Inhomogeneities in Spatially Extended Pattern Forming Systems

A THESIS
SUBMITTED TO THE FACULTY OF THE GRADUATE SCHOOL
OF THE UNIVERSITY OF MINNESOTA
BY

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IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

Arnd Scheel

June, 2015

Acknowledgements

I would like to express my deepest gratitude to my adviser Professor Arnd Scheel, whose guidance on both my research and career have helped me immensely. Thank you for your wisdom, patience, and support. I would also like to thank Professor Qiliang Wu for his help with the proof of Proposition 4 in Chapter 2. To Professors Daniel Spirn, Carme Calderer, and Duane Nykamp, thank you for serving as my committee members. To all the staff in the Mathematics Department, thank you so much for all your help.
A special thanks to my family. Words cannot express how grateful I am to my mother and father for all of the sacrifices that they have made on my behalf. To my sister María and my brother Juan, thank you for always believing in me. I would also like to thank my friends, it was wonderful to have you around for long discussions about our careers as mathematicians. Finally, I want to thank my husband James who was been with me throughout this process, cheering me on and making me smile.

Abstract

We study the effects of defects and impurities on pattern formation from the point of view of perturbation theory, and focus on three examples of spatially extended systems inspired by physical phenomena. We will look at striped patterns formed in Rayleigh Bénard convection, target patterns arising in chemical oscillations in dimension 3, and wave sources in large arrays of oscillators with nonlocal coupling. We explain why regular perturbation theory fails due to the presence of essential spectrum and show how Kondratiev spaces can help overcome this difficulty: the linearization at periodic patterns becomes a Fredholm operator, albeit with negative index. Finally, using far-field matching procedures we obtain the following results. In the case of Rayleigh-Bénard convection we show how impurities deform the striped pattern, in the case of chemical oscillations in dimension 3 we show that algebraically localized inhomogeneities do not give rise to target patterns, and lastly in the case of a large array oscillators with nonlocal coupling we show that defects generate wave sources.

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Chapter 1

Introduction

Patterns are ubiquitous in nature, from the spirals seen in slime mold to the chaotic granules that form on the photosphere of the sun. The study of pattern formation not only helps us understand naturally occurring patterns, but also to adapt this knowledge to applications such as tissue engineering, biomaterial fabrication [4, 5, 24], manufacturing of integrated circuits [11, 20], and even predicting crime hotspots [38, 37, 6].

Here we concentrate on non equilibrium systems and the formation of patterns via the competition of two mechanisms. A driving force that pushes the system out of equilibrium like flux of energy, momentum, or matter, and a dissipative force that opposes it, like viscous friction, heat conduction, or electrical resistance. Summarizing the interplay between these two forces via a parameter which describes their ratio, one can find a critical value of this quantity that signals the formation of a pattern. In essence, this critical value is an indication of the driving forces being strong enough to push the system out of equilibrium. Additionally, we work with spatially extended systems which generate patterns whose length scale is much smaller than the experiment's dimensions. We will therefore consider problems on the whole space $\mathbb{R}^n$, $n = 1, 2, 3$.

One of the difficulties when studying patterns in biological systems and applications comes from the fact that it is not always straight forward to identify all competing mechanisms, or the many parameters that influence the resulting pattern. As a result, it is difficult to develop and analyze a mathematical model for most pattern forming systems, and one resorts to examples which can be studied in the more controlled setting of a laboratory. The information derived from these simplified problems is still very relevant, as these systems exhibit rich phenomena that is also present in the naturally occurring patterns as well as being challenging to model mathematically. Moreover, there is good agreement between nature and experiment, which is in part due to how patterns form as a result of a homogenous state undergoing a bifurcation and the fact that the instabilities that lead to these bifurcations are universal. We will describe the relation between instabilities and model equations in more detail in Section 1.1.

In this work we focus on convective and oscillatory phenomena, and think of Rayleigh-Bénard convection and the Belousov-Zhabotinsky reaction as our main laboratory examples. In Rayleigh-Bénard convection a thin layer of liquid is placed between two plates which are held at different temperatures. The driving force is the temperature gradient which creates a flux of energy, and the dissipative forces are the friction and heat conduction which oppose the upward movement of the fluid, leading to stripes, spirals or chaotic patterns. The Belousov-Zhabotinsky reaction on the other hand is an autocatalytic oxidation/reduction reaction in which the chemical gradients play the role of the driving and dissipative forces. As the concentration of chemicals is varied, a steady homogenous state changes to a time dependent solution, in which chemicals of two different colors alternate and patterns form as a result of impurities and diffusion.

Although we are looking at specific examples, our hope is that the results presented here become useful in the study of more complicated systems such as neural networks, cardiac tissue, and in developing new materials. Indeed, there is a relation between oscillating chemical reactions and neural or cardiac tissue. One can describe the evolution of biological processes in these cells as a dynamical system with a stable limit cycle as a solution. One can then take the point of view that this limit cycle is an oscillator, so when a large number of cells with localized spatial coupling is considered the resulting system can be approximated as an oscillating chemical reaction. On the other hand, the convection rolls and hexagonal patterns seen in Rayleigh-Bénard experiments resemble the type of patterns that are present or desirable in certain materials.

The main question we try to answer in this work is how spatially localized defects, also here referred to as inhomogeneities, affect pattern formation in the examples mentioned above. In the case of convection rolls and stripe patterns, a defect corresponds to an impurity in one of the plates. In contrast, in an oscillating chemical reaction a defect represents an impurity in the chemical sample. Also, going back to the analogy between a field of oscillators and an oscillating chemical reaction, a defect can be viewed as a patch of oscillators which move with a higher frequency than the rest of the field. We will describe in Section 1.2 how we model these systems and in particular how we incorporate defects into these models. We will also derive the real and complex Ginzburg-Landau equations as normal forms, or amplitude equation, for Rayleigh-Bénard convection and for oscillating chemical reactions, respectively. We will see that viewed as normal forms, these equations provide solutions which approximate well the behavior of these phenomena when we are near the onset of pattern formation.

The rest of this thesis is organized as follows. In the remainder of this introduction we briefly explain the role of amplitude equations in describing pattern formation and justify why we can consider inhomogeneities as perturbations of these equations. We also address previous methods for studying the role of defects and discuss the perturbative approach that we will take. This discussion will lead us to the study of the Laplace operator and the role of the underlying Banach space in determining its Fredholm properties. In particular, we will see in Chapter 2 that while the Laplacian is not a Fredholm operator in translation invariant spaces, it is possible to recover this property in the setting of Kondratiev spaces. We will define these weighted spaces in more detail in Chapter 2 and prove this result in the case of dimension 1. In Chapter 3, we consider the effects of defects in striped patterns, in Chapter 4 we look at defects in three dimensional oscillatory media, and finally in Chapter 5 we consider the more complicated case of a large array of oscillators with nonlocal coupling

1.1 Modeling pattern formation

As mentioned before, the equations describing the competing mechanisms leading to pattern formation are complex. Nonetheless, the instabilities that give rise to pattern formation are universal and can be classified. The main assumption is that as the parameter relating the driving and dissipative forces is varied, the uniform steady solution becomes unstable and undergoes a bifurcation. To understand this bifurcation we study the stability of the uniform state and look at the spectrum of the linearized operator about this steady solution. This is equivalent to studying the behavior of planar waves,

$$u = e^{\omega t + ik\cdot x},$$

and finding conditions under which these solutions grow or decay. This information is encoded in the corresponding dispersion relation $\omega(k)$, so that if Re $(\omega(k)) < 0$ for all wave vectors k the solution is stable, and if Re $(\omega(k)) > 0$ the steady state is unstable.

In [9, 8] Cross and Hohenberg showed that in the case of dimension 1, pattern forming systems can be classified according to three types of instabilities. These instabilities in turn are characterized by the shape of the curve Re $\omega(k)$, see figures 1.1, 1.2 and 1.3, and can be further classified as stationary if Im $(\omega(k)) = 0$, or as oscillatory if Im $(\omega(k)) \neq 0$.

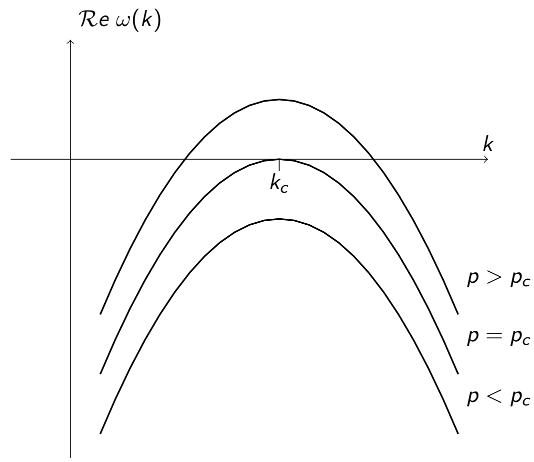

Figure 1.1: A type-I instability characterized by a symmetric band of wave numbers $k_- < k_c < k_+$ which become unstable. Examples: convection rolls.

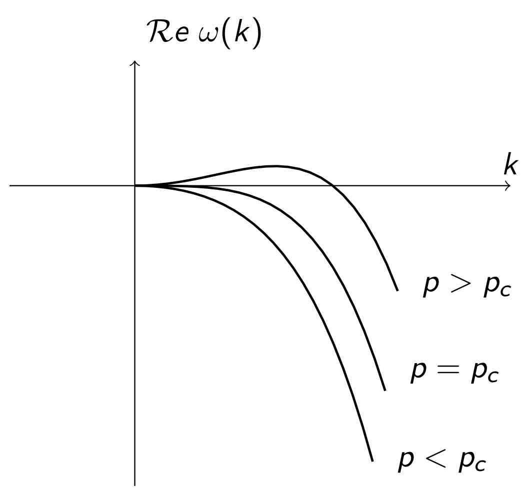

Figure 1.2: A type-II instability characterized by having critical wavenumber $k_c = 0$. This is a typical situation for systems with conservation laws.

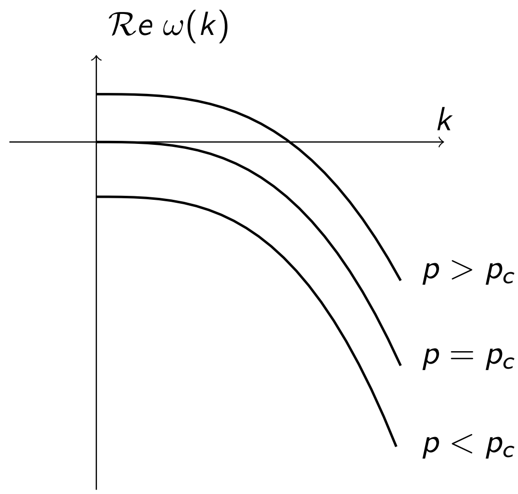

Figure 1.3: A type-III instability characterized by the maximum growth rate occurring at the critical wavenumber $k_c = 0$. Example: chemical oscillations.

The fact that most pattern forming systems experience one of the above instabilities at the onset of pattern formation means that it is possible to derive model equation which are capable of reproducing the same dynamics and phenomena seen in real experiments and in nature. To construct such an equation one can look at the shape of the dispersion curve, $\omega(k)$, to obtain information about the linear part, while symmetry considerations can be used to determine the nonlinear terms. So even though no physical laws were used to derive the model, we are still able to obtain qualitative properties of the system once the type of instability of the steady solution is known. This is the case of the Swift-Hohenberg equation

$$u_t = -(\Delta + 1)^2 u + \delta^2 u - u^3, \qquad (x, y) \in \mathbb{R}^2,$$

with $\delta \in \mathbb{R}$ small, which is widely used as a model equation for pattern forming systems exhibiting roll and spiral patterns as well as chaotic behavior.

Another use for this classification is for deriving amplitude equations or normal forms. These equations capture the dynamics of the modes closest to the unstable mode k_c, $(\omega(k_c) = 0)$, and provide good approximations to solutions of the full system near the onset of pattern formation. Their generality lies in the fact that the form of the amplitude equation is given by the instability and not by the type of phenomenon studied, and they are important because they are in general easier to analyze than the full equations. Moreover, properties specific to each system only appear in the coefficients of this equations, which can all be made equal to one by an appropriate scaling. Then whether we have a real amplitude equation or a complex amplitude equation comes from knowing if the underlying pattern is steady or time dependent, respectively. This leads to equations that are universal, and that can describe a wide range of phenomena.

1.2 Amplitude equations and inhomogeneities

In this section and in the rest of this Thesis we treat inhomogeneities as localized, algebraically decaying functions.

1.2.1 Oscillating chemical reactions

We start with oscillating chemical reactions and sketch the derivation of the complex Ginzburg-Landau equation as an amplitude equation for these systems. The derivation is based on the existence of different time scales which permits the use of the multiple scales formalism. From the derivation it will also become clear that a defect can be viewed as a perturbation of this equation. For more information regarding the derivation of the complex Ginzburg-Landau equation using the multiple scales formalisms see in [23, 31], and for a more detailed description of how one parametrizes the center manifold see [14].

We start from a general reaction-diffusion equation

$$U_t = \Delta U + F(U, \mu), \quad U \in \mathbb{R}^m, \quad \mu \in \mathbb{R}, \tag{1.1}$$

and assume that for negative values of the parameter μ there exist a steady solution, $U_*(\mu)$. Furthermore, we suppose that for $\mu = 0$ the Jacobian of F evaluated at $U_*(0)$ has two purely imaginary eigenvalues, $\lambda_{1,2} = \pm\omega_0$, which undergo a supercritical Hopf bifurcation, and that the rest of the eigenvalues have negative real part. Since $F(U, \mu)$ is assumed to be smooth, we know there exists a center manifold that can be parametrized using the center space E^c, that is by the eigenvectors corresponding to $\lambda = \pm\omega_0$. Moreover, our assumptions imply that all solutions which are not on this center manifold will converge exponentially to it, and we can therefore distinguish between two different time scales. If we now add back a slow spatial dependence, it is reasonable to expect that the coordinates of the center manifold are of the form

$$U_*(x, t; \varepsilon) = \varepsilon A(\varepsilon x, \varepsilon^2 t)\mathrm{e}^{-\mathrm{i}\omega_0 t} + c.c., x \in \mathbb{R}^n$$

where $\varepsilon = \sqrt{\mu}$ and A is a complex number.

Inserting this Ansatz back into (1.1) we obtain a hierarchy of equations at each order of ε, and find that at order $O(\varepsilon^3)$ the equations are solvable provided the amplitude A satisfies the complex Ginzburg-Landau equation,

$$A_t = (1 + i\alpha)\Delta A + A - (1 + i\gamma)A|A|^2. \tag{1.2}$$

Here, α and γ are parameters that are specific to the original reaction diffusion system.

We now consider the case in which an inhomogeneity is present in the medium. If we think of the above reaction diffusion equation as describing a Belousov-Zhabotinsky reaction occurring on a petri dish, an inhomogeneity represents a small impurity in this dish. Such an inhomogeneity will affect the concentration of chemicals in a localized region and can be modeled as a perturbation of equation (1.1). Because we know that spiral waves and target patterns occur as a result of introducing impurities in these chemical reactions, we expect that the effects of such a perturbation should be captured by the amplitude equation as well. It therefore seems reasonable to pick a scaling for the perturbative term in such a way that it appears at order $O(\varepsilon^3)$ in the multiple scales analysis and hence it also becomes a perturbation of the amplitude equation.

If on the other hand, we take the point of view that the spatial variables indicate the position of an oscillator whose dynamics are described by the nonlinearities of the reaction diffusion equations, then these equations represent an infinite number of such oscillators which are locally coupled. In this case, an inhomogeneity represents a localized patch of oscillators with a larger frequency. Recall that A is a complex number and that we may write it as $A = se^{i\phi}$. It is then straight forward to see that an impurity can be represented as a perturbation of the amplitude equation of the form $i\varepsilon g(x)A$,

$$A_t = (1 + i\alpha)\Delta A + A - (1 + i\gamma)A|A|^2 + i\varepsilon g(x)A. \tag{1.3}$$

1.2.2 Rayleigh-Bénard convection

In the case of the Rayleigh-Bénard experiment in two dimensions, we model the formation of stripe patterns using the Swift-Hohenberg equation

$$u_t = -(\Delta + 1)^2 u + \delta^2 u - u^3, \qquad (x, y) \in \mathbb{R}^2,$$

which is known to possess periodic patterns of the form

$$u(x,y) = u_*(kx;k), u_*(\xi + 2\pi; k) = u_*(\xi; k),$$

with $k \sim 1$, and small δ.

Again one can use a multiple scales argument to arrive at an amplitude equation for this model. In the case of the (isotropic) Swift-Hohenberg equation, one finds the Newell-Whitehead-Segel equation

$$A_T = -(\partial_X - \mathrm{i}\partial_{YY})^2 A + A - A|A|^2,$$

using an Ansatz

$$u(x,y,t) = \delta A(\delta x, \sqrt{\delta} y, \delta^2 t)\mathrm{e}^{\mathrm{i}x} + c.c.,$$

and expanding to order δ^3, as a solvability condition [9, 31]. There are several difficulties with the Newell-Whitehead-Segel equations and their validity as an approximation [36], related to the fact that the original equation is isotropic, while the expansion singles out a preferred wave vector, here the vector $k = (1,0)^T$. The situation is simplified in anisotropic pattern-forming systems such as

$$u_t = -(\Delta + 1)^2 u + \partial_{yy} u + \delta^2 u - u^3,$$

where a similar Ansatz leads to the isotropic (sic!) Ginzburg-Landau equation

$$A_T = \Delta A + A - A|A|^2,$$

possibly after rescaling X and Y. As before, impurities can be idealized as local inhomogeneities, $\varepsilon g(x,y)$, in the Swift-Hohenberg model, which are then carried over in the multiple scales argument and appear at order δ^3. We therefore obtain the perturbed Ginzburg-Landau equation

$$A_T = \Delta A + A - A|A|^2 + \varepsilon g(x,y). \tag{1.4}$$

In Chapter 3 we will use equation (1.4) to study defects in striped patterns and in Chapter 4 we will use equation (1.3) to study pacemakers in chemical oscillations. In the last chapter, Chapter 5, we will consider instead a field of oscillators with nonlocal coupling, but because the equations we use are not motivated by the above arguments we will defer its derivation to the introduction of said chapter.

1.3 Methods

Having looked at different approaches to modeling pattern forming systems we now consider the different methods used to study them.

1.3.1 Spatial dynamics

In the case of two and one dimensions, localized perturbations of the Belousov-Zhabotinsky reaction behave like pacemakers, generating traveling waves that move away from the defect and that create a target pattern [23, 42, 41]. The radial symmetry of these solutions suggest setting up the problem in radial coordinates and to study the relevant reaction diffusion equations using methods from spatial dynamics. Indeed, this approach was used in [21], where the authors study the effect of a localized and radially symmetric perturbation of a reaction diffusion system. In particular they look for coherent structures, which are defined as radially symmetric solutions $u_c(r,t)$ which are time periodic, i.e. $u_c(r, t + \frac{2\pi}{\omega_c}) = u_c(r,t)$ for some frequency $\omega_c > 0$, and which are localized in the sense that the solutions approach a wave train, $u_*(kr - \omega_c t + \phi(r); k + \phi'(r))$ as $r \to \infty$, for certain wave number k. Then a classification of these coherent structures can be done using their group velocity at the far field, $c_g(k) = \nabla\omega(k)$, so that the coherent structure u_c is a

- *source*, if $c_g(k) > 0$, and wave trains move away from the defect forming a target pattern.
- *sink*, if $c_g(k) < 0$ and wave trains move toward the defect.
- *contact defect*, if $c_g(k) = 0$ and waves trains die out in the far field.

In particular, their results show that in dimensions $n \leq 2$, and depending on the sign of the perturbation, the system is able to support sources and produce a target pattern. If on the other hand the sign of the inhomogeneity changes, only sinks and contact defects may occur. For dimensions $n > 2$ only contact defects and sinks are supported by the system, meaning that no target patterns are seen in dimensions 3 or higher.

As mentioned before the methods used to study this problem rely on spatial dynamics. In [21] the authors start from a perturbed reaction diffusion equation

$$u_t = D\Delta u + f(u) + \varepsilon g(x), x \in \mathbb{R}^n$$

with D a diagonal positive definite matrix, ε small, and $g \in C^\infty$, a localized function with algebraic decay, that is $|g(x,u)| = O(|x|^{-2-\beta})$ as $|x| \to \infty$ and $\beta > 0$. They also assume that for $\varepsilon = 0$ the system posses an asymptotically stable oscillation, $u_* = u_*(-\omega_c t)$, which is also homogeneous in space. Further assumptions are made on the spectrum of the associated period map in order to guarantee that the stable oscillation is orbitally stable. In particular, it is assumed that the spectrum lies inside the unit circle except for a quadratic tangency at 1. Using the radial symmetry of the system, coherent structures $u_c(r,t) = u(r - \omega t)$, satisfy the following equation

$$-\omega u_t = D\left(u_{rr} + \frac{n-1}{r}u_r\right) + f(u) + \varepsilon g(x).$$

Reversing the role of space and time, and considering time periodic solutions, $t \in [0, 2\pi)$, one can rewrite the above equation as a system

$$u_r = v \tag{1.5}$$

$$v_r = -\frac{n-1}{r}v - D^{-1}(\omega u_t + f(u) + \varepsilon g(x), \tag{1.6}$$

where $r \in [0,\infty)$ and the right hand side of the above equation defines an operator with domain $H^1_{per}(0,2\pi) \times H^{1/2}_{per}(0,2\pi)$ over the base space $H^{1/2}_{per}(0,2\pi) \times L^2_{per}(0,2\pi)$. This choice of base space is done so that no restrictions on the growth of the nonlinearity are necessary, and so that when viewed as a Nemitskii operator the nonlinearity maps back into the space $L^2_{per}(0,2\pi)$.

Because coherent structures converge to traveling waves in the far field, it seems reasonable to split the interval $(0,\infty)$ into two parts and try to connect solutions which are near u_* on the interval $[0,R]$ and are bounded in backward "time" , to solutions bounded on $[R,\infty)$, near u_*, and which also correspond to different coherent structures. Then given any finite R, the proof relies on finding on the interval $[0,R]$ a center unstable manifold W^{uc} containing u_*, and finding on the interval $[R,\infty)$ a center stable manifold W^{cs} also containing u_*. Then one needs to show that these two manifolds intersect transversely along the solution u_* and analyze the reduced vector field on this center manifold. The coherent structures are found as equilibria at infinity for the system in the unbounded domain $[R,\infty)$ using blow up techniques and a Dulac map. Finally, conditions are derived for situations when coherent structures connect to solutions on $[0,R]$ which are bounded backwards in "time".

As stated before, with this procedure it is possible to obtain a complete picture of the types of phenomena present in reaction diffusion systems when an inhomogeneity is present. The slight disadvantage of the above method is that it applies only to defects which are also radially symmetric and with a specified algebraic decay. One of the goals of this project is to remove these two restrictions and study more general inhomogeneities.

1.3.2 Perturbation theory

As mentioned earlier, we look at the problem of defects in pattern forming systems from the point of view of perturbation theory by describing them as perturbations of the amplitude equations. Writing the problem in a very general form, we want to find solutions to

$$\mathcal{L}u + N(u) + \varepsilon g(x) = 0,$$

where $\mathcal{L} : X \to Y$ represents the linearization of our amplitude equations about a stationary base pattern, $N(u)$ represents the nonlinear terms, and $g(x)$ corresponds to the algebraically localized inhomogeneity. Ideally we would justify the existence of solutions

$$u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3 + \cdots,$$

where u_0 represents the stationary pattern, via the Implicit Function Theorem and proceed to obtain the first order approximation u_1. Unfortunately, since we work in unbounded domains and the amplitude equations involve the Laplace operator, the linearization $\mathcal{L} : X \to Y$ is not an invertible operator when viewed in the setting of Sobolev spaces, or any translation invariant Banach space. One could try to set up the equations in a large bounded domain and use the fact that the linear operator has compact resolvent in order to carry out the perturbation analysis, but as the following simple example on the interval $[-L, L]$ shows, the results would then be valid only for $\varepsilon \sim \mathrm{O}(1/L^2)$, since the eigenvalues of the operator accumulate near zero when the domain is large.

Example: Suppose we consider stripe patterns in a Rayleigh-Bénard experiment and suppose we model the dynamics of pattern formation under the influence of a defect using the perturbed one dimensional real Ginzburg-Landau equation on a bounded domain,

$$A_t * = \Delta A + A - A|A|^2 + \varepsilon g(x), \quad x \in [-L, L]. \tag{1.7}$$

Again we consider algebraically localized inhomogeneities and we assume L is large. We look for steady solutions near a stable pattern $A_*(x) = \sqrt{1-k^2}\mathrm{e}^{\mathrm{i}x}$, and for simplicity of exposition we assume the wavenumber $k = 0$. Using polar coordinates $A = s\mathrm{e}^{\mathrm{i}\phi}$, we look for steady patterns of the form $A(x) = (1 + s(x))\mathrm{e}^{\mathrm{i}\phi(x)}$, which satisfy the following system of equations obtained by separating real and imaginary parts of equation (1.7),

$$\partial_t s = \partial_{xx} s - 2s - 3s^2 - s^3 - (\partial_x \phi)^2(1 + s) + \varepsilon \mathrm{Re}\,(g\mathrm{e}^{-\mathrm{i}\phi}), \tag{1.8}$$

$$\partial_t \phi = \partial_{xx}\phi + \frac{2\partial_x s \partial_x \phi}{1 + s} + \varepsilon \mathrm{Im}\,(g\mathrm{e}^{-\mathrm{i}\phi}). \tag{1.9}$$

Suppose we consider the linear part of the above equations as an operator $\mathcal{L} : H^2(-L, L) \times H^2(-L, L) \to L^2(-L, L) \times L^2(-L, L)$ defined by

$$\mathcal{L} = \begin{bmatrix} \partial_{xx} - 2 & 0 \\ 0 & \partial_{xx} \end{bmatrix}$$

and with Neumann boundary conditions. Then $\mathcal{L}$ is self adjoint, with a nontrivial kernel, Ker $= \{(0, 1)\}$, and eigenvalues given by

$$\sigma(\mathcal{L}) = \left\{ -\left(\frac{n\pi}{L}\right)^2,\ -2 - \left(\frac{\pi n}{L}\right)^2 : n = 0, 1, 2, \cdots \right\}.$$

Consequently, $\mathcal{L}$ is a Fredholm operator with index zero and we are able to use Lyapunov-Schmidt reduction to find the first order approximations for s and ϕ.

Letting $I - P : L^2(-L, L) \times L^2(-L, L) \to \mathrm{Ker}(\mathcal{L})$, denote the projection onto the kernel of $\mathcal{L}$ and writing

$$u = (s, \phi), \quad u = u_0 + u_0^{\perp},$$

with $u_0 \in \mathrm{Ker}(\mathcal{L})$ and $u_0^{\perp} \in \mathrm{Ker}(\mathcal{L})^{\perp}$, we are able to decompose the above equations into an invertible operator and a finite dimensional problem. Because we have a one dimensional kernel we must impose additional conditions to obtain unique solutions, so we let $(I - P)u = 0$ which implies $u_0 = 0$. Similarly, because we have a one dimensional cokernel we must consider solutions of the form $u = u_0^{\perp} + c\chi$, where χ is a smooth function satisfying $(I - P)\mathcal{L}\chi \neq 0$, $P\mathcal{L}\chi = 0$. The result is a system with the following abstract form

$$P\mathcal{L}u_0^{\perp} + PN(u_0^{\perp} + c\chi; \varepsilon) = 0 \tag{1.10}$$

$$(I - P)\mathcal{L}c\chi + (I - P)N(u_0^{\perp} + c\chi; \varepsilon) = 0 \tag{1.11}$$

where N represents the nonlinearities of equations (1.8), (1.9).

Notice that this system of equations is solvable and that we are able to obtain solutions $(u, c) = (u_0^{\perp}(\varepsilon), c(\varepsilon))$ using the Implicit Function Theorem. However, the eigenvalues of the invertible operator $\tilde{\mathcal{L}} = P\mathcal{L}|_{\mathrm{Ker}\,(\mathcal{L})^{\perp}}$ are

$$\sigma(\tilde{\mathcal{L}}) = \left\{ -\left(\frac{n\pi}{L}\right)^2,\ -2 - \left(\frac{\pi n}{L}\right)^2 : n \in \mathbb{N} \right\},$$

so that the norm of its inverse, $\tilde{\mathcal{L}}^{-1}$, is bounded by $\|\tilde{\mathcal{L}}^{-1}\| < \dfrac{L^2}{\pi^2}$. Therefore, the series

$$s = \varepsilon s_1 + \varepsilon^2 s_2 + \cdots$$
$$\phi = \varepsilon\phi_1 + \varepsilon^2\phi_2 + \cdots ,$$

converge only if the parameter ε is of oder $\mathrm{O}(1/L^2)$. Because the length scale of the patterns we consider are small compared to the length, L, of the interval, the results obtained through this procedure are valid for regimes of the parameter that are too small to be significant.

A similar situation arises when we consider the reaction diffusion system mentioned in Section 1.3.2. In this case one looks at the period map associated with the linearization of these equations about the asymptotically stable orbit. By perturbing the equation we would like to justify the persistence of this orbit and compute its first order approximation. Again the problem for large and bounded domains is that the spectrum of the period map has eigenvalues which accumulate near 1, implying that the results found would only be valid for values of $\varepsilon \sim \mathrm{O}(1/R^2)$, where R is the experiment's radius .

Instead, the approach we take in this work is to consider the problem in the whole space $\mathbb{R}^n$, $n = 1, 2, 3$. Part of the difficulty with this point of view is that the linearization is not invertible, since in this case zero is embedded in the essential spectrum. This would normally preclude us from using perturbation analysis. However, it is possible to overcome this difficulty by setting up the amplitude equations in Kondratiev spaces as the linearization becomes a Fredholm operator in these weighted spaces. We describe these spaces in more detail in the following chapter and in Chapters 3, 4, and 5 we apply these notions to three examples of pattern forming systems: convection rolls, chemical oscillations in 3-dimensional media, and oscillators with nonlocal coupling.

Chapter 2

Weighted spaces

This chapter is intended as a summary of theorems and results that describe weighted spaces and their properties. These results are the basis for the analysis in the following chapters and will allow us to conclude Fredholm properties of the linearized operators considered therein.

2.1 Weighted Sobolev spaces

Throughout the Chapter we will use the symbol $W^{s,p}_{\gamma}$ to denote weighted Sobolev spaces, which we define as the completion of $C_0^{\infty}(\mathbb{R}^n)$ under the norm

$$\|u\|_{W^{s,p}_{\gamma}} = \sum_{|\alpha|\leq s} \|\langle \mathbf{x}\rangle^{\gamma} D^{\alpha} u\|_{L^p}.$$

Here, $\langle \mathbf{x}\rangle = (1+|\mathbf{x}|^2)^{1/2}$, $\mathbf{x} = (x,y)$, $\gamma \in \mathbb{R}$, and s is a positive integer. Note that $W^{s,p}_{\beta} \subset W^{s,p}_{\alpha}$ whenever $\alpha < \beta$. Also note that for $p = 2$, the Fourier transform maps $\mathcal{F} : W^{s,2}_{\gamma} \to W^{\gamma,2}_{s}$, where one can consider this mapping as a definition of the fractional Sobolev spaces $W^{\gamma,2}$. We also write $W^{-s,q}_{-\gamma}$, $1/p + 1/q = 1$, for the dual of $W^{s,p}_{\gamma}$, so that the statement for $p = 2$ on Fourier images can be extended to negative values of γ.
We start with a generalization of the invertibility of the operator $\Delta - I$ to weighted spaces.

Proposition 1 *The operator $\Delta - a : W^{2,p}_{\gamma} \to L^p_{\gamma}$ is invertible for all real numbers $a > 0$ and $p \in [1,\infty)$.*

Before jumping into the proof, let us recall the following theorem from Kato [19], which we will use.

Theorem 1 (Kato, p. 370) *Let $T(\gamma)$ be a family of compact operators in a Banach space X which are holomorphic for all $\gamma \in D \subset \mathbb{C}$. Call γ a* singular point *if 1 is an eigenvalue of $T(\gamma)$. Then either all $\gamma \in D$ are singular points or there are only finitely many singular points in each compact subset of D.*

Proof of Proposition 1. It is straightforward to see that the following diagram commutes,

$$\begin{array}{ccc} W^{2,p}_\gamma & \xrightarrow{\Delta - a} & L^p_\gamma \\ \Big\downarrow{\scriptstyle \langle \mathbf{x} \rangle^\gamma} & & \Big\downarrow{\scriptstyle \langle \mathbf{x} \rangle^\gamma} \\ W^{2,p} & \xrightarrow{\mathcal{L}(\gamma)} & L^p \end{array}$$

where

$$\begin{aligned} \mathcal{L}(\gamma)u &= \langle \mathbf{x} \rangle^\gamma (\Delta - a) \langle \mathbf{x} \rangle^{-\gamma} u \\ &= (\Delta - a)u - 2\gamma \langle \mathbf{x} \rangle^{-2} x \cdot \nabla u + \gamma(\gamma - 2) \langle \mathbf{x} \rangle^{-4} |x|^2 u - n\gamma \langle \mathbf{x} \rangle^{-2} u \\ &= (\Delta - a)u - R(\gamma)u. \end{aligned}$$

Since $R(\gamma) : W^{2,p} \to L^p$ can be approximated in L^p by a sequence of compactly supported continuous functions, it is a compact operator and hence $\mathcal{L}(\gamma)$ has Fredholm index zero. Furthermore, the operator $T(\gamma) : W^{2,p} \to W^{2,p}$ defined by $T(\gamma) = (\Delta - a)^{-1} R(\gamma)$ is compact and holomorphic for all $\gamma \in \mathbb{C}$. We will use this fact and Kato's Theorem to show that the operator $\mathcal{L}(\gamma)$ is invertible. The result of the proposition then follows since the diagram commutes.

First, observe that if $u \in \ker \mathcal{L}(\beta)$ then u must also be in $\ker \mathcal{L}(\alpha)$ for all $\alpha \leq \beta$. This is a consequence of the commutativity of the diagram and the inclusions $W^{2,p}_\beta \subset W^{2,p}_\alpha$, which hold for all $\alpha \leq \beta$.

Now, to reach a contradiction suppose that $\beta \in \mathbb{R}$ is a singular point of $T(\gamma)$ so that there exists $u \neq 0 \in W^{2,p}$ such that $(\Delta - a)^{-1} \mathcal{L}(\beta) u = 0$. Since $(\Delta - a) : W^{2,p} \to L^p$ is one to one, then u has to be in the kernel of $\mathcal{L}(\beta)$. From the above discussion we can infer that if β is a singular point then all points $x \in (-\infty, \beta]$ are also singular. In particular $[\alpha^*, \beta]$, for any $\alpha^* < \beta$, is a compact subset of $\mathbb{C}$ with this property. Kato's Theorem then implies that all points in the complex plane are singular. However, since $\mathcal{L}(0) = (\Delta - a) : W^{2,p} \to L^{2,p}$ is one to one, $\gamma = 0$

cannot be singular. Consequently $T(\gamma)$ has no singular points in $\mathbb{R}$ and $(\Delta - a) : W^{2,p}_\gamma \to L^p_\gamma$ is an isomorphism for all $\gamma \in \mathbb{R}$. ■

Here, we explicitly write a specific instance of Proposition 1 in the one dimensional case, as we will use it in Chapter 5.

Proposition 2 *The operators* $1 - \partial_{xx} : W^{s+2,p}_\gamma \to W^{s,p}_\gamma$ *and* $1 - \partial_x : W^{s+1,p}_\gamma \to W^{s,p}_\gamma$ *are isomorphism for all* $p \in (1, \infty)$*, all* $\gamma \in \mathbb{R}$ *and* $s \in \mathbb{N}$*.*

We also remark that $\partial^s_x : W^{s,p} \to L^p$ does not have closed range, as an explicit Weyl sequence construction shows, and that an argument as in the preceding proof implies the same statement for $\partial^s_x : W^{s,p}_\gamma \to L^p_\gamma$. A similar reasoning shows that this results is also true for the Laplace operator on dimensions $n \geq 2$ and proves the following lemma.

Lemma 3 *The operator* $\Delta_\gamma : W^{2,p}_\gamma \to L^p_\gamma$ *is not a Fredholm operator for* $p \in (1, \infty)$*.*

2.2 Kondratiev spaces

Kondratiev spaces were first introduced to study boundary value problems for elliptic equations in domains with conical points [22]. Nirenberg and Walker [33] later used these spaces to show that elliptic operators with coefficients that decay sufficiently fast at infinity have finite dimensional kernel when considered as operators between these weighted spaces. Lockhart and McOwen [25, 26, 27] built on these ideas to establish Fredholm properties for classes of elliptic operators. For example, Lockart [27] studied elliptic operators of the form $A = A_\infty + A_0$ in non-compact manifolds, where A_∞ represents a constant coefficient homogeneous elliptic operator of order m, and A_0 an operator of order at most m with coefficients that decay fast at infinity. More recently, Kondratiev spaces were used to study the Laplace operator in exterior domains [2] and similar weighted spaces were used in [32] to understand the Poisson equation in a 1 periodic infinite strip $Z = [0, 1] \times \mathbb{R}$.

Kondratiev spaces have also been used in the context of the Stokes (rather than the Laplace) operator, for the description of far field asymptotics in fluid problems, such as flows past obstacles; see [40] for the specific example of exterior domains in $\mathbb{R}^3$ and [30] for an application towards bifurcation theory.

We will denote Kondratiev spaces by $M^{s,p}_\gamma$ and define them as the completion of $C^\infty_0(\mathbb{R}^n)$ under the norm

$$\|u\|_{M^{s,p}_\gamma} = \sum_{|\alpha|\leq s} \|\langle \mathbf{x}\rangle^{\gamma+|\alpha|} D^\alpha u\|_{L^p},$$

where $\langle \mathbf{x}\rangle = (1 + |\mathbf{x}|^2)^{1/2}$, $\gamma \in \mathbb{R}$, $s \in \mathbb{N}$, and $p \in (1, \infty)$. Notice the embeddings $M^{s,p}_\gamma \hookrightarrow W^{s,p}_\gamma$, as well as $M^{s,p}_\gamma \hookrightarrow M^{s-1,p}_\gamma$.

The following theorem describes the behavior of the Laplacian in Kondratiev spaces. Its proof can be found in [28].

Theorem 2 *Let* $1 < p = \frac{q}{q-1} < \infty$, $n \geq 2$, *and* $\gamma \neq n/q + m$ *or* $\gamma \neq 2 - n/p - m$, *for some* $m \in \mathbb{N}$. *Then*

$$\Delta : M^{2,p}_{\gamma-2} \to L^p_\gamma,$$

is a Fredholm operator and

1. *for* $2 - n/p < \gamma < +n/q$ *the map is an isomorphism;*
2. *for* $+n/q + m < \gamma < n/q + m + 1$, $m \in \mathbb{N}$, *the map is injective with closed range equal to*

$$R_m = \left\{ f \in L^p_\gamma : \int f(y)H(y) = 0 \text{ for all } H \in \bigcup_{j=0}^{m} \mathcal{H}_j \right\};$$

3. *for* $2 - n/p - m - 1 < \gamma < 2 - n/p - m$, $m \in \mathbb{N}$, *the map is surjective with kernel equal to*

$$N_m = \bigcup_{j=0}^{m} \mathcal{H}_j.$$

Here, $\mathcal{H}_j$ *denote the harmonic homogeneous polynomials of degree* j.
On the other hand, if $\gamma = 2 - n/p - m$ *or* $\gamma = +n/q + m$ *for some* $m \in \mathbb{N}$, *then* Δ *does not have closed range.*

In the case of dimension 1, the above theorem can be extended to a more general class of operators. In particular, we have the following proposition which we will use in Chapter 5.

Proposition 4 *Let* m *and* l *be non negative integers, and* $p \in (1, \infty)$. *Then, the operator*

$$(1 - \partial_x)^{-\ell} \partial_x^m : M^{m,p}_{\gamma-m} \to W^{\ell,p}_\gamma,$$

is Fredholm for $\gamma + 1/p \notin \{1, \ldots, m\}$. *In particular,*

- *for $\gamma < 1 - 1/p$ it is surjective with kernel spanned by $\mathbb{P}_m$;*
- *for $\gamma > m - 1/p$ it is injective with cokernel spanned by $\mathbb{P}_m$;*
- *for $j - 1/p < \gamma < j + 1 - 1/p$, where $j \in \mathbb{N}$, $1 \leq j < m$, its kernel is spanned by $\mathbb{P}_{m-j}$ and its cokernel is spanned by $\mathbb{P}_j$.*

For $\gamma + 1/p \in \{1, \ldots, m\}$, the operator does not have close range. Here, $\mathbb{P}_m$ is the m-dimensional space of all polynomials with degree less than m.

The proof is split into several lemmas. We first consider the case $k = 1$, $m = 0$ and $\gamma > 1 - 1/p$, and establish Fredholm properties for

$$\partial_x : M^{1,p}_{\gamma-1} \to L^p_\gamma.$$

We then treat the case $\gamma < 1 - 1/p$ in a similar fashion and conclude by showing that ∂_x is not Fredholm for $\gamma = 1 - 1/p$. The results for general m and k follow easily from elemantary calculus and additivity of the index for composition of Fredholm operators. In what follows we will denote the subspace spanned by the constants as $\mathbb{P}_0$.

Lemma 5 *Let $p \in (1, \infty)$ and $\gamma > 1 - 1/p$. Then, the operator*

$$\partial_x : M^{1,p}_{\gamma-1} \to L^p_\gamma,$$

is Fredholm with index -1 and cokernel spanned by $\mathbb{P}_0$.

Proof. Let $\gamma > 1 - 1/p$ and let $C^\perp$ denote the orthogonal complement of $\mathbb{P}_0$, that is

$$C^\perp = \{f \in L^p_\gamma : \int f = 0\},$$

a closed subspace of L^p_γ given our restrictions on γ.
Notice first that $\mathrm{Rg}\,(\partial_x) = C^\perp$, since any solution to the equation $\partial_x u = f$ solves

$$u(x) = \int_{-\infty}^x \partial_x f(y) \mathrm{d}y,$$

and $u(x) \to 0$ for $x \to \infty$. Also, the kernel of ∂_x is trivial since constants do not belong to $M^{1,p}_{\gamma-1}$ with our restriction on γ. It remains to show that the inverse, given through the formula

$$\begin{array}{rccc} \partial_x^{-1}: & C^\perp & \to & M^{1,p}_{\gamma-1}, \\ & f & \mapsto & \int_\infty^x f(y)\mathrm{d}y = \int_{-\infty}^x f(y), \end{array}$$

is bounded; note that both integration formulas differ by $\int_{\mathbb{R}} f = 0$ since $f \in C^{\perp}$. Establishing the required bounds follows a strategy similar to the one used in Chapter 5, Lemma 38. We restrict to $x > 0$ without loss of generality and show

$$\|u\|_{L^p_{\gamma-1}(0,\infty)} < \frac{1}{\gamma - 1 + 1/p}\|f\|_{L^p_\gamma(0,\infty)}.$$

This is accomplished after substituting

$$x = \mathrm{e}^\tau, \tau \in \mathbb{R}, \qquad w(\tau) = \mathrm{e}^{\bar{\gamma}\tau}u(\mathrm{e}^\tau), \qquad g(\tau) = \mathrm{e}^{(\bar{\gamma}+1)\tau}f(\mathrm{e}^\tau), \tag{2.1}$$

with $\bar{\gamma} = \gamma - 1 + 1/p$, and estimating the exponential convolution kernel. We omit the straightforward details which are similar to the proof of Lemma 38 but easier. ■

Lemma 6 *Let $p \in (1, \infty)$ and $\gamma < 1 - 1/p$. Then, the operator*

$$\partial_x : M^{1,p}_{\gamma-1} \to L^p_\gamma,$$

is Fredholm with index 1 and kernel spanned by $\mathbb{P}_0$.

Proof. One readily verifies the claim on the kernel so that it is sufficient to verify that the operator is onto. Therefore, we define

$$\begin{array}{rccc} \partial_x^{-1}: & L^p_\gamma & \to & M^{1,p}_{\gamma-1}, \\ & f & \mapsto & \int_0^x f(y)\mathrm{d}y. \end{array}$$

Clearly, ∂_x^{-1} is a right inverse. Using the coordinate transformations (2.1), one obtains once again a convolution operator with exponentially localized kernel and finds the desired estimates in a straightforward fashion. ■

Finally, we show that for $\gamma = 1 - 1/p$, ∂_x does not have closed range.

Lemma 7 *Let $p \in (1, \infty)$ and $\gamma = 1 - 1/p$. Then*

$$\partial_x : M^{1,p}_{\gamma-1} \to L^p_\gamma$$

does not have closed range.

Proof. One readily finds that kernel and cokernel are trivial, yet the operator cannot be Fredholm since operators for nearby values of γ can be viewed as compact perturbations, for which the Fredholm index jumps. As a consequence, the range cannot be closed. ■

Chapter 3

Striped patterns

3.1 Introduction

Periodic, stripe-like patterns emerge in a self-organized fashion in a variety of experiments, ranging from Rayleigh-Bénard convection to open chemical reactors. Such regular, periodic patterns are usually studied in domains with idealized periodic boundary conditions, where existence and stability can be readily obtained using classical methods of bifurcation theory. But as we saw in Chapter 1, a perturbation argument is only valid for very small values of the perturbation. So instead in the present chapter we study the effects of local impurities in Swift-Hohenberg-like systems on $\mathbb{R}^2$. More precisely, we focus on the somewhat simpler case of the isotropic Ginzburg-Landau equation modeling anisotropic pattern-forming systems, with an added localized inhomogeneity,

$$A_T = \Delta A + A - A|A|^2 + \varepsilon g(x, y). \tag{3.1}$$

The structure of this chapter is as follows. In the following subsection we state the main result of this chapter. Next, in Section 3.2 we summarize the procedure leading up to the main result, first explaining the difficulties encountered when analyzing the linearization of equation (3.2) about stripe patterns, and then describing the linear operator T that we use to overcome these difficulties. In Section 3.3 we use the results from Chapter 2 to explore the Fredholm properties of this last operator and show that by adding logarithmic corrections we can obtain an invertible operator $\hat{T}$. Section 3.4 establishes properties of the nonlinearity in our functional analytic setting. We show that the full operator $\hat{F}$ is well defined, continuously differentiable, with invertible linearization $\hat{T}$. Finally, in Section 3.5, we prove our main result using the Implicit Function Theorem.

3.1.1 Main results

To state our main results, we consider the stationary solutions of (3.1),

$$0 = \Delta A + A - A|A|^2 + \varepsilon g(x, y). \tag{3.2}$$

For $\varepsilon = 0$, the system possesses "stripe patterns"

$$A(x, y) = \sqrt{1-k^2}\mathrm{e}^{\mathrm{i}kx},$$

for wavenumbers $|k| < 1$. Those solutions are linearly stable for $|k| < 1/\sqrt{3}$ and unstable for $|k| > 1/\sqrt{3}$. The instability mechanism is known as the Eckhaus (sideband) instability. We are now ready to state our main result.

Theorem 3 *Fix k with $|k| < 1/\sqrt{3}$ and suppose that $g \in W^{2,2}_\beta$ for some $\beta > 2$. Then there exists an $\varepsilon_0 > 0$ and a family of solutions to* (3.2)*,*

$$A(x, y; \varepsilon, \varphi) = S(x, y; \varepsilon, \varphi)\mathrm{e}^{\mathrm{i}\Phi(x,y;\varepsilon,\varphi)}, \quad |\varepsilon| < \varepsilon_0,$$

with $A(x, y; 0, \varphi) = \sqrt{1-k^2}\mathrm{e}^{\mathrm{i}(kx+\varphi)}$. Moreover, $A(x, y; \varepsilon, \varphi)$ is smooth in all variables and satisfies the following expansions in x, y for fixed ε, φ,

$$S(x, y; \varepsilon, \varphi) \to \sqrt{1-k^2} \tag{3.3}$$

$$\Phi(x, y; \varepsilon, \varphi) - kx - \frac{c(\varepsilon, \varphi)}{2k\sqrt{1-k^2}}\log(\alpha x^2 + y^2) \to \Phi_\infty(\varepsilon) + \varphi, \tag{3.4}$$

for $|\mathbf{x}| \to \infty$, with $\alpha = \frac{1-k^2}{1-3k^2}$, for some smooth function $c(\varepsilon, \varphi)$ with expansion

$$c(\varepsilon, \varphi) = \varepsilon c_1(\varphi) + \mathrm{O}(\varepsilon^2),$$

where

$$c_1(\varphi) = \frac{\sqrt{1-3k^2}}{\pi(1-k^2)}\iint \mathrm{Im}\,[g(x, y)\mathrm{e}^{-\mathrm{i}(kx+\varphi)}]\mathrm{d}x\mathrm{d}y.$$

Remark 8 *1. Our approach gives more detailed expansions than stated. In fact, we obtain a decomposition of S and Φ into a localized part that is smooth in ε, uniformly in $\mathbf{x}$, and an explicit logarithmic far-field correction with coefficient $c(\varepsilon, \varphi)$; see the Ansatz* (3.12). *Also, in the class of functions with this particular form, the solutions described in the theorem are locally unique.*

2. *The expression for $c_1(\varepsilon, \varphi)$ is reminiscent of a projection onto the kernel. Indeed, the integral represents the scalar product $(u, v) = \mathrm{Re} \int u\bar{v}$ of the perturbation εg and the "kernel" of the linearization induced by the infinitesimal phase rotation, $\frac{\mathrm{d}}{\mathrm{d}\varphi}\mathrm{e}^{\mathrm{i}kx+\varphi}$. Vanishing of such a scalar product indicates persistence of solutions in problems where the linearization is Fredholm, such as in the Melnikov analysis for homoclinic bump-like solutions.*

   Note that $c_1(\varphi)$ is periodic in φ and vanishes at $\varphi = \arg\left(\iint g(x, y)\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}x\mathrm{d}y\right)$, so that it possesses at least 2 zeroes (counting multiplicity). Assuming that $c_1(\varphi_*) = 0$, $c_1'(\varphi_*) \neq 0$, we can find $\varphi_*(\varepsilon)$ so that

$$c(\varepsilon, \varphi_*(\varepsilon)) = 0.$$

   Inspection of the expansion in the theorem shows that φ is the phase shift of the underlying pattern in the far field. For these specific values of φ, the correction in the far field to the periodic pattern is bounded and small, while for other values of φ the correction is unbounded in the phase. We interpret this result by referring to $\varphi_*(\varepsilon)$ as the selected phase. *In other words, introducing inhomogeneities induces a selected phase shift of the primary pattern which accommodates stationary solutions without logarithmic corrections. Numerical simulations confirm this phenomenon, with a diffusive spread of the phase shift in the domain. It would be interesting to establish this diffusive convergence to a selected phase analytically.*

3. *We believe that similar results could be obtained in space dimensions 1 and 3. In one space dimension, the analysis is easier since ODE methods can be used to analyze stationary solutions. In fact, the analysis reduces to a Melnikov analysis for the intersection of center-stable and center-unstable manifolds. In this one-dimensional context, the analysis also immediately carries over to the Swift-Hohenberg equation. On the other hand, the 3-dimensional case is easier than the 2-dimensional case considered here since the corrections needed to compensate for negative Fredholm indices are decaying like* $1/|\mathbf{x}|$.

In fact, the Laplacian is invertible for suitable weights γ in Kondratiev spaces in $\mathbb{R}^3$. We refer the reader to [21], where dynamical systems methods were used to analyze the complex-coefficient Ginzburg-Landau equation in space dimensions 1, 2, 3, exhibiting a related dependence of far field corrections on the dimension. Note however that the analysis there uses exponential localization of $g(\mathbf{x})$ and is not easily extended beyond radial symmetry.

3.2 Outline of proof

Recall that we are interested in solutions of

$$0 = \Delta A + A - A|A|^2 + \varepsilon g(x,y), \tag{3.5}$$

for localized g and ε small, close to the solutions at $\varepsilon = 0$, $A_*(x) = \sqrt{1-k^2}\mathrm{e}^{\mathrm{i}kx}$, $|k|^2 < 1/3$. Since the case $k = 0$ is in fact easier, we will assume in the following that $k > 0$. A reasonable Ansatz then is

$$A(x,y,\varepsilon) = (\sqrt{1-k^2} + s(x,y;\varepsilon))\mathrm{e}^{\mathrm{i}(kx+\phi(x,y;\varepsilon))},$$

with new variables s, ϕ, which solve

$$\Delta s + (s+\tau) - (s+\tau)(k^2 + 2k\partial_x\phi + |\nabla\phi|^2) - (s+\tau)^3 + \varepsilon \mathrm{Re}\,(g\mathrm{e}^{-\mathrm{i}(kx+\phi)}) = 0 \tag{3.6}$$

$$\Delta\phi + \frac{2k\partial_x s}{s+\tau} + \frac{2\nabla s\cdot\nabla\phi}{s+\tau} + \frac{\varepsilon \mathrm{Im}\,(g\mathrm{e}^{-\mathrm{i}(kx+\phi)})}{s+\tau} = 0, \tag{3.7}$$

where we set $\tau = \sqrt{1-k^2}$. Linearizing at $\varepsilon = 0, s = 0, \phi = 0$, we obtain the operator

$$L\begin{bmatrix} s \\ \phi \end{bmatrix} = \begin{bmatrix} \Delta - 2\tau^2 & -2k\tau\partial_x \\ \frac{2k}{\tau}\partial_x & \Delta \end{bmatrix}\begin{bmatrix} s \\ \phi \end{bmatrix}. \tag{3.8}$$

The results from Chapter 2, and in particular Theorem 2, suggest that we should require that $\phi \in M^{2,p}_\gamma$ and that (3.7) holds in $L^p_{\gamma+2}$. Then $\phi_x \in W^{1,p}_{\gamma+1}$ and, using the linearization of (3.6) with Proposition 1, this suggests $s \in W^{3,p}_{\gamma+1}$ and $s_x \in W^{2,p}_{\gamma+1}$. This however is not sufficient localization for (3.7), where s_x enters, and which we assumed to be satisfied in $L^p_{\gamma+2}$.
In other words, the coupling terms, which are absent for $k = 0$, prohibit the simple use of Sobolev spaces for s and Kondratiev spaces for ϕ. Roughly speaking, the coupling destroys the linear scaling invariance in the ϕ-equation, which is necessary at least at infinity in results

on Fredholm properties in Kondratiev spaces, which intrinsically mix regularity and localization properties. We intend to address these problems more generally in future but focus here on a simple an direct construction that circumvents the problem by extending the system and introducing appropriate norms for derivatives.

Consider therefore $T : \mathcal{D} \subset \mathcal{X} \to \mathcal{R} \subset \mathcal{Y}$,

$$T\begin{bmatrix} s \\ \psi \\ \theta \\ u \\ v \\ w \end{bmatrix} = \begin{bmatrix} \Delta - a & -1 & 0 & 0 & 0 & 0 \\ 0 & \Delta & 0 & b & 0 & 0 \\ 0 & 0 & \Delta & 0 & b & 0 \\ 0 & -\partial_{xx} & 0 & \Delta - a & 0 & 0 \\ 0 & 0 & -\partial_{xx} & 0 & \Delta - a & 0 \\ 0 & -\partial_{yy} & 0 & 0 & 0 & \Delta - a \end{bmatrix}\begin{bmatrix} s \\ \psi \\ \theta \\ u \\ v \\ w \end{bmatrix}, \tag{3.9}$$

were $\quad \psi = 2k\tau\partial_x\phi, \quad \theta = 2k\tau\partial_y\phi, \quad a = 2\tau^2, \quad b = 4k^2,$

$$\mathcal{X} = W^{2,2}_{\gamma} \times M^{2,p}_{\gamma} \times M^{2,p}_{\gamma} \times L^p_{\gamma+2} \times L^p_{\gamma+2} \times L^p_{\gamma+2},$$

$$\mathcal{Y} = L^p_{\gamma} \times L^p_{\gamma+2} \times L^p_{\gamma+2} \times W^{-2,p}_{\gamma+2} \times W^{-2,p}_{\gamma+2} \times W^{-2,p}_{\gamma+2}.$$

Here, $W^{-k,p}_{\gamma}$ denotes the dual of $W^{k,p}_{\gamma}$. We also define the closed subspaces

$$\mathcal{D} = \Big\{X = (s, \psi, \phi, u, v, w) \in \mathcal{X} : u = \partial_{xx}s, \quad v = \partial_{xy}s, \quad w = \partial_{yy}s, \quad \partial_y\psi = \partial_x\theta\Big\},$$

$$\begin{aligned}\mathcal{R} = \Big\{&Y = (f_1, f_2, f_3, f_4, f_5, f_6) \in \mathcal{Y} : \int f_2 = \int f_2 \cdot y = \int f_3 = \int f_3 \cdot x = 0, \\ &f_4 = \partial_{xx}f_1, f_5 = \partial_{xy}f_1, f_6 = \partial_{yy}f_1, \quad \text{and} \quad \partial_y f_2 = \partial_x f_3\Big\}.\end{aligned}$$

The second and third components of T in (3.9) are obtained by taking the x and y derivatives of the phase equation (3.7), respectively. The last three components are obtained by taking the second partial derivatives of the amplitude equation (3.6) with respect to xx, xy, and yy.

We will see in Section 3.3 that the linear operator $T : \mathcal{D} \to \mathcal{R}$ is a Fredholm operator of index -1 for optimal choices of weights, indicating a missing parameter in the far field. We therefore add a single degree of freedom in the far field through the variable $\hat{c} \in \mathbb{R}$ via the Ansatz

$$\begin{aligned} s &= \hat{s} + \hat{c}P_1, & u &= \hat{u} + \hat{c}\partial_{xx}P_1, \\ \psi &= \hat{\psi} + \hat{c}\partial_x P_2, & v &= \hat{v} + \hat{c}\partial_{xy}P_1, \\ \theta &= \hat{\theta} + \hat{c}\partial_y P_2, & w &= \hat{w} + \hat{c}\partial_{yy}P_1, \end{aligned}$$

where

$$P_1 = \frac{1-\alpha}{2b\alpha}\partial_x[\chi \log(\alpha x^2 + y^2)], \qquad P_2 = \frac{1}{2}\chi \log(\alpha x^2 + y^2), \tag{3.10}$$

$b = (2k)^2$, $\alpha = \frac{1-k^2}{1-3k^2}$, and χ is a smoothed version of the indicator function $\chi_{|\mathbf{x}|>1}$. Substituting this Ansatz into (3.6),(3.7) and linearizing, we find an operator $\hat{T} : \mathcal{D} \times \mathbb{R} \to \mathcal{R}$, given by

$$\hat{T}\xi = \begin{bmatrix} \Delta - a & -1 & & & & & \Delta P_1 \\ & \Delta & & b & & & \Delta P_2 + b\partial_{xx}P_1 \\ & & \Delta & & b & & \Delta P_2 + b\partial_{xy}P_1 \\ & -\partial_{xx} & & \Delta - a & & & \Delta\partial_{xx}P_1 \\ & & -\partial_{xx} & & \Delta - a & & \Delta\partial_{xy}P_1 \\ & -\partial_{yy} & & & & \Delta - a & \Delta\partial_{yy}P_1 \end{bmatrix} \begin{bmatrix} \hat{s} \\ \hat{\psi} \\ \hat{\theta} \\ \hat{u} \\ \hat{v} \\ \hat{w} \\ \hat{c} \end{bmatrix}, \tag{3.11}$$

where again $a = 2\tau^2$, and $b = 4k^2$. We will show in the Section 3.3 that this operator is invertible.

Recapitulating, we are lead to consider the Ansatz

$$A(x, y; \varepsilon, \varphi) = \left(\sqrt{1-k^2} + s(x, y; \varepsilon, \varphi) + c(\varepsilon, \varphi)P_1(x, y)\right) \mathrm{e}^{\mathrm{i}\left(kx + \phi(x,y;\varepsilon,\varphi) + \frac{c(\varepsilon,\varphi)}{2k\sqrt{1-k^2}}P_2(x,y)\right)}, \tag{3.12}$$

with P_1 and P_2 as in (3.10), with additional equations for the derivatives

$$u = \partial_{xx}s, \quad v = \partial_{xy}s, \quad w = \partial_{yy}s, \quad \psi = (2k)\partial_x\phi, \quad \theta = (2k)\partial_y\phi.$$

We obtain a nonlinear equation

$$\hat{F}_{\varepsilon,\varphi} = 0, \qquad \hat{F}_{\varepsilon,\varphi} : \mathcal{D} \times \mathbb{R}^3 \to \mathcal{R}; \tag{3.13}$$

see Subsection 3.4 for a more detailed description of this nonlinear equation. The advantage of this subtle reformulation of the problem is encoded in the following result, which establishes applicability of the standard Implicit Function Theorem and is the key ingredient to the proof of Theorem 3.

Theorem 4 *Let $p = 2$, $\gamma \in (0, 1)$, and $g \in W^{2,p}_{\beta}$, with $\beta > \gamma + 2$. Then, the operator $\hat{F}_{\varepsilon,\varphi}$: $\mathcal{D} \times \mathbb{R}^3 \to \mathcal{R}$ is of class C^∞. Furthermore, for fixed φ and at $\varepsilon = 0$, its derivative is given by the invertible operator $\hat{T} : \mathcal{D} \times \mathbb{R} \to \mathcal{R}$.*

The proof of this theorem will occupy the next two sections. In Section 3.3 we show the fact that T is a Fredholm operator and that $\hat{T}$ is invertible, and in Section 3.4 we show that the operator $\hat{F}$ is of class C^∞.

3.3 The linear operator

In this section we consider the linear operators $T : \mathcal{D} \to \mathcal{R}$ and $\hat{T} : \mathcal{D} \times \mathbb{R} \to \mathcal{R}$ defined in (3.9) and (3.11), respectively. We first prove that for $p = 2$ and $\gamma \in (0, 1)$ the operator $T : \mathcal{X} \to \mathcal{Y}$ is a Fredholm operator of index -6. Then, we show that restricting the domain and range to $\mathcal{D}$ and $\mathcal{R}$ turns T into a Fredholm operator of index -1. Finally, using a bordering lemma, we prove that the operator $\hat{T}$ is invertible. Throughout, we assume $\gamma \in (0, 1)$ and $0 < |k| < \dfrac{1}{\sqrt{3}}$.

Proposition 9 *The operator $T : \mathcal{X} \to \mathcal{Y}$ defined in (3.9) is a Fredholm operator with index $i = -6$ and trivial kernel. The cokernel is spanned by*

$$\{(0, a, 0, b, 0, 0)^T e_j^*, (0, 0, a, 0, b, 0)^T e_j^*,\ j = 1, 2, 3\}, \qquad e_1^* = 1, e_2^* = x, e_3^* = y. \tag{3.14}$$

Proof. First, notice that due to the lower block-triangular structure of T, it is enough to consider the restriction $\tilde{T}$ to the variables ψ, θ, u, v, which we write in the form

$$\tilde{T}\xi = L\xi + bB\xi, \tag{3.15}$$

where $b = 4k^2$, $a = 2(1 - k^2)$,

$$L = \begin{bmatrix} \Delta & 0 & 0 & 0 \\ 0 & \Delta & 0 & 0 \\ -\partial_{xx} & 0 & \Delta - a & 0 \\ 0 & -\partial_{xx} & 0 & \Delta - a \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \text{and} \quad \xi = \begin{bmatrix} \psi \\ \theta \\ u \\ v \end{bmatrix}.$$

We need to show that

$$\tilde{T} : \tilde{\mathcal{X}} = M_\gamma^{2,2} \times M_\gamma^{2,2} \times L_{\gamma+2}^2 \times L_{\gamma+2}^2 \longrightarrow \tilde{\mathcal{Y}} = L_{\gamma+2}^2 \times L_{\gamma+2}^2 \times W_{\gamma+2}^{-2,2} \times W_{\gamma+2}^{-2,2},$$

is a Fredholm operator of index $i = -6$. Since $\tilde{T}$ is block-diagonal with respect to (ψ, u) and (θ, v), it is sufficient to show that the restriction to (ψ, u) is Fredholm with index -3. For $b = 0$, the claim now follows directly from Theorem 2 and Proposition 1, due to the lower triangular structure and the fact that, with our choice of γ, p, the Laplacian is Fredholm with index -3. We will address the more difficult situation $b \neq 0$, next.

In order to establish the desired Fredholm properties, we need to solve

$$\Delta\psi + bu = f_1 \tag{3.16}$$

$$\partial_{xx}\psi + (\Delta - a)u = f_2, \tag{3.17}$$

for f_1, f_2 in a codimension-3 subspace of $L^2_{\gamma+2} \times W^{-2,2}_{\gamma+2}$, with bounds on $(\psi, u) \in M^{2,2}_\gamma \times L^2_{\gamma+2}$. Denote by $I - Q$ a projection on the range of the Laplacian, so that $\int (I-Q)f = \int x(I-Q)f = \int y(I-Q)f = 0$. We can then decompose

$$\Delta\psi + bu = (I-Q)f_1 \tag{3.18}$$

$$\partial_{xx}\psi + (\Delta - a)u = (I-Q)f_2, \tag{3.19}$$

and $Qu = \frac{1}{b}Qf_1 = -\frac{1}{a}Qf_2$, exhibiting the 3 solvability conditions (3.14). We next solve (3.19) for u, substitute in (3.18), to obtain

$$\mathcal{L}\psi = (I-Q)f_1 - (\Delta - a)^{-1}(I-Q)f_2 =: f, \qquad \mathcal{L} = [\Delta + b(\Delta - a)^{-1}\partial_{xx}],$$

where $f = (I-Q)f$. It therefore remains to show that $\mathcal{L} : M^{2,2}_\gamma \to (I-Q)L^2_{\gamma+2}$ is invertible. We therefore factor

$$\mathcal{L}\psi = \mathcal{M}\left(\Delta - \frac{b}{a}\partial_{xx}\right)\psi,$$

$$M^{2,2}_\gamma \xrightarrow{\Delta - \frac{b}{a}\partial_{xx}} (I-Q)L^2_{\gamma+2} \xrightarrow{\mathcal{M}} (I-Q)L^2_{\gamma+2}$$

By Theorem 2 the operator $\left(\Delta - \dfrac{b}{a}\partial_{xx}\right) : M^{2,2}_\gamma \to R_m$ is invertible, since it is conjugate to the Laplacian by a simple x-rescaling operator. It is therefore sufficient to establish that $\mathcal{M}$ is an isomorphism of $(I-Q)L^2_{\gamma+2}$. Consider therefore the associated Fourier symbol

$$\hat{\mathcal{M}}(k,l) = \frac{k^2 + l^2}{k^2 + l^2 - \frac{b}{a}k^2} - \frac{bk^2}{(k^2 + l^2 + a)(k^2 + l^2 - \frac{b}{a}k^2)}.$$

Exploiting that $k^2 < \frac{1}{3}$ so that $1 - \dfrac{b}{a} > 0$, it is straightforward to see that

$$\sup_{(k,l)\in\mathbb{R}^2} |\hat{\mathcal{M}}(k,l)| + |\hat{\mathcal{M}}(k,l)^{-1}| < \infty,$$

so that $\mathcal{M}$ is an isomorphism of L^2. We next show that $\mathcal{M}$ is an isomorphism on $(I-Q)L^2_j$, $j = 2,3$, which by interpolation theory gives the desired result. Equivalently, we need to show boundedness of the multiplication operator $\hat{\mathcal{M}}$ on the subspace of H^j, $j = 2,3$ consisting of functions functions f with $f(0) = 0$ and, in case $j = 3$, $\nabla f(0) = 0$. Since $\hat{\mathcal{M}}(k,l) = a + \mathrm{O}(k^2 + l^2)$ near $k = l = 0$, we readily find that $\|D^\alpha \hat{\mathcal{M}}|_{L^\infty} + \|D^\alpha(\hat{\mathcal{M}}^{-1})|_{L^\infty} < \infty$ for all indices

$|\alpha| \le 2$, which proves that $\hat{\mathcal{M}}$ is an isomorphism on H^2. For the case $j = 3$, we use that $\|D^\alpha(\mathbf{k}\hat{\mathcal{M}})|_{L^\infty} + \|D^\alpha(\mathbf{k}\hat{\mathcal{M}}^{-1})|_{L^\infty} < \infty$ for all indices $|\alpha| = 3$, which readily implies that $\hat{\mathcal{M}}$ is an isomorphism on $H^3 \cap \{f(0) = 0\}$. One can also readily check that the range of $\mathcal{M}$ and $\mathcal{M}^{-1}$ is indeed contained in $\mathrm{Rg}\,(I - Q)$, which concludes the proof. ■

Corollary 10 *The operator $T : \mathcal{D} \to \mathcal{R}$ defined by (3.9) is Fredholm with index -1 and cokernel $\partial_y f_2 + \partial_x f_3$.*

Proof. Inspection of T shows that the range of the restriction of T to $\mathcal{D}$ is actually contained in $\mathcal{R}$, which implies that $T : \mathcal{D} \to \mathcal{R}$ is injective and the range is closed (T is semi-Fredholm). We need to show that the cokernel is one-dimensional.
Take $f \in \mathrm{Rg}\,(T) \subset \mathcal{X}$, with $f_4 = \partial_{xx} f_1,\ f_5 = \partial_{xy} f_1,\ f_6 = \partial_{yy} f_1$, and $\partial_y f_2 = \partial_x f_3$. By construction of T, notably having taken derivatives of equations for s and ϕ, and by injectivity, $T^{-1} f$ satisfies $u = \partial_{xx} s$, $v = \partial_{xy} s$, $w = \partial_{yy} s$, and $\partial_y \psi = \partial_x \theta$. As a consequence, the cokernel of $T : \mathcal{D} \to \mathcal{R}$ is a subset of the cokernel of $T : \mathcal{X} \to \mathcal{Y}$. Inspecting the cokernel in (3.14) and the definition of $\mathcal{R}$, we see that $(e_j^*, f_4) = (e_j^*, f_5) = 0$, so that the integral conditions in the definition of $\mathcal{R}$ represent precisely 4 of the 6 conditions on the co-kernel. One of the remaining conditions, $\int f_2 \cdot x = \int f_3 \cdot y$ is a consequence of $\partial_y f_2 = \partial_x f_3$, whereas $\int f_2 \cdot x$ can be readily seen to act non-trivially. As a consequence T is Fredholm of index -1 as claimed, and the cokernel is spanned by $(0, x, 0, 0, 0, 0)^T$ or, equivalently, $(0, x, y, 0, 0, 0)^T$. ■

We next consider the operator $\hat{T} : \mathcal{D} \times \mathbb{R} \to \mathcal{R}$ defined by (3.11). Recall that

$$\begin{array}{rclcrcl} s &=& \hat{s} + \hat{c} P_1, && u &=& \hat{u} + \hat{c}\partial_{xx} P_1, \\ \psi &=& \hat{\psi} + \hat{c} P_2, && v &=& \hat{v} + \hat{c}\partial_{xy} P_1, \\ \theta &=& \hat{\theta} + \hat{c} P_3, && w &=& \hat{w} + \hat{c}\partial_{yy} P_1. \end{array} \tag{3.20}$$

Here,

$$\begin{aligned} P_1(x, y) &= \frac{(1-\alpha)}{2b\alpha}\partial_x[\chi \ln(\alpha x^2 + y^2)], \\ P_2(x, y) &= \frac{1}{2}\partial_x[\chi \ln(\alpha x^2 + y^2)], \\ P_3(x, y) &= \frac{1}{2}\partial_y[\chi \ln(\alpha x^2 + y^2)], \end{aligned}$$

with $\alpha = \dfrac{1-k^2}{1-3k^2}$, $\quad b = (2k)^2$, $\quad$ and $\quad \chi \in C^\infty(\mathbb{R}^2)$ defined by

$$\chi(x,y) = \begin{cases} 0 & \text{if} \quad 0 \le \sqrt{\alpha x^2 + y^2} \le 1/2 \\ 1 & \text{if} \quad 1 \le \sqrt{\alpha x^2 + y^2} \end{cases} .$$

To show $\hat{T} : \mathcal{D} \times \mathbb{R} \to \mathcal{R}$ is invertible we will need the following lemma.

Lemma 11 *The operator*

$$M : \mathbb{R} \to \mathcal{R}, \quad c \mapsto \begin{bmatrix} \Delta P_1 & \Delta P_2 + b\partial_{xx}P_1 & \Delta P_3 + b\partial_{xy}P_1 & \Delta\partial_{xx}P_1 & \Delta\partial_{xy}P_1 & \Delta\partial_{yy}P_1 \end{bmatrix}^T c,$$

is well-defined and its range satisfies

$$\iint \left[\Delta P_2 + b\partial_{xx}P_1\right] \cdot x\mathrm{d}x\mathrm{d}y = \iint \left[\Delta P_3 + b\partial_{xy}P_1\right] \cdot y\mathrm{d}x\mathrm{d}y \neq 0.$$

Proof. First notice that the smooth functions P_i, for $i = 1, 2, 3$, are bounded in compact sets and behave like $\dfrac{1}{|\mathbf{x}|}$ for large values of $|\mathbf{x}|$ so that the range of M is indeed a subset of $\mathcal{Y}$. From the definition, it is not difficult to check that the operator M maps into the desired space $\mathcal{R}$. We need to show that

$$\iint \left[\Delta P_2 + b\partial_{xx}P_1\right] \cdot x\mathrm{d}x\mathrm{d}y = \iint \left[\Delta P_3 + b\partial_{xy}P_1\right] \cdot y\mathrm{d}x\mathrm{d}y \neq 0.$$

Straightforward calculations, using the rescaling $X = \sqrt{\alpha}x$, $Y = y$, show that

$$\Delta P_2 + b\partial_{xx}P_1 = \frac{\sqrt{\alpha}}{2}\Delta_{X,Y}\left[\frac{\partial}{\partial X}\left(\chi \cdot \ln(X^2+Y^2)\right)\right],$$

where we write $\Delta_{X,Y} = \partial_{XX} + \partial_{YY}$. Therefore,

$$\begin{aligned} \iint [\Delta P_2 + b\partial_{xx}P_1] \cdot x\mathrm{d}x\mathrm{d}y &= \iint \left[\frac{\sqrt{\alpha}}{2}\Delta_{X,Y}\left[\frac{\partial}{\partial X}\left(\chi \cdot \ln(X^2+Y^2)\right)\right]\right] \cdot X dX dY \\ &= \iint \left[\frac{\sqrt{\alpha}}{2}\Delta_{X,Y}\left(\chi \cdot \ln(X^2+Y^2)\right)\right] dX dY \\ &= \sqrt{\alpha}\pi. \end{aligned}$$

Similarly, using the same rescaling, it can be shown that

$$\Delta P_3 + b\partial_{xy}P_1 = \frac{1}{2}\Delta_{X,Y}\left[\frac{\partial}{\partial Y}\left(\chi \cdot \ln(X^2+Y^2)\right)\right],$$

and consequently

$$\iint \left[\Delta P_3 + b\partial_{xy} P_1\right] \cdot y\mathrm{d}x\mathrm{d}y = \iint \left[\frac{1}{2}\Delta_{X,Y}\left[\frac{\partial}{\partial Y}\left(\chi \cdot \ln(X^2+Y^2)\right)\right]\right] \cdot Y\sqrt{\alpha}dXdY$$
$$= \sqrt{\alpha}\pi.$$

■

Corollary 12 *The operator $\hat{T} : \mathcal{D} \times \mathbb{R} \to \mathcal{R}$ is invertible.*

Proof. Notice that $\hat{T} = [TM]$, where $T : \mathcal{D} \to \mathcal{R}$ is the Fredholm operator of index -1 described in Corollary 10, and $M : \mathbb{R} \to \mathcal{R}$ is defined in Lemma 11. A bordering lemma implies that $\hat{T} : \mathcal{D}\times\mathbb{R} \to \mathcal{R}$ is a Fredholm operator of index 0. Lemma 11 implies that $\mathrm{Rg}\,(M) \not\subset \mathrm{Rg}\,(T)$, so that $\hat{T}$ is onto, hence invertible. ■

3.4 The nonlinear map

In this section, we show by a series of lemmas that the nonlinear problem (3.13) is well defined and continuously differentiable. We first give explicit expressions for each component of the nonlinearities. We then state and prove several lemmas that will help us show that the nonlinearity is well defined. Finally, we show that the nonlinearities are of class C^∞.

The following expressions represent each component of the operator $\hat{F}_{\varepsilon,\varphi}$ announced in (3.13):

$$\begin{aligned}
\hat{F}_1(\xi;\varepsilon,\varphi) =& \Delta s - 2\tau^2 s - (s+\tau)\left(\frac{\psi}{\tau} + \frac{1}{(2k\tau)^2}(\psi^2+\theta^2)\right) - (s^3+3s^2\tau) + \varepsilon\mathrm{Re}\,[ge^{-\mathrm{i}(kx+\phi(\varphi))}],\\
\hat{F}_2(\xi;\varepsilon,\varphi) =& \Delta\psi + \frac{(2k)^2\tau u + 2(u\psi + v\theta + s_x\psi_x + s_y\psi_y)}{s+\tau} - \frac{(2k)^2\tau(s_x)^2 + 2s_x(s_x\psi + s_y\theta)}{(s+\tau)^2}\\
&+ \frac{\varepsilon\mathrm{Im}\,[\partial_x(ge^{-\mathrm{i}(kx+\phi(\varphi))})]}{s+\tau} - \frac{\varepsilon s_x\mathrm{Im}\,[ge^{-\mathrm{i}(kx+\phi(\varphi))}]}{(s+\tau)^2},\\
\hat{F}_3(\xi;\varepsilon,\varphi) =& \Delta\theta + \frac{(2k)^2\tau v + 2(v\psi + w\theta + s_x\theta_x + s_y\theta_y)}{s+\tau} - \frac{(sk)^2\tau s_x s_y + 2s_y(s_x\psi + s_y\theta)}{(s+\tau)^2}\\
&+ \frac{\varepsilon\mathrm{Im}\,[\partial_y(ge^{-\mathrm{i}(kx+\phi(\varphi))})]}{s+\tau} - \frac{\varepsilon s_y\mathrm{Im}\,[ge^{-\mathrm{i}(kx+\phi(\varphi))}]}{(s+\tau)^2},
\end{aligned}$$

$$
\begin{aligned}
\hat{F}_4(\xi;\varepsilon,\varphi) =& \Delta u - 2\tau^2 u - u\left(\frac{\psi}{\tau} + \frac{1}{(2k\tau)^2}(\psi^2+\theta^2)\right) - 2s_x\left(\frac{\psi_x}{\tau} + \frac{2}{(2k\tau)^2}(\psi_x\psi + \psi_y\theta)\right)\\
&- (s+\tau)\left(\frac{\psi_{xx}}{\tau} + \frac{2}{(2k\tau)^2}\left(\psi_{xx}\psi + \psi_{xy}\theta + |\nabla\psi|^2\right)\right)\\
&- (6s(s_x)^2 + 3s^2 u + 6\tau(s_x)^2 + 6\tau s u) + \varepsilon \mathrm{Re}\,[\partial_{xx}(g e^{-\mathrm{i}(kx+\phi(\varphi))})],\\
\hat{F}_5(\xi;\varepsilon,\varphi) =& \Delta v - 2\tau^2 v - v\left(\frac{\psi}{\tau} + \frac{1}{(2k\tau)^2}(\psi^2+\theta^2)\right) - s_x\left(\frac{\theta_x}{\tau} + \frac{2}{(2k\tau)^2}(\theta_x\psi + \theta_y\theta)\right)\\
&- s_y\left(\frac{\psi_x}{\tau} + \frac{2}{(2k\tau)^2}(\psi_x\psi + \psi_y\theta)\right) - (6ss_xs_y + 3s^2v + 6\tau s_xs_y + 6\tau s v)\\
&- (s+\tau)\left(\frac{\theta_{xx}}{\tau} + \frac{2}{(2k\tau)^2}\left(\theta_{xx}\psi + \psi_{yy}\theta + \theta_x\psi_x + \theta_y\psi_y\right)\right) + \varepsilon \mathrm{Re}\,[\partial_{xy}(g e^{-\mathrm{i}(kx+\phi(\varphi))})],\\
\hat{F}_6(\xi;\varepsilon,\varphi) =& \Delta w - 2\tau^2 w - w\left(\frac{\psi}{\tau} + \frac{1}{(2k\tau)^2}(\psi^2+\theta^2)\right) - 2s_y\left(\frac{\theta_x}{\tau} + \frac{2}{(2k\tau)^2}(\theta_x\psi + \theta_y\theta)\right)\\
&- (s+\tau)\left(\frac{\psi_{yy}}{\tau} + \frac{2}{(2k\tau)^2}\left(\theta_{xy}\psi + \theta_{xy}\theta + |\nabla\theta|^2\right)\right)\\
&- (6s(s_y)^2 + 3s^2 w + 6\tau(s_y)^2 + 6\tau s w) + \varepsilon \mathrm{Re}\,[\partial_{yy}(g e^{-\mathrm{i}(kx+\phi(\varphi))})].
\end{aligned}
$$

Here, $\varphi \in \mathbb{R}$, $\tau = \sqrt{1-k^2}$, $k^2 < \dfrac{1}{3}$, and the variable $\xi = (s,\psi,\theta,u,v,w)$ is given by the formulas in (3.20), so that we can actually consider $\hat{F}$ as an operator on $\mathcal{D}\times\mathbb{R}$ for fixed ε,φ.
Since $(2k\tau)\nabla\phi = \langle\psi,\theta\rangle$ we define ϕ by

$$
\phi(x,y;\varepsilon,\varphi) = \phi_{\mathrm{bd}} + \phi_{\log}, \tag{3.21}
$$

where

$$
\begin{aligned}
\phi_{\mathrm{bd}}(x,y;\varepsilon,\varphi) =& \varphi + \frac{1}{2k\tau}\int_{t=0}^{1}\left(\hat{\psi}(tx,ty;\varepsilon)x + \hat{\theta}(tx,ty;\varepsilon)y\right)\\
\phi_{\log}(x,y;\varepsilon,\varphi) =& \frac{1}{2k\tau}\int_{t=0}^{1}(P_2(tx,ty)x + P_3(tx,ty)y) = \frac{1}{2k\tau}\chi\log(\alpha x^2 + y^2).
\end{aligned}
$$

The following lemma shows that ϕ_{bd} is a well-defined function.

Lemma 13 *If $\hat{\psi},\hat{\theta} \in M^{2,2}_{\gamma}$ then for fixed ε and φ, the function $\phi_{\mathrm{bd}}(x,y;\varepsilon,\varphi)$ is well defined, bounded, continuous, and approaches a constant $\varphi + \Phi_\infty(\varepsilon)$ as $|\mathbf{x}| \to \infty$.*

Proof. Note that ϕ is continuous since $\hat{\psi},\hat{\theta} \in M^{2,p}_{\gamma} \subset BC^0$. The decay estimate from Lemma 25 at the end of this chapter guarantees that for large $|\mathbf{x}|$ we have $|\hat{\psi}|, |\hat{\theta}| \le C|\mathbf{x}|^{-\gamma-1}$. Therefore, the integrals converge as $|\mathbf{x}| \to \infty$. ■

The next four lemmas will help us show that the operator $\hat{F}_{\varepsilon,\varphi} : \mathcal{D} \times \mathbb{R}^3 \to \mathcal{R}$ is well defined.

Lemma 14 *There exists $C > 0$ so that for all $u \in L^p_\gamma$ with $Du \in L^p_{\gamma+1}$, $\langle \mathbf{x} \rangle^\gamma u \in W^{1,p}$ and,*

$$\|\langle \mathbf{x} \rangle^\gamma u\|_{W^{1,p}} \leq C\|u\|_{L^p_\gamma} + \|Du\|_{L^p_{\gamma+1}}.$$

Proof. We need to show that $D(\langle \mathbf{x} \rangle^\gamma u) \in L^p$. We compute

$$D(\langle \mathbf{x} \rangle^\gamma u) = Du \cdot \langle \mathbf{x} \rangle^\gamma + D\langle \mathbf{x} \rangle^\gamma \cdot u = Du \cdot \langle \mathbf{x} \rangle^\gamma + \gamma u \mathbf{x}(1 + |\mathbf{x}|^2)^{\frac{\gamma-2}{2}}.$$

Since $Du \in L^p_{\gamma+1} \subset L^p_\gamma$, we conclude that $Du \cdot \langle \mathbf{x} \rangle^\gamma \in L^p$. Furthermore, since $|\mathbf{x}|^p \leq (1 + |\mathbf{x}|^2)^{p/2}$,

$$|u \cdot D\langle \mathbf{x} \rangle^\gamma| \leq |u||\mathbf{x}|\langle \mathbf{x} \rangle^{(\gamma-2)} \leq |u|\langle \mathbf{x} \rangle^{(\gamma-1)} \leq |u|\langle \mathbf{x} \rangle^\gamma \in L^p.$$

This implies that $D(\langle \mathbf{x} \rangle^\gamma u) \in L^p$ and we obtain $\langle \mathbf{x} \rangle^\gamma u \in W^{1,p}$. ■

Lemma 15 *For $\gamma > 0$, we have the continuous embeddings $M^{2,2}_\gamma \hookrightarrow W^{2,2}_\gamma \hookrightarrow W^{2,2} \hookrightarrow BC^0$.*

Proof. The first embedding is due to Lemma 14, the second a consequence of $\gamma > 0$, and the last a classical Sobolev embedding in dimension 2. ■

Lemma 16 *For $\gamma > 0$ there exists $C > 0$ such that for all f, g with $\langle \mathbf{x} \rangle^{\gamma+1} f, \langle \mathbf{x} \rangle^{\gamma+1} g \in W^{1,p}$,*

$$\|fg\|_{L^p_{\gamma+2}} \leq C\|\langle \mathbf{x} \rangle^{\gamma+1} f\|_{W^{1,p}}\|\langle \mathbf{x} \rangle^{\gamma+1} g\|_{W^{1,p}}.$$

Proof. By Cauchy-Schwartz,

$$\|fg\langle \mathbf{x} \rangle^{(\gamma+2)}\|_{L^p} \leq \|f\langle \mathbf{x} \rangle^{(1+(\gamma/2))}\|_{L^{2p}}\|g\langle \mathbf{x} \rangle^{(1+(\gamma/2))}\|_{L^{2p}}$$

which proves the lemma using $\gamma > 0$ and the Sobolev embedding $W^{1,p} \hookrightarrow L^{2p}$, in $n = 2$. ■

Fredholm properties of the Laplacian imply in particular the following more basic estimate [33].

Lemma 17 *[33, Theorem 3.1] If $u \in L^p_\gamma$ and $\Delta u \in L^p_{\gamma+2}$ then $u \in M^{2,p}_\gamma$ and there exists a constant C such that*

$$\|u\|_{M^{2,p}_\gamma} \leq C\left(\|u\|_{L^p_\gamma} + \|\Delta u\|_{L^p_{\gamma+2}}\right).$$

Remark 18 *Notice that* $u, w \in L^2_{\gamma+2}$ *so that* $\Delta s = u + w \in L^2_{\gamma+2}$, *by the above lemma we have that, within the closed subset* $\mathcal{D} \subset \mathcal{X}$, $s \in M^{2,2}_{\gamma}$ *with uniform bounds in terms of* s, u, w.

To prove Theorem 4, the following three lemmas establish that each component of the operator $\hat{F}_{\varepsilon,\varphi} : \mathcal{D} \times \mathbb{R}^3 \to \mathcal{R}$ is well defined. Throughout, we use the standing assumptions $0 < \gamma < 1$ and $g \in W^{2,2}_{\beta}$, with $\beta > \gamma + 2$.

Lemma 19 *The component* $\hat{F}_1 : \mathcal{D} \times \mathbb{R}^3 \to L^2_{\gamma}$ *is well defined.*

Proof. We can rewrite

$$\hat{F}_1(\xi; \varepsilon, \varphi) = \Delta s - 2\tau^2 s - \psi - \frac{s\psi}{\tau} - (s + \tau)(\frac{1}{(2k\tau)^2}(\psi^2 + \theta^2)) - (s^3 + 3s^2\tau) + \varepsilon \mathrm{Re}\, [g\mathrm{e}^{-\mathrm{i}(kx+\phi(\varphi))}].$$

A short calculation shows that $\Delta s - 2\tau^2 s - \psi = (\Delta - 2\tau^2)\hat{s} - \hat{\psi} - c\Delta P_1$ is the first component of $\hat{T}$, thus well defined. Consider next the term $s\psi = (\hat{s} + cP_1)(\hat{\psi} + cP_2)$.
Notice that $\hat{s}\psi$ is in L^2_{γ} since both P_2 and $\hat{\psi}$, are bounded by Lemma 15 and $\hat{s} \in L^2_{\gamma}$. Since $\hat{\psi} \in L^2_{\gamma}$ and since P_1 is bounded, $\hat{\psi}P_1 \in L^2_{\gamma}$ as well. Notice also that the product $P_1 \cdot P_2$ is bounded in compact sets and behaves like $\dfrac{1}{|\mathbf{x}|^2}$ for large values of $|\mathbf{x}|$, hence it belongs to L^2_{γ} provided $\gamma < 1$. This shows the term $s\psi$ is in the desired space.
Using similar arguments, it is easy to check that the functions $\psi^2, \theta^2, s^3, s^2$ and $s(\psi^2 + \theta^2)$ are in L^2_{γ}. Finally, by Lemma 13 we know ϕ is a bounded continuous function so that we can conclude $\mathrm{e}^{-\mathrm{i}(kx+\phi(\varphi))}$ is a well defined function in L^∞. This implies that the term $\mathrm{Re}\, [g\mathrm{e}^{-\mathrm{i}(kx+\phi(\varphi))}]$ is in L^2_{γ} since $g \in L^2_{\beta} \subset L^2_{\gamma}$. ■

Lemma 20 *The component* $\hat{F}_2 : \mathcal{D} \times \mathbb{R}^3 \to L^2_{\gamma+2}$ *is well defined.*

Proof. Since we are trying to find solutions near $\xi = 0$ we can assume s/τ is close to zero. We can therefore write

$$\begin{aligned}\Delta\psi + (2k)^2\frac{\tau u}{s + \tau} &= \Delta\psi + (2k)^2\left(u + \frac{su}{\tau + su}\right)\\ &= \Delta\hat{\psi} + (2k)^2(\hat{u} + c\Delta P_2 + (2k)^2\partial_{xx}P_2) + (2k)^2\frac{(\hat{u} + c\partial_{xx}P_1))(\hat{s} + cP_1)}{\tau + (\hat{u} + c\partial_{xx}P_1))(\hat{s} + cP_1)}.\end{aligned}$$

Notice that the terms $\Delta\hat{\psi} + (2k)^2\hat{u} + c[\Delta P_2 + (2k)^2\partial_{xx}P_2]$, represent the second component of the linear operator $\hat{T} : \mathcal{D} \times \mathbb{R} \to \mathcal{R}$, hence are well defined. It is now straightforward to see that the remaining nonlinear terms are contained in $L^2_{\gamma+2}$. In terms of localization, the most dangerous term is $P_1\partial_{xx}P_1$, which can be bounded as

$$\int |P_1\partial_{xx}P_1|^2 \langle \mathbf{x}\rangle^{2(\gamma+2)} \leq \int \left|\frac{1}{r^4}\right|^2 r^{2(\gamma+2)} r\mathrm{d}r < \infty,$$

since $\gamma < 1$.

We next treat the remaining nonlinearities. Since $s \in L^\infty$, we only need to show that the numerators in $\hat{F}_2$ are in $L^2_{\gamma+2}$. It is not hard to mimic the above arguments to show that the terms $u\psi$ and $v\theta$ are in $L^2_{\gamma+2}$, so we will treat the term $s_x\psi_x$ first. Using the formulas in (3.20) we see that

$$s_x\psi_x = (\hat{s}_x + c\partial_x P_1)(\hat{\psi}_x + c\partial_x P_2).$$

By Lemma 17 and Remark 18 we know that $\hat{s} \in M^{2,2}_{\gamma}$. Therefore, $\hat{s}_x, \hat{\psi}_x \in W^{1,2}_{\gamma+1}$ and we can apply Lemma 16 to conclude that $\hat{s}_x\hat{\psi}_x \in L^2_{\gamma+2}$. The remaining terms are easily seen to be in the correct space.

Similar arguments show that the functions $(s_x)^2, s_y\psi_y, s_xs_y$ are in the correct space and that, since ψ and θ are bounded by Lemma 15, $s_x(s_x\psi + s_y\theta) \in L^2_{\gamma+2}$.

Finally, because we are assuming that g is in the space $W^{2,2}_{\beta}$, with $\beta > \gamma + 2$, $s_x \in L^2_{\gamma+1}$, and because the terms ψ and $\mathrm{e}^{-\mathrm{i}(kx+\phi(\varphi))}$ are bounded,

$$-\frac{\varepsilon s_x \mathrm{Im}\,[g\mathrm{e}^{-\mathrm{i}(kx+\phi(\varphi))}]}{(s+\tau)^2} + \frac{\varepsilon \mathrm{Im}\,[\partial_x(g\mathrm{e}^{-\mathrm{i}(kx+\phi(\varphi))})]}{s+\tau} \in L^2_{\gamma+2}.$$

Here, we used the fact that $\psi = (2k\tau)\phi$ so that $\partial_x(g\mathrm{e}^{-\mathrm{i}(kx+\phi(\varphi))}) = [g_x - ig(k+\psi/(2k\tau))]\mathrm{e}^{-\mathrm{i}(kx+\phi(\varphi))}$.

■

Lemma 21 *The component $\hat{F}_3 : \mathcal{D}\times\mathbb{R}^3 \to L^2_{\gamma+2}$ is well defined.*

Proof. The proof is almost identical to the proof of Lemma 20 and is omitted here. ■

Finally, we show

Lemma 22 *The component $\hat{F}_4 : \mathcal{D}\times\mathbb{R}^3 \to W^{-2,2}_{\gamma+2}$ is well defined. Moreover, the nonlinear part of $\hat{F}_4$ actually belongs to $L^2_{\gamma+2}$.*

Proof. We can rewrite $\hat{F}_4$ as

$$\Delta u - 2\tau^2 u - \psi_{xx} - \frac{s\psi_{xx}}{\tau} - u\left(\frac{\psi}{\tau} + \frac{1}{(2k\tau)^2}(\psi^2+\theta^2)\right) - 2s_x\left(\frac{\psi_x}{\tau} + \frac{2}{(2k\tau)^2}(\psi_x\psi + \psi_y\theta)\right)$$

$$-(s+\tau)\left(\frac{2}{(2k\tau)^2}\left(\psi_{xx}\psi+\psi_{xy}\theta+|\nabla\psi|^2\right)\right)-(6s(s_x)^2+3s^2u+6\tau(s_x)^2+6\tau su)+\varepsilon\mathrm{Re}\,[\partial_{xx}(g\mathrm{e}^{-\mathrm{i}(kx+\phi(\varphi))})].$$

Notice that

$$\Delta u-2\tau^2u-\psi_{xx}=\Delta\hat{u}-2\tau^2\hat{u}-\hat{\psi}_{xx}+c\Delta(\partial_{xx}P_1),$$

is just the fourth component of the linear operator $\hat{T}$, thus well defined. Furthermore, notice that $\hat{\psi}_{xx}\in L^2_{\gamma+2}$. Since $\Delta\partial_{xx}P_1$ behaves like $\dfrac{1}{|\mathbf{x}|^5}$ for large $|\mathbf{x}|$ and is bounded in compact sets, these two term now also belong to $L^2_{\gamma+2}$. The arguments used to show that the remaining nonlinearites are in the space $L^2_{\gamma+2}$ are the same as the once used in the above lemmas, we will omit the details here. ■

Having shown the result for the operator $\hat{F}_4$ it is not hard to see that the operators $\hat{F}_5,\hat{F}_6$: $\mathcal{D}\times\mathbb{R}^3\to W^{-2,2}_{\gamma+2}$ are well defined.

Remark 23 *Since all the nonlinear terms are in $L^2_{\gamma+2}$, including $\hat{\psi}_{xx}$ and $\Delta\partial_{xx}P_1$, then $\Delta\hat{u}-(2\tau^2)\hat{u}\in L^2_{\gamma+2}$. This implies that for the solution, $\hat{u}\in W^{2,2}_{\gamma+2}$. The same observation holds for $\hat{v}$ and $\hat{w}$.*

In the next lemma we show that there exist a neighborhood $\mathcal{U}$ of $\xi\in\mathcal{D}\times\mathbb{R}$ such that the operator $\hat{F}_{\varepsilon,\varphi}:\mathcal{U}\times\mathbb{R}^3\to\mathcal{R}$ is smooth.

Lemma 24 *Let $0<\gamma<1$ and $g\in W^{2,2}_{\beta}$, with $\beta>\gamma+2$. Then the operator $\hat{F}_{\varepsilon,\varphi}:\mathcal{D}\times\mathbb{R}^3\to\mathcal{R}$ is of class C^∞ in a neighborhood the origin.*

Proof. Most nonlinear terms are defined via superposition (or Nemytskii) operators, via smooth algebraic functions, that are automatically smooth once well defined. We therefore concentrate on the term $g\mathrm{e}^{-\mathrm{i}\phi}$ and its derivatives. Recall that

$$\phi(x,y;\varepsilon,\varphi)=\phi_{\mathrm{bd}}+\phi_{\log},$$

where

$$\begin{aligned}\phi_{\mathrm{bd}}(x,y;\varepsilon,\varphi)=&\varphi+\frac{1}{2k\tau}\int_{t=0}^{1}\left(\hat{\psi}(tx,ty;\varepsilon)x+\hat{\theta}(tx,ty;\varepsilon)y\right)\\ \phi_{\log}(x,y;\varepsilon,\varphi)=&\frac{1}{2k\tau}\int_{t=0}^{1}(P_2(tx,ty)x+P_3(tx,ty)y)=\frac{1}{2k\tau}c\chi\log(\alpha x^2+y^2).\end{aligned}$$

In order to show smoothness, we factor

$$g\mathrm{e}^{-\mathrm{i}\Phi} = \left(g\langle \mathbf{x}^{\gamma+2-\beta}\rangle\right)\cdot \mathrm{e}^{-\mathrm{i}\Phi_{\mathrm{bd}}}\cdot\left(\langle \mathbf{x}^{\beta-\gamma-2}\rangle \mathrm{e}^{-\mathrm{i}\Phi_{\log}}\right) =: G_1\cdot G_2\cdot G_3.$$

Clearly, $G_1 \in L^2_{\gamma+2}$. By Lemma 13, $\int\psi, \int\theta\in L^\infty$, so that $G_2\in L^\infty$ is bounded as a superposition operator. It remains to show that G_3 is differentiable with values in L^∞. This can be readily established, showing that the derivative with respect to c is

$$\partial_c G_3 = \langle \mathbf{x}^{\beta-\gamma-2}\rangle \mathrm{e}^{-\mathrm{i}\Phi_{\log}}\chi\log(\alpha x^2+y^2),$$

hence bounded in L^∞. Higher derivatives are bounded for the same reasons, which establishes the claim. ∎

3.5 Expansions and proof of main result

In this last subsection we use Theorem 4 to prove Theorem 3 and derive the expansions for the stationary solutions to the perturbed Ginzburg-Landau equation near roll patterns.

Proof of Theorem 3. Recall the Ansatz

$$A(x,y;\varepsilon,\varphi) = (\sqrt{1-k^2}+s(x,y;\varepsilon,\varphi)+c(\varepsilon,\varphi)P_1(x,y))\mathrm{e}^{\mathrm{i}kx+\mathrm{i}\phi(x,y;\varepsilon,\varphi)+\mathrm{i}\frac{c(\varepsilon,\varphi)}{2k\sqrt{1-k^2}}P_2(x,y)}$$

where Φ was defined in (3.21) and P_1, P_2 in (3.10). From Theorem 4 we know there exists a neighborhood, $\mathcal{U}$, of $\mathcal{D}\times\mathbb{R}$ where the operator $\hat{F}_{\varepsilon,\varphi}$ is continuously differentiable with invertible derivative at the origin, $\varepsilon = 0$. The Implicit Function Theorem therefore guarantees the existence of solutions $\xi(\varepsilon,\varphi)$ near $\xi(0,\varphi) = 0$. In particular, we know that $s\in W^{2,2}_\gamma$, and $\psi,\theta\in M^{2,2}_\gamma$.

We define

$$S(x,y;\varepsilon,\varphi) = \left(\sqrt{1-k^2}+s(x,y;\varepsilon,\varphi)+c(\varepsilon,\varphi)P_1(x,y)\right)$$

and

$$\Phi(x,y;\varepsilon,\varphi) = kx+\phi.$$

Since $s(x,y;\varepsilon,\varphi)\in W^{2,2}_\gamma$, Lemma 15 ensures that if $s(x,y;\varepsilon,\varphi)\sim \mathrm{O}(\langle\mathbf{x}\rangle^{-\gamma})$. Also, by definition, $P_1(x,y)\sim\mathrm{O}(\langle\mathbf{x}\rangle^{-1})$, and

$$\lim_{\mathrm{x}\to\infty} S(x,y;\varepsilon,\varphi) = S_\infty = \sqrt{1-k^2}.$$

By Lemma 13, $\phi_{\mathrm{bd}} \to \varphi + \Phi_\infty(\epsilon)$ for $\mathbf{x} \to \infty$ so that

$$\Phi(x, y; \varepsilon, \varphi) - kx - \frac{c(\varepsilon, \varphi)}{2k\sqrt{1-k^2}} \log(\alpha x^2 + y^2) \to \Phi_\infty(\varepsilon) + \varphi,$$

as $|\mathbf{x}| \to \infty$.

To find an expression for $c(\varepsilon, \varphi)$ we expand $\xi = \varepsilon\hat{\xi} + \mathrm{o}(\varepsilon)$. Gathering terms of order ε results in the system $\hat{T}\hat{\xi} = \hat{f}$. Inspecting the second component of this system, we find

$$\Delta\hat{\psi} + b\partial_{xx}\hat{u} + \hat{c}\,[\Delta P_2 + b\partial_{xx}P_1] = \frac{1}{\sqrt{1-k^2}} \mathrm{Im}\,[(g_x - ikg)\mathrm{e}^{-\mathrm{i}(kx+\varphi)}].$$

Taking the scalar product with x and solving for $\hat{c}$, we obtain after integration by parts in x,

$$\hat{c} = \frac{\sqrt{1-3k^2}}{\pi(1-k^2)} \iint \mathrm{Im}\,[g\mathrm{e}^{-\mathrm{i}(kx+\varphi)}].$$

Hence $c(\varepsilon, \varphi) = \varepsilon c_1(\varphi) + \mathrm{o}(\varepsilon)$ with $c_1(\varphi) = \hat{c}$.

■

3.6 Decay estimates

We prove that functions in $M^{2,2}_\gamma$ are spatially localized in a pointwise sense. This result is used in Lemma 13 to ensure that phases are well defined.

Lemma 25 *If $f \in M^{2,2}_\gamma$ then $|f(x)| \le C\|f\|_{M^{2,2}_\gamma}\langle\mathbf{x}\rangle^{-\gamma-1}$ as $|\mathbf{x}| \to \infty$.*

Proof. Since $M^{2,2}_\gamma$ is the completion of C^∞_0 under the norm $\|\cdot\|_{M^{2,2}_\gamma}$, it suffices to show that the result holds for $f \in C^\infty_0$. In polar coordinates, we have, up to constants

$$\begin{aligned}\int |f(\theta, R)|^2 \mathrm{d}\theta &\le \int \left(\int_\infty^R |f_r(\theta, s)|\mathrm{d}s\right)^2 \mathrm{d}\theta = \int \left(\int_\infty^R s^{-\gamma-3/2} s^{\gamma+1}|f_r(\theta, s)|s^{1/2}\mathrm{d}s\right)^2 \mathrm{d}\theta \\ &\le \int \left(\int_\infty^R s^{-2(\gamma+3/2)}\mathrm{d}s\right)\left(\int_\infty^R s^{2(\gamma+1)}|f_r(\theta, s)|^2 s\mathrm{d}s\right)\mathrm{d}\theta \\ &\lesssim R^{-2(\gamma+3/2)+1} \int \int_\infty^R s^{2(\gamma+1)}|f_r(\theta, s)|^2 s\mathrm{d}s\mathrm{d}\theta,\end{aligned}$$

which gives

$$\|f(\cdot, R)\|_{L^2} \lesssim R^{-\gamma-1}\|\nabla f\|_{L^2_{\gamma+1}}. \tag{3.22}$$

Similarly,

$$\begin{aligned}\int |f_\theta(\theta,R)|^2 \mathrm{d}\theta &\leq \int \left(\int_\infty^R |f_{r\theta}(\theta,s)|\mathrm{d}s\right)^2 \mathrm{d}\theta = \int \left(\int_\infty^R s^{-\gamma-5/2} s^{\gamma+2} |f_{r\theta}(\theta,s)| s^{1/2} \mathrm{d}s\right)^2 \mathrm{d}\theta \\ &\leq \int \left(\int_\infty^R s^{-2(\gamma+5/2)} \mathrm{d}s\right)\left(\int_\infty^R s^{2(\gamma+2)} |f_{r\theta}(\theta,s)|^2 s \mathrm{d}s\right) \mathrm{d}\theta \\ &\lesssim R^{-2(\gamma+5/2)+1} \int \int_\infty^R s^{2(\gamma+2)} |f_{r\theta}(\theta,s)|^2 s \mathrm{d}s \mathrm{d}\theta.\end{aligned}$$

This gives

$$\|f_\theta(\cdot,R)\|_{L^2} \leq R^{-\gamma-2} \|f_{r\theta}\|_{L^2_{\gamma+2}}. \tag{3.23}$$

Combining (3.22) and (3.23) and using the interpolation inequality [1, Thm 5.9]

$$\|f(\cdot,R)\|_\infty^2 \leq \|f(\cdot,R)\|_{L^2} \|f(\cdot,R)\|_{H^1},$$

now proves the claim. ■

Chapter 4

Chemical oscillations in 3 dimensions

4.1 Introduction

This chapter is concerned with the effects of inhomogeneities in oscillatory media. As a prototype we study the complex Ginzburg-Landau equation,

$$A_t = (1 + i\alpha)\Delta A + A - (1 + i\gamma)A|A|^2, \tag{4.1}$$

which is known to approximate the phase and amplitude of modulation patterns in reaction diffusion systems near a supercritical Hopf bifurcation [3]. As we saw in Chapter 1, stationary in time inhomogeneities which produce a localized change in the phase of oscillations can be well modeled by the inclusion of a term $i\varepsilon g(x)A$ in (4.1). The effects of such inhomogeneities can vary dramatically depending on the sign of ε and the space dimension. This has been explored formally in the phase-diffusion approximation in [42], and for general reaction diffusion equations and radially symmetric inhomogeneities in [21]. Most notably, inhomogeneities can create wave sources in space dimension 1 and 2. In dimension 3 and radial geometry it was shown in [21] that sources are weak, that is, the wavenumber decays in the far field. In this chapter, we establish a similar result *without* the assumption of radial symmetry and without relying on spatial dynamics. In addition, we relax the assumption of spatial decay of $g(x)$.

To accomplish this task we hope to use the Implicit Function Theorem combined with the methods described in Chapter 1 and the results from Chapter 2. In addition to Kondratiev spaces, our method relies on weighted Sobolev spaces. We will see that for certain weights of the form $(1 + |x|^2)^{\delta/2}$ the linearization about steady solutions possesses a cokernel. We will therefore

consider an Ansatz which adds far field corrections and obtain as a result an invertible operator. This approach works well for weights with $\delta < 1/2$. However for $\delta > 1/2$ these correction terms prove to be problematic since they result in nonlinearities which are not well defined, i.e. they do no belong to the correct weighted space. The same is true in the two dimensional case for all weights that account for decaying inhomogeneities. We hope to address these issues in the future and restrict ourselves in the present work to the 3 dimensional case with $\delta < 1/2$.
We begin the analysis by considering the spatially homogeneous solution $A_*(t) = \mathrm{e}^{-\mathrm{i}\gamma t}$ of equation (4.1) and looking for approximations of the form $A(x,t) = (1 - s(x))\mathrm{e}^{-\mathrm{i}(\gamma t - \phi(x))}$. In Section 4.2 we will show, using Lyapunov-Schimdt reduction, that in dimension 3 it is possible to find solutions near A_*. The asymptotics for the function $\phi(x)$ will show that in the far field the wavenumber $k \sim \nabla\phi$ decays to zero and hence target patterns will not form. We state this result in the following Theorem:

Theorem 5 *Suppose $\delta \in (-1/2, 1/2)$, $g \in L^2_{\delta+2}$, and $1 + \alpha\gamma > 0$. Then, there exist $\varepsilon_0 > 0$ and smooth functions $S(x,\varepsilon)$ and $\Phi(x,t;\varepsilon)$ such that*

$$A(x,t;\varepsilon) = S(x,\varepsilon)\mathrm{e}^{\Phi(x,t;\varepsilon)}$$

is a family of solutions to (4.1) near $A = \mathrm{e}^{-\mathrm{i}\gamma t}$ for all $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$. Furthermore, for fixed $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ and t, the functions $S(x;\varepsilon)$ and $\Phi(x,t;\varepsilon)$ satisfy the following asymptotics in x,

$$\begin{aligned} |S(x,\varepsilon) - 1| \leq& C|x|^{-(\delta+2.5)}, \\ \Phi(x,t;\varepsilon) =& -\mathrm{i}\gamma t + \mathrm{i}\frac{c(\varepsilon)}{|x|}\left(1 + \mathrm{o}_1(1/|x|)\right), \end{aligned}$$

as $|x| \to \infty$, where $c(\varepsilon)$ is a smooth function satisfying the expansion $c(\varepsilon) = \varepsilon c_1 + O(\varepsilon^2)$. In particular,

$$c_1 = \frac{1}{4\pi(1+\alpha\gamma)}\int g \,\mathrm{d}x.$$

Remark 26 *1. Notice that we do not have asymptotic predictions for the amplitude, just an upper bound on the rate of its decay.*

2. The values of δ are related to the choice of spaces we make. In the case of $\delta \in (-1/2, 1/2)$ our analysis shows that the linearization about the steady solution $A = \mathrm{e}^{-\mathrm{i}\gamma}$ is a Fredholm operator of index $i = -1$. If we consider weights with $\delta \in (1/2 + m, 1/2 + 2m)$, for $m \in \mathbb{N}$, the linearization is again a Fredholm operator, but now with a larger co-kernel consisting *of harmonic polynomials of degree* $m-1$*. In this case, it seems reasonable to add to the Ansatz a series of correction terms which would span the cokernel of our linearization. In particular, these terms should consist of derivatives of the fundamental solution* $\frac{1}{|x|}$ *of all degrees up to* $m-1$*. The difficulty in this case is that this type of Ansatz results in a non-linear operator which is not well defined in* $L^2_{\delta+2}$ *(see Proposition 35). Nonetheless, because* $L^2_\alpha \subset L^2_\beta$ *for* $\beta < \alpha$*, if we consider a very localized inhomogeneity we can always assume it is in a space* $L^2_{\delta+2}$ *with* $-1/2 < \delta < 1/2$*. In other words, Theorem 5 holds for* $g \in L^2_\sigma$ *with* $\sigma > 3/2$*, and in this case we take* $\delta = 1/2$ *for the bounds of* $|S(x, \varepsilon) - 1|$*. However, for these values of* σ *it is still an open problem to determine if this bound is sharp.*

3. *In the case of* $\delta \in (-3/2, -1/2)$*, we can consider spaces which yield an invertible linearization. Our analysis then shows that the amplitude* $S(x, \varepsilon)$ *should obey the same decay as stated in Theorem 5, but we do not expect phase decay at order* $\mathrm{O}(1/|x|)$*. In fact, the coefficient of the leading order term,* $\int \mathrm{g} \mathrm{d}x$*, is not necessarily defined when g is in* $L^2_{\delta+2}, \delta < -1/2$*. Our result would only give decay associated with the function space* $M^{2,2}_\delta$ *(see Lemma 3.2).*

4. *Finally, we just point out that we are not interested in studying inhomogeneities with slow decay,* $g \sim |x|^{-\alpha} \alpha < 1$*, or that grow algebraically, and so we do not look at the case when* $\delta < -3/2$*.*

The predictions of Theorem 5 agree with the results found in [21], where the authors show that in the more general case of reaction diffusion equations and in dimensions 3 and higher, there exists only contact defects (the wave number $k \sim \nabla\phi \to 0$ in the far field) and obtain asymptotics for the wavenumber k,

$$k(r, \varepsilon) = \frac{M\varepsilon}{r^{n-1}}(\hat{c} + \mathrm{O}_{1/r}(1)),$$

where the notation $\mathrm{O}_y(1)$ means that these terms go to zero as $y \to 0$. This implies that for large values of $|x|$ and fixed ε we do not see a pace maker effect. Nonetheless, if we fix $|x|$ large we can approximate the group velocity, c_g, for the family of solutions $A(x, t; \varepsilon)$ in terms of ε:

$$c_g(\varepsilon) = 2(\alpha - \gamma)k \sim -2(\alpha - \gamma)\frac{\varepsilon c_1}{|x|^2}.$$

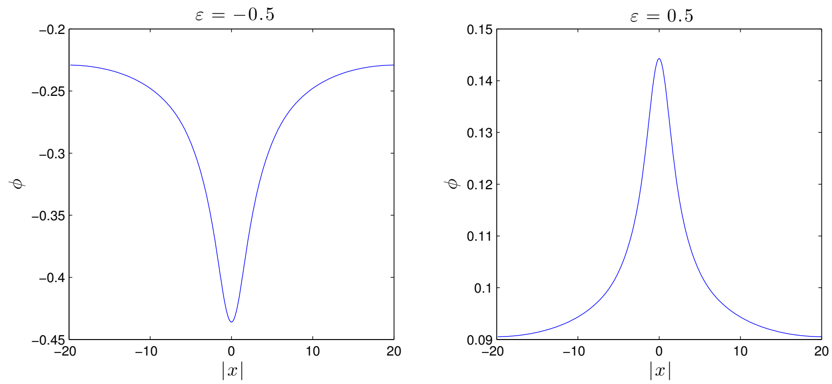

Figure 4.1: Plot of the phase vs. x-axis for the cross section $y = z = 0$. For the parameter values used in the simulation the expression $(\alpha - \gamma) < 0$. As a result, a negative phase gradient as $|x| \to \infty$ indicates a positive group velocity, whereas a positive phase gradient as $|x| \to \infty$ indicates a negative group velocity.

In particular, if $\varepsilon(\gamma - \alpha) \int g > 0$ then $c_g > 0$ and we obtain weak wave sources. These results were confirmed in numerical simulations with a cubic domain of length $l = 40$, parameter values $\alpha = 1, \gamma = 5$, and with the following inhomogeneity

$$g(x, y, z) = \frac{1}{(1 + 1/4(x - 10)^2 + 2(y - 10)^2 + (z - 10)^2)^{3.2/2}},$$

(see figure 4.1). All simulations were done with an exponential time differencing algorithm (ETDRK4) following the methods found in [18, 17].

This chapter is organized as follows. In Section 4.2 we give a proof of our main result and then in Section 4.3, we present numerical simulations of our results, in particular we show the decay rates for the amplitude and phase agree with our predictions.

4.2 Proof of Theorem 5

To facilitate the analysis we will split this section into four parts. In Subection 4.2.1 we describe how we set up the problem and how we obtain a linearization which is easier to work with. Next, in Subsection 4.2.2 we state conditions that allow us to use the Implicit Function Theorem and derive expansions for the amplitude and phase, effectively proving the results of Theorem 5. Finally, in the last two subsections we show that the linearization is invertible and the nonlinear operator associated to our problem is well defined.

4.2.1 Set up

We recall here our main equation, the complex Ginzburg-Landau equation in dimension 3,

$$A_t = (1 + i\alpha)\Delta A + A - (1 + i\gamma)A|A|^2 + i\varepsilon g(x)A, \tag{4.2}$$

where $g(x)$ is a localized real valued function and ε is small. In what follows we describe how we arrive at our linearization.
We pass to a corotating frame $A = e^{-i\Omega t}\tilde{A}$, so that $\tilde{A}$ satisfies the following equation,

$$\tilde{A}_t = (1 + i\alpha)\Delta\tilde{A} + (1 + i\Omega)\tilde{A} - (1 + i\gamma)\tilde{A}|\tilde{A}|^2 + i\varepsilon g(x)\tilde{A}. \tag{4.3}$$

At parameter values $\Omega = \gamma$ and $\varepsilon = 0$, the function $\tilde{A}_* = 1$ is a solution to (4.3) and the linearization about this constant solution is given by the following operator, T:

$$T\begin{bmatrix} s \\ \phi \end{bmatrix} = \begin{bmatrix} \Delta - 2 & -\alpha\Delta \\ \alpha\Delta - 2\gamma & \Delta \end{bmatrix}\begin{bmatrix} s \\ \phi \end{bmatrix}.$$

In Fourier space T can be represented by a matrix, $\mathcal{F}(T)(k)$, which at $k = 0$ has eigenvalues $\lambda_1 = -2$, and $\lambda_2 = 0$. This suggest that in order to simplify future computations we use the following change of coordinates,

$$\hat{s} = \gamma s, \quad \hat{\phi} = -\gamma s + \phi,$$

so as to diagonalize $\mathcal{F}(T)(0)$. The resulting operator that comes from the right hand side of the equations for $\hat{s}_t$ and $\hat{\phi}_t$, and which we label as $F : \mathcal{X} \times \mathbb{R} \to \mathcal{Y}$, is given by the following two components,

$$\begin{aligned} F_1(\hat{s}, \hat{\phi}) =&(1 - \alpha\gamma)\Delta\hat{s} - 2\hat{s} - \gamma\alpha\Delta\hat{\phi} - (\gamma + \hat{s})[|\nabla\hat{s}|^2 + 2\nabla\hat{s}\cdot\nabla\hat{\phi} + |\nabla\hat{\phi}|^2] - 2\alpha|\nabla\hat{s}|^2 \\ &- 2\alpha\nabla\hat{s}\cdot\nabla\hat{\phi} - \alpha\hat{s}(\Delta\hat{s} + \Delta\hat{\phi}) - \frac{3}{\gamma}\hat{s}^2 - \frac{1}{\gamma^2}\hat{s}^3, \end{aligned} \tag{4.4}$$

$$\begin{aligned} F_2(\hat{s}, \hat{\phi}) =&\left(\frac{\alpha}{\gamma} + \alpha\gamma\right)\Delta\hat{s} + (1 + \alpha\gamma)\Delta\hat{\phi} + (\gamma - \alpha + \hat{s})\left[|\nabla\hat{s}|^2 + 2\nabla\hat{s}\cdot\nabla\hat{\phi} + |\nabla\hat{\phi}|^2\right] \\ &+ \alpha\hat{s}(\Delta\hat{s} + \Delta\hat{\phi}) + 2\alpha|\nabla\hat{s}|^2 + 2\alpha\nabla\hat{s}\cdot\nabla\hat{\phi} + \frac{3\hat{s}^2}{\gamma} + \frac{\hat{s}^3}{\gamma^2} \\ &+ (\gamma + \hat{s})^{-1}\left[2|\nabla\hat{s}|^2 + 2\nabla\hat{s}\cdot\nabla\hat{\phi} - \hat{s}^2 - \frac{\hat{s}^3}{\gamma} - \frac{\alpha}{\gamma}\hat{s}\Delta\hat{s}\right] + \varepsilon g(x). \end{aligned} \tag{4.5}$$

We now introduce the following Ansatz for equation (4.2)

$$\tilde{A}(x,t,\varepsilon) = S(x,\varepsilon)\mathrm{e}^{\Phi(x,t,\varepsilon)}, \tag{4.6}$$

where

$$S(x,\varepsilon) = 1 + s(x,\varepsilon), \quad \Phi(x,\varepsilon) = -\mathrm{i}(\gamma t - \phi(x,\varepsilon)), \quad \phi(x,\varepsilon) = \tilde{\phi}(x,\varepsilon) + c(\varepsilon)\underbrace{\frac{\chi(|x|)}{|x|}}_{P},$$

and $\chi \in C^\infty(\mathbb{R})$ is a cut-off function equal to zero near the origin and equal to 1, for $|x| > 2$. This amounts to letting $\hat{\phi} = \tilde{\phi} + cP(x)$ in (4.4) and (4.5), and results in a nonlinear operator which we again label as $F : \mathcal{X} \times \mathbb{R}^2 \to \mathcal{Y}$. In the last section we show that there exists spaces $\mathcal{X}$ and $\mathcal{Y}$ such that F is well defined and smooth. We will also look at the properties of its linearization, $L : \mathcal{X} \times \mathbb{R} \to \mathcal{Y}$, in Subsection 4.2.3, but we explicitly write the form of this linear operator for future reference here

$$L\begin{bmatrix}\hat{s}\\ \tilde{\phi}\\ c\end{bmatrix} = \begin{bmatrix}(1-\alpha\gamma)\Delta - 2 & -\alpha\gamma\Delta & -\alpha\gamma\Delta P\\ (\alpha\gamma + \frac{\alpha}{\gamma})\Delta & (1+\alpha\gamma)\Delta & (1+\alpha\gamma)\Delta P\end{bmatrix}\begin{bmatrix}\hat{s}\\ \tilde{\phi}\\ c\end{bmatrix}.$$

We also clarify that in the rest of the paper we will write s instead of $\hat{s}$.

4.2.2 Main results: expansions for phase ϕ and amplitude s

For the remainder of the paper we let $\mathcal{X} = W^{2,2}_{\delta+2} \times M^{2,2}_{\delta}$ and $\mathcal{Y} = L^2_{\delta+2} \times L^2_{\delta+2}$. The next proposition, together with the Implicit Function Theorem, show the existence of solutions to (4.2).

Proposition 27 *Let $\delta \in (-1/2, 1/2)$ and let $g \in L^2_{\delta+2}$. Then the operator $F : W^{2,2}_{\delta+2} \times M^{2,2}_{\delta} \times \mathbb{R}^2 \to L^2_{\delta+2} \times L^2_{\delta+2}$ defined by* (4.4) *and* (4.5) *and the Ansatz* (4.6) *is smooth and its Fréchet derivative DF evaluated at $(s, \tilde{\phi}, c; \varepsilon) = 0$, is invertible.*

We leave the proof of this result for Subsection 4.2.3 and justify the expansions and decay rates of $S(x,\varepsilon)$ and $\Phi(x,t,\varepsilon)$ stated in Theorem 5. First, the decay rates follow from our choice of weighted spaces and the following two lemmas.

Lemma 28 *Let $\gamma > -3/2$. If $f \in M^{2,2}_{\gamma}$, then $|f(x)| \le C\langle \mathbf{x}\rangle^{-\gamma-3/2}$ as $|\mathbf{x}| \to \infty$.*

Proof. Since we define the space $M^{2,2}_\gamma$ as the completion of C_0^∞ under the norm $\|\cdot\|_{M^{2,2}_\gamma}$, it suffices to show the result for $f \in C_0^\infty$. Using polar coordinates we find that in dimension 3,

$$\begin{aligned}\int |f(\theta,R)|^2 \, \mathrm{d}\theta &= \int \left(\int_\infty^R |f_r(\theta,s)| \, \mathrm{d}s\right)^2 \, \mathrm{d}\theta = \int \left(\int_\infty^R s^{-(\gamma+2)}|f_r(\theta,s)|s^{\gamma+1} s \, \mathrm{d}s\right)^2 \, \mathrm{d}\theta \\ &\leq \int \left(\int_\infty^R s^{-2(\gamma+2)} \, \mathrm{d}s\right)\left(\int_\infty^R s^{2(\gamma+1)}|f_r(\theta,s)|^2 s^2 \, \mathrm{d}s\right) \mathrm{d}\theta \\ &\leq R^{-2(\gamma+2)+1}\|f_r\|_{L^2_{\gamma+1}}.\end{aligned}$$

Therefore $\|f(\cdot,R)\|_{L^2} \leq CR^{-\gamma-3/2}$. Similarly,

$$\begin{aligned}\int |f_\theta(\theta,R)|^2 \, \mathrm{d}\theta &= \int \left(\int_\infty^R |f_{\theta r}(\theta,s)| \, \mathrm{d}s\right)^2 \, \mathrm{d}\theta = \int \left(\int_\infty^R s^{-(\gamma+3)}|f_{\theta r}(\theta,s)|s^{\gamma+2} s \, \mathrm{d}s\right)^2 \, \mathrm{d}\theta \\ &\leq \int \left(\int_\infty^R s^{-2(\gamma+3)} \, \mathrm{d}s\right)\left(\int_\infty^R s^{2(\gamma+2)}|f_{\theta r}(\theta,s)|^2 s^2 \, \mathrm{d}s\right) \mathrm{d}\theta \\ &\leq R^{-2(\gamma+3)+1}\|f_{\theta r}\|_{L^2_{\gamma+2}}.\end{aligned}$$

Combining these results and using the interpolation inequality from [1, Thm 5.9],

$$\|f(\cdot,R)\|_\infty^2 \leq \|f(\cdot,R)\|_{L^2}\|f(\cdot,R)\|_{H^1},$$

shows the result of the claim. ■

The next lemma can be proven in a similar manner.

Lemma 29 *Let $\gamma > -1/2$. If $f \in W^{2,2}_\gamma$, then $|f(x)| \leq C\langle \mathbf{x}\rangle^{-\gamma-1/2}$ as $|\mathbf{x}| \to \infty$.*

Next, to show the expansion for the function $c(\varepsilon) = \varepsilon c_1 + \mathrm{O}(\varepsilon^2)$ stated in Theorem 5 we use Lyapunov-Schmidt reduction and the results of the next subsection, where we show that the vector $(0,1)^T$, spans the cokernel of the operator $\hat{L} : W^{2,2}_{\delta+2} \times M^{2,2}_\delta \to L^2_{\delta+2} \times L^2_{\delta+2}$ defined by the first two columns of L. If we assume expansions of the form $(s,\tilde{\phi},c)(x;\varepsilon) = \varepsilon(s_1,\tilde{\phi}_1,c_1)+\mathrm{O}(\varepsilon^2)$, we can obtain at order $\mathrm{O}(\varepsilon)$ an expression for the coefficient c_1:

$$\begin{aligned}-\int g \, \mathrm{d}x &= \int (\alpha\gamma + \frac{\alpha}{\gamma})\Delta s_1 + (1+\alpha\gamma)\Delta\tilde{\phi}_1 + c_1(1+\alpha\gamma)\Delta P \, \mathrm{d}x \\ &= -4\pi(1+\alpha\gamma)c_1 \\ c_1 &= \frac{\int g \, \mathrm{d}x}{4\pi(1+\alpha\gamma)},\end{aligned}$$

where the last two equalities follow from Theorem 2 and the fact that

$$\int \Delta\left(\frac{\chi(|x|)}{|x|}\right) dx = -4\pi.$$

4.2.3 The linear operator

In this subsection we prove Proposition 27, by decomposing the linear operator L as $L = [\hat{L}, M]$. First, we use the results from Chapter 2 to show that the operator, $\hat{L} : W^{2,2}_{\delta+2} \times M^{2,2}_{\delta} \to L^2_{\delta+2} \times L^2_{\delta+2}$, defined below, is Fredholm with index -1. Next, we show that the Ansatz (4.6) adds good far field corrections so that the linearization, $L : W^{2,2}_{\delta+2} \times M^{2,2}_{\delta} \times \mathbb{R} \to L^2_{\delta+2} \times L^2_{\delta+2}$ is an invertible operator. We define $\hat{L}$ explicitly for future reference:

$$\hat{L}\begin{bmatrix} s \\ \phi \end{bmatrix} = \begin{bmatrix} (1-\alpha\gamma)\Delta - 2 & -\gamma\alpha\Delta \\ \left(\gamma\alpha + \frac{\alpha}{\gamma}\right)\Delta & (1+\gamma\alpha)\Delta \end{bmatrix}\begin{bmatrix} s \\ \phi \end{bmatrix}. \tag{4.7}$$

Lemma 30 *Let* $\delta \in (-1/2, 1/2)$*, and* $1 + \gamma\alpha > 0$*. Then the linear operator* $\hat{L} : W^{2,2}_{\delta+2} \times M^{2,2}_{\delta} \to L^2_{\delta+2} \times L^2_{\delta+2}$*, defined by* (4.7) *is a Fredholm operator with index* $i = -1$ *and cokernel spanned by the vector* $(0, 1)^T$.

Proof. Assume

$$\begin{bmatrix} s \\ \phi \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}. \tag{4.8}$$

From the second component of L we obtain and equation for the variable ϕ,

$$\Delta\phi = \frac{g}{1+\alpha\gamma} - \frac{\alpha\gamma + \alpha/\gamma}{1+\alpha\gamma}\Delta s. \tag{4.9}$$

Since $1 + \alpha\gamma > 0$, we can insert the above expression for $\Delta\phi$ into the first line of equation (4.8) and solve for s:

$$s = [(1+\alpha^2)\Delta - 2(1+\alpha\gamma)]^{-1}(1+\alpha\gamma)f + [(1+\alpha^2)\Delta - 2(1+\alpha\gamma)]^{-1}\alpha\gamma g.$$

Next, we use the above result in (4.9) and obtain the following equation for ϕ:

$$\Delta\phi = [(1+\alpha^2)\Delta - 2(1+\alpha\gamma)]^{-1}[(1-\alpha\gamma)\Delta - 2]g + \Delta[(1+\alpha^2)\Delta - 2(1+\alpha\gamma)]^{-1}(1+\alpha\gamma)f.$$

Our goal is to show that the right hand side is in the range of $\Delta : M^{2,2}_{\delta} \to L^2_{\delta+2}$. It is clear that the term

$$\Delta[(1+\alpha^2)\Delta - 2(1+\alpha\gamma)]^{-1}(1+\alpha\gamma)f,$$

satisfies this requirement for any $f \in L^2_{\delta+2}$, given that it involves the Laplacian and that the operator $[(1+\alpha^2)\Delta - 2(1+\alpha\gamma)]^{-1} : L^2_{\delta+2} \to W^{2,2}_{\delta+2}$ is bounded.

The results from Theorem 2 and our assumption that $\delta \in (-1/2, 1/2)$ require us to show that if g has average zero, then the term

$$(1+\alpha^2)\Delta - 2(1+\alpha\gamma)]^{-1}[(1-\alpha\gamma)\Delta - 2]g$$

also has average zero. The result follows since the operator, $A : L^2_{\delta+2} \to L^2_{\delta+2}$ defined by

$$A = [(1+\alpha^2)\Delta - 2(1+\alpha\gamma)]^{-1}[(1-\alpha\gamma)\Delta - 2]$$

preserves this condition. To see this, notice that the condition $\int g = 0$ is equivalent to $\hat{g}(0) = 0$, where $\hat{g}$ denotes the Fourier transform of g. Moreover, since the Fourier symbol of A is given by

$$\hat{A}(k) = \frac{(1-\alpha\gamma)|k|^2 + 2}{(1+\alpha^2)|k|^2 + 2(1+\alpha\gamma)},$$

and $1+\alpha\gamma > 0$, then $\mathcal{F}(Ag)(0) = 0$ if and only if $g(0) = 0$. This proves the Lemma.

■

Remark 31 *Observe that the condition* $1+\alpha\gamma > 0$ *is also required for spectral stability, an indication that these methods are consistent with previous results.*

Remark 32 *If* $\delta \in (-3/2, -1/2)$ *the Laplace operator is invertible. A similar argument as in Lemma 30 then shows that for these values of* δ *the operator* $\hat{L} : W^{2,2}_{\delta+2} \times M^{2,2}_{\delta} \to L^2_{\delta+2} \times L^2_{\delta+2}$ *is invertible.*

Next, consider the Ansatz:

$$\phi = \tilde{\phi} + c\underbrace{\frac{\chi(|x|)}{|x|}}_{P},$$

where $\chi \in C^\infty(\mathbb{R})$ is defined as in the introduction. With this Ansatz, the linearization of $F : W^{2,2}_{\delta+2} \times M^{2,2}_{\delta} \times \mathbb{R} \to L^2_{\delta+2} \times L^2_{\delta+2}$ about the origin is given by the operator, $L : W^{2,2}_{\delta+2} \times M^{2,2}_{\delta} \times \mathbb{R} \to L^2_{\delta+2} \times L^2_{\delta+2}$,

$$L\begin{bmatrix} s \\ \tilde{\phi} \\ c \end{bmatrix} = \begin{bmatrix} (1-\alpha\gamma)\Delta - 2 & -\alpha\gamma\Delta & -\alpha\gamma\Delta P \\ (\alpha\gamma + \frac{\alpha}{\gamma})\Delta & (1+\alpha\gamma)\Delta & (1+\alpha\gamma)\Delta P \end{bmatrix} \begin{bmatrix} s \\ \tilde{\phi} \\ c \end{bmatrix}, \tag{4.10}$$

which we decompose as,

$$L = \begin{bmatrix} \hat{L} & M \end{bmatrix}.$$

Here, $\hat{L}$ is the same as (4.7) and $M : \mathbb{R} \to L^2_{\delta+2} \times L^2_{\delta+2}$ is given by

$$Mc = \begin{bmatrix} -\alpha\gamma\Delta P \\ (1+\alpha\gamma)\Delta P \end{bmatrix} c.$$

It is clear that the operator M is well defined since $\Delta P = \Delta\left(\frac{\chi(|x|)}{|x|}\right)$ has compact support. Notice as well that

$$\int_{\mathbb{R}^3} \Delta\left(\frac{\chi(|x|)}{|x|}\right) dx = -4\pi,$$

so that the range of M and the cokernel of L intersect. The Bordering lemma for Fredholm operators then shows that for $\delta \in (-1/2, 1/2)$, the operator $L : W^{2,2}_{\delta+2} \times M^{2,2}_{\delta} \times \mathbb{R} \to L^2_{\delta+2} \times L^2_{\delta+2}$ is invertible. This proves the following result.

Lemma 33 *Let* $\delta \in (-1/2, 1/2)$ *and* $1 + \alpha\gamma > 0$*. Then the operator* $L : W^{2,2}_{\delta+2} \times M^{2,2}_{\delta} \times \mathbb{R} \to L^2_{\delta+2} \times L^2_{\delta+2}$*, defined by* (4.10) *is an invertible operator.*

In order to finish the proof of Proposition 27 we just need to show that the full operator $F : W^{2,2}_{\delta+2} \times M^{2,2}_{\delta} \times \mathbb{R}^2 \to L^2_{\delta+2} \times L^2_{\delta+2}$ is well defined and smooth, justifying our assertion that $DF(0,0,0;0) = L$. This will be done in the following section.

4.2.4 Nonlinear terms

We now consider the full non-linear operator $F : M^{2,2}_{\delta} \times W^{2,2}_{\delta+2} \times \mathbb{R}^2 \to L^2_{\delta+2} \times L^2_{\delta+2}$, given by

$$\begin{aligned} F_1(s,\phi,c) =& (1-\alpha\gamma)\Delta s - 2s - \gamma\alpha\Delta\phi - (\gamma+s)[|\nabla s|^2 + 2\nabla s\cdot\nabla\phi + |\nabla\phi|^2] - 2\alpha|\nabla s|^2 \\ & - 2\alpha\nabla s\cdot\nabla\phi - \alpha s(\Delta s + \Delta\phi) - \frac{3}{\gamma}s^2 - \frac{1}{\gamma^2}s^3, \\ F_2(s,\phi,c) =& \left(\frac{\alpha}{\gamma} + \alpha\gamma\right)\Delta s + (1+\alpha\gamma)\Delta\phi + (\gamma - \alpha + s)\left[|\nabla s|^2 + 2\nabla s\cdot\nabla\phi + |\nabla\phi|^2\right] \\ & + \alpha s(\Delta s + \Delta\phi) + 2\alpha|\nabla s|^2 + 2\alpha\nabla s\cdot\nabla\phi + \frac{3s^2}{\gamma} + \frac{s^3}{\gamma^2} \\ & + (\gamma+s)^{-1}\left[2|\nabla s|^2 + 2\nabla s\cdot\nabla\phi - s^2 - \frac{s^3}{\gamma} - \frac{\alpha}{\gamma}s\Delta s\right] + \varepsilon g(x), \end{aligned}$$

We omitted the "hats" for ease of notation and use $\phi = \tilde{\phi} + cP$, with $P = \frac{\chi(|x|)}{|x|}$. With the help of the next lemma we show that F is well defined in the sense that all non-linear terms are in the space $L^p_{\delta+2}$.

Lemma 34 *Let $\delta \in \mathbb{R}$. If $f, g \in W^{1,2}_{\delta+1}$, then the product $fg \in L^2_{\delta+2}$.*

Proof. This lemma is a consequence of Hölder's inequality and the Sobolev embeddings. ■

Notice also that if $\delta > -2$, then $W^{2,p}_{\delta+2} \subset W^{2,p}$. Furthermore, if $p = 2$ we have $W^{2,2}_{\delta+2} \subset W^{2,2} \hookrightarrow BC(\mathbb{R}^3)$.

Proposition 35 *Let $\delta \in (-2, 1/2)$, and $g \in L^2_{\delta+2}$. Then the linear operator $F : W^{2,2}_{\delta+2} \times M^{2,2}_{\delta} \times \mathbb{R}^2 \to L^2_{\delta+2} \times L^2_{\delta+2}$ defined by* (4.4) *and* (4.5)*, is well defined and smooth.*

Proof. Since $\delta \in (-2, 1/2)$ the results form Lemma 34, and the embedding $W^{2,2}_{\delta+2} \subset BC(\mathbb{R}^2)$ suggest that all terms which do not involve the parameter c are in the space $L^2_{\delta+2}$. Since all derivatives of $\dfrac{\chi(|x|)}{|x|}$ are bounded, the only terms we need to worry about come from the expression $|\nabla\phi|^2$. Recall here that $\phi = \tilde{\phi} + cP$, with $P = \dfrac{\chi(|x|)}{|x|}$ and $\tilde{\phi} \in M^{2,2}_{\delta}$, so that

$$|\nabla\phi|^2 = |\nabla\tilde{\phi}|^2 + 2c\nabla\tilde{\phi} \cdot \nabla P + c^2|\nabla P|^2.$$

It is clear from Lemma 34 that the expression $|\nabla\tilde{\phi}|^2 \in L^2_{\delta+2}$. Also, because ∇P is bounded in compact sets and behaves like $\langle x\rangle^{-2}$ for large $|x|$, a straightforward calculation shows that $\nabla\tilde{\phi} \cdot \nabla P$ is in the desired space. Finally, since $\delta < 1/2$ the following integral converges

$$\int_{\mathbb{R}^3} |\nabla P|^4 \langle x\rangle^{2(\delta+2)}\, dx \leq \int_1^\infty r^{2(\delta+2)-8} r^2\, dr.$$

Given that all non-linear terms are defined via superposition operators of algebraic functions, they are smooth once well defined. This completes the proof.

■

4.3 Numerical results

For the numerical simulations we consider the perturbed complex Ginzburg-Landau equation in a co-rotating frame,

$$A_t = (1 + i\alpha)\Delta A + (1 + i\gamma)A - (1 + i\gamma)A|A|^2 + i\varepsilon g(x)A. \tag{4.11}$$

The initial condition is the steady state $A = 1$, and we take $\varepsilon = 0.5$ and define the inhomogeneity as,

$$g(x, y, z) = (1 + x^2 + y^2 + z^2)^{-\alpha}. \tag{4.12}$$

The domain is a cube of length $l = 40$ and the results are taken at time $T = 500$ for different values of α. Each value of α corresponds to a region in $\delta-$space for which the linearization $\hat{L}$ has different Fredholm properties (see Table 4.1).

$\hat{L}$ is	Invertible				Fredholm, $i = -1$			Fredholm, $i = -3$	
δ-range	$-3/2 < \delta < -1/2$				$-1/2 < \delta < 1/2$			$1/2 < \delta < \infty$	
α	1.2	1.3	1.4	1.5	1.6	1.8	2	2.2	2.4
m_ϕ	-0.602	0.508	-0.518	-0.714	-0.969	-1.020	-0.990	-1.080	-1.08

Table 4.1: The inhomogeneity, g, is in $L^2_{\delta+2}$ if $\alpha < -3/2 - (\delta + 2)$. The constant m_ϕ represents the decay rates for the phase ($\Phi(x, \varepsilon, t) \sim |x|^{m_\phi}$) found in the numerical simulations (see figures at the end of Section 4.3).

Table 4.1 illustrates for which values of δ our results are valid. We are not interested in inhomogeneities with $\alpha < 1$, since in this case our solutions blow up. For inhomogeneities with $1 < \alpha \leq 1.5$ we can pick $\delta \in (-3/2, -1/2)$. The result is that the linearization L is invertible and in this case we do not have far field corrections. Consequently, we cannot make predictions on the asymptotic decay of the phase, but we can say that the phase ϕ, viewed as a function of space alone, should satisfy the same properties as a function in $M^{2,2}_{\delta}$, i.e. $|\phi(x)| < C\langle x\rangle^{-\delta-3/2}$ (see Lemma 28). This is corroborated by our numerical simulations. On the other hand, for inhomogeneities with $\alpha > 1.4$, the numerical results confirm that the phase decays at order $O(1/|x|)$.

We conclude this short section with some plots (figure 4.2) that illustrate the results of Table 4.1. They depict the phase of solutions to (4.11) at the cross section $z = y = 0$ and for different values of α. All numerical simulations were done on Matlab using exponential time difference combined with an order four Runge-Kutta method. The domain size was $l = 40$, the grid size $N = 256$ and time step $h = 1$. For more details on the code see [18, 17].

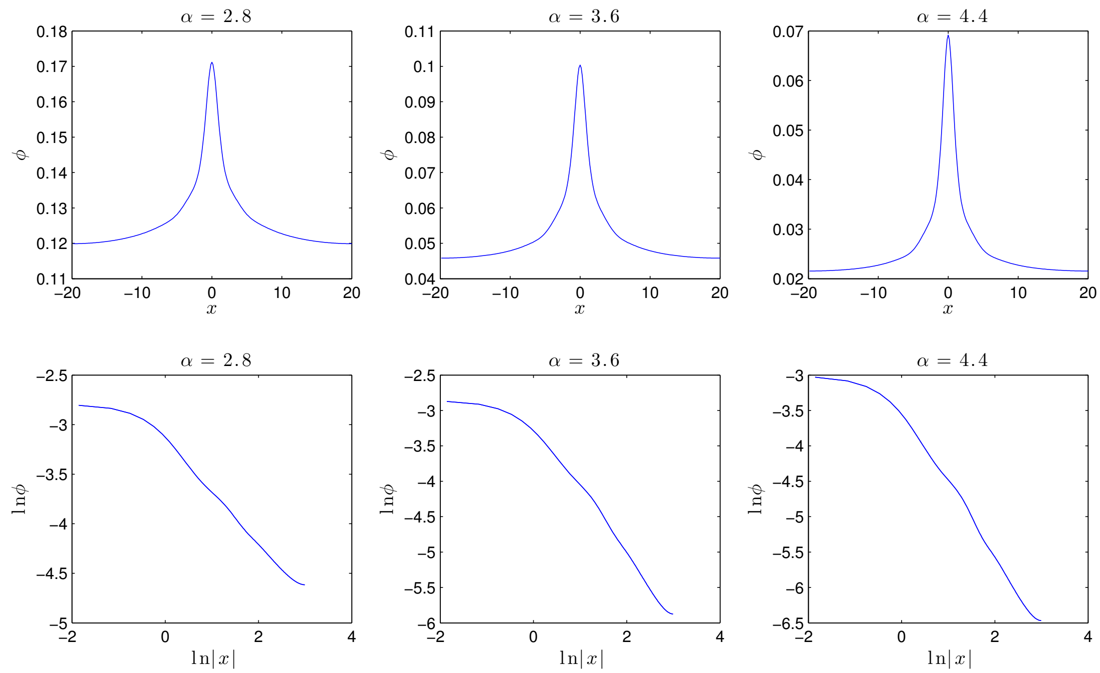

Figure 4.2: Plot of ϕ vs. x and $\ln\phi vs. \ln|x|$ at the cross section $z = 0, y = 0$ for values of $\alpha = 2.8, 3.6$, and $\alpha = 4.4$

Chapter 5

Pacemakers in an array of oscillators with nonlocal coupling

5.1 Introduction

The collective behavior in systems of coupled oscillators has attracted a tremendous amount of interest. Self-organized synchronization in large systems appears particularly dramatic when coupling effects are seemingly weak [44]. A substantial part of the work has been devoted to the study of such collections of oscillators in the strikingly simple and explicit Kuramoto model [23]. Synchronization and desynchronization as well as a plethora of more complex states have been found, and boundaries (or phase transitions) have been characterized [43, 7]. On the other hand, it is well known that the collective behavior may well depend on the type of coupling, as well as the type of internal dynamics at nodes. Of particular interest have been spiky oscillators in neuroscience with their quite different phase response curves, or phase-amplitude descriptions near Hopf bifurcations.

Our interest here focuses modestly on a rigorous description of pacemakers. Roughly speaking, we ask if and how a small collection of oscillators can influence the collective behavior of a large ensemble. This question has been addressed in numerous contexts. One observed dramatic influence manifests itself through the occurrence of target patterns. Phenomenologically, a faster (or slower) patch of oscillators entrains neighbors and a phase-lag gradient propagates through the medium according to an eikonal equation.

The analysis of such phenomena is notoriously complicated by the absence of spectral gaps in the linearization at the synchronized state, inherently related to the presence of a neutral phase in the medium. Standard perturbation analysis in a large collection of oscillators, based on an Implicit Function Theorem, is valid only for extremely small sizes of perturbations and fails to capture key phenomena. In an infinite medium, the range of the linearization is not closed, so that simple matched asymptotics cannot be justified. In fact, in an infinite medium (and also, with some corrections, in large media), one observes that the system relaxes to a frequency-synchronized state, but the collective frequency depends in unusual ways on the perturbation parameter. Characterizing, for instance, the deviation of the localized patch of oscillators from the ensemble background by ε, the collective frequency will change with $\omega \sim \varepsilon^2$ for $\varepsilon > 0$ and remain constant for $\varepsilon < 0$, in a one-dimensional medium. It is this general phenomenon that we are concerned with in this chapter.

One can ask questions of perturbative nature in many different circumstances. First, one can consider various types of oscillators, ranging from simple phase oscillators $\phi' = \omega$, over gauge-invariant phase-amplitude oscillators, $A' = (1 + \mathrm{i}\omega)A - (1 + \mathrm{i}\gamma)A|A|^2$, to general asymptotically stable periodic orbits $u_*(-\omega t)$ in an ODE $u' = f(u)$. On the other hand, one can look at simple scalar diffusive coupling, or, most generally, dynamics on networks. Our focus is on *simple* internal phase dynamics, but nonlocal coupling along a line. Previous results have studied phase-dynamics, formally derived from the complex Ginzburg-Landau equation for amplitude-phase oscillators [42], and general stable periodic orbits with diffusive coupling, but in a one-dimensional context [34] or with radial symmetry [21].

Phase dynamics can be derived and shown to approximate dynamics on long temporal and spatial scales [10]. A general form of the dynamics is

$$\phi_t = d\Delta\phi - \kappa|\nabla\phi|^2 + \omega_*, \qquad \phi \in \mathbb{R}/(2\pi\mathbb{Z}). \tag{5.1}$$

Substituting $\phi \to \phi + \omega_* t$, thus exploiting the phase invariance, we can assume that $\omega_* = 0$. The solutions $\phi \equiv \bar{\phi}$ correspond to the spatially synchronized state. One can show that this synchronized state is asymptotically stable under localized (L^1) perturbations, with decay rate given by the "effective viscosity" $\sim dt^{-n/2}$ [13].

Adding a localized inhomogeneity,

$$\phi_t = d\Delta\phi - \kappa|\nabla\phi|^2 + \omega_* + \varepsilon g(x), \qquad g(x) \to 0 \text{ for } |x| \to \infty, \tag{5.2}$$

will destroy the synchronized state. In particular, $g > 0$, compactly supported, encodes a localized patch of oscillators oscillating at a higher frequency $\omega_* + \varepsilon g(x)$. One can then ask if the system (5.2) possesses a periodic solution for $\varepsilon \neq 0$, and what the frequency of this solution would be. Therefore, note first that (5.2) possesses nontrivial solutions at $\varepsilon = 0$,

$$\phi(t, x) = k \cdot x - \omega t, \qquad \omega = \kappa |k|^2.$$

This family of solutions mimics periodic wave-trains $u_*(-\omega t + kx)$ in a reaction-diffusion context [34, 21] or plane-wave solutions $\sqrt{1-k^2}\mathrm{e}^{\mathrm{i}(-\omega t + k\cdot x)}$ in the complex Ginzburg-Landau equation. The coefficient κ (which could be scaled to $\kappa = 1$) therefore encodes nonlinear dispersion, that is, the dependence of nonlinear (here, affine) wave frequency on the spatial wavenumber. These waves travel in the direction of $k = \nabla\phi$, with group velocity $c_\mathrm{g} = 2\kappa k$. We are therefore interested in solutions $\phi(x - \omega t)$ for which $c_\mathrm{g} = 2\kappa\nabla\phi(x)\cdot x \geq 0$ for $|x| \to \infty$. In other words, we focus on waves "generated" by the inhomogeneity, rather than the effect of the inhomogeneity as a scattering object on waves sent in from infinity.

Equation (5.2) can be analyzed using a variety of methods. Hopf-Cole will conjugate the equation to a Schrödinger eigenvalue problem, where small eigenvalues can pop out of the edge of the essential spectrum depending on the sign of $\varepsilon \int g$ [21]. Studying eigenvalues of Schrödinger operators opens up an entirely different set of tools; see [39]. An approach that carries over to more general one-dimensional systems relies on rewriting (5.2) as an ODE,

$$\phi_x = u \tag{5.3}$$

$$u_x = \frac{\kappa}{d}u^2 - \omega - \varepsilon g(x). \tag{5.4}$$

When g is exponentially localized, one can compactify space $x = \operatorname{arctanh}\tau$, and study heteroclinic orbits in

$$u_x = \frac{\kappa}{d}u^2 - \omega - \varepsilon g(x(\tau)) \tag{5.5}$$

$$\tau_x = 1 - \tau^2. \tag{5.6}$$

We refer to [34, 21] for discussions of the corresponding heteroclinic bifurcation. Radial symmetry allows for a similar approach based on dynamical systems methods.

Our main result is concerned with the nonlocal equivalent of (5.2),

$$\phi_t = -\phi + G * \phi - (J' * \phi)^2 + \varepsilon g(x). \tag{5.7}$$

Here, G and J are symmetric convolution kernels, $x \in \mathbb{R}$, and $\int G = 1$. We are motivated by two aspects. First, nonlocal coupling is more realistic in most examples of coupled oscillator

problems; nonlocal kernels arise naturally in the limit of large networks [29]. Second, nonlocal problems are not immediately amenable to the type of dynamical systems approach described above and therefore pose interesting technical challenges. Indeed, the linearization at $\phi \equiv \bar{\phi}$ is not Fredholm as a closed operator on typical spaces and a more subtle analysis is necessary.
Linear problems similar to (5.7) have been studied in [12] using spaces of exponentially localized functions. As demonstrated in [15, 16], this approach is not easily viable in higher space dimensions. A more robust approach relies on algebraic weights and will be pursued here. As a side benefit, we are also able to allow algebraically localized inhomogeneities g. Such weak algebraic localization would cause problems even in the local version, since time compactifications would leave equilibria at infinity highly degenerate, necessitating for instance refined geometric blow-up methods.
In Section 5.2 will apply the results from Chapter 2 to analyze equation (5.2) when the inhomogeneity, g, is assumed to be algebraically localized. This represents a simpler version of the nonlocal case we want to understand, since the linearizations of equations (5.2) and (5.7) about the constant solution share the same Fredholm properties. Finally, in Section 5.3 the procedures used to find solutions for the local case are extended to solve the nonlocal problem with the following assumptions:

H1 **Diffusive Kernel:** The kernel G is continuous, even, exponentially localized with

$$\int G(x)\mathrm{d}x = 1, \quad G_2 := \int x^2 G(x)\mathrm{d}x > 0.$$

Moreover, we require that the Fourier transform satisfies $\hat{G} - 1 \leq 0$, which encodes linear stability of the synchronous state at $\varepsilon = 0$.

H2 **Nonlocal Transport:** The kernel J is exponentially localized, even, twice continuously differentiable, and

$$J_0 := \int J(x)\mathrm{d}x \neq 0.$$

H3 **Inhomogeneity:** The function g is algebraically localized, that is, for some $\sigma > 2$, we have

$$\int (\partial^j g(x))^2 (1 + x^2)^{\sigma+4} \mathrm{d}x < \infty, \qquad \text{for } j = 0, 1, 2.$$

Moreover, we assume that $g_0 := \int g(x)\mathrm{d}x \neq 0$, and define $g_1 := \int x g(x)\mathrm{d}x$.

Note that (5.7) possesses wave train solutions of the form $\phi(t,x) = kx - \omega t$, when the nonlinear dispersion relation

$$\omega = \omega_{\mathrm{nl}}(k) := J_0^2 k^2$$

is satisfied. We are interested in pacemakers (or sources), which, according to the above discussion and [10], resemble wave trains at $\pm\infty$ with outward pointing group velocity [34, 21]

$$\pm c_{\mathrm{g}}^{\pm} > 0, \qquad \text{where } c_g^{\pm} = 2J_0^2 k_{\pm}.$$

We are now ready to state the main result.

Theorem 6 *Consider the nonlocal eikonal equation* (5.7)*, under Hypotheses (H1)–(H3). Then, there exists* $\varepsilon_0 > 0$ *such that for all* $0 < |\varepsilon| < \varepsilon_0$ *and* $\mathrm{sign}(\varepsilon) = -\mathrm{sign}(g_0)$*, there exists a solution of the form*

$$\Phi(x,t;\varepsilon) = \phi(x,\varepsilon) + (\phi_0(\varepsilon) + k(\varepsilon)x)\tanh(x) - \omega_{\mathrm{nl}}(k(\varepsilon))t,$$

where ϕ_0, k *are* C^1 *with*

$$\phi_0'(0) = \frac{g_1}{G_2}, \qquad k'(0) = -\frac{g_0}{G_2},$$

and

$$|\phi(x;\varepsilon)| \to 0, \text{ for } |x| \to \infty, \text{ uniformly in } \varepsilon.$$

Note that the sign of $\partial_x \Phi$ is such that group velocities point outwards in the far field, since $\partial_x \Phi \sim \pm\varepsilon k'(0) = \pm\varepsilon g_0/G_2 > 0$ for $x \to \pm\infty$.

More precise statements on the dependence of ϕ on ε and x can be found in the proof. For instance, ϕ is C^1 in ε with values in $M_\sigma^{2,2}$, a Kondratiev space that we shall define below. One could also obtain higher-order expansions in ε by assuming more localization on g as we shall see from the proof. Additionally, the result readily generalizes to more general nonlinearities $f(J' * \phi)$, $f(u) = \mathrm{O}(u^2)$.

On the other hand, we do not believe that our assumptions on localization are sharp. But then again, critical localization is not fully understood, even in the simple conjugate problem of Schrödinger eigenvalue bifurcations from the essential spectrum; see for instance [35] and references therein.

Outline. The remainder of this chapter is organized as follows. Section 5.2 is concerned with a warm-up problem: we prove Theorem 6 in the (local) case of the eikonal equation, replacing nonlocal operators by differential operators as in (5.1). We then move to the proof of Theorem 6 in Section 5.3.

5.2 The eikonal equation — a local warm-up

We consider the local eikonal equation

$$\phi_t = \partial_{xx}\phi - (\partial_x\phi)^2 + \varepsilon g(x), \quad x \in \mathbb{R}, \quad \varepsilon > 0, \tag{5.8}$$

and look for wave sources in the spirit of Theorem 6. Of course, wave trains $\phi = kx - \omega t$ exist for $\varepsilon = 0$ and when $\omega = \omega_{\mathrm{nl}}(k) = k^2$.

Theorem 7 *Consider* (5.8)*, under Hypothesis (H3). Then, there exists* $\varepsilon_0 > 0$ *such that for all* $0 < |\varepsilon| < \varepsilon_0$ *and* $\mathrm{sign}(\varepsilon) = -\mathrm{sign}(g_0)$*, there exists a solution of the form*

$$\Phi(x, t; \varepsilon) = \phi(x, \varepsilon) + (\phi_0(\varepsilon) + k(\varepsilon)x)\tanh(x) - \omega_{\mathrm{nl}}(k(\varepsilon))t,$$

where ϕ_0, k *are* C^1 *with*

$$\phi_0'(0) = \frac{1}{2}g_1, \qquad k'(0) = -\frac{1}{2}g_0,$$

and

$$|\phi(x;\varepsilon)| \to 0, \text{ for } |x| \to \infty, \text{ uniformly in } \varepsilon.$$

Remark 36 *This result can be obtained directly using the Hopf-Cole linearizing transformation and results on eigenvalues of Schrödinger operators with small potentials; see [39, 21]. Our proof here is significantly more involved but lays the ground for the nonlocal result, Theorem 6.*

The proof of Theorem 7 is organized as follows. We find first-order approximations to solutions of (5.8) in Section 5.2.1. In Section 5.2.2, we formulate the problem of existence of Φ as finding the zeros of a nonlinear operator $F_b : M^{2,p}_\sigma \times \mathbb{R}^2 \to M^{2,p}_\sigma \times \mathbb{R}^2$, which we construct as the composition of a linear preconditioner $\mathcal{T}_b^{-1}$, and nonlinear operators $\tilde{N}_1$ and $\tilde{N}_2$. The additional variables $(a, b) \in \mathbb{R}^2$ stand for explicit far-field corrections, which will yield the terms including ϕ_0 and k in the expansion. We make b, which accounts for the most dramatic correction, explicit as an index. Our strategy is to show that F_b satisfies the conditions of the

Implicit Function Theorem. We therefore show bounded invertibility of $\mathcal{T}_b$ in Section 5.2.3 for $b \geq 0$. In Sections 5.2.4 and 5.2.5, we prove that $\mathcal{T}_b^{-1}$ and $b\mathcal{T}_b^{-1}$ are smooth and continuously differentiable with respect to the parameter b, for $b \geq 0$. We finally combine these results in Section 5.2.6 to prove Theorem 7. In the following, we will always assume $g_0 = \int g < 0$ and choose $\varepsilon > 0$.

5.2.1 First order approximations

To start with, notice that the linearization of (5.8) about the constant solution results in the Laplace operator which, according to Proposition 4 in Chapter 2, is Fredholm for $\gamma > 2 - 1/p$ on

$$\partial_{xx} : M^{2,p}_{\gamma-2} \to L^p_\gamma,$$

with index -2 and cokernel spanned by $\{1, x\}$. We therefore insert the Ansatz,

$$\phi(x,t) = \tilde{\phi}(x) + aS(x) + bxS(x) - b^2 t, \quad S(x) = \tanh(x),\ a, b \in \mathbb{R},$$

into (5.8) and obtain, dropping tildes,

$$0 = \partial_{xx}\phi - 2bS\partial_x\phi + a\partial_{xx}S + b\partial_{xx}(xS) + \varepsilon g - N(\phi, a, b), \tag{5.9}$$

where we gathered all nonlinear terms in

$$N(\phi, a, b) = (\partial_x\phi)^2 - b^2(1 - S^2) + 2(\partial_x\phi + bS)(a + bx)\partial_x S + (a + bx)^2(\partial_x S)^2. \tag{5.10}$$

We would like to consider the right-hand side of (5.9) as a map from $M^{2,2}_{\gamma-2} \times \mathbb{R}^2$ into L^2_γ for some $\gamma > 2 - 1/p$, and then use the Fredholm properties of the Laplacian along with Lyapunov-Schmidt reduction to find solutions. The main difficulty is that for $\phi \in M^{2,2}_{\gamma-2}$, we do not obtain $S\partial_x\phi \in L^2_\gamma$, so that the right-hand side is not well defined. Nonetheless, it is still possible to formally find a first order approximation for the solution, inserting an expansion of the form $\phi = \varepsilon\phi_1, a = \varepsilon a_1, b = \varepsilon b_1$. At leading order, we find

$$\partial_{xx}\phi_1 + a_1\partial_{xx}S + b_1\partial_{xx}(xS) = -g. \tag{5.11}$$

Computing the scalar products

$$\langle \partial_{xx}S, 1\rangle = 0, \quad \langle \partial_{xx}S, x\rangle = -2, \quad \langle \partial_{xx}(xS), 1\rangle = 2, \quad \langle \partial_{xx}(xS), x\rangle = 0, \tag{5.12}$$

shows that $\partial_{xx}S$ and $\partial_{xx}(xS)$, span the cokernel of $\partial_{xx} : M^{2,p}_{\gamma-2} \to L^p_\gamma$, and we obtain the following result.

Lemma 37 *For any $\gamma > 2 - 1/p$, the operator $A : M^{2,p}_{\gamma-2} \times \mathbb{R}^2 \to L^p_\gamma$ defined as*

$$A(\phi, a, b) = \partial_{xx}\phi + a\partial_{xx}S + b\partial_{xx}(xS),$$

is invertible.

In particular, given $g \in L^p_\gamma$ we can solve (5.11) and find $(\phi_1, a_1, b_1) \in M^{2,p}_{\gamma-2} \times \mathbb{R}^2$. Taking scalar products of (5.11) with $1, x$ and using (5.12), we also obtain $a_1 = \frac{1}{2}g_1$ and $b_1 = -\frac{1}{2}g_0$.

5.2.2 Construction of the nonlinear map F_b

In order to construct the map $F_b : M^{2,p}_\sigma \times \mathbb{R}^2 \times [0,\infty) \to M^{2,p}_\sigma \times \mathbb{R}^2$, we first introduce the space

$$\mathcal{D} = \{u \in M^{2,p}_\sigma : u_x \in L^p_{\sigma+2}\}, \tag{5.13}$$

and the linear operator $\mathcal{T}_b : \mathcal{D} \times \mathbb{R}^2 \to L^p_{\sigma+2}$,

$$\mathcal{T}_b(\rho, \alpha, \beta) = \partial_{xx}\rho - 2bS\,\partial_x\rho + \alpha\partial_{xx}S + \beta\partial_{xx}(xS).$$

A short calculation shows that inserting an Ansatz

$$\phi = \varepsilon(\phi_1 + \rho), \quad a = \varepsilon(a_1 + \alpha), \quad b = \varepsilon(b_1 + \beta),$$

into (5.8) gives the equation

$$\mathcal{T}_{\varepsilon(b_1+\beta)}(\rho, \alpha, \beta) - \varepsilon\tilde{N}_1(\rho, \alpha, \beta) - \varepsilon\bar{b}\tilde{N}_2 = 0,$$

where, $\tilde{N}_2 = 2S\,\partial_x\phi_1$, and the operator $\tilde{N}_1(\rho, \alpha, \beta)$ is defined in terms of N from (5.10),

$$\tilde{N}_1(\rho, \alpha, \beta) = N(\phi_1 + \rho, a_1 + \alpha, b_1 + \beta).$$

Now, suppose for the moment that, for b fixed, $\mathcal{T}_b : \mathcal{D} \to L^2_{\sigma+2}$ is bounded invertible. We may then precondition the equation with $\mathcal{T}_b$ and solve the equivalent system

$$F_b(\rho, \alpha, \beta; \varepsilon) = [I - \varepsilon\mathcal{T}_b^{-1}(\tilde{N}_1 + b\tilde{N}_2)](\rho, \alpha, \beta) = 0. \tag{5.14}$$

In particular, if $F_{\varepsilon(b_1+\beta)} : M^{2,2}_\sigma \times \mathbb{R}^2 \times [0,\infty) \to M^{2,2}_\sigma \times \mathbb{R}^2$ meets the conditions of the Implicit Function Theorem, we can conclude the existence of solutions to (5.8). We therefore will show that the operators

1. $\mathcal{T}_b^{-1}\tilde{N}_1 : M^{2,p}_\sigma \times \mathbb{R}^2 \to M^{2,p}_\sigma \times \mathbb{R}^2$, and
2. $b\mathcal{T}_b^{-1}\tilde{N}_2 : M^{2,p}_\sigma \times \mathbb{R}^2 \to M^{2,p}_\sigma \times \mathbb{R}^2$

are continuously differentiable with respect to b. We start proving that $\mathcal{T}_b : \mathcal{D} \to L^p_{\sigma+2}$ is invertible, next.

5.2.3 Invertibility of $\mathcal{T}_b^{-1} : L^p_\gamma \to M^{2,p}_{\gamma-2} \times \mathbb{R}^2$ for $\gamma > 2 - 1/p$

Consider $\mathcal{L}_b : \mathcal{D} \to L^p_\gamma$, defined through

$$\mathcal{L}_b\rho = \partial_{xx}\rho - 2bS\,\partial_x\rho. \tag{5.15}$$

We will see that $\mathcal{L}_b$ is Fredholm of index -2, and $\partial_{xx}S, \partial_{xx}(xS)$ span its cokernel. Throughout, we let $\mathrm{Rg}\,(\mathcal{L}_b)$ and $\mathfrak{I}^\perp(\mathcal{L}_b)$ denote range and cokernel of $\mathcal{L}_b$, and let $P : L^p_\gamma \to \mathrm{Rg}\,(\mathcal{L}_b)$ be a projection onto its range. Since we will be using linear Lyapunov-Schmidt reduction, it is useful to have an explicit definition of P. Notice that $\mathfrak{I}^\perp(\mathcal{L}_b) \subset L^q_{-\gamma}$ is spanned by

$$\psi_1^*(x) = \cosh(x)^{-2b}, \qquad \psi_2^*(x) = \int_0^x \left(\frac{\cosh(x)}{\cosh(y)}\right)^{-2b} \mathrm{d}y, \tag{5.16}$$

where $(\partial_x + 2bS(x))\psi_1^*(x) = 0,\quad (\partial_x + 2bS(x))\psi_2^*(x) = 1$. Using brackets $\langle\cdot,\cdot\rangle$ to express the relation between L^p_γ and its dual $L^q_{-\gamma}$, we pick ψ_1, ψ_2 such that $\langle\psi_i, \psi_j^*\rangle = \delta_{ij}$, $i, j = 1, 2$ and find

$$Pu = u - \langle u, \psi_1^*\rangle\psi_1 - \langle u, \psi_2^*\rangle\psi_2.$$

Notice as well that the functions $\psi_1^*(x)$ and $\psi_2^*(x)$ are in $L^q_{-\gamma}$ only for $b \geq 0$ — we will not be able to extend this analysis to $b < 0$. We are now ready to establish the Fredholm properties and bounds on inverses.

Lemma 38 *Let $p \in (1, \infty)$ and $\gamma > 1 - 1/p$. Then, the operator $\mathcal{L}_b : \mathcal{D} \to L^p_\gamma$, defined in (5.15) is Fredholm index -2, its co-kernel is given in (5.16), and the solution to $\mathcal{L}_b u = f$ satisfies bounds $\|u\|_{\mathcal{D}} \leq \dfrac{C}{b}\|f\|_{L^p_\gamma}$, with C independent of b and $f \in \mathrm{Rg}\,(\mathcal{L}_b)$.*

Proof. Since solutions to the ODE $\mathcal{L}_b\phi = 0$ are either constant or exponentially growing at infinity, $\mathcal{L}_b : \mathcal{D} \to L^p_\gamma$ has trivial kernel for $\gamma > 0$. We therefore only need to show that the range of $\mathcal{L}_b$ is closed to conclude that it is a Fredholm operator. We therefore examine the explicit solution formula

$$u(x) = \begin{cases} \int_{-\infty}^x \int_{-\infty}^y f(z)\left(\frac{\cosh(z)}{\cos(y)}\right)^{-2b} \mathrm{d}z\mathrm{d}y & \text{if} \quad x < 0 \\ \int_\infty^x \int_\infty^y f(z)\left(\frac{\cosh(z)}{\cos(y)}\right)^{-2b} \mathrm{d}z\mathrm{d}y & \text{if} \quad x \geq 0. \end{cases}$$

A direct calculation shows that the conditions $\langle f, \psi_i^*\rangle = 0$, $i = 1, 2$, guarantee continuity of u and u_x at $x = 0$. It remains to show that $u \in \mathcal{D}$. Therefore, we factor

$$\mathcal{L}_b u = (\partial_x - 2bS)\partial_x u = f,$$

and show that $(\partial_x - 2bS)^{-1} : \mathrm{Rg}\,(\mathcal{L}_b) \subset L^p_\gamma \to L^p_\gamma$ is bounded. Subsequently solving the Fredholm equation (Chapter 2, Proposition 4) $\partial_x u = (\partial_x - 2bS)^{-1} f$ gives a solution with $u_x, u_{xx} \in L^p_\gamma$ and, since $u \in L^p_{\gamma-1} \subset L^p_{\gamma-2}$, $u \in \mathcal{D}$.

Next, we establish uniform bounds $\|u\|_{\mathcal{D}} \leq \dfrac{C}{b}\|f\|_{L^p_\gamma}$. For $x \geq 0$,

$$|u_x(x)| = \left|\int_x^\infty f(y)\left(\frac{\cosh(y)}{\cosh(x)}\right)^{-2b}\mathrm{d}y\right| \leq \frac{1}{2}\int_x^\infty |f(y)|\mathrm{e}^{-2b(y-x)}\mathrm{d}y,$$

which gives, using $\langle x\rangle^\gamma\langle y\rangle^{-\gamma} \leq \langle x-y\rangle^{|\gamma|}$ and Young's inequality,

$$\|u_x\|_{L^p_\gamma[0,\infty)} \leq \frac{C}{b}\|f\|_{L^p_\gamma}.$$

A similar analysis in the case of $x < 0$ shows the bound $\|u_x\|_{L^p_\gamma(-\infty,0]} \leq \frac{C}{b}\|f\|_{L^p_\gamma}$, and the lemma follows from Proposition 4. ■

Lemma 39 *The operator $\mathcal{L}_b^{-1} : \mathrm{Rg}\,(\mathcal{L}_b) \subset L^p_\gamma \to M^{2,p}_{\gamma-2}$, is uniformly bounded in $b \geq 0$ provided $\gamma > 2 - 1/p$. Explicitly, we have*

$$\|u\|_{M^{2,p}_{\gamma-2}} \leq C\|f\|_{L^p_\gamma},$$

for all $f \in \mathrm{Rg}\,(\mathcal{L}_b)$ and all solutions $\mathcal{L}_b u = f$.

Proof. We use the fact that the operator $\partial_x^{-1} : M^{1,p}_{\gamma-1} \to M^{2,p}_{\gamma-2}$ is bounded for $\gamma > 2 - 1/p$, with bound depending only on the weight γ, and write $\mathcal{L}_b u = (\partial_x - 2bS)\partial_x u$. The result then follows, once we show that the operator $(\partial_x - 2bS)^{-1} : L^p_\gamma \to M^{1,p}_{\gamma-1}$ is uniformly bounded in b. Explicitly, we need to show that solutions to $(\partial_x - 2bS)v = f$ satisfy

$$\|v\|_{L^p_{\gamma-1}} \leq C\|f\|_{L^p_\gamma}. \tag{5.17}$$

We establish this estimate for $x > 0$, the other case being analogous. Set

$$x = e^\tau \quad \tau \in \mathbb{R}, \quad w = e^{\bar{\gamma}\tau}v(e^\tau), \quad g = e^{(\bar{\gamma}+1)\tau}f,$$

which gives

$$w_\tau - M(\tau)w = g, \qquad M(\tau) = \bar{\gamma} + 2bS\,e^\tau > \bar{\gamma}.$$

We find w as

$$w(\tau) = -\int_\tau^\infty g(s)e^{-\int_\tau^s M(\sigma)\mathrm{d}\sigma}\mathrm{d}s.$$

Since $-\int_\tau^s M(\sigma)\mathrm{d}\sigma \leq -\bar{\gamma}(s-\tau)$, we obtain, using again Young's inequality,

$$\|w(\tau)\|_{L^p} \leq \left(\int_{\mathbb{R}} \left[\int_\tau^\infty |g(s)| e^{-\bar{\gamma}(s-\tau)} \mathrm{d}s\right]^p \mathrm{d}x\right)^{1/p} \leq \bar{\gamma}^{-1} \|g(\tau)\|_{L^p}.$$

Setting $\gamma - 1 = \bar{\gamma} - \frac{1}{p}$ we find in the original variables

$$\|v\|_{L^p_{\gamma-1}[0,\infty)} \leq C\|f\|_{L^p_\gamma[0,\infty)},$$

which proves (5.17).

Finally, since we can write $v_x = f + 2bS\,v$, and since we have the bound $\|v\|_{L^p_\gamma} \leq \frac{C}{b}\|f\|_{L^p_\gamma}$ from Lemma 38, we are able to conclude that

$$\|v_x\|_{L^p_\gamma} \leq \|f\|_{L^p_\gamma} + 2b\|v\|_{L^p_\gamma} \leq C\|f\|_{L^p_\gamma}.$$

Consequently,

$$\|v\|_{M^{2,p}_{\gamma-1}} \leq C\|f\|_{L^p_\gamma}.$$

where C is a generic constant, independent of b, that can change from line to line. ■

We are now ready to show the invertibility of $\mathcal{T}_b : \mathcal{D} \times \mathbb{R}^2 \to L^p_\gamma$ with uniform bounds.

Lemma 40 *For $p \in (1,\infty)$ and $b \geq 0$, small, $\mathcal{T}_b : \mathcal{D} \times \mathbb{R}^2 \to L^p_\gamma$, defined through,*

$$\mathcal{T}_b(\rho,\alpha,\beta) = \mathcal{L}_b\rho + \alpha\partial_{xx}S + \beta\partial_{xx}(xS),$$

is invertible. Furthermore, solutions (ρ,α,β) to $\mathcal{T}_b(\rho,\alpha,\beta) = f$ satisfy,

$$\|(\rho,\alpha,\beta)\|_{\mathcal{D}\times\mathbb{R}^2} \leq \frac{C}{b}\|f\|_{L^p_\gamma}, \quad \|(\rho,\alpha,\beta)\|_{M^{2,p}_{\gamma-2}\times\mathbb{R}^2} \leq C\|f\|_{L^p_\gamma},$$

for $\gamma > 1 - 1/p$ and $\gamma > 2 - 1/p$, respectively, with constant C independent of b.

Proof. From Lemma 38 we know that $\mathcal{L}_b : \mathcal{D} \to L^p_\gamma$ is a Fredholm operator with index $i = -2$ and cokernel spanned by $\psi_1^*(x) = \cosh(x)^{-2b}$ and $\psi_2^*(x) = \int_0^x \left(\frac{\cosh(x)}{\cosh(y)}\right)^{-2b} \mathrm{d}y$. We will use this information together with the Bordering Lemma for Fredholm operators to show that $\mathcal{T}_b$ is invertible.

Let $R : \mathbb{R}^2 \to (\Im(\mathcal{L}_b))^\perp$ be defined as $R(\alpha,\beta) = \alpha\partial_{xx}S + \beta\partial_{xx}(xS)$ and write

$$\mathcal{T}_b(\rho,\alpha,\beta) = \mathcal{L}_b\rho + R(\alpha,\beta).$$

Since $\partial_{xx}S$ and $\partial_{xx}(xS)$ are exponentially localized functions, the operator R is well defined and bounded. Since scalar products of $\partial_{xx}S$ and $\partial_{xx}(xS)$ with ψ_1^* and ψ_2^* form an invertible 2×2 matrix, the range of R forms a complement to $\mathrm{Rg}\,(\mathcal{L}_b)$, and $\mathcal{T}_b : \mathcal{D} \to L^p_\gamma$ is invertible.
To obtain the desired bounds on ρ, decompose $\mathcal{T}_b(\rho, \alpha, \beta) = f$ into

$$P[\mathcal{L}_b\rho + \alpha\partial_{xx}S + \beta\partial_{xx}(xS)] = Pf,$$
$$(1 - P)[\alpha\partial_{xx}S + \beta\partial_{xx}(xS)] = (1 - P)f.$$

From the second expression we obtain bounds of the form,

$$|\alpha| \leq C\|f\|_{L^p_\gamma}, \quad |\beta| \leq C\|f\|_{L^p_\gamma}.$$

Then, the first equation and the results from Lemmas 38 and 39 show that

$$\|\rho\|_D \leq \frac{C}{b}\|\tilde{f}\|_{L^p_\gamma} \quad \text{and} \quad \|\rho\|_{M^{2,p}_{\gamma-2}} \leq C\|\tilde{f}\|_{L^p_\gamma},$$

where $\tilde{f} = f - \alpha\partial_{xx}S - \beta\partial_{xx}(xS)$. Finally, the desired bounds on the solution (ρ, α, β) follow from applying the triangle inequality to $\tilde{f}$, the bounds on $|\alpha|$ and $|\beta|$, and the fact that the functions $\partial_{xx}S, \partial_{xx}(xS)$ are exponentially localized. ■

The above lemma shows that the operator $\mathcal{T}_b^{-1} : L^p_\gamma \to M^{2,p}_{\gamma-2} \times \mathbb{R}^2$ is bounded linear and we have the following corollary to Lemma 40.

Corollary 41 *Let $b \geq 0$, $\gamma > 2 - 1/p$, and $p \in (1, \infty)$. Then $\mathcal{T}_b^{-1} : L^p_\gamma \to M^{2,p}_{\gamma-2} \times \mathbb{R}^2$ is linear, uniformly bounded in b.*

Roughly speaking, we have shown that we can achieve b-uniform bounds by giving away two degrees of localization. In the following, we show that giving away one or two more degrees of localization, we may even obtain continuity and differentiability in b. Eventually, we will compensate for the loss of localization by exploiting the fact that the nonlinearity gains localization.

5.2.4 Differentiability of $\mathcal{T}_b^{-1} : L^p_\gamma \to M^{2,p}_{\gamma-4} \times \mathbb{R}^2$ for $\gamma > 4 - 1/p$

We start by establishing continuity with respect to b.

Lemma 42 *Let $\gamma > 3 - 1/p$, $b \geq 0$, and $p \in (1, \infty)$ then, the operator $\mathcal{T}_b^{-1} : L^p_\gamma \to M^{2,p}_{\gamma-3} \times \mathbb{R}^2$ is Lipschitz in b in the operator norm topology.*

Proof. We show that

$$\left\|(\mathcal{T}_{b+h}^{-1} - \mathcal{T}_b^{-1})f\right\|_{M^{2,p}_{\gamma-3}\times\mathbb{R}^2} \leq C|h|\,\|f\|_{L^p_\gamma}.$$

Writing $(\rho,\alpha,\beta)|_b = \mathcal{T}_b^{-1}f$, $(\rho,\alpha,\beta)|_{b+h} = \mathcal{T}_{b+h}^{-1}f$, and $(\Delta\rho,\Delta\alpha,\Delta\beta) = (\rho,\alpha,\beta)|_{b+h} - (\rho,\alpha,\beta)|_b$, we have to show that

$$\|(\Delta\rho,\Delta\alpha,\Delta\beta)\|_{M^{2,p}_{\gamma-3}\times\mathbb{R}^2} \leq C|h|\,\|f\|_{L^p_\gamma}.$$

A short calculation shows that

$$\mathcal{T}_b(\Delta\rho,\Delta\alpha,\Delta\beta) = -2hS\,\partial_x\rho|_{b+h},$$

so that, using Lemma 40 with $\gamma - 1 > 2 - 1/p$, we find that

$$\|(\Delta\rho,\Delta\alpha,\Delta\beta)\|_{M^{2,p}_{\gamma-3}\times\mathbb{R}^2} \leq 2|h|C\|\partial_x\rho|_{b+h}\|_{L^p_{\gamma-1}} \leq 2C|h|\,\|f\|_{L^p_\gamma}, \tag{5.18}$$

where the last inequality follows again from $\mathcal{T}_{b+h}(\rho,\alpha,\beta) = f$ and Lemma 40 with $\gamma > 2 - 1/p$. This proves continuity. ■

We next use a weaker topology to conclude differentiability.

Lemma 43 *Let $\gamma > 4-1/p$, $b \geq 0$, and $p \in (1,\infty)$. Then $\mathcal{T}_b^{-1} : L^p_\gamma \to M^{2,p}_{\gamma-4}\times\mathbb{R}^2$ is differentiable in b with Lipschitz continuous derivative, with values in the operator norm topology.*

Proof. We abbreviate $R = \mathcal{T}_b^{-1}$ and define the candidate for the derivative,

$$\partial_b R|_b f = 2\mathcal{T}_b^{-1}S\,\partial_x(\mathcal{T}_b^{-1})^1 f,$$

where $(\mathcal{T}_b^{-1})^1 f$ denotes the first component ρ of the preimage $(\rho,\alpha,\beta) = \mathcal{T}_b^{-1}f$. Since $\gamma > 4-1/p$ the following diagram, together with Corollary 41 (with $\gamma-2 > 2-1/p$) and Proposition 4, shows that the composition

$$\mathcal{T}_b^{-1}S\,\partial_x(\mathcal{T}_b^{-1})^1 : L^p_\gamma \to M^{2,2}_{\gamma-4}\times\mathbb{R}^2,$$

is bounded for all $b \geq 0$.

$$L^p_\gamma \xrightarrow{(\mathcal{T}_b^{-1})^1} M^{2,p}_{\gamma-3} \xrightarrow{S\,\partial_x} M^{1,p}_{\gamma-2} \xrightarrow{\mathcal{T}_b^{-1}} M^{2,p}_{\gamma-4}\times\mathbb{R}^2, \tag{5.19}$$

We next show differentiability,

$$\|(R|_{b+h} - R|_b)f - h\partial_b R|_b f\|_{M^{2,p}_{\gamma-4}\times\mathbb{R}^2} = \mathrm{O}(h^2).$$

Indeed, we can bound the left-hand side by

$$\begin{aligned}
\|(R|_{b+h} - R|_b)f - h\partial_b R|_b f\|_{M^{2,p}_{\gamma-4}\times\mathbb{R}^2} &= \left\|2h\mathcal{T}_b^{-1}S\,\partial_x(\mathcal{T}_{b+h}^{-1})1f + 2h\mathcal{T}_b^{-1}S\,\partial_x(\mathcal{T}_b^{-1})^1 f\right\|_{M^{2,p}_{\gamma-4}\times\mathbb{R}^2} \\
&\leq 2|h|\ \left\|\mathcal{T}_b^{-1}S\,\partial_x\right\|_{M^{2,p}_{\gamma-3}\to M^{2,p}_{\gamma-4}\times\mathbb{R}^2}\ \left\|\left(\mathcal{T}_{b+h}^{-1} - \mathcal{T}_b^{-1}\right)^1 f\right\|_{M^{2,p}_{\gamma-3}} \\
&\leq 4|h|^2\ \left\|\mathcal{T}_b^{-1}S\,\partial_x\right\|_{M^{2,p}_{\gamma-3}\to M^{2,p}_{\gamma-4}\times\mathbb{R}^2}\ \|f\|_{L^p_\gamma},
\end{aligned}$$

where, because $\gamma > 4 - 1/p$, we are able to use (5.18) in the last inequality.

Next, we show that the derivative $\partial_b R$ is continuous with respect to b by proving that the following inequality holds

$$\|\partial_b R|_{b+h} f - \partial_b R|_b f\|_{M^{2,p}_{\gamma-4}\times\mathbb{R}^2} \leq C|h|\,\|f\|_{L^p_\gamma}.$$

We will use diagram (5.19) along with the following modified version to justify the choice of spaces in each step.

$$L^p_\gamma \xrightarrow{(\mathcal{T}_b^{-1})^1} M^{2,p}_{\gamma-2} \xrightarrow{S\,\partial_x} M^{1,p}_{\gamma-1} \xrightarrow{\mathcal{T}_b^{-1}} \mathcal{X} = M^{2,p}_{\gamma-4}\times\mathbb{R}^2. \tag{5.20}$$

The triangle inequality, the continuity of the operator $\mathcal{T}_b^{-1}S\,\partial_x : M^{2,p}_{\gamma-2} \to M^{2,p}_{\gamma-4}\times\mathbb{R}^2$, and the continuity in b of the operator $\mathcal{T}_b^{-1} : L^p_{\gamma-1} \to M^{2,p}_{\gamma-4}\times\mathbb{R}^2$ (since $\gamma - 1 > 3 - 1/p$) show that

$$\begin{aligned}
\|\partial_b R|_{b+h} f\ &-\partial_b R|_b f\|_{M^{2,p}_{\gamma-4}\times\mathbb{R}^2} \\
&= 2\left\|\mathcal{T}_{b+h}^{-1}S\,\partial_x(\mathcal{T}_{b+h}^{-1})^1 f - \mathcal{T}_b^{-1}S\,\partial_x(\mathcal{T}_b^{-1})^1 f\right\|_{M^{2,p}_{\gamma-4}\times\mathbb{R}^2} \\
&\leq 2\left[\left\|\mathcal{T}_{b+h}^{-1}S\,\partial_x\left(\mathcal{T}_{b+h}^{-1} - \mathcal{T}_b^{-1}\right)^1 f\right\|_{M^{2,p}_{\gamma-4}\times\mathbb{R}^2} + \left\|\left(\mathcal{T}_{b+h}^{-1} - \mathcal{T}_b^{-1}\right)(S\,\partial_x(\mathcal{T}_b^{-1})^1 f)\right\|_{M^{2,p}_{\gamma-4}\times\mathbb{R}^2}\right] \\
&\leq 2\left[\left\|\mathcal{T}_{b+h}^{-1}S\,\partial_x\right\|_{M^{2,p}_{\gamma-3}\to M^{2,p}_{\gamma-4}\times\mathbb{R}^2}\ \left\|\left(\mathcal{T}_{b+h}^{-1} - \mathcal{T}_b^{-1}\right)^1 f\right\|_{M^{2,p}_{\gamma-3}} + 2|h|\left\|S\,\partial_x(\mathcal{T}_b^{-1})^1 f\right\|_{L^p_{\gamma-1}}\right].
\end{aligned}$$

Then since we also have continuity in b of the operator $\mathcal{T}_b^{-1} : L^p_\gamma \to M^{2,p}_{\gamma-3}\times\mathbb{R}^2$, we obtain the desired result,

$$\begin{aligned}
\|\partial_b R|_{b+h} f \;-\partial_b R|_b f\|_{M^{2,p}_{\gamma-4}\times\mathbb{R}^2} \\
&\le 4\,|h| \left[\left\| \mathcal{T}^{-1}_{b+h} S\,\partial_x \right\|_{M^{2,p}_{\gamma-3}\to M^{2,p}_{\gamma-4}\times\mathbb{R}^2} \;\|f\|_{L^p_\gamma} + 2 \left\| S\,\partial_x (\mathcal{T}^{-1}_b)^1 f \right\|_{M^{1,p}_{\gamma+1}} \right] \\
&\le 4\,|h| \left[\left\| \mathcal{T}^{-1}_{b+h} S\,\partial_x \right\|_{M^{2,p}_{\gamma-3}\to M^{2,p}_{\gamma-4}\times\mathbb{R}^2} \;\|f\|_{L^p_\gamma} + 2 \left\| S\,\partial_x (\mathcal{T}^{-1}_b)^1 \right\|_{L^p_\gamma \to M^{1,p}_{\gamma-1}} \;\|f\|_{L^p_\gamma} \right].
\end{aligned}$$

■

5.2.5 Differentiability of $b\mathcal{T}_b^{-1} : L^p_\gamma \to M^{2,p}_{\gamma-3}\times\mathbb{R}^2$ for $\gamma > 3 - 1/p$

Unfortunately, we will need differentiability of $\mathcal{T}_b^{-1}$ in a stronger topology than the one used in the previous section. However, we can exploit that fact that the dangerous terms carry an additional factor b. The following two lemmas show that the extra factor b allows us to gain one degree of localization relative to the previous results.

Lemma 44 *Let $\gamma > 2 - 1/p$ for $p \in (1,\infty)$. Then $b\mathcal{T}_b^{-1} : L^p_\gamma \to M^{2,p}_{\gamma-2}\times\mathbb{R}^2$ is Lipschitz continuous in $b \ge 0$ in the operator topology.*

Proof. Similar to the preceding section, we pick $f \in L^p_\gamma$ and show that

$$\left\| \left[(b+h)\mathcal{T}^{-1}_{b+h} - b\mathcal{T}^{-1}_b \right] f \right\|_{M^{2,p}_{\gamma-2}\times\mathbb{R}^2} \le C|h|\|f\|_{L^p_\gamma}.$$

Equivalently, writing $(\rho,\alpha,\beta)|_b = b\mathcal{T}_b^{-1} f$ and $(\rho,\alpha,\beta)|_{b+h} = b + h\mathcal{T}^{-1}_{b+h} f$, we show that

$$\|(\Delta\rho,\Delta\alpha,\Delta\beta)\|_{M^{2,p}_{\gamma-2}\times\mathbb{R}^2} \le C|h|\|f\|_{L^p_\gamma}.$$

First, notice that the difference $(\Delta\rho,\Delta\alpha,\Delta\beta) = (\rho,\alpha,\beta)|_{b+h} - (\rho,\alpha,\beta)|_b$ solves the equation

$$\mathcal{T}_b(\Delta\rho,\Delta\alpha,\Delta\beta) = -2hS\,\partial_x\rho|_{b+h} + hf.$$

Since $\gamma > 2 - 1/p$, from Lemma 40 we know that the function $\rho|_{b+h}$ satisfies $\|\rho\|_{\mathcal{D}\times\mathbb{R}^2} \le \dfrac{C}{b+h}\|(b+h)f\|_{L^p_\gamma}$, where $\mathcal{D} = \{u \in M^{2,p}_{\gamma-2} : u_x \in L^p_\gamma\}$. Therefore,

$$\|(\Delta\rho,\Delta\alpha,\Delta\beta)\|_{M^{2,p}_{\gamma-2}\times\mathbb{R}^2} \le C\|-2hS\,\partial_x\rho|_{b+h} + hf\|_{L^p_\gamma} \le 3C|h|\,\|f\|_{L^p_\gamma}. \tag{5.21}$$

■

Lemma 45 *Let $\gamma > 3 - 1/p$ and $p \in (1,\infty)$. Then $b\mathcal{T}_b^{-1} : L^p_\gamma \to M^{2,p}_{\gamma-3} \times \mathbb{R}^2$ is differentiable in $b \geq 0$ with Lipschitz continuous derivative, with values in the operator norm topology.*

Proof. We again write $R|_b = b\mathcal{T}_b^{-1}$ and introduce the definition of the candidate for the derivative, $\partial_b R|_b = \mathcal{T}_b^{-1} + 2b\mathcal{T}_b^{-1} S\,\partial_x(\mathcal{T}_b^{-1})^1$. Following the proof of Lemma 43, and since $\gamma > 3-1/p$, we find uniform bounds for this operator. We next show that that for $f \in L^p_\gamma$ we have

$$\|(R|_{b+h} - R_b)\, f - h\partial_b R|_b f\|_{M^{2,p}_{\gamma-3}\times\mathbb{R}^2} = \mathrm{O}(h^2).$$

Using the fact that $\left[(b+h)\mathcal{T}_{b+h}^{-1} - b\mathcal{T}_b^{-1}\right] f = b\mathcal{T}_b^{-1} f - 2h\mathcal{T}_b^{-1} S\,\partial_x \mathcal{T}_{b+h}^{-1})^1 f$ we can rewrite the left-hand side of the above expression as

$$\begin{aligned}\|(R|_{b+h} - R_b)\, f\ &-h\partial_b R|_b f\|_{M^{2,p}_{\gamma-3}\times\mathbb{R}^2}\\ &= \left\|\left[(b+h)\mathcal{T}_{b+h}^{-1} - b\mathcal{T}_b^{-1}\right] f - h\left[\mathcal{T}_b^{-1} - 2b\mathcal{T}_b^{-1} S\,\partial_x(\mathcal{T}_b^{-1})1\right] f\right\|_{M^{2,p}_{\gamma-3}\times\mathbb{R}^2}.\end{aligned}$$

Now, recall the inequality (5.21), which for $\gamma > 2-1/p$ shows the continuity in b of the operator $b\mathcal{T}_b^{-1} : L^p_\gamma \to M^{2,p}_{\gamma-2} \times \mathbb{R}^2$. This result, together with the linearity of $\mathcal{T}_b^{-1} S\,\partial_x : M^{1,p}_{\gamma-2} \times \mathbb{R}^2 \to M^{2,p}_{\gamma-3} \times \mathbb{R}^2$ (for $\gamma - 1 > 2 - 1/p$), shows that

$$\begin{aligned}\|(R|_{b+h} - R_b)\ f - h\partial_b R|_b f\|_{M^{2,p}_{\gamma-3}\times\mathbb{R}^2}&\\ \leq &\left\|\left[h\mathcal{T}_b^{-1} f - 2h\mathcal{T}_b^{-1} S\,\partial_x((b+h)\mathcal{T}_{b+h}^{-1})^1 f\right] - h\left[\mathcal{T}_b^{-1} f - 2\mathcal{T}_b^{-1} S\,\partial_x(b\mathcal{T}_b^{-1})^1 f\right]\right\|_{M^{2,p}_{\gamma-3}\times\mathbb{R}^2}\\ \leq &2|h| \left\|\mathcal{T}_b^{-1} S\,\partial_x\left[(b+h)\mathcal{T}_{b+h}^{-1} - b\mathcal{T}_b^{-1}\right]^1 f\right\|_{M^{2,p}_{\gamma-3}\times\mathbb{R}^2}\\ \leq &C|h|^2 \left\|\mathcal{T}_b^{-1} S\,\partial_x\right\|_{M^{2,p}_{\gamma-2}\to M^{2,p}_{\gamma-3}\times\mathbb{R}^2} \|f\|_{L^p_\gamma},\end{aligned}$$

as desired. The final step is to prove that the derivative $\partial_b R$ is Lipschitz in b. For $f \in L^p_\gamma$ we show that

$$\left\|(\partial_b R|_{b+h} - \partial_b R|_b) f\right\|_{M^{2,p}_{\gamma-3}\times\mathbb{R}^2} \leq C|h|\,\|f\|_{L^p_\gamma}.$$

Using the triangle inequality we obtain a first bound,

$$\begin{aligned}\left\|(\partial_b R|_{b+h} - \partial_b R|_b) f\right\|_{M^{2,p}_{\gamma-3}\times\mathbb{R}^2}&\\ \leq &\left\|\left[\mathcal{T}_{b+h}^{-1} - 2(b+h)\mathcal{T}_{b+h}^{-1} S\,\partial_x(\mathcal{T}_{b+h}^{-1})^1\right] f - \left[\mathcal{T}_b^{-1} - 2b\mathcal{T}_b^{-1} S\,\partial_x(\mathcal{T}_b^{-1})^1\right] f\right\|_{M^{2,p}_{\gamma-3}\times\mathbb{R}^2}\\ \leq &\left\|\left(\mathcal{T}_{b+h}^{-1} - \mathcal{T}_b^{-1}\right) f\right\|_{M^{2,p}_{\gamma-3}\times\mathbb{R}^2} + 2\left\|\left[(b+h)\mathcal{T}_{b+h}^{-1} - b\mathcal{T}_b^{-1}\right]\left(S\,\partial_x(\mathcal{T}_{b+h}^{-1})^1\right) f\right\|_{M^{2,p}_{\gamma-3}\times\mathbb{R}^2}\\ &+ 2|b| \left\|\mathcal{T}_b^{-1} S\,\partial_x\left(\mathcal{T}_{b+h}^{-1} - \mathcal{T}_b^{-1}\right)^1 f\right\|_{M^{2,p}_{\gamma-3}\times\mathbb{R}^2}.\end{aligned}$$

Notice that because $\gamma - 1 > 2 - 1/p$, we can use again inequality (5.21) to conclude that the operator $b\mathcal{T}_b^{-1} : L^p_{\gamma-1} \to M^{2,p}_{\gamma-3} \times \mathbb{R}^2$ is continuous in b. Furthermore, since $S\partial_x(\mathcal{T}_{b+h}^{-1})^1 : L^p_\gamma \to M^{1,p}_{\gamma-1}$ is linear, we can bound the second term in this last inequality by

$$\left\|\left[(b+h)\mathcal{T}_{b+h}^{-1} - b\mathcal{T}_b^{-1}\right]\left(S\partial_x(\mathcal{T}_{b+h}^{-1})^1\right) f\right\|_{M^{2,p}_{\gamma-3}\times\mathbb{R}^2} \leq C|h|\, \|S\partial_x(\mathcal{T}_{b+h}^{-1})1 f\|_{L^p_{\gamma-1}} \leq C|h|\,\|f\|_{L^p_\gamma} \quad (5.22)$$

On the other hand, since $\gamma - 1 > 2 - /p$ the operator $\mathcal{T}_b^{-1} S\partial_x : M^{2,p}_{\gamma-2} \to M^{2,p}_{\gamma-3} \times \mathbb{R}^2$ is bounded and we have that

$$\left\|\mathcal{T}_b^{-1} S\partial_x \left(\mathcal{T}_{b+h}^{-1} - \mathcal{T}_b^{-1}\right)^1 f\right\|_{M^{2,p}_{\gamma-3}\times\mathbb{R}^2} \leq \left\|\mathcal{T}_b^{-1} S\partial_x\right\|_{M^{2,p}_{\gamma-2}\to M^{2,p}_{\gamma-3}} \left\|\left(\mathcal{T}_{b+h}^{-1} - \mathcal{T}_b^{-1}\right)^1 f\right\|_{M^{2,p}_{\gamma-2}}.$$

In particular, for $\gamma > 2 - 1/p$ Lemma 40 shows that

$$\|(\mathcal{T}_{b+h}^{-1} - \mathcal{T}_b^{-1}) f\|_{M^{2,p}_{\gamma-2}\times\mathbb{R}^2} \leq |h|\, \|S\partial_x \rho|_{b+h}\|_{L^p_\gamma} \leq \frac{C|h|}{b}\|f\|_{L^p_\gamma}.$$

Hence,

$$\left\|\mathcal{T}_b^{-1} S\partial_x \left(\mathcal{T}_{b+h}^{-1} - \mathcal{T}_b^{-1}\right)^1 f\right\|_{M^{2,p}_{\gamma-3}\times\mathbb{R}^2} \leq \frac{C}{b}|h|\,\|f\|_{L^p_\gamma}. \quad (5.23)$$

Finally, since $M^{2,p}_{\gamma-2} \subset M^{2,p}_{\gamma-3}$, Lemma 44 and the bounds (5.22)–(5.23) show that

$$\left\|(\partial_b R|_{b+h} - \partial_b R|_b) f\right\|_{M^{2,p}_{\gamma-3}\times\mathbb{R}^2} \leq C|h|\,\|f\|_{L^p_\gamma},$$

which concludes the proof of the lemma. ■

5.2.6 Proof of Theorem 7

We conclude the proof of Theorem 7. The following proposition makes precise the way in which we will apply the Implicit Function Theorem.

Proposition 46 *Under assumption* **H3** *and with ϕ_1, a_1, and b_1 as in Section 5.2.1, there exists $\varepsilon_0 > 0$ such that the operator $F_{\varepsilon(b_1+\beta)} : (M^{2,2}_\sigma \times \mathbb{R}^2) \times [0, \varepsilon_0) \to M^{2,2}_\sigma \times \mathbb{R}^2$, defined by*

$$F_{\varepsilon(b_1+\beta)}(\rho, \alpha, \beta; \varepsilon) = [I - \varepsilon\mathcal{T}^{-1}_{\varepsilon(b_1+\beta)}(\tilde{N}_1 + (b_1+\beta)\tilde{N}_2)](\rho, \alpha, \beta),$$

is of class C^1 in (ρ, α, β) and ε. Moreover, its Fréchet derivative $D_{(\rho,\alpha,\beta)}F_{\varepsilon(b_1+\beta)}$ is the identity at $(\rho, \alpha, \beta; \varepsilon) = 0$, hence invertible.

In order to prove proposition 46 we first show differentiability of the nonlinearity $\tilde{N}_1$.

Lemma 47 *The operator* $\tilde{N}_1 : M^{2,2}_\sigma \times \mathbb{R}^2 \to L^2_{\sigma+4}$ *defined by*

$$\begin{aligned}\tilde{N}_1(\rho,\alpha,\beta) =& (\partial_x\phi_1 + \partial_x\rho)^2 - (b_1+\beta)^2(1-S^2) + (a_1+\alpha+(b_1+\beta)x)^2(\partial_x S)^2 \\ &+ 2(\partial_x\phi_1 + \partial_x\rho + (b_1+\beta)S)(a_1+\alpha+(b_1+\beta)x)\partial_x S,\end{aligned}$$

is smooth for $\sigma > 2$ *.*

Proof. Since $\tilde{N}_1$ is a bilinear Nemitskii operator, the result of the proposition follows once we show $\tilde{N}_1 : M^{2,2}_\sigma \times \mathbb{R}^2 \to L^2_{\sigma+4}$ is well defined and bounded as a multilinear map. To that end, first notice that the terms involving $(1-S^2)$ and $\partial_x S$ are exponentially localized, hence belong to $L^2_{\sigma+4}$.
It remains to show that $(\partial_x\phi_1 + \partial_x\rho)^2 \in L^2_{\sigma+4}$. Since $\rho \in M^{2,2}_\sigma$ this term is of the form f^2, with $f \in M^{1,2}_{\sigma+1}$. In particular, $f \cdot \langle x\rangle^{\sigma+1} \in W^{1,2}$, and, by Sobolev embeddings we have $f \cdot \langle x\rangle^{\sigma+1} \in C^0_\mathrm{b}$. Using $\sigma > 2$ now gives the desired bound,

$$\|f^2\|_{L^p_{\sigma+4}} \leq \|f\langle x\rangle^{\sigma+2}\|_{C^0_\mathrm{b}}\|f\|_{L^2_3} \leq \|f\|_{M^{1,2}_{\sigma+1}}\|f\|_{M^{1,2}_{\sigma+1}}.$$

■

Proof of Proposition 46. The results from Subsections 5.2.4 and 5.2.5 (choosing $\gamma = \sigma + 2 > 4 - 1/2$), as well as Lemma 47, and the fact that $\tilde{N}_2 = 2S\partial_x\phi_1 \in L^2_{\sigma+3}$, show that the compositions

1. $\mathcal{N}_1 = \mathcal{T}^{-1}_{\varepsilon(b_1+\beta)}\tilde{N}_1(\rho,\alpha,\beta) : M^{2,2}_\sigma \times \mathbb{R}^2 \to M^{2,2}_\sigma \times \mathbb{R}^2$, and
2. $\mathcal{N}_2 = \varepsilon(b_1+\beta)\mathcal{T}^{-1}_{\varepsilon(b_1+\beta)}\tilde{N}_2(\rho,\alpha,\beta) : M^{2,2}_\sigma \times \mathbb{R}^2 \to M^{2,2}_\sigma \times \mathbb{R}^2$,

are continuously differentiable in a neighborhood of the origin. It is here that we encounter the strong localization of the inhomogeneity stated in hypothesis (H3), which allow us to obtain $\tilde{N}_2 \in L^2_{\sigma+3}$ and use the results from Section 5.2.5 . Here, we are using $\varepsilon > 0$ and $b_1 > 0$, so that $\varepsilon(b_1+\beta) \geq 0$ for $\varepsilon \geq 0$ and $|\beta| < b_1$. Inspecting the dependence on ε, we also readily conclude continuous differentiability in $\varepsilon \geq 0$. At $\varepsilon = 0$, we only have the identity map, which is bounded invertible, so that the Implicit Function Theorem can be applied near the trivial root $(\rho,\alpha,\beta;\varepsilon) = 0$. ■

5.3 Nonlocal array of oscillators

We now return to the problem of nonlocal coupling, (5.7), and the proof of Theorem 6. Throughout, we will assume that $\int J = J_0 = 1$, and $\int x^2 G(x) = G_2 = 1$, possibly after rescaling x and ϕ.

We will see that under Hypothesis (H1), the linearization of (5.7) about the constant solution has similar Fredholm properties as the second derivative. As in the case of the eikonal equation in Section 5.2, we look for solutions of the form

$$\phi(x) = \tilde{\phi}(x) + a\tanh(x) + bx\tanh(x) - b^2 t. \tag{5.24}$$

and analyze the resulting equation, dropping tildes

$$\begin{aligned} 0 =&(-I + G*)(\phi + a\tanh(x) + bx\tanh(x)) + \varepsilon g(x) - 2b(J' * \phi)(J' * x\tanh(x)) \\ &- 2a(J' * \phi)(J' * \tanh(x)) - 2ab(J' * \tanh(x))(J' * x\tanh(x)) \\ &- b^2[(J' * x\tanh(x))^2 - 1] - a^2(J' * \tanh(x))^2 - (J' * \phi)^2. \end{aligned} \tag{5.25}$$

The remainder of this section is organized as follows. In Section 5.3.1, we collect properties of the convolution kernels G and J and establish some elementary properties of the associated linear operators. Section 5.3.2 sets up the proof, introducing first-order approximations and the nonlinear equation that we will solve using the Implicit Function Theorem. We start the proof of our main result, Theorem 6, in Section 5.3.3, subject to several propositions that establish smoothness of nonlinearities and linear operators, which are provided in Sections 5.3.4 and 5.3.5.

5.3.1 Properties of the convolution kernels G and J

The linear part, $-I+G*$, represents nonlocal diffusive coupling in the following sense. Consider the Fourier symbol $-1 + \hat{G}(\ell)$, which is an analytic function on $\ell \in \mathbb{R} \times \mathrm{i}(-\delta, \delta)$, for some δ sufficiently small, due to the exponential localization of G. Moreover, by (H1),

$$-1 + \hat{G}(\ell) < 0, \text{ for } \ell \neq 0 \quad -1 + \hat{G}(\ell) = -\frac{1}{2}G_2\ell^2 + \mathrm{O}(\ell^4), \quad -1 + \hat{G}(\ell) = -\ell^2\hat{G}_b(\ell), \tag{5.26}$$

with $G_b(x)$ exponentially localized, continuously differentiable, and $0 \neq \hat{G}_b(\ell) =: \hat{G}_{-1}^{-1}(\ell)$. The Fourier multiplier $\hat{G}_{-1}(\ell)$ gives rise to an order-two pseudo-differential operator and we formally write

$$G_{-1} * u = (1 - \partial_{xx})(\tilde{G} * u), \qquad G_{-1} * (G_b * u) = u.$$

Here $\tilde{G}$ is an order zero pseudo-differential operator.
Analyticity and exponential localization of $\hat{G}$ give uniform exponential decay of derivatives, which then readily implies bounded mapping properties in algebraically localized spaces, which we summarize below.

Lemma 48 *The convolution operators* $(-I+G): H^2_\gamma \to H^2_\gamma$ *and* $(-I+G): M^{2,2}_\gamma \to H^2_{\gamma+2}$ *are bounded for all* $\gamma \geq 0$.

We note that the inverse of $(-I+G)$ is unbounded, due to the vanishing Fourier symbol at $\ell = 0$. We therefore introduced the kernel G_b in (5.26) through its Fourier symbol. Considerations analogous to Lemma 48 give the following result.

Lemma 49 *The convolution operator* $G_b : L^2_\gamma \to H^2_\gamma$ *is an isomorphism for all* $\gamma \geq 0$.

Similar statements also hold for the convolution operator $J' * u$. Since J is twice continuously differentiable and exponentially localized, we find bounded mapping properties between algebraically localized spaces while gaining one derivative.

Lemma 50 *The convolution operator* $J' : L^2_\gamma \to H^1_\gamma$ *is bounded for all* $\gamma \geq 0$.

5.3.2 Leading-order Ansatz and linear preconditioning

We are interested in finding steady solutions to equation (5.25). Equivalently, we want to find zeros of the operator defined by its right-hand side. From the previous section, we know that the linear part, $-I + G*$, can be written as the local operator ∂_{xx}, up to an invertible convolution operator G_b. Preconditioning with the inverse, G_{-1}, we therefore find a local linear part, but a now slightly more complicated, nonlocal nonlinearity. We will see that the basic strategy of the proof of Theorem 7 is still applicable. We first find leading-order approximations to the solutions of equation (5.25) using the properties of the Laplace operator in Kondratiev spaces.

First order approximations. We scale $\phi = \varepsilon\phi_1, a = \varepsilon a_1, b = \varepsilon b_1$, and find from (5.25) at $\mathrm{O}(\varepsilon)$,

$$0 = \partial_{xx}\phi_1 + a_1\partial_{xx}S + b_1\partial_{xx}(xS) + G_{-1} * g, \quad S = \tanh(x). \tag{5.27}$$

The results from Lemma 37, together with Lemma 49 and our assumption that the function g is in the space $H^2_{\sigma+4}$, show that solutions to equation (5.27) satisfy $\phi_1 \in M^{2,2}_{\sigma+2}$ and

$$a_1 = \frac{1}{2}\int x(G_{-1} * g)\,\mathrm{d}x = \frac{g_1}{G_2}, \quad b_1 = -\frac{1}{2}\int G_{-1} * g\,\mathrm{d}x = -\frac{g_0}{G_2}.$$

As we announced earlier, we will set $G_2 = J_0 = 1$, from now on.

Solution Ansatz. We set $\phi = \varepsilon(\phi_1 + \rho),\ a = \varepsilon(a_1 + \alpha),\ b = \varepsilon(b_1 + \beta)$, and insert this Ansatz into (5.25). Applying the pseudo-differential operator G_{-1} and dividing by ε gives

$$0 = \tilde{F}_\varepsilon(\rho,\alpha,\beta) := \mathcal{T}_{\varepsilon(b_1+\beta)}(\rho,\alpha,\beta) - \varepsilon N_1(\rho,\alpha,\beta) - 2\varepsilon(b_1+\beta)N_2(\rho), \tag{5.28}$$

where

$$\begin{aligned} N_1(\rho,\alpha,\beta) =& G_{-1} * \tilde{N}_1(\phi_1+\rho, a_1+\alpha, b_1+\beta),\\ \tilde{N}_1(\phi,a,b) =& b^2[(J' * xS)^2 - 1] + a^2(J' * S)^2 + (J' * \phi)^2 + 2a(J' * \phi)(J' * S)\\ &+ 2ab(J' * S)(J' * xS) + 2b(J' * \phi)(J' * xS - S), \end{aligned} \tag{5.29}$$

and

$$\begin{aligned} N_2(\rho) =& G_{-1} * (SJ' * (\phi_1 + \rho)) - S\partial_x\rho\\ =& G_{-1}(SJ' * \phi_1) + (G_{-1} - \delta) * (SJ' * \rho) + S(J - \delta) * \partial_x\rho. \end{aligned} \tag{5.30}$$

Here δ represents the Dirac delta distribution.
Preconditioning with the linear part, we may rewrite the equation $\tilde{F}_\varepsilon(\rho,\alpha,\beta) = 0$ as

$$0 = F_{\varepsilon(b_1+\beta)}(\rho,\alpha,\beta;\varepsilon) := [I - \varepsilon\mathcal{T}^{-1}_{\varepsilon(b_1+\beta)}(\tilde{N}_1 + 2(b_1+\beta)N_2)](\rho,\alpha,\beta).$$

We look at $F_{\varepsilon(b_1+\beta)}$ as a nonlinear map to which we would like to apply the Implicit Function Theorem near the trivial solution $(\rho,\alpha,\beta;\varepsilon) = 0$. In particular, we need to show that, for $\sigma > 2$, both,

(i) $T^{-1}_{\varepsilon(b_1+\beta)}N_1 : M^{2,2}_\sigma \times \mathbb{R}^2 \to M^{2,2}_\sigma \times \mathbb{R}^2$, and

(ii) $\varepsilon(b_1+\beta)T^{-1}_{\varepsilon(b_1+\beta)}N_2 : M^{2,2}_\sigma \times \mathbb{R}^2 \to M^{2,2}_\sigma \times \mathbb{R}^2$,

are C^1 in $\varepsilon, \rho, \alpha, \beta$, for $\varepsilon \geq 0$, and using that $b_1 > 0$.

5.3.3 Proof of Theorem 6

We first concentrate on the operator $T^{-1}_{\varepsilon(b_1+\beta)}N_1$ from (i). In Lemma 51 we show that $N_1 : M^{2,2}_\sigma \times \mathbb{R}^2 \to L^2_{\sigma+4}$ is smooth. We then use the fact that for $\sigma + 4 > 4 - 1/2$ the operator $T^{-1}_b : L^2_{\sigma+4} \to M^{2,2}_\sigma$ is linear and C^1 in $b \geq 0$; see Section 5.2.

Lemma 51 *For $\sigma > 2$, the operator $N_1 : M^{2,2}_\sigma \times \mathbb{R}^2 \to L^2_{\sigma+4}$, defined in (5.29), is continuously differentiable in a neighborhood of the origin.*

Proof. Since $G_{-1} : H^2_{\sigma+4} \to L^2_{\sigma+4}$ is bounded and $\tilde{N}_1$ quadratic, it is sufficient to show that $\tilde{N}_1 : M^{2,2}_\sigma \times \mathbb{R}^2 \to H^2_{\sigma+4}$ is bounded. This is immediately clear for all terms except for $(J' * (\phi_1 + \rho))^2$, due to the exponential localization of S' and $(1 - S^2)$. Since J defines a bounded convolution operator, and since $J' * (\phi_1 + \rho) = J * (\partial_x \phi_1 + \partial_x \rho)$, boundedness of this remaining term follows as in Lemma 47. ■

To show continuous differentiability of $\varepsilon(b_1 + \beta)T^{-1}_{\varepsilon(b_1+\beta)}N_2(\rho) : M^{2,2}_\sigma \times \mathbb{R} \to M^{2,2}_\sigma \times \mathbb{R}^2$, we first decompose,

$$N_2(\rho) = D\bar{N}_2(\rho) + G_{-1} * S(J' * \phi_1), \text{ where } D = \partial_x(1 - \partial_x)^{-1},$$

where $\bar{N}_2$ will be made explicit, later. We then show that $\bar{N}_2$ is C^1, Section 5.3.4, and that $bT^{-1}_b D : L^2_{\sigma+2} \to M^{2,2}_\sigma \times \mathbb{R}^2$ is C^1 in $b \geq 0$, with values in the space of operators with norm topology; Section 5.3.5.
Now, hypothesis (H3) implies that the term $\phi_1 \in M^{2,2}_{\sigma+2}$ and so $G_{-1} * S(J' * \phi_1) \in L^2_{\sigma+3}$. Then using Lemma 45 with $\gamma = \sigma + 3$ shows that $\varepsilon(b_1 + \beta)T^{-1}_{\varepsilon(b_1+\beta)}G_{-1} * S(J' * \phi_1)$ is C^1 in β.
Summarizing, we need to show

(i) $\bar{N}_2 : M^{2,2}_\sigma \to L^2_{\sigma+2}$ is continuously differentiable; see results from Section 5.3.4 with $\gamma = \sigma + 2$;

(ii) $bT^{-1}_b D : L^2_{\sigma+2} \to M^{2,2}_\sigma \times \mathbb{R}^2$ is continuously differentiable in $b \geq 0$; see results from Section 5.3.5, with $\gamma = \sigma + 2$.

Theorem 6 then follows in a completely analogous fashion to the proof of Theorem 7.

5.3.4 Decomposition of $N_2(\rho)$ and smoothness of $\bar{N}_2 : M^{2,2}_{\sigma} \to L^2_{\sigma+2}$, for $\sigma + 2 > 0$

We first recall the definition of $N_2(\rho)$,

$$N_2(\rho) = G_{-1}(SJ' * \phi_1) + (G_{-1} - \delta) * (SJ' * \rho) + S(J - \delta) * \partial_x \rho.$$

We will next show that $(J - \delta) = DJ_2$, $(G_{-1} - \delta) = DG_{-2}$, with $D = \partial_x(1 - \partial_x)^{-1}$, and establish operator norm bounds on J_2 and G_{-2}. Preparing for the proof, notice that, for $f \in M^{s,2}_{\gamma}$,

$$\hat{f} \in H^{\gamma}, \quad k\hat{f} \in H^{\gamma+1}, \quad \cdots, \quad k^s \hat{f} \in H^{\gamma+s}.$$

Lemma 52 *For $\gamma > 0$, the convolution operator $(J - \delta)$ can be written as the (commutative) product of $\partial_x(1 - \partial_x)^{-1}$ and J_2, where $J_2 : H^1_{\gamma} \to H^1_{\gamma}$ is bounded. In particular, the composition*

$$M^{1,2}_{\gamma-1} \xrightarrow{\partial_x(1-\partial_x)^{-1}} H^1_{\gamma} \xrightarrow{J_2} H^1_{\gamma},$$

is bounded.

Proof. Boundedness of $\partial_x(1 - \partial_x)^{-1}$ was shown in Chapter 2, Proposition 4. To show boundedness of J_2, we prove the result for γ integer, and conclude the general result by interpolation. We define

$$\hat{J}_2(k) = \frac{1 - ik}{ik}(\hat{J}(k) - 1).$$

Since we normalized $J_0 = 1$, $\hat{J}(k) = 1 + \mathrm{O}(k^2)$, so that $\hat{J}_2$ is analytic and decays at infinity. [1]
Now, suppose $f \in L^2_{\gamma}$. The properties of $\hat{J}_2$ then imply that $\hat{J}_2 \hat{f} \in H^{\gamma}$ and therefore $J_2 : L^2_{\gamma} \to L^2_{\gamma}$ is a bounded convolution operator. Since the convolution commutes with derivatives, we also conclude boundedness on H^1_{γ}. ■

We consider the pseudo-differential operator operator $G_{-1} - \delta$, next.

Lemma 53 *Let $\gamma > 0$ then, the convolution operator $(G_{-1} - \delta)$ can be written as $(G_{-1} - \delta) = \partial_x(1 - \partial_x)^{-1} G_{-2}$, where $G_{-2} : H^2_{\gamma} \to L^2_{\gamma}$ is bounded. In particular, the composition*

$$H^2_{\gamma} \xrightarrow{G_{-2}} L^2_{\gamma} \xrightarrow{\partial_x(1-\partial_x)^{-1}} L^2_{\gamma},$$

is bounded.

[1] Since $\hat{J}_2$ is of order $\mathrm{O}(k)$ near the origin, we could improve the result slightly, here.

Proof. The proof is similar to the proof of Lemma 52, exploiting that $\hat{G}_{-1}(0) = 1$, noticing the normalization $G_2 = 1$. We omit the straight forward adaptation. ■

As a corollary to the two preceding lemmas, we have established the following decomposition.

Corollary 54 *Let $D = \partial_x(1-\partial_x)^{-1}$ then, we can write*

$$N_2(\rho) = D\bar{N}_2(\rho) + N_3,$$

where

1. *the operator $\bar{N}_2 : M^{2,2}_\sigma \to L^2_{\sigma+2}$ defined by*

   $$\bar{N}_2(\rho) = G_{-2} * (SJ' * \rho) + SJ_2 * \partial_x\rho,$$

   is bounded for $\sigma + 2 > 0$;

2. *the constant $N_3 = G_{-1} * S(J' * \phi_1) + [S, D](J_2 * \partial_x\rho)$ lies in $L^2_{\sigma+2}$. In particular, the term $[S, D](J_2 * \rho)$ is exponentially localized.*

Proof. A straightforward calculation shows that the decomposition of $N_2(\rho)$ is as stated in the Corollary. Item (i) follows from Lemmas 52 and 53. In particular, notice that, because $\rho \in M^{2,2}_\sigma$, the function $J' * \rho$ satisfies

$$J' * \rho \in H^1_\sigma, \quad J' * \partial_x\rho \in H^1_{\sigma+1}, \quad J' * \partial_{xx}\rho \in H^1_{\sigma+2},$$

and, since the pseudo-differential operator with G_{-2} is of order two, the term $G_{-2} * (SJ' * \rho)$ belongs to $L^2_{\sigma+2}$.

To establish (ii), we only need to show that the commutator $[S, D](J_2 * \partial_x\rho)$ is exponentially localized, since it was shown already at the end of Subsection 5.3.3 that the term $G_{-1} * S(J' * \phi_1)$ lies in $L^2_{\sigma+3}$.

In what follows we use the fact that $(1-\partial_x)^{-1} : H^2_\eta \to L^2_\eta$ is a bounded invertible operator between exponentially weighted spaces[2] , H^1_η, where

$$H^s_\eta = \{f \in L^2 : f(x)\mathrm{e}^{\eta\langle x\rangle} \in H^s, \quad \eta \in \mathbb{R}\}. \tag{5.31}$$

[2] We use the subscript η to denote exponential weights, and the subscript γ to denote algebraic weights and indicate which weights are referred to when confusion is possible.

Now, let $f = J_2 * \partial_x\rho$ and examine

$$\begin{aligned} [S, D]f =& S\partial_x(1-\partial_x)^{-1}f + \partial_x(1-\partial_x)^{-1}Sf \\ (1-\partial_x)[S, D]f =& \partial_x S(1-\partial_x)^{-1}\partial_x f - \partial_x S f. \end{aligned}$$

Since the right-hand side of this last equality belongs to H^1_η, invertibility of $(1-\partial_x)^{-1}$ implies that the commutator $[S, D]f$ is exponentially localized as well. ■

5.3.5 Differentiability of $bT_b^{-1}D : L^2_\gamma \to M^{2,2}_{\gamma-2}$ for $\gamma > 3/2$

To prove continuous differentiability of $bT_b^{-1}D : L^2_\gamma \to M^{2,2}_{\gamma-2} \times \mathbb{R}^2$ we first establish Lipschitz continuity. Therefore, define

$$Y = M^{2,2}_{\gamma-2} \cap M^{1,2}_{\gamma-1}, \qquad \|f\|_Y = \|f\|_{L^2_{\gamma-1}} + \|f_x\|_{H^1_\gamma}.$$

Note that $D : M^{2,2}_{\gamma-2} \to Y$ is bounded.

Lemma 55 *Fix $\gamma > 3/2$. Then, the operator $bT_b^{-1}D : L^2_\gamma \to Y \times \mathbb{R}^2$ is uniformly bounded and Lipschitz continuous in the parameter $b \geq 0$.*

Proof. We show Lipschitz continuity; uniform bounds can be established in a similar fashion. For $f \in L^2_\gamma$, we need to obtain the following estimate,

$$\left\|(b+h)T^{-1}_{b+h}Df - bT_b^{-1}Df\right\|_{Y\times\mathbb{R}^2} \leq C|h|\,\|f\|_{L^2_\gamma}.$$

We define the auxiliary function g_b, which will monitor commutators between T_b and D. For this, first write

$$T_b(\rho, \alpha, \beta) = \partial_{xx}\rho + 2bS\partial_x\rho + \alpha\partial_{xx}S + \beta\partial_{xx}(xS) = bDf,$$

and set

$$b\partial_x g_b = 2b\partial_x(S\rho) - 2bS\partial_x\rho + \alpha(2\partial_{xxx}S - \partial_{xx}S) + \beta(2\partial_{xxx}(xS) - \partial_{xx}(xS)).$$

One readily notices that the right-hand side of this identity is exponentially localized and verifies that its average vanishes:

$$\begin{aligned} \int b\partial_x g_b &= -\int(\partial_{xx}\rho + 2bS\partial_x\rho + \alpha\partial_{xx}S + \beta\partial_{xx}(xS))\mathrm{d}x + 2\left[\int \partial_{xxx}(\alpha S + \beta xS)\mathrm{d}x\right] \\ &= -\int T_b(\rho, \alpha, \beta)\mathrm{d}x + 2\left[\int \partial_{xxx}(\alpha S + \beta xS)\mathrm{d}x\right] \\ &= -\int bDf\ \mathrm{d}x + 2\left[\int \partial_{xxx}(\alpha S + \beta xS)\mathrm{d}x\right]. \end{aligned}$$

As a consequence,

$$bg_b = 2bS\rho - 2b(\partial_x)^{-1}S\partial_x\rho + \alpha(\partial_{xx}S - D^{-1}\partial_{xx}S) + \beta(\partial_{xx}(xS) - D^{-1}\partial_{xx}(xS)),$$

is well defined and bounded in terms of $bT_b^{-1}Df$. A short calculation shows that

$$bg_b = T_b(D^{-1}\rho,\alpha,\beta) - D^{-1}T_b(\rho,\alpha,\beta),$$

so that

$$\begin{aligned}(\rho,\alpha,\beta)_b &= bT_b^{-1}Df = bD^1T_b^{-1}(f+g_b)\\ (\rho,\alpha,\beta)_{b+h} &= (b+h)T_{b+h}^{-1}Df = (b+h)D^1T_{b+h}^{-1}(f+g_{b+h})\end{aligned}$$

where we used the shorthand $D^1(\rho,\alpha,\beta) = (D\rho,\alpha,\beta)$. The result of the lemma then follows if we can show that

$$\left\|(b+h)D^1T_{b+h}^{-1}(f+g_{b+h}) - bD^1T_b^{-1}(f+g_b)\right\|_{Y\times\mathbb{R}^2} \le C|h|\,\|f\|_{L^2_\gamma}.$$

With this goal in mind, we we use the triangle inequality to obtain

$$\begin{aligned}&\left\|(b+h)D^1T_{b+h}^{-1}(f+g_{b+h}) - bD^1\ T_b^{-1}(f+g_b)\right\|_{Y\times\mathbb{R}^2}\\ &\qquad\le \left\|(b+h)D^1T_{b+h}^{-1}(f+g_{b+h}) - bD^1T_b^{-1}(f+g_{b+h})\right\|_{Y\times\mathbb{R}^2}\\ &\qquad\quad+ \left\|bD^1T_b^{-1}(f+g_{b+h}) - bD^1T_b^{-1}(f+g_b)\right\|_{Y\times\mathbb{R}^2}.\end{aligned}$$

Uniform bounds on $D^1T_b^{-1} : L^2_\gamma \to Y\times\mathbb{R}^2$, which follow immediately from Corollary 41 (with $\gamma > 2 - 1/p = 3/2$) and the definition of Y, allow us to simplify this estimate further,

$$\begin{aligned}&\left\|(b+h)D^1T_{b+h}^{-1}(f+g_{b+h}) - bD^1T_b^{-1}(f+g_b)\right\|_{Y\times\mathbb{R}^2}\\ &\qquad\le \left\|(b+h)D^1T_{b+h}^{-1}(f+g_{b+h}) - bD^1T_b^{-1}(f+g_{b+h})\right\|_{Y\times\mathbb{R}^2}\\ &\qquad\quad+ C|b|\,\|g_{b+h} - g_b\|_{L^2_\gamma}.\end{aligned}$$

The next step is to show that the operator $bD^1T_b^{-1} : L^2_\gamma \to Y\times\mathbb{R}^2$ is continuous in b and that the difference $g_b \in L^2_\gamma$ is Lipschitz in b. To show the continuity in b of $bD^1T_b^{-1}$ we let

$$(\rho,\alpha,\beta)|_b = bD^1T_b^{-1}f, \quad (\rho,\alpha,\beta)|_{b+h} = (b+h)D^1T_{b+h}^{-1}f,$$

and show the inequality,

$$\|(\rho,\alpha,\beta)|_{b+h} - (\rho,\alpha,\beta)|_b\|_{Y\times\mathbb{R}^2} \le C|h|\|f\|_{L^2_\gamma}.$$

Letting $\psi = D^{-1}\rho$, the expression $(\rho, \alpha, \beta) = bD^1 T_b f$ can be written as $T_b(\psi, \alpha, \beta) = bf$, and a short calculation shows that the difference between $(\psi, \alpha, \beta)|_b$ and $(\psi, \alpha, \beta)|_{b+h}$, denoted by $(\Delta\psi, \Delta\alpha, \Delta\beta)$, satisfies the equations

$$T_b(\Delta\psi, \Delta\alpha, \Delta\beta) = hf - 2hS\,\partial_x\psi|_{b+h}.$$

Now, since the function $\psi|_{b+h}$ is a solution to $T_{b+h}(\psi, \alpha, \beta) = (b+h)f$, the results from Lemma 40 with $\gamma > 1 - 1/p = 1/2$ and $\mathcal{D} = \{u \in M^{2,2}_{\gamma-2} : u_x \in L^2_\gamma\}$, show that

$$\|\psi\|_{\mathcal{D}} \leq \frac{C}{|b+h|}\|(b+h)f\|_{L^2_\gamma}.$$

In particular, we see that $\partial_x\psi|_{b+h}$ is in the space L^2_γ and therefore, solutions to

$$T_b(\Delta\psi, \Delta\alpha, \Delta\beta) = hf - 2hS\,\partial_x\psi|_{b+h},$$

with $\gamma > 2 - 1/p = 3/2$, satisfy the inequality

$$\|(\Delta\psi, \alpha, \Delta\beta)\|_{M^{2,2}_{\gamma-2}\times\mathbb{R}^2} \leq C|h|\,\|f - 2S\,\partial_x\psi|_{b+h}\|_{L^2_\gamma} \leq Ch\|f\|_{L^2_\gamma}. \tag{5.32}$$

Lastly, because the operator D^{-1} is linear, the difference $\Delta\psi = D^{-1}\Delta\rho$ is in $M^{2,2}_{\gamma-2}$ and we may conclude that $\Delta\rho \in Y$. Therefore,

$$\|\Delta\rho, \Delta\alpha, \Delta\beta)\|_{Y\times\mathbb{R}^2} \leq Ch\|f\|_{L^2_\gamma},$$

as desired. Finally, we show that

$$|b|\,\|g_{b+h} - g_b\|_{L^2_\gamma} \leq C|h|\,\|f\|_{L^2_\gamma}.$$

Notice that, writing $\psi_b = D^{-1}\rho_b$,

$$T_b(\psi_b, \alpha_b, \beta_b) = b(f + g_b), \qquad T_{b+h}(\psi_{b+h}, \alpha_{b+h}, \beta_{b+h}) = (b+h)(f + g_{b+h}).$$

Subtracting both equations and using the triangle inequality, we find that

$$\begin{aligned}
|b|\,\|g_{b+h} - g_b\|_{L^2_\gamma} \leq & h\|f + g_{b+h}\|_{L^2_\gamma} + \|T_{b+h}(\psi_{b+h}, \alpha_{b+h}, \beta_{b+h}) - T_b(\psi_b, \alpha_b, \beta_b)\|_{L^2_\gamma} \\
\leq & h\|f + g_{b+h}\|_{L^2_\gamma} + \|T_{b+h}(\psi_{b+h}, \alpha_{b+h}, \beta_{b+h}) - T_b(\psi_{b+h}, \alpha_{b+h}, \beta_{b+h})\|_{L^2_\gamma} \\
& + \|T_b(\psi_{b+h}, \alpha_{b+h}, \beta_{b+h}) - T_b(\psi_b, \alpha_b, \beta_b)\|_{L^2_\gamma} \\
\leq & h\|f + g_{b+h}\|_{L^2_\gamma} + h\|2S\,\partial_x\psi_{b+h}\|_{L^2_\gamma} + \|h(f + 2S\,\partial_x\psi_{b+h})\|_{L^2_\gamma} \\
\leq & h\|f + g_{b+h}\|_{L^2_\gamma} + h\|2S\,\partial_x\psi_{b+h}\|_{L^2_\gamma} + hC\|f\|_{L^2_\gamma} \\
\leq & h\|f\|_{L^2_\gamma},
\end{aligned}$$

where we used that $\|\partial_x\psi_b\|_{L^2_\gamma} \le \|\psi_b\|_{\mathcal{D}} \le \|f+g\|_{L^2_\gamma}$ from Lemma 40 ($\gamma > 1/2$). This completes the proof. ■

We are now ready to show the differentiability of the operator $bT_b^{-1}D : L^2_\gamma \to M^{2,2}_{\gamma-2} \times \mathbb{R}^2$.

Lemma 56 *Fix $\gamma > 3/2$. Then the operator $bT_b^{-1}D : L^2_\gamma \to M^{2,2}_{\gamma-2} \times \mathbb{R}^2$ is differentiable in the parameter $b \ge 0$ with Lipschitz continuous derivative.*

Proof. We abbreviate $R = bT_b^{-1}D$ and recall the notation $(T_b^{-1})^1 f$ for the first component of $T_b^{-1}f$. We first define the candidate for the derivative,

$$\partial_b R|_b f = 2bT_b^{-1}S\,\partial_x(T_b^{-1})^1 Df + T_b^{-1}Df,$$

and show that

$$\|(R|_{b+h} - R|_b)f - h\partial_b R|_b f\|_{M^{2,2}_{\gamma-2}\times\mathbb{R}^2} = \mathrm{O}(h^2).$$

A short calculation shows that the difference

$$R|_{b+h}f - R|_b f = (\rho,\alpha,\beta)|_{b+h} - (\rho,\alpha,\beta)|_b = (\Delta\rho, \Delta\alpha, \Delta\beta)$$

satisfies the equation $T_b(\Delta\rho, \Delta, \alpha, \Delta\beta) = -2hS\,\partial_x\rho|_{b+h} + hDf$. Therefore,

$$\begin{aligned}
&\|(R|_{b+h} - R|_b)f - h\partial_b R|_b f\|_{M^{2,2}_{\gamma-2}\times\mathbb{R}^2} \\
&\quad= \left\| \left[-2h(b+h)T_b^{-1}S\,\partial_x(T_{b+h}^{-1})^1 Df + hT_b^{-1}Df\right] - \left[-2hbT_b^{-1}S\,\partial_x(T_b^{-1})^1Df - hT_b^{-1}Df\right] \right\|_{M^{2,2}_{\gamma-2}\times\mathbb{R}^2} \\
&\quad= |2h| \left\| T_b^{-1}S\,\partial_x \left[(b+h)T_{b+h}^{-1} - bT_b^{-1}\right]^1 Df \right\|_{M^{2,2}_{\gamma-2}\times\mathbb{R}^2} \\
&\quad\le |2h| \left\| T_b^{-1}S\,\partial_x \right\|_{Y\to M^{2,2}_{\gamma-2}\times\mathbb{R}^2} \left\| \left[(b+h)T_{b+h}^{-1} - bT_b^{-1}\right]^1 Df \right\|_Y,
\end{aligned}$$

where the last inequality follows from the continuity of the operators

$$L^2_\gamma \xrightarrow{\left[(b+h)T_{b+h}^{-1} - bT_b^{-1}\right]^1 D} Y \xrightarrow{S\,\partial_x} L^2_\gamma \xrightarrow{T_b^{-1}} M^{2,2}_{\gamma-2} \times \mathbb{R}^2.$$

Using the results from Lemma 55, where we showed that for $\gamma > 3/2$ the operator $bT_b^{-1}D : L^2_\gamma \to Y \times \mathbb{R}^2$ is continuous with respect to the parameter b, we see that

$$\|(R|_{b+h} - R|_b)f - h\partial_b R|_b f\|_{M^{2,2}_{\gamma-2}\times\mathbb{R}^2} \le C|h^2|\,\|f\|_{L^2_\gamma} \tag{5.33}$$

as desired.This proves differentiability. It remains to establish continuity of the derivative,

$$\|(\partial_b R|_{b+h} - \partial_b R|_b)f\|_{M^{2,2}_{\gamma-2}\times\mathbb{R}^2} \le C|h|.$$

We split the difference into

$$\begin{aligned}
\|(\partial_b R|_{b+h} \ &-\partial_b R|_b)f\|_{M^{2,2}_{\gamma-2}\times\mathbb{R}^2} \\
&= \left\|(b+h)T^{-1}_{b+h} S\,\partial_x (T^{-1}_{b+h})^1 Df - bT_b^{-1} S\,\partial_x (T_b^{-1})^1 Df\right\|_{M^{2,2}_{\gamma-2}\times\mathbb{R}^2} \\
&\le 2|b+h| \left\|\left[T^{-1}_{b+h} - T^{-1}_b\right]\left(S\,\partial_x (T^{-1}_{b+h})\right)^1 Df\right\|_{M^{2,2}_{\gamma-2}\times\mathbb{R}^2} + 2\left\|T_b^{-1} S\,\partial_x\left[(b+h)T^{-1}_{b+h} - bT_b^{-1}\right]^1 Df\right\|_{M^{2,2}_{\gamma-2}\times\mathbb{R}^2} \\
&\le 2|b+h|\frac{C\,|h|}{|b+h|}\left\|S\,\partial_x (T^{-1}_{b+h})^1 Df\right\|_{L^2_\gamma} + 2\left\|T_b^{-1} S\,\partial_x\right\|_{Y\to M^{2,2}_{\gamma-2}\times\mathbb{R}^2}\left\|\left[(b+h)T^{-1}_{b+h} - bT_b^{-1}\right]^1 Df\right\|_Y .
\end{aligned}$$

This last inequality follows from the continuity of the operator $T_b^{-1} : L^2_\gamma \to \mathcal{D}$, see Lemma 40 with $\gamma > 1/2$, and the boundedness of the composition (5.33). Then, using again the results from Lemma 55 with $\gamma > 3/2$, we find that

$$\begin{aligned}
\|(\partial_b R|_{b+h} - \partial_b R|_b)f\ &\|_{M^{2,2}_{\gamma-2}\times\mathbb{R}^2} \\
&\le 2C|h|\left\|S\,\partial_x (T^{-1}_{b+h})^1 Df\right\|_{L^2_\gamma} + 2h\left\|T_b^{-1} S\,\partial_x\right\|_{Y\to M^{2,2}_{\gamma-2}\times\mathbb{R}^2}\|f\|_{L^2_\gamma},
\end{aligned} \tag{5.34}$$

which shows Lipschitz continuity of the derivative and concludes the proof. ■

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