Mathematical Identity and the Use of High-Leverage Thinking Moves During Problem-Solving Activities

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Dedication

This dissertation is dedicated to my parents and grandparents for their love of learning. It is also dedicated to our four kids Erik, Joe, Anna, and Veronique. It is dedicated to Erik for his courage to dream big and work hard to achieve those dreams, to Joe for following his dreams with passion and enthusiasm, to Anna for living life focused on serving God and serving others with graceful beauty, and to Veronique for her heart for serving others. Finally, and most importantly, this dissertation is dedicated to the love of my life, Sue.

Abstract

This case study explored the relationship between a student's mathematical identity and their understanding of statistical concepts through four individuals in an AP Statistics course in a Midwest metropolitan suburban high school. A primary framework was used to examine the connections between mathematical identity and thinking moves during problem-solving activities. Within the primary framework, one secondary framework was used to investigate mathematical identities though two lenses: (1) *current* identities, which are identities in the form of stories, generally told in the present tense, about the actual state of affairs; and (2) *designated* identities, which are identities in the form of stories, told in the future tense or in a way that expresses aspirations or needs about a state of affairs expected to be the case either presently or in the future. Moreover, a second secondary framework was used to reveal mathematical understanding through the demonstration of thinking moves during problem-solving activities. In the end, the research framework guided the investigation of the association between the mathematical identities (i.e., current identity and designated identity) and the use of thinking moves during problem-solving activities. The results indicate that while there were differences between the patterns of thinking moves used during the group activities, there was little difference between the thinking moves used on the individual work on selected exam questions. During the group activities, individuals who had mostly positive feelings and experiences related to mathematics, who viewed themselves as confident students of mathematics, and who saw themselves as central members of the mathematics classroom demonstrated more extensive use of the eight thinking moves and the use of thinking moves that reside in all three thinking categories. In addition, approaches to learning that are focused on understanding the material (i.e., a substantial approach) and are consistent with *discourse-for-oneself* status are linked to the use of a larger variety of thinking moves and the use of thinking moves which reside in all three thinking categories. An important implication of the research is that classroom teachers need to create learning environments that nurture vibrant student relationships with mathematics as well as develop mathematical understandings of concepts and ideas.

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Chapter 1

Introduction

Learning is more than a process that involves the attainment of skills and knowledge in some abstract sense. Instead, the act of learning transforms who we are and what we can do; as a result, learning is an experience of identity (Wenger, 1998). Wenger (1998) maintains that through the learning process, *ways of being* are explored in such a way that individuals can see the possibilities for new or transformed identities (p. 263). Consequently, learning becomes meaningful to the individual and provides a powerful source of personal and social energy, because of its interconnected relationship with the process of identity formation (Wenger, 1998).

Many researchers have also argued that there is a strong association between learning mathematics and identity (e.g., Anderson, 2007; Boaler & Greeno, 2000; Boaler, William, & Zevenbergen, 2000; Grootenboer & Zevenbergen, 2008; Kane, 2012; Varelas, Martin, & Kane, 2012). Boaler and Greeno (2000) posit that mathematics classrooms are *figured worlds* where students learn mathematics and develop mathematical identities and a sense of who they are as learners; these developed identities are interwoven with mathematical understanding and the approaches to teaching and learning in the classroom. In a similar way, Anderson (2007) maintains that experiences and relationships in the mathematics classroom have a symbiotic relationship with identity; experiences and relationships influence the identity of individuals and are in turn influenced by the identity of the individual. The present research study further examines these rich connections between mathematical understanding and mathematical identity.

Rationale

In order to develop a more helpful understanding of the mathematics education landscape, future research needs to consider issues of identity in addition to issues related to achievement and ability (Boaler et al., 2000; Kane, 2012; Varelas et al., 2012; Martin, 2000, 2009). Boaler and Greeno (2000) found that students liked or disliked mathematics not just for reasons related to the mathematical content, but also for how their interaction with the material lined up with their identity. For example, if a student has constructed an identity as a learner that views learning as active, creative, thoughtful, and social, they will most likely connect with and enjoy a flexible, student-centered learning environment where the study of mathematics allows for social interaction and the opportunity to wrestle with mathematical concepts. On the other hand, the same student may reject pedagogies in the mathematics classroom that are rigid, procedural, involve significant memorization, and expect students to simply regurgitate information (Boaler & Greeno, 2000). In addition, students engage in the mathematics classroom in a more positive manner when they are able to see connections between their success in the mathematics classroom and social and economic success in the future. Conversely, students tend not to engage with the mathematics environment in a constructive manner when they view mathematics negatively (Martin 2000, 2009).

These and similar findings have led many researchers and experts in mathematics education to argue that issues of identity are linked to critical issues in mathematics education. One of these issues is the significant number of successful students who drop out of mathematics all together, that is, do not enroll in optional mathematics courses in high school or study mathematics at the undergraduate level (Boaler & Greeno, 2000; Boaler et al., 2000; Grootenboer & Zevenbergen, 2008), despite the fact that mathematics is a gatekeeper to economic and educational opportunities (Anderson, 2007). In their work with AP calculus students, Boaler and Greeno (2000) found that a significant number of successful mathematics students did not enjoy mathematics and that students tended to reject mathematics if the classroom environment and pedagogies were not in sync with their mathematical and academic identities. In addition, learners who did enjoy learning mathematics did so because they were good at it or saw it as a way to gain access to a career or college, and not because they enjoyed the mathematics itself (Boaler et al., 2000). In such an environment, success and failure are often defined in terms of survival, rather than in terms of thoughtful and thorough understanding of the mathematical concepts and ideas. As a result, many successful students in the top track who have the ability and are nicely positioned to study mathematics at the next level choose not to do so.

In order to address some of these concerns, the present research study examines the dynamic relationship between mathematical identity and mathematical understanding. One objective of this exploration of the relationship between identity and understanding is to provide an effective avenue through which educators can become more knowledgeable about the mathematical learning experience and learn how to effectively address equity related issues in the mathematics classroom (Aguirre, Mayfield-Ingram, & Martin, 2013). Moreover, the present research study can inform the creation of learning environments that are based on knowledge of each student, the breadth of their experiences, and their relationship with mathematics. To effectively study this synergetic relationship between identity and understanding, the research present research study builds on the existing body of research; this background research is briefly discussed in the next section.

Background of the Problem

In the present research, the connections between mathematical understanding and mathematical identity are explored. For this reason, the research related to the phenomena of mathematical understanding and mathematical identity is briefly considered; the research that addresses mathematical understanding is considered first. Understanding is the result of an intricate process that requires various and diverse forms of thinking (Ennis, 1996; Facione & Facione, 1992; Ritchhart, Church, & Morrison, 2011; Tishman, Jay, & Perkins, 1993). In addition, mathematical understanding is a complex and dynamic process, a process that cannot be fully described through grades and standardized test scores (Gutierrez, 2000, 2007; Martin, 2000, 2009; Nasir, 2002). Accordingly, in certain circumstances, it may be more effective to measure student understanding through the investigation of student thinking.

The examination of student thinking in the classroom is essential as development of thinking nurtures and supports understanding; the thinking behaviors of individuals also reveal creative and critical approaches to learning and thought (Swartz & Perkins, 1990). In addition, when an individual develops understanding of an idea or a concept it is the result of a complex process that requires a variety of patterns and arrangements of thinking behaviors (Ritchhart et al., 2011). From this perspective, understanding is viewed as a consequence of thinking, rather than a component of thinking. Consequently, it makes more sense to think about levels or quality of thinking within thinking categories rather than in terms of hierarchal levels of thinking (e.g., the six levels of thinking as outlined by Bloom and Krathwohl (1956)). Based on their research, Ritchhart et al. (2011) compiled a list of different types of thinking that they contend lead to a comprehensive understanding; this list of thinking moves is put forward by the authors not as an exhaustive list, but rather as a place to begin the exploration of how thinking leads to understanding. These eight high-leverage thinking moves are: (a) observing closely and describing what's there, (b) building explanations and interpretations, (c) reasoning with evidence, (d) making connections, (e) considering different viewpoints and perspectives, (f) capturing the heart and forming conclusions, (g) wondering and asking questions, and (h) uncovering complexity and going below the surface of things (p. 14). Based on these findings, the high-leverage thinking moves (Ritchhart et al., 2011) are used as the theoretical framework for the present research study to organize and examine student understanding in a mathematical environment.

Through participation and performance in mathematical environments, individuals develop identities, beliefs, and dispositions related to mathematics (Aguirre et al., 2013). Mathematical identity, often expressed in the form of stories or narratives, can reveal how an individual views themselves, as well as how they are perceived by others in terms of their performance in the mathematics classroom. As a result, identity can be viewed through the stories of individuals as they describe who they are, who they are not, and who they want to become (Aguirre et al., 2013). In addition, our personal narratives are our identity and these narratives can be used effectively to build explanations for and interpret the reality of situations and circumstances (Bruner, 1991). Because educational investigations are based on experiences, it makes sense to study them through the narratives of the individuals involved (Clandinin & Connelly, 2000; Kaasila, 2007). As a result, narrative inquiry is used as a research methodology in this current research study to unpack the complexities of experiences in an educational setting. In the end, the research in relation to mathematical understanding and mathematical identity provide the foundation for the current research study.

Purpose

In order to inform the development of effective learning environments, it is important to investigate the connections between a student's mathematical identity and their thinking during problem-solving activities. The research discussed above indicates that issues of identity are closely related to learning in the mathematics classroom; experiences with mathematics can impact mathematical identity which in turn can influence the learning of mathematics. In addition, it is also evident that the examination of thinking moves can reveal mathematical understanding; learners that are able to make use of a number of different thinking behaviors often have a more sophisticated and complete understanding of the material.

The purpose of this study is to reveal and describe the connections between mathematical identity and the use of thinking moves used during problem-solving activities. There are several reasons why knowing more about those connections is essential. First, the findings may be a resource for teachers looking for approaches to instruction that have the potential to engage and encourage all students (Kane, 2012). In addition, this type of exploration may inform the selection of resources and instructional approaches that are pertinent in terms of student success in the mathematics classroom (Varelas et al., 2012). Finally, this research may highlight, for researchers, educators, and students, how mathematical understanding and thinking are connected to the lives of the students, the teachers, and to the culture and the community in which they live.

Research Questions

This present research study examines the association between the mathematical identities of students in an AP Statistics classroom and their use of high-leverage thinking moves used in problem-solving activities. The research question and the subquestions were designed so that they reflect the line of inquiry and take into account the context, the fluidity, and the complexity of the phenomena (i.e., mathematical identity and thinking moves) being investigated (Stake, 1995; Yin, 2009). In addition, the questions are directed at the research, they are *issue* questions that encourage interaction with the problems of the inquiry, and they focus the attention on the personal and social contexts (Stake, 1995; Yin, 2009).

The following research question is explored: *To what degree and in what ways are the narratives students construct for their mathematical identities related to the use of high-leverage thinking moves during problem-solving activities?* To address this overarching question, the following two subquestions are investigated.

A. To what degree and in what ways are the narratives students construct for their *current* mathematical identities related to the use of high-leverage thinking moves during problem-solving activities?

B. To what degree and in what ways are the narratives students construct for their *designated* mathematical identities related to the use of high-leverage thinking moves during problem-solving activities?

As indicated in the two subquestions, the relationship between thinking moves and mathematical identity are viewed through two lenses: (1) current identities, and (2) designated identities. Current identities are identities that are the stories and narratives told by individuals that reflect the current situation or circumstances (Graven & Buytenhuys, 2011; Sfard & Prusak, 2005). In contrast, designated identities are stories and narratives told by individuals regarding their ambitions and the situations and circumstances they expect to be in at the present time or in the future (Sfard & Prusak, 2005).

Overview of Study

A qualitative case study is used to examine the research questions. One goal of the present research is to reveal how learners interpret and make meaning of their experiences in regard to their mathematical identity; the phenomenon of mathematical identity is explored through the narratives of the individuals. A second goal is to study student understanding through the thinking moves displayed during problem-solving activities. The overall goal of this qualitative case study is to understand the association between these two phenomena (i.e., the thinking moves displayed by the learner, and the learner's mathematical identity).

The field research took place in an AP Statistics class in a suburban school in a Midwest metropolitan area. The class, made up of 30 juniors and seniors, was a one credit, yearlong course that met every day for 54 minutes. Of those 30 students, 23 students returned the consent forms and agreed to be part of the overall study. In addition, four students (i.e., Anna, Ashley, Erik, and Mitch; these are pseudonyms) were selected from the pool of 23 students to be involved with the more detailed portion of the research. For consistency purposes, the four students selected were in the first semester of their senior year. Moreover, two sets of cases were chosen in such a manner to predict comparable results as well as predict dissimilar results for anticipated reasons. The first set of students, Erik and Mitch, both had a positive to strong positive mathematical identity and both were currently enrolled in AP Calculus and AP Statistics. The second set of individuals, Anna and Ashley, both had a negative to mixed mathematical identity and both had successfully completed Algebra II or Algebra III prior to taking AP Statistics.

In terms of data collection, multiple sources of evidence are used to triangulate the data to potentially produce more accurate and convincing conclusions (Yin, 2009). The sources of data that are used to explore mathematical identity include: (a) a student survey, (b) three teacher interviews, (c) selected student reflections completed at the conclusion of each activity, and (d) two student interviews. In a similar way, triangulation of data is used to examine the thinking moves of students during problem-solving activities. The following sources of data are considered: (a) student conversations during group activities, (b) selected student reflections at the conclusion of each activity, and (c) student responses to selected exam questions.

Organization of Study

This paper is organized into six chapters. Chapter 2 reviews the relevant literature related to mathematical identity, thinking categories and moves, statistical thinking, and the use of statistical activities to reveal student thinking. Chapter 3 describes the research methodology; this includes a description of the data collection instruments and procedures as well as the research design and data analysis for the phenomena under investigation (i.e., mathematical identity and thinking moves). Chapter 4 describes the data analysis and results for each of the cases, while Chapter 5 presents a cross case analysis and a cross case discussion. Finally, Chapter 6 includes a summary, the limitations, the implications, the areas of future research, and concluding thoughts.

Chapter 2

Review of Literature

Reflection on the interconnected relationships between mathematical identity and learning in the mathematics classroom provides an important lens through which educators can become more knowledgeable about the learning experience and learn how to effectively address equity related issues (Aguirre et al., 2013). Aguirre et al. contend that equity-based approaches to teaching include the creation of learning environments that are based on knowledge of each student, the breadth of their experiences, and their dynamic relationship with mathematics. In this regard, the research presented here investigated to what degree and in what ways the narratives students construct for their current and designated mathematical identities are related to the use of high-leverage thinking moves during problem-solving activities. This chapter examines two major areas of literature, learning as an experience of identity and perspectives on thinking, which are both foundational to the theoretical framework that weaves together the concepts of identity construction and the use of high-leverage thinking moves. First, the literature related to (a) identity and learning, (b) identity and learning mathematics, and (c) using narrative as a means to study mathematical identity, are considered. Second, the research on thinking is investigated through: (a) cognitive development, (b) dispositional perspectives on thinking, (c) understanding and thinking, (d) statistical thinking, and (e) models and model-based reasoning.

Conceptual Framework

For the present dissertation study, a two-layer conceptual framework (i.e., one primary and two secondary frameworks) was developed to explore to what degree and in what ways the narratives students construct for their mathematical identities are related to the use of high-leverage thinking moves during problem-solving activities. First, a primary framework was used to examine the relationship between the mathematical identities of the students and their mathematical understanding. This primary framework was informed by a pedagogical and research framework created by Varelas et al. (2012) that to study the interaction between content learning and identity construction. Within the primary framework, a secondary framework was used to investigate mathematical identities though two lenses: (1) *current* identities, which are identities in the form of stories, generally told in the present tense, about the actual state of affairs; and (2) *designated* identities, which are identities in the form of stories, told in the future tense or in a way that expresses aspirations or needs about a state of affairs expected to be the case either presently or in the future. This mathematical identity framework was built on identity work of Sfard and Prusak (2005). Moreover, another secondary framework was used to reveal mathematical understanding through the demonstration of thinking moves during problem-solving activities (i.e., group work on statistical activities and individual work on exam questions). This thinking move framework is comprised of eight thinking moves (Ritchhart et al., 2011) that are categorized according to three thinking categories (Ritchhart, 2001). In the end, the research framework guided the investigation of the association between the mathematical identities (i.e., current identity and designated identity) and the use of thinking moves during problem-solving activities.

Identity and Learning

Learning is a social and cultural activity that is interconnected with the identity of the learner. Wenger (1998) argues that because learning transforms who we are and what we can do, it is an experience of identity; we obtain information and skills not in some abstract sense, but in order to develop or build an identity. In addition, Wenger (1998) maintains that through education, *ways of being* can be examined in such a way that individuals can see the possibilities for new or transformed identities (p. 263). Consequently, Wenger reasons that learning becomes meaningful and a source of personal and social energy as a result of the identity formation that takes place in the learning process. Extending this, Varelas et al. (2012) define learning as an activity that consists of social and cultural aspects where individuals impact, and are impacted by, ideas, artifacts, and other individuals, as they engage in sociohistorical and cultural environments (p. 324). Varelas et al. argue that learning takes place, then, when an individual is able to build a partnership between the ability to build a sense of who they are as a learner through participating in disciplinary activities and the capability to construct knowledge in conjunction with others (Varelas et al., 2012). Furthermore, connections with the community are a critical component of educating students; an important component of teaching is the ability to connect learning to the lived experiences of the community, the community resources, and community knowledge (Civil, 2007). Together, these findings suggest that the interactions between the individual, the culture, the community, and the environment, have a clear impact on learning.

In regard to identity formation and its relationship to learning, Wenger (1998, 2010) outlines three modes of identification. The first mode, engagement, describes the negotiation of meaning through active involvement; meaning can be made through individual engagement or interaction with others and can be the result of participation, or lack of participation. Imagination, the second mode of identification, is described as using experiences to make connections with, and develop images of, the world. Wenger (1998) indicates that this can be helpful to gain new perspectives, to consider new opportunities, or to reflect on present circumstances. For example, an individual may imagine playing at a desired concert venue, when practicing arpeggios and scales on a piano, to consider new possibilities (Wenger, 1998). The third mode of identification is alignment; in this mode individuals strive, through a two-way process, to be part of, and conform to, a larger framework and contribute to bigger initiatives by synchronizing their efforts and organizing their actions (Wenger, 1998, p. 174). If a group member does their part, through alignment, they can become part of something bigger. Thus, engagement, imagination, and alignment are three modes of belonging that describe the relationship between identity and learning.

In order to better unpack the idea of learning in relationship to identity, Wenger (1998) constructs a theory that describes the learning process as social participation. Wenger's social theory of learning has four components: (a) practice, learning as doing; (b) meaning, learning as experience; (c) identity, learning as becoming; and (d) community, learning as belonging (p. 4). In essence, this social participation is the process of developing an identity through active involvement in a social community. From this perspective, learning is viewed as who a person is and who they are becoming through a collaborative connection between their experiences as a learner and their social interactions in the community in which they reside (Wenger, 2010).

With a similar objective, Gee (2001) created an analytic lens for educational research, to explore both theoretical and practical issues, that considers identity through the issues of access, networking, and experience. He defines identity in terms of how an individual is viewed within a given context, that is, what “kind of person” (Gee, 2001, p.1) they are. Gee outlines four ways in which identity can be viewed: (1) the nature-identity, (2) the institution-identity, (3) the discourse-identity, and (4) the affinity-identity. The first, nature-identity, refers to a current state determined by nature over which an individual has no control. Nature-identity gains traction only when the individual allows the identity to take on meaning through the acceptance of the perspectives of other people, groups, or institutions, or acceptance of their own self talk. The second way to view identity, according to Gee, is as an institutional-identity, where authorities within institutions drive how an individual determines what “kind of person” they are based on their position within that institution. The third, discourse-identity, is described as an individual trait such as charisma, and is given power through the discourse with others. Finally, affinity-identity is characterized by experiences shared as part of an affinity group, a group that may have little in common besides their shared interest and may be spread over great distances. By developing these four perspectives of how identities can be viewed, Gee has taken a concept, identity, which has numerous definitions and interpretations in the literature, and defined it in a way that it can be used as an analytic tool for educational research. As the above examples have shown, the relationships between identity and learning are complex and important. In a similar manner, mathematics learning and identity are also connected in the literature. The next section considers this link.

Identity and learning mathematics. An individual’s identity in relation to their relationship and experiences with mathematics tightly corresponds to their learning in mathematical environments. Hawkins (2003) describes learning as the interactive relationship between a teacher, a learner, and the content; it is through the content that the teacher and the learner can meaningfully connect with each other. Within this framework, trust between the teacher and the learner is built through the learner coming to believe that the teacher has something important to offer in terms of the acquisition of knowledge, ideas, and understanding. In a similar way, Grootenboer and Zevenbergen (2008) contend that the interactions within the classroom context between the teacher, the student, and the mathematics make up the learning environment. Although all three components are an essential part of the learning process in the mathematics classroom, the Grootenboer and Zevenbergen emphasize that the relationship between the student and the mathematics is key in terms of building robust mathematical identities. The classroom community plays an important role, but generally only a temporary one. In contrast, the relationship between the identity of the learner and the mathematics is a lasting one. In this section, that complex and multifaceted relationship between the learner and the mathematics is considered.

There are a variety of ways that the learning of mathematics can be viewed; Anderson (2007) outlines three ways. First, some view learning mathematics as primarily the manipulation of symbols, the development of procedural fluency, and the use of efficient algorithms. A second viewpoint, championed by documents outlining mathematics standards such as the *Principles and Standards for School Mathematics* (NCTM, 2000), places a larger emphasis on conceptual understanding. A third perspective views the learning of mathematics as a community venture, where some students are central members of the community and others are marginalized. In terms of this third perspective, many researchers have argued that there is a strong tie between learning mathematics and identity (e.g., Anderson, 2007; Boaler & Greeno, 2000; Boaler et al., 2000; Grootenboer & Zevenbergen, 2008; Kane, 2012; Martin, 2000, 2009; Varelas et al., 2012).

Boaler (2000) posits that mathematics classrooms are spaces where students learn mathematics as well as develop mathematical identities and a sense of who they are as learners of mathematics. In her investigation of 76 students in six English schools, Boaler found that there was an association between the material and concepts learned and the pedagogical approaches to teaching and learning in the classroom. For example, traditional classroom experiences, that relied mostly on teacher directed instruction and focused on procedural approaches, memorization of facts, and use of efficient algorithms, reduced student achievement and limited the student use of mathematics outside of the classroom. On the other hand, students experienced rewarding mathematical experiences, in terms of achievement and the ability to transfer knowledge to other communities, when the classroom environments incorporated a variety of approaches, collaborative work, and activities that allowed students to make their own meaning of the mathematical concepts and ideas being investigated.

In a related study, Boaler and Greeno (2002) examined the mathematical identity of 48 AP Calculus students across six Northern California schools; this was part of a study investigating the complexities of mathematical confidence. The research found that students developed identities that were interconnected to their understanding of mathematical concepts and the presentation of mathematical ideas. In addition, there was evidence that suggested that the students liked or disliked mathematics not just for reasons related to the mathematical content, but also related to how their interaction with the material lined up with their identity. For example, students that construct an identity as a learner that views learning as active, creative, thoughtful, and social, will most likely connect with, and enjoy, a flexible, student-centered learning environment where mathematical investigations allow for social interaction and the opportunity to wrestle with mathematical concepts. On the other hand, the same student may reject pedagogies in the mathematics classroom that are rigid, procedural, involve significant memorization, and simply expect students to regurgitate information.

The findings of Boaler and Greeno (2000) can be further supported through the work of Cobb and Hodge (2007). In order to describe the identity of students in a mathematics classroom, Cobb and Hodge developed three identity related categories: core identity, personal identity, and normative identity. A *core* identity refers to how a learner views themselves currently and in the future. A *personal* identity describes the person a student is becoming in a given mathematics classroom. The third identity is the *normative* identity. To be a successful part of a mathematics classroom, learners need to fulfill certain duties and responsibilities that are part of the classroom culture; this can be achieved through developing a normative identity. In the Boaler and Greeno (2000) research, students experienced estrangement from mathematics when they perceived differences between their core identities and their normative identities. So considering both studies together, students found it troubling when their perspectives of who they were as a person and a learner, and who they wanted to become, did not line up with the identity needed to be a successful member of the classroom community.

Research by Martin (2000), specifically a study that involved African American middle school students in an urban setting, has also shown strong ties between identity and success and failure in the mathematics classroom. In the 2000 study, two different groups of students were interviewed. The first group consisted of students who viewed mathematics in a positive light; the students liked their teachers, embraced long term goals that they realized were tied to success in the mathematics classroom, and believed in themselves as mathematicians, or doers of mathematics. These students were also able to see that their success in the mathematics classroom was tied to future success both socially and economically. On the other hand, the second group generally viewed mathematics negatively and did not engage with the environment of the mathematics classroom in a positive manner. Likewise, in a later study, Martin (2009) found that the perspective of being a doer of mathematics was strongly tied to success in the mathematics classroom. Taken together, his research indicates that there are robust connections between the racial, cultural, and academic identities of students and their success in the mathematics classroom.

In a similar way, Anderson (2007) maintains that experiences and relationships in the mathematics classroom have an interactive partnership with identity; experiences and relationships influence the identity of individuals and are influenced by the identity of the individual. Drawing on the work of others (e.g., Gee, 2000; Wenger, 1998), Anderson reasons that there are four faces of identity related to the learning of mathematics: engagement, imagination, alignment, and nature. In order to learn more about how the faces of identity affect learning and how students construct their identity in a mathematics classroom, Anderson conducted a study with 54 rural high school students.

Fourteen juniors and seniors were part of the study, seven were enrolled in an advanced mathematics course, Precalculus or Calculus, and the other seven were not enrolled in a mathematics class. Anderson found that the first face of identity, engagement, which is related to world experiences and involvement and interaction with others, plays a significant role in the identity formation of a student. Anderson contends that how students view themselves in terms of a capable learner of mathematics is strongly tied to their engagement in the mathematics learning environment. Anderson also studied the second face of identity, imagination, which refers to one's self image and to one's relationship between life experiences and mathematics. According to Anderson (2007), mathematical identity is strongly influenced by how a student understands, or imagines, the future role of mathematics in their everyday life, in their post-secondary education, and in their career. Anderson also examined alignment, the third face of identity, which builds on the first two faces of identity, and refers to how students make sense of, and arrange, their institutional boundaries and energies. He found that if a student feels that mathematics will be a significant part of their future educational experience and their career, the student will align those goals with the mathematical learning opportunities they pursue; these decisions will affect how they see themselves and how others see them as learners of mathematics. Finally, Anderson discussed the *nature* identity, which is described as things that we are born with, such as gender and skin color. According to Anderson, although research indicates that the ability to learn mathematics is not linked to genetics, factors such as gender, race, and ethnicity, can affect how individuals are seen as mathematical learners. As noted previously, the theoretical framework for the current research, at least in term of a learner’s relationship with mathematics, is partially built on three of Anderson’s faces of identity: engagement, imagination, and alignment.

A final important point in regard to identity and its relationship to learning mathematics is that the influences on our identity are complex. For example, Tonso (2003) argues that identity is formed from both how we view ourselves and how others view us. Similarly, Grootenboer, Smith, and Lowrie (2006) argue that the use of multiple perspectives, to unpack the issues surrounding identity in mathematics education, is needed. As a result, Grootenboer et al. define identity in terms of two distinct lenses. The first lens is the use of an individual’s own eyes, that is, the knowledge an individual has of oneself and how they see themselves. The second lens is how others see and recognize that individual. Together, the research discussed (e.g., Anderson, 2007; Boaler, 2000; Boaler & Greeno, 2000; Martin, 2000, 2009) suggests a robust relationship between mathematical identity and the learning of mathematics. Due to the fact that the present research study examines the *mathematical* identity of students working in an AP Statistics classroom, it is important to discuss the relationship between mathematical identity and statistics. This is done in the next section.

Mathematical identity and statistics. In this dissertation research, mathematical identity is understood to be an individual’s relationship with mathematics and statistics. Franklin et al. (2005) make the point that mathematics is different from statistics due to the fact that statistical literacy is built though activities such as collection of data, data exploration, and interpretation of the results; these activities are often context dependent and require little in terms of formal mathematics. However, Franklin et al. contend that this does not indicate that, from an educational standpoint, statistics should not be a part of mathematics. For this reason, this researcher views statistics and mathematics, as does Shaughnessy (2010), as the twin sisters of the mathematical sciences, where mathematics deals with certainty and statistics and probability deals with the ability to reason within the context of uncertainty. Therefore, the term mathematical identity refers to a student's identity with respect to the mathematical sciences, that is, the relationship a student has with mathematics and statistics. In order to explore this mathematical and statistical identity in a research setting, the use of narrative as a tool to unpack the identity of individuals is considered next.

Narrative as a means to study mathematical identity. The examination of the narratives of individuals can be an effective way to reveal their identity in relation to their experiences and relationships with mathematics. Bruner (1991) contends that there are many commonalities between the interwoven concepts of narrative thought and narrative dialogue. He indicates that the objective of narrative inquiry is to acquire a better understanding of human behavior; this can be done, in part, through the ability of narrative inquiry to reveal the ways in which individuals think about and how they come to know and understand various ideas and concepts. For these reasons, narratives are an effective way of constructing and interpreting reality; in essence, our personal narratives are our identity (Bruner, 1991). In a similar way, Aguirre et al. (2013) explain that individuals develop identities, beliefs and dispositions, which are related to their participation and performance in mathematical, as well as more general, contexts. Mathematical identity, often expressed in the form of stories or narratives, can reveal aspects of how an individual views themselves, as well as how they are perceived by others in terms of their performance in the mathematics classroom. As a result, aspects of identity can be viewed through the stories of individuals as they describe who they are, who they are not, and who they want to become (Aguirre et al., 2013).

For these reasons, Clandinin and Connelly (2000) maintain that because educational investigations are based on experiences, it makes sense to study them through the narratives of the individuals involved. In fact, Clandinin and Connelly argue that the use of narrative inquiry is the most effective manner in which to unpack the experiences of individuals. Likewise, Kaasila (2007) contend that there is a close relationship between identity and narratives, and posits that narrative inquiry, as a research methodology, is an excellent way to unpack the complexities of experiences, especially in the education setting. To examine this approach, Kaasila made use of narrative inquiry as a research methodology to investigate preservice teacher beliefs related to mathematics. In that research, Kaasila explored the narratives, both in terms of the content of the narrative and in terms of the form the narrative took, that is, the way in which the stories were described. The study revealed that the examination of the structure of the narratives was an effective way to unpack the central events in the stories (Kaasila, 2007).

Informed by these perspectives regarding the relationship between narrative inquiry and identity, Sfard and Prusak (2005) set out to create an operational definition of identity. As others do, Sfard and Prusak maintain that the identity of an individual is critical in terms of the success or failure of the learning process in the mathematics classroom. However, they claim that previous definitions of identity (e.g., Gee, 2001; Wenger, 1998) are not necessarily useful in a practical sense when carrying out educational research. Recall that Gee (2000) described identity as what “kind of person” (p.1) an individual is, and Wenger (1998) indicated that *ways of being* can reveal the possibilities for new or transformed identities in the educational setting. Building on these ideas, and while keeping practical research objectives in mind, Sfard and Prusak define identity as the narratives of and the stories told by an individual.

In their research, Sfard and Prusak (2005) investigated learning using identity, as revealed through the narratives of individuals, as a tool. Informed by the work others (e.g., Ben-Yehuda, Lavy, Linchevski, & Sfard, 2005; Sfard, 2001), Sfard and Prusak argue that because thinking involves language, it can be described as a form of discourse, whether or not the communication is verbal. From this perspective, mathematical learning results when learners meaningfully change their mathematical discourse. In one study regarding the mathematical identity of students, Sfard and Prusak followed nineteen Grade 11 students for an entire year in an advanced mathematics class. Ten of the students were native Israelis and the other nine students were immigrants from the former Soviet Union. The research found that the native Israeli students tended to produce work that followed all the required directions and fulfilled the prescribed tasks; going through the learning process was done for the teacher and for the sake of completing the process, nothing more. Sfard and Prusak referred to this type of learning as *ritualized* learning, where the mathematical discourse was initiated by the teacher and was for the teacher. In contrast to this, the immigrants apparently completed the assigned tasks for themselves, rather than the teacher, often showing little or no work, and often not completing the required tasks. The learning process for these students was about transformational learning, making sense of the material in a meaningful and lasting manner. Sfard and Prusak referred to this type of learning as *substantial* learning, where the students moved from a *discourse-for-others* status to a *discourse-for-oneself* status.

The research found that this type of learning has a lasting effect on an individual's thinking, that is, an individual's discourse with oneself (Sfard & Prusak, 2005).

As a part of their work, Sfard and Prusak (2005) defined two types of identities, *actual* identities and *designated* identities. Later, according to Graven and Buytenhuys (2011), Sfard started using the term *current* identity, instead of the term *actual* identity, as it better described the state of affairs. For this reason, the term *current* identity is used in this dissertation study. Current identities, which are identities in the form of stories, are generally factual statements told in the present tense about current situations or circumstances. Examples of declarations that illustrate current identity, i.e., who an individual believes they are in the moment, are: I am a good at mathematics, I am a capable problem solver, and I am not a good test taker. On the other hand, designated identities, which also take the form of stories, are often told in the future tense, or in a way that expresses aspiration, assurance, or need, about a situations or circumstances expected to be the case either presently or in the future (Sfard & Prusak, 2005). Examples of declarations that illustrate designated identity are: I have to spend more time on my mathematics homework, I want to be an engineer, and I should meet with the teacher to go over my questions about the review assignment. The idea of a designated identity is in line with the work of Ezzy (1998), who, building on Ricoeur's theory of narrative identity, contends that a person's life story is a combination of remembered and anticipated events. In addition, Ezzy maintains that narrative identities are constructed through a partnership between internal dialogue and engagement with others.

Eaton and O Reilly (2009) also used a narrative identify research methodology in their investigation of the mathematical identity of preservice teachers in Northern Ireland.

The project, named the Mathematical Identity of Student Teachers (MIST), studied mathematical identity with a protocol that made use of open-ended prompts to encourage participants to share their mathematical story. The results of this survey were then used to frame questions for a focus group with the participants. The MIST protocol was designed in such a way to achieve a balance between being too prescriptive on one hand, and providing no direction at all on the other. Specifically, Eaton and O Reilly were hoping to learn the about the stories of the individuals by making use of open-ended questions without using too many directives, while at the same time providing some framework through the use of stimuli to get the thinking and reflecting process started. The MIST protocol used in the research also included some questions about the mathematical background of the participants and some Likert scale style questions related to the mathematical attitudes of the participants.

As part of the MIST research (Eaton & O Reilly, 2009), seven major themes emerged in the narratives of the preservice teachers: (a) harnessing student teachers' mathematical identity as a tool for self-reflection, (b) the role played by key figures in the formation of mathematical identity, (c) ways of working in mathematics, (d) how learning mathematics compares with learning in other subjects, (e) the nature of mathematics, (f) "right" and "wrong" in mathematics, and (g) mathematics as a rewarding subject (p. 231). Eaton and O Reilly (2009) indicate that the themes uncovered in the narrative were not always distinct; in many cases narratives had attributes that were part of two or more themes. In a later study, Eaton, McCluskey, and O Reilly (2011) learned that they were able to gain important knowledge regarding the educational experience of the participants as a result of using the MIST protocol as an investigative tool. In addition, the MIST survey effectively shed light on the mathematical identity of the participants in terms of their narrative (Eaton et al., 2011). Furthermore, the narrative tool has the dual capability of being able to be used by teachers to learn about the mathematical identities of students as well as the capability of being used by students to learn more about their own mathematical identity, that is, their relationship with mathematics, and the ways they learn mathematics (Eaton et al., 2011).

Building on the research related to the MIST protocol, Eaton, Horn, Liston, Oldham, and O Reilly (2013) developed a slightly different protocol which they called the Mathematical Identity using Narrative as a Tool (MINT). While incorporating all the components of the MIST survey, the MINT survey also includes a third part that asks the participants to reflect on any insights gained related to their mathematical identity as a result of completing the protocol. Eaton et al. used the MINT protocol in their study of preservice teachers, who had an emphasis in mathematics, in five colleges in Ireland. In that setting, Eaton et al. found the instrument to be effective in terms of revealing important information regarding how the preservice teachers felt about mathematics, as well as how they learned mathematical ideas and concepts. In summary, the findings in this section indicate that narrative, as measured using the MINT protocol, can be an effective research methodology to explore the identity of students in a mathematics classroom. The student identities revealed using narrative as a tool have the potential to impact the dynamics during the group work component of the problem-solving activities. The complex interplay between various elements that influence the individual achievement and the group dynamics during cooperative group work are considered next.

Group work in the mathematics classroom. Learning mathematics through interactions in small groups is a complex endeavor that has many benefits, but also laced with pitfalls. Many researchers have documented the benefits of cooperative work in small groups in the mathematics classroom (e.g., Boaler, 2008; Good, Reys, Grouws, & Mulryan, 1990; Slavin, 1990) including opportunities to share their thinking and an increased focus on higher-level thinking skills. While sharing one's thinking is an essential part of learning mathematics in a group setting, members sometimes are forced into peripheral roles or choose not to participate. For example, when working in a group setting there are times when members do not participate in a balanced manner (Boaler, 2008), do not work well together (Slavin, 1990), and do not engage in effective discussions with peers (Barron 2003; McCrone, 2005; Ryve, 2006; Sfard; 2001). Although there are many contributing factors to ineffective group work, how a student is identified and positioned within a group can influence the contributions they make (Esmonde, 2009). In the end, factors related to group composition such as sex, status, and achievement level of the student all have the potential to impact group dynamics and achievement. Each of these elements is explored in more detail. While the research related to group work is very broad and points to a variety of influences on group and individual success, the basis for the selection of the three elements examined here is result of the focus of the present research study.

In group settings, the sex of the students has the potential to impact their participation level and their contributions. For example, Webb (1984) found in her work with middle school mathematics students that in gender-balanced groups, females and males exhibit similar levels of achievement and interaction in small group work. On the other hand, in unbalanced groups in terms of the sex of the students, males displayed higher achievement levels than females (Webb, 1984). In addition, the pattern of interactions between students was different in the unbalanced groups. For instance, in the groups where females outnumbered the males, the females were inclined to interact more with the male; in groups where there were more males, the males tended to ignore the females. This is similar to the findings of Peterson and Fennema (1985) who contend that females are more dependent in cooperative group situations and tended to rely on others and the findings of Mulryan (1992) who argues that girls that are categorized as low achieving may not benefit from working in cooperative small groups due to passivity and lack of involvement. Moreover, Webb (1982) found that when females asked questions in small group interactions it was often due to their uncertainty and insecurity about the navigating the complex dynamics present in group settings. Finally, researchers (e.g., Mulryan, 1992, 1995; Webb, 1984) have found that boys are more likely to start discussions in mathematical environments.

General attribution theory may also shed light on the impact of the performance during group work in relation to the sex of the students. Attribution theory considers whether an individual attributes their success to stable causes (i.e., ability and task) or unstable causes (i.e., effort and luck). In addition, explanations of causality in terms of external factors (i.e., task and luck) and internal factors (i.e., ability and effort) are also important. According to attribution theory, males, more than females, attribute expected success to stable causes and as a result have increased expectations in regard to achievement and performance (Deaux, 1976). In contrast, females, more than males, attribute success to unstable causes, a perspective that is associated with a lack of confidence in repeated successful performances (Deaux, 1976). It is noteworthy that in a field that is perceived to be masculine (e.g., many students see mathematics classrooms as environments that are dominated by males; Mendick, 2005; Rodd & Bartholomew, 2006), these sex-related differences related to attribution theory can be amplified (Wolleat, Pedro, Becker, & Fennema, 1980). The dynamics in relation to what an individual attributes their success can also lead to a phenomenon known as *learned helplessness*, where failure in learning environments is perceived as expected and unavoidable. Learned helplessness is more often displayed in women, than in men, and results when success is attributed to external or unstable factors or when failure is attributed to internal or stable factors. (Dweck, Davidson, Nelson, & Enna, 1978).

Finally, a phenomenon known as stereotype threat has also been shown to affect the mathematics performance of women, especially when dealing with advanced mathematical material and material at the limits of their skill set (Spencer, Steele, & Quinn, 1999). When working in mathematical environments, there is the potential for women to be seen as having inferior capabilities through the lens of negative stereotypes (Spencer et al., 1999). This phenomenon can result in women feeling apprehensive and as a result reduce their ability to perform mathematically or cause them to drop out of mathematics. In this regard, Spencer et al. (1999) contend that stereotype threat, and the potential resulting underperformance of women in mathematics, may lead women to disidentify with mathematics and avoid mathematics all together.

The achievement level and status of a student may also affect their level of achievement and their willingness to participate in small group settings. For example, higher achieving students (Mulryan, 1994) often dominate the discourse during group activities and learn more as a result (Good et al., 1999; Esmonde, 2009). Esmonde (2009) found that more competent students (i.e., in terms of achievement) generally guide groups, while less competent students tend to follow the instructions of others. In addition, students that are perceived by their peers or their teacher to be low achieving are often marginalized in group settings (Esmonde, 2009). Similarly, Cohen and Lotan (1997) found that high-status students control and gain the most in terms of mathematical understanding during small group interactions. Cohen and Lotan contend that the two status characteristics that seem to matter most in classroom settings where students know each other are peer perception of mathematical achievement (i.e., how many classmates identify a student as smart) and popularity in regard to number of friends a student has in the classroom.

Simply placing students in cooperative groups does not necessarily lead to increased performance in mathematics (Good et al., 1990). In this regard, the use of accountability systems, performance goals, and group contracts can help groups collaborate effectively (Katzenbach & Smith, 2006; Tanner, Chatman, & Allen, 2003). Clearly, there are many complex demands placed on students during group work including the ability to navigate social dynamics, to unpack the mathematics concepts and ideas present, and to work through the expectations of the task as laid out by the teacher. In the end, the group configuration can affect the individual contributions of the group members and the overall achievement of the group as they address those demands (Esmonde, 2009).

The overall objective of the present research study is to explore the overarching research question: *To what degree and in what ways the narratives students construct for* *their mathematical identities are related to the use of high-leverage thinking moves during problem-solving activities*? As discussed above in regard to mathematical identity, the literature suggests that there is an interdependent relationship between identity and learning mathematics. Not only does an individual's identity impact how they learn mathematics, how an individual interacts with the mathematical environment can impact their identity. In addition, in order to explore an individual's mathematical identity in the research setting, the use of narrative as a tool can be effective. Up to this point, the review of literature considered the first part of the overarching research question (i.e., the narratives students construct for their mathematical identities). In the next part of the literature review, the second aspect of the research question (i.e., thinking moves and thinking dispositions) is considered.

Thinking Moves

Many researchers and educators have emphasized the importance of being aware of and exploring the thinking of students in educational settings as understanding is a result of diverse forms of thinking. For example, Ritchhart (2015) contends that making thinking visible is critical as it can shed light on what students understand, how they understand, and their misconceptions. For this reason, Ritchhart argues that the development of thinking should be a primary instructional objective along with the development of knowledge, skills, and concepts. Similarly, the findings of Ironside (2006) confirm the importance of teaching thinking behaviors based in contextual situations in addition to teaching content. In the end, it is important to examine and understand student thinking in classroom settings due to the fact that thinking nurtures and supports understanding as well as creative and critical approaches to learning and thought (Swartz and Perkins, 1990). Moreover, the use of traditional methods to examine student understanding in mathematical environments can be problematic. For instance, many researchers (e.g., Gutierrez, 2000, 2007; Martin, 2000, 2009; Nasir, 2002; Stanovich & West, 2008) argue that mathematical understanding is a complex endeavor and one that cannot be measured by static outcomes such as standardized test scores and grades. As a result, in the present dissertation study student thinking was examined to reveal understanding using a thinking move research framework comprised of eight thinking moves (Ritchhart et al., 2011) organized into three thinking categories (Ritchhart, 2001). This section provides the theoretical background for this framework. First, the literature related to dispositional perspectives on thinking as well as the literature related to the associations between understanding and thinking are explored. In addition, due to the setting of the research (i.e., an AP Statistics classroom), the literature in the areas of statistical thinking and model-based reasoning are reviewed.

Dispositional perspectives on thinking. The research related to thinking makes it clear that good thinking is more than a set of skills and that thinking dispositions are an essential part of describing how an individual thinks about something. In his seminal work related to critical thinking, Glaser (1941) conducted a number of controlled experiments that shed light on the complexities of the critical thinking process. Glaser's findings indicated that students need to be taught in a manner that helps them develop thinking dispositions and critical thinking skills as they did not generally develop the ability to think in a critical fashion or develop thinking dispositions just through the study of various subject matters. Moreover, he found that that thinking and thinking disposition are only somewhat related to general intelligence; to think effectively individuals need to have baseline intelligence, but once that baseline intelligence is reached, the effect of further intelligence on thinking abilities is not significant in comparison to other factors (Glaser, 1941). In the end, Glaser stresses the importance of persistence and perseverance in all aspects of the critical thinking process; this includes a having a vigilant mindset in order to be aware of situations that require critical thinking as well as the determination to maintain focus as one goes through the complex processes involved with critical thinking.

More recently, numerous researchers have confirmed and supported the work of Glaser (1941, 1942). In this regard, Tanner (2009) posits that critical thinking is comprised of skills and thinking processes, as well as the disposition to make use of those skills in appropriate manner. Similarly, Tishman et al. (1993) contend that, in addition to being strategic and having cognitive abilities, capable thinkers need to have thinking dispositions. These thinking dispositions are described as tendencies to “explore, inquire, seek clarity, take intellectual risks, and think critically and imaginatively” (Tishman et al., 1993, p. 148). In this regard, Perkins, Jay, and Tishman (1993) identify seven dispositions, made up of abilities, sensitivities, and inclinations, that are key contributors to good thinking: (a) the disposition to be broad and adventurous, (b) the disposition towards sustained intellectual curiosity, (c) the disposition to clarify and seek understanding, (d) the disposition to be strategic, (e) the disposition to be intellectually careful, (f) the disposition to seek and evaluate reasons, and (g) the disposition to be metacognitive (Tishman et al., 1993, p. 148). Effective thinkers generally use various combinations all seven dispositions when understanding is present (Tishman et al., 1993).

While thinking dispositions have been shown to be an important part of thinking in a critical manner, research has shown that these dispositions may be for the most part independent of cognitive ability. For instance, Stanovich and West (2008) indicate that cognitive ability is independent of important thinking components such as being able to make sense of evidence using an unbiased approach and to make use of several perspectives when thinking or reasoning; as a result, they claim that measures such as standardized tests (e.g., Scholastic Aptitude Test (SAT)) may not effectively measure the ability to think in a critical fashion. Similarly, Matthews and Lowe (2011) confirm that engagement in critical thinking is the result of much more than cognitive ability. Furthermore, Matthews and Lowe suggest that a disposition of critical thinking can result in the capacity to transfer the critical thinking strategies and knowledge to new situations if the critical thinking processes are learned in such a way that it facilitates transfer and generalization and that the individual has the self-confidence to participate in the critical thinking environment. In the end, in addition to having the necessary skills and capabilities, strong thinkers need to be able to determine when it is appropriate to use those abilities and have the fortitude and sense to follow through and make use of those abilities (Tishman et al., 1993). These aspects of thinking and thinking disposition are foundational to developing a framework to examine student thinking as a way to reveal mathematical understanding. The literature used to build this framework is reviewed next.

Understanding and thinking. To investigate the mathematical understanding of students in this dissertation research, a framework based on eight thinking moves (Ritchhart et al., 2011) and three thinking categories (Ritchhart, 2001) was used. Before this framework is discussed in more detail, some of the issues associated with attempting to unpack the processes of thinking and understanding in an educational setting are considered. For instance, when an individual develops understanding, it is the result of a complex process that requires different forms of thinking (Ennis, 1996; Ritchhart, 2015; Ritchhart et al., 2011; Tishman et al., 1993). As a result, it can often be difficult to assess, measure, and evaluate student thinking or thinking dispositions (Ennis, 1996; Stanovich & West, 2008; Tishman & Andrade, 1995). For example, Stanovich and West (2008) argue that because cognitive skills have been shown to be independent of many critical thinking components, tests like the SAT that measure cognitive skills are not able to measure important aspects of critical thinking abilities. Similarly, the use of standardized tests and school grades to measure students' understanding in the classroom is problematic as many researchers (e.g., Gutierrez, 2000, 2007; Martin, 2000, 2009; Nasir, 2002) contend that learning mathematics is a multifaceted and dynamic endeavor and one that cannot be measured by a static outcome such as achievement. This is true in relation to true-false questions, multiple-choice tests, and other traditional assessments as well as in relation to performance-based assessments (Ennis, 1996; Tishman & Andrade, 1995). Taken together, these findings indicate a need to consider alternatives to measuring student understanding in the mathematics classroom.

To effectively measure understanding, it is important to determine what it means of understand something. In order to do this, the interconnected relationship between thinking and understanding is examined. According to Harel (2008), the interrelated combination of *ways of understanding* and *ways of thinking* make up mathematical knowledge. Harel describes this symbiotic relationship between the ways of understanding and the ways of thinking through the use of the *duality principle* which is built on two key premises: (a) understanding is the result of the thinking behaviors an individual possesses, and (b) thinking behaviors are a consequence of the advancement of understanding. This duality principle not only outlines a relationship, but it describes the two phenomena (i.e., ways of thinking and ways of understanding) as synergistic change agents. For instance, Harel points out that a revision in the way an individual understands something produces a revision in their related thinking behaviors, and vice versa. As a result, by examining an individual's ways of understanding, certain cognitive characteristics (i.e., ways of thinking) related to the mental act can be seen. Moreover, an individual's cognitive products (e.g., statements and actions) reveal their ways of understanding related to a given mental act (Harel, 2008).

Similarly, Ritchhart et al. (2011) argue that understanding is a consequence of thinking and not a component of thinking. Ritchhart et al. maintain that an individual develops understanding of an idea or a concept as a result of a complex process that requires diverse forms of thinking; as a result, it is critical to consider context and purpose when discussing thinking and what it means to think. Consequently, Ritchhart et al. indicate it makes more sense to think about levels or quality of thinking within thinking categories rather than in terms of the six hierarchal levels of thinking as outlined by Anderson and Krathwohl (2001). For instance, an individual could *classify* something in a superficial manner or in a thorough manner. Moreover, a learner could *evaluate* something in a shallow and artificial manner or in an insightful and detailed way. In the end, the work of Harel (2008) and Ritchhart et al. support the premise that understanding is revealed through thinking behaviors or ways of thinking.

Accordingly, Ritchhart et al. (2011) compiled a list of eight different types of thinking that they contend lead to a comprehensive understanding. Ritchhart et al. explain that this list of thinking moves is not an exhaustive list, but rather a place to begin the exploration of how thinking leads to understanding. The eight high-leverage thinking moves are: (a) *observing closely and describing what is there*, that is, noticing features and various components and being able to describe them complexly and thoroughly; (b) *building explanations and interpretations*, for example, developing theories and hypotheses in science; (c) *reasoning with evidence* to develop new positions or to support current positions in an appropriate manner; (d) *making connections* between prior knowledge and experiences and new ideas; (e) *considering different viewpoints and perspectives* to ensure a more thorough understanding; (f) showing an understanding of the big ideas in play through *capturing the heart and forming conclusions*; (g) *wondering and asking questions* at the outset to create an avenue for learning and engagement and throughout the learning process to deepen understanding, and aid in the (h) *uncovering complexity and going below the surface of things* (p. 14). Ritchhart et al. emphasize that, for thorough understanding, all eight thinking moves must be present. For this reason, the use of the high-leverage thinking moves serves as an effective framework to measure a student's mathematical understanding in the present research study.

In addition to the use of the various types of individual, high-leverage thinking moves, the current research also made use of the categorization work Ritchhart (2001) in regard to these thinking moves. Ritchhart explored seven lists (AAAS, 1989; Costa & Kallick, 1992; Ennis, 1987, 1991, 1996; Facione, Sanchez, Facione, & Gainen, 1995; Meier, 1995; Paul, 1991, 1993; Perkins et al., 1993) related to thinking dispositions, intellectual virtues, and habits of mind. In order to synthesize the information in these lists, Ritchhart took three steps. First, he identified the three broad themes of developing a deep understanding, stimulating creativity, and inspiring curiosity. He then created six categories of dispositions in order to identify individuals who investigate, inquire, and reason with evidence, and are open-minded, curious, and metacognitive (Ritchhart, 2001). Finally, Ritchhart developed three super categories by analyzing (a) the behaviors and the products that resulted from the dispositions, (b) the purpose of the behaviors, and (c) the essence of the thinking involved in regard to the dispositions. The three resulting categories (i.e., critical thinking, creative thinking, and mental management and awareness) are used to organize the eight high-leverage thinking moves (see Table 2.1).

Table 2.1.

Thinking Move Research Framework

Thinking category	Thinking move
Creative thinking	• Considering different viewpoints and perspectives • Wondering and asking questions • Uncovering complexity and going below the surface of things
Mental management and awareness	• Building explanations and interpretations • Capturing the heart and forming conclusions • Making connections
Critical thinking	• Observing closely and describing what's there • Reasoning with evidence

In the end, the literature reviewed above was the basis for the development of high-leverage thinking moves framework that was used to investigate the thinking moves of students during problem-solving activities. Because this framework was used in an AP

Statistics class where the students worked on statistical activities, it is also important to explore the literature in relation to statistical thinking. This is done in the next section.

Statistical thinking. Statistical thinking is a complex and dynamic phenomena that is characterized by multiple domains. Biehler (1999) describes statistical thinking as a *statistical culture*, a culture that can be constructed through teachers acting as mentors and facilitators, as well as making use of collaborative and interactive approaches in the statistics classroom. Creating such a culture would include developing specific traits such understanding the role of data, nurturing more general traits such as inquisitiveness, scientific approaches, and writing, and promoting systems related to beliefs and values. In defining statistical thinking in this manner, Biehler was informed by recent research in mathematics education that suggests that learning can be viewed as a gradual acquisition of the characteristics and customs of a mathematical culture. In this section, the research related to statistical thinking is examined, including the overarching theme of the consideration of the variability and uncertainty and statistical thinking domains such as data in context, types of supporting knowledge, and informal and formal representations.

Statistics is a discipline that is in place to support other disciplines with the structure, tools, and procedures necessary for dealing with data (Cobb & Moore, 1997). This support is needed due to the fact that, in order to successfully navigate investigations and various forms of inquiry, many disciplines need to deal with the "omnipresence of variability" (p. 801). This variability can take on a number of different forms including: (a) measurement variability, variability resulting when repeated measurements vary due to measurement instruments or what is being measured; (b) natural variability, due to difference in measurements across individuals or things; (c) induced variability, the deliberate introduction of variability to permit the comparison with natural variability; (d) sampling variability, that is, the variation from sample to sample; and (e) chance variability (Franklin et al., 2005). In essence, an individual can deal with and make sense of variability, which may arise in various situations, activities, and problems, through statistical thinking.

In defining statistical thinking, Cobb and Moore (1997) argue that mathematical thinking and statistical thinking are different, not just because of the content studied in each of the disciplines, but because a different kind of thinking is involved. Cobb and Moore explain that this difference in thinking results from the fact that when dealing with data in statistics, individuals need to consider numbers in a context. Likewise, Garfield and Gal (1999) contend that data are simply numbers in a given context; context drives the actions taken, context provides meaning, and context forms the foundation for interpreting the results in statistical thinking. In a similar way, other researchers posit that in order to engage in statistical thinking one must make sense of variability through consideration of data in context and have a formal or informal knowledge of that context (Chance, 2002; Garfield & Gal, 1999; Pfannkuch & Rubick, 2002).

Groth (2005) studied patterns of statistical thinking in a qualitative study involving 15 students that ranged in grade level from freshman in high school to freshman in college. The research explored patterns in student approaches to statistical activities in various contexts related to the interplay between context knowledge and statistical knowledge. Groth found that the determination of some students to get, or to focus on, the “right” answer using formal algorithms, tools, or textbooks, stifled the statistical thinking of some students as they were unable to see or make sense of important connections between the context and the data. As a consequence of the research, Groth warns against the uncritical use of statistical information or procedures without consideration of context and stresses the importance finding a balance between intuition and the use of formal statistical tools and procedures. These findings make clear the importance of tying statistical thinking to the context of the data.

In addition to the necessity for dealing with uncertainty and the importance of understanding the data in context, statistical tools and statistical knowledge are also an important aspect of statistical thinking. Cobb and Moore (1997) contend that knowledge of statistical tools, the ability to select an appropriate one, and the ability to logically consider the results are all important aspects of working with data and essential for statistical understanding. Likewise, Wild and Pfannkuch (1999) suggest that statistical knowledge is one of the raw materials, along with the context knowledge and information for data, for statistical thinking. Moreover, the appropriate use of statistical knowledge and procedures (delMas, 2004), the application of understanding to realistic situations, the analysis of research studies, and the capability to generalize classroom knowledge to new situations are all ways that individuals can demonstrate the ability to think in a statistical manner (delMas, 2002). Furthermore, statistical thinking involves selecting appropriate models or building statistical models to simulate data and making inferences and judgments about the data (Garfield, delMas, & Zieffler, 2012; Wild & Pfannkuch, 1999). Pfannkuch and Rubick (2002) point out that this could include complex models, such as regression models, as well as more simplistic statistical models, like statistical graphs.

Wild and Pfannkuch (1999) developed a four-dimensional framework for statistical thinking that incorporates a number of the aspects that have been discussed. The first dimension, the investigative cycle, has five components including defining the problem, planning the design and data management, collecting and managing the data, analysis and exploration of data, and the sharing of interpretations, ideas, and conclusions. The second dimension describes the general types of thinking that could be part of statistical thinking such as strategic planning and consideration of constraints, modeling, seeking explanations, and application of techniques and problem-solving tools. In addition, Wild and Pfannkuch outline five types of statistical thinking: (1) recognition of the need for data, (2) transnumeration, (3) consideration of variation, (4) reasoning with statistical models, and (5) integrating the statistical and contextual. The interrogative cycle, Dimension 3, includes the following components: generating, seeking, interpreting, criticizing, and judging. The final dimension is made up of dispositions such as being open-minded, skeptical, inquisitive, logical, and persistent. An essential aspect of statistical thinking takes place when an individual is able to mentally move back and forth between the context of the data and the statistical knowledge related to the problem or situation (Wild & Pfannkuch, 1999).

Pfannkuch (2000) examined the four dimensional model developed by Wild and Pfannkuch (1999) through two different lenses: (a) assessment work in an undergraduate introductory statistics course, and (b) observation of graduate and undergraduate classes working on a statistical activity that required the use of data. In the first case, the research made use of a structured activity; as part of this activity, students were asked to construct and compare graphs and then complete a modeling exercise. In the second case, the researcher had students complete an unstructured activity; in this activity students constructed graphs and made predictions based on a data set that listed the times between eruptions of the geyser Old Faithful. This activity, which included the examination of data sets and the creation of graphs, gave students a glimpse into the variability of data, encouraged them to look beyond anecdotal evidence, and helped them understand the importance of interpreting multiple representations of the data and making connections between them (Pfannkuch, 2000). The research reinforced the fact that the appropriate application of new knowledge in various structured statistical situations informs statistical thinking. In addition, the research suggested that thinking in a statistical manner included uncovering and making sense of patterns related to variability, using modeling to make predictions, and developing and pursuing questions in order to gain the knowledge necessary to make meaning of the contextual knowledge related to the data. Finally, the research found that the four dimensional model (Wild & Pfannkuch, 1999) effectively assessed general statistical thinking abilities.

Together, these findings indicate that there are a number of statistical domains that allow individuals to deal with and make sense of variability and uncertainty. These statistical domains, at least in part, are: (a) data in context and contextual knowledge; (b) other formal or informal knowledge including statistical knowledge, mathematical knowledge, and logical thinking; and (c) formal and informal representations including tables, graphs, modeling, and statistical models (delMas et al., 2014). In order to effectively reveal these various aspects of statistically thinking in an educational research setting, the selection of appropriate statistical activities is essential. The selection of these activities is discussed next.

Statistical activities. Statistical activities were chosen for the present research study that had the potential to encourage thinking that incorporated the various statistical domains discussed in the previous section. Specifically, the selection of the activities for the current research was informed by a perspective that views learning through a constructivist lens and teaching through a Vygotskian lens (Fuson, Smith, & Lo Cicero, 1997). From this perspective, students use conceptual structures to make sense of their experiences and teachers support students as they construct helpful conceptual structures and, when students are ready, help them construct more advanced structures and methods (Fuson et al, 1997). In the classroom, teachers learn about how a student is thinking through formative feedback in order to provide the necessary assistance to help the student make meaning of the educational experience. This is in alignment with statistics education, where research across multiple disciplines indicates that developing statistical reasoning is best done through activities which take into account previous knowledge and help children build on that knowledge (Garfield & Ben-Zvi, 2009).

To effectively create an environment that has the potential to reveal student thinking during statistical activities, two existing frameworks were combined. The first framework is the Statistical Reasoning Learning Environment, where teachers act as facilitators to encourage students to build on their current knowledge through: (a) hands-on, collaborative activities; (b) discussions with peers, (c) and individual reflective activities (Garfield & Ben-Zvi, 2009). The Statistical Reasoning Learning Environment framework was used in conjunction with a conceptual framework based on the Guidelines for Assessment and Instruction in Statistics Education (GAISE) for introductory statistics courses at the college level. The college version of the document was used due to the fact that students who get a passing score on the AP Statistics exam often receive college credit. The report outlines six recommendations: (1) emphasize statistical literacy and develop statistical thinking; (2) use real data; (3) stress conceptual understanding, rather than mere knowledge of procedures; (4) foster active learning in the classroom; (5) use technology for developing concepts and analyzing data; and (6) use assessments to improve and evaluate student learning (Aliaga et al., 2010). Together, these frameworks informed the selection of activities that were suitable for completion in a group setting, which encouraged discussion with peers, which made use of real data, that focused on conceptual understanding, and that required the use of technology. As a result, activities for the current research were chosen that were meaningful to students and that exposed their thinking and thinking dispositions related to their conceptual understanding of statistics. A similar approach was taken in selecting the fourth activity (i.e., a MEA) used as a group problem-solving activity in the present research study. This is discussed next.

Models and model-based reasoning. The development of a model is an effective way for students to make sense of their experiences and reveal their thinking. Nersessian (2008) maintains that the ability for mental modeling provides a cognitive foundation for model-based reasoning. She indicates that conceptual change is possible through mental modeling because producing a mental model not only helps with reasoning, but also aids in the development of new ways of conceptual representation. Yoon and Thompson (2007) identify two approaches to developing modeling skills: (a) guiding students through the modeling process by initially teaching modeling skills, and (b) developing modeling abilities through a reflective analysis of the modeling task experience. They point out that an essential component of modeling activities is the revision cycle, a cycle that is completed multiple times as learners build, test, revise, and retest models.

Lesh, Doerr, Carmona, & Hjalmarson (2003) promote a models and modeling perspective that is built on the assumption that individuals can make use of models to make sense of their experiences. According to Lesh et al, models are defined as conceptual systems that are made up of representational media (e.g., spoken language, written symbols, and concrete materials). Once an individual creates a model, it can be used for “constructing, describing, explaining, manipulating, predicting or controlling systems that occur in the world” (Lesh et al., 2003, p. 214). This approach shifts the emphasis from the teacher facilitation of shared meaning to the creation of a learning environment in which students can, through numerous cycles of analysis, choose, examine, categorize, and revise a varied set of community ideas developed by students (Lesh et al., 2003). One way to accomplish this is thorough the use of MEAs; these activities are described in the next section.

Model-eliciting activities. MEAs are open-ended, real-world, complex problems which promote the development and testing of models. MEAs were originally research instruments created to: (a) make student reasoning, thinking, and problem-solving abilities visible; (b) allow students access to simulations based in reality; and (c) help with the identification of talents and skills not typically assessed by standardized exams (Moore, 2008). In regard to these objectives, Lesh and Doerr (2003) indicate that it is through the modeling experience in MEAs that students’ thinking and interpretations are visible through the conceptual tools. In addition, when working on MEAs, students are placed in realistic problem-solving situations where they are able to display understandings, abilities, and talents, across many disciplines, beyond the type of intelligence generally measured by standardized exams. These include the ability to: (a) apply problem-solving skills, (b) work in groups, (c) complete complex tasks, and (d) make use of appropriate technologies (Lesh & Doerr, 2003). As a result, MEAs can be used as a formative assessment tool to determine student weaknesses and student strengths related to conceptual understanding (Zawojewski, Hjalmarson, Bowman, & Lesh, 2008).

When working traditional problems, learners primarily use symbolic representations to make meaning of mathematical concepts and ideas (Lesh & Doerr, 2003). In contrast, an important aspect of MEAs is that they support students in their creation of mathematical descriptions of meaningful and relevant situations. As a result, students are often able to “mathematize” situations they encounter in the activity. When completing modeling activities such as MEAs, learners often move between representational media; this representational fluency is critical in understanding conceptual systems (Lesh & Doerr, 2003). Another important aspect of MEAs is that the modeling process often involves multiple cycles, where learners can engage with the components of the cycles in a variety of manners and sequences (Lesh & Doerr, 2003). The opportunity for learners to have extended time with a problem situation as well as repetitive cycles of engagement with their ideas, promotes student learning (Hjalmarson, Diefes-Dux, & Moore, 2008). Consequently, these finding indicate that it is essential to guide students during modeling in order to help them develop stable conceptual systems and put them into situations where they are able to test and refine alternative ways of understanding (Lesh & Doerr, 2003), Moore, Miller, Lesh, Stohlmann, and Kim (2013) investigated model development, including the roles of representations and representational fluency, through the implementation of the *Human Thermometer* MEA. The study, which took place at a large Midwestern public university, examined to the work of 16 teams of three to four engineering students over four class periods in a junior-level heat transfer course. Two research questions were explored, using the Lesh Translation Model as a theoretical framework: “(1) How do engineering students use multiple representations and representational fluency to develop engineering models during a complex modeling task? (2) What role do representations and representational fluency play in conceptual development during a complex modeling task?” (p. 142). Moore et al. found that the use of multiple representations assisted in model development, helped overcome misconceptions, avoided simplistic understandings, and increased the ability of students to reveal their conceptual understanding. While MEAs are an effective way to develop representational fluency in students, the researchers found that the movement between and within the representational domains was a complex endeavor for learners and required support from teachers or fellow students. As a result, teachers that are comfortable with a more traditional pedagogical approach may benefit from working with peers who have experience implementing MEAs in a classroom setting (Moore et al., 2013).

In a different study, English and Mousoulides (2011) studied the incorporation of engineering experiences in the elementary and middle school classrooms. In the 3-year longitudinal study of middle school students, the researchers examined the ability of learners to develop models through the use of the *Water Shortage Problem* MEA. The research highlighted the importance of the ability of students to integrate the combination of qualitative and quantitative data when developing models, to make decisions, and to form conclusions. In addition, English and Mousoulides point out that the simplification of engineering problems may be necessary in early explorations and model development exercises; more data and information can then be incorporated in later iterations of the model. Finally, the research showed that some students did not make use of all the available data in designing their model. Consequently, English and Mousoulides suggest that teachers encourage students to make use of all available data, as well as search for and make use of additional data and resources during the completion of modeling activities.

In related research, Rea-Ramirez and Núñez-Oviedo (2008) examined the capabilities of inner city middle school students to create and make use of conceptual models in an after school biology program. The researchers found that the students, mostly from low-income Hispanic families, used model-based reasoning skills to significantly increase their understanding of course material. Specifically, successful students, in terms of responses on open response style problems, applied complex, dynamic, and causal models. One of the most significant findings of the study was that the use of teacher or student generated flow diagrams may help students explain causal relationships (Rea-Ramirez & Núñez-Oviedo, 2008). To aid in this process, the researchers recommend that the teachers discuss the use of the flow diagrams with students and provide support as students make use of the diagrams. In addition, students can benefit from working with their models, through multiple reasoning cycles, even if their model is incomplete. Finally, the research indicated that shared reasoning or co-

construction of knowledge, between the students and the teacher, is an essential aspect of producing conceptual change in the science classroom. Overall, the literature reviewed above indicates that the use of MEAs can be effective tools to reveal student thinking, student reasoning, and student problem-solving abilities in group situations. Because the present research study took place in a statistics class, the use of MEAs in a statistics classroom is considered next.

Use of MEAs in statistics. The use of MEAs in a statistics classroom can be very effective in helping students unpack statistical concepts and reveal their statistical reasoning and thinking. Gal and Garfield (1997) maintain that statistical concepts and ideas are often misunderstood and difficult to teach, to learn, and to understand. Reform efforts have encouraged teachers to include more interactive activities and placed a greater emphasis on conceptual understanding though engagement with real data; at the same time, teachers of introductory statistics courses have been urged to include less coverage of theory, fewer formulaic manipulations, and fewer follow the recipe type problems (Gal & Garfield, 1997). In a similar way, Garfield and Ben-Zvi (2007) indicate that misconceptions and misunderstandings about statistical inference and reasoning are widespread and that there are inconsistencies related to how students transfer statistical reasoning from one concept to the next. To counteract such trends, Garfield and Ben-Zvi recommend, among other things, an active and collaborative learning environment. In addition, Park, delMas, Zieffler, and Garfield (2011), drawing on the invention for transfer work done by Schwartz, Sears, and Chang (2007), indicate that simply teaching new information is not enough to help students overcome incomplete and incorrect prior knowledge. Instead, students must confront and work through their misconceptions if new learning is to take place. One way to do this is to make use of well-designed activities, such as MEAs, to confront student misconceptions and misunderstandings (Park et al., 2011).

MEAs are excellent tools for assessing student statistical knowledge and reasoning abilities; the use of MEAs can be used in the statistics classroom to create cognitive dissonance as well as provide meaningful formative feedback (Hjalmarson, Moore, & delMas, 2011; Park et al., 2011). This is in part due to the fact that, when working on MEAs, students are placed in realistic problem-solving situations where they are able to display numerous abilities and strengths, beyond the type of intelligence generally measured by standardized exams (Park et al., 2011). Specifically, students are able to: (a) apply problem-solving skills, (b) work in groups, (c) complete complex tasks, and (d) make use of appropriate technologies. Furthermore, MEAs have the potential to address the six GAISE guidelines for teaching statistics (Hjalmarson et al., 2011). For these reasons, the use of an MEA in the current research was a suitable way (a) to illuminate the thinking moves used by students in a problem-solving situation, (b) to reveal a variety of student intelligences and abilities, (c) to have students wrestle with essential statistical concepts, and (d) to determine student weaknesses and strengths related to conceptual understanding (Zawojewski et al., 2008). Together, these findings provide justification for using an MEA to reveal student thinking in the current research.

Summary

The present research study examined the degree and in what ways the narratives that students construct for their mathematical identities are related to the use of high-leverage thinking moves during problem-solving activities. To provide a foundation for this research, the literature related to mathematical identity and the use of thinking moves to gauge student understanding was examined. The research related to both these phenomenon are summarized in this section.

One objective of this research was to determine how individuals understand and make meaning of their experiences in regard to mathematical identity. The research makes it clear that there is a strong association between learning mathematics and identity (Anderson, 2007; Boaler & Greeno, 2000; Boaler et al., 2000; Grootenboer & Zevenbergen, 2008; Kane, 2012; Varelas et al., 2012). In addition, one of the most effective ways to investigate student identity is through their narratives; narrative inquiry as a research methodology is an excellent way to unpack the complexities of the experiences of students in an educational setting (Clandinin & Connelly, 2000; Eaton & O Reilly, 2009; Eaton et al., 2011; Eaton et al., 2013; Kaasila 2007; Sfard & Prusak, 2005). Therefore, the narratives of the students, as revealed primarily through a student survey and during two student interviews, were used to explore their mathematical identity. In turn, these narratives were used to develop rich descriptions of the designated and current identity of the students.

A second objective of this research study was to explore student understanding of statistical concepts and ideas; to do this, the high-leverage thinking moves (Ritchhart et al., 2011) were used as a research framework. Because there are voices in mathematics educational research that argue that the consideration of quantitative measures such as standardized test scores do not give a complete picture of whether a student is successfully engaging with the material (Gutierrez, 2000, 2007; Martin, 2000, 2009; Nasir, 2002), this research investigated student understanding of mathematical ideas and concepts through the thinking moves they exhibited. A significant number of researchers have found that critical thinking dispositions are an essential part of describing how an individual thinks about something (Ennis, 1996). In terms of the learning of mathematical concepts and ideas, understanding is a consequence of thinking, and not a component of thinking; for thorough understanding, learners must be able to engage in a variety of thinking behaviors (Ritchhart et al., 2011). In that regard, Ritchhart et al. (2011) developed eight high-leverage thinking moves that they argue are essential to deep understanding: (a) observing closely and describing what is there, (b) building explanations and interpretations, (c) reasoning with evidence, (d) making connections, (e) considering different viewpoints and perspectives, (f) capturing the heart and forming conclusions; (g) wondering and asking questions, (h) uncovering complexity and going below the surface of things (p. 14). All eight of these thinking moves are generally present when individuals show a thorough understanding of ideas and concepts (Ritchhart et al., 2011). As a result, the thinking moves displayed during exams and group activities were used to explore the understanding of statistical concepts and ideas during problem-solving activities.

In order to reveal the thinking moves of the students, the statistical activities used were carefully chosen; the research related to these activities was also reviewed. First, to inform the selection of activities, the literature regarding statistical thinking was considered. Statistical thinking is essential for engaging with variability and uncertainty in situations, problems, and questions (Chance, 2002; Franklin et al., 2005; Garfield & Gal, 1999; Cobb & Moore, 1997; Pfannkuch & Rubick, 2002). In addition, statistical thinking requires the need for data in context, content knowledge, statistical tools, and statistical knowledge (Wild & Pfannkuch, 1999). Finally, the selection of appropriate models, or the building of statistical models, to simulate data and making inferences about the data, is a critical component of statistical thinking (Garfield et al., 2012; Wild & Pfannkuch, 1999); learners could make use of more complex models, like regression models, as well as more simplistic statistical models, like statistical graphs (Pfannkuch & Rubick, 2002).

In addition, the research related to activities that reveal statistically thinking was explored. First, the use of MEAs was explored; MEAs are modeling activities that can be used in the classroom to make student thinking visible during problem-solving activities (Lesh & Doerr, 2003; Moore, 2008). In addition, when working on MEAs, students are placed in realistic problem-solving situations where they are able (a) apply problem-solving skills, (b) work in groups, (c) complete complex tasks, and (d) make use of appropriate technologies (Lesh & Doerr, 2003). As a result, MEAs are used as a tool to display and unpack conceptual understanding of students (Zawojewski et al., 2008).

In addition to a MEA, three other statistical activities were used in this research. In order to create an environment that fosters the display of student thinking during statistical activities, two existing frameworks were combined. Those two frameworks are the Statistical Reasoning Learning Environment and the Guidelines for Assessment and Instruction in Statistics Education for introductory statistics courses at the college level. Based on those two frameworks, activities were selected that were suitable for completion in a group setting, which encouraged discussion with peers, which made use of real data, that focused on conceptual understanding, and that required the use of technology. As a result, activities for this research study were chosen that were meaningful to students and that exposed their thinking moves related to their conceptual understanding of statistics.

In Chapter 3 the research methodology, including the data collection instruments, the data collection procedures, the research design, and the data analysis methods, are outlined. Chapter 4 covers the various aspects of the data analysis in terms of each of the four cases, while Chapter 5 contains the cross case analysis. Finally, Chapter 6 includes the cross case discussion, the limitations, the implications, and the areas of possible future research.

Chapter 3

Research Methodology

A growing body of research suggests that the racial, academic, and disciplinary identities may have a significant impact on a student's ability to navigate the complicated social interactions in the classroom and the tasks related to unpacking mathematical content (Anderson, 2007; Boaler & Greeno, 2000; Boaler et al., 2000; Grootenboer & Zevenbergen, 2008). Mathematical experiences and social relations have a symbiotic relationship with identity; experiences and relationships influence the identity of individuals and are in turn influenced by the identity of the individual (Anderson, 2007). In this regard, research is needed to further describe this dynamic relationship between identity and understanding in the mathematics classroom; researching understanding in an educational setting can be a difficult task, because when an individual develops understanding, it is the result of a complex process that requires diverse forms of thinking (Ennis, 1996; Facione, Sanchez, & Facione, 1994; Ritchhart et al., 2011; Tishman et al., 1993). Consequently, it may be helpful to investigate the thinking moves of students, where understanding is viewed as a consequence of thinking, and not a component of thinking (Ritchhart et al., 2011). Therefore, the research presented here examined the association between the narratives students construct for their mathematical identities and the thinking moves displayed during problem-solving activities.

A qualitative case study was chosen to investigate the research questions; the unit of analysis was the thinking moves exhibited by students during the completion of group activities. Because the research focused on mathematical identities and the use of high-leverage thinking moves, a qualitative case study approach was appropriate for several reasons. First, one focus of this research was on how individuals interpreted and made meaning of their experiences in regard to mathematical identity. This phenomenon of mathematical identity, which is complex and shifting, was examined through the narratives of the individuals as the use of narrative inquiry is one of the most effective ways to study the experiences of individuals in an educational setting (Clandinin & Connelly, 2000; Kaasila, 2007). Second, because there are voices in mathematics educational research that argue that the consideration of quantitative measures such as standardized test scores do not give a complete picture of whether a student is successfully engaging with the material (Gutierrez, 2000, 2007; Martin, 2009), this research study investigated student understanding of mathematical ideas and concepts through the thinking moves they exhibited. A qualitative approach fits both these phenomena extremely well as the methodology is flexible and responsive to phenomena that are fluid (Merriam, 2009; Patton, 2002). In addition, it is suitable because the inquiry focused on the thinking moves and the mathematical identities of the participants, rather than on the products produced by the participants (Bogdan & Biklen, 2003; Merriam, 2009). A qualitative study is also proper as the objective of the research is not about predicting what might happen related to the phenomena, but to understand them (Patton, 2002). Finally, such an approach is appropriate in this instance due to the fact that qualitative research produces a product that is "richly descriptive" (Patton, 2002) and can deepen the reader's knowledge of the phenomenon (Merriam, 2009).

Chapter Overview

This chapter covers the research design, the approach to data collection, and the data analysis plan. First, the overview of the study is outlined. This is followed by a description of the participants, the setting, and the data sources. Next, each of the data collection instruments is outlined. The instruments used related to mathematical identity include the MINT Protocol, three teacher interviews, two student interviews, and the student reflections at the conclusion of each activity. The instruments used related to the high-leverage thinking moves include the discussions during the five group activities, the student reflections at the conclusion of each activity, and the student responses on selected exam questions. After the data collection instruments are described, the plan to analyze each of the data sources is explained. Finally, the validity, the reliability, and the generalizability of the research are discussed.

Overview of Study

The research presented here was designed to address the following question: To what degree and in what ways are the narratives students construct for their mathematical identities related to the use of high-leverage thinking moves during problem-solving activities? To address this overarching question, the following two questions were examined.

A. To what degree and in what ways are the narratives students construct for their *current* mathematical identities related to the use of high-leverage thinking moves during problem-solving activities?

B. To what degree and in what ways are the narratives students construct for their *designated* mathematical identities related to the use of high-leverage thinking moves during problem-solving activities?

To address subquestion A, the use of the thinking moves used during problem-solving activities and the student narratives in regard to their *current* mathematical identities were explored. Current identities are identities that are the stories and narratives of individuals that are generally told in the present tense and that reflect current circumstances (Graven & Buytenhuys, 2011; Sfard & Prusak, 2005). To address subquestion B, the use of thinking moves during problem-solving activities and the student narratives in regard to their *designated* mathematical identities are examined. Designated identities are stories and narratives of individuals regarding their ambitions and the situations they expect to be in at the present time or in the future (Sfard & Prusak, 2005). In the following sections, these three aspects of the research, thinking moves, current identity, and designated identity, are explained in more detail.

Thinking moves. First, in relation to both subquestions, the thinking moves displayed during group activities and on selected exam questions were examined. In addition, student reflections completed at the conclusion of each activity were used to confirm the thinking moves observed during the group discussions. The group activities are described in the section *Data Collection Instruments for Thinking Moves* and can be found in full in the appendices indicated below. The five activities used in the research were the: (1) *On Time Arrival* MEA (Chamberlin & Chamberlin, 2001; see Appendix A), (2) the *Race and the Death Penalty* activity (Bock, Velleman, & DeVeaux, 2007; see Appendix B), (3) the *Backhoes and Forklifts* activity (Bock et al., 2007; see Appendix C), (4) the *ESP* activity (Bock et al., 2007; see Appendix D), and (5) *The Spread of a Rumor* activity (Starnes, Yates, & Moore, 2003; see Appendix E).

The purpose of using these activities was to reveal student thinking. Activities were selected that were suitable for completion in a group setting, which encouraged discussion with peers, which made use of real data, that focused on conceptual understanding, and that required the use of technology. As a result, activities for this research study were chosen that were meaningful to students and that exposed their thinking moves related to their conceptual understanding of statistics.

Once the thinking moves were determined, the eight high-leverage thinking moves (Ritchhart et al., 2011) were used as a framework to organize the data. Specifically, the framework provided a structure to describe the how the students were thinking about the various concepts and ideas they encountered in the activities. The eight thinking moves are: (a) *observing closely and describing what is there*, that is, noticing features and various components and being able to describe them complexly and thoroughly; (b) *building explanations and interpretations,* for example, developing theories and hypotheses in science; (c) *reasoning with evidence* to develop new positions or to support current positions in an appropriate manner; (d) *making connections* between prior knowledge and experiences and new ideas; (e) *considering different viewpoints and perspectives* to ensure a more thorough understanding; (f) showing an understanding of the big ideas in play through *capturing the heart and forming conclusions*; (g) *wondering and asking questions* at the outset to create an avenue for learning and engagement and throughout the learning process to deepen understanding, and aid in the (h) *uncovering complexity and going below the surface of things* (Ritchhart et al., 2011, p. 14; see Chapter 2 for a more complete description of the high-leverage thinking moves). By definition, for an individual to exhibit a thorough understanding, all eight thinking moves are generally present (Ritchhart et al., 2011). For this reason, the high-leverage thinking moves serve as an effective framework to analyze a student's mathematical understanding. The methods used to study the current identity of each of the students are considered in the next section.

Current identity. To address subquestion A, the association between the current identity and the thinking moves was studied. To start, the narratives students constructed were analyzed in relation to their current mathematical identities; the narratives were revealed through the use of student interviews, teacher interviews, activity reflection responses, and a student survey. This analysis was done in two different ways. First, the data were organized using seven themes (Eaton & O Reilly, 2009): (a) the changing nature of mathematics as experienced from early childhood to now, (b) the balance between challenge and interest, (c) the role played by key figures in the formation of mathematical identity, (d) how learning mathematics compares with learning in other subjects, (e) ways of studying mathematics and working in mathematics, (f) persistence/perseverance with mathematics, and (g) mathematics as a rewarding subject. After the data were organized according to the seven themes, the data were then placed into five mathematical identity levels according to the strength of the student experiences and feeling about mathematics (i.e., strong positive, positive, mixed, negative, and strong negative). In the next section, the methods used to study the designated identity of the students are outlined.

Designated identity. To address subquestion B, the association between the designated identity and the thinking moves was studied. The designated identity was explored through the narratives of each of the students through the development of thorough descriptions related to their experiences and feelings about mathematics. The narratives were built using student responses on a survey and activity reflections, student interviews, and teacher interviews.

In summary, the present study research explored the connections between mathematical identity and thinking moves (Figure 3.1 summarizes the data sources used).

Figure 3.1 Associations Between Mathematical Identity and Thinking Moves

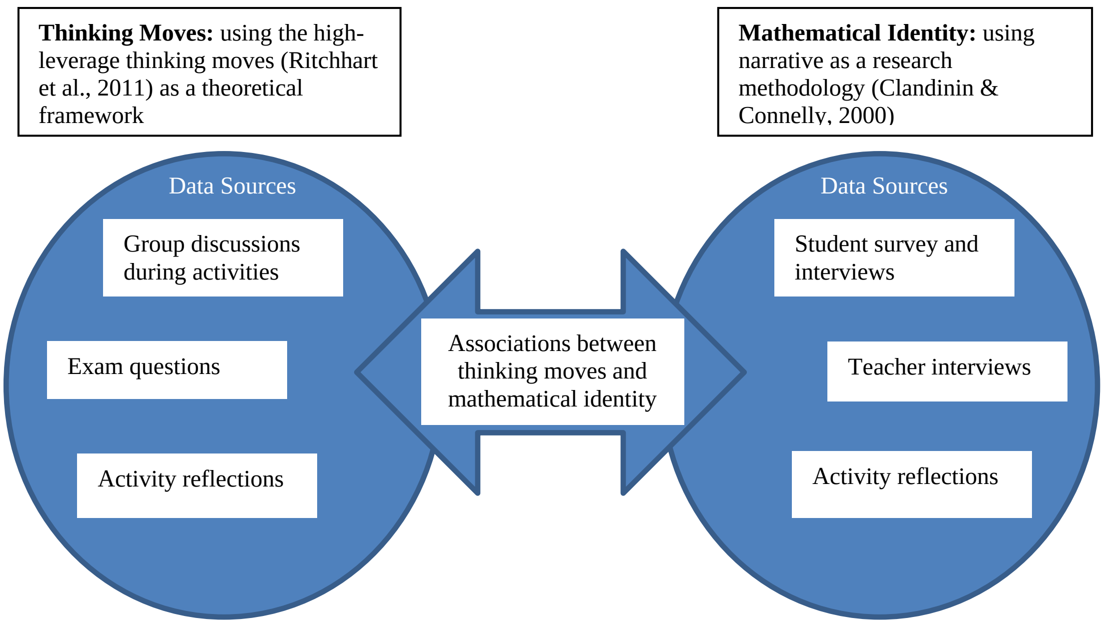

Figure 3.1. Associations between mathematical identity and thinking moves.

Specifically, the exploration of the association between the thinking moves used during problem-solving activities and the *current* mathematical identity of the student addressed subquestion A. In a similar way, the investigation of the links between the thinking moves and the student narratives related to their *designated* mathematical identity addressed subquestion B. Once the connections between these phenomena (i.e., thinking moves, current mathematical identity, and designated mathematical identity) were explored, the data were compared across cases to look for patterns. In the next section, the descriptions of the participants and how they were selected are discussed.

Participants

This research study was conducted over one semester in an AP Statistics classroom in a suburban high school in the Midwest with 9.1% students of color and 7.2% students on Free and Reduced Lunch. This classroom consisted of 30 students in their junior or senior year. Of those, 23 students returned the consent forms and agreed to be part of the overall study. In addition, four students were selected from the pool of 23 students to be involved with the more detailed portion of the research. The class contained a wide variety of students in terms of their interest in mathematics, their mathematical identity, their mathematical abilities, and their background in mathematics. This was due, at least in part, to the fact that AP Statistics was the only statistics course offered at the school.

In order to learn more about the demographics and characteristics of the students, the MINT protocol (Eaton et al., 2013; see Table 3.1) was given the second week of the semester. The MINT protocol, consisting of three sections, measures the mathematical identity of an individual through their narrative. In Section 1, students provided demographic information including gender, institution, grade in school, age, mathematics courses being taken presently, and mathematics classes taken prior to the 2014-2015 school year. In the second section, students responded to the following eight 5-point Likert (i.e., strongly agree to strongly disagree) prompts to get them thinking about their identity related to mathematics (Aiken, 1974; Dutton, 1954; Fennema & Sherman, 1976; Macnab & Payne, 2003; Tapia & Marsh, 2004): (1) mathematics is a challenging subject,

(2) mathematics is more difficult than other subjects, (3) I have had an overall positive experience in mathematics, (4) mathematics is irrelevant to everyday life, (5) I find mathematics intimidating, (6) I'll need a good understanding of mathematics for my future work, (7) mathematics is interesting, and (8) I feel competent in mathematics. In addition to providing data for the research, the Section 2 questions were also intended to stimulate the student narrative in Section 3. Finally, Section 3 is composed of open ended prompts; in this section, students completed Part 1 before being presented with Part 2 so that they were able to write about their experiences in an unstructured fashion and reflect on their experience from an additional viewpoint (Eaton & O Reilly, 2009; Eaton et al., 2011).

Table 3.1.

MINT Survey Prompts (Section 3)

	Prompts/ Questions
Part 1	1. Think about your total experience of mathematics. 2. Tell us about the dominant features that come to mind.
Part 2	1. Now think carefully about all stages of your mathematical journey from primary school (or earlier) to high school mathematics. Consider: (a) Why you chose to study AP Statistics (b) Influential people (c) Critical incidents or events (d) Your feelings or attitudes about mathematics (e) How mathematics compares to other subjects (f) Mathematical content and topics 2. With these and other thoughts in mind, describe some further features of your relationship with mathematics over time.
Part 3	What insight, if any, have you gained about your own attitude towards mathematics and studying the subject as a result of completing the questionnaire?

According to the data from the MINT protocol, the mathematical background of the students in the course was diverse; some students were concurrently taking AP Calculus while others had only completed Algebra II or Algebra III. There was also a wide range in terms of how students described their mathematical identity. For example, some students indicated that they were very confident regarding their mathematical abilities and their responses revealed positive experiences and feelings related to mathematics. On the other hand there were many students who indicated that they did not really enjoy mathematics, had bad experiences with mathematics in the past, and did not feel competent in mathematics.

In the third week of the study, using the results from the MINT protocol and from a teacher interview, four cases (i.e., Anna, Ashley, Erik, and Mitch; these are pseudonyms) were selected in a manner to promote external validity. First, to create a *literal replication* (Yin, 2009) individuals were chosen in order to predict comparable results. In this regard, both students in a pair were chosen to have a similar mathematical background and a similar mathematical identity. In terms of the first set of students, Erik and Mitch both had a positive to strong positive mathematical identity and both had a strong mathematical background, in terms of past coursework, as both students were currently enrolled in AP Calculus and AP Statistics. The second set of individuals, Anna and Ashley, both had a negative to mixed mathematical identity and both had a more moderate background, in terms of past mathematics coursework, as both had completed Algebra II or Algebra III prior to taking AP Statistics.

Second, to create a a *theoretical replication* (Yin, 2009) the two sets of cases were also chosen in order to predict dissimilar results for anticipated reasons. The differences between the two sets of cases exist in terms of mathematical background (i.e., enrolled in AP Calculus versus completion of Algebra II or Algebra III) and mathematical identity (i.e., positive versus negative). Finally, for consistency purposes, all four students were in the first semester of their senior year. In the next section, the setting of the research is described.

Setting

The classroom portion of the research took place during the fall semester in a suburban school in a Midwest metropolitan area. The AP Statistics class, made up of 30 juniors and seniors, was a one credit, yearlong course that met every day for 54 minutes. The classroom was set up with individual desks lined up in six rows; the teacher taught from the front of the classroom and often made use of an interactive whiteboard. A moderate amount of the instruction was lectured-based, but the teacher included time for student questions and class discussions. At the end of most periods, homework problems from the textbook were assigned. Students then had a chance at the beginning of the next class period to ask questions about any homework related problems.

There were also opportunities for students to work in a collaborative setting. For example, students had the opportunity to work with partners on certain quizzes during the semester; the partners both received the highest score of the two quizzes. In addition, students often worked in groups of three or four to complete statistical activities related to course content. Each group member received the same score based on the work of the group. When working with others, students sometimes had the chance to choose their own partner or group, while other times the partner or group members were assigned.

The topics covered during the first semester of the course were: (a) exploring data, analyzing categorical data, and displaying and describing quantitative data; (b) modeling distributions, describing locations in a distribution and in normal distributions; (c) describing relationships, scatterplots, least square regression lines, and correlation; (d) designing studies, sampling and surveying, experiments, and using studies wisely; (e) probability, randomness, simulation, rules, conditional probability, and independence; and (f) random variables, discrete, continuous, combining, binomial, and geometric.

Having described the participants involved with and setting of the present research, the data sources are now addressed.

Data Sources

Multiple data sources were used to unpack the research question: *To what degree and in what ways are the narratives students construct for their mathematical identities related to the use of high-leverage thinking moves during problem-solving activities?* To address the first subquestion (i.e., to what degree and in what ways are the narratives students construct for their *current* mathematical identities related to the use of high-leverage thinking moves during problem-solving activities?), the current mathematical identity and the thinking moves of the four students were examined. To address the second subquestion (i.e., to what degree and in what ways are the narratives students construct for their *designated* mathematical identities related to the use of high-leverage thinking moves during problem-solving activities?), the designated mathematical identity and the thinking moves of the four students were studied. To promote construct validity and to potentially produce more accurate and convincing conclusions, triangulation of data (Yin, 2009) was used to inform decisions regarding sources of evidence. As a result, multiple sources of data were used to explore the use of high-leverage thinking moves, the current mathematical identity of the students, and the designed mathematical identity of the students. In the next section, those data collection instruments and procedures are each addressed in detail.

Data Collection Instruments and Procedures

This research study was conducted over 12 months from May 2014 to May 2015. In early September, letters were distributed to the students who were enrolled in the AP Statistics course. The letters included consent forms to be signed and a research narrative describing the purpose and expectations of the research (Erickson, 1986). Two copies of the consent forms were distributed; one copy was to be signed and returned, and the second copy was to be kept for record keeping purposes. Twenty-three of the consent forms were signed and returned; at that point the MINT protocol was given to students. The MINT protocol was used, in part, to recruit the four cases for the research study. Initial analysis started in mid-September as emerging patterns drove fieldwork decisions and the latter parts of the research were used to confirm or disconfirm potential patterns or themes that emerged (Patton, 2002).

To inform data collection, two principles of data collection (Yin, 2009) were used to address issues of construct validity: triangulation of data and maintaining a chain of evidence. In terms of the first principle, multiple sources of evidence were used to triangulate the data related to mathematical identity and the use of thinking moves. The following sources of data were used to explore mathematical identity: (a) the MINT protocol, (b) the student interview protocol, (c) the teacher interview protocol, and (d) the student activity reflection prompts. In addition, the sources of data used to investigate thinking moves were: (a) discussions during group activities, (b) student activity reflection questions, and (c) the student responses on selected exam questions. A summary of which data sources are used to address each of the research subquestions is found in Table 3.2. Note that the data sources are categorized by current mathematical identity, designated mathematical identity, and high-leverage thinking moves.

Table 3.2.

Data Sources Used to Address Research Subquestions

Research subquestion	**Data sources**
A. To what degree and in what ways are the narratives students construct for their *current* mathematical identities related to the use of high-leverage thinking moves during problem-solving activities?	Current mathematical identity • MINT protocol • Three teacher interviews • Two student interviews • Student activity reflections at the conclusion of each activity High-leverage thinking moves • Group discussions during model-eliciting activity and three other group activities ▪ Discussions during the activities ▪ Written work and final product produced by the group ▪ Researcher field notes for the observations of each of the activities • Student activity reflections at the conclusion of each activity • Student responses on selected questions on exams (tied to each of the five activities)
B. To what degree and in what ways are the narratives students construct for their *designated* mathematical identities related to the use of high-leverage thinking moves during problem-solving activities?	Designated mathematical identity • MINT Protocol • Three teacher interviews • Two student interviews • Student activity reflections at the conclusion of each activity High-leverage thinking moves • Group discussions during model-eliciting activity and three other group activities ▪ Transcripts of discussions during the activities ▪ Written work and final product produced by the group ▪ Researcher field notes for the observations of each of the activities • Student activity reflections at the conclusion of each activity • Student responses on selected questions on exams (tied to each of the five activities)

Timeline for data collection. The classroom research was conducted over the first semester in an AP Statistics course. The research focused on the mathematical identity of students and their use of high-leverage thinking moves when working through the five group activities. The MINT survey was given during the second week of class, and was used, in part, to select the four students for the case study. To explore the mathematical identities of these four cases, the students were interviewed twice during the semester and the teacher was interviewed three times. Finally, to explore the thinking moves of the students, their work on the five group activities, their work on selected responses to exam questions, and their activity reflection responses were considered. See Table 3.3 for a timeline of the data collection events in the classroom.

Table 3.3.

Timeline for Classroom Research

Issue	Weeks 2 & 3	Week 4	Week 5	Week 6	Week 12	Week 14	Weeks 15 & 16
Math identity	• MINT Survey • Teacher interview 1			• Student interview 1 • Teacher interview 2			• Student interview 2 • Teacher interview 3
High-Leverage-thinking moves		• Group work on Airline MEA • Student activity reflection	Exam questions related to Activity 1 & Activity 2	• Group work on activity 2 • Student activity reflection	• Group work on Activity 3 • Student activity reflection • Exam questions related to Activity 3	• Group work on Activity 4 • Student activity reflection	Exam questions related to Activity 3 & Activity 4

The data sources used to unpack the mathematical identity of the students are considered first.

Data collection instruments for mathematical identity. In this section, the following data collection instruments are discussed: (a) the MINT protocol, (b) the student interviews, (c) the teacher interviews, and (d) the activity reflection prompts.

MINT protocol. The MINT protocol was given to students in the second week of the semester. In addition to assisting in the selection of cases, the MINT protocol was one of the three tools that were used to determine the current and designated identity of the cases. The MINT protocol, consisting of three sections, measures the mathematical identity of an individual through their narrative (see Appendix F for the entire MINT protocol). In Section 1, students provided demographic information including gender, institution, grade in school, age, mathematics courses being taken presently, and mathematics classes taken prior to the 2014-2015 school year. In Section 2, students responded to eight 5-point Likert prompts to get them thinking about their identity related to mathematics. Finally, in Section 3 students, responding to open-ended prompts, wrote about their mathematical experiences in an unstructured fashion.

Student interviews. In addition to the MINT protocol, two semi-structured interviews were conducted with each of the four students to further explore current and designated mathematical identity. In these interviews students had a chance to share their mathematical experiences and discuss their relationship with mathematics. The first set of student interviews were administered during Week 6 of the semester and the second set of student interviews were administered during Week 15 and Week 16 of the semester. The primary questions for the student interviews, which were read to the students, made use of the following prompts adapted from Varelas et al. (2012):

(a) I usually give my best effort in the mathematics classroom if …,

(b) I give less than my best effort in the mathematics classroom if ...,

(c) . . . shows that I am good at doing mathematics,

(d) . . . shows that doing mathematics is hard for me,

(e) Mathematics is an important part of my school (out-of-school) life because …,

(f) Mathematics is not a significant part of my school (out-of-school) life because ...,

(g) It's important for me to show my teacher and classmates that I am good in mathematics because …, and

(h) Some things that I do in the mathematics classroom which mathematicians do (or don't do)?

In addition to the primary questions, additional follow up questions were added at the time of the interview as needed (e.g., when the interviewer did not have a clear understanding of the student response). The interviews were audio recorded; the audio files were then transcribed into Word documents.

Teacher interviews. Three semi-structured interviews were conducted with the teacher to learn more about the current and designated mathematical identities of each of the four students. The first interview, which took place during week three of the semester, was used, along with the MINT protocol, to select the four cases. The second teacher interview took place during Week 6 and the third teacher interview took place during Week 16 of the semester. The following prompts, adapted from Varelas et al. (2012), were used for the primary questions to explore the mathematical identities of each case:

(a) This student sees herself/himself as a more (or less) central member of the mathematics classroom when …,

(b) This student sees herself/himself as competent (or not) in mathematics when …,

(c) This student sees herself/himself as a success (or not) in the mathematics classroom when …, and

(d) How does this student support (or not support) peers in academic tasks?

In addition to the primary questions, additional follow up questions were added at the time of the interview as needed (e.g., when a question was not fully answered). The interviews were audio recorded; the audio files were then transcribed into Word documents.

Student activity reflection prompts. The final source of data used to explore current mathematical identity was the student reflections completed at the conclusion of each activity. The activity reflection prompts were:

(1) What statistical concepts and skills did you use in this activity?

(2) Did this activity change how you think about yourself as a mathematician?

(3) Describe the thinking or problem-solving strategies that you used as part as this activity.

(4) Were there any thinking or problem-solving strategies that you thought of, but did not share with your group during the activity?

At the conclusion of each activity the students were given a one page document that contained the four questions; the students completed the written reflections on their own and returned them at the end of the class period or at the beginning of the next class period. In regard to current identity, only Question 2 was considered; the question asked students to consider whether the activity changed how they viewed themselves as a mathematician. The first prompt was used to get students to reflect back on the statistical activity, while Question 3 and Question 4 were related to the thinking moves exhibited by the student. This concludes the discussion of the data collection instruments used related to explore the mathematical identity of the four students. In total, four data sources were used to explore the *current* identity and three data sources were used to examine the *designated* identity of the four students.

In addition to the consideration of mathematical identity, the present research also studied the thinking moves of the students. Each of the data sources related to thinking moves is discussed in the next section.

Data sources for thinking moves. Three data sources were used to investigate the thinking moves of the four students: (a) the discussions during the five group activities, (b) the student activity reflections, and (c) selected exam question responses. Each of these data sources were analyzed using thinking moves as a framework (as described in Chapter 2). The eight high-leverage thinking moves are: (a) observing closely and describing what is there, (b) building explanations and interpretations, (c) reasoning with evidence, (d) making connections, (e) considering different viewpoints and perspectives, (f) capturing the heart and forming conclusions, (g) wondering and asking questions, and (h) uncovering complexity and going below the surface of things (Ritchhart et al., 2011, p. 14). Within that framework, the three thinking categories (Ritchhart, 2001), (a) creative thinking, (b) critical thinking, and (c) mental management and awareness, were used to organize the eight thinking moves.

The thinking move framework was used to explore the thinking of the four students in two ways. First, the thinking moves displayed during the activities and on the exam questions were organized according to the eight high-leverage thinking moves. In addition, thick and detailed descriptions of the thinking moves exhibited were developed, using the actual dialogue of the students, in order to help the reader develop a solid understanding of the thinking used by each of the four students (Merriam, 2009; Patton, 2002).

Three data sources were used to reveal the student thinking: (a) the discussions during the group activities, (b) the student activity reflections, and (c) the responses to selected exam questions linked to each of the five group activities. Each of these sources of data is considered in more detail in the following sections.

Group discussions during activities. The purpose of this research study is to determine to what degree and in what ways the mathematical identity of students is related to their use of high-leverage thinking moves during problem-solving activities. In order to do this, the researcher made use of statistical activities that had the potential to reveal student thinking. Activities were selected that were suitable for completion in a group setting, which encouraged discussion with peers, which made use of real data, that focused on conceptual understanding, and that required the use of technology.

By making use of these criteria, activities were chosen that were meaningful to students and that exposed their thinking moves related to their conceptual understanding of statistics. The five activities used in the research were the: (1) *On Time Arrival* MEA (see Appendix A), (2) the *Race and the Death Penalty* activity (see Appendix B), (3) the *Backhoes and Forklifts* activity (see Appendix C), (4) the *ESP* activity (see Appendix D), and (5) *The Spread of a Rumor* activity (see Appendix E). Each of these activities is discussed in more detail in the following paragraphs (see Table 3.4 for the embedded content for each of the group activities).

Table 3.4.

Embedded Content in the Group Activities

Activity	Topic links
On Time Arrival MEA	• Outliers, mean, median • Drawing boxplots and histograms • Comparing two data sets using graphics, measures of center, and measures of spread
Race and the Death Penalty	• Two-way and three-way contingency tables
Backhoes and Forklifts	• Experimental Design
ESP	• Design and testing of a simulation
The Spread of a Rumor	• Design and testing of a simulation • Conditional probability

The *On Time Arrival* MEA provides on time arrival data for five different airlines and challenges students to make use of their knowledge of measures of center and measures of spread to rank the airlines in order of on time departures from O'Hare Airport from the most likely to have flights that depart on time to the least likely to have flights that depart on time. In addition, students are asked to develop a ranking process that can be used by the company in the future to rank airlines when similar data are available.

The *Race and the Death Penalty* activity has students engaging with data organized in a 3-way table related to murder convictions and death penalty sentences in Philadelphia categorized by race. The students wrestle, perhaps unknowingly, with Simpson's Paradox and are asked to write a newspaper article discussing the association between race and death penalty sentences in the United States. The *Backhoes and Forklifts* activity requires students to create an experimental design; students are asked to discuss possible sources of bias, perform and explain the randomization procedures, and create a diagram of their experimental design. In the *ESP* activity, students are asked to design an experiment to solve a problem and then simulate that experiment, using at least 20 trials, with a calculator or random number table. In *The Spread of a Rumor* activity, students are asked to consider a model of rumor spreading, contemplate the reasonableness of the assumptions related to the model, and then change the assumptions of the model if needed. In addition, the group members are asked to create a simulation and then simulate, both by hand and with a calculator, 20 repetitions of the experiment. Finally, the group is asked to determine how the model could be modified to investigate a measles epidemic.

The five activities were administered during Week 4, Week 6, Week 12, and Week 14, respectively. Students worked in groups of three or four on the activities and produced one final product to turn in for assessment. The group discussions were audio recorded and then transcribed. Two supporting mechanisms, the written group work and the researcher field notes produced during the activity, were used to support the audio recording of the group discussions during the activities; these documents were helpful in terms of verification and making sense of various parts of the student discussions. These written group work documents were considered to be supporting documents, because they were jointly produced by students and therefore did not independently reveal the individual thinking moves of the students; even the written work produced individually by students during group discussions was often the result of a collaborative effort. The researcher field notes were used to note any irregularities during the discussion and to catch any inconsistencies with the group discussion transcripts. The field notes also were used to record anything noteworthy in term of group dynamics.

Student activity reflection prompts. The second source of data used to explore student thinking moves was the responses to Question 3 and Question 4 of the activity reflections that were completed by students at the conclusion of each task. Question 3 was metacognitive in nature as it gave students a chance to reflect on their own thinking. Question 4 asked students to consider whether there was any thinking or problem-solving strategies that they thought of, but did not share with their group during the activity. The activity reflection prompts that students completed at the conclusion of each of the five activities are:

(1) What statistical concepts and skills did you use in this activity?

(2) Did this activity change how you think about yourself as a mathematician?

(3) Describe the thinking or problem-solving strategies that you used as part as this activity.

(4) Were there any thinking or problem-solving strategies that you thought of but did not share with your group during the activity?

Note that the first question was to get students to think back on the activity as it encouraged them to reflect on the concepts and skills used, while Question 2 was used to study mathematical identity. In addition to the transcripts of the group discussions and the responses to the reflection prompts, the responses on selected exam questions were also used as a data source. This data source is discussed next.

Exam questions. The final source of data used to explore student thinking was the student responses to selected questions on chapter exams. Selected exams had a problem or set of problems that corresponded to one of the five activities completed by students. Table 3.5 outlines how each activity is linked by topic to certain exam questions.

Table 3.5.

Exam Exercises Categorized According to Group Activities

Activity	Exam questions	Topic links
On Time Arrival MEA	Exam 1 Problems 13, 15	• Outliers, mean, median • Drawing boxplots and histograms • Comparing two data sets using graphics, measures of center, and measures of spread
Race and the Death Penalty	Exam 1 Problem 17	• Two-way and three-way contingency tables
Backhoes and Forklifts	Exam 4 Problem 12	• Experimental Design
ESP	Exam 5 Problem 11	• Design and testing of a simulation
The Spread of a Rumor	Exam 5 Problems 11, 13	• Design and testing of a simulation • Conditional probability

The problems chosen were open-ended and many required explanations or justifications; this was done to make use of problems that had the potential to reveal student thinking.

In total, three data sources were used to display student thinking: (a) the transcripts of the discussions during the group activities, (b) the student responses on the activity reflections, and (c) student responses on selected exam questions linked to each of the five group activities. In addition, four data collection instruments were used to produce data related to mathematical identity: (a) the MINT protocol, (b) the student interviews, (c) the teacher interviews, and (d) the activity reflection prompts. How each of these data sources was used in the data analysis process is detailed in the next section.

Research Design and Data Analysis

The data analysis for this case study was built on four high quality analysis principles, one general analytic strategy, and four specific analysis techniques (Yin, 2009). First, four high quality analysis principles (Yin, 2009) were interwoven throughout the analysis by insuring that: (a) all the evidence was analyzed, (b) all major rival interpretations were considered, (c) the focus remained on the most important issue of the case study, and (d) the researcher's prior and expert knowledge was used. Second, the analysis was guided by the general analytic strategy *relying on theoretical propositions* that underpin the case study (Yin, 2009). The proposition in this study, to what degree and in what ways are the narratives students construct for their mathematical identities related to the use of high-leverage thinking moves during problem-solving activities, was foundational to the research as it: (a) assisted in the selection of data to be analyzed, (b) informed case study organization, and (c) illuminated alternative explanations to be explored. Finally, two specific analysis techniques, pattern matching and cross-case synthesis, were used. To address internal validity, pattern matching was used to compare the empirically based patterns with the anticipated patterns as well as with patterns predicted by alternative explanations (Yin, 2009). Cross-case analysis was also used to explore the multiple cases in an effort to generalize and to "build abstractions across cases" (Merriam, 1988, p. 154). In this analysis, where each case was treated as an individual study, codes were used to assign meaning to the action and events found in the descriptive research data collected (Patton, 2002).

Maintaining a chain of evidence was also used as part of the case study to increase the reliability of the information (Yin, 2009). If done properly, an external observer should be able follow the derivation of evidence in either direction; the reader should be able to trace the evidence from the initial questions to the conclusion as well as from the conclusions back to the research questions. In this research study, the chain of evidence was maintained by insuring that: (a) the case study report included adequate citations; (b) the case study database outlined the evidence collected, as well as how the data were collected; (c) the circumstances of data collection were consistent with the case study protocol; and (d) the protocol and the initial research questions were meaningfully connected (Yin, 2009). For these reasons, the chain of evidence was properly maintained. In the end, four high quality analysis principles, one general analytic strategy, and four specific analysis techniques, were used as a foundation for data analysis.

In the following sections, how the data were analyzed is outlined. First, the researcher's background and prior knowledge is addressed in more detail. In addition, the methods of analysis related to current and designated mathematical identities are considered. Finally, the analysis methods of the high-leverage thinking moves used during the group activities are explored.

Researcher background. Many researchers (D'Ambrosio et al., 2013) make a strong case for positioning oneself in the research. In that regard, this researcher is an educator that has worked in the fields of mathematics, statistics, and science. He spent 22 years teaching a variety of mathematics, statistics, and science courses in grades 7-12 in the public school system. For the past seven years the researcher has worked as an Associate Professor in the Education Department at a private liberal arts university teaching a variety of courses related to mathematics, physics, technology, and statistics. He was drawn to the topics of mathematical identity and the use of various thinking behaviors in the mathematics classroom through the voices of Boaler (1997), Ritchhart et al. (2011), and Martin (2009).

One aspect that was interwoven throughout the analysis was the *use of the researcher's own prior, expert knowledge* (Yin, 2009). In this regard, the researcher has experience in all three of the major areas addressed in this research: (a) mathematical identity, (b) thinking moves in the classroom, and (c) statistical thinking. He has read about, researched, and taught about mathematical identity over the last seven years, deliberately incorporating issues of identity into his undergraduate and graduate methods of teaching mathematics courses. In addition, in the courses which he teachers he has made extensive use of Visible Thinking Routines (Ritchhart et al., 2011) and places an emphasis on what thinking looks like in that given content area. In his methods courses, he has helped preservice teachers view the use of thinking routines both through the lens of a teacher and the lens of a student. Finally, he has taught statistical concepts and ideas in a variety of environments including at the middle school level, the high school level (e.g., International Baccalaureate (IB) courses), and at the undergraduate level. In the next section the mathematical identity analysis is considered.

Mathematical identity analysis. There were four sources of data, related to mathematical identity, that were analyzed: (a) the MINT protocol responses, (b) the student interview transcripts, (c) the teacher interview transcripts, and (d) the student activity reflection responses to Question 2. In the following sections, the analysis for each of these data sources is discussed in more detail.

MINT protocol for case selection. Initially the MINT protocol was used as part of the case selection process to determine the mathematical identity of the students. This was accomplished by a thorough examination of the data using an open coding system. To assign meaning to the descriptive data collected in Section 3, codes, in the form of labels, were used to identify chunks of text based on meaning (Miles & Huberman, 1994). This researcher entered the research process with defined codes, but kept an open mind if codes needed to be modified or discarded. Because there was a single researcher, transcripts were check-coded using a bar of a 90% code-recode consistency rate (Miles & Huberman, 1994).

Initially, the data were organized into seven MINT themes: (a) the changing nature of mathematics as experienced from early childhood to now, (b) the balance between challenge and interest, (c) the role played by key figures in the formation of mathematical identity, (d) how learning mathematics compares with learning in other subjects, (e) ways of studying mathematics and working in mathematics, (f) persistence/perseverance with mathematics, and (g) mathematics as a rewarding subject (Eaton & O Reilly, 2009). The coding information in relation to the seven themes is displayed in Table A1. To illustrate how the coding was done, Anna's and Sam's responses from Section 3 (Part 1) are examined (the names have been changed and some grammar and spelling issues have been corrected to make it easier to read). In Example 1, Anna referred to mathematics as a subject that takes some time and something that requires work (see Table 6). For this reason, it was coded with the general code PWM: persistence/perseverance with mathematics. In terms of the individual code, Anna was making these comments in the context of a possible future career in nursing. As a result, this chunk of code was coded with the individual code ABLE: attention beyond experience as learner. The final result was the code PWM: ABLE.

In Example 2, because Sam described his feelings about mathematics as a rewarding subject, this response was coded with the general code MRS: mathematics as a rewarding subject (see Table 3.6). In terms of individual code, the response was coded ALE: attention to experience as a learner, due to the fact that Sam described his feelings in relationship to feedback on a class test.

Table 3.6.

Coding for MINT Protocol (Section 3 Part 1)

	Theme: general codes	Individual codes	Sample response from student
Example 1: Anna	PWM: Persistence and perseverance with mathematics	ABLE: Attention beyond experience as learner	Math seems to be a subject that clicks eventually. My career in the future, most likely nursing right now, will utilize this course. I know math is important in the future that is why I work at it. (Anna)
Example 2: Sam	MRS: Mathematics as a rewarding subject	ALE: Attention to experience as learner	One of the best feelings I've gotten from math had been spending a lot of time on a hard problem, then getting my test back and seeing that I got it right. (Sam)

After the data were organized into seven MINT themes, they were then placed into five mathematical identity categories according to strength (i.e., strong positive, positive, mixed, negative, and strong negative). This organizational structure is shown in Table 3.7; student statements are organized according to the five levels of mathematical identity strength in terms of the theme of *mathematics as a rewarding subject*. Notice that a strong negative mathematical identity is characterized by completely negative feelings and experiences related to mathematics. These feelings that are revealed in Ashley's statement, “I do not enjoy math.” In contrast, a strong positive mathematics identity is represented by Mitch’s strong, absolute statement, “I enjoy math because it makes me think critically.” Between those two extremes is the statement by Rose, “But I enjoy learning math, because the rewards of understanding and having that ‘light bulb’ moment outweigh the difficulty and confusion.” While Rose clearly had mixed feelings and experiences related to mathematics, she clearly indicated that the positives outweigh the negative aspects.

Table 3.7.

Categorizing Mathematical Identity According to the Theme Mathematics as a Rewarding Subject

Strength	Feelings and experiences	Sample response from student
Strong Positive	Positive feelings and experiences related to mathematics	• I enjoy math because it makes me think critically. (Mitch) • I think math is awesome, it’s one of my favorite subjects. (Kylie)
Positive	Mostly positive feelings and experiences related to mathematics	• Overall, I enjoy learning math, but I have to work very hard in order to understand it. It has been one of my most difficult subjects. (Rose) • But I enjoy learning math, because the rewards of understanding and having that ‘light bulb’ moment outweigh the difficulty and confusion. (Rose)
Mixed	Mixed feelings and experiences related to mathematics, some positive and some negative	• I have found that I like mathematics, yet I always feel like I cannot progress at a faster rate. (Nate) • I did not care about mathematics in middle school. As time has progressed and I continue my journey, I began to like it with more passion. (Nate)
Negative	Mostly negative feelings and experiences related to mathematics	• I find math a little fun. But mainly, I’m always struggling and getting confused by the subject. Maybe math just isn’t my thing. (Jake)
Strong Negative	Negative feelings and experiences related to mathematics	• I do not enjoy math really, I always second guess myself. (Ashley)

This discussion demonstrated how the MINT protocol was used as part of the case selection process to determine the mathematical identity of the students. In the next section, the use of the MINT protocol to develop a mathematical idenity baseline for the four individuals is explained.

MINT protocol for identity baseline. In addition to being used as part of the case selection process, the MINT survey was also used in the early part of the data analysis process to establish a baseline for the current mathematical identity and the designated mathematical identity. Once the chunks of code were categorized by the seven MINT themes, they were organized according to identity type (i.e., current identity or designated identity); transcripts were coded using the DI for designated identity and CI for current identity. Current identities are identities that are the stories and narratives of individuals that are generally told in the present tense and that reflect current circumstances (Graven & Buytenhuys, 2011; Sfard & Prusak, 2005). In contrast, designated identities are stories and narratives of individuals regarding their ambitions and the situations they expect to be in at the present time or in the future (Sfard & Prusak, 2005).

In order to more fully describe the characteristics of current and designated identities, samples of student statements are used (see Table 3.8). Current identity is illustrated in Tia's first statement, "I like math" as it is an actual assertion from her narrative about her feelings about mathematics. In contrast, when Tia explained in her narrative that, "I could see myself doing something with math in the future" as a result of the influence of her father's profession, she was describing a possible aspiration, something she perhaps expected to be the case at that time or in the future. This is an aspect of Tia's designated identity.

Table 3.8.

Students' Current and Designated Identities Arranged by MINT Themes

Identity type	Theme	Student statement
Current Identity	MRS: Mathematics as a rewarding subject	• I like math. (Tia)
Designated Identity	KF: The role played by key figures in the formation of mathematical identity	• My dad is an accountant and I could see myself doing something with math in the future. (Tia)
Designated Identity	PWM: Persistence/perseverance with mathematics	• If I want to do well in math I have to put time into math and study more. (Tia)
Current Identity	LMC: How learning mathematics compares with learning in other subjects	• Compared to other subjects, mathematics currently comes easily. (Nate)
Current Identity	KF: The role played by key figures in the formation of mathematical identity	• Although teachers like Mr. Cox taught me to like mathematics, I believe my interest in computer programming sparked the fire. (Nate)
Designated Identity	MRS: Mathematics as a rewarding subject	• I took AP statistics because it is to me the mathematics course that most pertains to my life as a whole. (Nate)

In summary, the MINT protocol was used to assist in case selection and to develop a mathematics identity baseline for the four students in regard to current and designated identity. Once the identity baselines were established, three other sources of data related to mathematical identity were used in the analysis, (a) the student interview transcripts, (b) the teacher interview transcripts, and (c) the activity reflection responses. These data sources are considered in the next section.

Interview transcripts and activity reflections. After the baseline mathematical identity constructs were formed through the analysis of the MINT protocol data, the teacher and student interview transcripts were analyzed. In addition, the activity reflection responses on Question 2 were analyzed in terms of the current mathematical identity of the students. The analysis of these data sources is explored according to the identity type (i.e., current identity, designated identity). The current identity analysis is considered first.

Current identity. The analysis of the data related to current identity was used to investigate subquestion A: To what degree and in what ways are the narratives students construct for their *current* mathematical identities related to the use of high-leverage thinking moves during problem-solving activities? The current identity analysis was done in multiple ways. Initially, the baseline for the current mathematical identity for each of the four students was developed using the MINT protocol data. A similar procedure was used to analyze the student interview data; they were organized according to the seven MINT themes and according to identity strength. This type of organization allowed for conclusions based on the distribution of statement across those two categories (i.e., MINT themes, identity strength).

Once the data were organized in this manner, rich descriptions related to the current mathematical identity of each of the four students were developed. Whenever possible these descriptions made use of the words of the students in order to create a product that was “richly descriptive” (Patton, 2002) and had the potential to deepen the reader’s knowledge (Merriam, 2009) of the current mathematical identity of each case.

To provide a structural framework for these descriptions, the MINT themes were used to organize the descriptions.

Finally, the responses of each student on Question 2 on the reflections completed at the conclusion of each task were considered in regard to their current mathematical identity. Question 2 asked the students to consider whether the activity changed how they viewed themselves as a mathematician.

The data were analyzed during the data collection process; the researcher looked for changes in the patterns over time. As a pattern emerged in one case, it was examined across the other cases. To do this, word tables were created and studied to uncover themes that appeared; once completed the collection of tables was used to develop cross-case conclusions (Yin, 2009) regarding current identities. The data analysis in terms of designated identity is considered in the next section.

Designated identity. The data analysis associated with the designated identity was used to investigate subquestion B: To what degree and in what ways are the narratives students construct for their *designated* mathematical identities related to the use of high-leverage thinking moves during problem-solving activities? To analyze the designated identity for the four students, the following data sources were used: (a) the responses on the MINT protocol, (b) the transcripts of the student interviews, and (c) the transcripts of the teacher interviews. Using these data, thorough and rich descriptions were developed to deepen the reader's knowledge (Merriam, 2009) of the designated mathematical identity of each of the four individuals; whenever possible the analyses made use of the students' own words. In addition, the themes were used as frameworks to provide structure to the descriptions.

The data were analyzed during the data collection process; the researcher looked for changes in the patterns over time. As a pattern emerged in one case, it was examined across the other cases. To do this, word tables were created and studied to uncover themes that appeared; once completed the collection of tables was used to develop cross-case conclusions (Yin, 2009) regarding designated identities.

In summary, there were four sources of data, related to mathematical identity, that were analyzed: (a) the MINT protocol responses, (b) the student interview transcripts, (c) the teacher interview transcripts, and (d) the student activity reflection responses to Question 2. In the following section, the analysis of the data associated with the thinking moves of the students is discussed.

Thinking moves analysis. To explore the thinking moves of the students, three sources of data were considered: (a) the transcripts of group discussions during the five group activities, (b) the student responses to Question 3 and Question 4 of the reflections that were completed at the conclusion of each activity, and (c) the student responses on selected exams. The analysis of each of these sources of data is discussed in this section; the analyses of the group discussion transcripts and the exam question responses are considered first.

Discussion transcripts and exam question responses. The data collected from the discussions during the group activities and from the exam question responses were analyzed using an open coding system. The high-leverage thinking moves, identified by Ritchhart et al. (2011), were used as the theoretical framework. Within that framework, the three categories described by Ritchhart (2001), (a) creative thinking, (b) mental management and awareness, and (c) critical thinking, were used to organize the thinking moves (see Table 3.9).

Table 3.9.

High-Leverage Thinking Moves Organized by Thinking Categories

Thinking category	High-Leverage thinking move
Creative thinking	• Considering different viewpoints and perspectives • Wondering and asking questions • Uncovering complexity and going below the surface of things
Mental management and awareness	• Building explanations and interpretations • Capturing the heart and forming conclusions • Making connections
Critical thinking	• Observing closely and describing what is there • Reasoning with evidence

The transcripts from discussions that took place during the first activity, the *On Time Arrival* MEA, were used to construct a baseline for the high-leverage thinking moves construct. Once the baseline for the thinking construct was determined, the researcher looked for changes in the patterns over time. As a pattern emerged in one case, it was examined across the other cases. To do this, word tables were created and examined to reach cross-case conclusions (Yin, 2009) related to the use of high-leverage thinking moves during problem-solving activities.

To illustrate how the high-leverage thinking moves framework was used, the transcript of a conversation between students as they worked through the first activity is analyzed (see Table 3.10). The analysis starts with the consideration of an exchange between Anna and Ricky. Prior to this conversation, the group had looked over the data table that lists the number of minutes late for 30 flights for each of the five airlines; the group was deciding whether to use the mean, median, or both to solve the problem (the data table for the *On Time Arrival* MEA is shown in Table E1).

Table 3.10.

Student Statements Organized by Thinking Categories and Thinking Moves

Student statements	Thinking category	High-Leverage thinking move
Anna: So, we have five airlines, should we find the average for each airline? Like the number of minutes late? Because they all have the same amount of flights.	Creative Critical	• Wondering and asking questions • Observing closely and describing what's there
Ricky: Yeah.		
Anna: So, should we add them all up for each one and then find the mean for each one? Then we will be able to see, on average, how many, how long each plane was late.		
Ricky: Yeah, that can work, um, what I am thinking however, is do we want to actually find the correct median time or do we want to find a time that can be skewed?	Creative	• Wondering and asking questions • Uncovering complexity and going below the surface of things
Anna: I think we need to find the average because, the actual average because it is not, or we could do both I guess.	Creative	• Considering different viewpoints and perspectives

In her first statement, Anna asked a question about whether the group should find the average; this was labeled as a *wondering and asking questions* thinking move which is part of the *creative thinking* category. In addition, Anna's final comment in her first statement, "Because they all have the same amount of flights," was labeled as a *observing closely and describing what's there* thinking move. This shifted Anna's thinking from the creative thinking category to the critical thinking category. Ricky, who was working in the creative thinking category, moved the conversation forward in his second statement by asking Anna, through questioning, to consider a different perspective (i.e., answer the question by using the median values). He stated, "... do we want to actually find the correct median time or do we want to find a time that can be skewed?" At the end of this exchange, both Anna and Ricky were working in the creative thinking category. Ricky started to *uncover the complexity of the problem* with his statement that showed the understanding that the mean is not resistant to outliers, while Anna, who stated that, "or we could do both I guess," was open to considering Ricky's perspective.

In addition to the analyses of the transcripts from the group discussions during the activities and the exam question responses, the activity reflection responses were also analyzed. This analysis is discussed in the next section.

Activity Reflection Responses. The student responses to Question 3 and Question 4 of the activity reflections that were completed by the students at the conclusion of each statistical activity were used to confirm and supplement the discussion transcript data and the exam responses data. Question 3, *Describe the thinking or problem-solving strategies that you used as part of this activity*, was used to verify if the thinking moves and problem-solving strategies used by the student matched what the student thought they had contributed during the activity discussions. Question 4, *Were there any thinking or problem-solving strategies that you thought of, but did not share with your group during the activity?*, was used to confirm that the student had shared all their thinking and problem-solving approaches related to the activity during the group discussion.

In summary, through the data analysis related to the thinking moves of the students, the researcher attempted to create a product that was "richly descriptive" (Patton, 2002) and had the potential to deepen the reader's knowledge of the phenomenon

(Merriam, 2009). To do this effectively, the analyses included rich descriptions of the data as well as the students' own words.

In addition, using the baseline for the thinking construct that was determined from the transcripts of the student discussions during the first activity, the researcher looked for changes in the patterns over time through examination of the word tables; as a pattern emerged in one case, it was examined across the other cases. As a result, cross-case conclusions related to the use of high-leverage thinking moves during problem-solving activities were formed.

This section provided an overview of the data analysis related to the mathematical identity and the thinking moves of the four students. Once the analyses were complete related to these two phenomena, the results were considered in terms of the research question. This is discussed in the next section.

Overall Analysis

During the initial portion of the research, the baseline constructs were developed for the current mathematical identity, the designated mathematical identity, and for the use of high-leverage thinking moves construct. Once the baseline constructs were created, the researcher examined the patterns in regard to the two research subquestions:

A. To what degree and in what ways are the narratives students construct for their *current* mathematical identities related to the use of high-leverage thinking moves during problem-solving activities?

B. To what degree and in what ways are the narratives students construct for their *designated* mathematical identities related to the use of high-leverage thinking moves during problem-solving activities?

As data collection and data analysis continued, when patterns emerged in one case, it was examined across the other cases. Word tables were created and examined to reach cross-case conclusions (Yin, 2009) in an effort to determine the degree to which and in what ways the narratives students construct for their mathematical identities were related to the use of high-leverage thinking moves during problem-solving activities. As a result, conclusions were reached related to the overarching question and the two subquestions. The generalizability of those conclusions is addressed in the next section.

General Nature of the Conclusions Justifiable From this Research

Although issues of construct validity, internal validity, external validity, and reliability have all been addressed throughout Chapter 3, external validity, the extent to which conclusions are generalizable, is now considered in more detail. In many case studies, the results found and the conclusions made are not generalizable (Cohen, Manion, & Morrison , 2007). Moreover, case studies can be biased, selective, and subjective as cross-checking can be difficult; even if reflexivity is addressed, observer bias can be a problem (Cohen et al., 2007). Furthermore, case study results offer an inadequate basis for statistical generalization, and if not designed correctly, case studies can lack rigor (Yin, 2009).

That being said, issues of external validity can be addressed through research design. First, if conclusions are to be generalizable, research must be designed, as the present research student is, in such a way that the notion of internal validity is adequately addressed (Guba & Lincoln, 1985). If internal validity is present, case studies, if planned correctly, can achieve analytic generalization through replication logic in multiple case studies (Yin, 2009). In this research study, elements of external validity, as stated early, were included in the research design in terms of literal replication and theoretical replication. If results are similar for all cases, there will be a strong case for the initial propositions. On the other hand, if there are conflicting results, a different set of cases need to be used to retest the propositions. There is an additional approach that was incorporated in this present research study in an effort to increase the external validity of case study research: the use of “thick” descriptions, through detailed descriptions of findings and results, which allow readers to determine if the transfer to their context is appropriate (Guba & Lincoln, 1985). These issues of validity, reliability, and generalizability will be discussed further, in a contextual manner, in the next two chapters.

Summary

This chapter covered the research design, the data collection plan, and the data analysis plan for the present research. Initially, the participants, the setting, and the data sources were discussed. Next, the various data collection instruments were described. In terms of mathematical identity, this included the MINT Protocol, the three teacher interviews, the two student interviews, and the student reflections at the conclusion of each activity. In regard to the thinking moves of students, this included group discussions during the five group activities, the student reflections at the conclusion of each activity, and the responses on selected exams questions. That was followed by an explanation of how each of the data sources were analyzed and the approach used to determine the degree to which and in what ways the narratives students construct for their mathematical identities are related to the use of high-leverage thinking moves during problem-solving activities. Finally, validity, reliability, and the generalizability of the research were discussed. The conceptual framework for the current research is summarized in Figure 3.2.

Figure 3.2. Conceptual Framework for Research

Association between

The mathematical identity of students	The thinking moves used during problem solving activities

Math Identities	
	Actual identities: indentities in the form of stories told in the present tense about the actual state of affairs.
	Designated identities: indentities in the form of stories told in the future tense or in a way that expresses aspriations.

High-Leverage Thinking Moves	
	Eight high-leverage thinking moves
	The thinking categories: creative thinking, mental management and awareness, and crtical thinking.

Math Identities Data Collection	
	MINT Protocol
	Two interviews of each of the four cases
	Three interviews of the teacher
	Student activity reflections completed at the conclusion of each activity

Thinking Moves Data Collection	
	Discussions during five group activities
	Student activity reflections completed at the conclusion of each group activity
	Responses on selected exams questions (tied to each of the five group activities)

Math Identities Data Analysis	
	Constant comparative analysis
	Open coding
	Pattern matching and cross-case analysis (Yin, 2009)

Thinking Moves Data Analysis	
	Constant comparative analysis
	Open coding and analysis using the thinking moves
	Pattern matching and cross-case analysis (Yin, 2009)

Overall conclusions regarding the degree to which and the ways in which the narratives that students construct for their mathematical identities are related to the use of high-leverage thinking moves during problem-solving activities.

Figure 3.2. Conceptual framework used to research the associations between mathematical identity and thinking moves.

This chapter outlined the research methodology, including the data collection instruments, the data collection procedures, the research design, and the data analysis methods. In the following chapters the various aspects of the data analysis are described and the results of the data analysis are shared. In Chapter 4 the data related to the mathematical identity and the use of the high-leverage thinking moves are analyzed in regard to each of the four cases. This case analysis lays the foundation for the cross case analysis and the cross case discussion found in Chapter 5. Finally, Chapter 6 contains the limitations, the implications, and the areas of possible future research.

Chapter 4

Case Analysis and Results

The current research examined the association between an individual's use of high-leverage thinking moves used during problem-solving activities and their current and designated mathematical identities. One objective is to unpack student understanding through the thinking moves exhibited during the completion of group activities and exams. A second objective of the study is to shed light on how individuals interpret and make meaning of their experiences in regard to their mathematical identity; this phenomenon is studied through the narratives of the individuals. This chapter analyzes the thinking moves and the mathematical identity data of the students. First, the data related to the thinking moves of the four students are analyzed. Next, the data associated with the mathematical identity of each of the four students are examined. In each instance, the analyses are organized according to four cases (i.e., Anna, Erik, Mitch, and Ashley; these names are pseudonyms). These analyses lay the foundation needed to explore the research question and the two research subquestions; the cross case analysis and discussion are found in Chapter 5.

Thinking Moves Analysis

The use of the high-leverage thinking moves as research framework was discussed in Chapter 2. As a reminder, many experts maintain understanding is the result of a complex process that requires various and diverse forms of thinking (Ennis, 1996; Facione et al., 1994; Ritchhart et al., 2011; Tishman et al., 1993). Accordingly, the eight high-leverage thinking moves (Ritchhart et al., 2011) are used as the theoretical framework. Within that framework, the three categories (Ritchhart, 2001), (a) creative thinking, (b) critical thinking, and (c) mental management and awareness, are used to organize the eight thinking moves (see Figure 4.1 for the organization of the thinking moves within the three thinking categories).

Figure 4.1. Organization of Thinking Moves According to Thinking Categories

Thinking category	Thinking move
Creative thinking	• Considering different viewpoints and perspectives • Wondering and asking questions • Uncovering complexity and going below the surface of things
Mental management and awareness	• Building explanations and interpretations • Capturing the heart and forming conclusions • Making connections
Critical thinking	• Observing closely and describing what's there • Reasoning with evidence

Figure 4.1. Organization of thinking moves according to thinking categories

This section investigates the thinking moves of the four students through the analysis of the group discussions during five activities and the student responses to selected exam questions (see Chapter 3 for a more detailed explanation of this methodology). The thinking moves framework is used to explore the thinking moves data in two ways. First, the thinking moves displayed during the activities and on the exam questions are organized according the eight thinking moves and the three thinking categories. In addition, thick and detailed descriptions of the student thinking moves are compiled, using the actual dialogue of the students, in order to help the reader develop a solid understanding of the thinking displayed by each of the four students (Merriam, 2009; Patton, 2002).

In the first subsection, the transcripts of the discussions for each of the group activities are considered. The five activities are: (1) *On Time Arrival* MEA (see Appendix A), (2) the *Race and the Death Penalty* activity (see Appendix B), (3) the *Backhoes and Forklifts* activity (see Appendix C), (4) the *ESP* activity (see Appendix D), and (5) *The Spread of a Rumor* activity (see Appendix E). The purpose of these group activities was to reveal student thinking. Activities were selected that were suitable for completion in a group setting, which encouraged discussion with peers, which made use of real data, that focused on conceptual understanding, and that required the use of technology. As a result, the activities chosen for the current research had the potential to expose student thinking moves related to their conceptual understanding of statistics. In the second subsection, the student thinking moves on the exam questions are explored. The problems selected were open-ended, and many required explanations or justifications, in order to make the thinking moves of the students visible. In addition, the exam questions were chosen so that each activity was linked to one or two of the questions. The thinking moves of Anna are considered first.

Anna. This section investigates the thinking moves of Anna through the analysis of the group discussions during the five group activities and the student responses on selected exam questions.

Group activities. In this section, Anna's contributions during the group activities are used to explore her thinking moves. Overall, Anna only made partial use of the eight high-leverage thinking moves during the various group activities. Through the five activities, she used a total of 13 high-leverage thinking moves; four of 8 thinking moves were represented. In addition, the majority of the thinking moves used by Anna resided in the *creative thinking* category (i.e., 9 out of 13). Furthermore, her most commonly used thinking move, wondering and asking questions, was used a total of five times. Finally, none of the thinking moves in *the mental management and awareness* category (i.e., building explanations and interpretations, capturing the heart and forming conclusions, and making connections) were used by Anna (see Table 4.1 for a summary of Anna's use of high-leverage thinking moves during the activities).

Table 4.1.

Anna: High-Leverage Thinking Moves During Group Activities Summary

Category	High-Leverage thinking move	Total
Creative thinking	• Considering different viewpoints and perspectives	1
	• Wondering and asking questions	5
	• Uncovering complexity and going below the surface of things	3
Mental management and awareness	• Building explanations and interpretations	0
	• Capturing the heart and forming conclusions	0
	• Making connections	0
Critical thinking	• Observing closely and describing what is there	4
	• Reasoning with evidence	0

Selected conversations during the group activities are used to explore Anna's thinking moves in a more thorough manner. This analysis starts with the consideration of a conversation between the group members Anna, Jake, and Mitch as they worked through the *On Time Arrival* MEA (some grammar issues have been corrected in the transcripts to make the text easier to read). The *On Time Arrival* MEA provides on time arrival data for five different airlines and challenges students to make use of their knowledge of measures of center and measures of spread to rank the airlines in order of on time departures from O'Hare Airport (i.e., from the most likely to have flights that depart on time to the least likely to have flights that depart on time). In addition, students are asked to develop a ranking process that can be used by the company in the future to rank airlines when similar data are available.

Anna made use of the thinking moves *observing closely and describing what's there* and *wondering and asking questions* in the following exchange. Prior to this, the group had looked over the data table that lists the number of minutes late for 30 flights for each of the five airlines; the group is deciding whether to use the mean, median, or both to solve the problem. In addition to making a decision about what measure of center to use, the group was also trying to decide deal with five lists of 30 numbers each. In this regard, Anna, using the high-leverage thinking move *wondering and asking questions* that resides in the *creative thinking* category, asked the question "And are we doing box and whisker plots?" Ritchhart et al. (2011) argue that questioning can propel learning and understanding; in this case Anna's question assisted the group in finding a way to visually represent the data using a statistical graph. In the end, this question was important for two reasons. First, it got the group to represent the data in a visual manner which was essential in this case based on the fact that while all five data sets have approximately the same mean, the distributions are very different. In addition, the type of statistical graph used has the potential to impact how the problem is unpacked and analyzed. Anna's question is shown in the second line in the exchange below.

Mitch: Are you guys going to use your calculators, or are you just going to…

Anna: Yeah. And are we doing box and whisker plots?

Jake: Yeah.

Anna: Because then on the graph…

Jake: We can show it on here.

Anna: We can show it, whatever is more pushed over from zero.

Jake: Yeah.

Anna: So we should all use the same distance on the graph.

Jake: Yeah, window.

Anna also displayed the thinking move *observing closely and describing what's there* on two different occasions during this exchange. In the later part of the conversation, Anna stated, "We can show it, whatever is more pushed over from zero." This statement displayed the use of the thinking move *observing closely and describing what's there*, and shifted Anna's thinking to the *critical thinking* category. A critical component of understanding a concept is noticing and describing the various parts and features of the concept under consideration (Ritchhart et al., 2011). In this case, Anna pointed out that box and whisker plots would reveal the different distributions of the data sets and would be helpful in ranking the airlines in terms of on time arrival. This same thinking move is exhibited in her last statement of the exchange, "So we should all use the same distance on the graph." In this instance, Anna made the point that it is important to have a common window on the graphing calculator. This observation shared by Anna was critical in terms of getting the team to realize that in order to effectively compare the box plots of the five airlines, it was necessary to plot them using a common window on the graphing display calculator. In both these examples, Anna noticed and described key features of the ideas being explored.

Anna also used this same thinking move, *observing closely and describing what's there*, later in the discussion during the *On Time Arrival* MEA (this exchange is shown below). As can be seen in his statement, "I don't think that can be right though," Jake was struggling with the idea that the median of one of the airline data sets could be zero. In response, Anna stated, "They have a lot of zeros." In this instance, Anna noticed and described an important feature (i.e., a large percentage of the 30 numbers were zeros) of the data set that was being examined. It should be noted that Anna's observation was possible because of her knowledge that the median in this instance is the middle value of an ordered data set. The exchange between Anna and Jake is shown below.

Jake: I don't think that can be right though. Because, I mean, I guess it . . .

Anna: They have a lot of zeros.

Jake: Yeah, so the middle number could be zero.

In the exchange above, Anna's thinking move may have helped Jake realize that a median of zero makes sense in this case.

The patterns seen related to Anna's use of the high-leverage thinking moves in the *Race and the Death Penalty* activity, were similar to the ones observed in the *On Time Airline* activity. In this activity, the group examined the data regarding the number of murder convictions that resulted in the administration of the death penalty in the Philadelphia courts from 1893-1993; the data were organized in a three-way table in terms of the race of the defendants and the race of the murder victims. The task was to use these data to determine if the justice system in the Philadelphia court system was colorblind in the administration of the death penalty during the given time period.

In the exchange below, Anna again displayed the thinking move *observing closely and describing what's there*. In this instance, the group was contemplating whether the use of a two way bar graph or a segmented bar graph was more appropriate. In the fourth line of the exchange, Anna made the observation that more comparisons were needed to address the question being studied; she noticed and described an important aspect related to the understanding of the complex relationships involved. Specifically, Anna was referring to the fact that two comparisons (i.e., White defendants and White victims, Black defendants and Black victims) were not being considered. The dialogue between the four students is shown below.

Courtney: Okay yeah I think we should do one of those two way …

Anna: But then the two way bar graph …

Ashley: But that's not going to even out with numbers though.

Anna: No, the segmented bar graph isn't going to do anything because if we are not including the Whites killed by Whites and the Blacks killed by Blacks then it's not really doing anything for us.

Erik: Yeah.

In this instance, Anna's thinking (i.e., observing closely and describing what's there) helped the group realize why using the segmented bar graph in this manner was not an effective way to represent the data.

In the next exchange, Anna exhibited the thinking moves *considering different viewpoints and perspectives* and *observing closely and describing what's there.* In this conversation, the group was still trying to determine whether the use of a segmented bar graph was an appropriate way to visually represent the data in this case; there are two different perspectives being offered. On one hand, Courtney was arguing that a segmented bar graph would be appropriate (see the first line of the dialogue). In contrast, Ashley viewpoint was that the use of a segmented bar graph was not appropriate due to the fact that the percentages do not add up to 100%. With two different perspectives on the table, Anna was able to consider both viewpoints as part of her thinking process (i.e., considering different viewpoints and perspectives). Anna stated, "Yeah, they don't add up to 100%. It would be like 25.2%." In this case, Anna realized that it did not make sense to use a segmented bar graph in this instance. In addition, she noticed and described the exact percentage (i.e., 25.2%) to support the case made by Ashley; this observation was essential in understanding a key aspect of the creation and use of a segmented bar graph (i.e., observing closely and describing what's there). The exchange between the four students is shown below.

Courtney: No, it's not that though. I'm saying if we had the segmented bar graph and we had the Black defendant and the White defendant.

Erik: Oh, okay.

Courtney: And then we had them both go up.

Ashley: But is it not even 100%, because it doesn't add up to 100%.

Anna: Yeah, they don't add up to 100%. It would be like 25.2%.

In the above exchange, Anna displayed thinking moves in the *creative thinking* and *critical thinking* categories.

Overall, Anna only made partial use of the eight high-leverage thinking moves during the five group activities. The four moves used by Anna were (a) considering different viewpoints and perspectives, (b) wondering and asking questions, (c) uncovering complexity and going below the surface of things, (d) observing closely and describing what is there; the most commonly used thinking move, wondering and asking questions, was used a total of five times. In addition, almost 70% of the thinking moves constructed by Anna resided in the *creative thinking* category.

To further explore the thinking moves, Anna's responses on selected exams questions were analyzed and are considered next.

Exam question responses. The thinking moves of the students were also studied by examining their responses on selected exam questions. Anna made use of a number of different high leverage thinking moves when completing the selected exam questions (see Table 4.2). For example, Anna used three different thinking moves on Problem 12 (Exam 4). First, Anna was able to create a block design by *uncovering complexity and going below the surface of things.* In addition, she *built explanations and interpretations* in her answer *making connections* between the context of the problem and the statistical procedures and tools.

Table 4.2.

Anna: Answers to Selected Exam Problems

Exam problem	Anna's answers	Thinking move (thinking category)
Exam 4 (#12c)	**Question:** Assume that 600 men and 500 women suffering from high blood pressure are available for the study. Describe a design for the experiment. Be sure to include a description of how you assign individuals to the treatment group. (Starnes, Yates, & Moore, 2011) **Answer:** First, I would block my subjects by gender in order to take their initial BP and randomize into two groups labeling men 001-600 and women 01-500. I would then use Table 2 and pick a line before I open the table and write that down and choose the first 300 distinct numbers between 001-600 to be the men using treatment of ACE and the remaining men to be the control or placebos to decrease the effects of lurking variables and be more credible. Then I would do the same randomization for women, but first 250 numbers, no repeats between 01-250. Use ACE and remove to be control. Then I would take second BP and compare results.	• Uncovering complexity and going below the surface of things (creative thinking) • Building explanations and interpretations (mental management and awareness) • Making connections (mental management and awareness)
Exam 5 (#11c,d)	**Question 11c**: You want to estimate the probability that exactly three of the next four calls are for medical assistance. Describe the design of a simulation to estimate the probability. Explain clearly how you will use the partial table of random digits below to carry out your simulation. (Starnes et al., 2011) **Answer 11c:** I would first label 2 charts: numbers 01-81 to be medical calls and 82-00 other calls. Then choosing a line in Table D to collect data before I open it up. Looking at the line (repeats okay) boxing the first four two digit numbers. After those numbers are selected I would look and see what category they fit in (medical or non-medical) from labeling above. Next, I would put results as a fraction then repeating this simulation to see out of 10 trials how many sets had exactly ¾ medical calls and compare. **Question 11d:** Carry out five trials of your simulation. (Starnes et al., 2011) **Answer 11d:** 60% chance that exactly the next three out of four calls will be medical.	• Uncovering complexity and going below the surface of things (creative thinking) • Building explanations and interpretations (mental management and awareness) • Making connections (mental management and awareness) • Capturing the heart and forming conclusions (mental management and awareness)
Exam 5 (#13)	**Question:** Suppose you school is in the midst of a flu epidemic. The probability that a randomly-selected student has the flu is 0.35, and the probability that a student who has the flu also has a fever is 0.90. But there are other illnesses making the rounds, and the probability that a student who doesn't have the flu does have a high fever (as a result of some other ailment) is 0.12. Suppose a student walks into the nurse's office with a high fever. What is the probability that she has the flu? (Starnes et al., 2011) **Answer:** [Researcher's comment: the tree diagram was drawn correctly] 0.315/0.393 = 80.2% chance	• Uncovering complexity and going below the surface of things (creative thinking) • Observing closely and describing what's there (critical thinking) • Building explanations and interpretations (mental management and awareness)

In Problem 11 (Exam 5), Anna was able to create and use a simulation to estimate the probability that exactly 3 of the next 4 calls were for medical assistance. She *built explanations and interpretations* due to the fact that she was able to *uncover the complexity* of the problem and *made a connection* between the simulation and the context of the problem. As a result, she was able to *capture the heart and form a conclusion* related to the question. In all, four different thinking moves were displayed in answering this question.

Finally, in Problem 13 (Exam 5), through the creation of a statistical representation Anna displayed three different thinking moves. First, she was able to *uncover the complexity and go below the surface of things* as she sifted through the numbers and organized them in a helpful manner. In addition, by *observing closely*, she was able to create a visual statistical representation, a tree diagram, to *describe what was there*. Finally, Anna used the tree diagram to interpret the data, using the thinking move *building explanations and interpretations*.

On these selected exam questions, Anna displayed a total of 11 thinking moves, five different thinking moves, and four new thinking moves as compared to her group thinking moves during the group activities. In addition, her thinking moves resided in all three thinking categories.

Summary. During the statistical activity discussion, Anna's thinking moves were in the creative and critical thinking categories. Perhaps one of the reasons that none of the thinking moves displayed by Anna were part of the third thinking category of *mental management and awareness* was because she did not verbally link the statistical knowledge and logical thinking to the data in context (It should be noted that this could also be because others in the group verbalized this before Anna had a chance). For example, during the *On Time Airline* MEA Anna did not contributed to the discussion regarding what the outliers mean to travelers in terms of their ability to make connections or why it makes sense to use the mean or the median in that context. According to Pfannkuch and Rubick (2002), ". . . the integration of statistical knowledge and contextual knowledge is an identifiable element of statistical thinking" (p. 5). Based on that premise, one might wonder whether Anna exhibited any signs of statistical thinking during this activity.

The thinking moves displayed by Anna when completing the exam questions were different; she used thinking moves that resided in all three thinking categories. For this reason, her thinking was statistical in nature as it displayed the integration of statistical knowledge and contextual knowledge (Pfannkuch & Rubick, 2002) and demonstrated the appropriate use of statistical knowledge and procedures (delMas, 2004). In the end, while Anna did make use of thinking moves that displayed statistical thinking during the group activities, there is evidence to suggest that she did think statistically during the completion of the selected exam questions. The thinking moves of Erik are explored in the next section.

Erik. This section investigates the thinking moves of Erik through the analysis of the discussions during the group activities and Erik's responses to selected exam questions. Each of these sources of data are analyzed using the high-leverage thinking moves (Ritchhart et al., 2011) and three thinking categories (Ritchhart, 2001), (a) creative thinking, (b) mental management and awareness, and (c) critical thinking. First, the data collected during the group activities are analyzed.

Group activities. In this section, Erik's contributions during the group activities are used to explore his thinking moves. Overall, Erik displayed the eight high-leverage thinking moves during the various group activities. Through the five activities, Erik displayed a total of 24 thinking moves; the thinking moves were spread across all three thinking categories. Specifically, eight of the thinking moves displayed were in the *creative* category, 12 of the moves resided in the *mental management and awareness* category, and four of the moves fell in the *critical thinking* mode. In addition, two thinking moves were only used once: (a) considering different viewpoints and perspectives, and (b) reasoning with evidence. A summary of Erik's high-leverage thinking moves used during the activities is displayed in Table 4.3.

Table 4.3.

Erik: High-leverage Thinking Moves During Group Activities Summary

Category	High-Leverage thinking move	Total
Creative thinking	• Considering different viewpoints and perspectives	1
	• Wondering and asking questions	2
	• Uncovering complexity and going below the surface of things	5
Mental management and awareness	• Building explanations and interpretations	5
	• Capturing the heart and forming conclusions	4
	• Making connections	3
Critical thinking	• Observing closely and describing what is there	3
	• Reasoning with evidence	1

Selected conversations during the group activities are used to explore Erik's thinking moves in a more thorough manner; Erik's work on the *On Time Airline* MEA is considered first. For the *On Time Airline* MEA, Erik was working with Siiri, Sam, and Jordan. In the statement shown below, Erik displayed three thinking moves: (a) *observing closely and describing what's there,* which is in the critical thinking category, (b) *uncovering complexity and going below the surface of things*, which resides in the creative thinking category, and (c) *making connections*, which falls in the mental management and awareness category. At this point in the discussion, the group was considering how to make sense of the data sets that list the number of minutes late for 30 flights for each of the five airlines.

Erik: So, I think what we should do is, I think we should instead of averaging it out as the minutes late, ignore that. Because it doesn't matter how late they are after a certain point, because you're going to miss your flight. And all you care about is if you make your flight or miss your flight.

In the first part of the statement, Erik argued that the group should rule out simply averaging the numbers due to the fact that taking the average of the entire list does not make sense in this case. Erik explained that if you are late enough to miss your connecting flight, it does not really matter, within reason, how many more minutes you are late. Erik's comments reflect a variety of thinking moves. First, Erik was using informal logical thinking to consider the data in their context; the contextual knowledge in this instance is the knowledge about the how late the flight can be and still be considered on time in a practical sense. In doing so, he was able to *make connections* between the numbers in the data sets and the context of airline travel. In general, making connections are critical in the thinking process as that process helps individuals tie pervious knowledge and experiences to current situations and applications (Ritchhart et al., 2011).

Because he was able to make these connections, Erik noticed and described an important aspect of the problem under investigation (i.e., observing closely and describing what's there). In this regard, he explained that after some point the lateness of an individual no longer matters in terms of making a connecting flight. In addition, Erik was able to recognize the fact that finding the average for each data set would not be helpful in this case due to the context of the problem (i.e., uncovering complexity and going below the surface of things). This recognition, that sometimes it is important to search for the complexity that lies underneath the surface rather than take a simplistic approach, is an essential aspect of developing understanding (Ritchhart et al., 2011).

Later in the discussion of the *On Time Arrival* MEA, Erik displayed the thinking move *building explanations and interpretations*. This can be seen in his three statements in the exchange below in which Erik shared an idea for a model to represent the situation. In this instance, Erik used his contextual knowledge about what it means to be late in regard to catching a connecting flight to help build an explanation and interpret the problem presented in the activity. For example, he coded data that represented a flight 10 or more minutes late with a "1" and flights less than 10 minutes late with a "0." This is shown in the dialogue below.

Erik: So, I would assign the scores that are late with a one and a score that is on time with a score of zero just so it's easier to calculate.

Siiri: And then average out . . .

Erik: And then average that out. But, I would make the cut off point different than zero because if a plane is 10 minutes late you aren't going to miss your flight

Sam: Yeah.

Erik: So I would make the cut off like 10 minutes so any score below 10 minutes late you don't miss your flight so that gets a score of zero. Anything more than 10 minutes late you give that a score of 1 because you miss your flight.

As the group continued to work through the *On Time Airline* MEA, Erik displayed the thinking move *building explanations and interpretations* and also made use of the thinking move *capturing the heart and forming conclusions*. In his statements, Erik is sharing explanations and interpretations related to the system the group created to solve the problem. For example, in the first statement Erik developed an explanation related to the system that that the group had created. He stated that it, "controls for outliers and finds consistently the best service." In addition, later in the exchange he shared a model that he had developed that would work for the company, or another company, to deal with a similar situation in the future. In essence, using his statistical and contextual knowledge Erik was able to build an explanation and interpretation related to the question under examination. Interestingly, in their work with the *On Time Airline* MEA, Chamberlin and Moon (2005) indicated that students generally found more creative models when they took a more liberal approach to defining an on time flight (i.e., as did Erik's group), than the students who defined on time flights as zero minutes late (see the two exchanges below).

Erik: The goal is to avoid missing meetings. We created a system that controls for outliers and finds the consistently best service. We assigned flights that were late less than 10 minutes a score of zero.

A few moments later:

Sam: Yeah same, what did you come up with?

Erik: $\dfrac{(1)(NL) + (0)(NO)}{NL+NO}$

Jordan: Write it down.

Siiri: What's that?

Erik: It's one times the number of late, plus zero times the number of on time, over the number of on time plus the number of late.

Taken together, Erik's statements displayed the thinking move *capturing the heart and forming conclusions*. In creating a model that links the statistical knowledge and contextual knowledge in a way that controls for outliers and defines a late flight in a contextually relevant manner, Erik demonstrated that he was aware of the big ideas in play and understood the essence of the problem. These aspects are essential in regard to demonstrating the thinking move *capturing the heart and forming conclusions* (Ritchhart et al., 2011).

Finally, Erik's use of four thinking moves during the ESP activity is explored: (a) building explanations and interpretations, (b) reasoning with evidence, (c) making connections, and (d) capturing the heart and forming conclusions. In this task, an individual who claims that he has ESP tries to correctly match 10 envelopes, each with a name card in them, with the 10 individuals. The group was responsible for creating a simulation to help with the investigation the given problem. At this point in the discussion, the group was trying to determine whether the individual really does have ESP by looking at how many correct matches need to be made in order for the result to be statistically significant. In his three statements below, Erik was helping the group understand that, in general, certain results are not unusual and others are if certain randomization principles are considered. For example, Erik used the simulation results (i.e., reasoning with evidence) to build an explanation that, in this case, four matches or above may be unusual considering chance and the fact that some other factor may be in play (e.g., the person in question may have ESP). In doing so, he was *building explanations and interpretations* related to the activity (see exchange below).

Erik: Wouldn't we just, couldn't we actually just say he needs four because there's less than a 5% chance he gets four, but there's exactly a 5% chance he gets three.

Mitch: Wait, what?

Sarah: I don't get that.

Erik: He would need to get four in order to have ESP and it would need to be repeatable. Because there is less than a 5% chance that he gets four so there is some chance he gets four.

Sarah: But how are we doing the math of the 5%?

Erik: One over 20. So there's one dot on the 3, divided by 20 dots total. There is a 5% chance he gets three. So if he got three it would just be on the border of not being statistically significant. So anything more than three and it's statistically significant.

Taken together, Erik's comments indicate that he displayed the thinking move *capturing the heart and forming conclusions*. First, his comments captured the essence of the exercise in regard to statistical significance and matters of chance. In addition, he was able to form a conclusion about how many correct matches (i.e., four) the individual would have to make in order to be labeled as an individual who has ESP. In the end, the combination of these thinking moves is evidence that Erik was thinking statistically. For example, Erik used the data from the simulation to *reason with evidence* about statistical significance. Erik's thinking also showed that he was able to *make important connections* between the context of the problem and statistical knowledge and procedures. This evidence, taken together, suggests that Erik may have been engaging in statistical thinking (delMas, 2004; Pfannkuch & Rubick, 2002).

Summary. During the group activities, Erik displayed a total of 24 thinking moves; all eight high-leverage thinking moves were represented. The two most common thinking moves used by Erik *were uncovering complexity and going below the surface of things* and *building explanations and interpretations*. In regard to the thinking categories, half of the moves resided in the *mental management and awareness* category, a third of the thinking moves displayed were in the *creative* category, and a sixth of the moves fell in the *critical thinking* mode.

To further explore the thinking moves, Erik's responses on selected exams questions were analyzed and are considered next.

Exam question responses. The thinking moves of the students were also studied by examining their responses on selected exam questions; the exam questions were chosen so that each activity was linked to one or two of the questions. Erik made use of a variety of thinking moves when working on the selected exam problems (see Table 4.4). For example, on Problem 17 (Exam 1), Erik made use of three high-leverage thinking moves: (a) observing closely and describing what is there, (b) uncovering complexity and going below the surface of things, and (c) building explanations and interpretations. To build a solid explanation of the relationship between age and opinion, he noticed and unpacked subtle and complex aspects in the data that allowed him fully describe the situation.

Table 4.4.

Erik: Answers to Selected Exam Problems

Exam problem	Erik's answers	Thinking move (thinking category)
Exam 1 (#17)	**Question:** [Using the given two-way contingency table], discuss the relationship between age and opinion of the band in two or three sentences. (Starnes et al., 2011) **Answer:** When evaluating the relationship between age and opinion of the band coastal surface, one finds 15-18 year olds are the most likely to like coastal surface, also people in the 19-22 range have the most likely people to dislike the band and have the lowest level of uncertainty. There is not a large difference in liking the band from 11-14 and 19-22 year olds. Uncertainty does not change significantly from 11-14 to 15-18 year olds.	• Uncovering complexity and going below the surface of things (creative thinking) • Observing closely and describing what's there (critical thinking) • Building explanations and interpretations (mental management and awareness)
Exam 4 (#12c)	**Question:** Assume that 600 men and 500 women suffering from high blood pressure are available for the study. Describe a design for the experiment. Be sure to include a description of how you assign individuals to the treatment group. (Starnes et al., 2011) **Answer:** In this experiment men and women will be blocked in different groups. At Point one on both diagrams men and women will be randomly assigned, they will be assigned by being labeled in the range 001-600 for men and 001-500 for women. The first 300, or 250, numbers without repeats in table D will be taken. At Point 2 the real treatment will be assigned and at Point 3 the Placebo will be assigned. At Point 4 a person will take the groups blood pressure, this tester will not know who received the treatments. Then finally at Point 5 data will be collected and analyzed.	• Uncovering complexity and going below the surface of things (creative thinking) • Building explanations and interpretations (mental management and awareness) • Making connections (mental management and awareness)
Exam 5 (#11c,d)	**Question 11c**: You want to estimate the probability that exactly three of the next four calls are for medical assistance. Describe the design of a simulation to estimate the probability. Explain clearly how you will use the partial table of random digits below to carry out your simulation. (Starnes et al., 2011) **Answer 11c:** I would assign the numbers 00 through 80 to mean medical assistance call. The numbers 81-99 would mean not medical assistance. I would take these numbers with repeats because each event is independent. I would then pick the first four numbers in the range out of the random digit table. I would repeat this five times. **Question 11d:** Carry out five trials of your simulation. (Starnes et al., 2011) **Answer 11d:** Based off my simulations I found there is a 0.6 chance that ¾ of the calls will be for medical assistance.	• Uncovering complexity and going below the surface of things (creative thinking) • Building explanations and interpretations (mental management and awareness) • Making connections (mental management and awareness) • Capturing the heart and forming conclusions (mental management and awareness)

In addition, on Problem 12 (Exam 4), Erik made use of three different thinking moves. First, his ability to *uncover complexity and go below the surface of things* helped him select and create a block design. As he worked through the problem he exhibited the thinking move *building explanations and interpretations*. In addition, he was able to *make connections* between the contextual knowledge and the statistical knowledge and procedures. Furthermore, in Problem 11 (Exam 5), Erik designed and ran a simulation to estimate the probability that exactly three of the next four calls are for medical assistance. As part of this, Erik displayed four different thinking moves. In order to create the simulation, Erik *uncovered the complexity and went below the surface of things* to *build explanations and interpretations* related to the question. In addition, Erik was able to *make connections* between the contextual knowledge and the statistical knowledge and as a result was able to *capture the heart* and *form a conclusion* based on the simulation.

Overall, Erik displayed thinking moves in all three categories during the completion of the selected exam questions. In all, he made use of a total of 11 thinking moves and five different thinking moves. The three thinking moves not used were considering different viewpoints and perspectives, wondering and asking questions, and reasoning with evidence.

Summary. Erik's thinking moves, both during the group activities and on the selected exam questions, were spread out across all eight high-leverage thinking moves and all three thinking categories. In addition, there was evidence in many of Erik's statements that he was thinking in a statistical manner. For example, during the *On Time Airline* MEA, Erik helped his group see the connection between their statistical knowledge and the context of the problem; as a result Erik appeared to be engaging in statistical thinking as defined earlier (delMas, 2004; Pfannkuch & Rubick, 2002). On another occasion in the same activity, Erik developed, using statistical and contextual knowledge, a statistical model that represented the data in context. Erik's ability to “generalize knowledge obtained from classroom examples to new and somewhat novel situations” (delMas, 2002, p. 6) through the development of a model that would work for the company, or another company, to deal with a similar situation in the future, is also an example of statistical thinking. Finally, in the *ESP* activity Erik helped create a simulation and used it to make inferences about the data. This building of statistical models to simulate data and make judgments about the data is further evidence of statistical thinking (Garfield, delMas, & Zieffler, 2012; Wild & Pfannkuch, 1999). Overall, Erik appeared very comfortable using all eight of the high-leverage thinking moves on both the exams and during the activities; his ability to think in a statistical manner was displayed on multiple occasions. In the next section, the thinking moves of Mitch are explored.

Mitch. This section investigates the thinking moves of Mitch through the analysis of discussions during the group activities, and Mitch’s responses to selected exam questions. Each of these sources of data is analyzed using the high-leverage thinking moves (Ritchhart et al., 2011) and three thinking categories: (a) creative thinking, (b) mental management and awareness, and (c) critical thinking (Ritchhart, 2001).

Group activities. In this section, Mitch’s contributions during the group activities are used to explore his thinking moves. Overall, Mitch made extensive use of the eight high-leverage thinking moves during the various group activities. During the group activities, Mitch made use of a total of 19 high-leverage thinking moves; seven out of the 8 thinking moves and all three thinking categories were represented. In terms of distribution of the thinking moves across the categories, over half of Mitch’s thinking moves were in the creative thinking category. The most common thinking move used was *wondering and asking questions*, which was used a total of six times. The only thinking move not displayed during the group activities was *capturing the heart and forming conclusions* (see Table 4.5 for a summary of Mitch's high-leverage thinking moves used during the group activities).

Table 4.5.

Mitch: High-leverage Thinking Moves During Group Activities Summary

Category	High-Leverage thinking move	Total
Creative thinking	• Considering different viewpoints and perspectives	1
	• Wondering and asking questions	6
	• Uncovering complexity and going below the surface of things	4
Mental management and awareness	• Building explanations and interpretations	3
	• Capturing the heart and forming conclusions	0
	• Making connections	2
Critical thinking	• Observing closely and describing what is there	2
	• Reasoning with evidence	1

Selected conversations during the group activities are used to explore Mitch's thinking moves in a more thorough manner. For example, during the *ESP* activity Mitch made use of the thinking move *wondering and asking questions*. In this task, the objective is to determine whether an individual who claims they have ESP actually does. The person who believes that they have ESP is given 10 envelopes that each contain the name of one of the experimental subjects and attempts to correctly match each envelope with the test subject. Prior to the conversation between Mitch and Erik below, the group had been wrestling with the questions: How many matches do we expect by chance? How do we determine if the result, in terms of the number of matches, is statistically significant? At this point in the conversation the group was discussing the concept of statistical significance in relation to the simulation they had done. Mitch, through the use of the thinking move *wondering and asking questions*, moved the group in a productive direction by getting them to consider what it means, in terms of percentage, for something to be statistically significant in the current context.

Mitch: Do you know what percentage you need for it to be statistically significant?

Erik: Yeah, it's 5% I think, but it needs to be less than 5% that it happens by chance.

Mitch: Okay.

The thinking move exhibited by Mitch put his thinking in the *creative thinking* category and resulted in Erik sharing one possible level of significance that could be used by the group.

In the next exchange, Mitch made use of two different thinking moves *wondering and asking questions* and *uncovering the complexity and going below the surface of things*. At this point in the discussion, Erik, Mitch and Nicole were attempting to figure out how to set up the simulation that can help them determine the average number of matches over ten trials. In this regard, Mitch built on the ideas that Erik shared in his first statement regarding the development of a simulation. Mitch did this through asking two questions to inquire about the importance of the order of the numbers in setting up the simulation. This use of the thinking move *wondering and asking questions* is seen in the following exchange.

Erik: When we're running the experiment we need to decide what numbers we would need to pick in order, to get it right. If we did this trial 20 times, I'm trying to think how we would actually set it up. A random number…on the calculator?

Nicole: Without actually doing it?

Erik: Yes.

Nicole: Okay.

Mitch: Do we have to set it up where, like, the order it comes out? Like, it has to be the same order each time?

Erik: Yeah. What order would it have to be?

In this instance, Mitch's creative thinking moves benefited the group in regard to the creation of a simulation that would eventually be used to solve the problem under consideration.

In the next discussion, Mitch demonstrated the use of the thinking moves *uncovering the complexity and going below the surface of things* and *making connections.* At this point in the activity, the group continued to discuss the creation of an appropriate simulation. In Mitch's first two statements, he encouraged the group to create a list of digits from 01 to 10 to represent the10 individuals and then compare that list to the list of random numbers. In the second statement he made the point that in order to count as a match the random number would have to match the number of the individual or in his own words: "It'd have to fall on its own number." This dialogue is seen below.

Erik: Oh, so we're saying that . . .

Mitch: 01 to 10. But then, we put it in order and then we put in the random integers.

Erik: Um.

Mitch: It'd have to fall on its own number.

Erik: That's good. That's actually, yep. That's exactly what we need to do.

Mitch: Yeah.

Through his statements before and during the above conversation, Mitch revealed a key *connection* between the numbers the simulation produced and the context of the problem (i.e., matching the 10 names in the envelopes with the correct individuals). This is also significant in terms of statistical thinking as Mitch was able to tie the statistical knowledge and procedures to the contextual knowledge.

Finally, Mitch's contributions during *The Spread of a Rumor* activity are considered. In this activity, Mitch worked with the Tia, Courtney, and Joe to investigate the spread of a rumor. The problem scenario starts by one person sharing a rumor with another randomly selected person in the first time segment. In the second time segment, those two individuals share the same rumor with two randomly selected people. This continues on with the exception that if one of the randomly selected individuals has already heard the rumor, that branch of the rumor stops.

In the following excerpt, Mitch made use of the high-leverage thinking move *uncovering the complexity and going below the surface of things.* In his two statements, Mitch explained to Joe and Courtney how the probability generally increases from one round to the next. His careful inspection of the situation allowed him to dig into the complex nature of the changing probability from one time segment to the next (see the dialogue below).

Mitch: So each time someone gets picked, then the next increment the probability of a person getting picked increases so we have to show that.

Courtney: Wouldn't it decrease?

Joe: It decreases.

Courtney: Because the probability…

Mitch: What are you talking about? There are less students and more people who are picking.

The conversation about the changing probabilities continues in the following exchange between Mitch and Courtney shown below. In this discussion, Mitch displayed the use of the thinking moves *building explanations and interpretations* and *reasoning with evidence*. In this instance, Mitch used actual numbers from the problem (i.e., reasoning with evidence) to help make his point about the probability increasing from one segment to the next. Through his four statements, Mitch carefully constructed an explanation of how the probability increases from round 1 to round 2, ultimately finding the probability that the two individuals successfully choose a person they can share the rumor with (i.e., building explanations and interpretations).

Mitch: Well, in the next one there's going to be two kids that are already selected, so there is a sample size of 23, and then there's . . .

Courtney: Then there are two people that could have told it now, told the rumor, so there are 2 out of…

Mitch: So 2 in 20.

Courtney: And then the next time…

Mitch: A third chance…

Courtney: But, then the next time…

Mitch: You got to let me, let me finish talking. It's because two people are picking during the next increment, it's 2 over 23.

In the two excerpts above related to *The Spread of a Rumor* activity, Mitch's thinking resided in all three thinking categories (i.e., creative, mental management and awareness, and critical) as he helped the group find the probability of a person hearing the rumor in the second round.

Summary. Mitch made extensive use of the eight thinking moves as 7 out of the 8 high-leverage thinking moves were represented. The most common thinking move used, *wondering and asking questions*, was used a total of six times. In terms of distribution of the thinking moves across the categories, 58% of Mitch's thinking moves were in the *creative thinking* category, 26% of Mitch's thinking moves were in the *creative thinking*, and 26% of Mitch's thinking moves were in the *mental management and awareness* category.

To further explore the thinking moves, Mitch's responses on selected exams questions were analyzed and are considered next.

Exam question responses. The thinking moves of the four students were also studied by examining their responses on selected exam questions. Mitch made use of a variety of thinking moves when working on the selected exam questions (see Table 4.6). For example, on the Problem 13 (Exam 1), this instance, Mitch *reasoned with evidence*, using the median and maximum values to conclude that body temperature of females is higher than males. In addition, he *observed closely and described what was there* in regard to the similarities of the mean and the standard deviations of the two data sets. On Problem 15 (Exam 1), Mitch made use of two high-leverage thinking moves. First, using the thinking move *reasoning with evidence*, Mitch reasoned that the median and the IQR should be used to represent the data based on the evidence provided by the histogram (i.e., it was skewed to the left). In addition, Mitch's thinking in this case allowed him to *capture the heart and form a conclusion* regarding what measure of center and measures of spread would be best in this case.

Table 4.6.

Mitch: Answers to Selected Exam Problems

Exam problem	Mitch's answers	Thinking move (thinking category)
Exam 1 (#13)	**Question:** [Using a dot plot and a data table], write a few sentences comparing the body temperatures of healthy males and females. (Starnes et al., 2011) **Answer:** The average body temperature is 98.6, but the body temperature of individuals varies. The body temperature between males and females also vary. By observing the median and maximum values of female body temps, I concluded that females have a higher body temperature than males. On the other hand, they have similar means and standard deviations.	• Reasoning with evidence (critical thinking) • Observing closely and describing what's there (critical thinking)
Exam 1 (#15)	**Question:** Based on your histogram [from Question 14], what numerical measures of center and spread would be best to use for this distribution? Explain your choice. (Starnes et al., 2011) **Answer:** It would be best to use the median and IQR because the data skews the graph to the right.	• Reasoning with evidence (critical thinking) • Capturing the heart and forming conclusions (mental management and awareness)
Exam 5 (#11c,d)	**Question 11c**: You want to estimate the probability that exactly three of the next four calls are for medical assistance. Describe the design of a simulation to estimate the probability. Explain clearly how you will use the partial table of random digits below to carry out your simulation. (Starnes et al., 2011) **Answer 11c:** By using table D on line 77, I will select four numbers from a range of 00-99. If the number is between 00-80, the call will be for medical assistance. If the number is between 81-99, the call will not be medical related. I selected 70,39,87,28. I will repeat this multiple times to estimate the probability that exactly 3 of the 4 calls are for medical assistance. For my first repetition, 3 of the 4 calls were medical related. **Question 11d:** Carry out five trials of your simulation. (Starnes et al., 2011) **Answer 11d:** [Only one trail of the simulation was completed.]	• Uncovering complexity and going below the surface of things (creative thinking) • Building explanations and interpretations (mental management and awareness) • Making connections (mental management and awareness)

In addition, on Problem 11 (Exam 5), Mitch displayed three different thinking moves. First, in order to create a simulation to estimate the probability that exactly 3 of the next 4 calls are for medical assistance, Mitch *uncovered the complexity and went below the surface of things*. This allowed Mitch to *build explanations and interpretations* related to the question and *make connections* between the statistical knowledge and the context of the problem. However, because Mitch did not complete the simulation he was not able to *capture the heart* and *form a conclusion* about the topic under consideration.

Overall, Mitch made use of thinking moves on 14 occasions and displayed 6 of the 8 high-leverage thinking moves. In addition, these thinking moves were fairly evenly distributed across the three thinking categories. Finally, the thinking moves *considering different viewpoints and perspectives* and *wondering and asking questions* were not exhibited during the completion of the selected exam questions.

Summary. Overall, Mitch made broad use of the thinking moves in both the activity and the exam settings. Taken together, Mitch was able to demonstrate all eight high-leverage thinking moves; the use of these thinking moves were fairly evenly spread out over the three thinking categories, although a larger percentage of his thinking resided in the creative thinking category. The most common thinking move used by Mitch was *wondering and asking questions*; Mitch used this thinking move to support others in their understanding by getting them to consider various ideas and perspectives. An example of this was Mitch's question during the *ESP* activity: "Do we have to set it up where, like, the order it comes out? Like, it has to be the same order each time?" This question played a critical role the development of the simulation the group would later use to answer the question under consideration.

There is also significant evidence that Mitch thought in a statistical manner during the group activities and on his responses to the selected exam questions. During the *ESP* activity, Mitch made the point that in the simulation, in order for a match to occur, the random number would have to match the number of the individual. Mitch stated, "It'd have to fall on its own number." For example, the list of the numbers 2746939*8*18 has one match with the control list of numbers 1234567*8*910, at the eighth spot. In addition, Mitch's thinking promoted the creation of a box plot that provided a visual representation of the number of matches in each set of 10. Mitch's thinking revealed a key connection between the numbers the simulation produced and the context of the problem; this is significant in terms of statistical thinking as Mitch was able to tie the statistical knowledge and procedures to the contextual knowledge. His thinking also prompted the creation of a statistical representation of the data, in this instance a dot plot, which the group used to make judgments about the data. This ability, to build statistical models to simulate data and make inferences about the data, is reflective of statistical thinking (Garfield et al., 2012; Wild & Pfannkuch, 1999). Overall, Mitch demonstrated the use of a wide variety of thinking moves. In addition, there is evidence that he displayed statistical thinking during the exams and group activities. The thinking moves displayed by Ashley are considered in the next section.

Ashley. This section investigates the thinking moves of Ashley through the analysis of the discussions during the group activities and Ashley's responses to selected exam questions. Each of these sources of data is analyzed using the high-leverage thinking moves (Ritchhart et al., 2011) and three thinking categories: (a) creative thinking, (b) mental management and awareness, and (c) critical thinking (Ritchhart, 2001).

Group activities. In this section, Ashley's contributions during the group activities are used to explore her thinking moves. Overall, Ashley only made partial use of the eight high-leverage thinking moves during the various group activities. Through the group activities, Ashley used a total of 15 thinking moves; three of 8 thinking moves were represented. Ashley's most commonly used thinking move, *wondering and asking questions,* was used a total of nine times. In terms of the distribution of the thinking moves across the three thinking categories, 10 of the thinking moves resided in the *creative thinking* category, five were part of the *critical thinking* category, and none of the thinking moves fell in the *mental management and awareness* category (see Table 4.7 for a summary of Ashley's use of high-leverage thinking moves during the activities).

Table 4.7.

Ashley: High-leverage Thinking Moves During Group Activities Summary

Category	High-Leverage thinking move	Total
Creative thinking	• Considering different viewpoints and perspectives	0
	• Wondering and asking questions	9
	• Uncovering complexity and going below the surface of things	1
Mental management and awareness	• Building explanations and interpretations	0
	• Capturing the heart and forming conclusions	0
	• Making connections	0
Critical thinking	• Observing closely and describing what is there	5
	• Reasoning with evidence	0

Selected conversations during the group activities are used to explore Ashley's thinking moves in a more thorough manner. The thinking moves displayed by Ashley during the *On Time Airline* MEA are considered first. During this activity, Ashley made use of the thinking moves *observing closely and describing what's there* and *wondering and asking questions.* Prior to the exchange below, the group was deciding how to rank the five airlines from the most likely to have flights that depart on time to the least likely to have flights that depart on time. They were wrestling with the fact that all five data sets have similar means, but all are affected differently by the presence of or lack of possible outliers. In the first part of her statement, Ashley was *observing closely and describing what was there* when she made a reference to the existence of several large, possible outliers. In this instance, she noticed and shared an important aspect of the data sets, a component that plays a key role in the analysis. The conversation between Ashley, Mitch, and Jake is shown below.

Mitch: So, if we want to rank these from most likely to least likely, how should we write that? Should we go…?

Ashley: Do you think we should use…you know how there are a couple of huge outliers? Do you think we should keep those in the picture?

Jake: Are you thinking about taking it out?

Mitch: Because like how we can back up our answer, the chances that…

Jake: We kind of do because when this one with the 125 outlier, we use the median to describe the spread which doesn't include the outlier.

In the second part of her statement, Ashley asks a question about whether the outliers found in several of the data sets should be considered in the analysis (i.e., the thinking move *wondering and asking questions*). Ashley's thinking, in the form of a question, encouraged the group to evaluate their approach in a more thoughtful manner. For example, following Ashley's question, Jake initially wondered whether the outliers should be taken out. He then realized and shared with the group that the median is resistant to outliers (i.e., "we use the median to describe the spread which doesn't include the outlier."). In all, Ashley made use of two thinking moves during the above exchange with Jake and Mitch.

In the next exchange with Jake, Ashley made use of the thinking move *wondering and asking questions*. In this situation, Ashley's question helped Jake and the group realize, that in order to effectively compare the five boxplots representing the airline data, it might be helpful to use the same window on their graphing display calculators.

Jake: So we should all draw our box plots and we can compare them?

Ashley: What are you guys going to go by so they all look the same?

Following Ashley's question, the group spent some time looking over the five box plots on their graphing display calculators and decided on an appropriate window that would allow for an effective comparison.

In the next conversation that took place during *The Race and the Death Penalty* activity, Ashley displayed *observing closely and describing what's there* and *wondering and asking questions*. In this activity, the group examined the data regarding the number of murder convictions that resulted in the administration of the death penalty in the Philadelphia courts from 1893-1993; the data were organized in a three-way table in terms of the race of the defendants and the race of the murder victims. The task was to use these data to determine if the justice system in the Philadelphia court system was colorblind in the administration of the death penalty during the given time period.

In this conversation, the group was trying to figure out how to display the data in a visual manner. At the beginning of the exchange, Courtney was arguing, and Erik agreed, that a segmented bar graph should be used. However, Ashley's statement (i.e., "it doesn't add up to 100%") helped the group realized that the use of a segmented bar graph may not be appropriate because one of the requirements is that each bar will show 100% of the discrete values. In this instance, Ashley displayed the thinking move *observing closely and describing what's there* when she noticed and described an important component of a segmented bar graph (see the conversation between the four students below).

Courtney: No, it's not that though. I'm saying if we had the segmented bar graph and we had the Black defendant and the White defendant.

Erik: Oh, okay.

Courtney: And then we had them both go up.

Ashley: But is it not even 100%, because it doesn't add up to 100%.

Anna: Yeah, they don't add up to 100%. It would be like 25.2%.

Ashley: So what other visuals have we learned?

Ashley displayed an addition thinking move (i.e., *wondering and asking questions*) in the final statement of the exchange. Her question (i.e., "So what other visuals have we learned?") got the group to consider what type of statistical graph would best represent the relationships in the data. This was important as it shifted the group's focus from the use of a segmented bar graph to other possible visual representations that could be used to effectively display the data.

Finally, Ashley worked with Nate and Kate on the *Backhoes and Forklifts* activity is explored. This task is based on the story that a heavy equipment manufacturer introduces new models of their backhoes and forklifts only to discover that an unacceptably high number of their customers needed to seek repairs of the machines' hydraulic systems within the first few weeks of operation. The company engineers believe that an additive poured into the hydraulic oil may greatly extend the number of hours these machines can be used before repairs become necessary. The task of the students is to design an experiment to test the theory (i.e., that the new additive significantly extends the life of the machines) using up to 20 newly manufactured machines.

In the exchange below, Ashley demonstrated the use of the thinking move *uncovering complexity and going below the surface of things*. At this point in the activity, the students were discussing how to best design the experiment to test the hydraulic oil additive. Taken together, Nate's two comments suggest that he initially thought the best approach was a match pairs design. Ashley's thinking in the dialogue below moves the group in a different direction as she prodded them to begin the process of setting up a randomized block design.

Nate: I think it is safe to say that we can use a randomized sample, a randomized experiment. We're not going to use any blocks. So I can use the drawing, the diagram.

Ashley: Let's split them into groups of 10.

Kate: And then 10 would receive the additive.

Ashley: Like this would be one treatment and that would be one treatment.

Nate: Yeah. You're right. I was thinking of matched pairs.

Ashley: And one will get the additive and one will not.

Her three statements in the above dialogue described how the block design would be set up. First, two groups of 10 would be used in the experimental design (the students would discuss the random selection of the two groups at a later time). Next, one of the experimental groups of 10 would get the new hydraulic fluid additive and the other experimental group of 10 would not. Ashley's thinking in this instance (i.e., uncovering complexity and going below the surface of things) helped the group recognize the complex nature of the situation under examination; this is an essential aspect of understanding (Ritchhart et al., 2011).

Summary. Overall, Ashley only made partial use of the eight high-leverage thinking moves during the four problem-solving activities. The three moves used by Anna were: (a) wondering and asking questions, (b) uncovering complexity and going below the surface of things, and (c) observing closely and describing what is there. The most commonly used thinking move was *wondering and asking questions*, which was used a total of nine times. In regard to the distribution across the thinking categories, two-thirds of Ashley's thinking moves were in the *creative thinking* category, one-third of Ashley's thinking moves were in the *creative thinking*, and none of her thinking moves were in the *mental management and awareness* category.

To further explore the thinking moves, Ashley's responses on selected exams questions were analyzed and are considered next.

Exam question responses. Ashley made use of a variety of thinking moves during the completion of the selected exam questions (see Table 4.8). For example, Ashley made use of three different thinking moves on Problem 12c (Exam 4). First, Ashley's response, which describes the selection and creation of a block design, exhibited her ability *to uncover complexity and go below the surface of things*. Second, her response demonstrated the ability to *build explanations and interpretations* related to the block design. Finally, Ashley displayed the thinking move *making connections*, as she was able to make ties between the statistical procedures and the context of the problem.

Table 4.8.

Ashley: Answers to Selected Exam Problems

Exam problem	Ashley's answers	Thinking move (thinking category)
Exam 4 (#12c)	**Question:** Assume that 600 men and 500 women suffering from high blood pressure are available for the study. Describe a design for the experiment. Be sure to include a description of how you assign individuals to the treatment group. (Starnes et al., 2011) **Answer:** Starting with all 1,100 subjects, divide/block them into groups men=600 and women=500. Then label all men form 001-600 and label all women 001-500. Then using a line from table D starting with men take the 3 digit numbers in the 001-600 range never repeating a number until you get 300 men. They will receive the ACE, then take next 300 men doing same as above who will get Placebo. For women number 001-500, then using a line in table D randomly assign 250 women in range 001-500, never repeating a number to ACE. Do same for Placebo group.	• Uncovering complexity and going below the surface of things (creative thinking) • Building explanations and interpretations (mental management and awareness) • Making connections (mental management and awareness)
Exam 5 (#11c,d)	**Question 11c**: You want to estimate the probability that exactly three of the next four calls are for medical assistance. Describe the design of a simulation to estimate the probability. Explain clearly how you will use the partial table of random digits below to carry out your simulation. (Starnes et al., 2011) **Answer 11c:** To carry out this simulation number subjects/callers 00-80, these will be the people who need medical assistance when called then number, 81-99 the people who do not need medical assistance. Then using table D, taking 2 digit numbers in the range 00-99 with no repeats take 4 numbers and see if 3 of the 4 numbers are for medical assistance. Repeat more than once. **Question 11d:** Carry out five trials of your simulation. (Starnes et al., 2011) **Answer 11d:** From the data above for every 4 call we did average about 3 or 4 out of 4 calls were for medical assistance.	• Uncovering complexity and going below the surface of things (creative thinking) • Building explanations and interpretations (mental management and awareness) • Making connections (mental management and awareness)

In addition, on Problem 11 (Exam 5), Ashley also displayed three different thinking moves. First, Ashley used the thinking move *uncovering the complexity and going below the surface of things* in order to create a simulation to estimate the probability that exactly 3 of the next 4 calls are for medical assistance. In addition, her response demonstrated that she was able to *build explanations and interpretations* related to the question and *make connections* between the context of the problem and the statistical procedures. However, Ashley's response indicated that she was not able to *capture the heart and form an* appropriate *conclusion* about the topic under consideration.

In all, Ashley displayed 4 of the 8 high-leverage thinking moves and a total of 12 thinking moves. In addition, these thinking moves resided in all three thinking categories.

Summary. The thinking moves displayed by Ashley during the group activities looked very different than the thinking moves she demonstrated on her responses on selected questions on chapter exams. For example, on the exam questions, Ashley's thinking moves were part of all three thinking categories, while during the group activities Ashley was only able to make use of thinking moves in the creative thinking and critical thinking categories. In addition, 14 out of 15 of the high-leverage thinking moves exhibited during the activities were either the thinking move *wondering and asking questions* or *observing closely and describing what's there*. In contrast, the 12 thinking moves Ashley displayed during the exams were evenly spread across four of the high-leverage thinking moves. Perhaps one of the most significant differences was the fact that half of the thinking moves demonstrated by Ashley on her exam responses were in the mental management and awareness thinking category; none of her thinking moves during the activities resided in that category.

There was also a significant difference between Ashley's statistical thinking during the group activities and on her exam responses. On one hand, there was no evidence in Ashley's statements to suggest that she engaged in statistical thinking during the discussions related to the group activities. For instance, Ashley displayed thinking moves that connected the statistical and procedural knowledge and the contextual knowledge, an important component of statistical thinking (Pfannkuch & Rubick, 2002).

Conversely, there was evidence to suggest that Ashley engaged in statistical thinking on her exam responses as her thinking moves revealed connections between statistical knowledge and the context of the problem. In addition, there was evidence that Ashley generalized the knowledge gained in class to the new situations she encountered in the exam questions, which is one way that individuals can demonstrate the ability to think in a statistical manner (delMas, 2002).

Thinking moves summary. The use of high-leverage thinking moves by Anna, Erik, Mitch, and Ashley was investigated. This was done through the analysis of the group discussions during five group activities and the student responses to selected exam questions. Each of these sources of data was analyzed using the high-leverage thinking moves (Ritchhart et al., 2011) and three thinking categories (Ritchhart, 2001), (a) creative thinking, (b) mental management and awareness, and (c) critical thinking. During the various group activities, Ashley and Anna both made partial use of the eight thinking moves, while Erik and Mitch made extensive use of the thinking moves. Moreover, within the group activities, the thinking moves of Ashley and Anna reside in the creative and critical thinking categories, while the thinking moves of Erik and Mitch reside in all three thinking categories. In contrast, the patterns of thinking moves displayed on the exam questions were fairly similar for the four students.

In the next section, the data analysis and results for the current and designated mathematical identity of the four students are examined. These analyses build the foundation that is used in Chapter 5 to address the research question and the two subquestions.

Mathematical Identity Analysis

Aguirre et al. (2013) argue that individuals develop identities, beliefs and dispositions, which are related to their participation and performance in mathematical, as well as more general, contexts (see Chapter 2 for a more thorough discussion of mathematical identity). Mathematical identity, often expressed in the form of stories or narratives, can reveal how an individual views themselves, as well as how they are perceived by others in terms of their performance in the mathematics classroom. As a result, identity can be viewed through the stories of individuals as they describe who they are, who they are not, and who they want to become (Aguirre et al., 2013). In fact, the use of narrative inquiry is one of the most effective methods to examine the experiences of individuals in an educational setting (Clandinin & Connelly, 2000).

This section explores the mathematical identities of Anna, Erik, Mitch, and Ashley through consideration of their narratives. In the first subsection, the data associated with current mathematical identity are examined. In the second subsection, the designated mathematical identity data are analyzed. Each of the two subsections will be organized according the four students.

Current mathematical identity analysis. This section examines the current mathematical identity of the four cases (i.e., Anna, Erik, Mitch, and Ashley) using the data from the MINT protocol responses, the student interviews, and the teacher interviews.

Anna. The current identity of Anna is explored in relation to her current identity strength and two of the MINT themes, *ways of studying and working in mathematics* and *persistence and perseverance with mathematics* are used as structural frameworks.

Current identity strength. The data collected from the MINT protocol and the two interviews with Anna were used to explore the patterns in her current identity. The data were placed into five current mathematical identity categories related to strength of experience or feelings (i.e., strong positive, positive, mixed, negative, and strong negative; see Table 4.9). Anna's responses displayed mixed feelings and experiences related to mathematics. An example of one of Anna's statements that displayed a strong positive mathematical identity was, "I have to think hard at this class, but that is one of the reasons I like it." On the other hand, the statement, "Math isn't my favorite subject, but statistics I like," suggested a mixed mathematical identity as it displayed mixed feelings and experiences related to mathematics. Taken together, Anna's statements suggest that she had a mixed mathematical identity (i.e., mixed feelings and experiences related to mathematics, some positive and some negative).

Table 4.9.

Anna: Current Identity Summary

Theme	Strong positive	Positive	Mixed	Negative	Strong negative
The changing nature of mathematics as experienced from early childhood to now	0	1	2	1	0
The balance between challenge and interest	0	1	2	1	0
The role played by key figures in the formation of mathematical identity	0	0	2	0	0
How learning mathematics compares with learning in other subjects	0	0	1	0	0
Ways of studying and working in mathematics	0	4	5	2	0
Persistence and perseverance with mathematics	0	3	3	0	2
Mathematics as a rewarding subject	1	2	3	3	0
Total Responses	1	11	18	7	2

This summary of Anna's statements related to her current mathematical identity suggests that Anna reported a wide range of experiences and feelings, from strong negative to strong positive, related to mathematics. It is noteworthy that the majority of her statements, 36 out of 39, fall into the middle three categories (i.e., positive, mixed, and negative). Finally, almost 50% of her statements fall in the mixed category. In all, this evidence suggests the Anna had a mixed current mathematical identity. To explore this in more detail, detailed descriptions of Anna's current identity are developed in the next section.

Current identity descriptions. In addition to examining identity strength, rich descriptions related to the current mathematical identity of Anna were developed. Two of the MINT themes, *ways of studying and working in mathematics* and *persistence and perseverance with mathematics* are used as structural frameworks.

In relation to the theme *ways of studying and working in mathematics*, Anna saw herself as student who lacked confidence in her mathematical abilities and viewed herself as student who was sometimes intimidated by the difficult concepts, the long chapters, and the amount of information to be learned. Anna's lack of confidence is portrayed in her narrative as she discussed the difficulty of learning in the mathematics classroom. For example, she stated that, "You definitely have to work at it. There have been times where I know what's going on, but it's sometimes hard to put the different concepts together." She added that it can be difficult because, "the big long chapters show that there are lots of things to learn and how everything is kind of accumulated, it adds onto each other. There's just a lot to know." These examples and others reveal that Anna saw herself as someone who lacked confidence in her mathematical abilities. Her teacher confirmed

Anna’s lack of confidence in the mathematics classroom. Mr. Cox stated that, “I wouldn’t say, you know, she’s feeling incredibly confident.” Later he added, “Anna is not a central figure and prefers to sit back. She will answer if called upon, but she’s not going to offer an opinion.” In summary, this evidence suggests that in relation to the theme *ways of studying and working in mathematics*, Anna saw herself as student who lacked confidence in the mathematics classroom.

In terms of the theme *persistence and perseverance with mathematics*, Anna’s current mathematical was bifurcated. On one hand, Anna viewed herself as a student who sometimes lacked persistence in homework related issues. This perspective is reflected in the comment, “Typically I have my notebook open and use the textbook and my notes and the examples in the book to help me through the problem. I’ll check my answers in the back if it’s an odd day. If it’s even, we have to wait.” This statement indicates that Anna was using examples in the book and her notes to help her unpack the concepts and ideas present in the homework problems. However, this also indicates a lack of persistence in making use of other avenues to check if the homework is done correctly or in finding other sources for help with understanding the material. This same attitude is displayed in a different statement, “Sometimes when I’m so stumped, there’s not much I can do other than wait for the next day when we go over the homework questions.” Taken together, these statements suggest that in regard to homework Anna viewed herself as a student who sometimes lacked persistence in homework related issues. Similarly, this current identity is also revealed in Anna’s statements related to learning mathematics. She indicated that she sometimes found mathematics challenging as she has difficulty, “just grasping the concepts at times.” She went on to say that “you don’t want to spend too much time going back and reviewing what you've done." In the end, Anna viewed herself as someone who did not always understand the material and as someone who was not always willing to spend the time necessary to relearn or become more comfortable with the material.

Her teacher confirmed this current identity related to the learning of the material. He stated, "She's not as willing to get the extra help if she's struggling." Mr. Cox expanded on this,

> I wouldn't say that she has [strived] for excellence. I think it's not easy for her, but in the same sense, she's okay being average. I don't think she's shooting to be outstanding. If she's struggling, then she's struggling and well, that what's meant to happen rather than to get answers to those questions.

Together, these statements of Anna and Mr. Cox suggest that Anna viewed herself as a student who sometimes lacked persistence.

In contrast, Anna's narrative also suggests that she viewed herself as an individual who had a strong work ethic and who persevered when the material was difficult. For example, Anna stated that when she is, "confused about something, I'm trying to kind of work the hardest and overcome it." In a similar way, Anna indicated that, "I think it's important just to be able to take your own notes and kind of go into more detail on things that he explains in class or maybe if you're confused, look into it more." These statements portray a person who takes notes, looks into things further, and works hard to overcome obstacles such as confusing material. In addition, this narrative, of an individual who puts in the necessary time outside of class, is also seen in this statement by Anna: "I'll have to work on it if I want to be good at it. I'm going to have to work on it, especially this class." She added, "Even if you listen really hard for the 55 minutes of class you're going to still have to do things on your own time." These statements and others paint a picture of a mathematics student who viewed herself as an individual who persisted when confronted with challenging material.

This identity was confirmed by Anna's teacher. Mr. Cox indicated that he noticed a change in Anna's approach to her studies during the semester. He stated that, "After parent conferences, she came and talked to me and admitted that she should be doing more, that she wasn't doing enough outside of class. And so she's actually stepped up her efforts and the results have been better." This statement suggests Anna was aware that her effort was lacking and that, to successfully learn the material, she needed to work harder. As a result, Anna started to put more time and effort into the course. That being said, Mr. Cox did add, "I do think there's still more that she could be doing." Overall, Anna had a diverse current mathematical identity in regard to the theme *persistence and perseverance with mathematics*. In the next section, the current mathematical identity of Erik is examined.

Erik. In this section, the current mathematical identity of Erik is explored using his responses on the MINT protocol, and his narrative during the two interviews. In addition, the transcripts from the teacher interviews are used to confirm or disconfirm Erik's current mathematical identity.

Current identity strength. The data collected from the MINT protocol and the two interviews with Erik were used to explore the patterns in his current identity. Once all Erik's statements regarding current mathematical identity were collected, they were organized according to the strength (i.e., strong positive, positive, mixed, negative, and strong negative; see Table 4.10). These results show a current mathematical identity that was made up of a wide range of experiences and feelings related to mathematics from strong positive to negative. An example of one of Erik's statements that displayed a strong positive current mathematical identity is, "The dominant features of math are a combination of satisfaction and intrigue." This statement is a factual assertion that indicated that Erik saw himself as a student who gained satisfaction and was intrigued by the study of mathematics. These results indicate that the majority of Erik's statements (i.e., 27 out of 46) related to current mathematical identity reside in the strong positive category. In addition, 28% of the statements from the interviews fell into the negative and mixed category.

Table 4.10.

Erik: Current Identity Summary

Theme	Strong positive	Positive	Mixed	Negative	Strong negative
The changing nature of mathematics as experienced from early childhood to now	2	0	0	0	0
The balance between challenge and interest	3	0	1	1	0
The role played by key figures in the formation of mathematical identity	0	0	0	0	0
How learning mathematics compares with learning in other subjects	0	1	0	0	0
Ways of studying mathematics and working in mathematics	9	2	5	3	0
Persistence and perseverance with mathematics	5	2	0	1	0
Mathematics as a rewarding subject	8	1	2	0	0
Total Responses	27	6	8	5	0

Overall, the data suggest that Erik's current mathematical identity was in the positive range (mostly positive feelings and experiences related to mathematics). To examine his current identity in more detail, the following analysis includes rich and thorough descriptions that include the words of Erik and his teacher.

Current identity descriptions. In this section, Erik's current mathematical identity is analyzed using the following two MINT themes as a structural framework: (a) ways of studying mathematics and working in mathematics, and (b) persistence and perseverance with mathematics. Within each of these themes, his narratives are explored in order to reveal his feelings and experiences related to mathematics.

In terms of the theme *ways of studying mathematics and working in mathematics*, Erik saw himself as a curious student who was often compelled to dig deeper into the topic at hand as long as he could dictate the terms of his learning. For example, he explained that, "I'm actually not very into real life situations, I just prefer analytical mathematics." On another occasion, he mentioned that he did not like applied mathematics. His narrative suggests that the quality of the experience with mathematics was dependent, at least to an extent, on what type of mathematics was involved. Erik described, in detail, how learning in mathematics, for him, was connected with the drive and the motivation that a topic or concept provided. His teacher, Mr. Cox, confirmed that while Erik had the propensity to do extra work and that he really enjoyed learning, he also made it clear that Erik was, "going to learn on what he wants to learn on." In summary, Erik saw himself as a student who enjoyed learning mathematics on his own terms.

In relation to the theme *persistence and perseverance with mathematics*, Erik's narrative suggests that he saw himself as someone whose work ethic was linked to his curiosity about the topic being studied. For example, he stated that, "In Calculus we are working on related rates right now and it's difficult, but I'm putting a lot of effort into that." He added that he was putting the effort in because he was interested in the topic of related rates. This approach clearly embodied a strong work ethic and a willingness to put in the necessary time. In contrast, Erik saw himself as a mathematics student that shied away from too much work. For example, he stated that, related to homework for mathematics class, "I enjoy it, but it's kind of hard to sit down and, you know, do an hour or two of math." This current identity was confirmed in Mr. Cox's narrative as well. According to his teacher, Erik "can fall behind or display a lack effort at times. He's not someone who is going to kill himself, he lives a pretty happy, relaxed life, but he enjoys learning." Taken together, Erik's and Mr. Cox's statements indicate that Erik's current mathematical identity related to persistence and perseverance was situational and often related to interest. In the next section, the current mathematical identity of Mitch is explored.

Mitch. In this section, the current mathematical identity of Mitch is explored through the consideration of his responses on the MINT protocol, and the transcripts of the two interviews. In addition, the data from the teacher interviews are used to confirm or disconfirm Mitch's current identity.

Current identity strength. The data collected from the MINT protocol and the two interviews with Mitch were used to explore the patterns in his current identity. Once the data were collected, they were arranged by strength of identity (i.e., strong positive, positive, mixed, negative, and strong negative; see Table 4.11). Mitch's responses were primarily strong, positive statements. An example of one of Mitch's statements that displays strong positive mathematical identity is, "I enjoy math because it makes me think critically." In addition, he stated that, "mathematics is my favorite subject." Clearly, these statements reflect positive feelings and experiences related to mathematics and indicate that Mitch views himself as a person who enjoys the subject due to the fact that it makes him think in a critical manner. In all, 25 out of the 37 statements reflected a strong positive current mathematical identity, while 7 out of 37 statements were either negative or mixed.

Table 4.11.

Mitch: Current Identity Summary

Theme	Strong positive	Positive	Mixed	Negative	Strong negative
The changing nature of mathematics as experienced from early childhood to now	3	0	0	0	0
The balance between challenge and interest	4	0	2	1	0
The role played by key figures in the formation of mathematical identity	2	0	0	0	0
How learning mathematics compares with learning in other subjects	1	0	0	0	0
Ways of studying mathematics and working in mathematics	5	2	1	0	0
Persistence and perseverance with mathematics	4	0	0	2	0
Mathematics as a rewarding subject	6	3	0	1	0
Total Responses	25	5	3	4	0

Overall, these data suggest that Mitch had a positive current mathematical identity (i.e., mostly positive feelings and experiences related to mathematics). Mitch's current mathematical identity is explored in more detail in the next section. This exploration includes rich descriptions as well as Mitch's own words and the words of his teacher.

Current identity descriptions. In this section, Mitch's current mathematical identity is examined through his narrative as revealed through the student and teacher interviews. Two themes will be used as a structural framework to unpack Mitch's current mathematical identity: (a) ways of studying mathematics and working in mathematics, and (b) theme persistence and perseverance with mathematics.

First, in terms of studying and working in mathematics, Mitch saw himself as a highly capable and committed student. For example, Mitch stated, "I have a dedicated attitude towards math." In addition, Mitch's current mathematical identity is revealed, in part, in his response to a question about what things show that he is good at mathematics. He explained, "My grades and I'm currently taking my 5th and 6th math class of my high school career." This narrative, the fact that he gets good grades and has taken six mathematics classes in high school, suggests that Mitch identified himself as a strong, gifted mathematics student. This overall current identity was confirmed by his teacher who indicated that, "Mitch, I think has confidence in his abilities in math. I would say he would feel pretty competent." These and other statements reveal that Mitch views himself as a confident mathematics student.

Mitch's strong positive current mathematical identity related to learning mathematics was also portrayed in his willingness to be a participant in the mathematics classroom both in terms of helping his classmates and answering questions in class. Mitch explained that, "I would help my peers a lot, in Geometry my freshman year, and sometimes in Calculus." In addition to seeing himself as an accomplished mathematics student that can support his peers, his narrative suggests that he viewed himself as intelligent and as a committed member of the mathematics classroom. He stated, "I raise my hand because, when I know the answers, I like to provide…or I like to prove that I'm smart to my teacher and the kids in my class." In addition, in terms of being an active participant in classroom discussion, Mitch added, “It lets everyone know that you’re participating and that you’re showing that you’re involved and committed to learning mathematics and it kind of gives you a sense of feeling that you’re doing something right if everyone else knows.” Mitch’s current identity was confirmed by his teacher who explained that, “He has a small group of friends in the back that I would think that he’s interacting with and he’s probably one of the more gifted of his friends and so he’s helping them along the way.” Taken together, these narratives suggest that, in terms of the theme *ways of studying mathematics and working in mathematics*, Mitch had a robust current mathematical identity.

Although he saw himself as a very capable mathematics student, Mitch’s perspective of his *persistence and perseverance with mathematics* was somewhat mixed. On one hand, Mitch viewed himself as someone who perseveres through challenging mathematics problems. He explained, “But if it’s a problem that I can work through but it’s also difficult, I like it, because it keeps me interested.” This identity was also reflected in his willingness to do extra credit in Geometry and Algebra II. He stated that he did the extra credit, “Definitely because I wanted to. Not for the grade.” He went on to explain, “If I’m doing really well in this class maybe the extra credit is a little bit harder. Gives me a little bit more of an interest.” These statements suggest that Mitch perceived himself as someone who exceeded expectations through determination, not to get a better grade, but out of the need to be challenged or pursue an interest.

In contrast to this identity of a persistent individual, Mitch occasionally saw himself as a student who did not persevere. Mitch made it clear in some of his comments that he did not always put in the effort needed. He explained that his lack of commitment is revealed by, “Not doing my homework, and not reviewing before tests, not paying attention in class, like talking to friends instead.” In a similar way, this perspective can be seen in his explanation of how he decides how to get his questions answered in class. He explained, “It depends how lazy I am that day. It’s either like, is there someone close enough to me that will help me through it or do I need to get up and go talk to the teacher?” In this case, Mitch saw himself as a student who could sometimes be lazy in the pursuit of mathematical understanding.

Clearly, Mitch’s current mathematical identity related to his work ethic and determination in the mathematics classroom was complex. Ultimately, how Mitch viewed himself as a student of mathematics is captured in his story about his transformation in Calculus. He explained that during the first few weeks of the semester he was not putting in the time necessary to be successful. In terms of learning the material in Calculus, he explained, “It gives me a lot of trouble, but then I changed my schedule and I ended up starting [*sic*] coming in after school for an hour, and I rose my test scores on the third test by like 30%.” He went on to say that his extra effort made a difference as, “I went from like failing both my tests to getting a 70% on the third one.” This example makes it clear that Mitch’s current mathematical identity is mixed. This dichotomous identity was confirmed by his teacher. In regard to Mitch, he stated,

> He’s a pretty smart, gifted student; he runs into problems when he doesn’t work as hard. Some of that’s because of, some of his peer group is not very academic. Yet he is. So because of that, they kind of drag him down; on certain days and nights, he doesn’t get his stuff done, and it’s not as polished. But he’s gifted. He’s a gifted math student.

In addition, Mr. Cox stated that, “I would think he’s pretty confident, but I think he sometimes lacks the outside work. So because of that, he may not feel as confident because he hasn’t done the practice.” So while Mitch saw himself as a very capable student who could succeed when he put in the effort, he also viewed himself as a student who did not always approach the learning of mathematics in a productive manner. In the end, these narratives reveal that Mitch has a diverse current identity in relation to the theme *persistence and perseverance with mathematics*. In the next section, the current identity of Ashley is considered.

Ashley. In this section, the current mathematical identity of Ashley is explored using the data from the responses on the MINT protocol and the two student interviews. In addition, the transcripts of the teacher interviews are used to confirm or disconfirm Ashley’s current mathematical identity.

Current identity strength. The data collected from the MINT protocol and the two interviews with Ashley’s were used to explore the patterns in her current identity. First, the data were organized according to identity strength (i.e., strong positive, positive, mixed, negative, and strong negative; see Table 4.12). An example of one of Ashley’s statements that displayed strong negative mathematical identity is, “I do not enjoy math really. I always second guess myself.” In addition, the following statement by Ashley was placed into the negative category (it was identified as *mostly negative* due to Ashley’s use of the words *that well*): “I don’t test that well in math. I always second guess myself.” In total, while 14 out of 32 statements resided in the top three categories (i.e., strong positive, positive, and mixed), over half of the statements revealed a strong negative or a negative current mathematical identity.

Table 4.12.

Ashley: Current Identity Summary

Theme	Strong positive	Positive	Mixed	Negative	Strong negative
The changing nature of mathematics as experienced from early childhood to now	0	0	0	0	0
The balance between challenge and interest	0	0	0	2	0
The role played by key figures in the formation of mathematical identity	0	0	0	0	0
How learning mathematics compares with learning in other subjects	0	0	0	3	0
Ways of studying mathematics and working in mathematics	0	4	5	3	0
Persistence and perseverance with mathematics	0	2	1	3	0
Mathematics as a rewarding subject	1	1	0	1	6
Total Responses	1	7	6	12	6

Overall, the data suggest that Ashley had a negative mathematical identity (mostly negative feelings and experiences related to mathematics). The data show that Ashley had had a wide range of experiences and feelings, from strong positive to strong negative, related to mathematics. That being said, Ashley's narratives indicate that she viewed herself as an individual who did not really enjoy mathematics, did not feel very competent in mathematics, and found learning mathematics challenging. In the next section, Ashley's current mathematical identity is explored in more detail through the use of rich descriptions.

Current identity descriptions. In this section, Ashley's current identity is revealed through comprehensive descriptions. Two of the MINT themes are used as a structural framework to unpack Ashley's current identity: (a) ways of studying mathematics and working in mathematics, and (b) persistence and perseverance with mathematics.

First, related to the theme *ways of studying mathematics and working in mathematics*, Ashley saw herself as someone who lacked confidence in her ability to work and learn in the mathematics classroom. This current identity is reflected in her explanation of why AP Statistics is a difficult class for her. She stated:

> I struggled in the beginning of this class with learning how to do this AP Statistics. Instead of like just numbers, it's like describing your answer and how you got it. So I don't really get a lot of this stuff – like probability I'm not very good at.

This narrative suggests that she viewed herself as an individual who had a hard time describing her reasoning in the mathematics classroom. In addition, this description indicates she saw herself as someone that did not understand mathematics and was not good at it.

This lack of confidence in her ability to work and learn in the mathematics classroom is also revealed through Ashley's use of the phrase *I second guess myself* on several occasions. For example, she stated that, "I do not enjoy math really. I always second guess myself". In addition, she explained that, "I don't test that well in math. I always second guess myself." This lack of confidence was confirmed by her teacher, Mr. Cox. He stated that, "even when she does the work she's concerned and she finds herself double guessing her work at times." He added that, "she lacks confidence, lacks some confidence in her math ability." These statements by Ashley and Mr. Cox indicate that Ashley perceived herself as a student who lacked confidence and doubted herself in the mathematics classroom.

In addition, there is evidence that Anna did not see herself as a central member of the mathematic classroom. This is revealed in how she viewed herself in terms of asking questions and making contributions to class discussions. She explained that, 'I don't really add comments I guess. I ask questions about homework problems or such, but I don't like start discussions about the problem." This was confirmed by statements from her teacher Mr. Cox. He stated, "I would say she doesn't see herself as a central member of the class. I think she doesn't want to see herself fail, I don't think she wants to participate. You can tell when you ask her a question, the look in the eyes." He added that as a result, "she would prefer to kind of stay quiet and stay back." Finally, he emphasized that when she did ask questions they were, "questions on the side, whether in class or outside of class." Overall, in regard to the theme *ways of studying mathematics and working in mathematics*, Ashley saw herself as an individual who lacked confidence in her ability to understand the material and participate as a central member of the mathematics classroom.

In regard to the theme *persistence and perseverance with mathematics,* the narratives suggest that Ashley viewed herself as a mathematics student who sometimes persevered in order to learn the material, but also saw herself as someone who gave up too easily and got discouraged in the mathematics classroom. On one hand, she viewed herself as someone who gave up when she was confused. She indicated that, "when I get confused I just give up sometimes. And yeah, I just get discouraged after that." This current identity was confirmed by her teacher. He explained that, "When it's conceptually difficult, she can feel overwhelmed by things." This current identity is also revealed in her comments regarding whether she liked to work on assignments and activities that stretched her abilities and made her think hard. She indicated that, “No, not really, I like easier ones.” She went on to say that, “I don’t spend that much time on it, I guess sometimes. I like to do the easy things, I feel like.” These statements suggest that Ashley viewed herself as someone who did not persevere in the mathematics classroom.

In contrast, Ashley’s narrative also reveals that she saw herself as an individual with a strong work ethic. For example, this identity is revealed in the statement, “I like go over my old problems that I’ve like missed and do them again, see if I can get them again.” This persistence is also displayed in her perspective on making mistakes when working in mathematics. She stated that, “I feel like they help me more because they help me like figure out what I did wrong, and I can know like next time not to do that.” Finally, this current identity is reflected in her teacher’s statements. Mr. Cox indicated that, “She works extremely hard.” He went on to explain that this hard work included the willingness to put in extra time needed to learn the material. He pointed out, “I mean she’s in most mornings to ask questions. I know she’s working with a tutor outside of class right now, one of my students from last year.” Mr. Cox explained that this perseverance has paid off. He stated, “She’s had success again since she’s gotten the tutor, and she just really again put in a large amount of time.” These examples suggest that in relation to the theme *persistence and perseverance with mathematics,* Ashley’s current mathematical identity was bifurcated. At times, she showed signs of persistence in her approach to learning mathematics but also got discouraged when confronted with difficult or confusing material.

Current identity summary. This section explored the current mathematical identities of Anna, Erik, Mitch, and Ashley through the consideration of their narratives.

This analysis was done to prepare for the cross case analysis and discussion related to subquestion A (this is found in Chapter 5). First, the data from the MINT protocol and the student interviews were used to create a summary of the current identity strength. Next, rich descriptions of the current identity of each student were created using the student narratives and the data from the teacher interviews.

In terms of the theme *ways of studying mathematics and working in mathematics*, Ashley and Anna saw themselves as individuals who lacked confidence in their mathematical abilities and did not view themselves as central members of the mathematics classroom. On the other hand, Erik viewed himself as a capable mathematics student who liked to control the terms of his learning. Mitch also viewed himself as a strong mathematics student and as a central member of the mathematics classroom. In regard to the theme *persistence and perseverance with mathematics,* all four students had a mixed current mathematical identity. Each individual described instances where they were persistent, as well as situations where they lacked perseverance, in the mathematical environment. In the next section, the *designated* identities of Anna, Erik, Mitch, and Ashley are explored.

Designated mathematical identity analysis. This section explores the designated identity of the four students, Anna, Erik, Mitch, and Ashley. As stated previously in Chapter 2, designated identities are often told in the future tense, or in a way that expresses aspiration, assurance, or need, about a situations or circumstances expected to be the case either presently or in the future; these identities are revealed in the form of stories (Sfard & Prusak, 2005). Statements that illustrate designated identity are: I have to spend more time on my mathematics homework, I want to be an engineer, and I should meet with the teacher to go over my questions about the review assignment if I want to do well on the upcoming exam. The following data sources are used: (a) the responses on the MINT protocol, (b) the transcripts of the student interviews, and (c) the transcripts of the teacher interviews. In each case, the themes of *mathematics as a rewarding subject* and *ways of studying mathematics and working in mathematics* are used as a structural framework to unpack the designated mathematical identities. In order to deepen the reader's knowledge of the designated mathematical identity of each of the four cases, rich and detailed descriptions, that make use of the words of the students and their teacher, are included (Merriam, 2009; Patton, 2002). The designated identity of Anna is explored first.

Anna. Anna's designated mathematical identity in terms of the theme *mathematics as a rewarding subject* was comprised of various rewards related to education and a future career that she anticipated her studies in mathematics would bring. In her narrative, Anna shared three primary reasons why she considers mathematics to be a worthwhile subject. First, Anna saw AP Statistics as a vehicle to achieve success in her current and future science courses and in her future career. In regard to helping her in her science courses, Anna explained that, "In Advanced Biology, we used a slight amount of stats, and that is one of the reasons I choose to take this course." She also believed that learning statistics would help her in her profession. Anna explained that, "I know math is important in the future that is why I work at it. I'm going into nursing next year, so I really want to pass this AP test." Anna's narrative suggests that she saw statistics as something that would support her in her science classes, but also as a ticket that could give her access to her chosen career. Her designated identity was, at least in part, built on these anticipated rewards her studies in mathematics would bring either at that time or in the future.

In terms of the theme *ways of studying mathematics and working in mathematics*, Anna's approach to learning was both *substantial* and *ritualized* (Sfard & Prusak, 2005). On one hand, Anna felt like the group activities were useful in terms of learning statistics. She stated that, "I think activities help a lot." Anna went on to explain that the activities supported her learning of mathematics because the problems encountered offered new perspectives related to the content and helped her realize, "that there are other things that you can do with it other than just the textbook problems." These statements reveal that Anna expected it to be the case that working through these activities would be useful. In essence, Anna saw the activities as a way to support her learning and make use of her statistical knowledge in a different way; this type of learning is referred to as *substantial* learning, where the learning process is about making sense of the material in a meaningful and lasting manner.

Similarly, Anna viewed mistakes in the mathematics environment as something that could benefit her in terms of learning and understanding the material (i.e., a *substantial* learning perspective). In that regard, she stated, "Sometimes I think you learn better through your mistakes because you don't bypass it, you think a little harder. Like last night's quiz, you're like, 'Oh, I didn't even think to draw a graph with that.'" Her narrative suggests that Anna expected mistakes to promote learning and make her think in a more in depth manner. This same substantial learning perspective was revealed in Anna's discussion of how the understanding of mathematics is determined. She stated that understanding occurs when:

> You can teach someone what you're trying to learn. Like when someone has a question and you can answer it promptly and quickly and you know what you're talking about. That's what I feel like opposed to sometimes even if you get a question right on a test, you don't totally know what you're doing, you don't know if it's right, kind of feels right, but I feel like when you [understand you] are like, 'Oh, I can explain this to you.'

In this case, her dialogue clearly suggests that she viewed the learning process as a way to make sense of the material in a meaningful and lasting manner. Her expectations and aspirations in terms of mathematical learning and understanding was that it would bring about the ability to explain a concept or idea in a thorough and thoughtful manner. This approach to learning is described as a *discourse-for-oneself* status (Sfard & Prusak, 2005). In addition, in the above statement she downplayed the *discourse-for-others* status when she characterized learning as a process of teaching and explaining concepts to peers rather than providing a correct answer on a teacher generated exam.

In contrast, Anna's also displayed an approach to learning that was *ritualized*. Anna described a situation where she was working on an activity and thought the material would have been helpful in preparing for the test she had taken a few days earlier. Anna explained, "Oh this would have been nice to do before the test to help cement those problems in your head." This description of the activity as a means to do better on the test suggests that in this case Anna's approach to learning was *ritualized* because the mathematical discourse was initiated by the teacher and was for the teacher; this is also characterized as a *discourse-for-others* status. In other words, Anna's focus was not necessarily on learning for the sake of learning, but learning to do well on a teacher-generated exam. This attitude toward learning was also displayed in other circumstances as well. For example, when discussing whether she ever explored the material beyond what was expected, Anna stated, “I usually stick to the assignment.” These statements indicate that her objectives related to learning mathematics were often linked to methods generated by the teacher or were for the teacher. Overall, Anna’s narrative revealed patterns of *ritualized* learning and instances of *substantial* learning; clearly Anna’s designated mathematical identity related to the theme *ways of studying mathematics and working in mathematics* was complex. In the next section, the themes of *mathematics as a rewarding subject* and *ways of studying mathematics and working in mathematics* and are used as a structural framework to unpack Erik’s designated mathematical.

Erik. Erik displayed a *substantial* approach to learning and a *discourse-for-oneself* status in regard to the theme *mathematics as a rewarding subject*. During the interviews or on his MINT protocol, Erik never specifically mentioned preparation for college or a future career as reasons for learning mathematics or taking mathematics courses. Instead, Erik spoke of the importance of mathematics in a more holistic sense. He stated that, “Math teaches me how to think and that is vital in every aspect of life.” He went on to say that, “I’m not actually using the mathematical concepts, I’m using the skills I’ve developed while using these mathematical concepts. So, it’s not purely relatable, but it’s effective because of the techniques it has taught me.” These statements suggest that Erik expected that the skills and techniques acquired as well as learning “how to think” would play a vital role in all areas of his life. Again, this narrative indicates a substantial approach to learning. Erik did not see mathematics as a voucher to be used to get into a good university or to obtain a desirable career; instead, his wish was that his life would be transformed by the experiences in the mathematics classroom that enhanced his ability to think. This aspiration was an essential component of his designated mathematical identity related to the theme *mathematics as a rewarding subject*,

Similarly, Erik's approach to learning was *substantial* and one that was based on a *discourse-for-oneself* status in regard to the theme *ways of studying mathematics and working in mathematics*. For example, Erik spoke at length about his desire to study and learn the material presented in mathematics in order to discuss the concepts and ideas in a thoughtful manner. As part of this explanation, Erik stated that:

> I think being able to discuss the topic shows that you are good at math. I think there is a difference between being able to simply do the problem and being able to discuss the concepts behind the problem. I think that being able to do the problem shows that you're good at memorization and discussion of the problem shows you're good at math.

This narrative clearly reflects an approach to learning that is substantial and one that is based on a discourse-for-oneself standing. Erik goes on to discuss the relationship between understanding mathematics and being able to discuss the concepts and ideas with others. He stated that working to understand the material can result in "increasing the ability of discourse you can have with the rest of the class and with the teacher." As can be seen this narrative, Erik described this ability to discuss the material in terms of an aspiration, something that he expected to be the case in the future if he properly prepares; this was an essential component of his designated mathematical identity.

This aspect of his designated mathematical identity was confirmed by his teacher. Mr. Cox stated that, “He’s very gifted, so the discussion in class is very good.” In addition, Mr. Cox indicated that Erik adds to the class discussion, “with his questions. I think he probably has the best questions out of anybody as far as just seeing the little details and even things outside the classroom and how it connects to stuff.” These statements suggest that this wish to be able to discuss the material in a thoughtful manner in order to show his peers and his teacher that he is good at mathematics was an important part of his designed mathematical identity.

Erik’s substantial approach to learning mathematics was also revealed in his rationale for engaging in challenging mathematical material. One example of this was when he discussed how the topic of related rates in his Calculus class inspired him. He explained, “I went back and did a proof for one of the theorems and that obviously wasn’t necessary, but I did it because I was kind of interested in it.” This narrative suggests that Erik’s interest drove his learning; in this case, that drive was not linked to doing well on an assignment or receiving a high mark on an exam (i.e., a discourse-for-others). Erik’s aspiration, the expectation that working through challenging mathematics problems would quench his mathematical curiosity was an important component of his designated mathematical identity. This identity was confirmed by the comments of his teacher. Mr. Cox stated that when exploring topics introduced in class, Erik, “is going to go beyond the curriculum.” Taken together, these statements clearly reflect an approach to learning that is *substantial* and one that is based on a *discourse-for-oneself* status. In the next section, the designated identity of Mitch is considered.

Mitch. Mitch's designated mathematical identity is explored using the following themes as a structural framework: (a) mathematics as a rewarding subject, and (b) ways of studying mathematics and working in mathematics. To do this, Mitch's responses on the MINT protocol and the transcripts of his two interviews are analyzed. In addition, the transcripts of the teacher interviews are used to confirm or disconfirm Mitch's mathematical designated identity.

Mitch's designated mathematical identity in terms of the theme *mathematics as a rewarding subject* was built, in part, on the pursuit and expectation of future rewards. Mitch indicated that his decision to take AP Statistics was impacted by the importance of mathematics in regard to his plans for college and a future profession. He stated, in regard to taking AP Statistics, that, "It's important because I plan on going for a business major and/or engineering." Mitch explained that his brothers impressed on him the importance of mathematics and statistics in their study of engineering. He also described the influence his father had on his decision to take statistics. Mitch explained, "My dad owns a construction company, so this summer I might be doing a little bit of number crunching for him, or like financing for him." Finally, Mitch reflected on the significance of mathematics and its impact on his approach to learning mathematics. He stated, "I realized mathematics is very important, so I have a dedicated attitude towards math." Together, these statements suggest that Mitch's aspiration was that his committed approach to the study of mathematics would be rewarding as it would give him the ability to help out at his dad's construction company and would provide him the means to succeed in college and in a future career. Therefore, in this case Mitch's designated mathematical identity was the result of a *ritualized* approach to learning and a *discourse-for-others* status.

Similarly, Mitch's designated identity related to the theme *ways of studying mathematics and working in mathematics* was composed of a *ritualized* approach to learning and was consistent with a *discourse-for-others* status. According to his narrative, Mitch perceived that being good at mathematics was shown through his grades, what mathematics courses he had taken, and how many mathematics courses he had taken. He stated, "What shows I'm good at math is the fact that I'm in Calculus AP and AP Statistics my senior year with the fact that I've gotten pretty much A's and B's in all of my high school math classes." Mitch also explained that he thought taking two mathematics classes at once also showed that he was good at mathematics. In this regard he stated that, "This is my second time doubling up. I doubled up my freshman year, and now I'm doubling up my senior year." His narrative suggests that he placed value on taking hard mathematics classes, taking a large number of mathematics classes, and getting good grades; it was those things, rather than showing a thorough understanding of mathematical concepts and ideas, that were important. This viewpoint was a significant aspect of Mitch's designated mathematical identity.

A similar perspective on learning was also evident in Mitch's belief that high grades and the completion of homework demonstrated mathematical competence. He stated, "I'd have to say, a good example would be, my calculus class, starting 6th hour with Miss Moberg. Like in order to get even a normal grade, you have to get like a hundred and ten percent and like to get an A you have to get like a hundred and thirty." This statement indicates that Mitch believed that getting high percentages and earning a good grade were critical in terms of learning, rather than gaining a thorough understanding of the material. This same attitude about learning was also displayed in how Mitch talked about homework related issues. For example, he stated that another example of showing that he is good at mathematics, "Is doing your homework every day, and making sure that when you do your homework it's right." In addition, when discussing his approach to homework in his AP Calculus class, Mitch stated, "When I'd do my homework, I wouldn't even check if I did the problems right, but now with Miss Moberg, that's the most important thing to do to, check if your answers are right. I'd say getting them right is more beneficial." Again, these narratives suggest a ritualized approach to learning, where Mitch was focused on the performance on teacher developed items (e.g., homework assignment, exams) in relation to what it meant to understand and learn mathematics. Overall, Mitch had a *discourse-for-others* status in relation to the theme *ways of studying mathematics and working in mathematics*. Finally, the designated identity of Ashley is examined in the next section using the themes of *mathematics as a rewarding subject* and *ways of studying mathematics and working in mathematics* and as a structural framework.

Ashley. In regard to the theme *mathematics as a rewarding subject*, Ashley's designated identity was built on a *discourse-for-others* status where her focus was on what she could get out of the study of mathematics, rather than on the learning and understanding of mathematics. Her narrative suggests that she saw statistics as a ticket to access a business degree in college and or a voucher to access a career. She stated that, "I want to go into business, so I probably need some sort of stats for a business degree and it helps get you a job." Furthermore, she stated that, "I like math that can actually apply to life skills." Taken together, these statements indicate that her designated identity was built on a *ritualized* approach to learning.

Similarly, Ashley's narrative exposes a *ritualized* approach to learning and a *discourse-for-other*s status (i.e., completing tasks assigned by the teacher for the teacher) related to the theme *ways of studying mathematics and working in mathematics*. For example, this designated identity was revealed in how she approached her homework assignments in the AP Statistics classroom. She stated, "When I have like a lot of other homework, I don't do this homework because he doesn't check it." This statement suggests that Ashley viewed homework as something that was completed for the teacher or for a grade, not as an avenue to gain and understanding of the material. This *discourse-for-others* approach, where the mathematical discourse was initiated by the teacher and was for the teacher, was also evident in Ashley's approach to preparing for exams. Ashley indicated that her ability to work hard was reflected in the manner in which she reviews for tests. She indicated that when a test is approaching, "I go in for help a lot more and do my homework more and better, like thoroughly." She went on to say, "That's probably when I give my best effort, before tests to study for them." Later, she explained that this especially happens, "when I need a good grade on a test or something." These examples suggest that Ashley's approach to learning was ritualized; her main focus was on preparing for and doing well on tests, rather than on understanding the concepts or ideas. This approach was confirmed by her teacher, Mr. Cox, who indicated that Ashley did come in for help, especially before tests or quizzes.

Ashley's perception of her mathematical achievement also revealed a designated mathematical identity that was associated with a *ritualized* approach to learning. She explained that she never really achieved and indicated that, “in math, my grades for it are usually C+ or B- range.” Her statement indicates that she believed that her achievement was connected to the grades she had received and not linked to her understanding of the material. In addition, her perspective of learning was also revealing. Ashley indicated that she often went in for help because, “he’ll just like tell me what to do because usually if I get one step I can get the rest of the steps down.” This statement indicates that she viewed mathematics as a step-by-step process or following a recipe. Taken together, Ashley’s narrative indicates that a crucial component of her designated identity related to the theme *ways of studying mathematics and working in mathematics* was an approach to learning that was about completing tasks assigned by the teacher for the teacher.

Designated identity summary. The designated mathematical identity of each of the four students was investigated through their narratives. The following data sources were used: (a) the responses on the MINT protocol, (b) the transcripts of the student interviews, and (c) the transcripts of the teacher interviews. The designated identity of each individual was described using detailed descriptions and making use of the words of the student and the teacher. The MINT themes of *mathematics as a rewarding subject* and *ways of studying mathematics and working in mathematics* were used as structural frameworks to organize the analysis of the designated mathematical identity of each individual.

In terms of the theme *mathematics as a rewarding subject*, Anna, Ashley, and Mitch all had a ritualized approach to learning and a discourse-for-others status. In contrast, Erik had a substantial approach to learning and a discourse-for-oneself status. The data revealed similar results in relation to the theme *ways of studying mathematics* *and working in mathematics*. Erik's narrative suggested an approach to learning that was focused on learning and understanding the material. On the other hand, Ashley and Mitch both described an approach to learning that was designed by others and for others (i.e., a ritualized approach to learning and a discourse-for-others status). Finally, Anna had a more complex designated identity as her narrative suggested both a ritualized and a substantial approach to learning. In the next section, the thinking moves and the mathematical identity of each of the four cases is summarized.

Summary of the Cases

Each of the four cases (i.e., Anna, Erik, Mitch, and Ashley) is summarized in this section. The summary considers their thinking moves during the problem-solving activities (i.e., group activities and responses on selected exam questions) and their current and designated mathematical identities as revealed through their narratives.

Anna. Anna only made partial use of the eight high-leverage thinking moves during the five group activities and on the selected examination questions. That being the case, the patterns of thinking moves demonstrated in the two contexts were different. Throughout the five activities, Anna used a total of 13 thinking moves and four unique high-leverage thinking moves. In addition, almost 70% of the thinking moves used by Anna resided in the *creative thinking* category; the remaining moves were part of the *critical thinking* category. The thinking moves pattern displayed by Anna when completing the exam questions was different; she used five different thinking moves that resided in all three thinking categories. In addition, these thinking moves provided evidence that Anna's thinking on the exam questions was statistical in nature as it displayed the integration of statistical knowledge and contextual knowledge (Pfannkuch

& Rubick, 2002) and demonstrated her ability to use statistical knowledge and procedures in an appropriate manner (delMas, 2004).

In terms of her *current* mathematical identity, Anna had mixed feelings and experiences related to mathematics, some positive and some negative (i.e., mixed identity strength). On one hand, in certain circumstances Anna viewed herself as an individual who had a strong work ethic and who persevered when the material was difficult. However, in other situations Anna saw herself as a student who lacked persistence in homework related issues. In addition, Anna saw herself as an individual who lacked confidence in her mathematical abilities and who was sometimes intimidated by the difficult concepts, the long chapters, and the amount of information to be learned.

Finally, Anna had a dichotomous *designated* mathematical identity as her narrative revealed both *substantial* and *ritualized* approaches to studying and learning mathematics. In certain situations, Anna indicated that her focus was on understanding the material (i.e., discourse-for-oneself), while in other circumstances her objective was to complete tasks for the teacher that were created by the teacher (i.e., discourse-for-others). Anna also displayed a *ritualized* approach to learning as she described her motivation for engaging with mathematics in terms of the expected benefits (e.g., preparation for university coursework, access to certain careers), rather than in terms of understanding the material.

Erik. Overall, Erik made full use of the high-leverage thinking moves during the five group activities and his ability to think in a statistical manner was displayed on multiple occasions. Through the activities, Erik displayed a total of 24 thinking moves; the thinking moves were fairly evenly spread across all three thinking categories. In this regard, half of the moves resided in the *mental management and awareness* category, a third of the thinking moves displayed were in the *creative* category, and the remainder of the thinking moves fell in the *critical thinking* category. Erik also made extensive use of the thinking moves on the selected exam questions and his thinking moves were spread out across all three thinking categories. Furthermore, during the group activities and on the exams, there was evidence in many of Erik's statements that he was thinking in a statistical manner. For example, he made connections between statistical knowledge and contextual knowledge, developed a statistical model that represented data in context, and created a simulation that he used to make inferences about the data. These are all evidence of statistical thinking (Garfield et al., 2012; Wild & Pfannkuch, 1999).

In terms of identity, Erik had mostly positive experiences and feelings related to mathematics (i.e., positive current identity). For example, Erik saw himself as a curious student who was often compelled, when motivated by the topic, to explore the material in more detail. He also saw himself as someone with a strong work ethic if he found the concept being studied interesting. In contrast, Erik also viewed himself as a mathematics student who did not approach his mathematical studies in a serious manner if the topic was not of interest to him. In addition, Erik perceived himself as someone who lacked persistence when distracted by peers or when busy with other tasks.

Finally, Erik had a *designated* mathematical identity that was based on a substantial approach to learning. For example, Erik perceived mathematics as a rewarding subject due to the way it helped developed thinking skills that were applicable in many parts of his life. In addition, Erik's rationale for working and studying in the mathematics classroom was to engage with and understand challenging mathematical material so he could discuss the material with his peers and teacher. In the end, Erik's designated identity was built on a *discourse-for-oneself* status.

Mitch. Overall, Mitch displayed components of statistical thinking through the use of a wide variety of thinking moves during the five group activities and on the exam responses. During the group activities, Mitch displayed 7 out of the 8 thinking moves and all three thinking categories were represented. On the exam questions, Erik's thinking moves were also spread across all three thinking categories. In examining his thinking moves during the activities and on the exam questions, there is evidence that Mitch thought in a statistical manner. For example, he helped create a statistical representation of the data that he used to make judgments about the data. In another instance, Mitch linked the statistical knowledge and procedures to the contextual knowledge when he made a key connection between the numbers a simulation produced and the context of the problem. Both these examples illustrate the ability to think statistically (Garfield et al., 2012; Wild & Pfannkuch, 1999).

In terms of mathematical identity, Mitch had a positive current identity (i.e., mostly positive experiences and feelings related to mathematics). For example, Mitch saw himself as a proficient mathematics student and as a central member of the mathematics classroom based on the success he had in pervious mathematics courses, his ability to be a vital part of class discussions, and his ability to help his classmates. In addition, Mitch viewed himself as someone who persevered through challenging mathematics problems. While Mitch had a strong positive current identity in many situations, he occasionally saw himself as a student who lacked motivation and did not always put in the effort needed to complete homework and prepare for exams.

Finally, in regard to *designated* mathematical identity, Mitch had a *ritualized* approach to learning. His comments indicated that he placed a high value on taking difficult mathematics classes and getting good grades rather than on demonstrating an understanding of the mathematical material. In the end, Mitch had a discourse-for-others status as he was focused on completing tasks designed by the teacher and for the teacher.

Ashley. Overall, Ashley only made partial use of the eight high-leverage thinking moves during the five group activities. Specifically, Ashley used 3 out of the 8 thinking moves; her most commonly used thinking move was *wondering and asking questions* which she used a total of nine times. In regard to the distribution across the thinking categories, two-thirds of Ashley's thinking moves were in the *creative thinking* category, one-third of Ashley's thinking moves were in the *critical thinking* category, and none of her thinking moves were in the *mental management and awareness* category. While there was no evidence that Ashley engaged in statistical thinking during the group discussions, she did make connections between the statistical and procedural knowledge and the contextual knowledge on exam responses. In addition, Ashley generalized the knowledge gained in class to new situations encountered in the exam questions. These are both ways that individuals can demonstrate the ability to think in a statistical manner (delMas, 2002, Garfield et al., 2012; Wild & Pfannkuch, 1999).

In regard to mathematical identity, Ashley had a negative current identity (i.e., mostly negative experiences and feelings related to mathematics). For example, Ashley saw herself as an individual who lacked confidence in her ability to understand the material and participate as a central member of the mathematics classroom. Ashley also saw herself as a mathematics student who gave up too easily and got discouraged when faced with confusing or difficult material. On the other hand, Ashley viewed herself as a mathematics student who persevered in areas related to learning mathematics such as doing her homework and coming in for help when necessary.

Finally, with respect to designated mathematical identity, Ashley had a *ritualized* approach to learning. This was true in terms of how she described her aspirations related to mathematics as a rewarding subject and her expectations in regard to studying and working in mathematics. In both instances, Ashley's focus was on completing mathematical tasks designed by her teacher and for her teacher (i.e., a discourse-for-others status).

Chapter 4 Summary

This chapter covered the case analysis and results of the four students Anna, Ashley, Erik, and Mitch. The case analysis was divided into two parts. First, the data collected related to the thinking moves of the four students were analyzed. This analysis was done through consideration of the student thinking moves displayed during the group activities and on the selected exam questions. Second, the data associated with the mathematical identity of each of the four students were studied. The mathematical identity case analysis was divided into current identity and designated identity. The data associated with the current mathematical identity of each case were explored to lay the foundation need to explore subquestion A (i.e., *to what degree and in what ways are the narratives students construct for their current mathematical identities related to the use of high-leverage thinking moves during problem-solving activities?*); this exploration will take place in Chapter 5. The data sources used were the responses on the MINT protocol and the responses during the student and teacher interviews. In addition, the designated mathematical identity of each of the four students was examined to lay the foundation needed to address subquestion B (i.e., *to what degree and in what ways are the narratives students construct for their designated mathematical identities related to the use of high-leverage thinking moves during problem-solving activities?*); this examination will be done in Chapter 5. The data sources used were the responses on the MINT protocol and the responses on the student and teacher interviews.

This chapter analyzed the data related to each of the four cases in regard to their mathematical identity and their use of the high-leverage thinking moves. Chapter 5 contains the cross case analysis and the cross case discussion, while Chapter 6 includes the limitations, the implications, and the areas of possible future research.

Chapter 5

Cross Case Analysis and Discussion

The present research study examined the relationship between mathematical identity and the use of thinking moves during problem-solving activities. A primary objective of the study is to learn more about the relationship individuals have with mathematics in terms of their feelings and experiences. This phenomenon of mathematical identity is explored through the narratives of the individuals; the use of narrative inquiry is one of the most effective ways to study the experiences of individuals in an educational setting (Clandinin & Connelly, 2000; Kaasila, 2007). In order to determine the impact of mathematical identity on the understanding of mathematical concepts and ideas, the thinking moves displayed by students during problem-solving activities are examined. This approach is taken (i.e., the exploration of student thinking moves to gauge understanding) because understanding is viewed as a consequence of thinking, not as a component of thinking (Ritchhart et al., 2011). Moreover, the use of quantitative measures of mathematical understanding (e.g., standardized test scores, exam scores in school) do not give a complete picture of whether a student is successfully engaging with the material (Gutierrez, 2000, 2007; Martin, 2000, 2009; Nasir, 2002; Stanovich & West, 2008) as understanding is result of a complex process that requires diverse forms of thinking. In the end, the thinking moves patterns during problem-solving activities have the potential to reveal the depth and breadth of student understanding.

The overall goal of this qualitative study is to understand the connections between these two phenomena: the thinking moves displayed by students during problem-solving activities and the mathematical identity of the students. This objective is framed in the research question: To what degree and in what ways are the narratives students construct for their mathematical identities related to the use of high-leverage thinking moves during problem-solving activities? This overarching question is made up of two subquestions:

A. To what degree and in what ways are the narratives students construct for their *current* mathematical identities related to the use of high-leverage thinking moves during problem-solving activities?
B. To what degree and in what ways are the narratives students construct for their *designated* mathematical identities related to the use of high-leverage thinking moves during problem-solving activities?

Because the phenomena under consideration are fluid and complex, a qualitative case study was chosen. In Chapter 4, the data for each of the four cases (i.e., Anna, Ashley, Erik, Mitch) were analyzed in terms of three phenomena: (a) thinking moves, (b) current mathematical identity, and (c) designated mathematical identity. This chapter (Chapter 5) includes a cross case analysis and a cross case discussion of the four cases; this is done in relation to the research question and the two subquestions. Because the present research study was undertaken to promote a deeper understanding of the relationship between thinking moves and mathematical identity, the cross case analysis includes rich descriptions and makes use of the words of the students (Merriam, 2009; Patton, 2002).

In the first section, the analysis of the thinking moves used by the four students during problem-solving activities is summarized. Next, a cross case analysis related to the mathematical identities (i.e., current identity and designated identity) of the students is discussed. Finally, the results of the cross case analyses are used to answer subquestion A and subquestion B.

Cross Case Analysis of Thinking Moves

In this section, the thinking moves displayed by the four students are examined using the data from: (a) the discussions during the group activities, and (b) the responses to selected exam questions.

Thinking moves displayed during group activities. In the first part of this section, the thinking moves used by the four students during the group activities are explored in relation to the three thinking categories. After that, the high-leverage thinking moves displayed by the students are examined in regard to the following categories: (1) partial use of the high-leverage thinking moves, and (2) extensive use of the high-leverage thinking moves. The analysis starts with the consideration of thinking moves that reside in the *creative thinking* category.

Creative thinking moves. During completion of the group activities, all four students displayed thinking moves in the creative thinking category (see Table 5.1). The most common thinking move used in the creative thinking category was *wondering and asking questions*; in fact, almost 60% of the thinking moves displayed by all four students resided in this category. In addition, *wondering and asking questions* was the most commonly used thinking move used by Anna, Ashley, and Mitch within the creative thinking category, while it was Erik's second most used thinking move in this category.

Table 5.1.

Summary of Students' Thinking Moves During Group Activities That Reside in the Creative Thinking Category

	Thinking move		
Individual	Considering different viewpoints and perspectives	Wondering and asking questions	Uncovering complexity and going below the surface of things
Anna	1	5	3
Ashley	0	9	1
Erik	1	2	5
Mitch	1	6	4
Total	3	22	13

The thinking move *uncovering complexity and going below the surface of things* was also used by all four students. Of the three thinking moves in the creative thinking category, Erik used this thinking move the most. On the other hand, the least used thinking move was *considering different viewpoints and perspectives*. Anna, Erik and Mitch each used this thinking move once; Ashley did not display this thinking move at all. While all four students made extensive use of thinking moves that fall in the *creative* thinking category, that was not the case in relation to the thinking moves that reside in the *mental management and awareness* category. Those thinking moves are considered in the next section.

Mental management and awareness thinking moves. The use of the three thinking moves (i.e., building explanations and interpretations, capturing the heart and forming conclusions, and making connections) in the *mental management and awareness* thinking category were very different among the four cases (see Table 5.2). First, Anna and Ashley did not display any thinking moves in this category. In addition, Erik was the only one of the four cases that made use of all three of the thinking moves in this category. In total he exhibited 12 thinking moves that were fairly evenly distributed across the three thinking moves. Erik's most commonly used high-leverage thinking move was *building explanations and interpretations*. Mitch displayed five thinking moves in this category; he exhibited the thinking move *building explanations and interpretations* three times and the thinking move *making connections* twice. In addition, he did not display the thinking move *capturing the heart and forming conclusions*.

Table 5.2.

Summary of Students' Thinking Moves During Group Activities That Reside in the Mental Management and Awareness Thinking Category

	Thinking move		
Individual	Building explanations and interpretations	Capturing the heart and forming conclusions	Making connections
Anna	0	0	0
Ashley	0	0	0
Erik	5	4	3
Mitch	3	0	2
Total	8	4	5

Clearly, the use of thinking moves that reside in the *mental management and awareness* category was very different across the four students. In contrast, the patterns of thinking moves use in relation to the *critical thinking* category had more similarities. Those thinking moves are explored next.

Critical thinking moves. The use of the thinking moves that resided in the *critical thinking* category looked fairly similar among the four students (see Table 5.3). All four students made use of the thinking move *observing closely and describing what's there* during the completion of the group activities.

Table 5.3.

Summary of Students' Thinking Moves During Group Activities That Reside in the Critical Thinking Category

	Thinking move	
Individual	Observing closely and describing what's there	Reasoning with evidence
Anna	4	0
Ashley	5	0
Erik	3	1
Mitch	2	1
Total	14	2

On the other hand, the least used thinking move was *reasoning with evidence*. Erik and Mitch each used this thinking move once; Ashley and Anna did not use the thinking move at all.

Summary. Overall, there were different patterns of thinking moves used among the four cases in regard to the three thinking categories during group activities (see Table 5.4). For example, the majority of the thinking moves displayed by Anna, Ashley, and Mitch resided in the *creative thinking* category; in fact, 67% of Ashley's thinking moves, 69% of Anna's thinking moves, and 58% of Mitch's thinking moves fell into this category. On the other hand, the preponderance of Erik's thinking moves (i.e., 12 out of

24) fell into the *mental management and awareness* category. Another significance difference between the four cases was the number of thinking moves exhibited that resided in the *mental management and awareness* category. For example, Mitch and Erik made use of thinking moves in all three categories including the *mental management and awareness* category, while Ashley and Anna did not demonstrate the use any of thinking moves in this category during the group activities.

Table 5.4.

Summary of Students' Thinking Moves During Group Activities Arranged by Thinking Category

	Thinking category		
Individual	Creative thinking	Mental management and awareness	Critical thinking
Anna	9	0	4
Ashley	10	0	5
Erik	8	12	4
Mitch	11	5	3

Overall, the patterns of the eight high-leverage thinking moves used by the four students had commonalities and differences. The pattern of thinking moves used by Anna and Ashley looked very different from the pattern of thinking moves used by Erik; Mitch's pattern of thinking moves had common aspects with both Erik's pattern and the pattern displayed by Anna and Ashley. These similarities and differences are explored in the next section in more detail making use of the following two categories: (1) partial use of the high-leverage thinking moves, and (2) extensive use of the high-leverage thinking moves.

Partial use of the thinking moves. During the various group activities, Ashley and Anna both made partial use of the eight thinking moves (see Table 5.5 for a summary of the thinking moves used by the four students). Specifically, Anna exhibited 4 out of the 8 thinking moves and Ashley displayed 3 out of 8 of the thinking moves. The most common high-leverage thinking move used by both individuals was *wondering and asking questions*. In addition, the majority of Anna's thinking moves and Ashley's thinking moves resided in the *creative thinking* category; none of the thinking moves of Ashley or Anna fell in the *mental management and awareness* category.

Table 5.5.

Summary of Students' Thinking Moves Used During Group Activities

		Total			
Thinking category	High-Leverage thinking move	Anna	Ashley	Erik	Mitch
Creative thinking	Considering different viewpoints and perspectives	1	0	1	1
	Wondering and asking questions	5	9	2	6
	Uncovering complexity and going below the surface of things	3	1	5	4
Mental management and awareness	Building explanations and interpretations	0	0	5	3
	Capturing the heart and forming conclusions	0	0	4	0
	Making connections	0	0	3	2
Critical thinking	Observing closely and describing what's there	4	5	3	2
	Reasoning with evidence	0	0	1	1
Totals		13	15	24	19

Furthermore, there was no evidence to suggest that during the group activities Ashley or Anna displayed thinking moves that would indicate that they thought in a statistical manner. In this regard, neither student demonstrated that they connected their statistical knowledge and logical thinking to the data in context; making this connection is a distinguishable component of statistical thinking (Pfannkuch & Rubick, 2002). For example, during the *On Time Airline* MEA neither Anna nor Ashley ever contributed to the discussion regarding what the outliers mean to travelers in terms of their ability to make connections or why it makes sense to use the mean or the median in that context. This and other similar examples suggest that Anna and Ashley did not demonstrate identifiable elements of statistical thinking during the group activities.

Extensive use of the thinking moves. Both Mitch and Erik made extensive use of the eight high-leverage thinking moves during the group activities. Erik made use of all eight of the thinking moves and Mitch made use of 7 out of the 8 thinking moves during the activities (see Table 5.5). In addition, Erik displayed a total of 24 thinking moves, while Mitch made use of 19 thinking moves. That being the case, the thinking moves of Mitch and Erik were distributed differently across the three thinking categories. For example, eight of the thinking moves displayed by Erik were in the *creative thinking* category, twelve of the moves resided in the *mental management and awareness* category, and four of the moves fell in the *critical thinking* mode. In terms of the pattern of the thinking moves used by Mitch, over half of his thinking moves were in the creative thinking category. Finally, the two most commonly used thinking moves used by Erik were: (a) uncovering complexity and going below the surface of things, and (b) building explanations and interpretations. In contrast, the most common thinking move used by

Mitch was *wondering and asking questions*.

In addition, there was evidence in many of Erik's and Mitch's statements that they were thinking in a statistical manner. For example, during the *On Time Airline* MEA, Erik developed, using statistical and contextual knowledge, a statistical model that represented the data in context that would work for the company, or another company, to deal with a similar situation in the future. Similarly, Mitch's thinking during the *ESP* activity stimulated the creation of a box plot that provided a visual representation of the number of matches in each set of 10. In this instance, Mitch's thinking revealed a key connection between the numbers the simulation produced and the context of the problem. These examples of building statistical models to simulate data and make inferences about the data and of being able to link statistical knowledge and procedures to the contextual knowledge reveal components of statistical thinking (Garfield et al., 2012; Wild & Pfannkuch, 1999). Overall, Mitch and Erik demonstrated the use of a wide variety of thinking moves. In addition, there is evidence that they displayed statistical thinking during the group activities.

In regard to the type of high-leverage thinking moves that were used during the group activities, Anna's thinking and Ashley's thinking were significantly different than the thinking of Mitch and Erik. The thinking moves of Ashley and Anna resided mainly in the *creative* and *critical* thinking categories, while the thinking moves of Mitch and Erik resided in all three thinking categories (i.e., critical, creative, and mental management and awareness). In the next section, the thinking moves displayed by the four cases when completing exam questions are explored.

Thinking moves displayed during exams. In contrast to the thinking moves used during the group activities, the thinking moves demonstrated on the exam question responses were fairly similar for the four students (see Table 5.6 for a summary of the high-leverage thinking moves used during exams). The four thinking moves primarily used by the students were: (a) uncovering complexity and going below the surface of things, (b) building explanations and interpretations, (c) making connections, and (d) observing closely and describing what's there (note that the two thinking moves, *wondering and asking questions* and *considering different viewpoints and perspectives*, were not able to be used based on the format of the exam questions). Ashley only used those four moves, while Anna and Erik both used one additional thinking move (i.e., capturing the heart and forming conclusions). Mitch made use of two additional thinking moves; he used the thinking move *capturing the heart and forming conclusions* once and the thinking move *reasoning with evidence* twice. Overall, the number of thinking moves and the types of thinking moves used during completion of the selected exam questions were fairly similar across the four cases.

Table 5.6.

Summary of Students' Thinking Moves Displayed During Exams

		Total			
Thinking category	High-Leverage thinking moves	Anna	Ashley	Erik	Mitch
Creative thinking	Considering different viewpoints and perspectives	N/A	N/A	N/A	N/A
	Wondering and asking questions	N/A	N/A	N/A	N/A
	Uncovering complexity and going below the surface of things	3	3	3	3
Mental management and awareness	Building explanations and interpretations	3	3	3	3
	Capturing the heart and forming conclusions	1	0	1	1
	Making connections	2	3	2	3
Critical thinking	Observing closely and describing what's there	2	3	2	2
	Reasoning with evidence	0	0	0	2
Totals		11	12	11	14

However, the pattern of thinking moves displayed during the completion of selected exam questions looked very different than the pattern of thinking moves exhibited during the group activities. The differences in these patterns are analyzed and the connections to statistical thinking are considered in the two next sections. This analysis is divided into two categories: (a) different patterns of thinking moves between group activities and exams, and (b) similar patterns of thinking moves between group activities and exams.

Different patterns of thinking moves. The thinking moves displayed by Anna and Ashley during the group activities looked very different than the thinking moves they demonstrated on the selected exam questions (see Table 5.7). For example, on the exam questions Anna and Ashley made more extensive use of the high-leverage thinking moves than they did during the group activities. In addition, both students made use of thinking moves that resided in all three thinking categories on the exam questions; recall that during the group activities they only made use of thinking moves in the *creative thinking* and *critical thinking* categories. Furthermore, there is evidence that indicates that Anna and Ashley engaged in statistical thinking on exam responses, while they did not demonstrate aspects of statistical thinking during the group activities. For example, on the exam questions they made connections between statistical knowledge and contextual knowledge, they generalized knowledge gained in class to the new situations they encountered, and they demonstrated the appropriate use statistical knowledge and procedures. These are all ways in which individuals can demonstrate thinking in a statistical manner (delMas, 2002, 2004; Pfannkuch & Rubick, 2002).

Table 5.7.

Summary of Students' Thinking Moves Used During Problem-Solving Activities

		Thinking category		
Student	Problem-Solving activity	Creative thinking	Mental management and awareness	Critical thinking
Anna	Group activities	9	0	4
	Exam responses	3	6	2
Ashley	Group activities	10	0	5
	Exam responses	3	6	3
Erik	Group activities	8	12	4
	Exam responses	3	6	2
Mitch	Group activities	11	5	3
	Exam responses	3	7	4

In the end, while Anna and Ashley did not demonstrate components of statistical thinking moves during the group activities, there is evidence to suggest that they did think statistically during the completion of the selected exam questions. The patterns of thinking moves displayed by Erik and Mitch during the group activities and on the exam questions are explored in the next section.

Similar patterns of thinking moves. Erik and Mitch had similar patterns in terms of the high-leverage thinking moves used during group activities and demonstrated on responses to selected exam questions. Mitch made use of all six of the possible thinking moves on the exam questions, while Erik displayed 5 out of 6 of the possible thinking moves (recall that the two thinking moves, *wondering and asking questions* and *considering different viewpoints and perspectives*, were not able to be used based on the format of the exam questions). This was similar to the distribution of thinking moves displayed by Erik and Mitch during the group activities where Erik made use of all eight thinking moves and Mitch made use of 7 out of 8 of the thinking moves. In addition, Erik's and Mitch's thinking moves, both during the group activities and on the selected exam questions, were spread out across all three thinking categories. Finally, as during the group activities, there was evidence in Erik's and Mitch's responses on the selected exam questions that they were able to think in a statistical manner.

This completes the cross case analysis of the use of high-leverage thinking moves used during group activities and on the selected exam questions. The cross case analysis in regard to mathematical identity is explored in the following section.

Cross Case Analysis of Mathematical Identity

In order to address the primary research question and the two research subquestions, a cross case analysis related to the mathematical identity of the four students was completed. This cross case analysis was divided into two parts based on the two research subquestions; the first part examined the *current* mathematical identity and the second part addressed the *designated* mathematical identity.

Current mathematical identity. This section is a cross case analysis related to subquestion A: To what degree and in what ways are the narratives students construct for their *current* mathematical identities related to the use of high-leverage thinking moves during problem-solving activities? Current identities are identities that are the stories and narratives of individuals that are generally told in the present tense and that reflect current circumstances (Graven & Buytenhuys, 2011; Sfard & Prusak, 2005). This section explores the current mathematical identity across the four cases (i.e., Anna, Erik, Mitch, and Ashley) through their narratives as revealed in the data from the MINT protocol responses, the student interviews, and the teacher interviews. Initially, the data from the MINT protocol and the student interviews were organized according to five current mathematical identity categories related to strength of experience or feelings (i.e., strong positive, positive, mixed, negative, and strong negative). The summary of this distribution organized according to strength is shown in Table 5.7.

Table 5.8.

Summary of Students' Current Identity

	Total responses by category				
Student	Strong Positive	Positive	Mixed	Negative	Strong Negative
Anna	1	11	18	7	2
Ashley	1	7	6	12	6
Erik	27	6	8	5	0
Mitch	25	5	3	4	0

According to this current identity summary, Ashley displayed a negative current mathematical identity, while Anna displayed a mixed current mathematical identity. On the other hand, Mitch and Erik both displayed positive current mathematical identities. Therefore, for this analysis, the current identity of Anna and Ashley are examined together and the current identity of Erik and Mitch are considered together. This exploration is structured using two of the MINT themes: (a) persistence and perseverance with mathematics, and (b) ways of studying mathematics and working in mathematics.

Persistence and perseverance with mathematics. Using the theme *persistence and perseverance with mathematics*, this section explores the similarities and differences across the four cases. This is done using examples of positive current identity and examples of negative current identity.

Positive current identity. While the narrative of all four students revealed a strong positive current identity in matters of persistence, there were differences in how the students viewed themselves with regard to this theme. On one hand, Ashley's narrative and Anna's narrative suggest that they viewed themselves as students who persevere in mathematics in how they approach their homework and in getting help when they do not understand the material. Conversely, Erik and Mitch saw themselves as persistent in seeking out and solving problems that offered challenge and intrigue. The narratives of Ashley and Anna are considered first.

Anna is a mathematics student who views herself as an individual who has a strong work ethic and who perseveres when the homework is confusing. For example, Anna stated that when she is, "confused about something, I'm trying to kind of work the hardest and overcome it." In a similar way, Anna indicated that, "I think it's important just to be able to take your own notes and kind of go into more detail on things that he explains in class or maybe if you're confused, look into it more." These statements portray a person who takes notes, looks into things further, and works hard to overcome obstacles such as confusing material. This identity was confirmed by Anna's teacher. Mr. Cox indicated that he noticed a change in Anna's approach to her studies during the semester. He stated that, "After parent conferences she came and talked to me and admitted that she should be doing more, that she wasn't doing enough outside of class. And so she's actually stepped up her efforts and the results have been better." This statement suggests Anna was aware that her effort was lacking and that to successfully learn the material she needed to work harder.

Ashley's narrative also reveals that she saw herself as an individual with a strong work ethic. For example, this identity is revealed in the statement, "I like go over my old problems that I've like missed and do them again, see if I can get them again." This persistence is also displayed in her perspective on making mistakes when working in mathematics. She stated that, "I feel like they help me more because they help me like figure out what I did wrong and I can know like next time not to do that." Finally, this current identity is reflected in her teacher's statements. Mr. Cox indicated that, "she's in most mornings to ask questions. I know she's working with a tutor outside of class right now, one of my students from last year." Overall, these narratives suggest that Ashley viewed herself as a mathematics student who persevered in order to learn the material.

In terms of persistence and perseverance, Erik and Mitch viewed themselves in a slightly different manner. For example, Mitch viewed himself as someone who perseveres through challenging mathematics problems. He explained, "But if it's a problem that I can work through but it's also difficult, I like it because it keeps me interested." This example and others suggests that Mitch perceived himself as someone who exceeded expectations through determination, not to get a better grade, but out of the need to be challenged or to pursue an interest. In a similar way, Erik's narrative suggests that he saw himself as someone whose work ethic was tied to his curiosity in the topic being studied. He stated that, "In Calculus, we are working on related rates right now and it's difficult, but I'm putting a lot of effort into that." He added that he was putting the effort in because he was interested in the topic of related rates. This approach clearly embodied a strong work ethic and a willingness to put in the necessary time based on interest in the topic.

These narratives related to perseverance suggest that Anna and Ashley saw themselves differently than Erik and Mitch. For example, Anna and Ashley saw themselves as students who had a strong work ethic in relation to the completion of homework and seeking out assistance when necessary (e.g., coming in before school for help, working with a tutor). In contrast, Erik and Mitch viewed themselves as students who persevered in mathematical environments that offered challenge and intrigue. This contrast between cases is also seen in areas related to *lack of* persistence; these issues are explored in the next section.

Negative current identity. At times, all four students viewed themselves as students who lacked persistence in the mathematical environment. According to their narrative, Ashley and Anna both saw themselves as students who lacked perseverance related to mathematics in areas related to homework and the difficulty of the material. Conversely, Erik and Mitch saw themselves as lacking persistence in mathematics when the interest or challenge was not there. The narratives of Ashley and Anna are considered first.

Ashley's and Anna's narratives in terms of the theme *persistence and perseverance with mathematics* indicate that they each viewed themselves as a student who gave up when confronted with confusing or difficult material. For example, Ashley stated that, "when I get confused I just give up sometimes. And yeah, I just get discouraged after that." This current identity was confirmed by her teacher. He explained that, "When it's conceptually difficult, she can feel overwhelmed by things." These statements suggest that Ashley viewed herself as someone who did not persevere in the mathematics classroom when the material was confusing or difficult. Similarly, Anna saw herself as a mathematics student that lacked persistence when faced with challenging or confusing concepts. She indicated that she sometimes finds mathematics challenging as she has difficulty "just grasping the concepts at times." She went on to say that, "you don't want to spend too much time going back and reviewing what you've done." This statement and others suggest that while Anna viewed herself as someone who does not always understand the material, she also saw herself as someone who was not always willing to spend the time necessary to relearn or become more comfortable with the material.

While Anna and Ashley both tied their lack of perseverance to working through confusing or challenging material, Mitch and Erik viewed their lack of persistence slightly differently. Mitch made it clear in some of his comments that he did not always put in the effort needed. He explained that his lack of commitment is revealed by, "not doing my homework, and not reviewing before tests, not paying attention in class, like talking to friends instead." His teacher, Mr. Cox, confirmed this. He stated, "I would think he's pretty confident, but I think he sometimes lacks the outside work. So because of that he may not feel as confident because he hasn't done the practice." These narratives reveal that Mitch viewed himself as a mathematics student who did not always have the work ethic to complete the homework, review for exams, or to learn the material.

In a similar way, Erik also saw himself as a mathematics student who sometimes avoided too much work. For example, he stated that in relation to homework for mathematics class, "I enjoy it, but it's kind of hard to sit down and, you know, do an hour or two of math." This current identity was confirmed in Mr. Cox's narrative as well. According to his teacher, Erik "can fall behind or display a lack effort at times. He's not someone who is going to kill himself. He lives a pretty happy, relaxed life, but he enjoys learning." Taken together, Erik's and Mr. Cox's statements indicate that Erik's current mathematical identity related to persistence and perseverance was situational and often related to interest.

Current identity summary in relation to persistence and perseverance with mathematics. In terms of current identity, all four students had a dichotomous view of themselves as mathematics students related to *persistence and perseverance in mathematics*. There are times when they saw themselves as a student with a strong work ethic, but they also described situations where they lacked persistence in the study of mathematics. In addition, within those perspectives related to current identity, Ashley and Anna viewed themselves differently than did Erik and Mitch in some respects. For example, Anna and Ashley viewed their persistence in terms of the effort (e.g., completing homework, coming in for help, and preparing for tests) needed to overcome difficult and confusing material. In contrast, Erik and Mitch saw their perseverance based on their connection with the content in terms of challenge, interest, and curiosity. While Anna and Ashley saw themselves as students who might give up when faced with confusing concepts, Erik and Mitch viewed themselves as students who might not even engage with the material if they did not find it interesting or had other things (e.g., talking to friends, relaxing) that they wanted to do instead. On the other hand, Anna and Ashley sometimes viewed themselves as mathematics students who would persevere through hard work, coming in before school, and working with a tutor to prepare for tests. This was in contrast to Mitch and Erik who saw themselves as students who would persist in situations where they were challenged or intrigued by the mathematics concepts being studied. The next section compares the current identity of the four cases in terms of the theme *ways of studying and working in mathematics*.

Ways of studying and working in mathematics. Using the theme *ways of studying and working in mathematics*, this section explores the similarities and differences across the four cases in regard to their current identity. As with the theme *persistence and perseverance in mathematics*, there were differences between the four cases. In terms of studying and working in mathematics, Ashley and Anna both saw themselves as students who lacked confidence in their mathematical abilities and did not see themselves as a central member of the mathematics classroom. In contrast, Erik and Mitch viewed themselves as confident mathematics students who played a central role in the mathematics classroom. The cases of Anna and Ashley are considered first.

Negative current identity. Both Anna and Ashley viewed themselves as mathematics students who lacked confidence and were on the margin in the mathematics classroom in relation to the theme, *ways of studying and working in mathematics*. When learning in the mathematics classroom, Anna saw herself as someone who lacked confidence in her mathematical abilities. She stated that, “You definitely have to work at it. There have been times where I know what’s going on, but it’s sometimes hard to put the different concepts together.” This and other examples suggest that she saw herself as a student who was sometimes intimidated by the difficult concepts, the long chapters, and the amount of information to be learned. Her teacher confirmed Anna’s lack of confidence in the mathematics classroom. Mr. Cox stated that, “I wouldn’t say, you know, she's feeling incredibly confident." Overall, this evidence suggests that in relation to the theme *ways of studying and working in mathematics*, Anna saw herself as student who lacked confidence in the mathematics classroom.

In a similar manner, Ashley saw herself as someone who lacked confidence in her ability to work and learn in the mathematics classroom. This lack of confidence is revealed in Ashley's use of the phrase *I second guess myself* on several occasions. For example, she stated that, "I do not enjoy math really. I always second guess myself." In addition, she explained that, "I don't test that well in math. I always second guess myself." This lack of confidence was confirmed by her teacher, Mr. Cox. He stated that, "even when she does the work, she's concerned, and she finds herself double guessing her work at times." He added that, "she lacks confidence, lacks some confidence in her math ability." These statements and others indicate that Ashley perceived herself as a student who lacked confidence in her abilities and did not see herself as a central member of the mathematics classroom.

Overall, Ashley and Anna viewed themselves as students who lacked confidence in the *ways of studying mathematics and working in mathematics.* In addition, their narratives indicate that they viewed themselves as students that were not a key part of the mathematics classroom. In contrast, the current identity of Mitch and Erik were very different; their current mathematical identities are considered in the next section.

Positive current identity. Both Mitch and Erik viewed themselves as confident mathematics students who were significant members of the mathematics classroom. First, in terms of learning mathematics, Mitch saw himself as a highly capable and committed student. For example, Mitch stated, "I have a dedicated attitude towards math." This overall current identity was confirmed by his teacher who indicated that, “Mitch, I think has confidence in his abilities in math. I would say he would feel pretty competent.” Mitch’s current mathematical identity related to learning mathematics was also portrayed in his willingness to be a participant in the mathematics classroom; this was evident both in terms of helping his classmates and answering questions in class. Taken together, these narratives suggest that in terms of the theme, *ways of studying mathematics and working in mathematics*, Mitch had a strong positive current mathematical identity.

Erik’s current mathematical identity was also categorized as strong positive in relation to the theme *ways of studying mathematics and working in mathematics*. However, Erik saw himself as a student who enjoyed learning mathematics on his own terms. For example, he explained that, “I’m actually not very into real life situations, I just prefer analytical mathematics.” His narrative suggests that the quality of the experience with mathematics was dependent, at least to an extent, on what type of mathematics was involved. His teacher, Mr. Cox, confirmed that while Erik had the propensity to do extra work and that he really enjoyed learning, he also made it clear that Erik was, “going to learn on what he wants to learn on.” In summary, regarding the theme *ways of studying mathematics and working in mathematics,* Erik saw himself as a curious student who was often compelled to dig deeper into the topic at hand. However, he also saw himself as mathematics student who wanted to dictate the terms of his learning.

Current identity summary in relation to ways of studying and working in mathematics. The current mathematical identities of the four students were very different. Anna and Ashley saw themselves as marginal players who lacked confidence in mathematical environments related to *ways of studying and working in mathematics.* Both individuals described this lack of confidence in terms of their engagement with difficult and challenging concepts and ideas as well as taking part in class discussions. On the other hand, Mitch viewed himself as a central member of the mathematics classroom; he was willing to be an active part of class discussions and help peers when needed. In addition, he viewed himself as a confident student who was able to successfully navigate through a variety of challenging mathematics courses. Erik also saw himself as a self-assured mathematics student. However, his current identity appeared to be linked more to seeking out and solving challenging and interesting mathematical problems. A summary of those results is displayed in Table 5.8.

Table 5.9.

Summary of Students' Current Identity

Case	Current identity related to the theme: *ways of studying mathematics and working in mathematics*	Current identity related to the theme: *persistence and perseverance with mathematics*
Anna	Negative current identity ▪ Anna saw herself as student who lacked confidence in her mathematical abilities ▪ Anna viewed herself as student who was sometimes intimidated by the difficult concepts, the long chapters, and the amount of information to be learned	Dichotomous current identity ▪ Anna sometimes viewed herself as an individual who has a strong work ethic and who perseveres when the material is difficult. ▪ Anna sometimes viewed herself as a student who sometimes lacked persistence in homework related issues.
Ashley	Negative current identity ▪ Ashley saw herself as an individual who lacked confidence in her ability to understand the material and participate as a central member of the mathematics classroom.	Dichotomous current identity ▪ Ashley viewed herself as a mathematics student who sometimes persevered in order to learn the material. ▪ Ashley sometimes saw herself as someone who gave up too easily and got discouraged in the mathematics classroom when confused or faced with difficult material.
Erik	Dichotomous current identity ▪ Erik saw himself as a curious student who was often compelled, when motivated by the topic, to explore the material in more detail. ▪ Erik also viewed himself as mathematics student who did not approach his mathematical studies in a serious manner if the topic was not of interest to him.	Dichotomous current identity ▪ Erik sometimes saw himself as someone with a strong work ethic if his curiosity was piqued by the topic being studied. ▪ Erik sometimes viewed himself as someone who lacked persistence if the topic was not interesting to him or if he had better things to do.
Mitch	Strong positive current identity Mitch saw himself as a highly capable mathematics student and a central member of the mathematics classroom based on the success he had in pervious mathematics courses and his ability to be a central part of discussions and helping out peers in class.	Dichotomous current identity ▪ Mitch sometimes viewed himself as someone who persevered through challenging mathematics problems. ▪ Mitch occasionally saw himself as a student who did not always put in the effort needed to complete homework and prepare for exams.

This concludes the cross case analysis of the four cases (i.e., Anna, Ashley, Erik, and Mitch) in regard to their current mathematical identity. In the next section, the cross case analysis related to the designated identity of each case is discussed.

Designated mathematical identity. In this section, the designated mathematical identity of each of the four cases is investigated through their narratives. This exploration forms a foundation to answer the subquestion B: To what degree and in what ways are the narratives students construct for their *designated* mathematical identities related to the use of high-leverage thinking moves during problem-solving activities? The designated identity of each individual is described using detailed descriptions of the data and making use of the words of the student and the teacher. Two of the seven themes are used as a framework to organize the analysis of the designated mathematical identity of each individual: (a) mathematics as a rewarding subject, and (b) ways of studying mathematics and working in mathematics.

Mathematics as a rewarding subject. Erik's perspective of mathematics as a rewarding subject was different than the perspectives of Anna, Ashley, and Mitch. Erik's narrative suggests a more universal view of the rewards associated with learning mathematics; this can be described as *substantial learning* and as a *discourse-for-oneself* (Sfard & Prusak, 2005) where the approach to learning is directed by the learner and is for the learner. In contrast, the narratives of Anna, Ashley, and Mitch indicate a more pragmatic view of the rewards; their approach to learning resembles *ritualized learning* and a *discourse-for-others* status (Sfard & Prusak, 2005). A *discourse-for-others* approach is where learners engage in learning that is directed by others and is for others (e.g., the completion and the turning in of a set of homework problems given by the teacher). This section addresses the designated identity of the four students in terms of the theme *mathematics as a rewarding subject* and is divided into two subsections: (a) discourse-for-others status, and (b) discourse-for-oneself status.

Discourse-for-others status. Anna, Ashley, and Mitch all had a narrative that revealed a discourse-for-others status. For example, Anna saw AP Statistics as a vehicle to achieve success on the AP Statistics exam and in her future career. In this regard, Anna explained that "I know math is important in the future that is why I work at it. I'm going into nursing next year, so I really want to pass this AP test." Anna's narrative suggests that she saw statistics as something that would support in the college admissions process as well as a ticket that could give her access to her chosen career. Her designated mathematical identity was, at least in part, comprised of the rewards she anticipated her studies in mathematics would bring, either at that time or in the future.

In a similar manner, Ashley's narrative and Mitch's narrative both suggest that their designated identities were built on the aspiration that the study of mathematics would be rewarding as it would provide them with the means to succeed in college and in a future career. For example, Ashley stated that, "I want to go into business, so I probably need some sort of stats for a business degree, and it helps get you a job." In addition, she stated that, "I like math that can actually apply to life skills." These statements and others indicate that her designated identity was partially built on what she expected to get out of the study of statistics, rather than on the learning and understanding of statistics. Mitch also indicated that his decision to study statistics was impacted by his plans for college and a future profession. In regard to taking AP Statistics, he stated, "It's important because I plan on going for a business major and/or engineering." Mitch explained that his brothers impressed on him the importance of having a solid understanding of mathematics and statistics in their study of engineering. In the end, these statements indicate that Mitch saw statistics as a ticket to access a college degree and a voucher to access a career.

These narratives suggest that a significant aspect of the designated identity of Anna, Ashley, and Mitch was comprised of an approach to learning that was *ritualized* and was consistent with a *discourse-for-others* status. All three students discussed the learning of mathematics in terms of an aspiration to be prepared for university studies and a future career. One objective of this approach is to learn mathematics because it provides a ticket that others (e.g., universities, future employers) will ask for in the future; this approach is in contrast to an approach that is focused on understanding mathematical concepts and ideas. Ultimately, the focus was not on learning mathematics, but on the rewards that completing a mathematics class could bring. Erik's *discourse-for-oneself* status is considered in the next section.

Discourse-for-oneself status. Erik's designated mathematical identity related to the theme *mathematics as a rewarding subject* was different than the other three students. During the interviews or on his MINT protocol, Erik never specifically mentioned preparation for college or a future career as reasons for learning mathematics or taking mathematics courses. Rather, Erik spoke of the importance of mathematics in a more holistic sense. He stated that, "Math teaches me *how to think* and that is vital in every aspect of life." He went on to say that, "I'm not actually using the mathematical concepts, I'm using the skills I've developed while using these mathematical concepts. So, it's not purely relatable, but it's effective because of the techniques it has taught me." These statements suggest that Erik expected that the skills and techniques acquired as well as learning "how to think" would play a vital role in all areas of his life (i.e., a substantial approach to learning). In terms of the theme *mathematics as a rewarding subject*, this aspiration was an essential component of his designated mathematical identity. In the next section, the designated mathematical identity of the four students in terms of the theme *ways of studying mathematics and working in mathematics* are explored.

Ways of studying mathematics and working in mathematics. In regard to the theme *ways of studying mathematics and working in mathematics*, the perspectives of Anna, Ashley, Erik, and Mitch were different. Erik's narrative suggests a more *substantial* approach to learning and a *discourse-for-oneself* status. In contrast, the narratives of Ashley and Mitch indicate an approach to learning that was *ritualized* and aligned with a *discourse-for-others* status. Finally, Anna's narrative suggests a more complex approach to learning that was sometimes substantial and sometimes ritualized. This section addresses the designated identity of the four students in terms of the theme *ways of studying mathematics and working in mathematics* and is divided into two subsections: (a) discourse-for-others status, and (b) discourse-for-oneself status.

Discourse-for-others status. The narratives of Ashley and Mitch reveal an approach to learning that was ritualized and aligned with a *discourse-for-others* status. Ashley's and Mitch's designated identities were revealed in how they approached their studies in AP Statistics. For example, Ashley viewed her homework in the mathematics classroom as something that was completed for the teacher and for a grade, not as an avenue to gain an understanding of the material. Ashley stated, "When I have a lot of other homework, I don't do this homework, because he doesn't check it." In addition, her perspective of how she viewed learning was revealing. Ashley indicated that she often went in for help because, "he'll just like tell me what to do because usually if I get one step I can get the rest of the steps down.” These examples, as well as others, reveal a *ritualized* approach to learning, where the mathematical discourse was initiated by the teacher and was for the teacher (i.e., discourse-for-others status).

Mitch’s narrative suggests that he had a similar designated identity related to the theme *ways of studying mathematics and working in mathematics*. In this regard, Mitch perceived that being good at mathematics was shown through his grades, through the mathematics courses he had taken, and through the number of mathematics courses he had taken. For example, Mitch stated that, “What shows I’m good at math is the fact that I’m in Calculus AP and AP Statistics my senior year with the fact that I’ve gotten pretty much A’s and B’s in all of my high school math classes.” This example, as well as others, suggests that he placed a high value on taking hard mathematics classes, taking a large number of mathematics classes, and getting good grades; it was those things, rather than showing a thorough understanding of mathematical concepts and ideas, that were important. Mitch’s approach to learning in these instances was *ritualized* and was consistent with a *discourse-for-others* status. Overall, Ashley’s and Mitch’s belief that mathematical understanding was linked to the successful completion of tasks designed by the teacher and that were for the teacher was a critical component of their designated mathematical identity in terms of the theme *ways of studying mathematics and working in mathematics*.

Varied discourse. Anna’s narrative suggests a complex discourse in terms of the theme *ways of studying mathematics and working in mathematics*. In this regard, Anna had a *substantial* approach to learning in some circumstances, and a *ritualized* approach to learning in other situations. An example of her substantial approach to learning is illustrated in Anna's discussion of how the understanding of mathematics is determined. She stated that understanding occurs when:

> You can teach someone what you're trying to learn. Like when someone has a question and you can answer it promptly and quickly and you know what you're talking about. That's what I feel like opposed to sometimes even if you get a question right on a test, you don't totally know what you're doing, you don't know if it's right, kind of feels right, but I feel like when you [understand you] are like, 'Oh, I can explain this to you.'

This example suggests that she viewed the learning process as a way to make sense of the material in a meaningful and lasting manner. Her expectations and aspirations in terms of mathematical learning and understanding was that it would bring about the ability to explain a concept or idea in a thorough and thoughtful manner; this approach to learning reflects a *discourse-to-oneself* status. In addition, in the above statement she downplayed the *discourse-for-others* status when she characterized learning as a process of teaching and explaining concepts to peers rather than providing a correct answer on a teacher generated exam.

In contrast, Anna's narrative also suggests that she occasionally approached learning in a *ritualized* manner. This approach is illustrated in Anna's comments related to the completion of the group activities. In this regard, Anna explained, "Oh this would have been nice to do before the test to help cement those problems in your head." This description of an activity as a means to do better on the test suggests that the mathematical discourse was initiated by the teacher and was for the teacher (i.e., a *discourse-for-others* status). In other words, Anna's focus was not necessarily on learning for the sake of learning, but learning in order to do well on a teacher-generated exam. Overall, Anna's narrative revealed patterns of *ritualized* learning and instances of *substantial* learning related to the theme *ways of studying mathematics and working in mathematics*.

*Discourse-for-oneself***.** Erik's narrative suggests that in relation to the theme *ways of studying mathematics and working in mathematics*, his designated identity is characterized by a *substantial* approach to learning and is consistent with a *discourse-for-oneself* status. An illustration of this approach is revealed in his comments about how the topic of related rates in his Calculus class inspired him. He explained, "I went back and did a proof for one of the theorems and that obviously wasn't necessary, but I did it because I was kind of interested in it." This narrative suggests that Erik's interest drove his learning (i.e., a discourse-for-oneself); in this case, that drive was not linked to doing well on an assignment or receiving a high mark on an exam (i.e., a discourse-for-others). This *substantial* approach to learning is also revealed in statements Erik made related to his desire to study and learn the material presented in mathematics in order to discuss the concepts and ideas in a thoughtful manner. As part of this explanation, Erik stated that:

> I think being able to discuss the topic shows that you are good at math. I think there is a difference between being able to simply do the problem and being able to discuss the concepts behind the problem. I think that being able to do the problem shows that you're good at memorization and discussion of the problem shows you're good at math.

This narrative clearly reflects an approach to learning that is substantial and one that incorporates a desire to engage in mathematical activities in an effort to facilitate understanding. This is confirmed by his teacher, Mr. Cox, who indicated that Erik adds to the class discussion “with his questions.” He added, “I think he probably has the best questions out of anybody as far as just seeing the little details and even things outside the classroom and how it connects to stuff.” These examples and others suggest that Erik’s wish to engage with the material in a meaningful way and to understand it thoroughly was an important part of his designed mathematical identity. In the end, Erik’s approach to learning was *substantial* and was based on a *discourse-for-oneself* status.

Designated identity summary. The designated identities of the four cases were different. The narratives of Ashley and Mitch revealed an approach to learning that was *ritualized* and aligned with a *discourse-for-others* status in relation to both themes. Anna’s narrative indicated a varied approach to learning. For example, in regard to the theme *mathematics as a rewarding subject* Anna’s learning was *ritualized*, and she had a *discourse-for-others* status. In contrast, Anna’s approach to learning in relation to the theme *ways of studying mathematics and working in mathematics* was mixed; at times it was *substantial* and in other circumstances it was *ritualized*. Finally, Erik’s narrative revealed a *substantial* approach to learning and a *discourse-for-oneself* status in relation to both themes. A summary of the designated identity of the four cases is shown in Table 5.9.

Table 5.10.

Summary of Students' Designated Identity

Student	Designated identity related to the theme: *mathematics as a rewarding subject*	Designated identity related to the theme: *ways of studying mathematics and working in mathematics*
Ashley	▪ Discourse-for-others status ▪ Ritualized approach to learning	▪ Discourse-for-others status ▪ Ritualized approach to learning
Mitch	▪ Discourse-for-others status ▪ Ritualized approach to learning	▪ Discourse-for-others status ▪ Ritualized approach to learning
Anna	▪ Discourse-for-others status ▪ Ritualized approach to learning	▪ Discourse-for-oneself and discourse-for-others statuses ▪ Substantial and ritualized approaches to learning
Erik	▪ Discourse-for-oneself status ▪ Substantial approach to learning	▪ Discourse-for-oneself status ▪ Substantial approach to learning

This completes the cross case analyses of the four cases in regard to their thinking moves and their mathematical identity. Using these analyses, the final section of this chapter is a cross case discussion that answers the research question and the two subquestions.

Cross Case Discussion

The present research study explored the relationship between the mathematical identity and thinking during problem-solving activities. The following research question was addressed: To what degree and in what ways are the narratives students construct for their mathematical identities related to the use of high-leverage thinking moves during problem-solving activities? To address this overarching question, the following two questions were investigated.

a. To what degree and in what ways are the narratives students construct for their *current* mathematical identities related to the use of high-leverage thinking moves during problem-solving activities?

b. To what degree and in what ways are the narratives students construct for their *designated* mathematical identities related to the use of high-leverage thinking moves during problem-solving activities?

This section discusses the results of the cross case analyses using the two research subquestions as a structural framework.

Research subquestion A. In the various analyses, the evidence suggests that there is a significant difference between the patterns of thinking moves used during the group activities and on the individual work on selected exam questions.

For example, during the group activities the evidence indicates that a more positive *current* identity may be associated with the use of a wider range of high-leverage thinking moves. In addition, the data indicate that the positivity of the current mathematical identity may be linked to the type of thinking (i.e., creative thinking, critical thinking, mental management and awareness thinking). However, there appears to be no significant connection between the current identity of an individual and the use of thinking moves on the exam questions. Each of these claims is considered in more detail; the discussion starts with the exploration of the thinking moves of the four students.

Thinking moves used during problem-solving activities. The use of all eight different types of thinking moves by an individual may demonstrate a thorough understanding of a concept, topic, or idea (Ritchhart et al., 2011). These eight thinking moves are used as a framework to discuss the thinking moves of Anna, Ashley, Erik, and Mitch during problem-solving activities (i.e., group activities and individual work on exam questions). The student thinking is also explored using three thinking categories: critical thinking, creative thinking, and mental management and awareness. These three thinking categories are used, along with the eight high-leverage thinking moves, to structure the following discussion.

The thinking moves displayed on the selected exam questions are somewhat similar for the four students. This is evident in terms of the number of thinking moves used and the types of thinking moves used during completion of the exam questions. For example, in terms of the number of thinking moves, Anna used 11, Ashley and Erik used 12, and Mitch used 14. In addition, the thinking moves used by the four students reside in all three thinking categories and the pattern across the categories is similar.

That being the case, there was a small difference in regard to the number of different thinking moves used; Ashley used four, Anna and Erik used five, and Mitch used six (recall that based on the format of the exam questions, only 6 of the 8 thinking moves could be used).

While there were similarities between the four cases in relation to the thinking moves used on the exam questions, during the group activities the thinking moves patterns displayed by Anna and Ashley were somewhat different than those exhibited by Erik and Mitch. For example, Anna and Ashley each made partial use of the eight high-leverage thinking moves during the group activities. Specifically, Ashley used a total of 15 thinking moves; three of 8 thinking moves were represented. Similarly, Anna displayed a total of 13 thinking moves and the use of 4 out of the 8 thinking moves. The most common high-leverage thinking move used by both individuals was *wondering and asking questions*; Ashley displayed the move nine times, and Anna exhibited the move five times. Finally, neither Ashley nor Anna made use the thinking moves: (a) building explanations and interpretations, (b) capturing the heart and forming conclusions, and (c) making connections.

There were also similar patterns in terms of types of thinking (i.e., creative thinking, critical thinking, and mental management and awareness) Anna and Ashley displayed. First, the thinking of both individuals fell mainly in the *creative thinking* category; in fact, 67% of Ashley's thinking moves and 69% of Anna's thinking moves fell into this category. In addition, none of the thinking moves exhibited by the two individuals fell into the *mental management and awareness* category during the group activities.

In contrast, both Erik and Mitch exhibited a large range of thinking moves during the group activities. Over the course of the group activities, Erik made use of all eight thinking moves while Mitch displayed 7 out of 8 of the thinking moves; the only thinking move not used by Mitch was *capturing the heart and forming conclusions*. In addition, the thinking moves of Erik and Mitch were fairly evenly distributed across the eight thinking moves; all three thinking categories were represented in the thinking displayed by Erik and Mitch. In this regard, eight of the thinking moves displayed by Erik were in the *creative thinking* category, 12 of the moves resided in the *mental management and awareness* category, and four of the moves fell in the *critical thinking* category. Similarly, 11 of Mitch's thinking moves were in the *creative thinking* category, while five were in the *mental management and awareness* category, and three were in the *critical thinking* category.

Next, the patterns across the four cases related to current identity strength are considered, followed by a discussion of the links between the current identity and the use of the high-leverage thinking moves used during problem-solving activities. Following that, the association between the use of thinking moves and current identity in relation to two of the MINT themes (i.e., persistence and perseverance in mathematics, ways of studying and working in mathematics) is explored.

Current identity strength. The relationship between current mathematical identity and the use of high-leverage thinking moves during problem-solving activities (i.e., group activities and individual work on exam questions) is a complex one. On one hand, there is evidence to suggest that a move positive current identity may be associated with using a wider range of the eight high-leverage thinking moves during group activities. In addition, a strong positive current identity may be tied to the use of thinking moves in all three thinking categories. In contrast, the data indicate that there may be no connection between the current identity strength and the use of thinking moves displayed on individual exam responses. To justify these claims, the current identity strength of Anna and Ashley are discussed first, followed by a discussion of the current identity strength of Erik and Mitch.

Overall, Anna and Ashley had similar current identities. Anna had a mixed current mathematical identity (i.e., mixed feelings and experiences related to mathematics, some positive and some negative). The majority of her statements (i.e., 92%) fell into the middle three categories (i.e., positive, mixed, and negative) and 18 out of 39 resided in the mixed category. Similarly, Ashely displayed a mixed to a negative current mathematical identity. Ashley's distribution of statements across the strength categories was similar to Anna's; the majority of her statements (i.e., 81%) fell into the middle three categories. In addition, 12 out of 31 were in the negative category. This current identity was reinforced by their narratives which suggest that they viewed themselves as students who sometimes persisted in mathematical environments, but other times did not have the perseverance needed to be successful. In addition, both Anna and Ashley viewed themselves as students who often lacked confidence in the mathematics classroom.

In contrast, Mitch and Erik displayed strong positive to positive current mathematical identities. The distributions of their statements in regard to the strength of current mathematical identity were similar. First, the distributions ranged from negative to strong positive. In addition, both distributions were evenly spread across the middle three categories (i.e., positive, mixed, and negative). Finally, both individuals had over half of their statements in the strong positive category; almost 59% of Erik’s statements and 68% of Mitch’s statements were associated with a strong positive current mathematical identity. Their current mathematical identity was also revealed in a similar manner in their narratives. According to his narrative, Mitch saw himself as an extremely capable mathematics student who helped his classmates and was part of class discussions. Erik’s narrative suggests that he saw himself as an individual who enjoyed mathematics, enjoyed challenges, and went above and beyond the curriculum when he was interested in the topic. That being the case, Mitch and Erik also saw themselves as being persistent in mathematical matters only some of the time. Overall, Mitch and Erik had a positive current mathematical identity.

Taken together, this evidence indicates that a move positive current identity may be associated with the use of a wider range of high-leverage thinking moves during the group activities. In addition, a strong positive current identity may be linked to the use of thinking moves that reside in all three thinking categories. Specifically, the use of the thinking moves in *the mental management and awareness* category seems to be tied to a strong positive current mathematical identity. In contrast, the data suggest that negative and mixed current mathematical identities may be associated with more limited range of high-leverage thinking moves. In addition, a mixed current identity appears to be linked to the reliance on thinking moves that fall into the creative thinking category (e.g., the use of the thinking move *wondering and asking questions*). That being the case, there appears to be no link between current identity strength and the use of thinking moves on the exam questions as in that setting the four students were able to use a variety of thinking moves that reside in all three thinking categories.

These relationships are explored in more detail in the next two sections. Specifically, the associations between the use of thinking moves and current identity in relation to two of the MINT themes (i.e., persistence and perseverance in mathematics, ways of studying and working in mathematics) are explored.

Persistence and perseverance in mathematics. There is evidence that indicates a possible link between how an individual perseveres in a mathematical environment and their use of the eight high-leverage thinking moves during group activities. In terms of current identity related to persistence and perseverance, all four cases saw themselves as students with a strong work ethic, but they also described situations where they lacked persistence in the study of mathematics. In addition, within those perspectives, Ashley and Anna viewed themselves differently than did Erik and Mitch in some respects. For example, Anna and Ashley viewed their persistence in terms of the effort (e.g., completing homework, coming in for help, and preparing for tests) needed to overcome difficult and confusing material. In this regard, both students described themselves as mathematics students who persevered through hard work, through coming in before school to get help, and through working with a tutor to prepare for tests. However, Anna and Ashley also saw themselves as students who gave up when faced with confusing concepts. In contrast, Erik and Mitch linked their perseverance in mathematical environments to issues of challenge, interest, and curiosity. For example, they described themselves as students who persisted in situations where they were challenged or intrigued by the mathematics concepts being studied. On the other hand, they also viewed themselves as students who did not even engage with the material if they did not find it interesting or had other things that they wanted to do instead.

Taken together, this evidence suggests a possible link between how an individual perseveres in a mathematical environment and their use of a wider range of the eight high-leverage thinking moves during group activities. In essence, learners that viewed themselves as individuals who persevered when they encountered mathematical environments that were challenging and interesting made more extensive use of the eight thinking moves. In addition, these individuals made use of thinking moves that resided in all three thinking categories. In contrast, individuals who described themselves as students who persisted in situations (e.g., completing homework, coming in for help, and preparing for tests) where they needed to overcome difficult and confusing material made only partial use of the eight high-leverage thinking moves and did not display thinking moves that reside in the *mental management and awareness* category. However, these findings are only in relation to the thinking moves displayed during the group activities.

In regard to the exam questions, all four students made use of a similar number of thinking moves and thinking moves that reside in all three thinking categories. As a result, the overall performances of the four students on the selected exam questions were fairly similar.

In the next section, the associations between the use of thinking moves and current identity in relation to the MINT theme *ways of studying and working in mathematics* are explored.

***Ways of studying and working in mathematics*.** There is evidence that indicates a possible connection between a learner's confidence in mathematical situations and their use of the eight high-leverage thinking moves during group activities. The current mathematical identities of Anna and Ashley were very different from the current identities of Erik and Mitch in regard to the theme *ways of studying and working in mathematics*. Anna and Ashley saw themselves as marginal players who lacked confidence in mathematical environments related to *ways of studying and working in mathematics.* Both individuals described this lack of confidence in terms of their engagement with difficult and challenging concepts and ideas as well as taking part in class discussions. On the other hand, Mitch and Erik viewed themselves as central members of the mathematics classroom; they were willing to be an active part of class discussions and helped peers when needed. In addition, they viewed themselves as confident students who were able to successfully navigate through a wide variety of mathematical material. In the end, the current identity of Erik and Mitch appeared to be linked to seeking out and solving challenging and interesting mathematical problems.

Together, this suggests a possible connection between a learner's confidence in mathematics situations and their thinking moves used during group activities. Essentially, learners that viewed themselves as confident students of mathematics and as central members of the mathematics classroom made more extensive use of the eight thinking moves; these individuals also made use of thinking moves that resided in all three thinking categories. In contrast, individuals who described themselves as students who lacked confidence in their ability to understand mathematics and viewed themselves as marginal members of the mathematics classroom made only partial use of the eight thinking moves during group activities. In addition, these learners did not demonstrate thinking moves that reside in the *mental management and awareness* category during the group activities.

Summary for research subquestion A. Overall, this evidence indicates that a more positive current identity may be associated with student use of a wider range of high-leverage thinking moves during the group activities and thinking moves that reside in all three thinking categories. In addition, the evidence suggests that negative and mixed current mathematical identities may be associated with a more limited range of high-leverage thinking moves and thinking moves that reside mainly in the creative thinking category. It should be noted that these results are only with respect to thinking moves displayed during the group activities. During the exam question, all four students displayed a similar number of thinking moves and thinking moves that reside in all three thinking categories regardless of their current mathematical identity. The cross case discussion related to research subquestion B is considered in the next section.

Research subquestion B. This section makes use of the cross case discussion to answer subquestion B: To what degree and in what ways are the narratives students construct for their *designated* mathematical identities related to the use of high-leverage thinking moves during problem-solving activities? Designated identities are often told in the future tense, or in a way that expresses aspiration, assurance, or need, about a situations or circumstances expected to be the case either presently or in the future; these identities are revealed in the form of stories (Sfard & Prusak, 2005). In the various analyses, there is evidence to suggest that a designated identity in relation to a *substantial* approach to learning and a *discourse-for-oneself* is associated with the use of a wider range of high-leverage thinking moves during group activities. However, this relationship does not appear to be present during the completion of the exam questions. These relationships are explored in more detail in the next two sections. Specifically, the associations between the use of thinking moves and designated identity in relation to two of the MINT themes (i.e., mathematics as a rewarding subject, ways of studying and working in mathematics) are explored.

Mathematics as a rewarding subject. Erik's perspective of mathematics as a rewarding subject was different than the perspectives of Anna, Ashley, and Mitch. The narratives of Anna, Ashley and Mitch indicate that they viewed mathematics as a rewarding subject through a lens in which the approach to learning is directed by others and is for others; their approach to learning resembled *ritualized learning* and a *discourse-for-others* status. In contrast, Erik's narrative suggests a more universal view of the rewards associated with learning mathematics; this can be described as *substantial learning* and as a *discourse-for-oneself* where the approach to learning is directed by the learner and is for the learner.

These narratives of Anna, Ashley, and Mitch suggest that a significant aspect of their designated identity was comprised of an approach to learning that was *ritualized* and was consistent with a *discourse-for-others* status. In her narrative, Anna shared three primary reasons why she considered mathematics to be a worthwhile subject: (a) it is a vehicle to achieve success in her current and future science courses, (b) it would help her in terms of her college coursework, and (c) it would be beneficial in regard to career preparation (she wanted to be a nurse). In a similar manner, Ashley's and Mitch's narratives suggest that they saw mathematics as an admission ticket to college and an access voucher to a career; Ashley wanted to pursue a degree in business while Mitch was considering engineering or business. All three students discussed the learning of mathematics in terms of an aspiration to be prepared for university studies and a future career. The objective of this approach is to learn mathematics because it provides a ticket that others (e.g., universities, future employers) will ask for in the future. Taken together, these statements indicate that their designated identity was built on a perspective of mathematics as a gatekeeper to future collegiate and professional success, rather than on the learning and understanding of mathematics.

In contrast, Erik never indicated that he was learning mathematics to help him with university or career related matters. Instead, Erik expressed his reasons for learning mathematics in terms of the stimulation, satisfaction, and the intrigue it provided. In addition, Erik spoke of the importance of mathematics in a more holistic sense. He stated that, "Math teaches me *how to think* and that is vital in every aspect of life." Erik expected that learning "how to think" would play a vital role in all areas of his life; this narrative, the aspiration that his life would be transformed by the experiences in the mathematics classroom that enhanced his ability to think, indicates a substantial approach to learning. In the end, a significant aspect of Erik's designated mathematical identity is based on a substantial approach to learning and a *discourse-for-oneself* status.

This evidence suggests a possible link between the designated identity of an individual in terms of mathematics as a rewarding subject and the use of high-leverage thinking moves used during group activities. Specifically, a designated mathematical identity that is built primarily on a *discourse-for-oneself* and a *substantial* approach to learning may be linked to the amount of thinking moves that reside in *the mental management and awareness* category. For example, consider the percentage of thinking moves used during the group activities by each of the four cases that reside in the *mental management and awareness* category. Ashley and Anna both had no thinking moves that were part of this category, while 26% of Mitch's thinking moves resided in this category. On the other hand, 50% of Erik's thinking moves fell into the mental management and awareness category and he was the only individual to make use of all eight high-leverage thinking moves. It should be noted that this link is not evident in the data associated with the thinking moves displayed on the exam question responses. The connections between the use of thinking moves and designated identity in relation to the MINT theme *ways of studying and working in mathematics* are explored in the next section.

***Ways of studying and working in mathematics*.** In regard to the theme *ways of studying and working in mathematics*, Erik's narrative indicates that his designated mathematical identity was different than the other three students. One common strand that ran through Ashley's, Anna's, and Mitch's approach to learning mathematics was the focus was on grades, completing assignments, and preparing for exams; this approach suggested a *discourse-for-others* status as the learning was done for the teacher through the completion of items created by the teacher. In contrast, Erik displayed a substantial approach to learning and a *discourse-for-oneself* status. His narrative suggests that Erik's curiosity drove his learning (i.e., a discourse-for-oneself), rather than the desire to do well on an assignment or receive a good grade on an exam (i.e., a discourse-for-others). In addition, Erik's narrative indicates that he hoped that through the engagement in various mathematical activities he would be able to understand the material and be able to discuss it in a thoughtful manner. These aspirations are an essential aspect of his designated mathematical identity.

In the end, this evidence indicates a possible connection between the use of high-leverage thinking moves used during group activities and the designated identity of an individual in relation to *ways of studying and working in mathematics*. In this regard, a designated mathematical identity that is built primarily on a *discourse-for-oneself* and a *substantial* approach to learning may be associated with more extensive use of the eight high-leverage thinking moves during group activities. In addition, this approach to learning may also be linked to more extensive use of thinking moves during group activities that reside in *the mental management and awareness* category. Again, this connection is not present in the data related to the thinking moves used on the exam questions.

Chapter 5 Summary

The present research study investigated the research question: To what degree and in what ways are the narratives students construct for their mathematical identities related to the use of high-leverage thinking moves during problem-solving activities? This overarching question was explored through two subquestions; the results are summarized according to these two research questions.

Subquestion A asks, to what degree and in what ways are the narratives students construct for their *current* mathematical identities related to the use of high-leverage thinking moves during problem-solving activities? Overall, the evidence suggests that during the group activities a more positive current identity may be linked with the use of a larger variety of high-leverage thinking moves and thinking moves which reside in all three thinking categories. In addition, the evidence suggests that during the group activities more negative current mathematical identities may be associated with a more limited range of high-leverage thinking moves and thinking moves which primarily reside in the *creative thinking* category. In contrast, during the individual exam work the students, regardless of their current mathematical identity, made use of a wide assortment of thinking moves and displayed thinking moves that reside in all three thinking categories.

Subquestion B asks, to what degree and in what ways are the narratives students construct for their *designated* mathematical identities related to the use of high-leverage thinking moves during problem-solving activities? Overall, the evidence indicates that a designated identity that is characterized by a *substantial* approach to learning and a *discourse-for-oneself* is associated with the use of a wider range of high-leverage thinking moves and the use of thinking moves that reside in all three thinking categories. In addition, the evidence indicates that a *designated* mathematical identity characterized by a *discourse-for-others* and a *ritualized* approach to learning is associated with a more limited range of high-leverage thinking moves and thinking moves that reside primarily in the *creative thinking* category. On the other hand, during the individual exam work the students, regardless of their designed mathematical identity, made use of a wide variety of thinking moves and exhibited thinking moves that reside in all three thinking categories.

Overarching research question. The current research study indicates that there are complex relationships between the narratives students construct for their mathematical identities and the use of high-leverage thinking moves during problem-solving activities (i.e., group activities and exam questions). In this regard, the evidence suggests that there was a difference between the patterns of thinking moves used during the group activities and the patterns of thinking moves used on the individual work on selected exam questions. For example, a number of aspects of mathematical identity appear to be linked to the use of thinking moves during group activities. First, individuals who had mostly positive feeling and experiences related to mathematics, who viewed themselves as confident students of mathematics, and who saw themselves as central members of the mathematics classroom demonstrated more extensive use of the eight thinking moves and the three thinking categories. In addition, approaches to learning that are focused on understanding the material (i.e., a substantial approach) and are consistent with *discourse-for-oneself* status are linked to the use of a larger variety of thinking moves and the use of thinking moves which reside in all three thinking categories. See Figure 5.1 for a summary of these associations between mathematical identity and thinking moves during group activities.

Figure 5.1. Associations Between Mathematical Identity and Thinking Moves During Group Activities (Part 1)

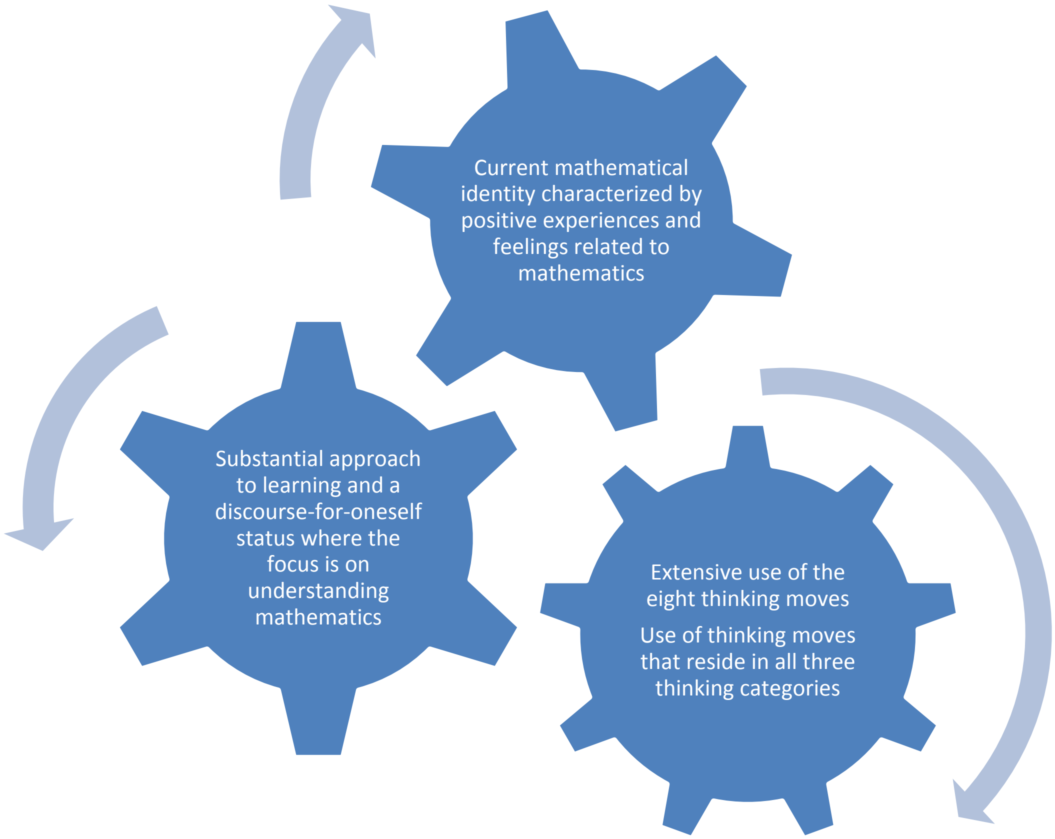

Figure 5.1. Associations between mathematical identity and thinking moves during group activities (Part 1). The gear on the top represents a positive current mathematical identity, while the gear on the left represents a designated identity comprised of a focus on understanding mathematics. The gear on the lower right represents extensive use of the thinking moves.

In contrast, mixed or mostly negative feelings and experiences related to mathematics appear to be associated with the limited use of the eight thinking moves during group activities and the use of thinking moves that are not part of the *mental management and awareness* during group activities. These negative feelings and experiences related to mathematics were revealed in the narratives of students who viewed themselves as lacking confidence in mathematical environments and saw themselves as marginal players in the mathematics classroom. In a similar manner, approaches to learning that are *ritualized* (e.g., learning environments designed by the teacher and products produced for the teacher) and are characterized by a *discourse-for-others* status are tied to the use of fewer of the eight thinking moves and to the use of thinking moves that reside primarily in the *creative* and *critical* thinking categories. See Figure 5.2 for a summary of these associations between mathematical identity and thinking moves during group activities.

Figure 5.2. Associations Between Mathematical Identity and Thinking Moves During Group Activities (Part 2)

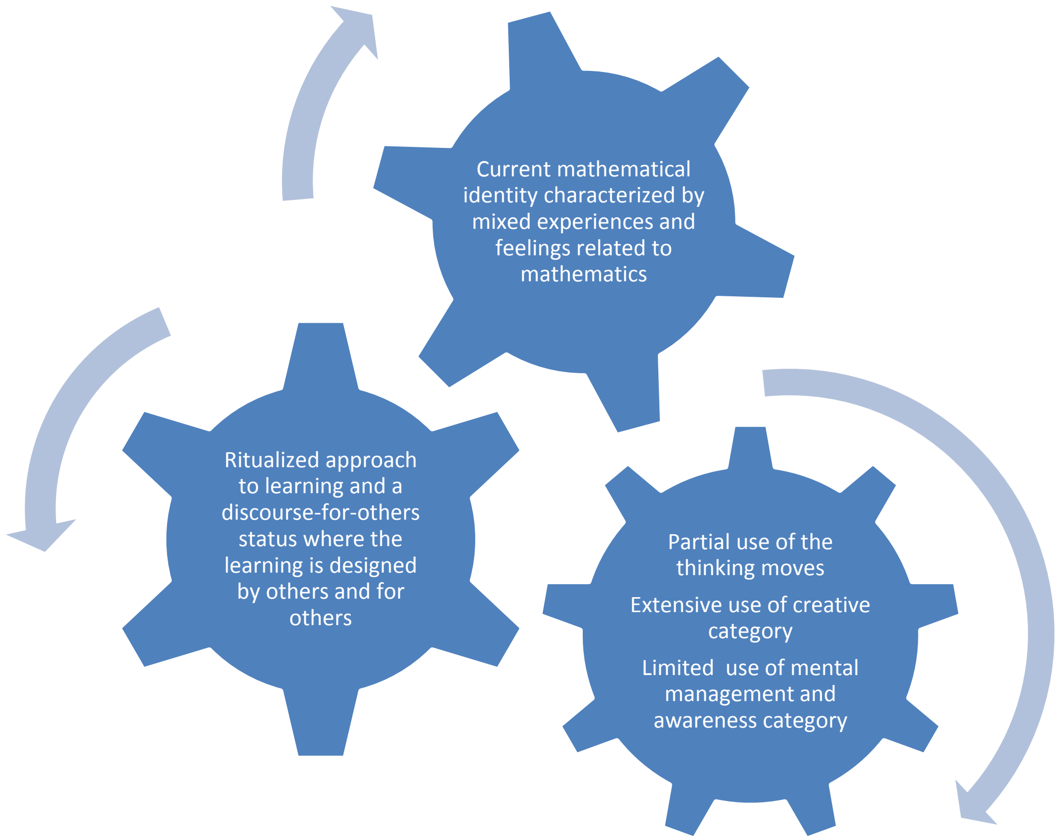

Figure 5.2. Associations between mathematical identity and thinking moves during group activities (Part 2). The gear on the top represents a mixed to negative current mathematical identity, while the gear on the left represents a designated identity comprised of an approach to learning that is designed by others and is for others. The gear on the lower right represents a limited use of the thinking moves.

While there is evidence of links between the narratives students construct for their mathematical identities and the use of thinking moves during group activities, this was not the case during the completion of exam questions. In this regard, there appeared to be no association between mathematical identity (i.e., current and designated) and the use of thinking moves during the individual exam work. Some of the possible reasons for these discrepancies in the findings related to the two types of problem solving activities (i.e., group activities and exam questions) are considered in Chapter 6. In addition, Chapter 6 includes the limitations, the implications, and the areas of possible future research.

Chapter 6

Discussion, Limitations, Implications, and Future Research

An individual's relationship with mathematics can have a powerful influence on opportunities to have meaningful experiences in the classroom that impact mathematical understanding. Consideration of students' mathematical identities can inform the creation of learning environments that address the needs of all students (Aguirre et al., 2013). The alignment of the classroom environment and pedagogies with the mathematical and academic identities of students is critical (Boaler & Greeno, 2000). Such an approach has the potential of addressing issues related to low achievement, underachievement, student dropout rates, and lack of enrollment in optional courses at the high school and university levels (Boaler & Greeno, 2000; Boaler et al., 2000; Grootenboer & Zevenbergen, 2008).

To further explore the impact of mathematical identity, this dissertation study examined identity in conjunction with the understanding of mathematical concepts. There are researchers in mathematics education (e.g., Gutierrez, 2000, 2007; Martin, 2000, 2009; Nasir, 2002; Stanovich & West, 2008) who argue that because the understanding of mathematics is a dynamic and complex process, understanding cannot be fully explored through standardized test scores and grades. Consequently, in certain circumstances it may make more sense to measure student understanding through the examination of students thinking moves as many experts (e.g., Ennis, 1996; Facione et al., 1994; Ritchhart et al., 2011; Tishman et al., 1993) contend that understanding is the result of a multifaceted process that requires diverse forms of thinking. Moreover, other researchers (e.g., Swartz & Perkins, 1990) point out that creative approaches to learning and thought are nurtured and supported by student thinking. Ultimately, rather than defining understanding as an aspect of thinking, understanding can be described as a consequence of thinking (Ritchhart et al., 2011). For these reasons, the current research and similar research in mathematics education is essential as it can unlock important aspects of the relationships between mathematical identity and understanding (Boaler et al., 2000).

Discussion

The purpose of this research was to study the relationships between mathematical identity and student thinking in mathematical environments. Specifically, the current research investigated to what degree and in what ways the narratives students construct for their *current* and *designated* mathematical identities are related to the use of high-leverage thinking moves during problem-solving activities (i.e., group activities and selected exam questions). In the course of the present dissertation research, there were a number of interesting findings in relation to the research question. For instance, a key finding is that the thinking moves patterns displayed during the group activities were very different than the thinking moves patterns displayed on the exam questions. For example, while the evidence reveals various connections between mathematical identity and the use of thinking moves *during group activities*, the same data display no noteworthy links between mathematical identity and the use of thinking moves *on the exam question responses*. As a result, the associations between mathematical identity and the use of thinking moves *during group activities* are explored in relation to the two research subquestions. This is followed by a discussion of the possible reasons why the thinking moves patterns were different in regard to two types of problem-solving activities (i.e., group activities vs. exam questions).

Research subquestion A during group activities. Research subquestion A states, to what degree and in what ways are the narratives students construct for their *current* mathematical identities related to the use of high-leverage thinking moves during problem-solving activities? Recall that current identities are revealed in the form of stories that use factual statements in the present tense about existing circumstances (Graven & Buytenhuys, 2011; Sfard & Prusak, 2005) and that a strong positive current identity is characterized by positive or mostly positive feelings and experiences related to mathematics. Individuals with a strong positive current identity view themselves as confident mathematics students and as central members of the mathematics classroom. The various analyses provide evidence that a strong positive current identity may be associated with a comprehensive use of the eight high-leverage thinking moves during group activities. In contrast, a mixed current identity may be linked to a limited use of the eight thinking moves during group activities. This mixed mathematical identity is comprised of mixed feelings and experiences in relation to mathematics, and is characterized by individuals who view themselves as lacking confidence in mathematical environments and who see themselves as peripheral members of the mathematics classroom. These findings are aligned with the work of Esmonde (2009) who indicates that a student's positional identity in relation to their mathematical capabilities impact their ability to interact in group discussions in an effective manner. For example, she found that more competent students generally guide groups, while less competent students tend to follow the instructions of others. In addition, students that are perceived by their peers or their teacher to be low achieving are often marginalized in group settings (Esmonde, 2009). In the end, these conclusions of Esmonde support the findings in this current research that a student's mathematical identity is associated with their role and contributions (i.e., the use of thinking moves) during group problem solving activities.

The present research study also indicates that mathematical identity is linked to the type of thinking (i.e., creative, critical, mental management and awareness) displayed during the group activities. For example, the evidence indicates that a strong positive mathematical identity may be associated with the use of thinking moves that reside in all three thinking categories. In addition, an identity comprised of perseverance in relation to challenging and interesting problems corresponds to the more extensive use of thinking moves in the *mental management and awareness* category. In contrast, the data suggest that a mixed mathematical identity may be associated with the use of thinking moves that reside in the *creative* and *critical* thinking categories. Moreover, the evidence indicates a current identity comprised of persistence when faced with difficult and confusing material may be linked to the use of the thinking moves that reside predominantly in the *creative* thinking category and infrequently, if ever, in the *mental management and awareness* thinking category.

These findings are consistent with the work of Boaler et al. (2000) in relation to what it means to learn mathematics. Boaler et al. contend that,

> Students do not just learn methods and processes in mathematics classrooms, they learn to *be* mathematics learners and their learning of content knowledge cannot be separated from their interactional engagement in the classroom, as the two mutually constitute one another at the time of learning (p. 380).

For instance, perhaps students who persevere in relation to challenging mathematical content display a wider range of thinking moves because they are able to learn the material *and* engage in an interactional manner. Boaler (1997) contends that this more open-ended and engaging approach to learning results in the increased ability to transfer mathematical knowledge to other situations. This may explain, in part, why the present research revealed a connection between certain mathematical identities (e.g., persistence in working through complex mathematical material) and the use of thinking moves that reside in the *mental management and awareness* category. Boaler's work (e.g., Boaler, 1997, 2008; Boaler et al., 2000) also supports the present research in that it indicates that persisting in relation to *learning methods and processes* is not enough to become a learner of mathematics as it shortchanges the *interactional engagement* in the classroom. For example, Boaler (1997) found that a more traditional approach to learning mathematics (e.g., memorization, focus on procedures) hindered students in applying their mathematics in other circumstances. The findings of Boaler shed light on and support the connections revealed by the present research, that is, that during group activities a strong positive *current* mathematical identity may correspond to the extensive use of thinking moves and a mixed *current* mathematical identity may be associated with the limited use of thinking moves.

Research subquestion B during group activities. Research subquestion B states, to what degree and in what ways are the narratives students construct for their *designated* mathematical identities related to the use of high-leverage thinking moves during problem-solving activities? Recall that designated identities are revealed in the form of stories that express aspirations, desires, or needs about circumstances expected to be the case presently or in the future (Sfard & Prusak, 2005). In the various analyses, there is evidence to suggest that a designated identity comprised of a *substantial* approach to learning and a *discourse-for-oneself* status is associated with the use of a more diverse pattern of thinking moves and thinking moves that reside predominantly in the *mental management and awareness* category (i.e., building explanations and interpretations, making connections, and capturing the heart and forming conclusions). In contrast, a designated mathematical identity that is built primarily on a *ritualized* approach to learning and a *discourse-for-others* status is associated with the limited use of the eight high-leverage thinking moves and thinking moves that reside primarily in the *creative* thinking category.

These links between designated identity and the use of thinking moves are similar to the findings of Sfard and Prusak (2005). In their study regarding the mathematical identity of students, Sfard and Prusak followed nineteen Grade 11 students for an entire year in an advanced mathematics class. The research found a tight connection between how well a student learned and the student's approach to learning in relation to their designated identity (i.e., ritualized versus substantial). Specifically, students with a substantial approach to learning engaged with the material in a manner that resulted in a more meaningful understanding of the mathematics, an understanding that went beyond simply regurgitating the information learned. In a similar way, the evidence in the current research indicates that individuals with a substantial approach to learning better demonstrated their understanding (i.e., understanding as a consequence of thinking; Ritchhart et al., 2011) through the more extensive use of thinking moves and the use of thinking moves that reside in the *mental management and awareness* thinking category.

Overall, the findings of the present study in terms of the two research subquestions and in relation to the group activities are consistent with and are supported by the work of Esmonde (2009), Boaler (1997, 2008; Boaler et al., 2000), and Sfard and Prusak (2005). First, the determination from the present research that a student's mathematical identity is closely linked with their contributions during the group activities (i.e., the use of thinking moves) is supported by Esmonde. For example, a strong positive current mathematical identity may be linked to a more significant role during group activities, a role that is revealed through a more extensive use of thinking moves. In addition, the work of Boaler supports the conclusion of the present research study that the current identity in relation to the approach to learning may be linked to the use of thinking moves during the group activities. Specifically, an identity characterized by persistence in working through interesting and complex mathematical material, as opposed to persistence needed to overcome confusing material, corresponds to a more extensive use of thinking moves during group activities. Finally, the work of Sfard and Prusak (2005) supports the findings of the current research that suggests that a *substantial* approach to learning, in contrast to a ritualized approach to learning, is associated with more extensive use of thinking moves and the use of more thinking moves that reside in the *mental management and awareness* thinking category. While the evidence in the present research suggests links between the use of thinking moves and mathematical identity during the group activities, there were no such associations found during the individual work on exam questions. That phenomena is explored next.

Group activities versus exam questions. One noteworthy finding was that for individuals with a mixed or negative current identity, the patterns of thinking moves used during the group activities were substantially different than the patterns of thinking moves used on the individual exam work. For instance, while these individuals made limited use of the thinking moves during group activities, they made extensive use of the thinking moves on the exam question responses. The possible factors that could account for this difference include: (a) the complex interactions between learning, identity, and the social construction of knowledge; (b) the degree to which the classroom pedagogies are aligned with the student's learning style; and (c) the interpersonal dynamics present during the group activities.

First, the social aspect of the co-construction of knowledge during the group activities may have been influenced by the mathematical identities of the students. Learning is a social activity that evolves out of the symbiotic relationship between the identity development through disciplinary activities and the construction of knowledge in conjunction with others (Anderson, 2007; Varelas et al., 2012). The group activities were, in essence, learning experiences. Researchers have confirmed the strong tie between learning mathematics, identity, and social interaction (e.g., Anderson, 2007; Boaler & Greeno, 2000; Grootenboer & Zevenbergen, 2008; Varelas et al., 2012). As a result of this complex relationship between identity and the social construction of knowledge, students who viewed themselves as lacking confidence in their mathematical abilities and saw themselves as peripheral members of the mathematics classroom may have been less likely to share their thinking in a group setting. On the other hand, the individual work on the exam questions was not a learning situation, but rather an assessment of learning. As a result, the identity characterized by a lack of confidence that was consistent with the limited use of thinking moves during group activities may not have carried over to the individual work on exams where the social component of learning was not a factor. Clearly, the two contexts of the problem-solving activities (i.e., individual work on exam questions and the group work on the statistical activities) in the current research study were very different. In this regard, Esmonde (2009) contends that identity is dependent on context and can change from one situation to another as one becomes proficient in that context. In the end, while the students, regardless of current identity strength, appeared to be proficient in the individual exam work, only the students with a strong positive current identity appeared to be proficient in the context of group activities.

Additionally, the degree of alignment between how students learn mathematical concepts and the pedagogies of the mathematical classroom could explain the difference in thinking moves in the individual and the group settings. Boaler and Greeno (2000) found that a student’s relationship with mathematics was associated with the mathematical content as well as how their identity as a learner lined up with the approaches to teaching and learning in the classroom. In this respect, in the present research study the students with a strong positive mathematical identity who displayed the use of a wide range of thinking moves may have had a learning style that better connected with the learning environment (i.e., an environment that was designed for social interaction and the opportunity to co-construct mathematical concepts) of the group activities. In contrast, the students in this dissertation study with a mixed to strong negative mathematical identity who made limited use of the thinking moves during the group activities may have rejected the social learning pedagogies of the group work as they preferred an approach that involved memorization, manipulation of symbols, and the regurgitation of information. This is consistent with research (e.g., Boaler & Greeno, 2000; Cobb & Hodge, 2007) that indicates that students experience estrangement from mathematics when they perceive differences between their core identities and their normative identities. In other words, students find it troubling when their perspective of who they are as a person and as a learner, and who they want to become, does not line up with the identity needed to be a successful member of the classroom community.

Moreover, the difference in the use of thinking moves between the group and individual settings in the present research study may have been the result of the role of certain factors related to group dynamics (i.e., gender, status, and achievement level) played in the willingness of students to share their thinking during the group activities. Esmonde (2009) indicates that sharing one's thinking is an essential part of learning mathematics in a group setting, but how a student is identified and positioned within a group can influence how they explain their thinking. As a result, certain students in this dissertation study may have felt more comfortable sharing their thinking with the group, while others did not, based on the composition and dynamics of the group; a common issue with group work is that certain members are sometimes marginalized and choose not to participate or are forced out by other members (Boaler, 2008; Slavin, 1990). The influence of the group dynamic factors of gender, status, and achievement level of the student are each explored in more detail.

The sex of the students may have been one of the factors that affected the use of thinking moves during the group activities. In the present research study, during the group activities the two cases that made limited use of the thinking moves were female while the two cases that made extensive use of thinking moves were male. In this regard, some researchers (e.g., Mulryan, 1992, 1995; Webb, 1984) have found that boys are more likely to start discussions. In addition, many students see mathematics classrooms as environments that are dominated by males (Mendick, 2005; Rodd & Bartholomew, 2006); this dynamic could be linked to the willingness of students to contribute during group work. Moreover, the female cases could have underachieved during the group work due to a phenomenon known as stereotype threat as they are one of the affected social groups (Spencer et al., 1999). Finally, general attribution theory many also shed light on the findings in regard to the sex of the students. Recall that according to attribution theory, males, more than females, attribute expected success to stable causes (i.e., ability and the task) and as a result have increased expectations in regard to achievement and performance (Deaux, 1976). In contrast, females, more than males, attribute success to unstable causes (i.e., effort and luck), a perspective that is associated with a lack of confidence in repeated successful performances (Deaux, 1976). Also, recall that in a field that is perceived to be masculine (e.g., mathematics), these sex-related differences can be amplified (Wolleat et al., 1980). In the present research study, the more extensive use of thinking moves during group activities by the males, as compared to the limited use of thinking moves by the females, appears to be supported by and aligned with the general attribution theory. Taken together, these various elements (e.g., influence of sex of student on contributions during class discussions, stereotype threat, and attributional theory) may shed light on the possible link between sex of the student and the use of thinking moves during the group activities.

The status and achievement level of the student may have also affected their willingness to explain their thinking during group activities. For example, higher achieving students (Mulryan, 1994) and high-status students (i.e., as defined by their mathematical ability as perceived by their classmates and their popularity in regard to number of friends in the classroom; Cohen & Lotan, 1997) often control the discourse during group activities and learn more as a result (Esmonde, 2009). In regard to high status and high achievement, the two students that made limited use of the thinking moves during the group activities had only progressed through Algebra II or Algebra III, while the other two students who made extensive use of the thinking moves during group activities were both enrolled in AP Calculus. These phenomena may also explain why the two students who had only completed Algebra II or Algebra III, who viewed themselves as lacking confidence in mathematical situations, and did not see themselves as central members of the mathematics classroom, were more likely to use the thinking move *wondering and asking questions*. These students may have felt more comfortable asking questions due to the fact that they thought this approach carried less risk and felt that their contributions were less likely to be dismissed or ridiculed if they were in the form of a question; recall that Webb (1982) found that when females asked questions in small group interactions, it was often due to their uncertainty and insecurity about the navigating the complex dynamics present in group settings. Clearly, there are many complex demands placed on students during group work including the ability to navigate social dynamics, to unpack the mathematics concepts and ideas present, and to work through the expectations of the task as laid out by the teacher. In the end, the group configuration can affect the individual contributions of the group members and the overall achievement of the group as they address those demands (Esmonde, 2009). In the end, there are a number of possible reasons why the thinking moves patterns were different on the exams than they were during the group activities including issues related to identity, approaches to teaching and learning, and group dynamics.

Summary in relation to the overarching research question. Overall, the findings of the present study in terms of the overarching research question suggest that mathematical identity is associated with the use of high-leverage thinking moves in certain problem-solving situations (i.e., during group activities) and not others (i.e., during individual exam work). Moreover, the findings in this dissertation study indicate that the extensive use of thinking moves in both individual and group settings can be linked with a strong positive mathematical identity characterized by confidence in mathematical discussions, enjoyment in self-directed mathematical exploration, and the persistence to investigate intriguing mathematical problems. Finally, in the present research study extensive use of thinking moves on exam questions, but not during the group activities, may be linked to mathematical identities characterized by perseverance in more traditional learning environments and a lack of confidence in discussion based situations. In the end, the *practices of mathematics* described by Boaler (1997) that allowed students to successfully transfer their mathematical learning to multiple situations are revealed in the identities of the students in the present research study who made extensive use of thinking moves in multiple problem-solving situations.

Limitations

The limitations of the present research study are considered in terms of the specific limitations related to this study and in terms of the general limitations of a qualitative study. Limitations to this research include the following: (a) the fact that the research subjects knew they were being observed, (b) the selection process of the four cases and the number of cases that were studied, and (c) the dynamics during the group activities. Each of these limitations is considered in more detail.

One possible limitation of the present research is that the students knew they were being followed as research subjects. As a result of being nervous, wanting to please the researcher, or wanting to affect the research in a certain manner, the answers provided during interviews and on surveys as well as their contributions during the problem-solving activities could have been affected. For example, the students may not have shared everything they were thinking during the group activities as a result of feeling uncomfortable with their comments being recorded. Additionally, students may not have stated their true thoughts during the interview as they were embarrassed of their opinions about mathematics or were not comfortable sharing their mathematical experiences with an individual that they did not know very well.

Other possible limitations are that only four cases were studied and that the selection process for the four cases resulted in cases that may not have been fully representative. In order to predict comparable results (i.e., a literal replication; Yin, 2009), both sets of students were chosen to have a similar mathematical background and a comparable mathematical identity. Moreover, to predict dissimilar results for anticipated reasons (i.e., theoretical replication; Yin, 2009), the two sets of cases were chosen to be different in relation to mathematical background (i.e., enrolled in AP Calculus versus completion of Algebra II or Algebra III) and mathematical identity (i.e., strong positive to mixed versus mixed to strong negative). This framework may have been a limiting factor in the selection process. For instance, there were no females who were taking AP Calculus and AP Statistics during the timeframe of the research. In addition, the only available students (i.e., had consented to be part of the present research study) with a mixed or negative mathematical identity that had only progressed through Algebra II or Algebra III were female. The end result was that the two cases with a strong positive to positive mathematical identity were male, and the two cases with a mixed to negative mathematical identity were female.

An additional possible limitation is that the group dynamics during the five statistical activities affected the display of thinking moves. For example, research has shown that students placed into groups do not always work well together (Slavin, 1990), that all students do not always have an equal chance to participate (Boaler, 2008), and that students are sometimes unable to engage in effective discussions with peers (Barron 2003; McCrone, 2005; Ryve, 2006; Sfard; 2001). Moreover, the use of accountability systems, performance goals, and group contracts can help groups collaborate effectively (Katzenbach & Smith, 2006; Tanner et al., 2003); yet, these team building components were not utilized in the classroom in which this research took place. In the end, the group dynamics may have affected the willingness or the ability of the students to share their thinking with the group.

Furthermore, there are possible limitations regarding the results of this research due to the qualitative nature of the case study. In this regard, qualitative case studies can be biased, selective, and subjective as cross-checking can be difficult. Additionally, observer bias can be a problem even if reflexivity is addressed (Cohen et al., 2007). Frequently the results found and the conclusions made in this type of qualitative case study are not generalizable and can lack rigor (Cohen et al., 2007; Yin, 2009). Finally, some or all of the possible associations discussed may be spurious relationships due to the failure to account for potential confounding variables. While every care was taken in the present research to overcome these limitations where possible, some of the above may have affected the outcomes of this study. The implications of the present research study are considered in next section.

Implications

Because the current research study found a number of possible associations between mathematical identity and the use of thinking moves during problem-solving activities, there are a number of implications in relation to these two phenomena. Additionally, there are implications to the findings that certain students did not make the same use of thinking moves during group activities that they did during individual work on exam questions. While the generalization of results and conclusions from qualitative research can be problematic, the use of "thick" descriptions through detailed explanations of findings and results in the present research study allows readers to determine if the transfer to their context is appropriate (Guba & Lincoln, 1985).

One important implication is that the promotion and the evaluation of student thinking in the mathematics classroom are necessary for educators interested in having students show understanding through the display of a wide range of thinking moves. In this regard, it may be helpful for teachers to consider desired thinking moves when writing educational objectives and when designing activities for teaching and learning. The introduction of these types of objectives would naturally lead to the explicit modeling of thinking moves, discussions with students about the relationship between thinking and understanding, and the intentional use of certain *thinking terms* in class discussions. For example, a teacher could model what it looks like to *make connections* in a statistics classroom or explain why it is important to *consider other viewpoints and perspectives* in mathematics. These are all important aspects of creating a culture of thinking in the classroom.

A related implication is that it is important for educators to provide multiple ways for students to demonstrate their thinking moves. The current research study showed that while some students displayed extensive use of thinking moves in group activities and on individual exam questions, other students made use of a wider range of thinking moves on exam questions than during group activities. For this reason, it is important for students to have multiple ways to show that they understand the material; specifically, students should have opportunities to work in group problem-solving situations as well as time to work individually. Moreover, this information indicates that teachers should create learning environments that emphasize the *process* of problem solving and not just finding answers. One way that educators could encourage students to value the problem solving process is through the design and use of problems that have either multiple entry points or have multiple solutions, or both. In addition, if the objective of the group activities was shifted from finding solutions to the necessity to have all group members engage in a wide range of thinking moves, would that change the collective patterns of thinking moves used? These all are important implications related to the display of thinking moves in the classroom.

An additional implication is that addressing issues related to mathematical identity can impact the thinking moves that take place in mathematical environments. While teaching mathematical content and helping students develop critical thinking skills is important, the awareness of students' mathematical identities and the openness to working with students to transform those identities are also crucial. For instance, to differentiate well, teachers need to take the time to get to know their students; getting to know students in relation to their mathematical identity may help teachers leverage the classroom environment in such a way to encourage the use of a wider range of thinking moves and thinking moves in all three thinking categories. Moreover, there are implications in relation to mathematical identity and group dynamics. While researchers (e.g., Boaler, 2008; Good et al., 1990; Slavin, 1990) have documented the effectiveness of cooperative small group work in the mathematics classroom, the current research suggests that the consideration of mathematical identity may be an essential component of building effective group dynamics. Failure to do so may create an environment where students appear to be engaged, but where only certain students are able to display a wide range of thinking moves. Recommendations for areas of future research are considered next.

Recommendations for Future Research

The exploration of how the narratives that students construct of their mathematical identity are related to the use of thinking moves during problem-solving activities illuminated a number of areas for future research. These areas of future research include the consideration of the differences of thinking moves displayed during group activities and on exam questions, the examination of possible associations between the use of MEAs and the range of thinking moves displayed, and an exploration of the connections between the eight high-leverage thinking moves and statistical thinking.

One key finding was that the students with a mixed or negative *current* identity had substantially different patterns of thinking moves for the group activities as compared to the individual exam questions. These students performed very similarly to the students with a positive current identity on the individual exam questions; however, during the group activities, these students displayed a more limited use of the thinking moves.

Further research could explore this difference in multiple ways. For instance, the thinking moves exhibited on responses to exam questions could be explored in relation to the mathematical identity of the student. In such an exploration, it would be important to make use of exam questions that have the potential to solicit the use of all eight high-leverage thinking moves. Note that the thinking moves *wondering and asking questions* and *considering different viewpoints and perspectives* were not accessible on the exam questions in the current research. Moreover, additional research could examine in what ways group dynamics relate to the thinking moves used during problem-solving activities. For example, one could examine how the strategic selection of group members and the intentional building of cooperative groups using research-based methods (e.g., Good et al., 1990; Katzenbach & Smith, 2006; Tanner et al., 2003) affect the use of thinking moves displayed during group activities. Furthermore, the relationship between the mathematical identity of students; other factors such as sex, status, and achievement level; and the use of high-leverage thinking moves in individual and social learning environments needs further exploration.

Future research could also investigate how effective the use of MEAs is to elicit various high-leverage thinking moves compared to more traditional activities. For example, it would be interesting to see if all eight high-leverage thinking moves are always used by students to develop models during a complex modeling task. In addition, it would be interesting to see how the thinking moves displayed during MEAs progress in regard to the interplay between the three thinking categories. For instance, do the thinking moves used at different points during the MEA reside equally in all three thinking categories, or do the thinking moves build from the *creative* and *critical* categories at the beginning towards the *mental management and awareness* category at the completion of the activity.

Finally, it may be beneficial to pursue research in regard to the connections between the use of the eight high-leverage thinking moves and statistical thinking as many researchers (e.g., Franklin et al., 2005; Garfield, 2002; Shaughnessy, 2010) have stressed the importance of addressing statistical literacy, statistical reasoning, and statistical thinking in the educational system. For instance, one could explore the interplay between the thinking moves used and the various statistical thinking domains (e.g., data in context and contextual knowledge, statistical representations, and other knowledge including statistical knowledge, mathematical knowledge, and logical thinking; see delMas, 2014). One could also investigate the role the thinking moves in the *mental management and awareness* category play in regard to thinking in a statistical manner. Ultimately, there are a number of important areas for future research related to the mathematical identity and the use of high-leverage thinking moves.

Conclusion

If our goal is to create an educational system that provides a high quality mathematics education for all students, it is essential that we move away from the teaching of mathematics as a set of procedures and rules and the reliance on standardized tests to measure success. This researcher believes, as does Ritchhart (2015), that a quality education embodies the “promotion of the dispositions needed for students to become active learners and effective thinkers eager and able to create, innovate, and solve problems” (p. 34). As we pursue this goal, it is important that our mathematics classrooms are designed in such a way that they promote a culture where thinking is supported and stimulated. In essence, one of our main educational objectives has to be developing effective thinkers.

In exploring what it means to create a culture of thinking in the classroom, it is essential to consider the symbiotic relationship between the thinking moves displayed in mathematical environments and the mathematical identities of the students. In this regard, it is important to design curriculum and classroom environments that encourage a love for learning mathematics through the introduction of engaging and interesting problems and pedagogies that are aligned with the constructed identities of the learners (Boaler & Greeno, 2000). Students who see learning as innovative and interactive will likely connect with and enjoy a flexible, student-centered learning environment where the study of mathematics allows for social interaction and the opportunity to wrestle with mathematical concepts.

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Appendix A

On Time Arrival MEA

Table A1.

On Time Arrival MEA Problem Statement (Chamberlin & Chamberlin, 2001)

Problem statement	You team is part of a consulting firm that works with a number of local businesses. A company that does a significant amount of business in Venezuela has hired your team to help them select an airline to use. Last year the many of the executives who work for the company had miserable experiences when traveling to Venezuela and other locations around the world. In one case, the CEO missed her connecting flight to Barcelona because her flight into Reykjavik, Iceland was late. She ended up having to stay overnight in the airport. The company has identified five airlines with economical fares that fly from O'Hare Airport to Venezuela, but they are still in the process of identifying more airlines that fly to Venezuela. Most of the flights have a connecting flight in Mexico City. They are hoping to find the airline that has the smallest chance of departing late from O'Hare so that they are less likely to arrive late in Mexico City. They don't want to miss any of their connecting flights to Venezuela this year! In the table (see Table E2), you will find information about departure times for flights on the five airlines that the company has identified thus far. The departure times are for flights leaving from O'Hare Airport and scheduled to arrive in Mexico City.
Part I	Rank the five airlines in terms of most likely to be on time to least likely to be on time for departing from O'Hare Airport. As you rank the airlines, keep track of your process. Write a paragraph describing your ranking process that includes a justification for the ranking and procedure for the ranking based on statistics. In addition, for each airline include a paragraph explaining, using statistical language and terms, your reasoning for the ranking.
Part II	Describe your process in a letter to the company so that they may use a similar process to rank the additional airlines that they may identify at a later time. This detailed description should include justifications and procedures based on statistics.

Table A2.

Number of Minutes Late for Flights Departing from O'Hare Airport (Chamberlin & Chamberlin, 2001)

Sky Voyage Airline	**Central American Airlines**	**Mexico Express**	**Sudamerica Internacional**	**Southeast Airline**
5	15	9	0	0
0	9	5	25	5
20	4	5	0	0
5	0	5	9	9
0	0	125	0	40
6	14	10	0	0
0	20	5	4	5
0	15	10	0	25
15	16	0	35	10
0	0	4	0	30
0	0	10	0	12
7	15	10	10	0
0	10	10	5	0
5	10	9	55	10
40	25	7	0	9
4	5	12	0	5
0	20	5	0	0
0	15	0	17	27
0	11	10	5	11
0	12	7	0	0
3	0	13	65	30
60	5	0	5	5
5	0	0	0	0
0	30	10	0	4
7	4	5	2	40
0	5	4	0	0
0	10	6	0	15
123	10	5	75	0
0	25	7	0	6
5	4	5	0	9

Appendix B

Race and the Death Penalty Activity

Table B1.

Race and the Death Penalty Activity (Bock et al., 2007)

In 1976 the Supreme Court ruled that the death penalty does not violate the US. Constitution‘s ban on "cruel and unusual punishments." Since then many states have passed capital punishment statutes, and over 500 convicted murderers have been executed nationwide.

Capital punishment may be constitutional. but there continues to be a debate about whether or not it is fair. One of the major issues in this debate involves race - the race of both the defendant and the murder victim. The central question: is justice blind?

In 1998 the Death Penalty Information Center published *The Death Penalty in Black and White*, a study examining the sentences following 667 murder convictions in Philadelphia courts between 1983 and 1993. This 3-way table shows how many death sentences were given among all the murder convictions.

Death Sentences	Black Victim	White Victim	Total
Black Defendant	76 of 422	21 of 99	97 of 521
White Defendant	1 of 25	17 of 121	18 of 146
Total	77 of 447	38 of 220	115 of 667

Is our system of justice colorblind in the administration of the death penalty? Based upon the above information, write a newspaper article discussing the association between race and the death sentences in the United States. (Don’t forget the best analyses usually combine visual, numerical, and verbal descriptions.)

Appendix C

Backhoes and Forklifts activity

Table C1.

Backhoes & Forklifts Activity (Bock et al., 2007)

A heavy equipment manufacturer introduces new models of their backhoes and forklifts, and sells several of these machines. A few months later they discover a problem. An unacceptably high number of their customers needed to seek repairs of the machines' hydraulic systems within the first few weeks of operation. The company's engineers go to work on this problem, and soon think that they have found a solution. They believe that an additive poured into the hydraulic oil may greatly extend the number of hours these machines can be used before repairs become necessary. A few tests conducted under laboratory conditions indicate that this solution shows promise, but "real world" customer experience is needed before they can be sure. Impressed by these preliminary results, the company's management gives the research team the green light to test their theory using up to 20 newly manufactured machines that will be sold during the next few weeks. Clearly describe how you would design this experiment and outline your design with a diagram. Explain and perform your randomization procedure, showing the resulting assignments. Indicate any additional measures you would take to eliminate potential sources of bias. Explain how you would decide whether the additive has solved the problem. Be sure to use appropriate vocabulary throughout your description of the design.

Appendix D

ESP Activity

Table D1.

ESP Activity (Bock et al., 2007)

Your friend claims he "has ESP". Being properly skeptical, you decide to subject his claim to an experiment. Here is your design. You will get ten volunteers to sign their names on identical cards, and seal the cards in identical envelopes. You will then shuffle the pile of envelopes, and hand them to your friend. Using his alleged powers of extrasensory perception, he will distribute the envelopes back to the volunteers, trying to match each person with the one containing the proper signature. It will, of course, be quite stunning if, when the ten volunteers open their envelopes, they all find their own signatures. If that happens you will believe he really does have ESP – and contact James Randi about that million dollars! Since that's unlikely, before actually conducting this experiment you need to determine how well he needs to do to convince you that he does have some mystical insight. How many correct matches would you consider unusual (that is, statistically significant)? To find out, stimulate this experiment. You may use either your calculator or the random number table. Write a report in which you clearly explain your procedure, show the results of at least 20 trials, and state your conclusion.

Appendix E

The Spread of a Rumor Activity

Table E1.

The Spread of a Rumor Activity (Starnes et al., 2003)

Problem statement	Suppose there are 25 students in your math class. You just learned some juicy gossip about the teacher, and you are dying to tell someone. For purposes of the model, suppose that during a given time increment (say 2 minutes), you randomly select one person in the class to whom you will whisper the rumor. Then during the next time increment, the two of you who know the rumor each randomly selects someone in the class to whom you will whisper the rumor. Then each of the four of you who know the rumor randomly selects yet another person to tell, and so forth. But we adopt the rule that if someone is selected to hear the rumor who has already heard it, that branch of the rumor spreading stops, because nobody likes to hear stale gossip. Will all 25 students eventually hear the rumor? If not, what proportion of the class will hear the rumor over the long haul? Make a thoughtful conjecture. Then consider the following questions or tasks.
Questions	1. Do you think the assumptions made are reasonable? Comment. 2. How could you change the assumptions of this model to make it more interesting? 3. Discuss with your partner how to stimulate this rumor experiment *by hand* (i.e., using a table of random numbers or equivalent random number device, pencil and paper, etc.). Define a repetition. Then stimulate 20 repetitions of the experiment. 4. Once you have developed some insight into this problem, simulate the experiment using the computer or your TI-83. Do the results you get with the computer/calculator agree with the results you got by hand? How could you increase your confidence in your results? 5. Using the methods described in Section 5.3, could you write a TI-83 program to simulate this experiment? You may want to refer to the programs FREETHRO (text Exercise 5.58) or FLIP50 (text Exercise 2.19) as a model. Allow the user to determine the number of replications. Provide a listing of your program and the results of several representative executions. Based on your results from this investigation answer the question: On average, what proportion of the class will hear the rumor? 6. Discuss how you might modify the spreading rumors problem to study the problem of a measles epidemic. 7. Write a report that summarizes your finding to the questions above. Refer to the Special Problem Guidelines for guidance on preparing your report.

Appendix F

MINT Protocol

Table F1.

MINT Protocol Introduction, Section 1, and Section 2 (Eaton et al., 2013)

Introduction	The MINT Survey measures the mathematical identity of an individual through their narrative. Please complete the survey in order from Section 1 to Section 3. All data collected will be kept strictly confidential.
Section 1	Name: ___________________________ Gender: _______ Year in school: _______ Age: _______ High school mathematics classes taken prior to this course: Other math courses being taken this school year:
Section 2	Please respond to the following statements using the following scale 1- Strongly Agree 2- Agree 3- Neutral 4- Disagree 5- Strongly Disagree _______ Mathematics is a challenging subject _______ Mathematics is more difficult than other subjects _______ I have had an overall positive experience of mathematics _______ Mathematics is irrelevant to everyday life _______ I find mathematics intimidating _______ I'll need a good understanding of mathematics for my future work _______ Mathematics is interesting _______ I feel competent in mathematics

Table F2.

MINT Protocol Section 3 (Eaton et al., 2013)

Part 1	Think about your total experience of mathematics. Describe the dominant features (thoughts, feelings, and experiences) that come to mind.
Part 2	Now think carefully about all stages of your mathematical journey from elementary school (or earlier) mathematics to high school mathematics. Consider: (a) Why you chose to study AP Statistics (b) Influential people (c) Critical incidents or events (d) Your feelings or attitudes about mathematics (e) How mathematics compares to other subjects (f) Mathematical content and topics With these and other thoughts in mind, describe some further features (thoughts, feelings, and experiences) of your relationship with mathematics over time.
Part 3	What insight, if any, have you gained about your own attitude towards mathematics and studying the subject as a result of completing this questionnaire?