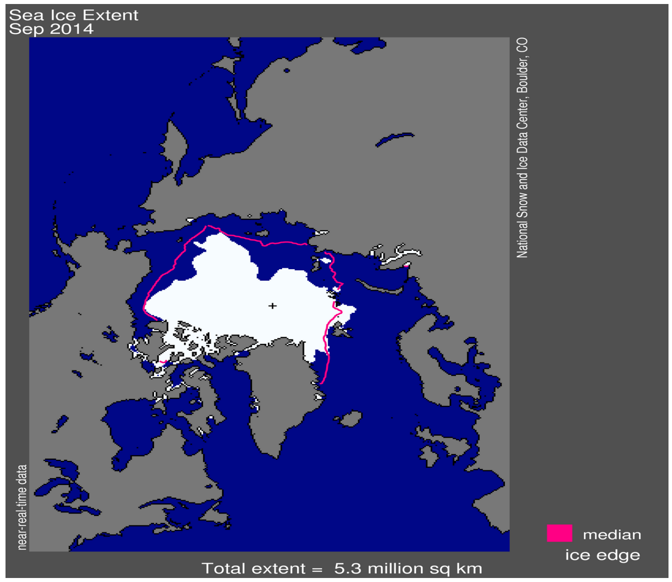


Figure 1.1: Arctic sea ice extent for September 2014 was 5.28 million square kilometers (2.04 million square miles). The magenta line shows the 1981 to 2010 median extent for that month. The black cross indicates the geographic North Pole.

from the surface of Earth, and affect the Earth's heat balance. This heat imbalance has a large impact on the Arctic Sea overall energy, causing more icecaps to melt.

IPCC, the Intergovernmental Panel on Climate Change, which is the leading international body for the assessment of climate change, gives a very comprehensive analysis and prediction for the Arctic icecaps in their 5th Assessment Report. They consider hundreds of natural and anthropogenic factors that may affect or be related to the Arctic icecaps. Additionally, they use a super computer and comprehensive math models to predict that the Arctic Sea could be icefree around the year 2050[1].

In comparison with IPCC's prediction methods and results, this thesis discusses much more concise math models, and only considers a few factors: atmospheric CO_2 concentration, the Arctic Sea openwater area, and the Arctic Sea icecap extent.

The prediction methods are divided by two perspectives in this thesis. Firstly, a preliminary math model is set up from the perspective of the Arctic Sea icecap extent. In this preliminary model, the basic idea is to analyze how the CO_2 concentration affects the Arctic Sea icecap extent. According to the given datasets of the CO_2 concentration vs. time and the Arctic Sea icecap extent vs. time, it is appropriate to use the linear or

Year of September Average Extent	Extent (million sq. km.)	Anomaly Relative to 1981-2010 Average (million sq. km.)	Anomaly Relative to 1981-2010 Average (%)	Anomaly Relative to Previous Record (million sq. km.)	Anomaly Relative to Previous Record (%)	Linear Trend Since 1979 (sq. km. per year)	Linear Trend Since 1979 Relative to 1981-2010 Average (% per decade)
2002	5.96	-0.56	-8.6	-0.17	-2.8	-51000	-7.8
2003	6.15	-0.37	-5.7	0.19	3.2	-52800	-8.1
2004	6.05	-0.47	-7.2	0.09	1.5	-54600	-8.4
2005	5.57	-0.95	-14.6	-0.39	-6.5	-59400	-9.1
2006	5.92	-0.6	-9.2	0.35	6.3	-60200	-9.2
2007	4.3	-2.22	-34	-1.27	-22.8	-71600	-11
2008	4.73	-1.79	-27.5	0.43	10	-77700	-11.9
2009	5.39	-1.13	-17.3	1.09	25.3	-78200	-12
2010	4.93	-1.59	-24.4	0.63	14.7	-80700	-12.4
2011	4.63	-1.89	-29	0.33	7.7	-84000	-12.9
2012	3.63	-2.89	-44.3	-0.67	-15.6	-91400	-14.0
2013	5.35	-1.16	-17.9	1.74	48.2	-89530	-13.7

September Average Extents, 2002-2013: Calculated by Walt Meier and Julienne Stroeve, National Snow and Ice Data Center. All values in table estimated based on the NSIDC Sea Ice Index.

Figure 1.2: The table of the Arctic icecap extent from the year 2002 to 2013.

the second order polynomial regressions to make the icecap extent prediction by observing the trend of data. According to the linear regression model of CO_2 concentration vs. icecap extent, the Arctic icecaps will vanish in the year 2076. By applying the second order polynomial regression model instead of the linear one, the icecap extent will be approximately zero in the year 2048.

The other icecap prediction method is from the aspect of the Arctic Sea openwater. Because there is a close correlation between openwater area and icecap melting rate, an advanced math model is set up in this method based on the relation. The basic idea of the advanced model is that the Earth's imbalanced heat causes a warmer ocean surface and more openwater area. Moreover, more openwater area causes a faster rate of openwater expansion. By applying the exponential regression model of the Arctic Sea openwater area vs. time, the residual of the model has a normal distribution. Therefore, it can be deduced that the Arctic Sea openwater area follows a normal distribution dataset. According to the probability density function of the Arctic Sea openwater area, the Arctic Sea would be icefree in the year 2035 with more than 95% probability.

The thesis is organized as follows:

- Chapter 2 sets up a preliminary model of the Arctic icecap extent and the CO_2 atmospheric concentration, briefly introduces a linear regression model, and makes a prediction about the Arctic icecaps.

- Chapter 3 sets up an advanced model – an exponential model of the Arctic openwater area vs. time, briefly introduce how to test whether the regression model is proper for the dataset, and makes a prediction of the Arctic openwater by using the normal distribution probability density function.
- Chapter 4 introduces the Arctic icecap extent prediction result in the IPCC's 5th Assessment Report, and compares the IPCC's model, considerations and results with those of this thesis.
- Chapter 5 briefly summarizes the prediction results and its impacts in the future.

Chapter 2

A Preliminary Prediction - When the Arctic will be icefree

2.1 A Preliminary Model

In 1950s, Charles David Keeling from the Scripps Institution of Oceanography began to record carbon dioxide (CO_2) concentrations in the atmosphere on the Mauna Loa Volcano, Hawaii Island, which is a good location to collect the data of CO_2 with relatively less interference from other factors such as vegetation, wind and weather.

Based on the Mauna Loa CO_2 data from 1959 to 2013[2], the annual mean CO_2 concentration (ppm) in the atmosphere have a roughly linearly increasing trend. It is natural to set up a linear regression model to analyse the atmosphere CO_2 concentration over the past 50 years.

$$y \sim \alpha_1 + \alpha_2 x \tag{2.1}$$

Equation (2.1) is the relation form of linear regression. Let y be the mean CO_2 concentrations in the given years, and x is the time measured in year, that is, $x \in \{1, 2, \cdots, 55\}$. The unknown parameters α_1 and α_2 are to be determined. If α_2 is positive, that indicates the CO_2 concentration is increasing in past 55 years.

Once the unknown parameters are obtained, it is possible to compute the residuals

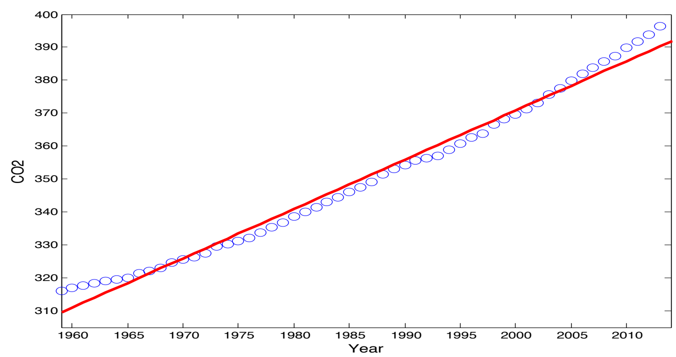


Figure 2.1: The linear regression plot of the atmospheric CO_2 concentration vs. time. The vertical axis is the CO_2 concentration in $ppm(micromole/mol)$, and the horizontal axis is year from 1979 to 2014.

$r = y - \hat{y}$ and the R^2, which is the coefficient of determination.

$$\boldsymbol{R^2 = 1 - \frac{\|r\|^2}{\|y - \bar{y}\|^2}} \tag{2.2}$$

Where $\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$.

In Figure 2.1, the parameters $\alpha_1 = 307.9993$ and $\alpha_2 = 1.4961$. Residue standard error is 2.8231, and $R^2 = 0.9866$. The parameter $\alpha_2 = 1.4961$ (positive) indicates that the CO_2 concentration at atmosphere is increasing over the past 55 years. The value of R^2 is close to 1, which indicates that every variation x with the corresponding y is well explained by this regression model.

Assuming the CO_2 concentration is the only factor influencing the Arctic icecap extent, the next coming question is, when the Arctic sea ice will be totally melted if there is not any improvement in the carbon emission reduction.

National Oceanic and Atmospheric Administration (NOAA), a scientific agency focusing on the oceans and the atmosphere, supplies the all year long's data[3] of Arctic Sea ice extent since 1979. It is wise to choose the September's extent data to analyse the trend, because in September, the end of summer in the Northern Hemisphere, the sea ice reaches its annual minimum extent.

Linear Regression is also used here to estimate the relation between the icecap extent and time, because the data trend of the Arctic icecap extent has an approximate linear shape. Based on the dataset[3] and Equation (2.1), assume $x \in \{0, 1, \ldots, 36\}$, and y is the icecap extent of North Pole in the given years.

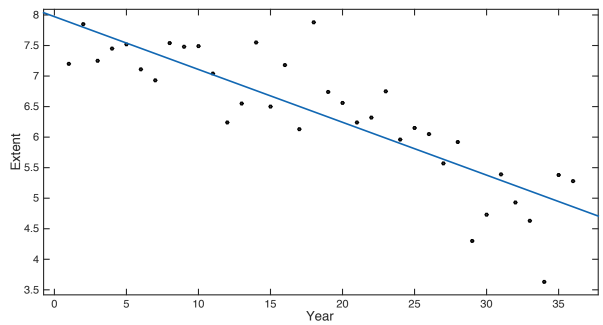


Figure 2.2: The linear regression of the Arctic icecap extent vs. time. The vertical axis is the Arctic Ocean icecaps extents in *million square kilometers*, the horizontal axis is year from 1979 to 2014.

In Figure 2.2, the red linear regression line provides all the parameters and residuals. The parameters $\alpha_1 = 7.972968$, and $\alpha_2 = -0.086497$. Residue standard error is 0.5663, and $R^2 = 0.7192$. The parameter $\alpha_2 = -0.086$ indicates that the North Pole icecaps are shrinking over the past 35 years. This model's result consists with the North Pole icecap photo from satellites as shown in Figure 1.1.

2.2 Prediction Results of the Preliminary Model

Now, it is ready to answer the question – when the Arctic will be ice-free. The brief idea of the prediction method is to combine the CO_2 concentration and the North Pole icecap extent data in one plot, and then make a linear regression to figure out the value of CO_2 concentration when the icecap extent is approximate zero. In Figure 2.3, it is shown that the North Pole would be icefree when the CO_2 concentration is 485.5796 ppm $(micromole/mol)$.

After that, plug this CO_2 concentration value into Figure 2.1, which is the linear regression plot of CO_2 concentration over the past 50 years. By using the correlation of the CO_2 concentration and time, the time of the Arctic Ocean being icefree is predictable. The prediction result is shown in Figure 2.4. The Arctic icecaps will be totally melted in approximate the year 2076.

In the Figure 2.4, based on the behavior of these data points, the quadratic regression

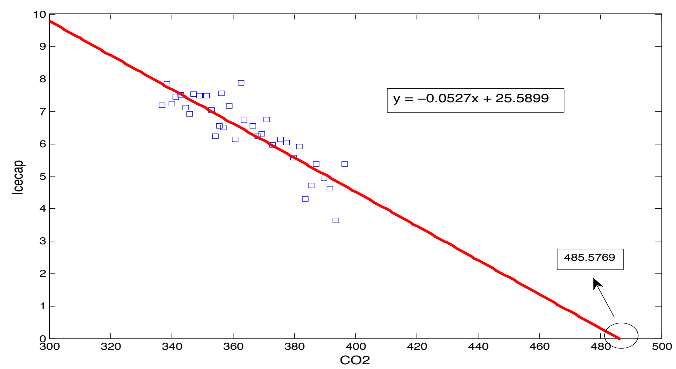


Figure 2.3: A linear regression plot of the Arctic icecap extent vs. CO_2 concentration, and it is obtained that the CO_2 concentration is 485.5769ppm when the Arctic is icefree.

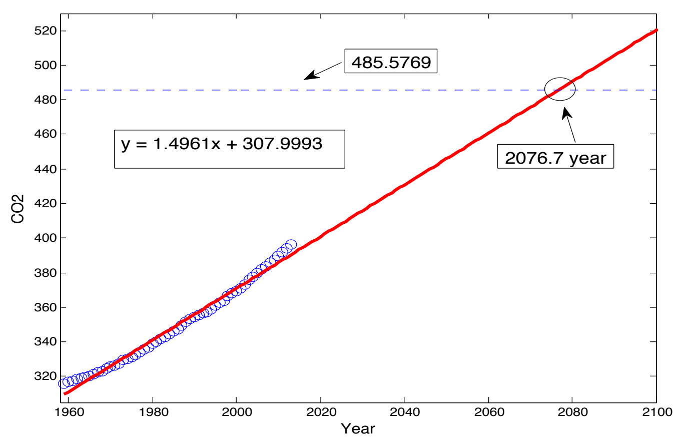


Figure 2.4: A linear regression of the CO_2 concentration vs. time, and it is obtained that the year of 2076 when the CO_2 concentration is 485.5796 ppm.

could be the other model choice for the Arctic Ocean icecap prediction. Because as shown in Figure 2.2 the tail of the data line tilts upward slightly, in this situation, the quadratic regression model could provide some radian on the curve. The quadratic regression plot is showing in 2.5.

The quadratic regression gives the year 2048 as the result, which is an earlier icefree time than the linear one. In the Figure 2.5, that day will come approximately 28 years earlier than the result shown in Figure 2.4.

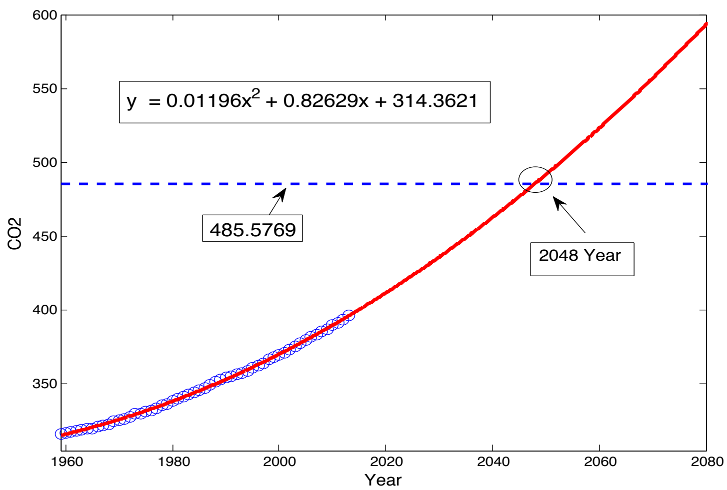


Figure 2.5: A quadratic regression of the CO_2 concentration vs. time,

Chapter 3

An Advanced Prediction - Perspectives from the Arctic Openwater And CO_2

3.1 Backgrounds

The total Arctic Ocean area, which is 14.06 $million\ km^2$, is equal to the icecap area plus the openwater area. Now is the time to analyse when the Arctic Sea will be ice free considering the Arctic openwater and atmospheric CO_2 concentration.

First of all , it is necessary to distinguish the icecap extent and the icecap area. A simplified way to think about extent versus area is to imagine a slice of Swiss cheese[4]. The extent would be a measure of the edge of the slice of cheese and all of the space inside it. Area would be the measure of where there is cheese only, not including the holes. For example, imagine there are three $25km \times 25km$ grid cells covered by 16%, 2% and 90% ice. If the threshold is 15%, then two of the three cells, the 16% and the 90%, would be considered 100% ice covered. Multiplying the grid cell areas by 100% sea ice yields the total of the three grid cells' icecap extent of $1250km^2$. To find the area of three $25km \times 25km$ grid cells covered by 16%, 2% and 90% ice, multiply the grid cell areas that are over 15% threshold by the percent of sea ice in those grid cells, and add up those products. The total area of sea ice in the three grid cells is $662km^2$.

Therefore, the extent is always larger than the area. Since the sea water is liquid and flowing, it is better to measure the Arctic Sea openwater by area instead of extent.

Another important concept to introduce is called albedo, which is the fraction of solar radiation reflected by a surface or object, often expressed as a percentage[5]. Snow-covered surfaces have a high albedo, the surface albedo of soils ranges from high to low, and vegetation-covered surfaces and oceans have a lower albedo. The Earth's planetary albedo changes mainly because of level of cloudiness, snow, ice, leaf area and land cover changes[6].

Surface	**Range of Albedo**
Fresh snow	0.80 to 0.90.
Old/melting snow	0.40 to 0.80
Desert sand	0.40
Grassland	0.25
Deciduous trees	0.15 to 0.18
Coniferous forest	0.08 to 0.15
Tundra	0.20
Ocean	0.07 to 0.10

Figure 3.1: The ranges of albedo of different Earth surfaces.

The albedo for the Arctic Sea has been deceasing over the past 30 years. The loss of the Arctic Sea ice is a particular concern. By exposing the ocean surface to sunlight, the open water absorbs the Sun's radiation and it warms up. Consequently, more ice therefore melts, which exposes more open water, which then melts more ice from underneath with respect to the time. Therefore, it is intuitive to choose a regression model, which should grow/decay relatively faster, to predict the trend of open water.

3.2 An Advanced Model

After knowing some basic background knowledge and relation of the icecaps and the openwater in the Arctic Ocean, it is sufficient to set up a mathematical model.

$$\frac{dA}{dt} = kA \tag{3.1}$$

$$A = openwater\ area,\ t = year,\ k = parameter$$

This ordinary differential equation describes that there is a correlation between the expansion rate of openwater area and the total openwater area in the Arctic Ocean. The basic idea is that the openwater has a higher albedo than the icecap, the openwater absorbs the heat from the Sun's radiation, and keeps the heat in storage. More openwater contains more heat energy and causes the melting rate of the icecap to increase, so that the openwater appears to be expanding at a increasing rate. That is, more total openwater area causes a faster rate of openwater expansion.

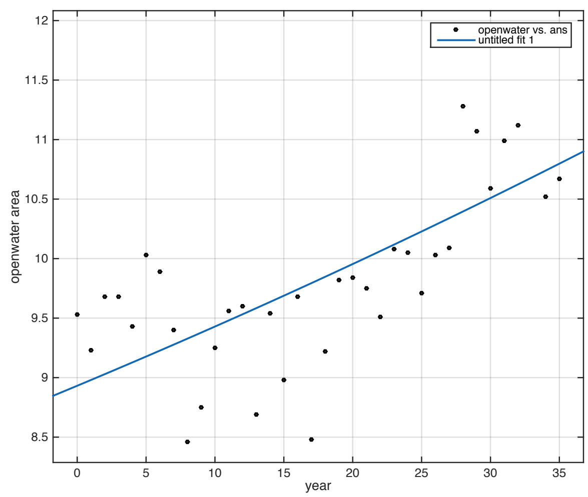


Figure 3.2: The exponential regression model $f(x) = 8.932e^{0.005421x}$ for the openwater area vs. year.

The solution to this ordinary differential equation:

$$\begin{gathered} A = ae^{bt} \\ where\ a = 8.932\ ,\ and\ b = 0.005421 \end{gathered} \tag{3.2}$$

The solution format provides an exponential relation between the Arctic openwater and the time which starts from the year 1979 to 2014.

In Figure 3.2, there are only 36 data points in the dataset. It is not sufficient to tell if the exponential curve fitting is proper or not, and it is also difficult to figure out which data point would be a noise. Therefore, it is necessary to test this exponential regression model for the dataset.

3.3 Model Tests

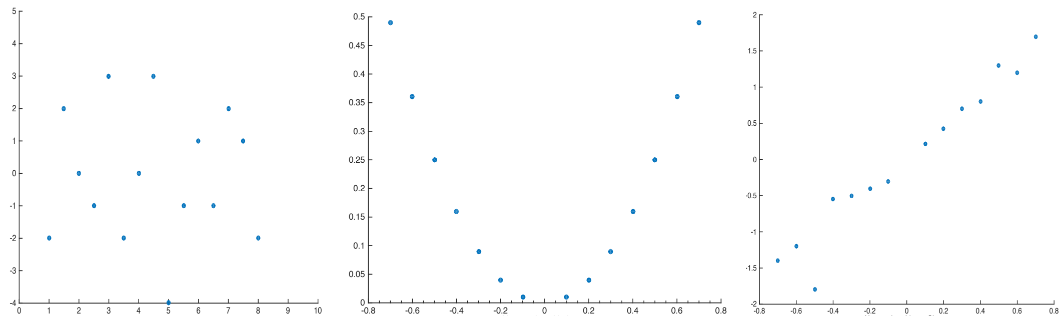
Is it a good model? How can we identify if a regression model is a good model for a given dataset? In general, there are two ways to test models. The first tester is R^2, which is called the coefficient of determination in statistics. $R^2, 0 \leq R^2 \leq 1$, measures the percentage of data that is the closest to the line of best fit, in other words, it tests how well the regression line represents the data. It equals the square of the correlation coefficient (r). For example, if $r = 0.922$ then $R^2 = 0.85$, which means that 85% of the total variation in y can be explained by the relationship between x and y, and the other 15% of the total variation in y remains unexplained. Even though the correlation coefficient (r) is a measure of the degree of linear correlation (dependence) between two variables, the R^2 can measure any model, not only the linear one.

The second tester is residual plot, also called residual analysis, which is a scatterplot of residual vs. x-data values (the residuals on the vertical axis, and the independent variable on the horizontal axis). Because a regression model is not always appropriate for the dataset, it should assess the appropriateness of the model by defining residuals and examining residual plot. The method is looking for no pattern in the residual plot. Residuals (e) are the difference between the observed value of the independent variable (y), and the predicted value ($\hat{y}$). Each data point has one residual.

$$\begin{aligned} Residual &= Observed\ Value - Predicted\ Value \\ e &= y - \hat{y} \end{aligned} \tag{3.3}$$

There are three typical patterns commonly shown in the residual plots. In Figure 3.3, plot (a) shows a fairly random pattern (no pattern), and it means the regression model is a good fit for the data. The plot (b) is a nonrandom U shape (which can be an inverted U shape), and the plot (c) also shows a nonrandom linear shape (which can be an inverted linear shape). The plot (b) and (c) patterns tell the regression models are not fitting the data very well, and one can use other regression models to fit datasets better.

After introducing the above two testers for identifying the appropriateness of a regression model, it is the time to check whether the openwater – year exponential regression model is proper for the dataset. In the openwater – year exponential regression



(a) Random Pattern (b) Nonrandom U Shape (c) Nonrandom Linear Shape

Figure 3.3: Three typical patterns in residual plots.

model, the residual plot shows a fairly random pattern. In other words, this regression model works well for the dataset from the perspective of residual analysis.

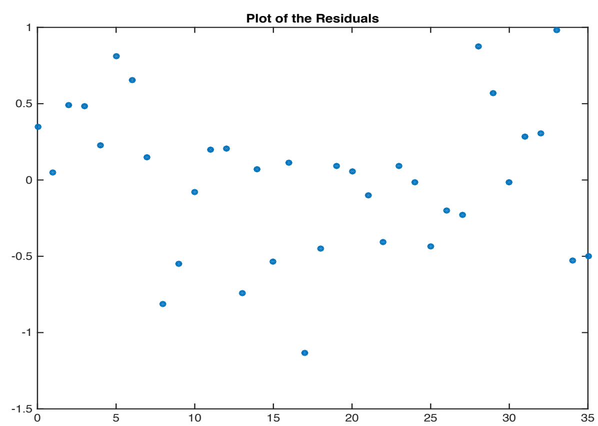


Figure 3.4: The residual plot of the preliminary exponential regression model. It basically shows a random pattern.

The R^2 is approximately 49%, which means that 49% of the total variation in the Arctic openwater area can be explained by the exponential relation between time and openwater area. The other 51% of the total variation in openwater area remains unexplained by this exponential regression model. Even though 49% seems that it does not strongly support this exponential regression model as well as the residual plot test, it can not undermine the choice of this model. The reason is R^2 has its limitations when it is used to predict behaviour at conditions for there are only 36 datapoint, and in the situations that the dataset contains many random components[7]. Furthermore, a higher R^2 does not necessarily indicate a good choice of a model as well.

For example, consider the Gaussian regression model below:

$$\begin{aligned} f(x) &= a_1 \times e^{-(\frac{x-b_1}{c_1})^2} + a_2 \times e^{-(\frac{x-b_2}{c_2})^2} \\ a_1 &= 6.049, b_1 = 35.75 \\ c_1 &= 14.34, a_2 = 9.537 \\ b_2 &= 0.1417, c_2 = 42.26 \end{aligned} \tag{3.4}$$

Following the two testers for the regression model as mentioned previously, the $R^2 = 73.3\%$, and the residual plot is shown in Figure 3.5:

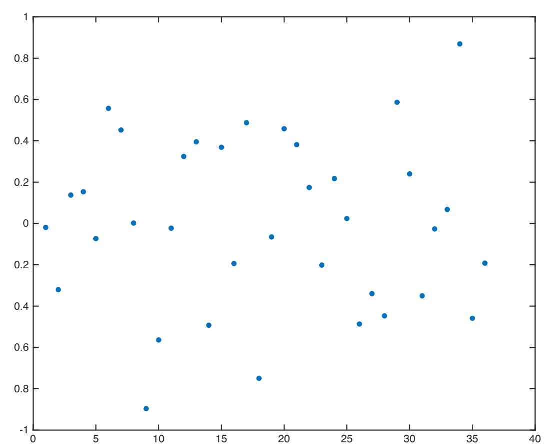


Figure 3.5: Gaussian regression model residual plot.

For this Gaussian regression model, which works on the dataset, the coefficient of determination is better than the exponential regression model, and the residual plot is a fairly random pattern. So this regression model satisfies the two testers. Does that mean the Gaussian regression is a proper model for the dataset? The answer is no. By observing the Gaussian regression function plots in Figure 3.6, the global trend is going up, but at the end of the plot, the local trend is going down. It is difficult to figure out what would happen in the future.

Figure 3.7 is a plot to predict what would happen in the next century (extend the time interval up to 300 years long) based on the Gaussian model. The plot shows that

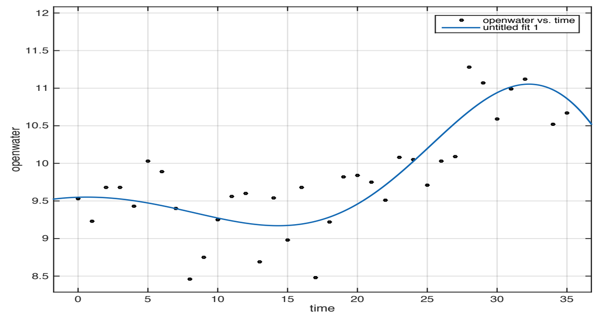


Figure 3.6: Gaussian regression model plot.

the Arctic openwater will sharply decrease in the next decades, which is inconsistent with the facts and the actual data trend. Therefore, the Gaussian regression model is not a proper method in this situation.

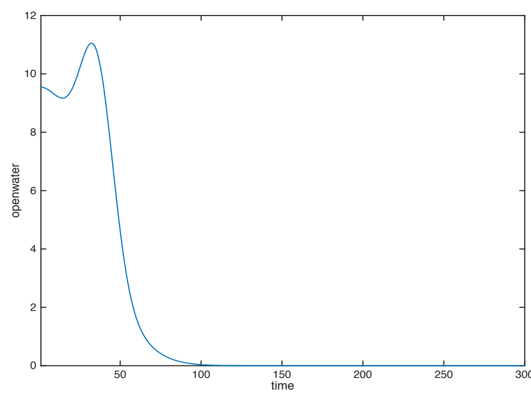


Figure 3.7: A trend prediction plot of the Arctic openwater based on the Gaussian model.

As for the exponential regression model, if the time x is extended up to 300 years, the plot – Figure 3.8 gives an opposite trend of the Arctic openwater, which compares to the Gaussian regression model. This trend is more reasonable according to the data trend in the past 30 years.

In summary, the original regression model presented in equation (3.2) is a viable model. It could be revised to fit the data better and make a more accurate prediction.

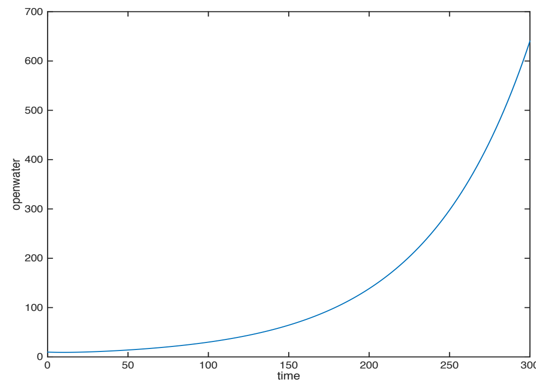


Figure 3.8: A trend prediction plot of the Arctic openwater based on the exponential model.

3.4 An Advanced Model With CO_2

In chapter 2, it discusses that the atmospheric CO_2 concentration is a key factor that has caused the icecaps to melt in the past 30 years, it is intuitive to add the CO_2 factor in the model equation (3.2).

$$\frac{dA}{dt} = k(at + b)A \tag{3.5}$$

A is the Arctic Ocean openwater area, t is year, a and b are the parameters from the CO_2 – year linear regression, and k is a parameter to be determined. The model means the expansion rate of openwater area is related to both the total openwater area in the Arctic Sea and the concentration of CO_2 in the atmosphere. As is mentioned previously, the openwater absorbs the heat from the Sun's radiation and keeps the heat in storage, more openwater contains more heat energy, and causes the icecaps to melt faster. In addition, the big density of greenhouse gases such as CO_2 acts just like an insulated coat that keeps the ocean's surface warm, and it breaks the Earth Heat Balance, so that the extra heat will cause more and more sea ice to melt. Therefore, the expansion rate of openwater is not only closely related to the existing openwater area, but it is also correlated with the concentration of CO_2 in the atmosphere.

Modeling of Math

In order to fix the parameters in (3.5), it is necessary to solve and reformat the equation.

$$
\begin{aligned}
&\frac{dA}{dt} = k(at+b)A \\
&\Rightarrow \int \frac{1}{A} dA = \int k(at+b)dt \\
&\Rightarrow ln(A) = k(\frac{a}{2}t^2 + bt) + c_0
\end{aligned}
\tag{3.6}
$$

$$\Rightarrow A = ce^{k(\frac{a}{2}t^2+bt)} \tag{3.7}$$

Now the exponential regression model of the Arctic Ocean open water vs. year turns out to be a more comprehensive form in comparison with the equation (3.2) The parameter k and c are to be determined in the following.

Math strategies in solving equation (3.7). The basic idea is to take the natural logarithm on both sides of the equation (3.7).

$$ln(A) = ln(c) + ln(e^{k(\frac{a}{2}t^2+bt)}) \tag{3.8}$$

Then follow the logarithm rules, equation (3.8) turns to be:

$$ln(A) = ln(c) + k(\frac{a}{2}t^2 + bt) \tag{3.9}$$

Now the equation (3.9) is a second order polynomial function with respect to the independent variable t. In order to fix the parameters k and c in the equation (3.7), it is necessary to figure out which parameters are known, and which are unknown. Let $P = \frac{a}{2}t^2 + bt$, and parameters a and b are already known in section 2.2, Figure (3.2), which is $a = 307.9993, b = 1.4961$, so that P is known.

$$ln(A) = ln(c) + kP \tag{3.10}$$

In the equation (3.10), A and P are known. Take the polynomial regression to fix the parameters k and c.

$$ln(A) = ln(c) + \frac{ak}{2}t^2 + (bk)t \tag{3.11}$$

let $K_1 = \frac{ak}{2}, K_2 = bk$, then $ln(A) = ln(c) + K_1t^2 + K_2t$. There is a ratio relation between K_1 and K_2, that is $\frac{K_1}{K_2} = \frac{a}{2b}$. Then take the equation (3.10) (the equation 3.11,

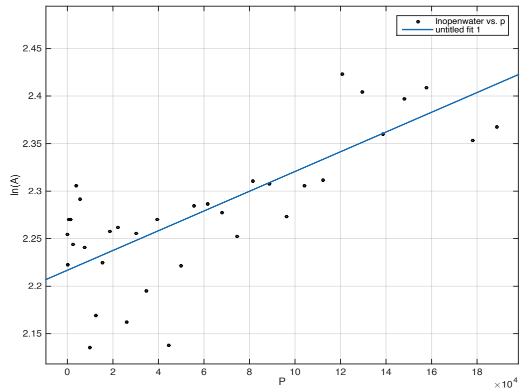


Figure 3.9: First order polynomial regression plot of $ln(A) = ln(c) + kP$

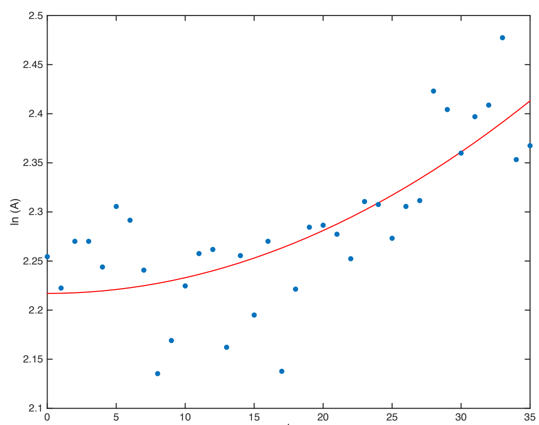


Figure 3.10: Second order ploynomial regression plot of $ln(A) = ln(c) + K_1t^2 + K_2t$.

which follows the ratio rule of the K_1 and K_2) into the polynomial regression model, the plot is shown in Figure 3.9.

Finally, we have $ln(c) = 2.217, k = 1.039 \times 10^{-6}$, and $c = e^{2.217} = 9.1798$. Then plug all of the known parameters back into the equation (3.9), and the second order polynomial regression plot is shown in Figure 3.10.

The $R^2 \approx 59.6\%$ in the second order polynomial regression model Figure 3.10, and the residual plot is a fairly random pattern in Figure 3.11.

So there is a big improvement on the coefficient of determination in this second order polynomial model $ln(A) = ln(c) + k(\frac{a}{2}t^2 + bt)$ comparing with the regression model

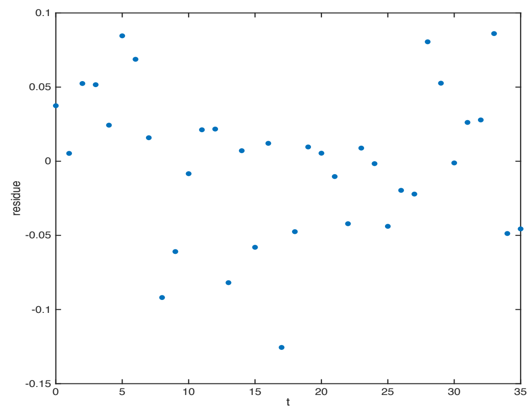


Figure 3.11: The residual plot of the second order polynomial regression of equation (3.11).

$f(x) = ae^{bx}$. The goal is to figure out the relation between the Arctic Sea openwater area and time, instead of the relation between the nature logarithm of openwater area and time. Therefore, after knowing all of the parameters a, b, k, and c, it is obtained the equation (3.7) which is shown in Figure 3.12.

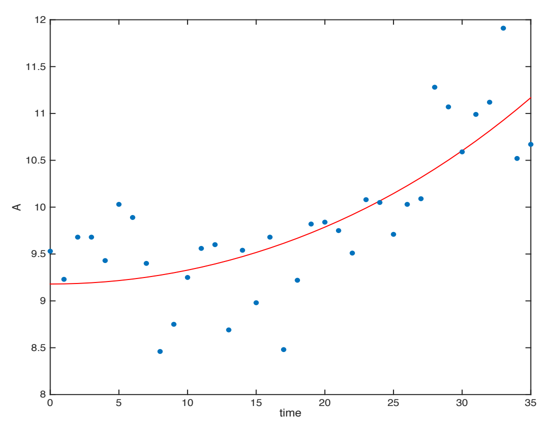


Figure 3.12: The exponential regression plot of $A = ce^{k(\frac{a}{2}t^2+bt)}$.

The vertical axis is the Arctic openwater area, and the horizontal axis is the time. To check how well this exponential regression works for the dataset, it is necessary to use the two testers: R^2 and the residual plot. The $R^2 \approx 62\%$, which means that 62% of the total variation can be explained by the exponential relationship between time and

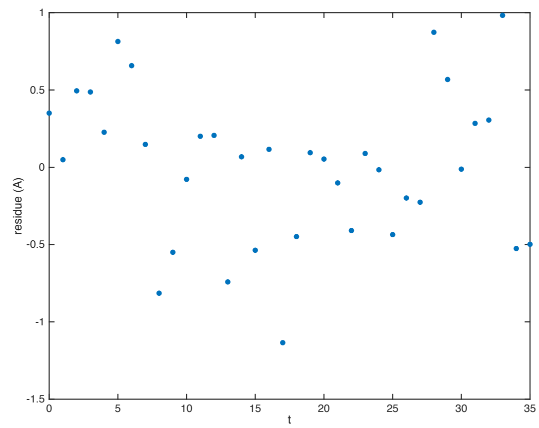


Figure 3.13: The residual plot of the exponential regression of equation (3.7).

openwater area, this is much higher than the model, which has 49% in R^2. The second tester a residual plot is a fairly random pattern, the residual plot is given in Figure 3.13.

Therefore, the revised exponential regression model $A = ce^{k(\frac{a}{2}t^2+bt)}$ is more appropriate for the dataset. Furthermore, it is convincing to make a prediction of the Arctic icecaps by using this advanced model.

3.5 Completeness of Model

Completeness of the model. In the previous pages, I show how to setup a math model to describe that how the CO_2 concentration and the total openwater area affect the rate of openwater expansion in the Arctic Ocean. As is known, the nature is in constant change, it is crucial to complete the model with randomness.

$$A = ce^{k(\frac{a}{2}t^2+bt)} + residuals \tag{3.12}$$

As shown in Figure 3.13, the distribution of residuals in equation (3.12) has a random spread, in other words, the residuals have a normal distribution. Then the equation (3.12) can be written as:

$$A = ce^{k(\frac{a}{2}t^2+bt)} + f(x, \mu, \sigma) \tag{3.13}$$

$f \sim N(\mu, \sigma)$ is the probability density function of residuals, μ is the mean of the

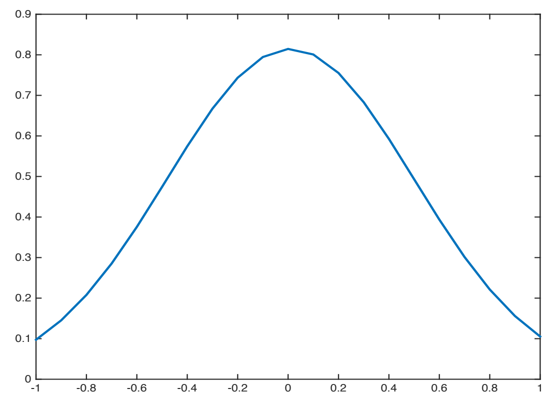


Figure 3.14: Plot of the residual probability density function.

residuals, and σ is the standard deviation. In Figure 3.14, the bell shape plot is the residual probability density function. It is obtained that $\mu = 0.009367, \sigma = 0.4897$.

3.6 Prediction Result

According to the normal distribution properties,

$$if\ f \sim N(\mu, \sigma),\ then\ A \sim N(ce^{k(\frac{a}{2}t^2+bt)} + \mu, \sigma) \tag{3.14}$$

Here A can be regarded as a normal distribution with mean $ce^{k(\frac{a}{2}t^2+bt)} + \mu$, and its standard deviation is σ, which is equal to 0.4897. At this point, the model can predict the area of the Arctic openwater with randomness in the next decades, or even next centuries. Specifically, for each given time t, we can have a specific normal distribution curve for the openwater area. The normal distribution probability density function is below:

$$f(x, \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\mu^2}} \tag{3.15}$$

The total area of the Arctic Ocean is 14.06 $million\ km^2$. When the Arctic Ocean has no ice, in the other words, the total openwater area is approximately equal to the Arctic Ocean area, it is the time for polar bears to look for a new habitat. The prediction plot is shown in Figure 3.15.

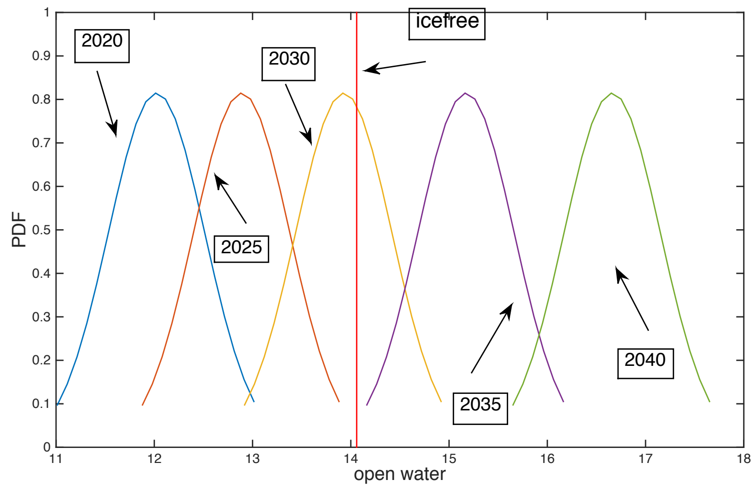


Figure 3.15: The final prediction plot by using the normal distribution properties. That indicates the Arctic Ocean will be icefree with above 95% chance in the year 2035.

In Figure 3.15, the horizontal axis is the area of the openwater of the Arctic Sea, the vertical axis is the value of the probability density function equation (3.15). The red vertical line marks the Arctic Ocean's total area 14.06 $million\ km^2$. For example, the blue bell curve is far from the red vertical line, that means there is little chance or probability for the Arctic Ocean to be icefree in year of 2020. The yellow bell curve stands for the year 2030, it crosses the red vertical line, and almost half of the area that is under the yellow bell curve is located on the right side of the red line, which means there are almost 50% probability to have a no ice Arctic Ocean in year of 2030. In the year 2035, there are approximately 100% chance of the Arctic Ocean would be icefree.

Chapter 4

Comparisons with IPCC's Result

4.1 The Prediction Results of IPCC 5th Assessment Report

The working group in IPCC contributing to the IPCC 5th Assessment Report considered many kinds of evidence of climate change by investigating data from the atmosphere, ocean, cryosphere, sea level, carbon, and other biogeochemical cycles[1]. They discovered that the anthropogenic increase in CO_2 emissions lead to an increase in the concentration of atmospheric CO_2 which is the main driver of climate change. They also provided some predictions about future global and regional climate change. The IPCC's prediction about the extent of Arctic Sea ice is the key point to put eyes on in this thesis.

In IPCC's prediction on the Arctic Sea icecap, the working group used a subset of models to create an assessment that most closely reproduced the climatological mean state and the 1979 to 2012 trend of the Arctic Sea ice extent. Their results show that a nearly icefree Arctic Ocean is likely to happen in September around the year 2050. This prediction result is shown in Figure4.1:

A new set of scenarios, the Representative Concentration Pathways (RPCs), was used for the new climate model simulation carried out under the framework of the Coupled Model Intercomparison Project Phase 5 (CMIP5) of the World Climate Research Programme. In all RCPs, the atmospheric CO_2 concentration is higher in 2100 relative

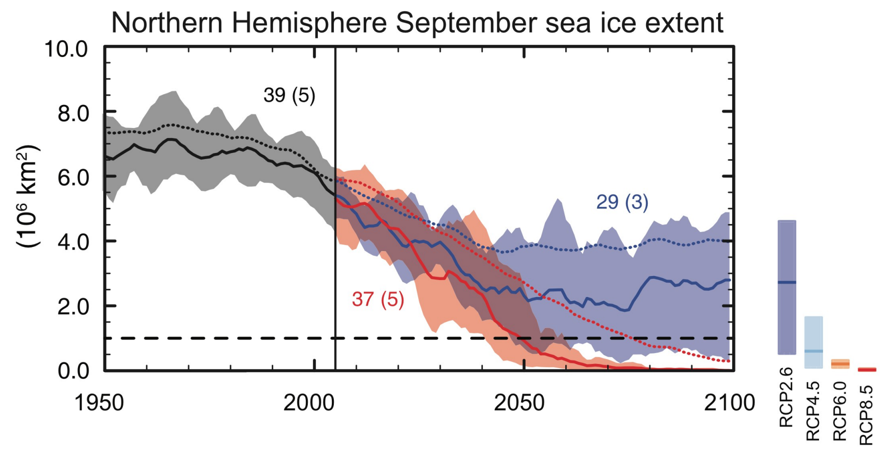


Figure 4.1: Northern Hemisphere Septermber sea ice extent (5-year running mean).

to present day as a result of a further increase of cumulative emissions of CO_2 to the atmosphere during the $21st$ century. IPCC explained the reason for using the RCP8.5: a projection of when the Arctic might become nearly icefree in September in the 21st century cannot be made with confidence for the other scenarios. Based on the explanation of IPCC, the dash line which stands for the $1.0 \times 10^6\ km^2$ icecap extent has a cross intersection with the red curve which is RCP8.5, which means the Arctic ocean will be nearly icefree around the year 2050.

4.2 Comparisons

This thesis only considers three factors, the Arctic icecap extent, openwater area, and CO_2 concentration. It incorporates no more than 40 data points, and uses some simple mathematical models to make a prediction. On the opposite, IPCC working group made the future climate prediction results by using a hierarchy of climate models ranging from simple climate models, to models of intermediate complexity, to comprehensive climate models and Earth system models. For the Arctic icecap extent prediction, IPCC's working group comprehensively analysed hundreds of datasets in order to consider almost every factor which might affect the Arctic icecaps. It included not only the icecap extent/ area and atmospheric CO_2 concentration, but also many other climate factors, such as global surface temperature, ocean surface pH, ocean heat content, and so on.

These mathematical models are analysing a great many data sets of different climate factors and the anthropogenic forces. Therefore, IPCC's Arctic icecap prediction models are much more complex than the openwater-CO_2-time model in this thesis. They predicted in the year 2050 the Arctic Ocean could be icefree, which is approximately 15 years later than the thesis's prediction.

Chapter 5

Impacts

As mentioned in the introduction, the polar icecaps help to regulate the global temperature, and play an important role in the global climate system. If the Arctic Ocean is going to be icefree, it will have multiple impacts. It will cause global temperature to rise, and accelerate the melting of the global ice sheets which hold enough water to raise sea levels. Then the coastline cities will disappear by the rising sea level.

In the thesis, the atmospheric CO_2 concentration is assumed to be the only factor affecting the rate of the Arctic Sea icecap melting, and these above mathematical models are not comprehensive enough to make a prediction of sophisticated Earth climate system. Therefore, the prediction results may be not convincing enough to make the public believe. But the comprehensive organisation IPCC already focuses on this issue, and gives a specific and concrete prediction about the future of the Arctic icecaps. Therefore, no matter whether the Arctic Ocean will icefree in the year of 2050 or the year of 2035, these results adequately catch people's attention. Only when more and more policy makers pay attention to the global warming and its implications can people realise the climate situation is going to be on brink of an abyss.

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